$\{\} \models (A \to (A \to B)) \to (A \to B)$ use deduction method.

assume temporarily

1. $A \to (A \to B)$ Deduction method Premise

// now show $A \to B$

2. A Deduction method Premise

// now show B

3. $A \to B$ 1, 2, mp
4. B 2, 3, mp
5. $A \to B$ 2–4 deduction metho (discharges assumption in line 2).
6. ~~$A \to (A \to B)$~~ 1–5 ded. method

$[A \to (A \to B)] \to (A \to B)$ (discharges line 1 assumption)

★ Do not cite lines 2–4 any more

$\{\} \models (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ use deduction method.

assume temporarily

1. $A \rightarrow (A \rightarrow B)$ Deduction method Premise

//now show $A \rightarrow B$

2. A Deduction method premise

//now show B

3. $A \rightarrow B$ 1,2, mp
4. B 2,3, mp
5. $A \rightarrow B$ 2–4 deduction metho (discharges assumption in line 2).

☆ Do not cite lines 2–4 anymore

6. ~~$A \rightarrow (A \rightarrow B)$~~ 1–5 ded. method

$[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$ (discharges line 1 assumption)

Oh, so show $\{A \to (A \to B), A\} \vdash B$

(actually: take premises $A \to (A \to B)$ and A, and show B)

1.

2.

3. $A \to B$ 1, 2, mp

4. B 2, 3, mp

What if φ is $A \to (A \to (A \to B))$

Take as premises A, A, and A and show B.

- ~~not going~~ to happen

<u>OR</u>

1. A ded method premise

// now show $A \to (A \to B)$

2. A ded method premise

// now show $A \to B$

3. A ded method premise

// now show B

4.

Show $\{A \to B,\ B \to C\} \models A \to C$

hypothesis $\equiv$ premise

i.e. Show $\big((A \to B) \wedge (B \to C)\big) \to (A \to C)$ is a tautology

or $(A \to B) \to \big((B \to C) \to (A \to C)\big)$

1. $A \to B$ — ded method premise
2. $B \to C$ — ded method ″
3. A — ded method ″

// prove C

Formalizing English Arguments

1. identify true/false statements and assign variables to them

$A \equiv$ Jack fetches pail

$B \equiv$ Jack works hard

$C \equiv$ Jack is paid

if A, B	$A \to B$
B if A	$A \to B$
B only if A	$B \to A$

2. translate argument using letters and Boolean combinations

A   A only if B   B only if C $\therefore$ C

$\{A, \; A \to B, \; B \to C\} \models C$?? yes

using either $\vdash$ or $\models$

Logics

To define a *logic*, answer three questions:

1. What are the *models*?
2. What are the *formulas*?
3. Which formulas are true in which models?

A logic is a formal system relating
syntax (formulas) and
semantics (models of the world).

Propositional Logic: Models

- Covered on the overhead projector

Propositional Logic: Formulas

- Covered on the overhead projector

Example formulas and non-formulas

- Covered on the overhead projector

Precedence in formulas

- See p. 6 in text
- Covered on the overhead projector

When is a formula true in a model?

- Each Boolean connective has a *truth table*

e.g.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	$\neg P$
T	F
F	T

What about the other connectives?

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Tricky cases

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

"P is the same as Q"

Using truth tables to evaluate formulas

- Given the truth tables for the connectives, it is easy to tell when a formula is true in a model.

The formula

			T F	T F	F F
A	B	C	$(A \vee B) \rightarrow (C \vee B)$	$A \vee B$	$C \vee B$
T	F	F	??? F	T	F

The model

Terminology

- If a formula Φ is true in a model *M*, we say *M* *is a model of* Φ.

Entailment

- Given some *premises*

$$A$$

$$B \rightarrow C$$

$$(A \wedge B) \rightarrow (D \vee \neg C)$$

$$B$$

- When are we willing to *conclude* some new query formula?

