Math 122a
Homework #2

Due: Tuesday, 9/19

Problem 1 *Computing fluid resistance*

In the coral reef model from class, we assumed the flow rate Q through a pipe satisfied $Q = P/R$, where P was the pressure and R was the fluid resistance. By Poiseuille's law, the resistance R can be calculated in terms of the pipe radius ρ:

$$R = \frac{8\mu L}{\pi \rho^4},$$

where μ is the fluid viscosity and L is the pipe length.

Let's try to accurately compute the fluid resistance of a certain kind of pipe that is surrounded by a solid outer core of radius 4 meters, so that in terms of the measurable outer radius x we have $\rho = x - 4$. If we choose parameters $L = 10$ and $\mu = \pi/80$ to make the prefactors cancel out, we're left with the formula

$$R(x)^{-1} = (x-4)^4 = x^4 - 16x^3 + 96x^2 - 256x + 256$$

for the inverse resistance.

Let's now explore the idea, expressed nicely in Charles Van Loan's Intro to Scientific Computing book, that *algorithms that are equivalent mathematically may behave very differently numerically.* Consider the polynomial $p(x) = R(x)^{-1}$ and the x-vector $x = (4.00001, 4.00002, 4.00003, ..., 4.00101)$.

(a) Plot $p(x)$ over x by evaluating p via its coefficients $1, -16, 96$, etc.

(b) Now, evaluate p via the expression $(x-4)^4$. Show both curves for $p(x)$ in the same figure using different colors.

(c) What do you observe and how can you explain your findings in terms of floating-point arithmetic? Algorithms are often constructed such that floating-point operations involve numbers of similar order of magnitude instead of numbers that vary greatly in size (whenever possible). Why?

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Problem 2 *Roundoff Errors in Action*

A famous example of roundoff error was a short-lived index devised at the Vancouver stock exchange. The index contained 1400 stocks listed on the exchange, and each stock was weighted equally in determining the value of the index (most other indexes are weighted so that large companies count more than small ones). At the time the index was started in January 1982, the sum of the initial selling prices of all 1400 stocks (the baseline sum) was rescaled to give the index an initial value of 1000.

Taken together, the stocks in this index underwent changes in price a total of 2800 times per day. Each time one of the stocks changed its price, the index was updated as follows:

$$\text{New Index} = \text{Old Index} \; + \; (\text{Change in Stock Price}) \; \frac{1000}{\text{Baseline Sum}}$$

Then, after each change, the index was truncated after the third decimal place. For example, if after a change in stock price the index stood at 735.32567, the computer simply dropped the last two digits, making it 735.325.

(a) Write a `python` function `trunc_x = truncate(x)` that truncates a number x after the third decimal place.

(b) Assume that the prices of the stocks initially varied randomly with a uniform distribution between \$0 and \$200. As discussed above, the sum of these initial prices is rescaled to give the index an initial value of 1000. To track the evolution of the index, draw a uniformly-distributed random value between -200 and 200 cents for each change in stock price. Write a program that computes the evolution of this index over time. Compute and plot the evolution of the index over 1 day and over 22 months (you may assume 30 days in a month). Briefly explain what you see. (*Hint:* The python library `random` with function `random.random()` will be useful to you.)

(c) Given the truncation procedure used in updating the index, by how many points on average would you expect the index to drop in 1 day and in 22 months?

(d) Make a suggestion for how to modify the procedure used to update the index. Implement your modification (you will need to write a new function to replace the function `truncate` from part (a)). Plot the evolution of your modified index over 1 day and over 22 months, and compare this to the evolution of the actual index. What do you observe?

(e) After 22 months, the actual index stood at 524.881. What can you infer about the evolution of the market during this time: was it a bull market or a bear market? Explain.

Peter Brown, the former chairman of the Vancouver stock exchange, and his colleagues knew something was awry when the index would not climb over 1000

in February even though the exchange was setting records for volume and value. So they started an investigation, and the mistake was found. Upon opening of the stock exchange on Monday, November 28, 1983, the index mysteriously went up by 574.081 points from Friday's close despite no changes in stock prices. The index was abandoned shortly thereafter.

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