Consider Let S be the set defined by

$0 \in S$

for any $n \in S$, $n+1 \in S$

---

So e.g. if $S_1 = \{5, 10\}$, $S = \{0, 6, 11\}$

for any set $S_1$, the above definition defines a set $S$;

so it defines a function from sets to sets s.t.

$S = f(S_1)$ e.g. $f(\{5, 10\}) = \{0, 6, 11\}$

---

We seek a set $S$ so that $S = f(S)$

i.e. $S$ is a fixed point of $f$.

but since $\mathbb{R} = f(\mathbb{R})$ $\mathbb{N} = f(\mathbb{N})$ $\emptyset = f(\emptyset)$ $\mathbb{Z} = f(\mathbb{Z})$

we need a way to choose. we take the smallest fixed point to be the defined set $S$. $S$ = least fixed point of $f$.

Construct the least fixed point of $f$
by unioning all of the "finite" approximations:

$\{0\} = f(\emptyset) \cup$
$\{0,1\} = ff(\emptyset) \cup$
$\{0,1,2\} = fff(\emptyset) \cup$
etc.

compactly $\bigcup_{i \in \mathbb{N}} f^i(\emptyset) = \mathbb{N}$
where $f^0(\emptyset) = \emptyset$

$f(\emptyset)$ is defined by $0 \in f(\emptyset)$ and for any $n \in \emptyset$, $n+1 \in f(\emptyset)$
$= \{0\}$
$ff(\emptyset)$ is defined by $0 \in f(f(\emptyset))$ and for any $n \in f(\emptyset)$, $n+1 \in ff(\emptyset)$
$= \{0,1\}$
etc

$S$ is the set s.t. for any $n \in S$, $n+1 \in S$.

$f(\mathbb{Z}) = \mathbb{Z}$ $\quad$ $f(\mathbb{N}) \neq \mathbb{N}$

$\| \quad \{1, 2, 3, \dots\}$

but $f(\emptyset) = \emptyset$ so $\emptyset$ is the dfp and $S = \emptyset$

in fact for all $i \in \mathbb{N}$ $f^i(\emptyset) = \emptyset$

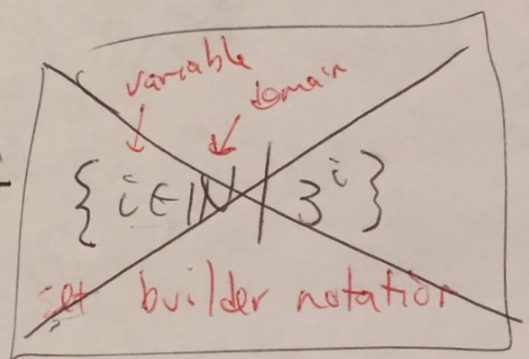
---

$S$ is set to have $1 \in S$ and $3n \in S$ for any $n \in S$

$f(\emptyset) = \{1\}$
$ff(\emptyset) = \{1, 3\}$
$f^3(\emptyset) = \{1, 3, 9\}$
$\vdots$

$\bigcup$ to get $\{3^i \mid i \in \mathbb{N}\}$

the powers of 3




↑ this is in the set for each way to choose $i$ ∈.

# Boolean Logic

Boole

## Defining a Logic

Step 0] What are your symbols? (the "language" of the logic)

reserved symbols:

| | | | |
|---|---|---|---|
| $\wedge$ | and | $\rightarrow$ | implies |
| $\vee$ | or | $\leftrightarrow$ | iff |
| $\neg$ | not | ( ) | |

a set of variables $V$ example $V = \{A, B, C\}$

Step 1] What are the allowed formulas?

SYNTAX

A legal formula is either

- a variable, or
- a Boolean combination of legal formulas

i.e. given legal formulas $\alpha$ and $\beta$, each of the following are also legal:

$\alpha \vee \beta$ $\alpha \wedge \beta$ $\alpha \rightarrow \beta$ $\alpha \leftrightarrow \beta$ $\neg\alpha$

$A$ $A \rightarrow B$

$\neg A$ $A \vee B$

$((\neg A) \wedge (\neg B))$

$f[\phi] = V$

$f^2[\phi] =$ variables and combinations of variables

$\vdots$

$f^i[\phi] =$ combinations built up from variables with max nesting depth $i-1$