Entailment (continued)

Let *P* be the conjunction of the premises:

$$A \wedge (B \rightarrow C) \wedge ((A \wedge B) \rightarrow (D \vee \neg C)) \wedge B$$

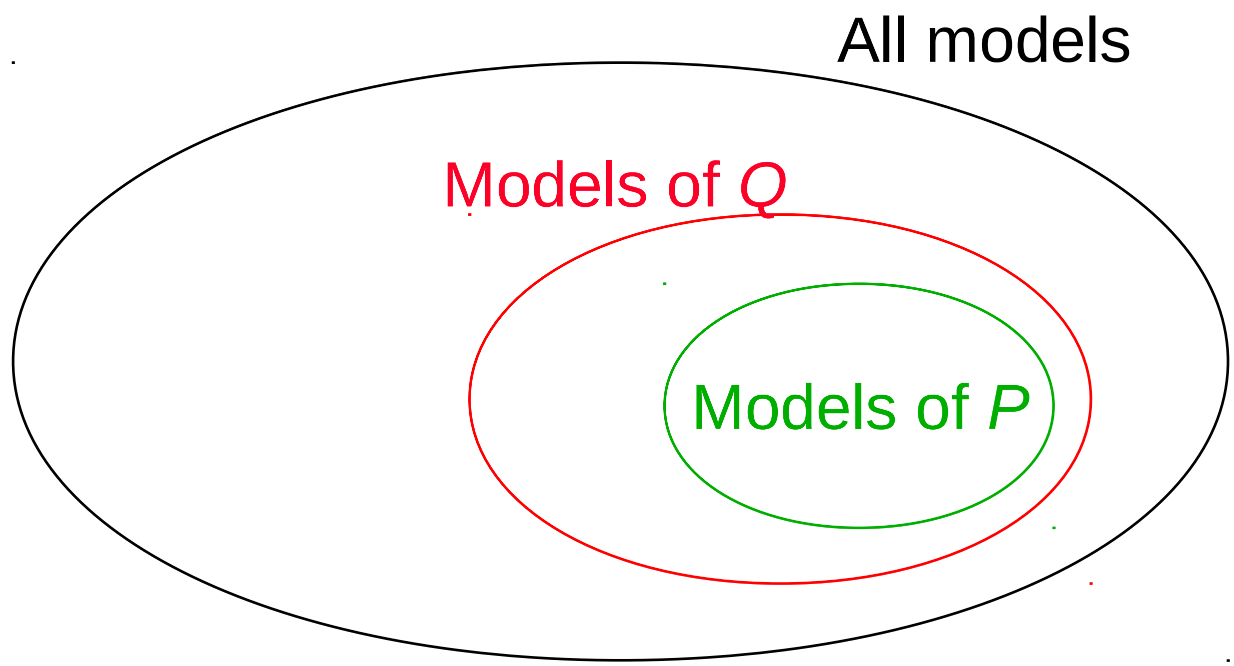
- Believing *P* is true involves committing to a class of models:
  - Those models that make *P* true

- <u>We should be willing to also believe any formula that is true in all those models.</u>

Entailment defined

We say that *P* *entails* *Q* whenever

- <u>Every</u> model making *P* true makes *Q* true.
- …written *P* |= *Q*



Tautologies and Contradictions

- A *tautology* is a formula that is true in <u>every</u> model. (also called a *theorem*)
  - for example, $(A \lor \neg A)$ is a tautology
  - …written $\models (A \lor \neg A)$
- A *contradiction* is a formula that is <u>false</u> in every model.
  - for example, $(A \land \neg A)$ is a contradiction
  - …written $\models \neg(A \land \neg A)$

