

The Reach-Avoid Problem for Constant-Rate Multi-Mode Systems^{*} ^{**}

Shankara Narayanan Krishna¹, Aviral Kumar¹,
Fabio Somenzi², Behrouz Touri², and Ashutosh Trivedi²

¹ Indian Institute of Technology Bombay, India.

² University of Colorado Boulder, USA.

Abstract. A constant-rate multi-mode system is a hybrid system that can switch freely among a finite set of modes, and whose dynamics is specified by a finite number of real-valued variables with mode-dependent constant rates. Alur, Wojtczak, and Trivedi have shown that reachability problems for constant-rate multi-mode systems for open and convex safety sets can be solved in polynomial time. In this paper we study the reachability problem for non-convex state spaces, and show that this problem is in general undecidable. We recover decidability by making certain assumptions about the safety set. We present a new algorithm to solve this problem and compare its performance with the popular sampling based algorithm rapidly-exploring random tree (RRT) as implemented in the Open Motion Planning Library (OMPL).

1 Introduction

Autonomous vehicle planning and control frameworks [15,19] often follow the hierarchical planning architecture outlined by Firby [9] and Gat [11]. The key idea here is to separate the complications involved in low-level hardware control from high-level planning decisions to accomplish the navigation objective. A typical example of such separation-of-concerns is proving the controllability property (vehicle can be steered from any start point to arbitrary neighborhood of the target point) of the motion-primitives of the vehicle followed by the search (path-planning) for an obstacle-free path (called the *roadmap*) and then utilizing the controllability property to compose the low-level primitives to follow the path (path-following). However, in the absence of the controllability property, it is not always possible to follow arbitrary roadmaps with given motion-primitives. In these situations we need to study a motion planning problem that is not opaque to the motion-primitives available to the controller.

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We study this motion planning problem in a simpler setting of systems modeled as constant-rate multi-mode systems [4]—a switched system with constant-rate dynamics (vector) in every mode—and study the reachability problem for the non-convex safety sets. Alur et al. [4] studied this problem for convex safety sets and showed that it can be solved in polynomial time. Our key result is that even for the case when the safety set is defined using polyhedral obstacles, the problem of deciding reachability is undecidable. On a positive side we show that if the safety set is an open set defined by linear inequalities, the problem is decidable and can be solved using a variation of cell-decomposition algorithm [21]. We present a novel bounded model-checking [7] inspired algorithm equipped with acceleration to decide the reachability. We use the Z3-theorem prover as the constraint satisfaction engine for the quadratic formulas in our implementation. We show the efficiency of our algorithm by comparing its performance with the popular sampling based algorithm *rapidly-exploring random tree* (RRT) as implemented in the *Open Motion Planning Library* (OMPL).

For a detailed survey of motion planning algorithms we refer to the excellent expositions by Latombe [16] and LaValle [17]. The motion-planning problem while respecting system dynamics can be modeled [10] in the framework of hybrid automata [1,12]; however the reachability problem is undecidable even for simple stopwatch automata [14]. There is a vast literature on decidable subclasses of hybrid automata [1,6]. Most notable among these classes are initialized rectangular hybrid automata [14], two-dimensional piecewise-constant derivative systems [5], timed automata [2], and discrete-time control for hybrid automata [13]. For a review of related work on multi-mode systems we refer to [3,4].

2 Motivating Example

Let us consider a two-dimensional multi-mode system with three modes m_1, m_2 and m_3 shown geometrically with their rate-vectors in Figure 1(a). We consider the reach-while-avoid problem in the arena given in Figure 1(b) with two rectangular obstacles \mathcal{O}_1 and \mathcal{O}_2 and source and target points x_s and x_t , respectively. In particular, we are interested in the question whether it is possible to move a point-robot from point x_s to point x_t using directions dictated by the multi-mode system given in Figure 1(a) while avoiding passing through or even grazing any obstacle.

It follows from our results in Section 5 that in general the problem of deciding reachability is undecidable even with polyhedral obstacles. However, the example considered in Figure 1 has an interesting property that the safety set can be represented as a union of finitely many polyhedral open sets (cells). This property, as we show later, makes the problem decidable. In fact, if we decompose the workspace into cells using any off-the-shelf cell-decomposition algorithm, we only need to consider the sequences of obstacle-free cells to decide reachability. In particular, for a given sequence of obstacle-free convex sets such that the starting point is in the first set, and the target point is in last set, one can write a linear program checking whether there is a sequence of intermediate states, one

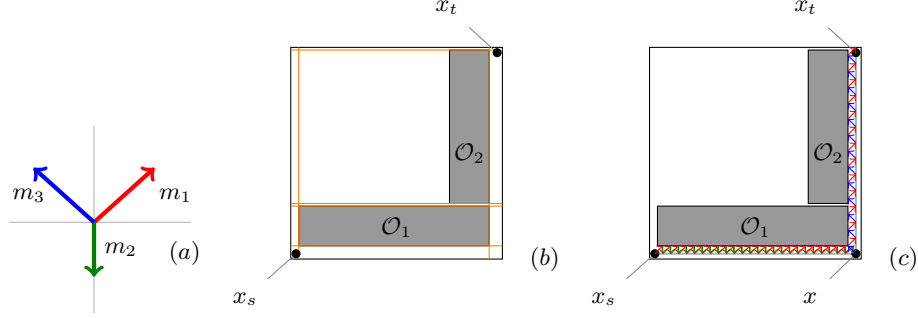


Fig. 1: a) A multi-mode system, b) an “L”-shaped arena consisting of obstacles \mathcal{O}_1 and \mathcal{O}_2 with start and target points x_s and x_t along with the cell-decomposition shown by orange lines, and c) a safe schedule from x_s to x_t .

each in the intersection of successive sets, such that these points are reachable in the sequence using the constant-rate multi-mode system. Our key observation is that one need not to consider cell-sequences larger than the total number of cells since for reachability, it does not help for the system to leave a cell and enter it again.

This approach, however, is not very efficient since one needs to consider all sequences of the cells. However, this result provides an upper bound on sequence of “meta-steps” or “bound” through the cells that system needs to take in order to reach the target and hint towards a bounded model-checking [7] approach. We progressively increase bound k and ask whether there is a sequence of points x_0, \dots, x_{k+1} such that $x_0 = x_s$, $x_{k+1} = x_t$, and for all $0 \leq i \leq k$ we have that x_i can reach x_{i+1} using the rates provided by the multi-mode system (convex cone of rates translated to x_i contains x_{i+1}) and the line segment $\lambda x_i + (1 - \lambda)x_{i+1}$ does not intersect any obstacle. Notice that if this condition is satisfied, then the system can safely move from point x_i to x_{i+1} by carefully choosing a scaling down of the rates so as to stay in the safety set, as illustrated in Figure 1.

Let us first consider $k = 0$ and notice that one can reach point x_t from x_s using just the mode m_1 , however unfortunately the line segment connecting these points passes through both obstacles. In this case we increase the bound by 1 and consider the problem of finding a point x such that the system can reach from x_s to x and also from x to x_t , and the line segment connecting x_s with x , and x with x_t do not intersect any obstacles. It is easy to see from the Figure 1 that it is indeed the case. We can alternate modes m_1, m_2 from x_s to x , and modes m_1, m_3 from x to x_t . Hence, there is a schedule that steers the system from x_s to x_t as shown in the Figure 1(c).

The property we need to check to ensure a safe schedule is the following: there exists a sequence of points $x_s = x_0, x_1, x_2, \dots, x_n = x_t$ such that for all $0 \leq \lambda \leq 1$, and for all i , the line $\lambda x_i + (1 - \lambda)x_{i+1}$ joining x_i and x_{i+1} does not intersect any obstacle \mathcal{O} . This can be thought of as a first-order formula of the form $\exists X \forall Y F(X, Y)$ where $F(X, Y)$ is a linear formula. By invoking the Tarski-

Seidenberg theorem we know that checking the satisfiability of this property is decidable. However, one can also give a direct quantifier elimination based on Fourier-Motzkin elimination procedure to get existentially quantified quadratic constraints that can be efficiently checked using theorem provers such as Z3 (<https://github.com/Z3Prover/z3>). This gives us a complete procedure to decide reachability for multi-mode systems when the safety set can be represented as a union of finitely many polyhedral open sets.

3 Problem Formulation

Points and Vectors. Let \mathbb{R} be the set of real numbers. We represent the states in our system as points in \mathbb{R}^n , which is equipped with the standard *Euclidean norm* $\|\cdot\|$. We denote points in this state space by x, y , vectors by \mathbf{r}, \mathbf{v} , and the i -th coordinate of point x and vector \mathbf{r} by $x(i)$ and $\mathbf{r}(i)$, respectively. The distance $\|x, y\|$ between points x and y is defined as $\|x - y\|$.

Boundedness and Interior. We denote an *open ball* of radius $d \in \mathbb{R}_{\geq 0}$ centered at x as $B_d(x) = \{y \in \mathbb{R}^n : \|x, y\| < d\}$. We denote a closed ball of radius $d \in \mathbb{R}_{\geq 0}$ centered at x as $\overline{B}_d(x)$. We say that a set $S \subseteq \mathbb{R}^n$ is *bounded* if there exists $d \in \mathbb{R}_{\geq 0}$ such that, for all $x, y \in S$, we have $\|x, y\| \leq d$. The *interior* of a set S , $\text{int}(S)$, is the set of all points $x \in S$, for which there exists $d > 0$ s.t. $B_d(x) \subseteq S$.

Convexity. A point x is a *convex combination* of a finite set of points $X = \{x_1, x_2, \dots, x_k\}$ if there are $\lambda_1, \lambda_2, \dots, \lambda_k \in [0, 1]$ such that $\sum_{i=1}^k \lambda_i = 1$ and $x = \sum_{i=1}^k \lambda_i \cdot x_i$. We say that $S \subseteq \mathbb{R}^n$ is *convex* iff, for all $x, y \in S$ and all $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$ and moreover, S is a *convex polytope* if there exists $k \in \mathbb{N}$, a matrix A of size $k \times n$ and a vector $\mathbf{b} \in \mathbb{R}^k$ such that $x \in S$ iff $Ax \leq \mathbf{b}$. A closed *hyper-rectangle* is a convex polytope that can be characterized as $x(i) \in [a_i, b_i]$ for each $i \leq n$ where $a_i, b_i \in \mathbb{R}$.

Definition 1. A (constant-rate) multi-mode system (MMS) is a tuple $\mathcal{H} = (M, n, R)$ where: M is a finite nonempty set of modes, n is the number of continuous variables, and $R : M \rightarrow \mathbb{R}^n$ maps to each mode a rate vector whose i -th entry specifies the change in the value of the i -th variable per time unit. For computation purposes, we assume that the real numbers are rational.

