Generation of Random Graphs: Applications in Financial Networks

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Abstract

This study, conducted within the framework of the Econophysics course (Fall 2024, EuroTeQ Virtual Mobility Program), explores the generation of random graph models and their applications in modeling financial networks. Random graphs, constructed via stochastic processes, offer a robust framework for analyzing the complex connectivity inherent in financial systems, such as interbank networks and asset correlation structures. We investigate three canonical models—the Erdős-Rényi, Barabási-Albert, and Watts-Strogatz models—focusing on their structural properties, including degree distributions, clustering coefficients, and path lengths. By applying these models to empirical financial data, we evaluate their efficacy in replicating network topologies, visualizing asset correlations, and simulating systemic risk dynamics, such as contagion. Our findings highlight the strengths and limitations of each model in capturing real-world financial network characteristics, providing insights into their utility for risk assessment and network resilience analysis. This work underscores the potential of random graph models to inform strategies for mitigating systemic vulnerabilities in financial systems.

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1 Introduction

Financial systems are characterized by intricate interconnections among economic entities, including banks, assets, and institutions, which collectively influence market stability and systemic risk. Representing these systems as networks, where nodes denote entities and edges signify interactions, facilitates the analysis of their structural and dynamic properties. Random graph models, rooted in probabilistic frameworks, have emerged as powerful tools in econophysics for simulating such networks, enabling researchers to study phenomena such as risk contagion and asset correlations. This report investigates the generation of random graphs and their application to financial network modeling, with a focus on three prominent models: the Erdős-Rényi (ER) model, which assumes uniform edge probability; the Barabási-Albert (BA) model, characterized by scale-free degree distributions via preferential attachment; and the Watts-Strogatz (WS) model, which captures small-world properties through high clustering and short path lengths.

The study is organized as follows. Section 1 provides a detailed exposition of the theoretical foundations of these random graph models, including their mathematical formulations and key topological properties. Section 2 explores their practical applications in financial contexts, specifically in visualizing asset correlation networks and modeling interbank lending structures. By fitting these models to empirical data, we assess their ability to replicate observed network characteristics and simulate risk propagation. Section 3 discusses criteria for selecting appropriate models based on network topology and application goals, concluding with implications for financial stability and directions for future research. This work aims to advance the understanding of financial network dynamics and support the development of robust strategies for systemic risk management.

2 Random Graph Generation Model

2.1 Generalities on Random Graphs

Let us start by defining what we mean by a random graph.

A graph can be viewed as an abstract structure composed of objects (the nodes) connected to each other by certain relationships. The presence of a relationship between node i and node j is characterized by the presence of an edge between nodes i and j.

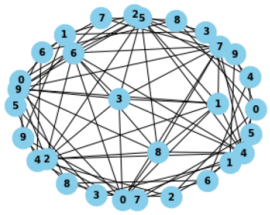


Figure 1: Example of a graph

A network is said to be random when it is obtained through a random process, meaning that the presence or absence of edges between node i and node j depends exclusively on a random process P.

Consider a network with N nodes. We now present the key characteristics of such a network:

- **Adjacency matrix:**
$$A = (A_{i,j})_{i,j\in\{1,\dots,N\}} \in \mathbf{M}_n(\mathbb{R})$$
such that $A_{i,j} = 1$ if there is an edge between nodes i and j, and $A_{i,j} = 0$ otherwise. For an undirected network, we always have $A_{i,j} = A_{j,i}$.

- **Degree of a node** i (k_i): the number of nodes connected to i:
$$k_i = \sum_j A_{i,j}$$
The **mean degree** is defined as:
$$\langle k \rangle = \frac{1}{N} \sum_{i=1}^{N} k_i$$

- **Local clustering coefficient** (C_i):
For a node i with k_i neighbors, it quantifies how many of the possible connections between these neighbors actually exist. It is defined as:
$$C_i = \frac{2 \times \text{Number of edges between neighbors of } i}{k_i \times (k_i - 1)}$$

- **Global clustering coefficient** (C):
An overall measure of clustering in the network. It is often defined as the average

of the local clustering coefficients:

$$C = \frac{1}{N} \sum_{i=1}^{N} C_i$$

High clustering coefficients indicate strong local connectivity, a property often observed in real-world networks.

- **Closeness centrality** $C_C(i)$:
  Quantifies how close a node is to all other nodes in the network:

$$C_C(i) = \frac{1}{\sum_{j \neq i} d(i,j)}$$

  where $d(i,j)$ is the shortest path distance between nodes i and j. Nodes with high closeness centrality can reach other nodes more quickly.

- **Degree distribution** $P(k)$:
  The fraction of nodes in the network with degree k. It is mathematically expressed as:

$$P(k) = \frac{N_k}{N}$$

  where N_k is the number of nodes with degree k.

There are several other characteristics, but we list here those most relevant to our study. We now focus on specific models for generating random networks.

2.2 The Erdős–Rényi (ER) model

An Erdős–Rényi network is a random network generated by independently connecting each pair of vertices with a given probability. There are two Erdős–Rényi models: the binomial model, denoted $\mathbb{G}(n,p)$, and the uniform model, denoted $\mathbb{G}(n,M)$.

Let $p \in [0,1]$ and n be an integer. Let $(X_{i,j})_{1 \leq i < j \leq n}$ be a family of independent Bernoulli random variables with parameter p, such that:

$$\mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0) = p.$$

Definition 1: The binomial Erdős–Rényi graph, denoted by $\mathbb{G}(n,p)$, is a random variable taking values in the set of graphs with n vertices, where the vertices are $1, 2, \ldots, n$, and the set of edges is $\{(i,j) \mid i < j, X_{i,j} = 1\}$. Then the number N_p of edges in $\mathbb{G}(n,p)$ follows a binomial distribution with parameters $\frac{n(n-1)}{2}$ and p:

$$\mathbb{P}(G) = p^{e_G}(1-p)^{\frac{n(n-1)}{2} - e_G}$$

where e_G is the number of edges in G.

In the original model proposed by Erdős and Rényi, the idea was to start from a set Σ of n points ($|\Sigma| = n$) and to uniformly choose a subset of M edges from the $\binom{n}{2}$ possible edges. This corresponds to the Erdős–Rényi uniform model $\mathbb{G}(n,M)$.

