AE_544_LecNote02__Newtonian_Mechanics__Ch02

> ## Disclaimers
>
> This repository contains markdown-based lecture notes for the Spring 2025 semester. These materials are intended solely for students enrolled in the course during this term. Mistakes may exist and will be corrected as they are identified, but the official textbook remains the ultimate reference for accuracy.

Newton's Laws

Newton's First law

Unless acted upon by a force, a particle will maintain a straight line motion with constant inertial velocity.

> ⓘ **This actually express a belief that there is an ideal and isolated inertial frame.**

Newton's Second law

Let the vector $\boldsymbol{F}$ be the sum of all forces acting on a particle having a mass m with the inertial position vector $\boldsymbol{r}$. Assume that $\mathcal{N}$ is an inertial reference frame, then

$$\boldsymbol{F} = \frac{^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(m\dot{\boldsymbol{r}}) \tag{2.1}$$

If the mass m is constant, then this result simplifies to the well known result

$$\boldsymbol{F} = m\ddot{\boldsymbol{r}} \tag{2.2}$$

> ⓘ **Note that all derivatives taken in Newton's second law must be inertial derivatives.**

> ⓘ **Without correctly formulated kinematics, the dynamical system description will be incorrect from the start. *The textbook authors* mention that a large fraction of errors made in practice have their origin in kinematics errors formulating $\ddot{r}$ and similar vector derivatives.**

Newton's Third law

If mass m_1 is exerting a force $\boldsymbol{F}_{21}$ on mass m_2, then the force $\boldsymbol{F}_{12}$ experienced by m_1 due to interaction with m_2 will be

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} \tag{2.3}$$

> ⚠ **Textbook convention for Free-Body Diagram (FBD)**
>
> The FBD should show all forces and moments acting on the system. We exclude from our FBDs acceleration vectors and so-called inertia forces that are subsets of the $m\ddot{\boldsymbol{r}}$ terms in Eq. (2.2) that may arise in rotating coordinate systems.
> (*TL;DR*: Only inertial and real forces/moments; no fictitious ones.)

Newton's Law of Universal Gravitation

Let the vector $\boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ describe the position of mass m_2 relative to mass m_1 as shown in Fig. 2.2. Then the mutually attractive gravitational force between the objects will be

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = \frac{Gm_1m_2}{|\boldsymbol{r}_{12}|^2} \cdot \frac{\boldsymbol{r}_{12}}{|\boldsymbol{r}_{12}|} \tag{2.4}$$

where $G \approx 6.6732 \times 10^{-11}\,\mathrm{m}^3/(\mathrm{s}^2 \cdot \mathrm{kg})$ is the universal gravity constant.

Example 2.4 Planar pendulum

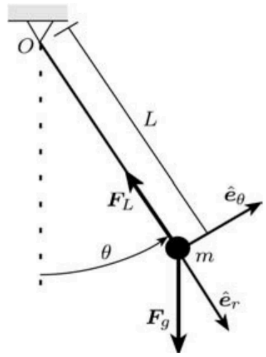

Fig. 2.6 Planar pendulum illustration.

A rotating frame $\mathcal{E}$: $\{\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta\}$

Position vector $\boldsymbol{r} = L\hat{\boldsymbol{e}}_r$

Inertial acceleration $\ddot{\boldsymbol{r}} = -L\dot{\theta}^2\hat{\boldsymbol{e}}_r + L\ddot{\theta}\hat{\boldsymbol{e}}_\theta$

Kinematics derivation >

$$
\begin{aligned}
\ddot{\boldsymbol{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{L}\hat{\boldsymbol{e}}_r + L\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{e}}_r\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{L}\hat{\boldsymbol{e}}_r + L(\dot{\theta}\hat{\boldsymbol{e}}_3)\times\hat{\boldsymbol{e}}_r\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{L}\hat{\boldsymbol{e}}_r + L\dot{\theta}\hat{\boldsymbol{e}}_\theta\right) \\
&= \left(\ddot{L}\hat{\boldsymbol{e}}_r + \dot{L}\dot{\theta}\hat{\boldsymbol{e}}_\theta\right) + \left(\dot{L}\dot{\theta}\hat{\boldsymbol{e}}_\theta + L\ddot{\theta}\hat{\boldsymbol{e}}_\theta + L\dot{\theta}(-\dot{\theta}\hat{\boldsymbol{e}}_r)\right) \\
&= \left(\cancel{\ddot{L}}^{0}\,\hat{\boldsymbol{e}}_r + \cancel{\dot{L}}^{0}\,\dot{\theta}\hat{\boldsymbol{e}}_\theta\right) + \left(\cancel{\dot{L}}^{0}\,\dot{\theta}\hat{\boldsymbol{e}}_\theta + L\ddot{\theta}\hat{\boldsymbol{e}}_\theta + L\dot{\theta}(-\dot{\theta}\hat{\boldsymbol{e}}_r)\right) && \text{(Constant length } L\text{)} \\
&= -L\dot{\theta}^2\hat{\boldsymbol{e}}_r + L\ddot{\theta}\hat{\boldsymbol{e}}_\theta
\end{aligned}
$$

FBD gives:

$$
\begin{aligned}
-mL\dot{\theta}^2 &= -F_L + mg\cos\theta \\
mL\ddot{\theta} &= -mg\sin\theta
\end{aligned}
$$

The tension force magnitude $F_L = mL\dot{\theta}^2 + mg\cos\theta$

The nonlinear differential equations of motion of the spherical pendulum mass m is

$$\ddot{\theta} = \frac{-g\sin\theta}{L}$$

Textbook problem 2.10

2.10 A massless cylinder is rolling down a slope with an inclination angle α under the influence of a constant gravity field. A mass m is attached to the cylinder and is offset from the cylinder center by $R/2$ as shown in Fig. P2.10.

