Introduction to SAS
Vectors and Matrices

Lecture 2

August 31, 2005

Multivariate Analysis

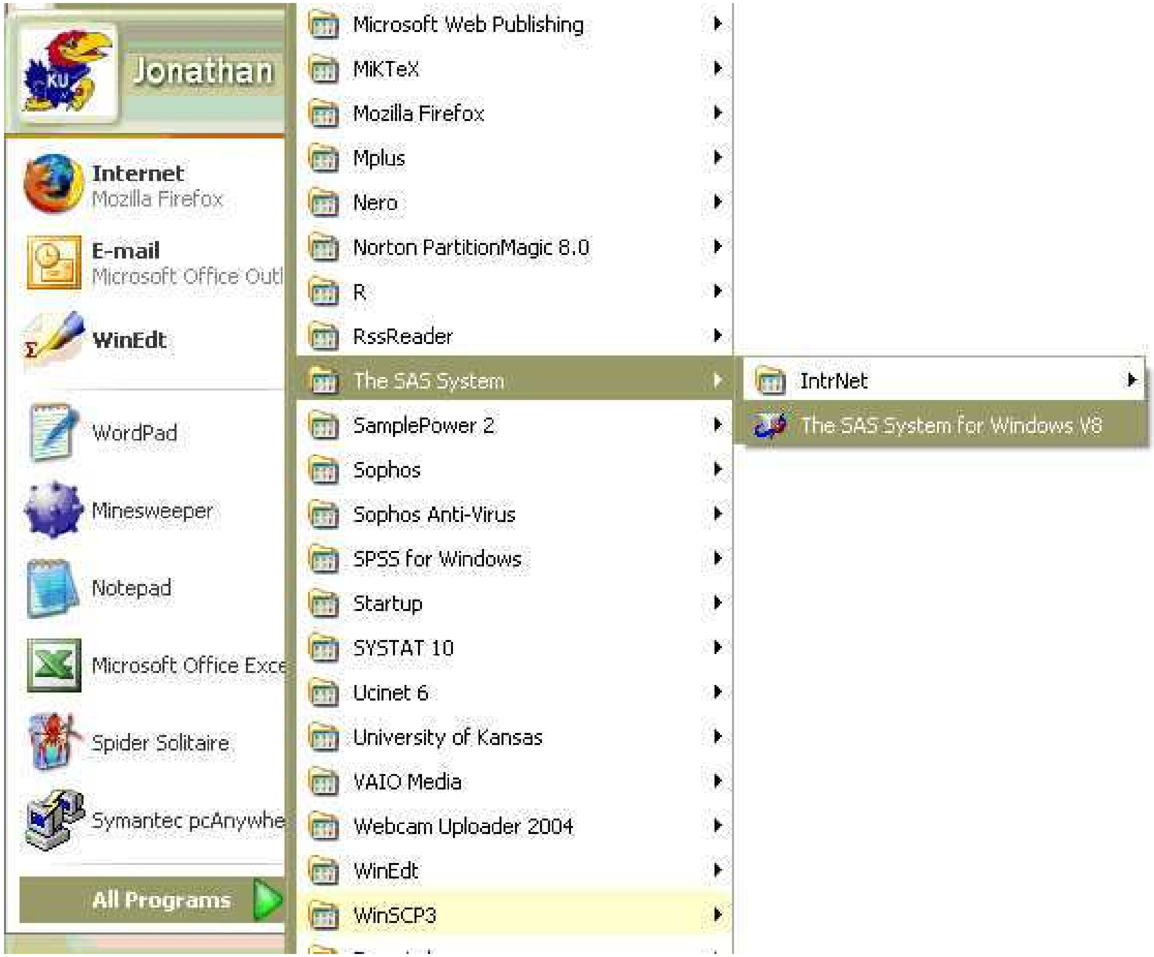
Today's Lecture

Overview
Today's Lecture
Introduction to SAS
Matrix Algebra
Algebra
Vector Definitions
Matrix Properties
Advanced Topics
Wrapping Up

- Introduction to SAS.
- Vectors and Matrices (Supplement 2A - augmented with SAS *proc iml*).