We will show that the two models are very close, before focusing exclusively on the binomial model. For a more complete proof, see [**?**].

Proposition: Let Γ be the set of all constructible edges from $\{1, \dots, n\}^2$, i.e., $|\Gamma| = N = \frac{n(n-1)}{2}$. Let Q be a convex property of subsets of Γ, and let $0 \leq M \leq N$. If $\mathbb{P}(\mathbb{G}(n, M/N) \in Q) \to 1$ as $n \to \infty$, then $\mathbb{P}(\mathbb{G}(n, M) \in Q) \to 1$.

Proof: Assume that $\frac{M(N-M)}{N} \to +\infty$. Let:

$$M_1 = \arg\max_{M' \leq M} \mathbb{P}(\mathbb{G}(n, M') \in Q), \quad M_2 = \arg\max_{M' \geq M} \mathbb{P}(\mathbb{G}(n, M') \in Q).$$

We then have:

$$\mathbb{P}\left(\mathbb{G}\left(n, \frac{M}{N}\right) \in Q\right) \leq \mathbb{P}\left(\mathbb{G}(n, M_1) \in Q\right) \mathbb{P}\left(|\mathbb{G}(n, M/N)| \leq M\right) + \mathbb{P}\left(|\mathbb{G}(n, M/N)| \geq M\right).$$

By the Central Limit Theorem:

$$\mathbb{P}(|\mathbb{G}(n, M/N)| \leq M) \to \frac{1}{2}.$$

Hence:

$$1 = \lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}(n, M/N) \in Q\right) \leq \frac{1}{2} \lim_{n\to\infty} \mathbb{P}(\mathbb{G}(n, M_1) \in Q) + \frac{1}{2},$$

which implies:

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{G}(n, M_1) \in Q) = 1.$$

Similarly:

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{G}(n, M_2) \in Q) = 1.$$

Since $M_1 \leq M \leq M_2$, it follows that:

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{G}(n, M) \in Q) = 1. \quad \square$$

We therefore assume that the properties of $\mathbb{G}(n, M)$ and $\mathbb{G}(n, p)$ are very similar whenever $p = \frac{2M}{n(n-1)}$. Consequently, we will only study the $\mathbb{G}(n, p)$ model.

Construction

1. Fix $d \in \mathbb{N}^*$, the dimension of the Gaussian random variables.
2. For each edge (i, j), simulate a Gaussian random variable $Y_{ij} \in \mathbb{R}^d$.
3. Choose a vector $\mathbf{u} \in \mathbb{S}^{d-1}$, the unit sphere in d-dimensional space.
4. Construct the graph:
   - Vertices: $\{1, \dots, n\}$.
   - Edges:
     $$\{(i, j) \mid \langle Y_{ij}, \mathbf{u} \rangle \geq r\},$$
     where $\langle Y_{ij}, \mathbf{u} \rangle$ is the dot product and $r \in \mathbb{R}$ is a threshold.

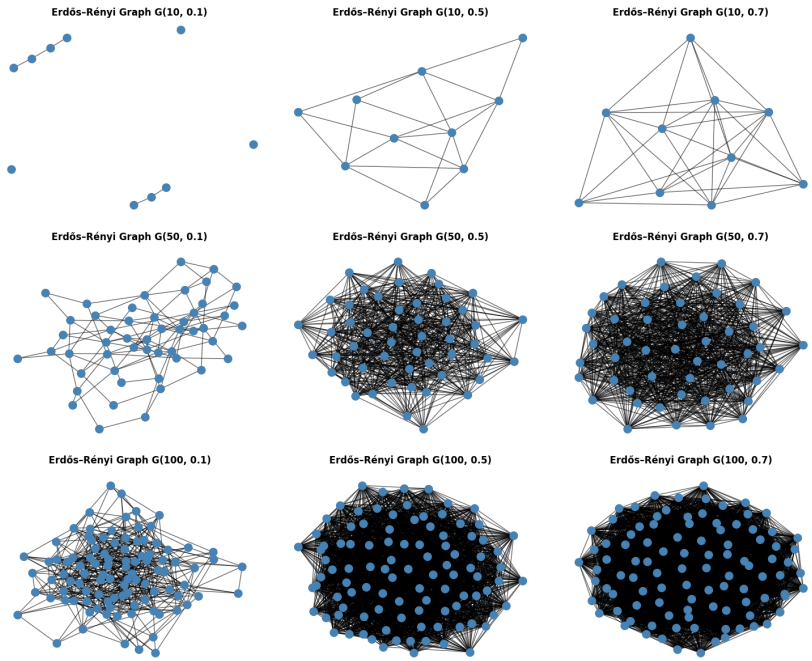


Figure 2: Erdős–Rényi graphs for different values of n and p.

5. Varying r yields different instances of the Erdős–Rényi graph $G(n, p)$.

Properties Let $n \in \mathbb{N}$, $p \in [0, 1]$. Given an integer l, the probability of generating a graph with l edges in the model $\mathbb{G}(n, p)$ is:

$$\mathbb{P}(G) = \binom{\binom{n}{2}}{l} p^l (1-p)^{\binom{n}{2}-l}.$$

The expected number of edges is:

$$\langle l \rangle = \binom{n}{2} p.$$

The mean degree of a vertex in a graph with l edges is $\langle k \rangle = \frac{2l}{n}$, thus:

$$\langle k \rangle = 2 \cdot \frac{\binom{n}{2} p}{n} = (n-1)p.$$

For large n, we have $\langle k \rangle \approx np$.

Definition: A giant component is a connected subgraph that contains a significant fraction of the total number of vertices. Two extreme cases:

- If $p = 0$: no edges, the graph is a set of isolated vertices. The size of the giant component is an **intensive property** (does not grow with n).
- If $p = 1$: all nodes connected — the graph is a clique. The size of the giant component is an **extensive property** (scales with n).

As p increases, the giant component's size shifts from intensive to extensive.

Let ν be the proportion of nodes not in the giant component, and $\mu = 1-\nu$ the proportion in it.