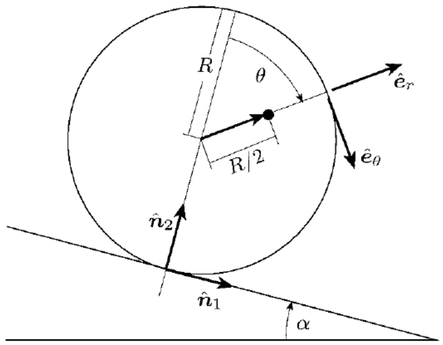

Fig. P2.10 Rolling cylinder with offset mass.

(a) Find the equations of motion of the mass m in terms of the angle θ.
(b) What is the normal force $\boldsymbol{N} = N\hat{\boldsymbol{n}}_2$ that the ground is exerting against the cylinder?

(a) EOM

Assuming no slip between the cylinder and the ground.

Inertial frame $\mathcal{M}$.

Translational moving frame $\mathcal{N}$.

Moving and rotating body frame $\mathcal{E}$.

Position vector

$$\boldsymbol{r} = \boldsymbol{r}_N + R\hat{\boldsymbol{n}}_2 + \frac{1}{2}R\hat{\boldsymbol{e}}_r$$

Velocity vector

$$\begin{aligned}
\boldsymbol{v} = \dot{\boldsymbol{r}} &= \frac{{}^{\mathcal{M}}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_N + R\hat{\boldsymbol{n}}_2 + \frac{1}{2}R\hat{\boldsymbol{e}}_r\right) \\
&= \dot{\theta}R\hat{\boldsymbol{n}}_1 + R\frac{{}^{\mathcal{M}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{n}}_2 + \frac{1}{2}R\frac{{}^{\mathcal{M}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{e}}_r \\
&= \dot{\theta}R\hat{\boldsymbol{n}}_1 + R\boldsymbol{\omega}_{\mathcal{N}/\mathcal{M}} \times \hat{\boldsymbol{n}}_2 + \frac{1}{2}R\boldsymbol{\omega}_{\mathcal{E}/\mathcal{M}} \times \hat{\boldsymbol{e}}_r \\
&= \dot{\theta}R\hat{\boldsymbol{n}}_1 + \mathbf{0} + \frac{1}{2}R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\
&= \dot{\theta}R(\cos\theta\hat{\boldsymbol{e}}_\theta + \sin\theta\hat{\boldsymbol{e}}_r) + \frac{1}{2}R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\
&= (\dot{\theta}R\cos\theta + \frac{1}{2}R\dot{\theta})\hat{\boldsymbol{e}}_\theta + \dot{\theta}R\sin\theta\hat{\boldsymbol{e}}_r
\end{aligned}$$

Acceleration vector

$$\begin{aligned}
\boldsymbol{a} = \dot{\boldsymbol{v}} &= \frac{{}^{\mathcal{M}}\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}R\hat{\boldsymbol{n}}_1 + \mathbf{0} + \frac{1}{2}R\dot{\theta}\hat{\boldsymbol{e}}_\theta\right) \\
&= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}R\hat{\boldsymbol{n}}_1\right) + \boldsymbol{\omega}_{\mathcal{N}/\mathcal{M}} \times \left(\dot{\theta}R\hat{\boldsymbol{n}}_1\right) + \frac{{}^{\mathcal{E}}\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}R\dot{\theta}\hat{\boldsymbol{e}}_\theta\right) + \boldsymbol{\omega}_{\mathcal{E}/\mathcal{M}} \times \left(\frac{1}{2}R\dot{\theta}\hat{\boldsymbol{e}}_\theta\right) \\
&= \ddot{\theta}R\hat{\boldsymbol{n}}_1 + \mathbf{0} + \frac{1}{2}R\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \frac{1}{2}R\dot{\theta}^2\hat{\boldsymbol{e}}_r \\
&= \ddot{\theta}R(\cos\theta\hat{\boldsymbol{e}}_\theta + \sin\theta\hat{\boldsymbol{e}}_r) + \mathbf{0} + \frac{1}{2}R\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \frac{1}{2}R\dot{\theta}^2\hat{\boldsymbol{e}}_r \\
&= \left(\ddot{\theta}R\cos\theta + \frac{1}{2}R\ddot{\theta}\right)\hat{\boldsymbol{e}}_\theta + \left(\ddot{\theta}R\sin\theta - \frac{1}{2}R\dot{\theta}^2\right)\hat{\boldsymbol{e}}_r
\end{aligned}$$