An example

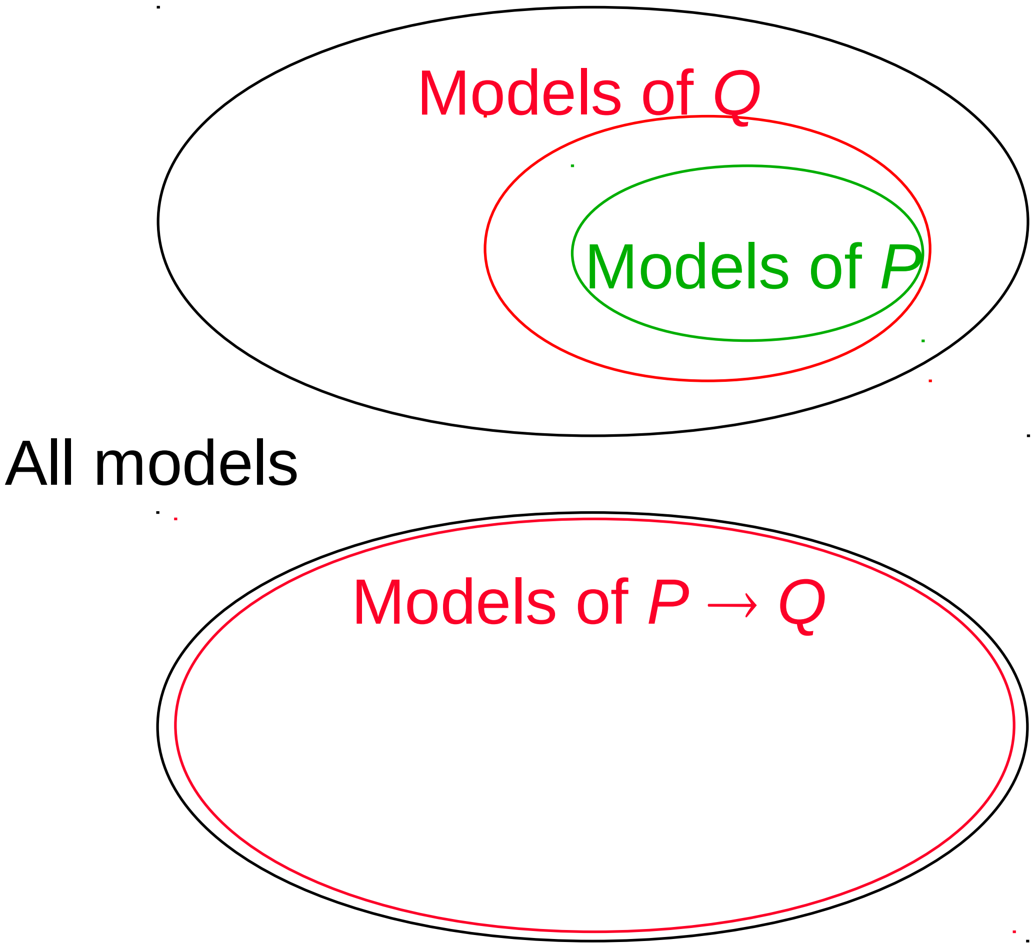
What is the difference between:

1. $P \models Q$ (P entails Q)

2. $\models P \rightarrow Q$ ($P \rightarrow Q$ is a tautology)

Expand the definition of |= for each.

Expanding the definition of |=



$P \models Q$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\models P \rightarrow Q$

<u>Surprise</u>: these are actually the same thing!
(this result is called the *deduction theorem*)

Demonstrating entailment

A straightforward argument for entailment requires checking all models (“model checking”)

- But there are exponentially many models, so this can be prohibitively expensive.
- For the worst case, it is believed there is no better approach
  - this is so unless P = NP (covered later)
- But we can give an an approach that often works better, called a <u>proof system</u>

Example of model checking

- Covered in class on overhead projector

A Proof System

- A *proof system* is a syntactic system for finding formulas entailed by your premises.
  - "syntactic" means manipulating syntax
    - i.e. manipulating formulas rather than models.

- A *proof* is a sequence of formulas, where each formulas in the sequence is either
  - a premise
  - a formula justified by previous formulas and a proof rule.

- The sequence "proves" the last formula.

Example proof rule

$$\frac{P \qquad P \rightarrow Q}{Q}$$

modus ponens (*mp*)

Given the formulas above the line, we can add the formula below the line to the proof.