Example 1. An example of a 2-dimensional multi-mode system $\mathcal{H} = (M, n, R)$ is shown in Figure 1(a) where $M = \{m_1, m_2, m_3\}$, $n = 2$, and the rate vector is such that $R(m_1) = (1, 1)$, $R(m_2) = (0, -1)$, and $R(m_3) = (-1, 1)$.

A *schedule* of an MMS specifies a timed sequence of mode switches. Formally, a *schedule* is defined as a finite or infinite sequences of *timed actions*, where a timed action $(m, t) \in M \times \mathbb{R}_{\geq 0}$ is a pair consisting of a mode and a time delay. A finite *run* of an MMS \mathcal{H} is a finite sequence of states and timed actions $r = \langle x_0, (m_1, t_1), x_1, \dots, (m_k, t_k), x_k \rangle$ such that for all $1 \leq i \leq k$ we have that $x_i = x_{i-1} + t_i \cdot R(m_i)$. For such a run r we say that x_0 is the *starting state*, while x_k is its *terminal state*. An *infinite run* of an MMS \mathcal{H} is similarly defined to be

an infinite sequence $\langle x_0, (m_1, t_1), x_1, (m_2, t_2), \dots \rangle$ such that for all $i \geq 1$ we have that $x_i = x_{i-1} + t_i \cdot R(m_i)$.

Given a finite schedule $\sigma = \langle (m_1, t_1), (m_2, t_2), \dots, (m_k, t_k) \rangle$ and a state x , we write $Run(x, \sigma)$ for the (unique) finite run $\langle x_0, (m_1, t_1), x_1, (m_2, t_2), \dots, x_k \rangle$ such that $x_0 = x$. In this case, we also say that the schedule σ steers the MMS \mathcal{H} from the state x_0 to the state x_k .

We consider the problem of MMS reachability within a given *safety set* S . We specify the safety set by a pair $(\mathcal{W}, \mathcal{O})$, where $\mathcal{W} \subseteq \mathbb{R}^n$ is called the *workspace* and $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_k\}$ is a finite set of *obstacles*. In this case the safety set S is characterized as $S_{\mathcal{W} \setminus \mathcal{O}} = \mathcal{W} \setminus \mathcal{O}$. We assume in the rest of the paper that $\mathcal{W} = \mathbb{R}^n$ and for all $1 \leq i \leq k$, \mathcal{O}_i is a *convex* (not necessarily closed) polytope specified by a set of linear inequalities.

We say that a finite run $\langle x_0, (m_1, t_1), x_1, (m_2, t_2), \dots \rangle$ is S -safe if for all $i \geq 0$ we have that $x_i \in S$ and $x_i + \tau_{i+1} \cdot R(m_{i+1}) \in S$ for all $\tau_{i+1} \in [0, t_{i+1}]$. Notice that if S is a convex set then for all $i \geq 0$, $x_i \in S$ implies that for all $i \geq 0$ and for all $\tau_{i+1} \in [0, t_{i+1}]$ we have that $x_i + \tau_{i+1} \cdot R(m_{i+1}) \in S$. We say that a schedule σ is S -safe from a state x , or is (S, x) -safe, if the corresponding unique run $Run(x, \sigma)$ is S -safe. Sometimes we simply call a schedule or a run safe when the safety set and the starting state are clear from the context. We say that a state x' is S -safe reachable from a state x if there exists a finite schedule σ that is S -safe at x and steers the system from state x to x' .

We are interested in solving the following problem.

Definition 2 (Reachability). *Given a constant-rate multi-mode system $\mathcal{H} = (M, n, R)$, safety set S , start state x_s , and target state x_t , the reachability problem $REACH(\mathcal{H}, S_{\mathcal{W} \setminus \mathcal{O}}, x_s, x_t)$ is to decide whether there exists an S -safe finite schedule that steers the system from state x_s to x_t .*

Alur *et al.* [4] gave a polynomial-time algorithm to decide if a state x_t is S -safe reachable from a state x_0 for an MMS \mathcal{H} for a convex safety set S . In particular, they characterized the following necessary and sufficient condition.

Theorem 1 ([4]). *Let $\mathcal{H} = (M, n, R)$ be a multi-mode system and let $S \subset \mathbb{R}^n$ be an open, convex safety set. Then, there is an S -safe schedule from $x_s \in S$ to $x_t \in S$, if and only if there is $\mathbf{t} \in \mathbb{R}_{\geq 0}^{|M|}$ satisfying: $x_s + \sum_{i=1}^{|M|} R(m_i) \cdot \mathbf{t}(i) = x_t$.*

A key property of this result is that if x_t is reachable from x_s without considering the safety set, then it is also reachable inside arbitrary convex set as long as both x_s and x_t are strictly in the interior of the safety set.

We study the extension of this theorem for the reachability problem with non-convex safety sets. A key contribution of this paper is a precise characterization of the decidability of the reachability problem for multi-mode systems.

Theorem 2. *Given a constant-rate multi-mode system \mathcal{H} , workspace $\mathcal{W} = \mathbb{R}^n$, obstacles set \mathcal{O} , start state x_s and target state x_t , the reachability problem $REACH(\mathcal{H}, S_{\mathcal{W} \setminus \mathcal{O}}, x_s, x_t)$ is in general undecidable. However, if the obstacle set \mathcal{O} is given as finitely many closed polytopes, each defined by a finite set of linear inequalities, then reachability is decidable.*

4 Decidability

We prove the decidability condition of Theorem 2 in this section.

Theorem 3. *For a MMS $\mathcal{H} = (M, n, R)$, a safety set S , a start state x_s , and a target state x_t , the problem $\text{REACH}(\mathcal{H}, S_{\mathcal{W} \setminus \mathcal{O}}, x_s, x_t)$ is decidable if \mathcal{O} is given as finitely many closed polytopes.*

For the rest of this section let us fix a MMS $\mathcal{H} = (M, n, R)$, a start state x_s and a target state x_t . Before we prove this theorem, we define cell cover (a notion related to, but distinct from the one of cell decomposition introduced in [16]).

Definition 3 (Cell Cover). *Given a safety set $S \in \mathbb{R}^n$, a cell of S is an open, convex set that is a subset of S . A cell cover of S is a collection $\mathcal{C} = \{c_1, \dots, c_N\}$ of cells whose union equals S . Cells $c, c' \in \mathcal{C}$ are adjacent if and only if $c \cap c'$ is non-empty.*

A channel in S is a finite sequence $\langle c_1, c_2, \dots, c_N \rangle$ of cells of S such that c_i and c_{i+1} are adjacent for all $1 \leq i < N$. It follows that $\cup_{1 \leq i \leq N} c_i$ is a path-connected open set. A \mathcal{C} -channel is a channel whose cells are in cell cover \mathcal{C} .

Given a channel $\pi = \langle c_1, \dots, c_N \rangle$, a multi-mode system $\mathcal{H} = (M, n, R)$, start and target states $x_s, x_t \in S$, we say that π is a *witness* to reachability if the following linear program is feasible:

$$\begin{aligned} \bigwedge_{0 \leq i \leq N} x_i \cdot (x_s = x_0 \wedge x_t = x_N) \wedge \left(1 \leq i < N \rightarrow x_i \in (c_i \cap c_{i+1}) \right) \wedge \quad (1) \\ \bigwedge_{1 \leq i \leq N, m \in M} t_i^{(m)} \cdot (t_i^{(m)} \geq 0) \wedge \bigwedge_{1 \leq i \leq N} \left(x_i = x_{i-1} + \sum_{m \in M} R(m) \cdot t_i^{(m)} \right). \end{aligned}$$

Lemma 1. *If S is an open safety set, there exists a finite S -safe schedule that solves $\text{REACH}(\mathcal{H}, S, x_s, x_t)$ if and only if S contains a witness channel $\langle c_1, c_2, \dots, c_N \rangle$ for some $N \in \mathbb{N}$.*

Proof. (\Leftarrow) If $\langle c_1, c_2, \dots, c_N \rangle$ is a witness channel, then for $0 < i \leq N$, x_{i-1} and x_i are in c_i . Theorem 1 guarantees the existence of a c_i -safe schedule for each i . The concatenation of these schedules is a solution to $\text{REACH}(\mathcal{H}, S, x_s, x_t)$.

(\Rightarrow) The run of a finite schedule that solves $\text{REACH}(\mathcal{H}, S, x_s, x_t)$ defines a closed, bounded subset P of S . Since S is open, every point $x \in P$ is contained in a cell of S . Collectively, these cells form an open cover of P . By compactness, then, there is a finite subcover of P . If any element of the subcover is entered by the run more than once, there exists another run that is contained in that cell between the first entry and the last exit. For such a run, if two elements of the subcover are entered at the same time, the one with the earlier exit time is redundant. Therefore, there is a subcover in which no two elements are entered by the run of the schedule at the same time. This subcover can be ordered according to the time at which the run enters each cell to produce a sequence that satisfies the definition of witness channel. \square

Lemma 2. *If S is an open safety set and \mathcal{C} a cell cover of S , there exists a witness channel for $\text{REACH}(\mathcal{H}, S, x_s, x_t)$ iff there exists a witness \mathcal{C} -channel.*

Proof. One direction is obvious. Suppose therefore that there exists a witness channel; let σ be the finite schedule whose existence is guaranteed by Lemma 1. The path that is traced in the MMS \mathcal{H} when steered by σ is a bounded closed subset P of S because it is the continuous image of a compact interval of the real line. (The time interval in which \mathcal{H} moves from x_s to x_t .) Since \mathcal{C} is an open cover of P , there exists a finite subset of \mathcal{C} that covers P ; specifically, there is an irredundant finite subcover such that no two cells are entered at the same time during the run of σ . This subcover can be ordered according to entry time to produce a sequence of cells that satisfies the definition of witness channel. \square

Lemma 3. *If \mathcal{O} is a finite set of closed polytopes, then a finite cell cover of the safety set S is computable.*

Proof. If \mathcal{O} is a finite set of closed polytopes, one can apply the vertical decomposition algorithm of [16] to produce a cell *decomposition*. Each cell C in this decomposition of dimension less than n that is not contained in the obstacles (and hence is entirely contained in S) is replaced by a convex open set obtained as follows. Let B be an n -dimensional box around a point of C that is in S . The desired set is the convex hull of the set of vertices of either C or B . \square

Proof (of Theorem 3). Lemmas 1–2 imply that $\text{REACH}(\mathcal{H}, S, x_s, x_t)$ is decidable if a finite cell cover of S is available. If \mathcal{O} is given as a finite set of closed polytopes, each presented as a set of linear inequalities, then Lemma 3 applies. \square

The algorithm implicit in the proof of Theorem 3 requires one to compute the cell cover in advance, and enumerate sequences of cells in order to decide reachability. We next present an algorithm inspired by bounded model checking [7] that implicitly enumerates sequences of cells of increasing length till the upper bound on number of cells is reached, or a safe schedule from the source point to the target point is discovered. The key idea is to guess a sequence of points x_1, \dots, x_N starting from the source point and ending in the target point such that for every $1 \leq i < N$ the point x_{i+1} is reachable from x_i using rates provided by the multi-mode system. Moreover, we need to check that the line segment connecting x_i and x_{i+1} does not intersect with obstacles, i.e.: $\forall_{0 \leq \lambda \leq 1} (\lambda x_i + (1 - \lambda)x_{i+1}) \notin \cup_{j=1}^k \mathcal{O}_j$. We write $\text{OBSTACLEFREE}(x_i, x_{i+1})$ for this condition. Algorithm 1 sketches a bounded-step algorithm to decide reachability for multi-mode systems that always terminates for multi-mode systems with sets of closed obstacles defined by linear inequalities thanks to Theorem 3.