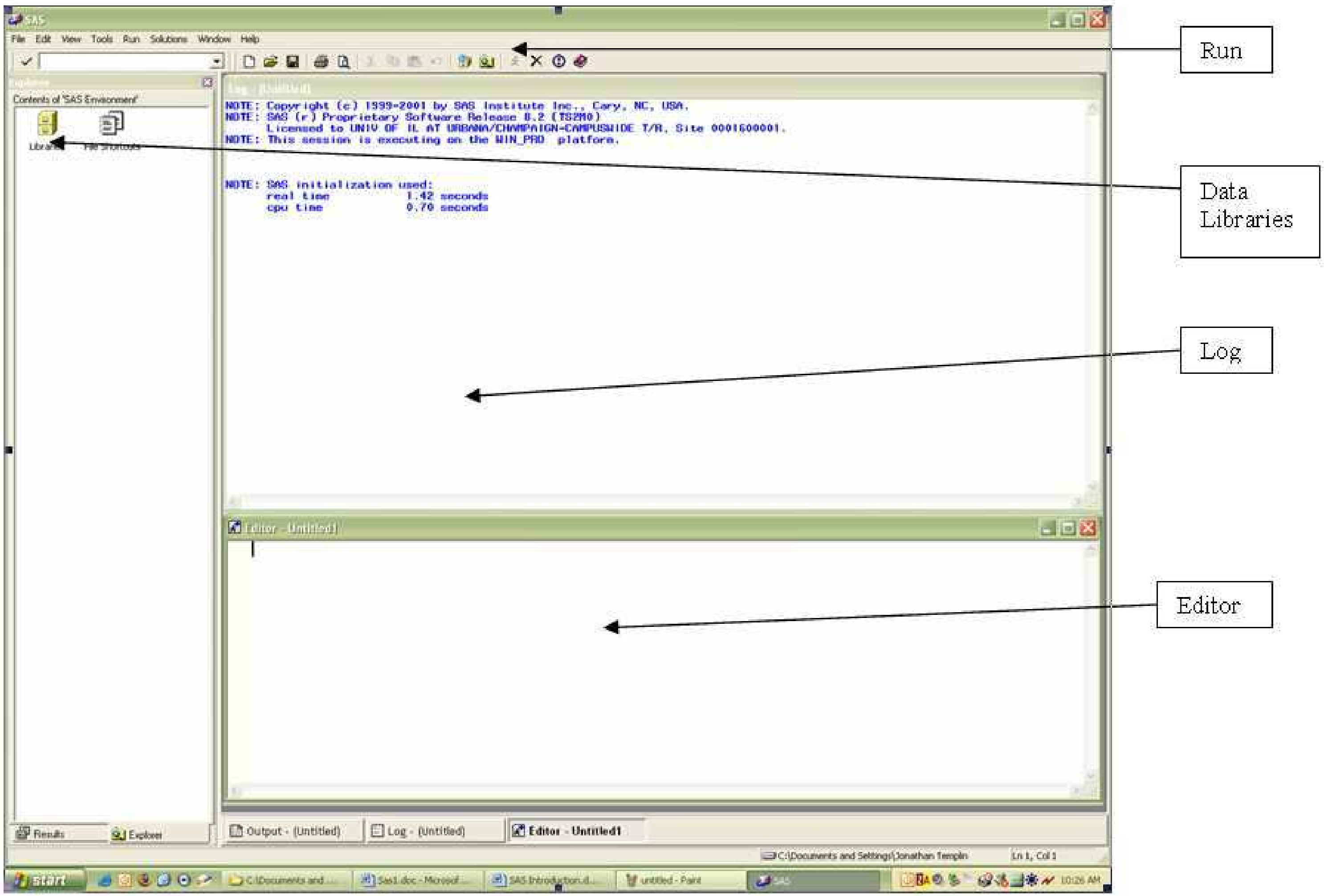
SAS

- To start SAS, on a Windows PC, go to Start...All Programs...The SAS System...The SAS System for Windows V8.



Main Program Window

- The SAS program looks like this (some helpful commands are shown with the arrows):



SAS Editor

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- To run SAS you must create a file of SAS code, which the SAS processor reads and uses to run the program.
- Simply type your SAS code into the Program Editor window.
- For our example today, we will create (and save to) a new SAS code file, so to do that be sure to have your curser inside of the editor window and go to File...Save...
- SAS code files usually end with the extension *.sas.

