Mathematical Induction

- for proving properties of the natural numbers $\boxed{\forall n \geq c \ P(n)}$ where $P(n)$ means "the property holds of n"

$\forall n \ n \geq c \rightarrow P(n)$

$1 + \cdots + n$

$n + \cdots + 1$

$\underbrace{(n+1) + \cdots + (n+1)}_{n \text{ times}}$

$n(n+1)$

$\forall k \ red(k) \rightarrow red(k+1)$

$\wedge$

$\cancel{P(c)}\ red(c)$

$\Longrightarrow \forall n \geq c \ red(n)$

First Principle of Mathematical Induction

inductive case

$$P(c) \wedge \left[\forall k \geq c \ \boxed{P(k)} \rightarrow P(k+1)\right] \longrightarrow \forall n \geq c \ P(n)$$

base case

induction hypothesis (IH)

e.g. $P(n) \equiv \boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

Show $\forall n \geq 0\ P(n)$ where $P(n) \equiv \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

by induction,

Base case has $n=0$, show $\left(P(0), \text{ i.e. }\right) \sum_{i=1}^{0} i = \frac{0(0+1)}{2}$

$0 = 0$ ✓

Consider arbitrary $k \geq 0$ and show $P(k) \to P(k+1)$ using deduction method

Assume $P(k)$, i.e. suppose for induction $\boxed{\sum_{i=1}^{k} i = \frac{k(k+1)}{2}}$ and show $P(k+1)$, i.e. $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

induction hypothesis

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k+1 = \underbrace{\frac{k(k+1)}{2}}_{\text{by IH}} + k+1 = \frac{(k+1)(k+2)}{2}$$

✓

Prove $\forall n \geq 1 \;\; n < 2^n$

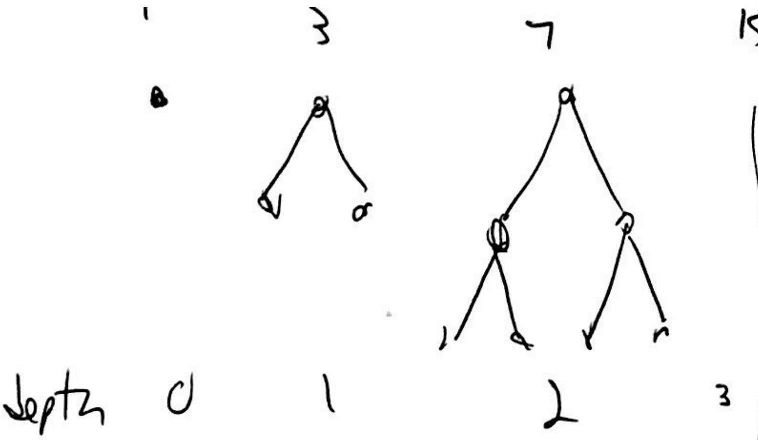
$\forall n \geq 1 \;\; P(n)$

$P(k) \equiv k < 2^k$

Proof

$1 < 2^1 = 2$ so property holds at 1.

Suppose $\boxed{k < 2^k}$ for induction then $k+1 \underset{\text{IH}}{<} 2^k + 1 \leq 2^k + 2^k = 2^{k+1}$



There are $\underline{2^{n+1}-1}$ nodes in a complete binary tree of depth n.

Show by induction.

There are $2^{0+1}-1=1$ node in a depth 0 tree, for base case.

Consider arb. $k \geq 0$ and suppose for induction there are $2^{k+1}-1$ nodes in a depth k binary tree. Then a depth $k+1$ binary tree has $2(2^{k+1}-1)+1$ nodes, as desired $\square$

$$\begin{gathered} 2(2^{k+1}-1)+1 \\ \| \\ 2^{k+2}-1 \end{gathered}$$