Infinite Sets

<u>Principle</u> 2 sets have same size if there is a pairing (bijection, or 1-1 onto mapping) between the sets using all of both exactly once..

$$|S| = |T|$$

<u>Principle</u>: A proper subset $S \subset T$ is smaller than T
not the whole set

$$\left.\begin{array}{c} S \subseteq T \\ \text{and} \\ S \neq T \end{array}\right\} \rightarrow |S| < |T|$$

Which is bigger?

$$\begin{array}{ccccccc} & \{\,0 & 1 & 2 & 3 & 4 & \cdots\,\} \\ \text{or} & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ & \{\,1 & 2 & 3 & 4 & 5 & \cdots\,\} \end{array}$$

So for infinite sets we keep the <u>pairing principle</u>

and abandon the proper subset principle:

$$|\mathbb{N}| = |\mathbb{N} - \{0\}|$$

Which is bigger? $\mathbb{N}$ vs. evens ?

$\{0, 2, 4, \cdots\}$

$$\begin{array}{cccccc} \{0 & 1 & 2 & 3 & 4 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ \{0 & 2 & 4 & 6 & 8 & \cdots \end{array}$$

Same size.

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we say $|S| \leq |T|$ if there is a pairing between S and a subset of T.

$$\left|\{0, 2, 4, 6 \cdots\}\right| \leq |\mathbb{N}|$$

$$\begin{array}{ccccccccc} 0 & & 2 & & 4 & & 6 & \cdots \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}$$

Thm if $|S| \leq |T|$ and $|T| \leq |S|$

then $|S| = |T|$. (Schroeder – Bernstein Theorem)

(if S can be paired w/ a subset of T and T " " " " " " " S then S and T can be fully paired off)

Show $|\mathbb{N}| = |\mathbb{Z}|$

$\mathbb{N} = \{ 0, \ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots$

$\mathbb{Z} = \{ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \ldots$

never get to negatives

so $|\mathbb{N}| \leq |\mathbb{Z}|$

$\mathbb{Z} = \{ 0 \quad -1 \quad 1 \quad -2 \quad 2 \quad -3 \quad 3 \quad \ldots$

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so $|\mathbb{N}| = |\mathbb{Z}|$