SAS Code Basics

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- For use in day-to-day statistical applications, SAS code consists of two components:
  - Data steps (where data input *usually* happens.
  - Proc steps (where statistical analyses *usually* happen.
- General exceptions to these rules exist.
- All statements terminate with a semicolon (this is usually where errors can occur).
- Commented code can begin with:
  - An asterisk (*) for single lines - terminated with a semicolon.
  - /* for multiple lines, terminated with an ending */.
- Enough talk...how about an example? Type the following into the SAS Editor:

First SAS Program

```
title ’First SAS Program’;
data example1;
input id gender $ age;
cards;
1 F 23
2 M 20
3 F 18
4 F .
5 M 19
6 M 21
7 M 25
8 F 24
9 F 20
10 M 22
;

proc print;
var id gender age;
run;
```

First SAS Program

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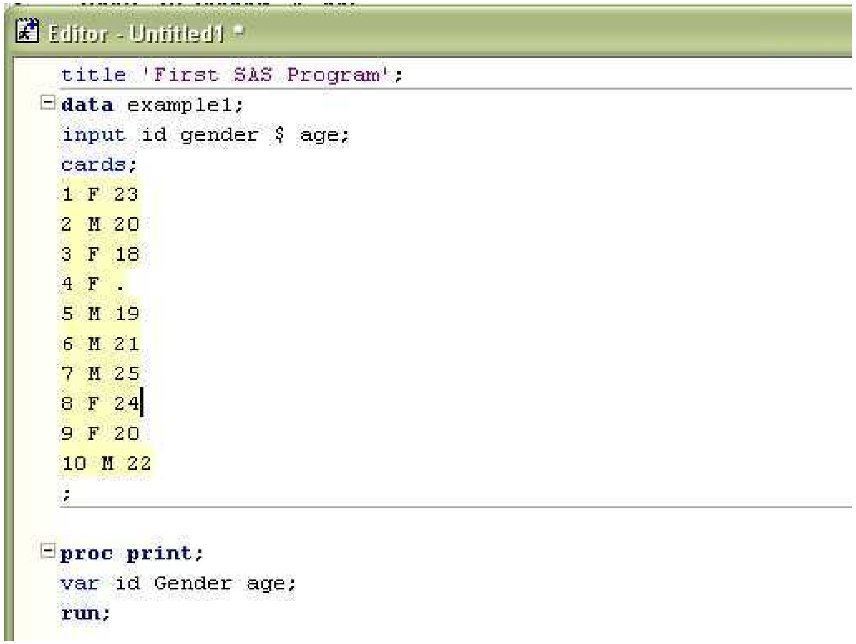
Notice the color scheme of the SAS Enhanced Editor (note: if you do not see color, do not panic, you may not have the Enhanced Editor installed).



- Items in light blue are command words (like “input” or “var”.
- Items in dark blue are procedural words (like “proc,” “data,” or “run”).

Run the SAS Program

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- To run the program you just entered, press the running man icon at the center of the top of the SAS main program.
- Once you press the “run” button, the log window will become active, giving you information about the program as it executes.
- In this window you will see errors (in red), or warnings (in green I think).
- As multivariate progresses, you will become aware of instances where warnings will be present because of problems in your analysis.
- Also now appearing is a new window called output...this is where the output of the procedure that was just run is displayed.

SAS Data Libraries

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- The data set you just entered is now part of a SAS data library that can be referenced at any point during the remainder of the program.
- The default SAS data library is called “work,” and can be accessed by clicking through the explorer window on the left hand side of the program.
- Double click on Libraries...Work...Example1 and you will see the data displayed in a grid.
- To familiarize you with SAS, here are some handouts (also available on the BlackBoard Site)...

SAS Procedures

- The bulk of the statistical work done in SAS is through procedure statements.

