Enumerating a set S as $S = \{a_0, a_1, a_2 \cdots\}$

shows $|S| = |\mathbb{N}|$ by $0 \longleftrightarrow a_0 \cdots i \longleftrightarrow a_i \cdots$

12	18	4	---	
2	4	6	8	----

A set is <u>countable</u> if it is finite or the same size as $\mathbb{N}$.

<u>Thm</u> The union of two countable sets is countable.

Pf. Given using that S_1 & S_2 are countable.

$S_1 = \{s_{1,0}\ s_{1,1}\ s_{1,2}\ s_{1,3} \cdots\}$

$S_2 = \{s_{2,0}\ s_{2,1}\ s_{2,2}\ s_{2,3} \cdots\}$

Show $S_1 \cup S_2$ is countable.

WLOG $S_1 \cap S_2 = \emptyset$.
(no overlap)

$S_1 \cup S_2 = \{s_{2,0}, s_{1,0}, s_{2,1}, s_{1,1}, s_{2,2}, s_{1,2}, \cdots\}$

What if I have infinitely many countable sets

$S_0 \quad S_1 \quad S_2 \quad S_3 \quad \cdots$ and take their union $\bigcup_{i \in \mathbb{N}} S_i$

Thm If each S_i is countable then $\boxed{\bigcup_{i \in \mathbb{N}} S_i}$ is too!

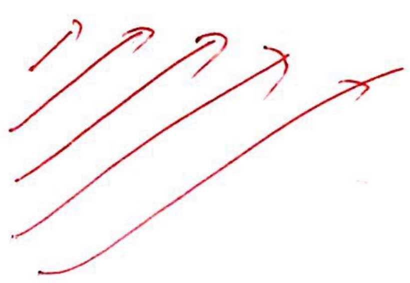
Then enumerate this

Pf. Given

$$
\begin{aligned}
S_0 &= \{ s_{00} \quad s_{01} \quad s_{02} \quad s_{03} \quad \cdots \} \\
S_1 &= \{ s_{10} \quad s_{11} \quad s_{12} \quad s_{13} \quad \cdots \} \\
S_2 &= \{ s_{20} \quad s_{21} \quad s_{22} \quad s_{23} \quad \cdots \} \\
S_3 &= \{ s_{30} \quad s_{31} \quad s_{32} \quad s_{33} \quad \cdots \} \\
&\vdots
\end{aligned}
$$

etc.

enumerate s_{ij} by the order of sum $i+j$ prefering big i to break ties.



Ex
rationals formed from numerator $x \in \mathbb{N}$ and denominator $y \in \mathbb{N} - \{0\}$ are countable.

Take $S_0 = \{\}$ $S_i = \{ x/i \mid x \in \mathbb{N} \}$ for $i > 0$

Each S_i is countable.

then $\bigcup_{i \in \mathbb{N}} S_i$ is the desired set and is also countable.

Thm (Cantor's) to prove: $\mathbb{R}$ is uncountable.

$\boxed{\text{Thm} \quad |P(S)| > |S| \text{ for any set } S.}$

$|S| > |T|$ means not $|S| \leq |T|$

proof technique: Diagonalization.

$\mathbb{N}$ $P(\mathbb{N})$ $P(P(\mathbb{N}))$ --- no biggest infinity

Thm There is no enumeration of $\mathbb{R}$.

Pf Consider an enumeration of numbers in $[0, 1)$

position

$$\begin{array}{lllllll}
0: & 0. & d_{00} & d_{01} & d_{02} & d_{03} & \cdots \\
1: & 0. & d_{10} & d_{11} & d_{12} & d_{13} & \cdots \\
2: & 0. & d_{20} & d_{21} & d_{22} & d_{23} & \cdots \\
3: & 0. & d_{30} & d_{31} & d_{32} & d_{33} & \cdots \\
 & \vdots & & & & & \ddots
\end{array}$$

choose digit c_i to be different from d_{ii}

Then $0.\, c_0\, c_1\, c_2\, c_3 \cdots$ cannot match any position i since $c_i \neq d_{ii}$