Counting Unions   Principle of Inclusion/Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
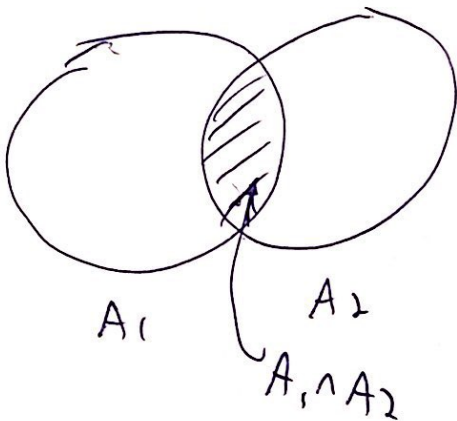
A_1 A_2

$A_1 \cap A_2$

How many numbers Less than 100,000
start w/ 5 or end with 09?
[no leading zeroes] $\underline{\quad\ \underline{0\ 9}}$

A_1 = #s starting w/ 5   and < 100,000

A_2 = #s ending w/ 09   "   "   "

$|A_1| = 11,111$ $|A_2| = 999$

$|A_1 \cap A_2| = 111$

$|A_1 \cup A_2| =$
$11111 + 999 - 111 = \underline{\underline{11999}}$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| \cdots$$

$$- |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$

$$+ |A_1 \cap A_2 \cap A_3|$$
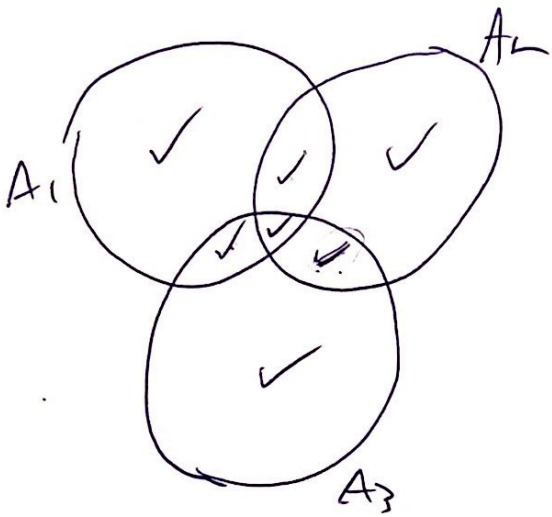
Generalized Inclusion / Exclusion

$$|A_1 \overset{\cup}{\cancel{\times}} A_2 \overset{\cup}{\cancel{\times}} \cdots \overset{\cup}{\cancel{\times}} A_n| = |A_1| + \cdots + |A_n|$$

$$- |A_1 \wedge A_2| - |A_1 \wedge A_3| - \cdots$$

$$+ |A_1 \wedge A_2 \wedge A_3| + |A_1 \wedge A_2 \wedge A_4| + \cdots$$

$$- |A_1 \wedge A_2 \wedge A_3 \wedge A_4| - \cdots$$

$$+$$

$$-$$

$$\vdots$$

$$\pm \; |A_1 \wedge A_2 \wedge \cdots \wedge A_n|$$

$k=1$

all pairs $k=2$

all triples $k=3$

all 4-way intersections

$$|A_1 \cup \cdots \cup A_n| =$$

$$\sum_{k=1}^{n} (-1)^{(k+1)} \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \wedge A_{i_2} \wedge \cdots \wedge A_{i_k}|$$

k'th row

sign of k'th row

How many functions are there from D, $|D| = n$, to
R, $|R| = 3$ that use all elements of R?

(i.e. the functions are "ONTO R")

D R

$x \longrightarrow A$

x B

x C

r

t

t

x
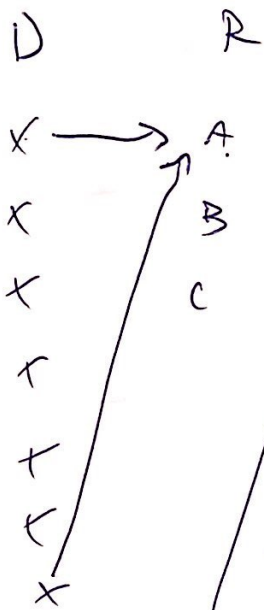
So, count complement.
The NOT ONTO functions.

$S_A \cup S_B \cup S_C$

$S_A =$ functions not using A

$S_B =$ "   "   "   " B

$S_C =$ "   "   "   " C

$$|S_A \cup S_B \cup S_C| = |S_A| + |S_B| + |S_C| - |S_A \cap S_B| - |S_B \cap S_C|$$
$$\qquad\qquad\qquad\qquad 2^n \qquad 2^n \qquad 2^n \qquad - |S_A \cap S_C| + |S_A \cap S_B \cap S_C|$$
$$\qquad\qquad\qquad 1 \qquad\qquad\qquad\qquad 1 \qquad\qquad 0$$

$$= 3 \cdot (2^n - 1)$$

$$|\text{ONTO functions}| = 3^n - 3(2^n - 1)$$

n choices to cover A $n-1$ choices to cover B $n-2$ choices
to cover C $(n)(n-1)(n-2) \, 3^{n-3} \implies$ over counts