What is the difference between $\rightarrow$ and $\models$?

$\rightarrow$ is a syntactic construct for building formulas, so it occurs in wffs.

$\models$ is used when we talk about logic, so it is meta-logical.

$\{A, A\rightarrow B\} \models C$ abbreviates an English sentence, but is not itself a logical formula.

$\Sigma \cancel{\rightarrow} \varphi$

<u>Deduction Theorem</u>:

$$\Sigma \models \alpha \rightarrow \beta \quad \text{iff} \quad \Sigma \cup \{\alpha\} \models \beta$$

Proof of $\Sigma \models \alpha \to \beta$ iff $\Sigma \cup \{\alpha\} \models \beta$

i.e. $\Sigma \models \alpha \to \beta$ implies $\Sigma \cup \{\alpha\} \models \beta$

and

* $\underline{\Sigma \cup \{\alpha\} \models \beta}$ implies $\underline{\Sigma \models \alpha \to \beta}$

Show *

Assume $\Sigma \cup \{\alpha\} \models \beta$. <u>Thus every model of $\Sigma \cup \{\alpha\}$ is a model of β.</u>

Show $\Sigma \models \alpha \to \beta$. i.e. Show every model of Σ is a model of $\alpha \to \beta$.

Consider an arbitrary model m making Σ true and show m is a model of $\alpha \to \beta$.

case 1. m makes α true, use assumption
- easy to show m makes $\alpha \to \beta$ true

case 2. m does not make α true.
- easy to show m makes $\alpha \to \beta$ true

Since checking models is intractable, we introduce line by line proof.

$\Sigma \vdash \varphi$ means φ has been "proven" from premises Σ.

we design "proven" so $\Sigma \models \varphi$ iff $\Sigma \vdash \varphi$

soundness: $\Sigma \vdash \varphi$ implies $\Sigma \models \varphi$

completeness: $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$ if φ is entailed ~~from~~ Σ then there is a "proof" for it from Σ

<u>modus ponens</u> : (mp)

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$ "can conclude"

$$\frac{A \wedge (B \vee C) \qquad A \wedge (B \vee C) \to A}{A}$$

(soundness easy to see) $\{A, B, A \to (B \to C)\} \models C$ by proof:

1. A premise
2. $A \to (B \to C)$ premise
3. $B \to C$ 1, 2, mp
4. B
5. C } 4, mp

so $\{A, B, A \to (B \to C)\} \vdash C$ then via soundness

$\{\} \models (A \to (A \to B)) \to (A \to B)$ use deduction method.

1. $A \to (A \to B)$ Deduction method Premise

// now show $\underline{A \to B}$

2. A Deduction method premise

// now show $\underline{B}$

3. $A \to B$ 1, 2, mp

4. $\underline{B}$ 2, 3, mp

5. $A \to B$ 2–4 deduction metho

6. $\cancel{A \to (A \to B)}$ 1–5 ded. method

$[A \to (A \to B)] \to (A \to B)$