Total external force is

$$
\begin{aligned}
\boldsymbol{F}_e &= (mg\sin\alpha\hat{\boldsymbol{n}}_1 - mg\cos\alpha\hat{\boldsymbol{n}}_2) + N\hat{\boldsymbol{n}}_2 \\
&= mg\sin\alpha\hat{\boldsymbol{n}}_1 + (N - mg\cos\alpha)\hat{\boldsymbol{n}}_2 \\
&= mg\sin\alpha(\cos\theta\hat{\boldsymbol{e}}_\theta + \sin\theta\hat{\boldsymbol{e}}_r) + (N - mg\cos\alpha)(-\sin\theta\hat{\boldsymbol{e}}_\theta + \cos\theta\hat{\boldsymbol{e}}_r) \\
&= (mg\sin\alpha\cos\theta - (N - mg\cos\alpha)\sin\theta)\hat{\boldsymbol{e}}_\theta + \big(mg\sin\alpha\sin\theta + (N - mg\cos\alpha)\cos\theta)\hat{\boldsymbol{e}}_r\big)
\end{aligned}
$$

$$
\begin{cases}
mg\sin\alpha\cos^2\theta - (N - mg\cos\alpha)\sin\theta\cos\theta = m\left(\ddot{\theta}R\cos\theta + \dfrac{1}{2}R\ddot{\theta}\right)\cos\theta \\
mg\sin\alpha\sin^2\theta + (N - mg\cos\alpha)\cos\theta\sin\theta = m\left(\ddot{\theta}R\sin\theta - \dfrac{1}{2}R\dot{\theta}^2\right)\sin\theta
\end{cases}
$$

$$
\begin{aligned}
mg\sin\alpha\cos^2\theta + mg\sin\alpha\sin^2\theta &= m\left(\ddot{\theta}R\cos\theta + \frac{1}{2}R\ddot{\theta}\right)\cos\theta + m\left(\ddot{\theta}R\sin\theta - \frac{1}{2}R\dot{\theta}^2\right)\sin\theta \\
g\sin\alpha &= \left(\ddot{\theta}R\cos\theta + \frac{1}{2}R\ddot{\theta}\right)\cos\theta + \left(\ddot{\theta}R\sin\theta - \frac{1}{2}R\dot{\theta}^2\right)\sin\theta \\
g\sin\alpha &= \left(R\cos^2\theta + \frac{1}{2}R\cos\theta + R\sin^2\theta\right)\ddot{\theta} - \frac{1}{2}R\sin\theta\dot{\theta}^2 \\
\left(R + \frac{1}{2}R\cos\theta\right)\ddot{\theta} &= g\sin\alpha + \frac{1}{2}R\dot{\theta}^2\sin\theta \\
(2R + R\cos\theta)\ddot{\theta} &= 2g\sin\alpha + R\dot{\theta}^2\sin\theta \\
\ddot{\theta} &= \frac{2g\sin\alpha + R\dot{\theta}^2\sin\theta}{R\,(2+\cos\theta)}
\end{aligned}
$$

Alternatively, keep F_e in $\mathcal{N}$ but convert $\boldsymbol{a}$ to $\mathcal{N}$

$$
\begin{aligned}
\boldsymbol{a} &= \ddot{\theta}R\hat{\boldsymbol{n}}_1 + \boldsymbol{0} + \frac{1}{2}R\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \frac{1}{2}R\dot{\theta}^2\hat{\boldsymbol{e}}_r \\
&= \ddot{\theta}R\hat{\boldsymbol{n}}_1 + \boldsymbol{0} + \frac{1}{2}R\ddot{\theta}(\cos\theta\hat{\boldsymbol{n}}_1 - \sin\theta\hat{\boldsymbol{n}}_2) - \frac{1}{2}R\dot{\theta}^2(\sin\theta\hat{\boldsymbol{n}}_1 + \cos\theta\hat{\boldsymbol{n}}_2) \\
&= \left(\ddot{\theta}R + \frac{1}{2}R\ddot{\theta}\cos\theta - \frac{1}{2}R\dot{\theta}^2\sin\theta\right)\hat{\boldsymbol{n}}_1 + \left(-\frac{1}{2}R\ddot{\theta}\sin\theta - \frac{1}{2}R\dot{\theta}^2\cos\theta\right)\hat{\boldsymbol{n}}_2
\end{aligned}
$$

Then directly, we have

$$
\begin{aligned}
mg\sin\alpha &= m\left(\ddot{\theta}R + \frac{1}{2}R\ddot{\theta}\cos\theta - \frac{1}{2}R\dot{\theta}^2\sin\theta\right) \\
g\sin\alpha &= \ddot{\theta}\left(R + \frac{1}{2}R\cos\theta\right) - \frac{1}{2}R\dot{\theta}^2\sin\theta \\
\ddot{\theta} &= \frac{\left(g\sin\alpha + \frac{1}{2}R\dot{\theta}^2\sin\theta\right)}{\left(R + \frac{1}{2}R\cos\theta\right)} = \frac{2g\sin\alpha + R\dot{\theta}^2\sin\theta}{(2R + R\cos\theta)}
\end{aligned}
$$

which is the same result.

(b) Supporting force vector

$$
\begin{aligned}
(N - mg\cos\alpha) &= m\left(-\frac{1}{2}R\ddot{\theta}\sin\theta - \frac{1}{2}R\dot{\theta}^2\cos\theta\right) \\
(N - mg\cos\alpha) &= m\left(-\frac{1}{2}R\left(\frac{2g\sin\alpha + R\dot{\theta}^2\sin\theta}{(2R + R\cos\theta)}\right)\sin\theta - \frac{1}{2}R\dot{\theta}^2\cos\theta\right) && \text{(plug in } \ddot{\theta}) \\
N &= \ldots && \text{(omitted)}
\end{aligned}
$$

Finally, don't forget the supporting force is a vector, $\boldsymbol{N} = N\hat{\boldsymbol{n}}_2$.