Example proof

- Premises: C, B, $B \rightarrow (C \rightarrow A)$
- Conclusion: A

1.	B	*premise*
2.	$B \rightarrow (C \rightarrow A)$	*premise*
3.	$C \rightarrow A$	1, 2, *mp*
4.	C	*premise*
5.	A	3, 4, *mp*

Proof Rules: Inference Rules

From	Can derive	Rule name	Abbr.
$P,\ P \rightarrow Q$	Q	Modus ponens	mp
$P \rightarrow Q,\ \neg Q$	$\neg P$	Modus tollens	mt
$P,\ Q$	$P \wedge Q$	Conjunction	con
$P \wedge Q$	$P,\ Q$	Simplification	sim
P	$P \vee Q$	Addition	add

Proof rules: Equivalence rules

Expression	Equivalent	Rule name	Abbr.
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	Commutative	comm
$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	Associative	ass
$\neg(P \vee Q)$ $\neg(P \wedge Q)$	$\neg P \wedge \neg Q$ $\neg P \vee \neg Q$	De Morgan	dm
$P \rightarrow Q$	$\neg P \vee Q$	Implication	imp
$\neg(\neg P)$	P	Double Neg.	dn
$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge$ $(Q \rightarrow P)$	Equivalence	equ

Another Example Proof

Show D from

A, B, B $\rightarrow$C, and (A $\wedge$ B) -> (D $\vee$ C')

- Covered in class on overhead projector

Deduction Method in Proofs

- The deduction theorem allows a natural method for proving implications.
  - Recall, the deduction theorem:
    $P \models Q$ if and only if $\models P \rightarrow Q$
- When proving $P \rightarrow Q$…
  - add P to premises and prove Q.
  - (you are proving $P \models Q$ instead)
- Repeat to prove $P \rightarrow (Q \rightarrow R)$
  - add P and Q to premises and prove R.

Example proof using deduction method

Prove $\models [A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$.

1.	$[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$	ded. method
	a. $A \rightarrow (A \rightarrow B)$	ded. method premise
	b. $A \rightarrow B$	ded. method
	i. A	ded. method premise
	ii. $A \rightarrow B$	1a, 1bi, mp
	iii. B	1bi, 1bii, mp

- Each ded. method line must be justified by a nested proof of its right-hand side which can use the left-hand side as a “ded. method premise”
- Proofs can only cite lines from enclosing nesting levels.
  - i.e. once a nesting level is closed, its lines cannot be cited

Formalizing English Arguments

To formalize an English argument:

1. Find the minimal *statements* in the argument and symbolize them with propositional letters A, B, …

2. Convert English connectives to propositional ones.

3. Give a proof of the conclusion using the premises.

Examples

Jack went to fetch a pail of water. Jack fetches a pail of water only if Jack works hard. Jack works hard only when Jack is paid. Therefore Jack was paid.

Minimal True/False Statements

<u>Jack went to fetch a pail of water</u>. <u>Jack fetches a pail of water</u> only if <u>Jack works hard</u>. <u>Jack works hard</u> only if <u>Jack is paid</u>. Therefore <u>Jack was paid</u>.

A = Jack fetches a pail of water
B = Jack works hard
C = Jack is paid

A. A only if B. B only if C. Therefore C.

Eliminating connectives

A. *A* only if *B*. *B* only if *C*. Therefore *C*.

becomes

Premises: A, $A \rightarrow B$, $B \rightarrow C$.
Conclusion: C.

Easy to prove that these premises entail this conclusion using rule mp.

Examples — a bad argument

Jack went to fetch a pail of water. Jack fetches a pail of water if Jack works hard. Jack works hard if Jack is paid. Therefore Jack was paid.

Premises: A, $B \rightarrow A$, $C \rightarrow B$.
Conclusion: C.

Premises do <u>not</u> entail conclusion…to show this, give a model of the premises that makes the conclusion false.

- A=T, B=T, C=F

Another example

Fish can walk. Fish can walk only if elephants can fly. Elephants can fly only if eggplants can talk. Therefore, eggplants can talk.

Is this a valid argument?

Examples where propositional logic fails

Every positive number is greater than zero. Five is a positive number. Therefore, five is greater than zero.

Minimal statements?

A = Every positive number is greater than zero.
B = Five is a positive number.
C = Five is greater than zero.

Premises: A, B. Conclusion: C.
Conclusion not entailed (consider A = B = T, C = F)

Our logic does not model the internal structure of the propositions