- Proc statements follow a flexible syntax that typically has the following:

```
proc -statement_name- data=-data_name- [options];
var [included variables];
[options];
run;
```

- The names and options are all found in the SAS manual, which is freely (shhh) available online at: http://www.id.unizh.ch/software/unix/statmath/sas/sasdoc/main.htm

SAS Procedures Example

- Using the Example1 data set, type the following into the editor:

```
proc sort data=example1;
by gender;
run;

proc univariate data=example1 plots;
by gender;
var age;
run;
```

- The univariate procedure produces univariate statistics, the manual entry can be at found:

  http://www.id.unizh.ch/software/unix/statmath/sas/sasdoc/proc/z0146802.htm

Matrix Introduction

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- Imagine you are interested in studying the cultural differences in student achievement in high school.
- Your study gets IRB approval and you work hard to get parental approval.
- You go into school and collect data using many different instruments.
- You then input the data into your favorite stat package, like SAS (or MS Excel).
- How do you think the data is stored?

Why Learn Matrix Algebra?

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- Nearly all multivariate techniques are founded in matrix algebra.
- Often times when statistics break down, the cause of the failure can be traced through matrix procedures.
- If you are trying to apply a new method, most of the technical statistical literature uses matrix algebra assuming a basic to advanced knowledge of matrices - you may need to read these articles.
- Have you seen:
  - $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$?
  - $\mathbf{\Lambda\Phi\Lambda}' + \mathbf{\Psi}$?
  - $\mathbf{F}_1\mathbf{F}_2\mathbf{F}_3\mathbf{L}\mathbf{F}_3^{-1}\mathbf{F}_2^{-1}\mathbf{P}(\mathbf{F}_2^{-1})'(\mathbf{F}_3^{-1})'\mathbf{L}'\mathbf{F}_3'\mathbf{F}_2'\mathbf{F}_1' + \mathbf{U}^2$?
- Warning: using matrix algebra lingo is a great way to end conversations or break up parties.

Definitions

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- We begin this class with some general definitions (from dictionary.com):
  - Matrix:
    1. A rectangular array of numeric or algebraic quantities subject to mathematical operations.
    2. The substrate on or within which a fungus grows.
  - Algebra:
    1. A branch of mathematics in which symbols, usually letters of the alphabet, represent numbers or members of a specified set and are used to represent quantities and to express general relationships that hold for all members of the set.
    2. A set together with a pair of binary operations defined on the set. Usually, the set and the operations include an identity element, and the operations are commutative or associative.

Proc IML

- To help demonstrate the topics we will discuss today, I will be showing examples in SAS proc iml.
- The Interactive Matrix Language (IML) is a scientific computing package in SAS that typically used for complicated statistical routines.
- Of course, other matrix programs exist - for many statistical applications MATLAB is very useful.
- SPSS and SAS both have matrix computing capabilities, but (in my opinion) neither are as efficient, as user friendly, or as flexible as MATLAB.
  - It is better to leave most of the statistical computing to the computer scientists.

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- Proc IML is a proc step in SAS that runs without needing to use a preliminary data step.
- To use IML, make sure the following are placed in a SAS code file.

```
proc iml;

reset print;

quit;
```

- The “reset print;” line makes every result get printed in the output window.
- The IML code will go between the “reset print;” and the “quit;”

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- Away from the definition, a matrix is simply a rectangular way of storing data.
- Matrices can have unlimited dimensions, however for our purposes, all matrices will be in two dimensions:
  - Rows
  - Columns
- Matrices are symbolized by **boldface** font in text, usually with capital letters.

$$\mathbf{A} = \begin{bmatrix} 4 & 7 & 5 \\ 6 & 6 & 3 \end{bmatrix}$$

Vectors

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- A vector is a matrix where one dimension is equal to size one.
  - Column vector: A column vector is a matrix of size $r \times 1$.
  - Row vector: A row vector is a matrix of size $1 \times c$.
- Vectors allow for geometric representations of matrices.
- The Pearson correlation coefficient is a function of the angle between vectors.
- Much of the statistical theory underlying linear models (ANOVA-type) can be conceptualized by projections of vectors (think of the dependent variable Y as a column vector).
- Vectors are typically symbolized by **boldface** font in text, usually with lowercase letters.