A system of particles

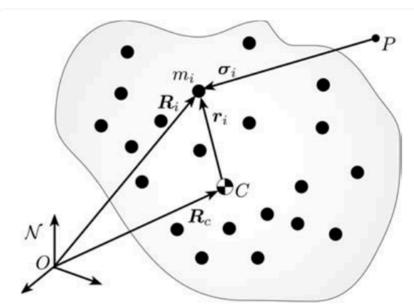

Fig. 2.8 System of N particles.

- A system of N particles, a finite number.
- Each has a constant mass m_i.
- Don't need to be rigid.

Using Newton's second law, the force acting on m_i can be broken down into two subsets of forces as

$$\boldsymbol{F}_i = m_i \ddot{\boldsymbol{R}}_i = \boldsymbol{F}_{iE} + \sum_{j=1}^{N} \boldsymbol{F}_{ij} \tag{2.39 and 2.40}$$

where $\boldsymbol{F}_{iE}$ is the external forces, and $\boldsymbol{F}_{ij}$ is the internal force due to the jth masses.

The total force vector $\boldsymbol{F}$ acting on the entire system of these N particles is given as

$$\boldsymbol{F} = \sum_{i=1}^{N} \boldsymbol{F}_i = \sum_{i=1}^{N} \boldsymbol{F}_{iE} \tag{2.41}$$

The total mass M is

$$M = \sum_{i=1}^{N} m_i \tag{2.42}$$

The center of mass position vector $\boldsymbol{R}_c$ is expressed in terms of the individual inertial mass position vectors $\boldsymbol{R}_i$ as

$$\boldsymbol{R}_c = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{R}_i \tag{2.46}$$

ⓘ **Definition of the center of mass**

The system center of mass position vector $\boldsymbol{R}_c$ is defined such that

$$\sum_{i=1}^{N} m_i \boldsymbol{r}_i = \sum_{i=1}^{N} m_i \left(\boldsymbol{R}_i - \boldsymbol{R}_c\right) = \mathbf{0} \tag{2.43}$$

$$M\boldsymbol{R}_c = \sum_{i=1}^{N} m_i \boldsymbol{R}_i \tag{2.45}$$

Why the center of mass is defined in this way, as a mass-weighted (m-weighted) position? Why not a mass-squared-weighted (m^2-weighted) position (just an arbitrary statement)?

- The net external force acting on the object produces a translational motion of the center of mass.
- The net external torque about the center of mass produces a rotational motion around the center of mass.

This allows us to describe the system's translational motion without worrying about the complexities of rotational motion.

By choosing the center of mass as the point where all the mass is concentrated, we can analyze the translational motion (linear acceleration) of the object separately from its rotational motion.

Superparticle theorem: The dynamics of the mass center of the system of N particles under the influence of the total external force vector F is the same as the dynamics of the superparticle M.

$$M\ddot{\boldsymbol{R}}_c = \sum_{i=1}^{N} m_i \ddot{\boldsymbol{R}}_i = \sum_{i=1}^{N} \boldsymbol{F}_i = \boldsymbol{F} \tag{2.45}$$

Think of the Earth-Moon system in the solar system as an example.

Kinetic Energy

The total kinetic energy of a system of N constant mass particles mi can therefore be written as

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{N}\left(m_i\dot{\boldsymbol{R}}_i\cdot\dot{\boldsymbol{R}}_i\right) && (2.49)\\
&= \frac{1}{2}\sum_{i=1}^{N}\left(m_i(\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{r}}_i)\cdot(\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{r}}_i)\right) && (\text{Use } \dot{\boldsymbol{R}}_i=\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{r}}_i)\\
&= \frac{1}{2}\sum_{i=1}^{N}\left(m_i(\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+2\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{r}}_i+\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i)\right) && (\text{Factor out } \dot{\boldsymbol{R}}_c)\\
&= \frac{1}{2}\sum_{i=1}^{N}\left(m_i\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c\right)+\sum_{i=1}^{N}\left(m_i\cdot\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{r}}_i\right)+\frac{1}{2}\sum_{i=1}^{N}(m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i)\\
&= \frac{1}{2}\left(\sum_{i=1}^{N}m_i\right)\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{R}}_c\cdot\left(\sum_{i=1}^{N}m_i\dot{\boldsymbol{r}}_i\right)+\frac{1}{2}\sum_{i=1}^{N}(m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i)\\
&= \frac{1}{2}\left(\sum_{i=1}^{N}m_i\right)\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{R}}_c\cdot\left(\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}(m_i\boldsymbol{r}_i)\right)+\frac{1}{2}\sum_{i=1}^{N}(m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i)\\
&= \frac{1}{2}\left(\sum_{i=1}^{N}m_i\right)\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{R}}_c\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}(m_i\boldsymbol{r}_i)\right)+\frac{1}{2}\sum_{i=1}^{N}(m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i)\\
&= \frac{1}{2}\left(\sum_{i=1}^{N}m_i\right)\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+\dot{\boldsymbol{R}}_c\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\cancelto{0}{\sum_{i=1}^{N}(m_i\boldsymbol{r}_i)}\right)+\frac{1}{2}\sum_{i=1}^{N}(m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i)
\end{aligned}
\tag{2.50}
$$

Finally, we have the simplest definition of kinetic energy, which clearly consists of two parts:

$$
T=\frac{1}{2}M\dot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+\frac{1}{2}\sum_{i=1}^{N}m_i\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i \tag{2.51}
$$

- The first term contains the system translational kinetic energy.
- The second contains the system rotation and deformation kinetic energy.