Notice that at line 2 of algorithm 1, we need to check the feasibility of the constraints system, which is of the form $\exists X \forall Y F(X, Y)$ where universal quantifications are implicit in the test for OBSTACLEFREE . If the solver we use to solve the constraints has full support to solve the \forall quantification, we can use that to solve the above constraint. In our experiments, we used the Z3 solver (<https://github.com/Z3Prover/z3>) to implement the Algorithm 1 and found

Algorithm 1: BOUNDEMOTIONPLAN($\mathcal{H}, \mathcal{W}, \mathcal{O}, x_s, x_t, B$)

Input: MMS $\mathcal{H} = (M, n, R)$, two points x_s, x_t , workspace \mathcal{W} , obstacle set \mathcal{O} , and an upper bound B on number of cells in a cell-cover.

Output: NO, if no safe schedule exists and otherwise such a schedule.

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1  $k \leftarrow 0$ ; while  $k \leq B$  do
2   Check if the following formula is satisfiable:
      
$$\bigvee_{1 \leq i \leq N} x_i \quad \bigvee_{1 \leq i \leq N, m \in M} t_i^{(m)} \text{ s.t. } (x_s = x_1 \wedge x_t = x_N) \wedge \bigwedge_{\substack{1 \leq i \leq N \\ m \in M}} t_i^{(m)} \geq 0 \wedge$$

      
$$\bigwedge_{i=2}^N \left( x_i = x_{i-1} + \sum_{m \in M} R(m) \cdot t_i^{(m)} \right) \wedge \bigwedge_{i=2}^N \text{OBSTACLEFREE}(x_{i-1}, x_i)$$