Let:

- A: "Node i is not connected to node j"
- B: "Node i is connected to node j"

By definition:

$$\nu = (\mathbb{P}(A) + \mathbb{P}(B)\nu)^{n-1} = (1 - p + p\nu)^{n-1}.$$

This simplifies to:

$$\nu = \left(1 - \frac{\langle k \rangle}{n-1}(1-\nu)\right)^{n-1} \approx e^{-\langle k \rangle (1-\nu)}.$$

We then solve:

$$\mu = 1 - e^{-\langle k \rangle \mu}.$$

This can be visualized by plotting $f(s) = 1 - e^{-\langle k \rangle s}$ and identifying intersections with $y = s$:

- If $\langle k \rangle \leq 1$: only solution is $s = 0$ — no giant component (**subcritical**).
- If $\langle k \rangle > 1$: solution $s > 0$ — giant component exists (**supercritical**).
- $\langle k \rangle = 1$ is the **critical point**.

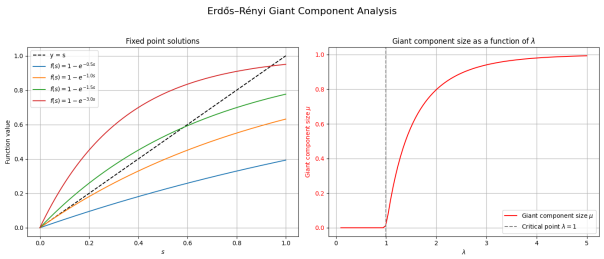


Figure 3: Analysis of the number of solutions of the equation $\nu = 1 - e^{\lambda\nu}$.

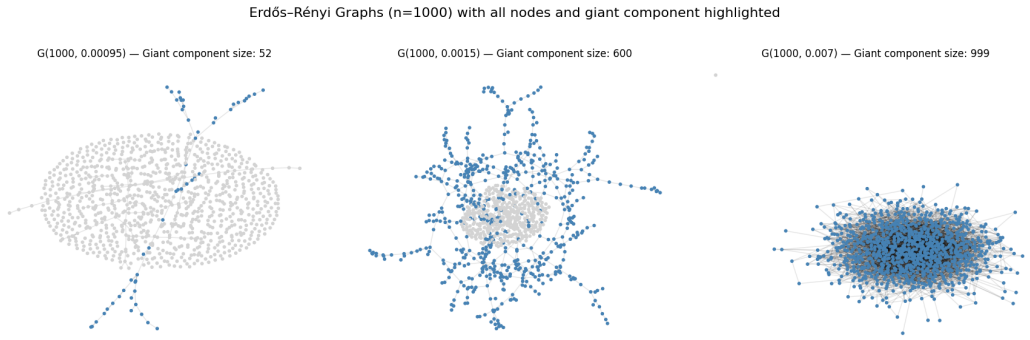


Figure 4: Erdős–Rényi graphs with increasing edge probability p.

For small p and large n, the degree distribution approximates a Poisson distribution with mean $\langle k \rangle$:

$$p(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}.$$

The global clustering coefficient is $C = p \approx \frac{\langle k \rangle}{n}$.

Definition: A social network is a network in which most nodes represent people.

Note: Triadic closure refers to the tendency of people with a common friend to become friends themselves, forming triangles.

The Erdős–Rényi model lacks two key properties of real social networks:

- It does not generate local clustering or triadic closure.
- It does not produce hubs — ER graphs follow a Poisson degree distribution, whereas many real-world networks follow a power-law distribution.

To address these limitations:

- The **Watts–Strogatz model** incorporates local clustering.
- The **Barabási–Albert model** generates networks with hubs.

2.3 The Watts–Strogatz (WS) model

The Watts–Strogatz (WS) model is a foundational model in the study of complex networks. Introduced in 1998 by Duncan Watts and Steven Strogatz, it aims to explain the *small-world phenomenon* observed in many real-world networks: nodes tend to be highly clustered like in regular lattices, yet the average shortest path between any two nodes remains small, as in random graphs. This dual behavior is not captured by purely regular or purely random networks.

Definition: A *small-world network* is a network that exhibits both high clustering and short average path length.

The WS model starts from a regular ring lattice and introduces randomness by rewiring edges with a certain probability. The model is defined by three parameters:

- N: the total number of nodes.
- k: each node is initially connected to its k nearest neighbors (assumed even).
- p: the probability of rewiring each edge.

Algorithm 1: Watts–Strogatz model generation algorithm

Input: Number of nodes N, number of neighbors k, rewiring probability p
Output: Generated graph G

Create a regular ring lattice G with N nodes;
for *each node* i **do**
    Connect i to its $k/2$ nearest neighbors on each side;
end
for *each edge* (i, j) *in* G **do**
    Generate a random number $r \in [0, 1]$;
    if $r < p$ **then**
        Disconnect the edge (i, j);
        Choose a random node $m \neq i$ such that (i, m) does not already exist;
        Connect i to m;
    end
end
return G;

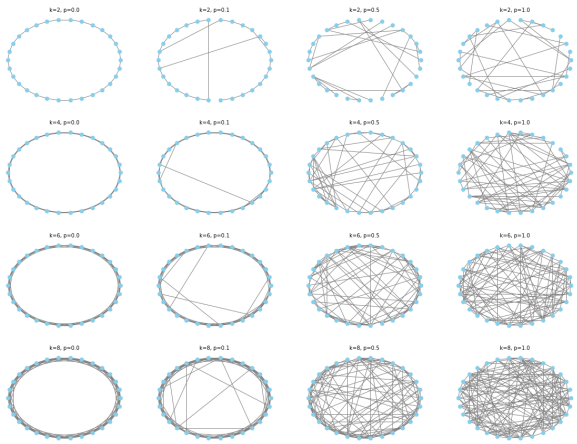


Figure 5: Watts–Strogatz graphs for different values of N, k, and p.