Scalars

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- A scalar is a matrix of size 1×1.
- Scalars can be thought of as any single value.
- The difficult concept to get used to is seeing a number as a matrix:

$$A = \left[\begin{array}{c} 2.759 \end{array} \right]$$

Matrix Elements

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- A matrix is composed of a set of elements, each denoted it's row and column position within the matrix.
- For a matrix **A** of size $r \times c$, each element is denoted by:

$$a_{ij}$$

  - The first subscript is the index for the rows: $i = 1, \ldots, r$.
  - The second subscript is the index for the columns: $j = 1, \ldots, c$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1c} \\ a_{21} & a_{22} & \ldots & a_{1c} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1} & a_{r2} & \ldots & a_{rc} \end{bmatrix}$$

Transpose

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- The transpose of a matrix is simply the switching of the indices for rows and columns.
- An element a_{ij} in the original matrix (in the i^{th} row and j^{th} column) would be a_{ji} in the transposed matrix (in the j^{th} row and the i^{th} column).
- If the original matrix was of size $i \times j$ the transposed matrix would be of size $j \times i$.

$$\mathbf{A} = \begin{bmatrix} 4 & 7 & 5 \\ 6 & 6 & 3 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} 4 & 6 \\ 7 & 6 \\ 5 & 3 \end{bmatrix}$$

Types of Matrices

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- Square Matrix: A matrix that as the same number of rows and columns.
  - Correlation and covariance matrices are examples of square matrices.
- Diagonal Matrix: A diagonal matrix is a square matrix with non-zero elements down the diagonal and zero values for the off-diagonal elements.

$$\mathbf{A} = \begin{bmatrix} 2.759 & 0 & 0 \\ 0 & 1.643 & 0 \\ 0 & 0 & 0.879 \end{bmatrix}$$

- Symmetric Matrix: A symmetric matrix is a square matrix where $a_{ij} = a_{ji}$ for all elements in i and j.
  - Correlation/covariance and distance matrices are examples of symmetric matrices.

Algebraic Operations

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- As mentioned in the definition at the beginning of class, algebra is simply a set of math that defines basic operations.
  - Identity
  - Zero
  - Addition
  - Subtraction
  - Multiplication
  - Division
- Matrix algebra is simply the use of these operations with matrices.

Matrix Addition

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- Matrix addition is very much like scalar addition, the only constraint is that the two matrices must be of the same size (same number of rows and columns).
- The resulting matrix contains elements that are simply the result of adding two scalars.

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \\ a_{41} + b_{41} & a_{42} + b_{42} \end{bmatrix}$$

IML Addition

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```
proc iml;
reset print;

A={10 15, 11 9, 1 -6};
B={5 2, 1 0, 10 7};
C=A+B;

quit;
```

Matrix Subtraction

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- Matrix subtraction is identical to matrix addition, with the exception that all elements of the new matrix are the subtracted elements of the previous matrices.
- Again, the only constraint is that the two matrices must be of the same size (same number of rows and columns).
- The resulting matrix contains elements that are simply the result of subtracting two scalars.

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \\ a_{31} - b_{31} & a_{32} - b_{32} \\ a_{41} - b_{41} & a_{42} - b_{42} \end{bmatrix}$$

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```
proc iml;
reset print;

A={10 15, 11 9, 1 -6};
B={5 2, 1 0, 10 7};
C=A-B;

quit;
```

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- Unlike matrix addition and subtraction, matrix multiplication is much more complicated.
- Matrix multiplication results in a new matrix that can be of differing size from either of the two original matrices.
- Matrix multiplication is defined only for matrices where the number of columns of the first matrix is equal to the number of rows of the second matrix.
- The resulting matrix as the same number of rows as the first matrix, and the same number of columns as the second matrix.

$$\underset{(r \times \mathbf{c})}{\mathbf{A}} \quad \underset{(\mathbf{c} \times k)}{\mathbf{B}} \quad = \quad \underset{(r \times k)}{\mathbf{C}}$$

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{bmatrix}$$

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{bmatrix}$$

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$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{23} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{bmatrix}$$

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proc iml;
reset print;