      if not satisfiable then  $k \leftarrow k + 1$ ;
3   else
4     Let  $\sigma$  be an empty sequence;
5     for  $i = 1$  to  $k - 1$  do
6        $\sigma = \sigma :: \text{REACH\_CONVEX}(\mathcal{H}, x_i, x_{i+1}, S)$ 
7   return  $\sigma$ ;
```

Algorithm 2: REACH_CONVEX(\mathcal{H}, x_s, x_t, S)

Input: MMS $\mathcal{H} = (M, n, R)$, two points x_s, x_t , convex, open, safety set S

Output: NO if no S -safe schedule from x_s to x_t exists and otherwise such a schedule.

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1  $t_1 = \min_{m \in M} \max \{ \tau : x_s + \tau \cdot R(m) \in S \}$ ;
2  $t_2 = \min_{m \in M} \max \{ \tau : x_t + \tau \cdot R(m) \in S \}$ ;
3  $t_{\text{safe}} = \min \{ t_1, t_2 \}$ ;
4 Check whether the following linear program is feasible:
      
$$x_s + \sum_{m \in M} R(m) \cdot t^{(m)} = x_t \text{ and } t^{(m)} \geq 0 \text{ for all } m \in M \quad (2)$$

5 if no satisfying assignment exists then return NO;
6 else
7   Find an assignment  $\{t^{(m)}\}_{m \in M}$ .
8   Set  $l = \lceil (\sum_{m \in M} t^{(m)}) / t_{\text{safe}} \rceil$ .
9   return the following schedule  $\langle (m_i, t_i) \rangle$  where
      
$$m_k = (k \bmod |M|) + 1 \text{ and } t_k = t^{(m_k)} / l \text{ for } k = 1, 2, \dots, l|M|.$$

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that the solver was unable to solve in some cases. Fortunately, the universal quantification in our constraints is of very special form and can be easily removed

using the Fourier-Motzkin elimination procedure, which results in quadratic constraints that are efficiently solvable by Z3 solver. In Section 6 we present the experimental results on some benchmarks to demonstrate scalability.

5 Undecidability

In this section we give a sketch of the proof of the following undecidability result.

Theorem 4. *Given a constant-rate multi-mode system \mathcal{H} , convex workspace \mathcal{W} , obstacles set \mathcal{O} , start state x_s and target state x_t , the reachability problem $\text{REACH}(\mathcal{H}, S_{\mathcal{W} \setminus \mathcal{O}}, x_s, x_t)$ is in general undecidable.*

Proof. (Sketch.) We prove the undecidability of this problem by giving a reduction from the halting problem for two-counter machines that is known to be undecidable [18]. Given a two counter machine \mathcal{A} having instructions $L = \ell_1, \dots, \ell_{n-1}, \ell_{halt}$, we construct a multi-mode system $\mathcal{H}_{\mathcal{A}}$ along with non-convex safety $S_{\mathcal{W} \setminus \mathcal{O}}$ characterized using linear constraints. The idea is to simulate the unique run of two-counter machine \mathcal{A} via the unique safe schedule of the MMS $\mathcal{H}_{\mathcal{A}}$ by going through a sequence of modes such that a pre-specified target point is reachable iff the counter machine halts.

Modes. For every increment/decrement instruction ℓ_i of the counter machine we have two modes \mathcal{M}_i and \mathcal{M}_{ik} , where k is the index of the unique instruction ℓ_k to which the control shifts in \mathcal{A} from ℓ_i . For every zero check instruction ℓ_i , we have four modes $\mathcal{M}_i^1, \mathcal{M}_i^2, \mathcal{M}_{ik}$ and \mathcal{M}_{im} , where k, m are respectively the indices of the unique instructions ℓ_k, ℓ_m to which the control shifts from ℓ_i depending on whether the counter value is > 0 or $= 0$. There are three modes $\mathcal{M}_{halt}, \mathcal{M}_{halt}^{c_1}$ and $\mathcal{M}_{halt}^{c_2}$ corresponding to the halt instruction. We have a special “initial” mode \mathcal{I} which is the first mode to be applied in any safe schedule.

Variables. The MMS $\mathcal{H}_{\mathcal{A}}$ has two variables $C = \{c_1, c_2\}$ that store the value of two counters. There is a unique variable $S = \{s_0\}$ used to enforce that mode \mathcal{I} as the first mode. For every increment or decrement instruction ℓ_i , there are variables w_{ij}, x_{ij} , where j is the index of the unique instruction ℓ_j to which control shifts from ℓ_i . We define variable $z_{i\#}$ for each zero-check instruction ℓ_i .

Simulation. A simulation of the two counter machine going through instructions $\ell_0, \ell_1, \ell_2, \dots, \ell_y, \ell_{halt}$ is achieved by going through modes $\mathcal{I}, \mathcal{M}_0, \mathcal{M}_{01}, \mathcal{M}_1$ or \mathcal{M}_1^1 or $\mathcal{M}_1^2 \dots, \mathcal{M}_y, \mathcal{M}_{y\ halt}$ in order, spending exactly one unit of time in each mode. Starting from a point x_s with $s_0 = 1$ and $v = 0$ for all variables v other than s_0 , we want to reach a point x_t where $w_{halt} = 1$ and $v = 0$ for all variables v other than w_{halt} . The idea is to start in mode \mathcal{I} , and spending one unit of time in \mathcal{I} obtaining $s_0 = 0, w_{01} = 1$ (spending a time other than one violates safety, see Lemma 4). Growing w_{01} represents that the current instruction is ℓ_0 , and the next one is ℓ_1 . Next, we shift to mode \mathcal{M}_0 , spend one unit of time there to obtain $x_{01} = 1, w_{01} = 0$. This is followed by mode \mathcal{M}_{01} , where x_{01} becomes 0, and one of the variables $z_{1\#}, w_{12}$ attain 1, depending on whether ℓ_1 is a zero

check instruction or not (again, spending a time other than one in $\mathcal{M}_0, \mathcal{M}_{01}$ violates safety, see Lemma 5).

In general, while at a mode \mathcal{M}_{ij} , the next instruction ℓ_k after ℓ_j is chosen by “growing” the variable w_{jk} if ℓ_j is not a zero-check instruction, or by “growing” the variable $z_{j\#}$ if ℓ_j is a zero-check instruction. In parallel, x_{ij} grows down to 0, so that $x_{ij} + w_{jk} = 1$ or $x_{ij} + z_{j\#} = 1$. The sequence of choosing modes, and enforcing that one unit of time be spent in each mode is necessary to adhere to the safety set as can be seen by Lemmas 5 and 6.

- In the former case, the control shifts from \mathcal{M}_{ij} to mode \mathcal{M}_j where variable x_{jk} grows at rate 1 while w_{jk} grows at rate -1, so that $x_{jk} + w_{jk} = 1$. Control shifts from \mathcal{M}_j to \mathcal{M}_{jk} , where the next instruction ℓ_g after ℓ_k is chosen by growing variable w_{kg} if ℓ_k is not zero-check instruction, or the variable $z_{k\#}$ is grown if ℓ_k is a zero-check instruction.
- In the latter case, one of the modes $\mathcal{M}_j^1, \mathcal{M}_j^2$ is chosen from \mathcal{M}_j where $z_{j\#}$ grows at rate -1. Assume ℓ_j is the instruction “If the counter value is > 0 , then goto ℓ_m , else goto ℓ_h ”. If \mathcal{M}_j^1 is chosen, then the variable x_{jm} grows at rate 1 while if \mathcal{M}_j^2 is chosen, then the variable x_{jh} grows at rate 1. In this case, we have $z_{j\#} + x_{jm} = 1$ or $z_{j\#} + x_{jh} = 1$. From \mathcal{M}_j^1 , control shifts to \mathcal{M}_{jm} , while from \mathcal{M}_j^2 , control shifts to \mathcal{M}_{jh} .

Continuing in the above fashion, we eventually reach mode $\mathcal{M}_{y\text{ halt}}$ where $x_{y\text{ halt}}$ grows down to 0, while the variable w_{halt} grows to 1, so that $x_{y\text{ halt}} + w_{\text{halt}} = 1$ (see Lemma 7 which enforces this).

Starting from x_s —which lies in the hyperplane H_0 given as $s_0 + w_{0j} = 1$ where ℓ_j is the unique instruction following ℓ_0 —a safe execution stays in H_0 as long as control stays in the initial mode \mathcal{I} . Control then switches to mode \mathcal{M}_0 , to the hyperplane H_1 given by $w_{0j} + x_{0j} = 1$. Note that $H_0 \cap H_1$ is non-empty and intersect at the point where $w_{0j} = 1$, and all other variables are 0. Spending a unit of time at \mathcal{M}_0 , control switches to mode \mathcal{M}_{0j} , and to the hyperplane H_2 given by $x_{0j} + w_{jk} = 1$ depending on whether ℓ_j is not a zero-check instruction. Again, note that $H_1 \cap H_2$ is non-empty and intersect at the point where $c_1 = 1, x_{0j} = 1$ and all other variables are zero. This continues, and we obtain a safe transition from hyperplane H_i to H_{i+1} as dictated by the simulation of the two counter machine. The sequence of safe hyperplanes lead to the hyperplane H_{last} given by $w_{\text{halt}} = 1$ and all other variables 0 iff the two counter machine halts. Appendix A.3 gives an example of a reduction from 2-counter machines. \square

6 Experimental Results

In this section, we discuss some preliminary results obtained with an implementation of Algorithm 1. In order to show competitiveness of the proposed algorithm, we compare its performance with a popular implementation of the RRT algorithm [17] on a collection of micro-benchmarks (some of these benchmarks are inspired by [20]).

Dimension	Arena Size	OMPLRRT		BoundedMotionPlan	
		Time(s)	Nodes	Time(s)	Witness Length
2	100×100	0.011	8	0.012	2
2	1000×1000	0.076	245	0.012	2
3	100×100	0.107	4836	0.183	2
3	1000×1000	1.9	1800	0.19	2
4	100×100	1.2	612	0.201	2
4	1000×1000	94.39	2857	0.206	2
5	100×100	3.12	778	2.69	2
5	1000×1000	149.4	2079	2.68	2
6	1000×1000	105	3822	15.3	2
7	1000×1000	319.63	2639	190.3	2

Table 1: Summary of results for the L shaped arena

6.1 Experimental Setup

Rapidly-exploring Random Tree (RRT) [17] is a space-filling data structure that is used to search a region by incrementally building a tree. It is constructed by selecting random points in the state space and can provide better coverage of reachable states of a system than mere simulations. There are many versions of RRTs available; we use the *Open Motion Planning Library (OMPL)* implementation of RRT for our experiments. The OMPL library (<http://ompl.kavrakilab.org>) consists of many state-of-the-art, sampling-based motion planning algorithms. We used the RRT API provided by the OMPL library. The results for RRT were obtained with a goal bias parameter set to 0.05, and obstacles implemented as `StateValidityCheckerFunction()` as mentioned in the documentation [22].