Properties

For a network generated using the WS model, the clustering coefficient C behaves as follows:

- When $p = 0$, the graph is regular, and the clustering coefficient is:

$$C_0 = \frac{3(k-2)}{4(k-1)} \longrightarrow \frac{3}{4} \quad \text{as } k \to \infty.$$

- For $0 < p < 1$, Barrat and Weigt (2000) showed that for $N \to \infty$, the clustering coefficient decays as:

$$C(p) \sim \frac{3(k-2)}{4(k-1)}(1-p)^3.$$

Thus:

- The clustering coefficient remains relatively high for small values of p.

- The average shortest path length ℓ decreases rapidly with even small increases in p.

The degree distribution $P(m)$ of the WS model can be approximated as:

$$P(m) = \sum_{x=\max\{2k-m,0\}}^{\min\{m,2k\}} \binom{2k}{x} p^x (1-p)^{2k-x} \frac{(kp)^{m-2k+x}}{(m-2k+x)!} e^{-kp},$$

where m is the degree of a node.

2.4 The Barabási-Albert (BA) model

The Barabási-Albert (BA) model is a fundamental model for generating *scale-free networks*, introduced by Albert-László Barabási and Réka Albert in 1999. It explains how many real-world networks—such as social networks, the internet, and citation networks—exhibit a power-law degree distribution.

The model is based on two key principles:

- **Growth:** The network starts with a small number of nodes and grows over time by adding new nodes.
- **Preferential Attachment:** New nodes are more likely to connect to nodes that already have a high degree—leading to a "rich-get-richer" effect.

This mechanism gives rise to hubs—nodes with a significantly larger number of connections than average.

Key Parameters:

- n: Total number of nodes in the final graph.
- m: Number of edges that each new node adds upon joining (attaching to m existing nodes).

Definition: A scale-free network is one in which the degree distribution follows a power law:

$$P(k) \sim k^{-\gamma}$$

Algorithm 2: Barabási-Albert Scale-Free Network Generation Algorithm

Input: Number of nodes N, number of edges m per new node
Output: Generated graph G

Initialize a fully connected graph with $m_0 \geq m$ nodes;
for *each time step* $t = 1$ *to* $N - m_0$ **do**
    Add a new node v_t with m edges;
    for *each edge of* v_t **do**
        Select an existing node v_i with probability $P(i) = \frac{k_i}{\sum_l k_l}$;
        Connect v_t to v_i;
    end
end
return G;

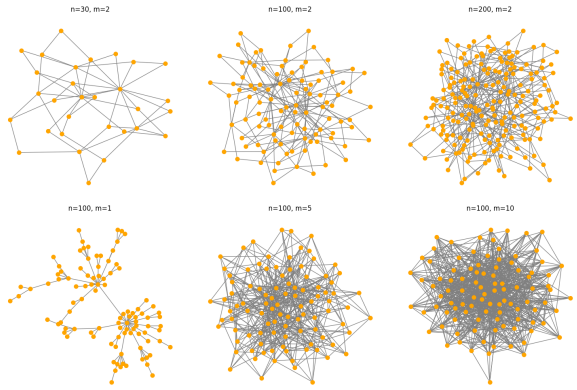


Figure 6: Barabási-Albert graphs for different values of n and m.

Properties

To derive the degree distribution, we use the *continuum approach*. Let $k_i(t)$ be the degree of node i at time t, introduced at time t_i. Initially, $k_i(t_i) = m$.

The probability that node i receives a connection is proportional to its degree:

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j} = \frac{k_i}{2mt}$$

Assuming continuous degree growth:

$$\frac{dk_i}{dt} = m \cdot \frac{k_i}{2mt} = \frac{k_i}{2t}$$

Solving this differential equation with the initial condition $k_i(t_i) = m$, we obtain:

$$k_i(t) = m \left(\frac{t}{t_i} \right)^{1/2}$$

The cumulative distribution is:

$$P(k_i(t) < k) = P\left(t_i > \frac{mt}{k^2} \right)$$

Assuming uniform node addition over time:

$$P(t_i > T) = 1 - \frac{T}{t + m_0} \quad \Rightarrow \quad P(k_i(t) < k) = 1 - \frac{mt}{k^2(t + m_0)}$$

Differentiating:

$$P(k) = \frac{d}{dk} P(k_i(t) < k) = \frac{2mt}{(t + m_0)} k^{-3} \Rightarrow P(k) \sim k^{-3}$$

This indicates a scale-free network with power-law exponent $\gamma = 3$.

An alternative derivation using the *master-equation approach* yields:

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \sim k^{-3}$$

This result confirms that the BA model generates scale-free networks at all times, not just asymptotically.

Degree Correlations

Let $N_{kl}(t)$ denote the number of node pairs with degrees k and l. Its evolution is governed by:

$$\frac{dN_{kl}}{dt} = (k-1)N_{k-1,l} - kN_{kl}\left(\frac{k}{kN(k)}\right) + (l-1)N_{k,l-1} - lN_{kl}\left(\frac{l}{kN(k)}\right)$$

Assuming $kkN(k) \to 2t$ and $N_{kl}(t) \to tn_{kl}$, we derive:

$$n_{kl} = \frac{4(l-1)}{k(k+1)(k+l)(k+l+1)(k+l+2)} + \frac{12(l-1)}{k(k+l-1)(k+l)(k+l+1)(k+l+2)}$$

This result shows that the joint distribution n_{kl} is **not** factorizable:

$$n_{kl} \neq n_k \cdot n_l$$

indicating nontrivial degree correlations in the BA model.

3 Applications in Financial Networks

3.1 Visualization of Asset Correlations

Asset correlation is a fundamental concept in finance that quantifies how two or more assets move in relation to each other. It plays a critical role in risk management, portfolio diversification, and asset allocation.

Definition: The correlation between two assets is typically measured via the Pearson correlation coefficient. Given a set of n assets over m time periods, let p_j^i denote the price of asset i at time j. The log-return of asset i at time j is defined by:

$$r_j^i = \log\left(\frac{p_j^i}{p_{j-1}^i}\right)$$

The log-returns can be organized as follows:

Date	$\mathbf{Asset}_1$	$\mathbf{Asset}_2$	$\mathbf{Asset}_3$	$\dots \mathbf{Asset}_n$
Date_1	r_1^1	r_1^2	r_1^3	$\dots r_1^n$
Date_2	r_2^1	r_2^2	r_2^3	$\dots r_2^n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Date_m	r_m^1	r_m^2	r_m^3	$\dots r_m^n$

Table 1: Log-returns by date and asset.

