Sample Exam

ECE 369

Name: ______________________________

Read all of the following information before starting the exam:

1. **NOTE:** Unanswered questions are worth 25% credit, rounded down. Writing any answer loses this "free" credit, which must be earned back by the quality of the answer. If you wish a question to be treated as unanswered, but you have written on it, clearly write "DO NOT GRADE" in the answer area. In a multi-part question, unanswered *parts* are worth 25%. This applies only to parts explicitly separated on the exam, you may not define "parts" of a question yourself.
2. Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
3. No calculators, or materials other than pen/pencil and blank paper are allowed except those we distribute during the exam.
4. Please keep your written answers brief; be clear and to the point. Points will be deducted for rambling and for incorrect or irrelevant statements.
5. This test has 6 problems, each of which is approximately equal in value, except where indicated. Multi-part problems divide the problem score approximately equally among the parts.
6. Good luck!

1. Consider adding one new "proof rule" to our proof system, called "converse." This rule allows the conclusion $P \to Q$ whenever we have a line $Q \to P$, for any formulas P and Q. You will need this rule once in the proofs for this question. Number each proof line you write in the different parts with new numbers so you can refer back between parts, constructing one long proof. In each part, show only the new line(s).

a. What happens to our proof system with this rule added? Explain, mentioning entailment.

b. Use the deduction method (without using the new rule) to prove $A \to A$.

c. Extend the part b proof to show $(\neg[A \wedge \neg A]) \vee [A \to A]$.

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d. Extend the part c proof to show $[A \wedge \neg A] \to [A \to A]$.

e. Extend the part d proof to show $[A \to A] \to [A \wedge \neg A]$.

f. Extend the proof from part e to a proof of $A \wedge \neg A$, a contradiction.

2. Expand the definition of $\models$ to say in English what the following assertions mean and then prove that they are equivalent assertions for any formulas P, Q, and R, or that they are different for some choices of P, Q, and R.

a. $P \models (Q \vee R)$

For all models where P is true, $Q \vee R$ is true.

b. $P \models Q$ or $P \models R$

Either Q is true for all models that make P true or R is true for all models that make P true.

c. Prove or refute that parts a and b are equivalent for all possible formulas P, Q, and R.

Although the two statements sound similar, they are not equivalent. Let $P = A \vee B$, $Q = A$ and $R = B$. Now, $P \models Q \vee R$ because $A \vee B \models A \vee B$. But $A \vee B \not\models A$ and $A \vee B \not\models B$. So, the two are not equivalent since we've found a counterexample.

3. Let D be the set $\{1, 2, 3\}$. Consider a predicate logic with one one-argument relation symbol $P(x)$ and one two-argument relation symbol $R(x, y)$, along with constants a and b.

a. How many elements are there in $D \times D$?

b. Give a model with domain D in which $\exists x R(x, a)$ is false.

c. How many models are there with domain D for which $\forall x \forall y \neg R(x, y)$ is true? Explain.

d. Give a model refuting $[\forall x P(x)] \rightarrow \exists y \exists x R(x, y)$.

4.

a. Prove or refute $[\forall x \exists y P(x,y)] \rightarrow [\exists y \forall x P(x,y)]$.

This can be engaged intuitively using the "likes" relation. The left hand side translates to everybody likes somebody. The right hand side translates to somebody is liked by everybody. The formula can be refuted by the following example.

Let $D = \{1,2,3\}$ and $P = \{(1,2),(2,3),(3,1)\}$.

b. Prove or refute $[\exists y \forall x P(x,y)] \rightarrow [\forall x \exists y P(x,y)]$.

When somebody is liked by everybody, then everybody likes that person. So this implication is true. Formally,

$$\forall x P(x,a) \text{ by e.i} \tag{1}$$
$$P(b,a) \text{ by u.i} \tag{2}$$
$$\exists y P(b,y) \text{ by e.g} \tag{3}$$
$$\forall x \exists y P(x,y) \text{ by u.g} \tag{4}$$

It can be verified that a similar attempt to prove the previous part will fail during the universal generalization step.

5. Suppose the professor is thinking of a mystery function f, known only to him, from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$.

Consider the following recursive definition of a set of numbers S. Any even natural number is in S. For any two numbers x and y in S, the number $f(x, y)$ is S. No other numbers are in S except those just specified.

Suppose you want to "prove" that the professor likes all the numbers in S. What two things would you need to know about which numbers the professor likes in order to prove this using the principle of mathematical induction for recursive definitions?

There are two facts needed to prove that the professor likes all numbers in S.

(1) All even numbers are liked by the professor (2) For x and y that are liked by the professor and in S, $f(x, y)$ is liked by the professor.

The following proof is not necessary for this problem, but is included to show how the proof would be done.

Proof: Let L be the set of numbers liked by the professor $S_0 = \{$ even numbers$\}$

$S_1 = \{z | z \in S_0 \text{ or } z = f(x, y) \text{ for} x, y \in S_0\}$

$S_2 = \{z | z \in S_1 \text{ or } z = f(x, y) \text{ for} x, y \in S_1\}$

$\ldots$

$\cup Si = S$ by the recursive definition of S

Base Case: S_0 is in L by (1)

Inductive Case. Assume that S_k is liked by the professor. Prove S_{k+1} is liked by the professor.

$S_{k+1} = \{z | z \in S_k \text{ or } z = f(x, y) \text{ for } x, y \in S_k\}$

For each z in S_{k+1}, if z is in S_k, then z is liked by the professor by the inductive hypothesis. Or if $z = f(x, y)$ for x, y in S_k, then since S_k is a subset of S, x and y are liked by the professor and in S. Thus $z = f(x, y)$ is liked by the professor by (2). This implies that S_{k+1} is liked by the professor.

By induction, the professor likes all numbers in S.

6. Prove by mathematical induction:

$$(1^1 \times 1!) \times (2^2 \times 2!) \times \cdots \times (n^n \times n!) = (n!)^{n+1}.$$

Base case: k=1, then L. H. S. $= (1^1 \times 1!) = (1!)^{1+1} = 1 =$ R. H. S.

Let the equation be true for $n = k$ so we have:

$$(1^1 \times 1!) \times (2^2 \times 2!) \times \cdots \times (k^k \times k!) = (k!)^{k+1}.$$

Now, for $n = k + 1$, the L. H. S. is

$$(1^1 \times 1!) \times (2^2 \times 2!) \times \cdots \times (k^k \times k!) \times ((k+1)^{k+1} \times (k+1)!)$$

$$\begin{aligned}
&= (k!)^{k+1} \times (k+1)^{k+1} \times (k+1)! \text{ (using the induction hypothesis)} \\
&= (k! \times k+1)^{k+1} \times (k+1)! \\
&= ((k+1)!)^{k+1} \times (k+1)! \\
&= ((k+1)!)^{(k+2)}
\end{aligned}$$

So we have the result by the principle of mathematical induction.