A={10 15, 11 9, 1 -6};
B={5 2, 1 0, 10 7};
C=A*T(B);
D=T(B)*(A);

quit;
```

Multiplication and Summation

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- Because of the additive nature induced by matrix multiplication, many statistical formulas that use:

$$\sum$$

can be expressed by matrix notation.

- For instance, consider a single variable X_i, with $i = 1, \ldots, N$ observations.
- Putting the set of observations into the column vector **X**, of size $N \times 1$, we can show that:

$$\sum_{i=1}^{N} X^2 = \mathbf{X}' \, \mathbf{X}$$

Matrix Multiplication by Scalar

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- Recall that a scalar is simply a matrix of size (1×1).
- Matrix multiplication by a scalar causes all elements of the matrix to be multiplied by the scalar.
- The resulting matrix has all elements multiplied by the scalar.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \qquad s \times \mathbf{A} = \begin{bmatrix} s \times a_{11} & s \times a_{12} \\ s \times a_{21} & s \times a_{22} \\ s \times a_{31} & s \times a_{32} \\ s \times a_{41} & s \times a_{42} \end{bmatrix}$$

Identity Matrix

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- The identity matrix is defined as a matrix that when multiplied with another matrix produces that original matrix:

$$\mathbf{A}\,\mathbf{I} = \mathbf{A}$$

$$\mathbf{I}\,\mathbf{A} = \mathbf{A}$$

- The identity matrix is simply a square matrix that has all off-diagonal elements equal to zero, and all diagonal elements equal to one.

$$\mathbf{I}_{(3\times 3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Zero Matrix

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- The zero matrix is defined as a matrix that when multiplied with another matrix produces the matrix:

$$\mathbf{A}\,\mathbf{0} = \mathbf{0}$$

$$\mathbf{0}\,\mathbf{A} = \mathbf{0}$$

- The zero matrix is simply a square matrix that has all elements equal to zero.

$$\mathbf{0}_{(3\times 3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix “Division”: The Inverse

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- Recall from basic math that:

$$\frac{a}{b} = \frac{1}{b}a = b^{-1}a$$

- And that:

$$\frac{a}{a} = 1$$

- Matrix inverses are just like division in basic math.

The Inverse

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- Subtraction
- Multiplication
- Identity
- Zero
- "Division"
- Singular Matrices

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- For a square matrix, an inverse matrix is simply the matrix that when pre-multiplied with another matrix produces the identity matrix:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- Matrix inverse calculation is complicated and unnecessary since computers are much more efficient at finding inverses of matrices.
- One point of emphasis: just like in regular division, division by zero is undefined.
- By analogy - not all matrices can be inverted.

Singular Matrices

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- A matrix that cannot be inverted is called a *singular* matrix.
- In statistics, common causes of singular matrices are found by linear dependence among the rows or columns of a square matrix.
- Linear dependence can be cause by combinations of variables, or by variables with extreme correlations (either near 1.00 or -1.00).

Linear Combinations

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- Vectors can be combined by adding multiples:

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_k\mathbf{x}_k$$

- The resulting vector, **y**, is called a linear combination.
- All for k vectors, the set of all possible linear combinations is called their span.

Linear Dependencies

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- A set of vectors are said to be linearly dependent if $a_1, a_2, \ldots, a_k$ exist, and:
  - $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_k\mathbf{x}_k = 0$.
  - $a_1, a_2, \ldots, a_k$ are not all zero.
- Such linear dependencies occur when a linear combination is added to the vector set.
- Matrices comprised of a set of linearly dependent vectors are singular.
- A set of linearly independent vectors forms what is called a basis for the vector space.
- Any vector in the vector space can then be expressed as a linear combination of the basis vectors.

Vector Length

- The length of a vector emanating from the origin is given by the Pythagorean formula:

$$L_{\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + \ldots + x_k^2}$$

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Inner Product

- The inner (or dot) product of two vectors **x** and **y** is the sum of element-by-element multiplication:

$$\mathbf{x}'\mathbf{y} = x_1 y_1 + x_2 y_2 + \ldots + x_k y_k$$

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Vector Angle

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- The angle formed between two vectors **x** and **y** is