From the log-return matrix, we compute the correlation matrix $A \in \mathbb{R}^{n\times n}$, where each entry is:

$$\text{Cor}_{i,j} = \frac{\sum_{k=1}^m \left(r_k^i - \bar{r}_i\right)\left(r_k^j - \bar{r}_j\right)}{\sigma_i \sigma_j}$$

Here, $\bar{r}_i$ is the mean of asset i's returns, and σ_i is its standard deviation.

An interesting extension would be to use a **weighted correlation**, where weights reflect, for example, asset-specific volatilities or recency:

$$\text{Corr}_w(X,Y) = \frac{\sum_{i=1}^m w_i (X_i - \bar{X}_w)(Y_i - \bar{Y}_w)}{\sqrt{\sum_{i=1}^m w_i (X_i - \bar{X}_w)^2} \cdot \sqrt{\sum_{i=1}^m w_i (Y_i - \bar{Y}_w)^2}}$$

where $\bar{X}_w = \frac{\sum_{i=1}^m w_i X_i}{\sum_{i=1}^m w_i}$ is the weighted mean.

We can visualize the correlation structure as a **graph**, where:

- Nodes represent assets.
- Edges represent pairwise correlations.

We applied this method to a dataset of 9 financial instruments:

$$\{\texttt{GOOG, TSLA, PX1, MSFT, DAX, .INX, AMZN, JPM, VTI}\}$$

based on daily closing prices from 01/01/2023 to 12/31/2023. The resulting correlation graph is shown in Figure 7.

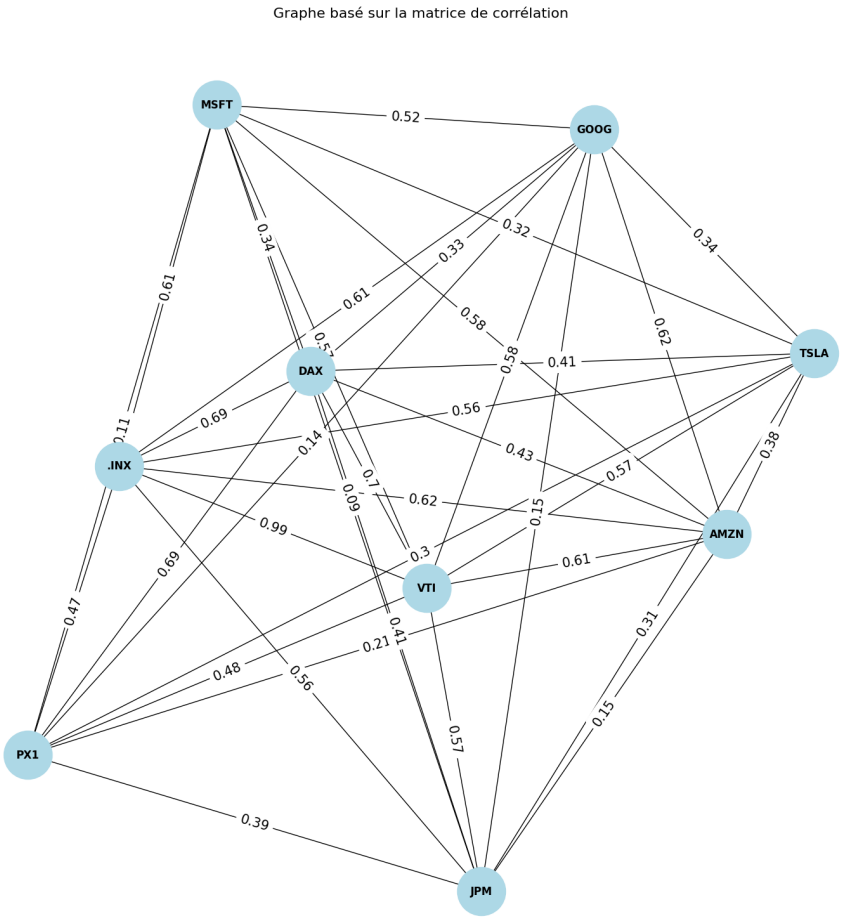


Figure 7: Correlation graph of selected assets based on Pearson correlations.

To enhance interpretability, we introduce a threshold s, retaining only the edges where $\text{Cor}_{i,j} > s$. This filtering yields a sparse graph that emphasizes the strongest relationships, from which we extract a **Minimum Spanning Tree (MST)** to preserve connectivity with minimal redundancy. The MST is shown in Figure 8.

Model Comparison

We now seek to model this financial network structure using canonical random graph models:

- Erdős–Rényi (ER) model
- Barabási–Albert (BA) model
- Watts–Strogatz (WS) model

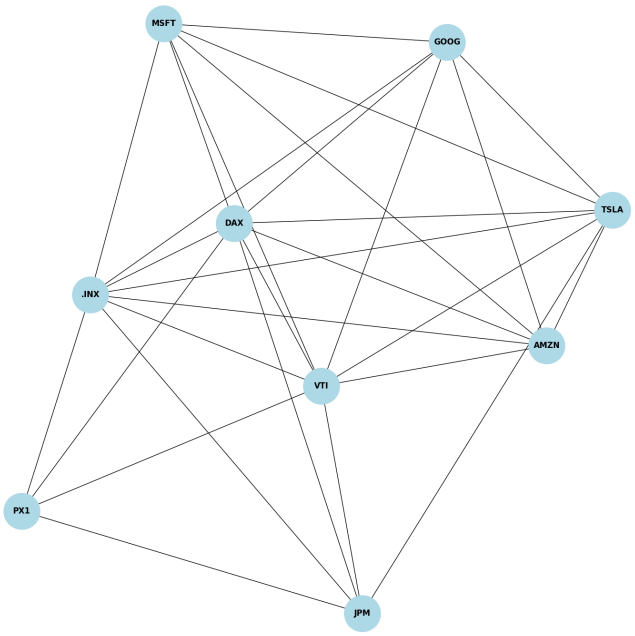


Figure 8: Minimum Spanning Tree derived from filtered correlation matrix.