We implemented our algorithm on the top of the Z3 solver [8]. The implementation involves coding formulae in FO-logic over reals and checking for a satisfying assignment. Our algorithm was implemented in Python 2.7. The OMPL implementation was done in C++. The experiments with Algorithm 1 and RRT were performed on a computer running Ubuntu 14.10, with an Intel Core i7-4510 2.00 GHz quadcore CPU, with 8 GB RAM. We compared the two algorithms by executing them on a set of microbenchmarks whose obstacles are hyper-rectangular, though our algorithm can handle general polyhedral obstacles. We considered the following microbenchmarks.

- **L-shaped arena.** This class of microbenchmarks contains examples with hyper-rectangular workspace and certain “L” shaped obstacles as shown in Figure 1. The initial vertex is the lower left vertex of the square (x_s) and the target is the right upper vertex of the square (x_t). Our algorithm can give the solution to this problem with bound $B = 2$ returning the sequence $\langle x_1, x, x_t \rangle$ as shown in the figure, while the RRT algorithm in this case samples most of the points which lie on the other side of the obstacles and if the control modes are not in the direction of the line segments x_1x and xx_t , then it grows in arbitrary directions and hits the obstacles a large number of times, leading

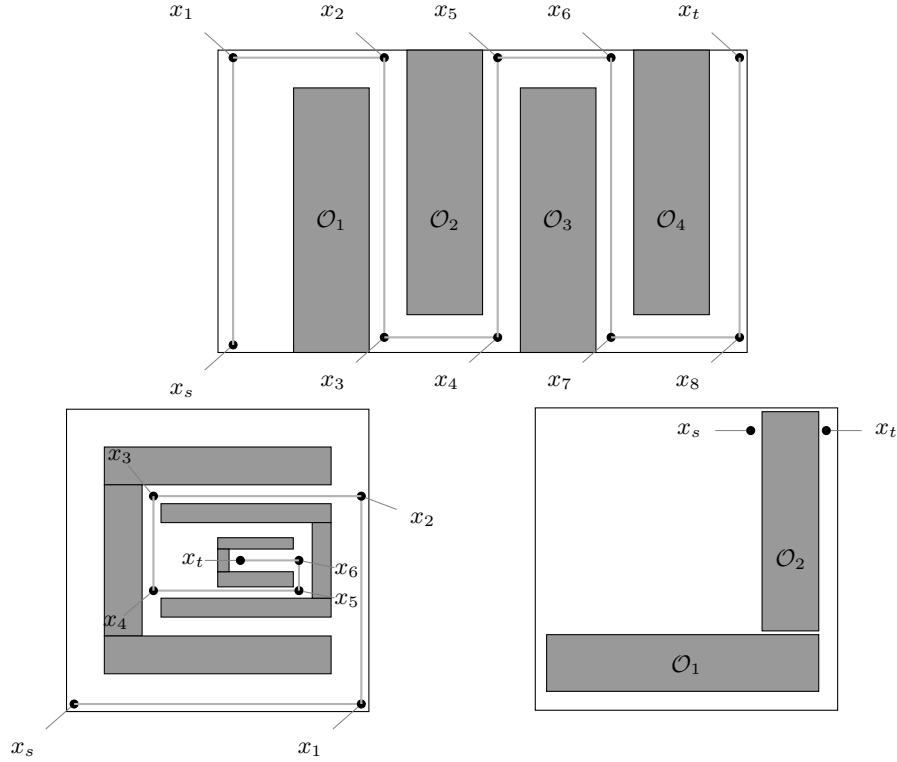


Fig. 2: a) Snake-shaped arena with four obstacles (left), b) Maze-shaped arena with three *C*-shaped patterns (middle) and c) modified L-shaped arena (right).

to a large number of iterations slowing the growth. We experimented with L-shaped examples for dimensions ranging from 2 to 7. In most of the cases, we found that the performance of the BOUNDEDMOTIONPLAN algorithm was better than that of OMPLRRT. Another important point to note is that RRT or other simulation-based algorithms do not perform well as the input size increases, which can be clearly seen from the running times obtained on increasing arena sizes in Table 1. Our algorithm worked better than RRT for higher dimensions (≥ 3).

- **Snake-shaped arena.** The name comes from the serpentine appearance of the safe sets in these arenas. The motivation to study these microbenchmarks comes from motion planning problems in regular environments. The arena has rectangular obstacles coming from the top and the bottom (as shown in Figure 2 for two dimensions) alternately. The starting point is the lower left vertex x_s and the target point is x_t . A sample free-path through the arena is also shown in the figure. RRT algorithm performs well for lower dimensions but fails to terminate for higher dimensions. The results for this class of

Dim.	Arena Size	Obstacles	OMPLRRT			BoundedMotionPlan	
			Time(s)	Nodes	Nodes in Path	Time(s)	Witness Length
2	350 × 350	3	3.56	13142	72	2.54	4
2	350 × 350	4	4.12	15432	96	4.23	5
2	3500 × 3500	3	4.79	15423	83	2.57	4
3	350 × 350	3	102.3	86314	67	96.43	4
3	3500 × 3500	3	100.22	1013	27	96.42	4

Table 2: Summary of results for the snake-shaped arena

Dim.	Arena Size	Obstacles	OMPLRRT			BoundedMotionPlan	
			Time(s)	Nodes	Nodes in Path	Time(s)	Bound
2	600 × 600	2	1.8	9500	60	1.3	4
2	6000 × 6000	3	23.5	11256	78	45.23	5
3	600 × 600	2	132.6	90408	71	120.3	5
3	6000 × 6000	3	1002.6	183412	93	953.4	5

Table 3: Summary of results for the maze-shaped arena

obstacles are summarised in Table 2. Experiments were performed for up to 3 dimensions and 4 obstacles.

- **Maze-shaped arena.** These benchmarks mimic the motion planning situations where the task of the robot is to navigate through a maze. We model a maze using finitely many concentric “C”-shaped obstacles with different orientations as shown in Figure 2. The task is to navigate from the lower left outer corner to the center point of the square. This kind of arena seems to be particularly challenging for the RRT algorithm and the growth of the tree seems to be quite slow. Also, the performance of our tool degrades as the bound increases due to a increase in the number of constraints, and hence, these examples require more time as compared to the other two microbenchmarks. However, as shown in Table 3, OMPLRRT and BoundedMotionPlan perform almost equally well, with the latter being slightly better.
- **Modified L-shaped obstacles.** These set of microbenchmarks contains a hyperrectangular workspace and 2 hyperrectangular obstacles arranged in a “L-shaped” fashion as shown in Figure 2. The initial vertex lies very close to one of the obstacles. The target vertex is the vertex very close to the start vertex but on the other side of the obstacle. Our algorithm can give the solution to this problem with bound $B = 3$ while RRT algorithm spends time in sampling from the bigger obstacle-free part of the arena. The results are summarised in Table 4.

The micro-benchmarks presented above involved the situations where the target point is reachable from the source point. It is interesting to see the performance of two algorithms in cases when there is no path from the source to target point. For the cases when an upper bound on cell-decomposition can be

Dimension	Arena Size	OMPLRRT			BoundedMotionPlan	
		Time	Nodes	Nodes in Path	Time	Bound
2	100 × 100	0.445	27387	40	0.126	3
2	1000 × 1000	2.57	38612	47	0.132	3
3	100 × 100	115.23	57645	71	92.1	3
3	1000 × 1000	675.62	183412	93	95.23	3
4	100 × 100	287.32	64230	65	283.23	3
4	1000 × 1000	923.45	192453	78	292.53	3
5	100 × 100	523.62	73422	69	534.45	3
5	1000 × 1000	1043	223900	72	533.96	3

Table 4: Summary of results for the modified L-shaped obstacles

Dimension	OMPLRRT		BoundedMotionPlan
	Time(s)	Nodes	Time(s)
2	500 (TO)	5301778	0.0088
3	500 (TO)	7892122	0.032
4	500 (TO)	4325621	0.056
5	500 (TO)	5624609	2.73
6	500 (TO)	4992951	18.34
7	500 (TO)	3765123	213.23

Table 5: Summary of results for the unreachable L-shaped obstacles.

imposed, our algorithm is capable of producing negative answer. Table 5 summarizes the performance of OMPLRRT and BoundedMotionPlan for L -shaped arenas when the target point is not reachable. The timeout for RRT was set to be 500 seconds, and it did not terminate until the timeout, which is as expected. On the other hand, BOUNDEDMOTIONPLAN performed well, with running times close to those when the target point is reachable.

Discussion. Our implementation of BOUNDEDMOTIONPLAN even though preliminary, compares favorably with a state-of-the-art implementation of RRT. BOUNDEDMOTIONPLAN, in addition, can naturally deal with restrictions on the dynamics of the MMS, that is, with systems such that the positive linear span of the mode vectors is not \mathbb{R}^n .

A trend observed in our experiments is that if a large fraction of the arena is covered by obstacles, then the probability of a randomly sampled point lying in the obstacle region is high and this makes RRT ineffective in this situation by wasting a lot of iterations. Another trend is that as the arena size increases, it becomes more difficult for RRT to navigate to the destination points even with higher values of goal bias.

Our algorithm performs better in situations when it terminates early (target reachable from source with shorter witnesses) while the performance of our algorithm degrades as the bound or the dimensions increases since the number

of constraints introduced by the Fourier-Motzkin like-procedure implemented in our algorithm grows exponentially with the dimension exhibiting the curse of dimensionality.

7 Conclusion

In this paper we studied the motion planning problem for constant-rate multi-mode system with non-convex safety sets given as a convex set of obstacles. We showed that while the general problem is already undecidable in this simple setting of linearly defined obstacles, decidability can be recovered by making appropriate assumption on the obstacles. Moreover, our algorithm performs satisfactorily when compared to well-known algorithms for motion planning, and can easily be adapted to provide semi-algorithms for motion-planning problems for objects with polyhedral shapes. While the algorithm is complete for classes of safety sets for which a bound on the size of a cell cover can be effectively computed, bounds based on cell decompositions of the safety set may be too large to be of practical use. This situation is akin to that encountered in bounded model checking of finite-state systems, in which bounds based on the radii of the the state graph are usually too large. We are therefore motivated to look at extensions of the algorithm that incorporate practical termination checks.

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A Proof of Theorem 4

In this section we present details of the proof of our undecidability theorem. We show undecidability of the motion planning problem by giving a reduction from the undecidable halting problem for two-counter machines.

Definition 4. A two-counter machine (Minsky machine) \mathcal{A} is a tuple (L, C) where: $L = \{\ell_0, \ell_1, \dots, \ell_{n-1}, \ell_{halt}\}$ is the set of instructions and $C = \{c_1, c_2\}$ is the set of two counters. There is a distinguished terminal instruction ℓ_{halt} called *HALT* and the instructions L are one of the following types:

- **increment.** $\ell_i : c := c + 1; \text{ goto } \ell_k,$
- **decrement.** $\ell_i : c := c - 1; \text{ goto } \ell_k,$
- **zero-test.** $\ell_i : \text{if } (c > 0) \text{ then goto } \ell_k \text{ else goto } \ell_m,$
- **Halt.** $\ell_{halt} : \text{HALT}.$

where $c \in C$, $\ell_i, \ell_k, \ell_m \in L$. Let I, D , and O represent the sets of increment, decrement and zero-check instructions, respectively.

A configuration of a two-counter machine is a tuple (ℓ, c, d) where $\ell \in L$ is an instruction, and c, d are natural numbers that specify the value of counters c_1 and c_2 , respectively. The initial configuration is $(\ell_0, 0, 0)$. A run of a two-counter machine is a (finite or infinite) sequence of configurations $\langle k_0, k_1, \dots \rangle$ where k_0 is the initial configuration, and the relation between subsequent configurations is governed by transitions between respective instructions. The run is a finite sequence if and only if the last configuration is the terminal instruction ℓ_{halt} . Note that a two-counter machine has exactly one run starting from the initial configuration. We assume without loss of generality that ℓ_0 is an increment instruction. The *halting problem* for a two-counter machine asks whether its unique run ends at the terminal instruction ℓ_{halt} . It is well known [18] that the halting problem for two-counter machines is undecidable.