The kinetic energy rate $\dot{T}$ is:

$$
\frac{dT}{dt}=M\ddot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{R}}_c+\sum_{i=1}^{N}m_i\ddot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i \tag{2.52}
$$

$$
\frac{dT}{dt}=\boldsymbol{F}\cdot\dot{\boldsymbol{R}}_c+\sum_{i=1}^{N}m_i\ddot{\boldsymbol{R}}_i\cdot\dot{\boldsymbol{r}}_i-\sum_{i=1}^{N}m_i\ddot{\boldsymbol{R}}_c\cdot\dot{\boldsymbol{r}}_i \tag{2.53}
$$

$$
\begin{aligned}
&=\boldsymbol{F}\cdot\dot{\boldsymbol{R}}_c+\sum_{i=1}^{N}m_i\ddot{\boldsymbol{R}}_i\cdot\dot{\boldsymbol{r}}_i-\ddot{\boldsymbol{R}}_c\cdot\left(\cancelto{0}{\sum_{i=1}^{N}m_i\dot{\boldsymbol{r}}_i}\right)\\
&=\boldsymbol{F}\cdot\dot{\boldsymbol{R}}_c+\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\dot{\boldsymbol{r}}_i
\end{aligned}
\tag{2.54}
$$

Now we try to *simplify the second term under certain assumptions*. If there are only conservative forces $\boldsymbol{F}_i$ are acting on m_i , which can be written as the gradient of a potential function $V_i(\boldsymbol{r}_i)$:

$$
\boldsymbol{F}_i=-\nabla V_i(\boldsymbol{r}_i)=-\frac{\partial V_i}{\partial\boldsymbol{r}_i} \tag{2.55}
$$

Defining the total conservative potential function V as

$$
V(\boldsymbol{r}_1,\boldsymbol{r}_2,\cdots,\boldsymbol{r}_N)=\sum_{i=1}^{N}V_i(\boldsymbol{r}_i)\,,
$$

and using the relationship

$$
\begin{aligned}
\mathrm{d}V_i(\boldsymbol{r}_i(t),t)&=\frac{\partial V_i}{\partial\boldsymbol{r}_i}\cdot\mathrm{d}\boldsymbol{r}_i+\cancelto{0}{\frac{\partial V_i}{\partial t}}\,\mathrm{d}t\,, && (\text{Total derivative, move } \mathrm{d}t \text{ then})\\
\frac{\mathrm{d}}{\mathrm{d}t}V_i(\boldsymbol{r}_i(t),t)&=\frac{\partial V_i}{\partial\boldsymbol{r}_i}\cdot\dot{\boldsymbol{r}}_i+\cancelto{0}{\frac{\partial V_i}{\partial t}}\,, && (\text{*Explicitly* independent of } t)
\end{aligned}
$$

then we have:

$$
\frac{\mathrm{d}}{\mathrm{d}t}V=\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}V_i=\sum_{i=1}^{N}\frac{\partial V_i}{\partial\boldsymbol{r}_i}\cdot\dot{\boldsymbol{r}}_i=-\sum_{i=1}^{N}\boldsymbol{F}_i\cdot\dot{\boldsymbol{r}}_i \tag{2.56}
$$

Is this derivative correct?
The order of summation and derivative operators is switched.

The derivative and summation are interchangeable if the summation involves terms that are differentiable with respect to t.

- The summation is finite (or absolutely convergent if infinite).
- Each term $V_i(\boldsymbol{r}_i, t)$ is differentiable with respect to t.

Now Eq. (2.54) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}T + \frac{\mathrm{d}}{\mathrm{d}t}V = \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c$$

- If the total force $\boldsymbol{F} = \mathbf{0}$, the total energy $E = T + V$ is conserved.
- If the total force $\boldsymbol{F} = -\nabla V_c(\boldsymbol{R}_c)$ is also a conservative force due to a potential function $V_c(\boldsymbol{R})_c$, the total system energy $E = T + V + V_c$ is conserved.

In other cases of general forces exerted, we need to integrate the kinetic energy rate $\dot{T}$ to find the work, which is:

$$T(t_2) - T(t_1) = \int_{\boldsymbol{R}_c(t_1)}^{\boldsymbol{R}_c(t_2)} \boldsymbol{F} \cdot d\boldsymbol{R}_c + \sum_{i=1}^{N} \int_{\boldsymbol{r}(t_1)}^{\boldsymbol{r}(t_2)} \boldsymbol{F}_i \cdot d\boldsymbol{r}_i \tag{2.60}$$

$$= \int_{t_1}^{t_2} \boldsymbol{F} \cdot \dot{\boldsymbol{R}}_c \cdot \mathrm{d}t + \sum_{i=1}^{N} \int_{t_1}^{t_2} \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i \cdot \mathrm{d}t \tag{2.59}$$

where the first integral is translational work done by the total force $\boldsymbol{F}$, and the second integral is the rotation and deformation work on each particle.