For each model, we:

1. Estimate the model parameters from the empirical MST graph.
2. Generate a synthetic graph using the fitted model.
3. Compare structural properties (degree distribution, clustering, diameter, assortativity, etc.) between the synthetic and real graph.

This analysis helps evaluate which generative model best captures the observed topology of financial networks and provides insights into the underlying mechanisms of asset interdependence.

3.1.1 Approximation Using the Erdős–Rényi Model

The Erdős–Rényi model $\mathbb{G}(n, p)$ generates random graphs by connecting each pair of n nodes independently with probability p. To approximate a given empirical network using this model, we estimate p by:

$$\hat{p} = \frac{2m}{n(n-1)} = \frac{\text{Number of observed edges}}{\binom{n}{2}}$$

Applying this formula to the asset correlation graph, we obtain:

$$\hat{p} = 0.83$$

We then generate a random graph $\mathbb{G}(9, 0.83)$, whose structure is visualized in Figure 9. Key structural statistics of the simulated graph are compared to those of the real network

in Table 3.

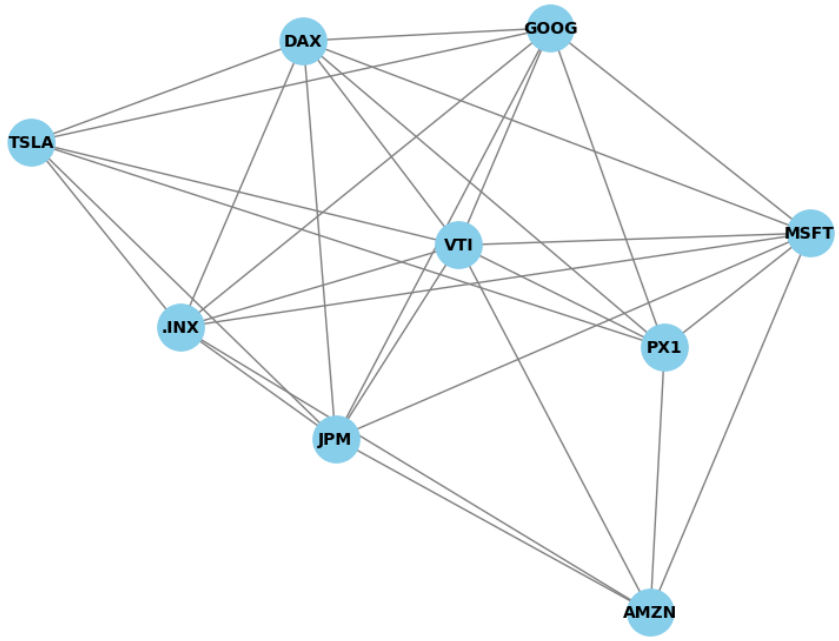


Figure 9: Erdős–Rényi graph $\mathbb{G}(9, 0.83)$ generated from estimated probability $\hat{p}$.

Metric	**Empirical Graph**	**Erdős–Rényi Graph**
Number of nodes (n)	9	9
Number of edges (m)	29	31
Average degree ($\langle k \rangle$)	6.44	6.89
Clustering coefficient	0.89	0.85
Average path length	1.19	1.14
Size of largest component	9	9

Table 2: Comparison between the real asset graph and an Erdős–Rényi graph $\mathbb{G}(9, \hat{p})$.

Discussion.

- The ER model appears to replicate several key features of the empirical graph closely, such as the average degree, clustering coefficient, and average path length.
- This result is somewhat surprising, as ER graphs typically have lower clustering in larger-scale networks. However, due to the small network size and high edge density, the ER model performs well in this case.
- Nonetheless, the ER model does not capture mechanisms such as preferential attachment or local clustering dynamics that are often present in real-world financial networks.
- In the next section, we examine whether a **Barabási–Albert model**, which introduces scale-free properties, better explains the observed topology.

3.1.2 Approximation Using the Barabási–Albert (BA) Model

The Barabási–Albert (BA) model generates networks through a preferential attachment mechanism, producing scale-free graphs whose degree distribution follows a power law of the form:

$$P(k) \sim k^{-\gamma},$$

where k is the node degree, and γ is the power-law exponent, typically close to 3 in many real-world networks.

Parameter Estimation Given the adjacency matrix A, we estimate the BA model parameters as follows:

1. **Degree calculation:** Compute the degree k_i of each node i:

$$k_i = \sum_j A_{ij}.$$

2. **Degree distribution:** Construct the empirical degree distribution $P(k)$, representing the frequency of nodes with degree k.

3. **Power-law fit:** Fit $P(k)$ to a power law by performing a linear regression on the log–log plot:

$$\log P(k) = -\gamma \log k + C,$$

   to estimate the exponent $\hat{\gamma}$.

4. **Estimating m:** In the BA model, each newly added node connects to m existing nodes, which implies an expected average degree:

$$\langle k \rangle \approx 2m.$$

   Thus, we estimate:

$$\hat{m} = \left\lfloor \frac{\langle k \rangle}{2} \right\rfloor.$$

Applying this procedure to the asset correlation network yields:

$$\hat{m} = 3, \quad \hat{\gamma} = 1.3,$$

where $\hat{\gamma}$ is positive, as expected for the degree exponent.

Generated BA Graph and Comparison Using the estimated parameter $\hat{m}$, we generated a BA graph $\mathbb{G}(n, \hat{m})$ with $n = 9$ nodes. The resulting network is shown in Figure 10, and its structural properties are summarized alongside the empirical graph in Table 3.