A.1 Reduction

Given a two counter machine \mathcal{A} having instructions $L = \ell_1, \dots, \ell_{n-1}, \ell_{halt}$, we construct a MMS $\mathcal{H}_{\mathcal{A}}$ having a number of modes and variables polynomial in n . The idea is to simulate the two-counter machine in the MMS by going through a sequence of modes such that a target point is reachable iff the counter machine halts. We will next present the details of our reduction by characterizing the set of modes, the set of variables, rates of variables in different modes, as well as the set of obstacles for the instance of multi-mode system corresponding to a given instance of counter machine.

- **Modes.** For every increment/decrement instruction $\ell_i \in I \cup D$, we have two modes \mathcal{M}_i and \mathcal{M}_{ik} , where k is the index of the unique instruction ℓ_k to which the control shifts in \mathcal{A} from ℓ_i . For every zero check instruction $\ell_i \in O$, we have four modes $\mathcal{M}_i^1, \mathcal{M}_i^2, \mathcal{M}_{ik}$ and \mathcal{M}_{im} , where k, m are respectively the

indices of the unique instructions ℓ_k, ℓ_m to which the control shifts from ℓ_i depending on whether the counter value is > 0 or $= 0$. There are three modes $\mathcal{M}_{halt}, \mathcal{M}_{halt}^{c_1}$ and $\mathcal{M}_{halt}^{c_2}$ corresponding to the halt instruction. We have a special “initial” mode \mathcal{I} which is the first mode to be applied in any safe schedule in our reduction. This property is ensured using a special variable s_0 and a careful definition of obstacles.

- **Variables.** The multi-mode system has two variables $C = \{c_1, c_2\}$ that store the value of two counters. There is a unique variable $S = \{s_0\}$, whose rate is 0 at all modes other than the mode \mathcal{I} and is used to enforce that mode \mathcal{I} is the first valid mode in the starting state. For $\ell_i \in I \cup D$, there are variables w_{ij}, x_{ij} , where j is the index of the unique instruction ℓ_j to which control shifts from ℓ_i . We write W for the set $\{w_{ij}, w_{i\text{halt}} : 0 \leq i, j \leq n \text{ and } \ell_i \notin O\}$ and $X = \{x_{ij} : 0 \leq i, j \leq n\}$. Also, we define a variable $z_{i\#}$ for every $\ell_i \in O$ and we write $Z = \{z_{i\#} : \ell_i \in O\}$. Hence the set of variables is

$$\mathcal{V} = X \cup W \cup Z \cup C \cup S \cup \{w_{halt}\}.$$

Let X_{halt} be the subset of X consisting of variables of the form $x_{i\text{halt}}$. Observe that the dimension of the MMS $|\mathcal{V}|$ is $\mathcal{O}(n^2)$, where n is the number of instructions in the two counter machine.

- **Intuition for Dynamics and Obstacles.** A simulation of the two counter machine going through instructions $\ell_0, \ell_1, \ell_2, \dots, \ell_y, \ell_{halt}$ is achieved by going through modes $\mathcal{I}, \mathcal{M}_0, \mathcal{M}_{01}, \mathcal{M}_1$ or \mathcal{M}_1^1 or $\mathcal{M}_1^2 \dots, \mathcal{M}_y, \mathcal{M}_{y\text{halt}}$ in order, spending exactly one unit of time in each mode. Starting from a point with $s_0 = 1$ and $v = 0$ for all variables v other than s_0 , we want to reach a point where $w_{halt} = 1$ and $v = 0$ for all variables v other than w_{halt} . The idea is to start in mode \mathcal{I} , and spending one unit of time in \mathcal{I} obtaining $s_0 = 0, w_{01} = 1$. Growing w_{01} represents that the current instruction is ℓ_0 , and the next one is ℓ_1 . Next, we shift to mode \mathcal{M}_0 , spend one unit of time there to obtain $x_{01} = 1, w_{01} = 0$. This is followed by mode \mathcal{M}_{01} , where x_{01} becomes 0, and one of the variables $z_{1\#}, w_{12}$ attain 1, depending on whether ℓ_1 is a zero check instruction or not.

In general, while at a mode \mathcal{M}_{ij} , the next instruction ℓ_k after ℓ_j is chosen by “growing” the variable w_{jk} if ℓ_j is not a zero-check instruction, or by “growing” the variable $z_{j\#}$ if ℓ_j is a zero-check instruction. In parallel, x_{ij} grows down to 0, so that $x_{ij} + w_{jk} = 1$ or $x_{ij} + z_{j\#} = 1$.

- In the former case, the control shifts from \mathcal{M}_{ij} to mode \mathcal{M}_j where variable x_{jk} is grows at rate 1 while w_{jk} grows at rate -1, so that $x_{jk} + w_{jk} = 1$. Control shifts from \mathcal{M}_j to \mathcal{M}_{jk} , where the next instruction ℓ_g after ℓ_k is chosen by growing variable w_{kg} if ℓ_k is not zero-check instruction, or the variable $z_{k\#}$ is grown if ℓ_k is a zero-check instruction.
- In the latter case, one of the modes $\mathcal{M}_j^1, \mathcal{M}_j^2$ is chosen from \mathcal{M}_j where $z_{j\#}$ grows at rate -1. Assume ℓ_j is the instruction “If the counter value is > 0 , then goto ℓ_m , else goto ℓ_h ”. If \mathcal{M}_j^1 is chosen, then the variable x_{jm} grows at rate 1 while if \mathcal{M}_j^2 is chosen, then the variable x_{jh} grows at rate 1. In this case, we have $z_{j\#} + x_{jm} = 1$ or $z_{j\#} + x_{jh} = 1$. From \mathcal{M}_j^1 , control shifts to \mathcal{M}_{jm} , while from \mathcal{M}_j^2 , control shifts to \mathcal{M}_{jh} .

Continuing in the above fashion, we eventually reach mode $\mathcal{M}_{y \text{ halt}}$ where $x_{y \text{ halt}}$ grows down to 0, while the variable w_{halt} grows to 1, so that $x_{y \text{ halt}} + w_{\text{halt}} = 1$. It remains to use the modes $\mathcal{M}_{\text{halt}}, \mathcal{M}_{\text{halt}}^c$ as many times to obtain $c_1 = 0, c_2 = 0$ and $w_{\text{halt}} = 1$.

- **Dynamics.** We will next define the rates of the variables in different modes.
 1. Variable rates at mode \mathcal{I} are such that $R(\mathcal{I})(s_0) = -1, R(\mathcal{I})(w_{0j}) = 1$, while $R(\mathcal{I})(v) = 0$ for all variables v other than s_0 and w_{0j} . Here j is the index of the unique instruction ℓ_j to which the control shifts from the initial instruction ℓ_0 (recall that ℓ_0 is an increment instruction, and hence control shifts deterministically to some ℓ_j).
 2. Assume $\ell_i \in I \cup D$ is an increment/decrement instruction for counter $c_1(c_2)$ and let ℓ_j be the resultant instruction. The rates of variables at mode \mathcal{M}_i are $R(\mathcal{M}_i)(w_{ij}) = -1, R(\mathcal{M}_i)(x_{ij}) = 1$, and $R(\mathcal{M}_i)(v) = 0$ for $v \neq c_1(c_2)$, while $R(\mathcal{M}_i)(c_1) = 1$ ($R(\mathcal{M}_i)(c_2) = 1$) if $\ell_i \in I$ and $R(\mathcal{M}_i)(c_1) = -1$ ($R(\mathcal{M}_i)(c_2) = -1$) if $\ell_i \in D$.
 3. For $\ell_i \in O$, the rates of variables in the modes $\mathcal{M}_i^1, \mathcal{M}_i^2, \mathcal{M}_{ik}, \mathcal{M}_{im}$ are
 - (a) $R(\mathcal{M}_i^1)(z_{i\#}) = -1, R(\mathcal{M}_i^1)(x_{ik}) = 1$, and we have $R(\mathcal{M}_i^1)(v) = 0$ for all other variables v .
 - (b) $R(\mathcal{M}_i^2)(z_{i\#}) = -1, R(\mathcal{M}_i^2)(x_{im}) = 1$, and we have $R(\mathcal{M}_i^2)(v) = 0$ for all other variables v .
 4. We have the modes \mathcal{M}_{ij} for $i \in I \cup D \cup O$. The rates of variables at mode $\mathcal{M}_{ij}, j \neq \text{halt}$ are
 - (a) $R(\mathcal{M}_{ij})(x_{ij}) = -1$
 - (b) If ℓ_j is not a zero check instruction, then there is a unique instruction ℓ_k to which the control will shift from ℓ_j . Then $R(\mathcal{M}_{ij})(w_{jk}) = 1$, while $R(\mathcal{M}_{ij})(v) = 0$ for all variables $v \notin \{w_{jk}, x_{ij}\}$.
 - (c) If ℓ_j is a zero check instruction, then we have $R(\mathcal{M}_{ij})(z_{j\#}) = 1$, while $R(\mathcal{M}_{ij})(v) = 0$ for all variables $v \notin \{z_{j\#}, x_{ij}\}$.
 5. The rates of variables at mode $\mathcal{M}_{i \text{ halt}}$ are as follows:

$$R(\mathcal{M}_{i \text{ halt}})(x_{i \text{ halt}}) = -1 \text{ and } R(\mathcal{M}_{i \text{ halt}})(w_{\text{halt}}) = 1$$

while all other variables have rate 0. The rates at modes $\mathcal{M}_{\text{halt}}, \mathcal{M}_{\text{halt}}^{c_1}, \mathcal{M}_{\text{halt}}^{c_2}$ are given by :

- $R(\mathcal{M}_{\text{halt}})(w_{\text{halt}}) = -1$ and $R(\mathcal{M}_{\text{halt}})(v) = 0$ for all other variables.
- $R(\mathcal{M}_{\text{halt}}^{c_1})(c_1) = -1, R(\mathcal{M}_{\text{halt}}^{c_1})(w_{\text{halt}}) = 1$ while $R(\mathcal{M}_{\text{halt}}^{c_1})(v) = 0$ for all other variables.
- $R(\mathcal{M}_{\text{halt}}^{c_2})(c_2) = -1, R(\mathcal{M}_{\text{halt}}^{c_2})(w_{\text{halt}}) = 1$ while $R(\mathcal{M}_{\text{halt}}^{c_2})(v) = 0$ for all other variables.

- **Workspace and Obstacles.** Instead of describing the obstacles directly, we describe its complement. i.e. the *safety set*. The safety set consists of points in the Euclidean space satisfying $\bigwedge_{t=a}^g \varphi_t$ where:
 (φ_a) *Init*:

$$\begin{aligned} 0 \leq y \leq 1 & \text{ for } y \in N \cup S \cup X \cup Z \\ y \geq 0 & \text{ for } y \in \{n_{\text{halt}}, c_1, c_2\} \end{aligned}$$

(φ_b) $Mutex(X)$:

$$\bigwedge_{i,j} \left(x_{ij} > 0 \Rightarrow \bigwedge_{\substack{(g \neq i) \vee \\ (f \neq j)}} x_{gf} = 0 \right)$$

(φ_c) $Mutex(W, Z)$:

$$\bigwedge_{i,j} \left(w_{ij} > 0 \Rightarrow \left(\bigwedge_{\substack{(g \neq i) \vee \\ (f \neq j)}} w_{gf} = 0 \wedge \bigwedge_g z_{g\#} = 0 \right) \right)$$

(φ_d) $Mutex(Z, W)$:

$$\bigwedge_i \left(z_{i\#} > 0 \Rightarrow \left(\bigwedge_{k \neq i} z_{k\#} = 0 \wedge \bigwedge_{g,f} w_{gf} = 0 \right) \right)$$

(φ_e) $Mutex(S, X)$:

$$s_0 > 0 \Rightarrow \bigwedge_{i,j} x_{ij} = 0$$

(φ_f) $Mutex(w_{halt}, X \cup W \cup Z \cup S)$:

$$w_{halt} > 0 \Rightarrow \bigwedge_{y \in \mathcal{V} \setminus (X_{halt} \cup C)} (y = 0)$$

(φ_g) $Sum(X_{halt}, w_{halt})$:

$$\bigwedge_i (x_{i \text{ halt}} > 0 \Rightarrow (x_{i \text{ halt}} + w_{halt} = 1))$$