Linear Momentum

The total linear momentum $\boldsymbol{p}$ of the system is defined as the sum

$$\boldsymbol{p} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} \left(m_i \dot{\boldsymbol{R}}_i\right) \tag{2.61}$$

We can write the total linear momentum expression in terms of the total system mass M and the center of mass inertial velocity vector $\boldsymbol{R}_c$

$$\boldsymbol{p} = \sum_{i=1}^{N} \left(m_i \dot{\boldsymbol{R}}_i\right) = \sum_{i=1}^{N} m_i(\dot{\boldsymbol{R}}_c + \dot{\boldsymbol{r}}_i) = \sum_{i=1}^{N} m_i \dot{\boldsymbol{R}}_c + \cancelto{\mathbf{0}}{\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i} = M\dot{\boldsymbol{R}}_c \tag{2.62}$$

Note

Notice that the center of mass is defined in a way such that this simplification is feasible.

The the time rate of change of the linear momentum of a particle system is equal to the total external force acting on the system:

$$\dot{\boldsymbol{p}} = \sum_{i=1}^{N} (m_i \ddot{\boldsymbol{R}}_i) = \sum_{i=1}^{N} \boldsymbol{F}_i = \boldsymbol{F}$$

Law of conservation of linear momentum: If no external force $\boldsymbol{F}$ is present, then the total system linear momentum vector $\boldsymbol{p}$ will be constant.

Angular Momentum

The total system angular momentum vector $\boldsymbol{H}$ about the point P (an arbitrary point) is given as the sum of all the single particle angular momentum vectors about this point:

$$\boldsymbol{H}_P = \sum_{i=1}^{N} \boldsymbol{\sigma}_i \times m_i \dot{\boldsymbol{\sigma}}_i \tag{2.66}$$

where $\boldsymbol{\sigma}_i = \boldsymbol{R}_i - \boldsymbol{R}_P$ is the relative position vector, with $\boldsymbol{R}_i$ and $\boldsymbol{R}_p$ standing for position vectors.

The rate of $\boldsymbol{H}_P$ is

$$
\begin{aligned}
\dot{\boldsymbol{H}}_P &= \sum_{i=1}^{N} \dot{\boldsymbol{\sigma}}_i \times m_i \dot{\boldsymbol{\sigma}}_i + \sum_{i=1}^{N} \boldsymbol{\sigma}_i \times m_i \ddot{\boldsymbol{\sigma}}_i && (2.67) \\
&= \cancelto{\mathbf{0}}{\sum_{i=1}^{N} \dot{\boldsymbol{\sigma}}_i \times m_i \dot{\boldsymbol{\sigma}}_i} + \sum_{i=1}^{N} \boldsymbol{\sigma}_i \times m_i (\ddot{\boldsymbol{R}}_i - \ddot{\boldsymbol{R}}_P) && \text{(cross-product)} \\
&= \sum_{i=1}^{N} \boldsymbol{\sigma}_i \times m_i \ddot{\boldsymbol{R}}_i - \left(\sum_{i=1}^{N} \boldsymbol{\sigma}_i m_i \right) \times \ddot{\boldsymbol{R}}_P && (2.68) \\
&= \sum_{i=1}^{N} \boldsymbol{\sigma}_i \times \boldsymbol{F}_i - \left(\sum_{i=1}^{N} \boldsymbol{R}_i m_i - \sum_{i=1}^{N} m_i \boldsymbol{R}_P \right) \times \ddot{\boldsymbol{R}}_P \\
&= \sum_{i=1}^{N} \boldsymbol{\sigma}_i \times \boldsymbol{F}_i - M(\boldsymbol{R}_c - \boldsymbol{R}_P) \times \ddot{\boldsymbol{R}}_P && \text{(center of mass)} \\
&= {\color{blue}\boldsymbol{L}_P} + {\color{red}M \ddot{\boldsymbol{R}}_P \times (\boldsymbol{R}_c - \boldsymbol{R}_P)} && (2.71)
\end{aligned}
$$

- The first term is defined as total external torque $\boldsymbol{L}_P$ applied to the system.
- The second term is the change due to the acceleration of the reference point $\boldsymbol{R}_P$ and the relative position.

If no external torque $\boldsymbol{L}_P$ is acting on the system of particles, then the total angular momentum vector $\boldsymbol{H}_P$ is constant.

> **Note**
>
> Notice that P is an arbitrary point in the derivation. But if we choose the center of mass as P, Eq. (2.71) can be simplified further.
>
> So, again, the center of mass is defined in a way such that this simplification is feasible.

Reduce to a Single Particle System

Just set $N = 1$ for the above discussions.