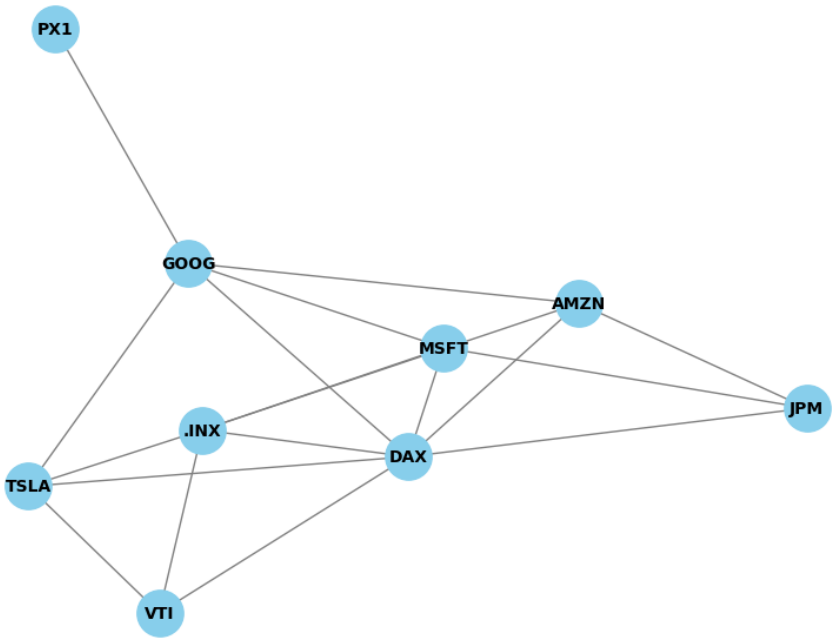


Figure 10: Barabási–Albert model $\mathbb{G}(9, 3)$ generated from estimated parameter $\hat{m} = 3$.

Metric	**Empirical Graph**	**BA Model Graph**
Number of nodes (n)	9	9
Number of edges (m)	29	18
Average degree ($\langle k \rangle$)	6.44	4
Clustering coefficient	0.89	0.51
Average path length	1.19	1.58

Table 3: Comparison of graph metrics between the real asset correlation network and the BA model $\mathbb{G}(9, 3)$.

Discussion

- The BA model generates a sparser network compared to the empirical one, as indicated by the smaller number of edges and lower average degree.
- The clustering coefficient is substantially lower in the BA graph, reflecting the known tendency of the BA model to produce networks with limited local clustering.
- The average path length is longer in the BA graph, consistent with its sparser connectivity.
- The estimated power-law exponent $\hat{\gamma} = 1.3$ deviates notably from the canonical value (3) expected in scale-free networks generated by the BA mechanism, suggesting that the real network does not follow a pure preferential attachment process.
- Consequently, the BA model does not adequately capture the structure of the observed asset correlation network.

3.2 Bank Networks

Bank networks refer to the interconnected relationships among financial institutions, particularly banks, which facilitate the flow of capital, payment systems, and financial services. These networks are critical to ensuring liquidity and stability across the banking system.

- **Interbank Lending and Borrowing**: Banks often lend money to each other on a short-term basis to manage liquidity and meet reserve requirements. These relationships are typically represented as a directed graph, where nodes correspond to banks and edges denote loans or borrowings.
- **Payment Systems**: Banks participate in payment systems to settle transactions. This includes domestic payment networks (e.g., ACH in the US, TARGET2 in Europe) and global systems (e.g., SWIFT for international transfers).

The structure of a bank network strongly influences risk propagation. Highly interconnected banks can provide liquidity but may also act as contagion channels if one fails. Dense networks facilitate liquidity flow but also increase the risk of cascading defaults, leading to systemic crises—commonly referred to as *contagion*.

Example Consider a simple lending network of five banks: A, B, C, D, and E:

- Bank A lends to B, C, and D.
- Banks B and C lend to D.
- Bank C lends to E.

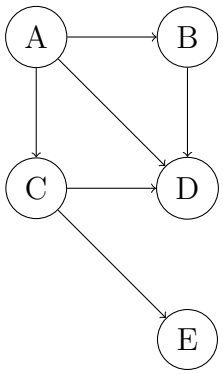


Figure 11: Interbank lending network among banks A, B, C, D, and E.

This network is centralized around bank A. If bank A defaults, it would significantly impact the entire network.

Historical Illustration Figure 12 illustrates correspondent relationships among 12 banks in 1862 and 1867. The hierarchical arrangement reflects the National Banking Association's reserve pyramid, with central reserve city banks at the top, reserve city banks in the middle, and non-reserve city banks at the bottom.

Contagion Examples In Figure 13, the contagion effect triggered by the American Exchange Bank's losses is shown. More centralized networks in 1867 experience stronger contagion.

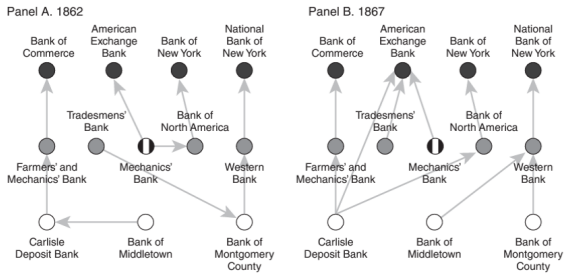


Figure 12: Interbank correspondent relationships among 12 banks in 1862 and 1867. Nodes are color-coded by bank type and arranged to reflect the three-tier reserve pyramid structure. Arrows indicate deposit relationships pointing to correspondent banks.(see [1])

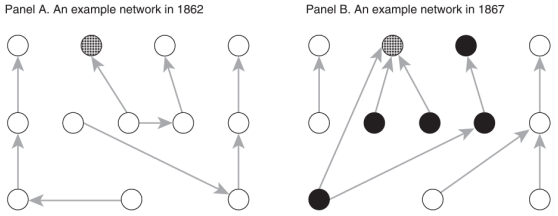


Figure 13: Degree of contagion when the American Exchange Bank experiences significant investment losses. Black nodes are heavily impacted; white nodes are not. The increased centralization in 1867 amplifies contagion effects. (see [1])

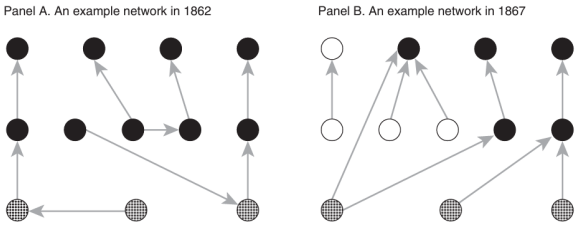


Figure 14: Contagion spread following an exogenous withdrawal shock affecting country banks. (see [1])

Figure 14 depicts a top-down crisis where country banks suffer an exogenous withdrawal shock, illustrating contagion propagation from peripheral to central banks.