An obstacle \mathcal{O} is thus one which satisfies $\bigvee_{t=a}^g \neg \varphi_t$. As an example, $\mathcal{O}_{i \text{ halt}} = x_{i \text{ halt}} > 0 \wedge (x_{i \text{ halt}} + w_{halt} \neq 1)$ is an obstacle obtained by negating φ_g . Note that the safety set thus defined is not necessarily an open set.

A.2 Correctness of the Reduction

We represent a point in the state space of the multi-mode system as a tuple of valuation to all variables with an arbitrary but fixed ordering. In our ordering, s_0 is the first variable, followed by c_1 and c_2 as the next two variables, and w_{halt} as the last variable in the tuple. For the multi-mode system constructed earlier, we show that starting from the initial point $(1, 0, 0, \dots, 0)$ it is possible to safely reach the target point $(0, 0, 0, \dots, 0, 1)$ if and only if the corresponding two counter machine halts.

We present a set of lemma to prove the correctness. In particular, our lemmas establish that

- The schedule begins in mode \mathcal{I} , and exactly one unit of time is spent in \mathcal{I} (Lemma 4),
- the order of choosing modes is decided by the sequence of instructions chosen in a correct simulation of the two counter machine (Lemmas 5, 6, 7), and
- starting from $s_0 = 1$ and $v = 0$ for all variables $v \neq s_0$, one can reach the point with $w_{halt} = 1$ and $v = 0$ for all variables $v \neq w_{halt}$ avoiding obstacles iff (1), (2) are true. This is shown by Lemma 8.

Lemma 4. *Any safe schedule must begin in mode \mathcal{I} and exactly one unit of time is spent at this mode.*

Proof. Observe that the starting point is such that $s_0 = 1$ and all other variables are 0. Assume if the schedule begins in a mode other than \mathcal{I} and spend t units of time there. If the schedule begins in some mode \mathcal{M}_i and spend time t , then we will obtain $w_{ij} = -t$ for some j , hitting the obstacle $\neg Init$ (violating φ_a). Similarly, if the schedule begins in mode \mathcal{M}_i^1 or \mathcal{M}_i^2 , then again we will obtain $z_{i\#} < 0$ hitting the obstacle $\neg Init$. Also, starting in mode $\mathcal{M}_{i\ halt}$ or \mathcal{M}_{halt}^c or \mathcal{M}_{halt} will give $x_{i\ halt} < 0$ or $c < 0$ or $w_{halt} < 0$, respectively, all of which will hit the obstacle $\neg Init$. Thus, any safe schedule must begin in mode \mathcal{I} . Now we show that each such schedule must spend exactly one unit of time in \mathcal{I} .

1. If more than one unit of time is spent at \mathcal{I} , then $s_0 < 0$ will violate φ_a .
2. Assume that a time $t < 1$ is spent at \mathcal{I} , and the control shifts to any other mode. Then we have $s_0 = 1 - t$ and $w_{0j} = t$ on entering that mode. Note that $w_{0j} + s_0 = 1$. If any time is spent at that mode, then we will either obtain $s_0 > 0$ and some $x_{ij} \neq 0$ (hits $\neg Mutex(S, X)$ and violates φ_e) or some $w_{jk}, z_{k\#} < 0$ (hits $\neg Init$ and violates φ_a).

The proof is now complete. \square

Lemma 5. *After spending 1 time unit in mode \mathcal{I} , any safe schedule must choose mode \mathcal{M}_0 followed by mode \mathcal{M}_{0j} (where ℓ_j is the unique instruction that follows ℓ_0 in the two counter machine) both for exactly 1 time unit.*

Proof. After spending one unit of time in \mathcal{I} , we have $s_0 = 0$ and $w_{0j} = 1$. We claim that the control will switch to mode \mathcal{M}_0 from \mathcal{I} .

1. Recall that $R(\mathcal{M}_0)(w_{0j}) = -1$, and $R(\mathcal{M}_0)(x_{0j}) = 1$, where j is the unique index of the instruction ℓ_j to which control shifts in the two counter machine from ℓ_0 . If control switches to \mathcal{M}_0 , and $0 \leq t \leq 1$ time is spent, then we have $w_{0j} = 1 - t$, $x_{0j} = t$. The resultant points are all safe; note that $w_{0j} + x_{0j} = 1$.
2. If control switches from \mathcal{I} to some \mathcal{M}_k , $k \neq 0$, or some \mathcal{M}_k^1 (or \mathcal{M}_k^2) or some \mathcal{M}_{gf} , then we have
 - $R(\mathcal{M}_k)(w_{kg}) = -1$, and $R(\mathcal{M}_k)(x_{kg}) = 1$ for some g , or
 - $R(\mathcal{M}_k^1)(z_{k\#}) = -1$, and $R(\mathcal{M}_k)(x_{kg}) = 1$ for some g .
 - $R(\mathcal{M}_{gf})(x_{gf}) = -1$.
 If $t > 0$ time is spent at \mathcal{M}_k , or \mathcal{M}_k^1 (\mathcal{M}_k^2) or \mathcal{M}_{gf} , then we obtain $w_{kg} < 0$ or $z_{k\#} < 0$ or $x_{gf} = -t < 0$, violating the safety requirement.

Next we claim that exactly one unit of time is spent at \mathcal{M}_0 before control switches to any other mode. By Lemma 4, we have $s_0 = 0, w_{0j} = 1$ on entering \mathcal{M}_0 .

1. It is easy to see that if time $t > 1$ is spent at \mathcal{M}_0 , the obstacle $\neg \text{Init}$ is hit, since $x_{0j} > 1$.
2. Assume now that t time is spent at \mathcal{M}_0 , and control switches to some mode. Then we have $w_{0j} = 1 - t, x_{0j} = t$ on exiting \mathcal{M}_0 .
 - If mode \mathcal{M}_{0j} is chosen after \mathcal{M}_0 , and a time $t' > 0$ is spent at \mathcal{M}_{0j} , then we obtain $x_{0j} = t - t', w_{0j} = 1 - t$. Also, we have $w_{jk} = t' > 0$ or $z_{j\#} = t' > 0$, depending on whether ℓ_j is not a zero check instruction having ℓ_k as its successor, or ℓ_j is a zero check instruction. In the former case, we obtain $w_{0j} > 0$ and $w_{jk} > 0$, violating φ_c , while in the latter case, we obtain $w_{0j} > 0$ and $z_{j\#} > 0$, again violating φ_c . However, if $t = 1$ time is spent at \mathcal{M}_0 , then there is no violation since we have $w_{0j} = 0, x_{0j} = 1 - t'$ and exactly one of $w_{jk} = t'$ or $z_{j\#} = t'$.
 - Assume that a time $t = 1$ is spent at \mathcal{M}_0 , but a mode other than \mathcal{M}_{0j} is chosen from \mathcal{M}_0 and a time $t' > 0$ is spent there.
 - If \mathcal{M}_k is chosen for some k , then we obtain $x_{kg} = t'$. This violates φ_b since $x_{kg} = t'$ and $x_{0j} = 1$.
 - If \mathcal{M}_{kg} is chosen for some $k, g \neq 0, j$, then we obtain $w_{gh} = t'$ or $z_{g\#} = t'$ along with $x_{kg} = -t'$. This violates φ_a since $x_{kg} = -t' < 0$.

Thus, we have seen that starting from \mathcal{I} , the control shifts to \mathcal{M}_0 and then to \mathcal{M}_{0j} in order, spending exactly one unit of time at \mathcal{M}_0 . We now argue that the time t' spent at \mathcal{M}_{0j} has to be 1. Assume $t' \neq 1$. Spending t' units of time at \mathcal{M}_{0j} results in $x_{0j} = 1 - t'$ and one of $w_{jk} = t'$ or $z_{j\#} = t'$.

1. If $t' > 1$, then we obtain $x_{0j} = 1 - t' < 0$, violating φ_a .
2. Assume $t' < 1$, and control switches from \mathcal{M}_{0j} to some \mathcal{M}_f . Let t'' time be spent at \mathcal{M}_f . If $f \neq 0$, then we obtain $x_{fg} = t'' > 0$ for some g , and $x_{0j} = 1 - t' > 0$ violating φ_b . If $f = 0$, then we obtain $x_{0j} = 1 - t' + t''$, but $w_{0j} = -t'' < 0$, violating φ_a .
3. Assume $t' < 1$, and control switches from \mathcal{M}_{0j} to some \mathcal{M}_f^1 or \mathcal{M}_f^2 . Let t'' time be spent at \mathcal{M}_f^1 (\mathcal{M}_f^2). Then we obtain $z_{f\#} = -t'' < 0$ violating φ_a .
4. Assume $t' < 1$, and control switches from \mathcal{M}_{0j} to some \mathcal{M}_{cd} . Then we have one of $z_{d\#} = t'' > 0$ or $w_{dh} = t'' > 0$ along with one of $w_{jk} = t' > 0$ or $z_{j\#} = t' > 0$, both which violate one of φ_c, φ_d .

If $t' = 1$, then we have $x_{0j} = 0$ and one of $w_{jk} = 1$ or $z_{j\#} = 1$. The proof is now complete. \square

Now if j was not a zero check instruction, and has ℓ_k as the successor of ℓ_j , then as seen above in Lemma 5 in the case of \mathcal{M}_0 and \mathcal{M}_{0j} , we can show that the control has to shift to \mathcal{M}_j from \mathcal{M}_{0j} . If j is a zero check instruction, then we claim that the control has to switch from \mathcal{M}_{0j} to one of \mathcal{M}_j^1 or \mathcal{M}_j^2 . Lemma 6 generalises this claim.

Lemma 6. *If the system is in mode \mathcal{M}_{gf} then any safe schedule must pick mode \mathcal{M}_f if f is not a zero check instruction. However, if f is a zero check instruction, then the next mode must be either of mode \mathcal{M}_f^1 or \mathcal{M}_f^2 . Any safe schedule must also spent 1 time unit at modes \mathcal{M}_{gf} and at \mathcal{M}_f , \mathcal{M}_f^1 , or \mathcal{M}_f^2 (as is the case).*

Proof. Assume that the system is in mode \mathcal{M}_{gf} , and assume that it followed a safe execution from the starting state. We know that $R(\mathcal{M}_{gf})(x_{gf}) = -1$. In this case, upon entering mode \mathcal{M}_{gf} , we must have $x_{gf} = 1$, while all variables except c_1, c_2 must be 0. There are two cases to consider.

1. $\ell_f \notin O$. Then $R(\mathcal{M}_{gf})(w_{fq}) = 1$ for some unique index q corresponding to the successor ℓ_q of ℓ_f . Spending $t = 1$ here results in $x_{gf} = 0$ and $w_{fq} = 1$, with no violation to the non-hitting zone. As seen in Lemma 5 for the case of \mathcal{M}_{0j} , a time t' spent at \mathcal{M}_{gf} is safe iff $t' = 1$ ($t' > 1$ violates safety immediately, while a switch in control from \mathcal{M}_{gf} with $t' < 1$ to any mode disallows spending time at the new mode).
2. $\ell_f \in O$. Then $R(\mathcal{M}_{gf})(z_{f\#}) = 1$. Spending t unit of time at \mathcal{M}_{gf} results in $x_{gf} = 1 - t$ and $z_{f\#} = t$.
 - (a) If $t > 1$, then we obtain $x_{gf} < 0$ violating φ_a .
 - (b) Assume $t < 1$, and control switches out of \mathcal{M}_{gf} .
 - If the next mode chosen is some \mathcal{M}_k , and $t' > 0$ units of time spent, then we obtain $w_{k*} = -t' < 0$ violating φ_a .
 - If the next mode chosen is some \mathcal{M}_k^1 or \mathcal{M}_k^2 with $k \neq f$, and t' units of time spent there, then we obtain $z_{k\#} = -t' < 0$ violating φ_a .
 - If the next mode chosen is some \mathcal{M}_{cd} , then we obtain $x_{cd} = -t' < 0$ violating φ_a .
 - If the next mode chosen is \mathcal{M}_f^1 or \mathcal{M}_f^2 , and $t' > 0$ units of time spent, then we obtain $z_{f\#} = t - t'$. However, we also obtain $x_{f*} = t' > 0$ and $x_{gf} = 1 - t > 0$ violating φ_b .
 - (c) Assume $t = 1$ unit of time is spent at \mathcal{M}_{gf} and control switches out of \mathcal{M}_{gf} . We then have $x_{gf} = 0$ and $z_{f\#} = 1$.