Extend to a Continuous System

For most case, summations can be safely and carefully converted to integrals.

$$
\mathrm{d}\boldsymbol{F} = \ddot{\boldsymbol{R}}\,\mathrm{d}m
$$

$$
\boldsymbol{F} = \int_B \mathrm{d}\boldsymbol{F}
$$

$$
M\,\boldsymbol{R}_C = \int_B \boldsymbol{R}\,\mathrm{d}m
$$

$$
T = \frac{1}{2} M \dot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \frac{1}{2} \int_B \dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}}\, dm
$$

$$
\frac{dT}{dt} = M \ddot{\boldsymbol{R}}_c \cdot \dot{\boldsymbol{R}}_c + \int_B \dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}}\, dm
$$

$$
\boldsymbol{H}_P = \int_B \boldsymbol{\sigma} \times \dot{\boldsymbol{\sigma}}\, dm
$$

$$
\dot{\boldsymbol{H}}_P = \boldsymbol{L}_P + M \ddot{\boldsymbol{R}}_P \times (\boldsymbol{R}_c - \boldsymbol{R}_P)
$$

Textbook problem 2.11

2.11 A ball m is freely rolling in the lower half of a sphere under the influence of a constant gravity field as shown in Fig. P2.11. The sphere has a constant radius r. Assume that $\dot{\phi}(t_0)$ is zero and that $\theta(t_0)$, $\dot{\theta}(t_0)$, and $\phi(t_0)$ are given.

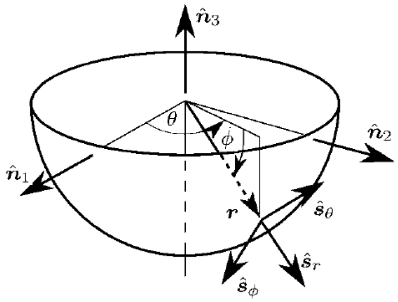

Fig. P2.11 Ball rolling inside a sphere.

(a) Find the equation of motion of the ball rolling without slip inside the sphere in terms of the spherical angle ϕ. *Hint: The angular momentum about the $\hat{n}_3$ axis is conserved.*

(b) What is the normal force that the wall of the sphere exerts onto the ball at any point in time?

(c) Since $\dot{\phi}(t_0) = 0$, the ball is starting out on an extrema. Find an expression in terms of θ_0, $\dot{\theta}_0$, and ϕ_0 that determines the other motion extrema where $\dot{\phi} = 0$. *Hint: Use conservation of energy.*

⚠ Be careful about that which of $\hat{s}_\phi$ and $\hat{s}_\theta$ is the second basis and why.

(a) EOM in terms of ϕ

Initial speed is horizontal, in $\hat{\boldsymbol{n}}_1$-$\hat{\boldsymbol{n}}_2$ plane.

Inertial frame $\mathcal{N}$.

Spherical frame $\mathcal{S}$: $\{\hat{\boldsymbol{s}}_r, \hat{\boldsymbol{s}}_\phi, \hat{\boldsymbol{s}}_\theta\}$.

Relationship among axes using geometry:

$$\begin{aligned}
\hat{\boldsymbol{s}}_r &= \cos\phi\cos\theta\hat{\boldsymbol{n}}_1 + \cos\phi\sin\theta\hat{\boldsymbol{n}}_2 - \sin\phi\hat{\boldsymbol{n}}_3 \\
\hat{\boldsymbol{s}}_\theta &= -\sin\theta\hat{\boldsymbol{n}}_1 + \cos\theta\hat{\boldsymbol{n}}_2 \\
\hat{\boldsymbol{s}}_\phi &= \sin\phi\cos\theta\hat{\boldsymbol{n}}_1 + \sin\phi\sin\theta\hat{\boldsymbol{n}}_2 - \cos\phi\hat{\boldsymbol{n}}_3
\end{aligned}$$

Or write in vectrix and DCM format:

$$\begin{Bmatrix} \hat{\boldsymbol{s}}_r \\ \hat{\boldsymbol{s}}_\theta \\ \hat{\boldsymbol{s}}_\phi \end{Bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & -\cos\phi \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix}$$

Angular velocity

$$\begin{aligned}
\boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} &= \dot{\theta}\hat{\boldsymbol{n}}_3 + \dot{\phi}\hat{\boldsymbol{s}}_\phi \\
&= \dot{\theta}(-\sin\phi\hat{\boldsymbol{s}}_r - \cos\phi\hat{\boldsymbol{s}}_\phi) + \dot{\phi}\hat{\boldsymbol{s}}_\theta \\
&= -\dot{\theta}\sin\phi\hat{\boldsymbol{s}}_r - \dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\phi + \dot{\phi}\hat{\boldsymbol{s}}_\theta
\end{aligned}$$

Position vector $\boldsymbol{r} = r\hat{\boldsymbol{s}}_r$.

Velocity vector:

$$\begin{aligned}
\dot{\boldsymbol{r}} &= r\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{s}}_r \\
&= r\frac{{}^{\mathcal{S}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{s}}_r + r\boldsymbol{\omega}_{\mathcal{E}/\mathcal{N}} \times \hat{\boldsymbol{s}}_r \\
&= r(-\dot{\theta}\sin\phi\hat{\boldsymbol{s}}_r - \dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\phi + \dot{\phi}\hat{\boldsymbol{s}}_\theta) \times \hat{\boldsymbol{s}}_r \\
&= r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta + r\dot{\phi}\hat{\boldsymbol{s}}_\phi
\end{aligned}$$

Acceleration vector:

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}(r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta + r\dot{\phi}\hat{\boldsymbol{s}}_\phi) \\
&= \frac{{}^{\mathcal{S}}\mathrm{d}}{\mathrm{d}t}(r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta + r\dot{\phi}\hat{\boldsymbol{s}}_\phi) + (-\dot{\theta}\sin\phi\hat{\boldsymbol{s}}_r - \dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\phi + \dot{\phi}\hat{\boldsymbol{s}}_\theta) \times (r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta + r\dot{\phi}\hat{\boldsymbol{s}}_\phi) \\
&= r\ddot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta - r\dot{\theta}\dot{\phi}\sin\phi\hat{\boldsymbol{s}}_\theta + r\ddot{\phi}\hat{\boldsymbol{s}}_\phi + \left(r\dot{\theta}^2\sin\phi\cos\phi\hat{\boldsymbol{s}}_\phi - \dot{\theta}r\dot{\phi}\sin\phi\hat{\boldsymbol{s}}_\theta - r\dot{\theta}^2\cos^2\phi\hat{\boldsymbol{s}}_r - r\dot{\phi}^2\hat{\boldsymbol{s}}_r\right) \\
&= -(r\dot{\theta}^2\cos^2\phi + r\dot{\phi}^2)\hat{\boldsymbol{s}}_r + (r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi})\hat{\boldsymbol{s}}_\phi + (r\ddot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta - 2r\dot{\theta}\dot{\phi}\sin\phi)\hat{\boldsymbol{s}}_\theta
\end{aligned}$$

The force vector is

$$\begin{aligned}
\boldsymbol{F} &= -N\hat{\boldsymbol{s}}_r - mg\hat{\boldsymbol{n}}_3 \qquad \text{(supporting + gravity)} \\
&= -N\hat{\boldsymbol{s}}_r - mg(-\sin\phi\hat{\boldsymbol{s}}_r - \cos\phi\hat{\boldsymbol{s}}_\phi) \\
&= (mg\sin\phi - N)\hat{\boldsymbol{s}}_r + mg\cos\phi\hat{\boldsymbol{s}}_\phi
\end{aligned}$$

Using Newton's 2nd law:

$$\begin{aligned}
mg\sin\phi - N &= -m(r\dot{\theta}^2\cos^2\phi + r\dot{\phi}^2) \\
g\cos\phi &= r\dot{\theta}^2\sin\phi\cos\phi + r\ddot{\phi} \\
0 &= r\ddot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta - 2r\dot{\theta}\dot{\phi}\sin\phi
\end{aligned}$$

If we can get an expression of $\dot{\theta}$ in terms of $(\phi, \dot{\phi}, \ddot{\phi})$, then we can get the EOM in terms of ϕ from the second equation above.

Using conservation of angular momentum along $\hat{\boldsymbol{n}}_3$:

$$\begin{aligned}
\boldsymbol{H}\cdot\hat{\boldsymbol{n}}_3 &= (\boldsymbol{r}\times m\boldsymbol{v})\cdot\hat{\boldsymbol{n}}_3 \\
&= mr\hat{\boldsymbol{s}}_r \times (r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta + r\dot{\phi}\hat{\boldsymbol{s}}_\phi)\cdot\hat{\boldsymbol{n}}_3 \\
&= (-mr\hat{\boldsymbol{s}}_r r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\phi + mr^2\dot{\phi}\hat{\boldsymbol{s}}_\theta)\cdot\hat{\boldsymbol{n}}_3 \\
&= mr^2\dot{\theta}\cos^2\phi \\
&= \boldsymbol{H}_0\cdot\hat{\boldsymbol{n}}_3 = mr^2\dot{\theta}_0\cos^2\phi_0
\end{aligned}$$

So,

$$\dot{\theta} = \frac{\cos^2\phi_0}{\cos^2\phi}\dot{\theta}_0$$

Finally,

$$\ddot{\phi} = g\cos\phi/r - \sin\phi\cos\phi\left(\frac{\cos^2\phi_0}{\cos^2\phi}\dot{\theta}_0\right)^2$$

(b) Normal supporting force

$$N = mg\sin\phi + m(r\dot{\theta}^2\cos^2\phi + r\dot{\phi}^2)$$

(c) Another extrema with $\dot{\phi} = 0$

$$\begin{aligned}
T + V &= \frac{m}{2}\|r\dot{\theta}\cos\phi\hat{\boldsymbol{s}}_\theta + r\dot{\phi}\hat{\boldsymbol{s}}_\phi\|^2 - mgr\sin\phi \\
&= \frac{m}{2}r^2\dot{\theta}^2\cos^2\phi + \frac{m}{2}r^2\dot{\phi}^2 - mgr\sin\phi
\end{aligned}$$

Plug in $\dot{\theta}$, $\dot{\phi} = 0$, $\phi_0, \theta_0, \dot{\theta}_0$ and we get the equation to solve for ϕ.

Rocket Problem

Study Section 2.6 Rocket Problem and try answer textbook problem 2.15.

2.15 Newton's second law for a particle of mass m states that $\boldsymbol{F} = \mathrm{d}/\mathrm{d}t(m\boldsymbol{v})$. If m is time varying, then one might expect $\boldsymbol{F} = \dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}}$ to be true. Explain why this logic is incorrect and does not lead to the correct rocket thrust equation.

Homework 01

Due on Feb 08. Check Canvas for detailed guidance on submissions.

Use office hours if you need.

Emails will be replied as timely as I could.