$$cos(\theta) = \frac{\mathbf{x}'\mathbf{y}}{\sqrt{\mathbf{x}'\mathbf{x}}\sqrt{\mathbf{y}'\mathbf{y}}}$$

- If $\mathbf{x}'\mathbf{y} = 0$, vectors **x** and **y** are perpendicular, as noted by $\mathbf{x}\perp\mathbf{y}$
- All basis vectors are perpendicular.

Vector Projections

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- The projection of a vector **x** onto a vector **y** is given by:

$$\frac{\mathbf{x}'\mathbf{y}}{L^2_{\mathbf{y}}}\mathbf{y}$$

- Through such projections, a set of linear independent vectors can be created from any set of vectors.
- One process used to create such vectors is through the Gram-Schmidt Process.

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The following are some algebraic properties of matrices:

- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ - Associative
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ - Commutative
- $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ - Distributive
- $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$
- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- $(cd)\mathbf{A} = c(d\mathbf{A})$
- $(c\mathbf{A})' = c\mathbf{A}'$

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The following are more algebraic properties of matrices:

- $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B}$
- $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
- $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
- For $\mathbf{x}_j$ such that $\mathbf{Ax}_j$ is defined:
  - $\sum_{j=1}^{n} \mathbf{Ax}_j = \mathbf{A}\sum_{j=1}^{n} \mathbf{x}_j$
  - $\sum_{j=1}^{n} (\mathbf{Ax}_j)(\mathbf{Ax}_j)' = \mathbf{A}\left(\sum_{j=1}^{n} \mathbf{x}_j\mathbf{x}_j'\right)\mathbf{A}'$

Advanced Matrix Functions/Operations

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- Decompositions

Wrapping Up

- We end our matrix discussion with some advanced topics.
- All of these topics are related to multivariate analyses.
- None of these will seem too straight forward today, but next week we will use some of these results to demonstrate properties of sample statistics.

Matrix Determinants

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Wrapping Up

- A square matrix can be characterized by a scalar value called a determinant.

$$\det \mathbf{A} = |\mathbf{A}|$$

- Much like the matrix inverse, calculation of the determinant is very complicated and tedious, and is best left to computers.
- What can be learned from determinants is if a matrix is singular.
- Matrices with determinants that are greater than zero are said to be "positive definite," a byproduct of which is that a positive matrix is non-singular.

Matrix Orthogonality

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Wrapping Up

- A square matrix (**A**) is said to be *orthogonal* if:

$$\mathbf{A}\mathbf{A}' = \mathbf{A}'\mathbf{A} = \mathbf{I}$$

- Orthogonal matrices are characterized by two properties:

1. The product of all row vector multiples is the zero matrix (perpendicular vectors).

2. For each row vector, the sum of all elements is one.

Eigenvalues and Eigenvectors

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Wrapping Up

- A square matrix can be decomposed into a set of eigenvalues and eigenvectors.

$$\mathbf{Ax} = \lambda\mathbf{x}$$

- From a statistical standpoint:
  - Principal components are comprised of linear combination of a set of variables weighed by the eigenvectors.
  - The eigenvalues represent the proportion of variance accounted for by specific principal components.
  - Each principal component is orthogonal to the next, producing a set of uncorrelated variables that may be used for regression purposes.

Spectral Decompositions

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Wrapping Up

- Imagine that a matrix **A** is of size $k \times k$.
- **A** then has:
  - k eigenvalues: λ_i, $i = 1, \ldots, k$.
  - k eigenvectors: $\mathbf{e}_i$, $i = 1, \ldots, k$ (each of size $k \times 1$.
- **A** can be expressed by:

$$\mathbf{A} = \sum_{i=1}^{k} \lambda_i \mathbf{e}_i \mathbf{e}_i'$$

- This expression is called the *Spectral Decomposition*, where **A** is decomposed into k parts.
- One can find $\mathbf{A}^{-1}$ by taking $\frac{1}{\lambda}$ in the spectral decomposition.

Final Thought

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- Final Thought
- Next Class

- The SAS learning curve is small, but once you have the basics, all you need is the manual and you can do almost anything.

- Proc IML is useful for matrices, but probably not so useful to you.
- Matrix algebra makes the technical things in life easier.
- The applications of matrices will be demonstrated throughout the rest of this course.

Next Time

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- Applications of matrix algebra in statistics (Chapter 2 section six and on).
- Geometric implications of multivariate descriptive statistics (more applications).