Modeling Contagion We can study contagion within the frameworks of random graph models introduced earlier. By fitting an appropriate random graph model to the real network structure, we can simulate contagion dynamics and assess systemic risk.

3.2.1 Contagion Model

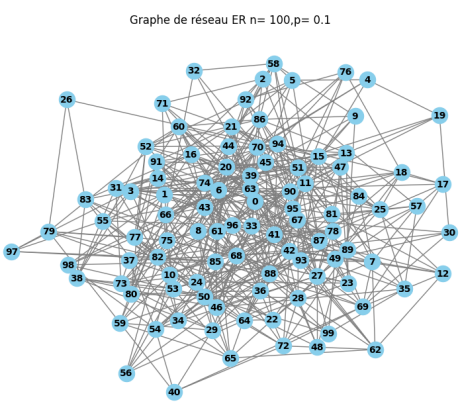
Contagion in networks is often modeled using epidemiological frameworks such as the **Susceptible-Infected-Recovered (SIR)** model or similar stochastic processes:

- **Susceptible (S)**: Nodes not yet infected.
- **Infected (I)**: Nodes currently infected and capable of transmitting contagion.
- **Recovered (R)**: Nodes recovered or immune, no longer susceptible.

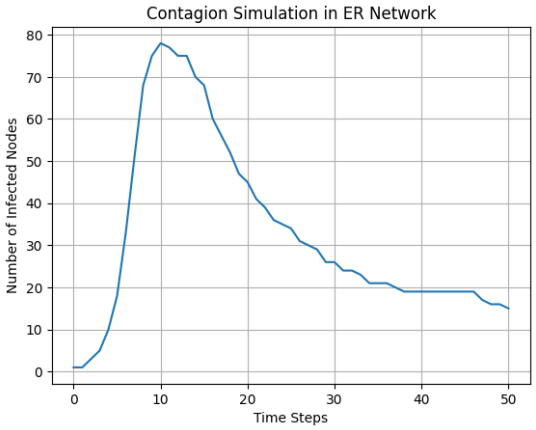
The process evolves as follows:

1. **Initial infection**: Some nodes start as infected.
2. **Spread**: Each infected node infects its neighbors with probability p, depending on the network structure.
3. **Recovery**: After a certain time, infected nodes recover.
4. **Iteration**: The process repeats until no susceptible or infected nodes remain.

Results can be visualized by plotting the number or proportion of infected nodes over time.

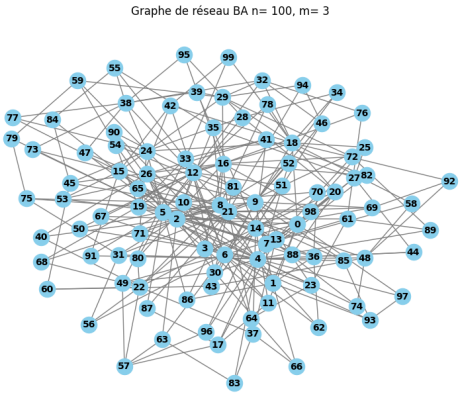


(a) ER Network $G(100, 0.1)$

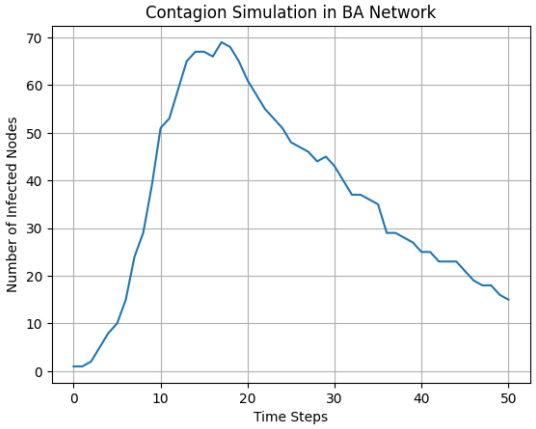


(b) Contagion simulation on ER

Figure 15: Contagion simulation in an Erdős-Rényi network

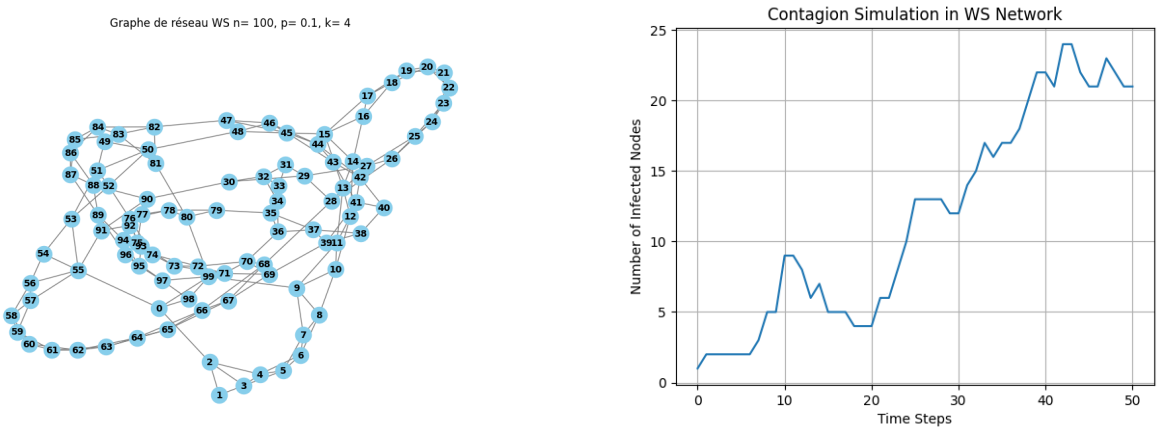


(a) BA Network $G(100, 3)$



(b) Contagion simulation on BA

Figure 16: Contagion simulation in a Barabási–Albert network



(a) WS Network $G(100, 0.1, 4)$ (b) Contagion simulation on WS

Figure 17: Contagion simulation in a Watts–Strogatz network

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