 - If the next mode chosen is some \mathcal{M}_k , and $t' > 0$ units of time spent, then we obtain $w_{k*} = -t' < 0$ violating φ_a .
 - If the next mode chosen is \mathcal{M}_j^1 or \mathcal{M}_j^2 , $j \neq f$, and $t' > 0$ units of time spent, then we obtain $z_{j\#} = -t' < 0$ violating φ_a .
 - If the next mode chosen is some \mathcal{M}_{cd} and $t' > 0$ units of time spent, then we obtain $x_{cd} = -t' < 0$, violating φ_a .
 - If the next mode chosen is \mathcal{M}_f^1 or \mathcal{M}_f^2 , and $t' > 0$ units of time spent, we obtain $z_{f\#} = 1 - t'$, along with $x_{f*} = t' > 0$. If ℓ_f is the instruction “if $c_1 > 0$, goto f_1 , else goto f_2 ”, then $*$ is either f_1 or f_2 . There is no violation to safety as long as $c_1 > 0$ and \mathcal{M}_f^1 is chosen, or $c_1 = 0$ and \mathcal{M}_f^2 is chosen when $t' \leq 1$. If $t' = 1$, then we obtain $z_{f\#} = 0$ and $x_{f*} = 1$. After spending $t' = 1$ unit of time at \mathcal{M}_f^1 or \mathcal{M}_f^2 , assume the control switches to \mathcal{M}_{f*} , and a time t'' is spent there. As seen in Lemma 5, it can be shown that there is no violation to safety as long as $t'' \leq 1$. In particular, it can be shown that if $t'' < 1$, and control switches out of \mathcal{M}_{f*} , safety is violated.

Thus, it can be seen that some obstacle is hit if a time other than one is spent at any mode, or if a mode violating the order of instructions in the two counter machine is chosen. Lemmas 4, 5 and 6 prove this. Assume now that the mode switching happens respecting the instruction flow in the two counter machine, and one unit of time is spent at each mode. It can be seen that two counter machine halts iff some mode $\mathcal{M}_{i \text{ halt}}$ is reached. After spending one unit of time at $\mathcal{M}_{i \text{ halt}}$, we obtain $w_{\text{halt}} = 1$, and all x variables 0. Note that by condition φ_f , no x variables other than $x_{i \text{ halt}}$ can be non-zero when $w_{\text{halt}} > 0$.

Lemma 7. *Any safe schedule, upon entering mode $\mathcal{M}_{i \text{ halt}}$, must spend exactly one unit of time.*

Proof. On entering $\mathcal{M}_{i \text{ halt}}$, we have $x_{i \text{ halt}} = 1$ and $w_{\text{halt}} = 0$. Assume $t < 1$ time is spent at $\mathcal{M}_{i \text{ halt}}$ and a mode change happens. Then we have $x_{i \text{ halt}} = 1 - t$ and $w_{\text{halt}} = t$.

- If we move to any $\mathcal{M}_k, \mathcal{M}_k^1, \mathcal{M}_k^2$ or \mathcal{M}_{cd} , and elapse a time $t' > 0$, we will have a safety violation due to some variable becoming negative (w_{k*} in the case of \mathcal{M}_k , x_{cd} in the case of \mathcal{M}_{cd} and $z_{k\#}$ in the case of $\mathcal{M}_k^1, \mathcal{M}_k^2$).
- Assume that we move to $\mathcal{M}_{\text{halt}}$ and spend $t' > 0$ time there. Then we obtain $w_{\text{halt}} = t - t'$. This violates φ_g since we have $x_{i \text{ halt}} + w_{\text{halt}} \neq 1$. Moving to $\mathcal{M}_{\text{halt}}^{c_1}, \mathcal{M}_{\text{halt}}^{c_2}$ also violates safety for the same reason.

However, if $t = 1$, then we have $w_{\text{halt}} = 1$ and all other x, n variables are 0. Moving to any mode other $\mathcal{M}_{\text{halt}}$ or $\mathcal{M}_{\text{halt}}^{c_1}, \mathcal{M}_{\text{halt}}^{c_2}$ will violate φ_f .

Lemma 8 (Correctness). *The target point $(0, 0, 0, \dots, 1)$ is safely reachable from the starting point $(1, 0, 0, \dots, 0)$ in multi-mode system $\mathcal{H}_{\mathcal{A}}$ iff the two counter machine \mathcal{A} halts.*

Proof. Assume that the two counter machine halts. Then starting from the initial instruction, we reach the instruction ℓ_{halt} . From the above lemmas, and the construction of the MMS, we know that starting from the initial mode \mathcal{I} , we will reach a unique mode $\mathcal{M}_{i \text{ halt}}$, spending one unit of time at all the intermediate modes. From Lemma 7, we also know that a time of one unit is spent at $\mathcal{M}_{i \text{ halt}}$, and that the only safe modes to goto from here are $\mathcal{M}_{\text{halt}}$ or $\mathcal{M}_{\text{halt}}^{c_1}, \mathcal{M}_{\text{halt}}^{c_2}$. The use of modes $\mathcal{M}_{\text{halt}}^{c_1}, \mathcal{M}_{\text{halt}}^{c_2}$ is just to obtain $c_1 = c_2 = 0$. Notice that w_{halt} stays non-zero and grows in these modes. Once we achieve $c_1 = c_2 = 0$, then we can visit $\mathcal{M}_{\text{halt}}$ and obtain $w_{\text{halt}} = 1$. Notice that when this happens, we will have all variables other than w_{halt} as 0. By our safety conditions, no x, w or z variable can stay non-zero when $w_{\text{halt}} > 0$. Even if we obtain $w_{\text{halt}} = 0$ by staying at $\mathcal{M}_{\text{halt}}$, we cannot visit any other mode, since atleast one variable will become negative and violate safety. Our starting configuration with $s_0 = 1$ ensured that we could start from \mathcal{I} and continue in a safe manner.

The converse, that is, any safe schedule starting from $(1, 0, 0, \dots, 0)$ and reaching $(0, 0, 0, \dots, 1)$ is possible only when the two counter machine halts by the construction of the MMS. \square

A.3 Undecidability: Example of the reduction

Consider an example of a two counter machine with counters c_1, c_2 and the following instruction set.

- $\ell_0 : c_1 := c_1 + 1; \text{ goto } \ell_1$
- $\ell_1 : c_1 := c_1 - 1; \text{ goto } \ell_2$
- $\ell_2 : \text{ if } (c_2 > 0) \text{ then goto } \ell_3; \text{ else goto } \ell_0$
- $\ell_3 : \text{ HALT.}$

Note that this machine does not halt. We now describe a multi-mode system that simulates this two counter machine. The modes and variables are as follows.

1. Variables $\{c_1, c_2\}$ correspond to the two counters, $\{w_{01}, w_{12}, z_{2\#}\}$ correspond to instructions $\ell_0, \ell_1, \ell_2, \ell_3$ and the switches between instructions, and variable w_3 corresponds to the halt instruction. We also have variables s_0 and $\{x_{ij} \mid 0 \leq i, j \leq 3\}$.
2. Mode \mathcal{I} : $R(\mathcal{I})(s_0) = -1$ and $R(\mathcal{I})(w_{01}) = 1$ and other variables have rate 0.
3. Modes $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2^1, \mathcal{M}_2^2, \mathcal{M}_{01}, \mathcal{M}_{12}, \mathcal{M}_{20}$ with rates
 - $R(\mathcal{M}_0)(w_{01}) = -1, R(\mathcal{M}_0)(x_{01}) = R(\mathcal{M}_0)(c_1) = 1$, and $R(\mathcal{M}_0)(v) = 0$ for all other variables v
 - $R(\mathcal{M}_1)(w_{12}) = -1, R(\mathcal{M}_1)(x_{12}) = 1, R(\mathcal{M}_1)(c_1) = -1$, and $R(\mathcal{M}_1)(v) = 0$ for all other variables v
 - $R(\mathcal{M}_2^1)(z_{2\#}) = -1 = R(\mathcal{M}_2^2)(z_{2\#}), R(\mathcal{M}_2^1)(x_{23}) = 1 = R(\mathcal{M}_2^2)(x_{20})$. All other variables have rate 0 in modes $\mathcal{M}_2^1, \mathcal{M}_2^2$.
 - $R(\mathcal{M}_{12})(z_{2\#}) = 1$, and $R(\mathcal{M}_{12})(v) = 0$ for all other variables v
 - $R(\mathcal{M}_{01})(x_{01}) = -1, R(\mathcal{M}_{01})(w_{12}) = 1$, and $R(\mathcal{M}_{01})(v) = 0$ for other v
 - $R(\mathcal{M}_{20})(x_{20}) = -1, R(\mathcal{M}_{20})(w_{01}) = 1$, and $R(\mathcal{M}_{20})(v) = 0$ for other v
4. Mode \mathcal{M}_{23} with $R(\mathcal{M}_{23})(x_{23}) = -1, R(\mathcal{M}_{23})(w_3) = 1, R(\mathcal{M}_{23})(v) = 0$ for other v . Modes $\mathcal{M}_3, \mathcal{M}_3^{c_1}, \mathcal{M}_3^{c_2}$ with $R(\mathcal{M}_3)(w_3) = -1$ and $R(\mathcal{M}_3)(v) = 0$ for all other variables v ; $R(\mathcal{M}_3^{c_i})(c_i) = -1, R(\mathcal{M}_3^{c_i})(w_3) = 1$.

The safety set is given by the conjunction of conditions (1)-(7).

- (1) $0 \leq w_{01}, w_{12}, x_{ij}, z_{2\#}, s_0 \leq 1, 0 \leq w_3, c_1, c_2$, (2) At any point, if some x_{ij} is non-negative, then all the other x_{kl} variables are 0, $i \neq k, j \neq l$. (3) At any point, if some w_{ij} is non-negative, then all other w_{kl} are zero, $i \neq k, j \neq l$ and $z_{2\#} = 0$. (4) At any point, if $z_{2\#} > 0$, then all the w_{ij} are 0. (5) At any point, if $s_0 > 0$, then all $x_{ij} = 0$. (6) At any point, if $w_3 > 0$, then variables x_{i3} are 0, and $c_1, c_2 = 0$. (7) At any point, if $x_{i3} > 0$, then $x_{i3} + w_3 = 1$. The obstacles are hence, the complement of this conjunction.

Starting from $s_0 = 1$ and $v = 0$ for $v \neq s_0$, a safe computation must start from \mathcal{I} , then visit in order modes $\mathcal{M}_0, \mathcal{M}_{01}, \mathcal{M}_1, \mathcal{M}_{12}, \mathcal{M}_2^1, \mathcal{M}_{20}$, and repeat this sequence spending one unit in each mode. This will not reach \mathcal{M}_3 .

We start from the point $s_0 = 1$ and $v = 0$ for all variables $v \neq s_0$. The safe set of values for s_0 is $[0, 1]$. As seen in Lemma 4, computation starts in mode \mathcal{I} , and one unit of time is spent there. This results in $s_0 = 0, w_{01} = 1$, and $v = 0$ for all other variables. The control then shifts to modes $\mathcal{M}_0, \mathcal{M}_{01}$ in order. One unit of time is spent in \mathcal{M}_0 , and we obtain $w_{01} = 0, x_{01} = c_1 = 1$, and $v = 0$

for all other variables v . This is followed by spending one unit of time in \mathcal{M}_{01} , obtaining $x_{01} = 0$, $w_{12} = 1$, $c_1 = 1$ and $v = 0$. The control shifts from \mathcal{M}_{01} to \mathcal{M}_1 , where one unit of time is spent. This results in $w_{12} = 0$, $x_{12} = 1$ and $c_1 = 0$, and $v = 0$ for all other variables v . Control then shifts to \mathcal{M}_{12} , where one unit of time is spent. This results in obtaining $z_{2\#} = 1$ and $c_1 = 1$, and $v = 0$ for all other variables. Control then shifts to \mathcal{M}_2^1 , and one unit of time is spent, resulting in $z_{2\#} = 0$, $x_{20} = 1$, $c_1 = 0$. This is continued by visiting mode \mathcal{M}_{20} , and we obtain $x_{20} = 0$, $w_{01} = 1$, after spending one unit of time in \mathcal{M}_{20} . Any other sequence of visiting modes, or spending times other than 1 in the visited modes will result in hitting an obstacle.

The computation now proceeds to modes $\mathcal{M}_0, \mathcal{M}_{01}, \mathcal{M}_1, \mathcal{M}_{12}, \mathcal{M}_2^1, \mathcal{M}_{20}$ in a loop. Note that the halt mode \mathcal{M}_3 is never reached.