

# AE\_544\_LecNote04\_\_Eulerian\_Mechanics\_\_Ch04

## Disclaimers

This repository contains markdown-based lecture notes for the Spring 2025 semester. These materials are intended solely for students enrolled in the course during this term. Mistakes may exist and will be corrected as they are identified, but the official textbook remains the ultimate reference for accuracy.

The rotational dynamics of a rigid body are often referred to as Eulerian mechanics, because Euler's equation  $\dot{\mathbf{H}} = \mathbf{L}$  and Euler's rotational equation of motion generally govern this field.

But it is actually just Newtonian Mechanics being specialized for the case of rigid body dynamics.

### Review

Newton's second law:

$$\mathbf{F} = m\mathbf{a}$$

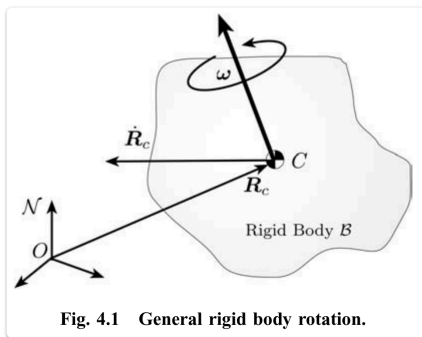
Euler's rotational equation of motion

$$\mathbf{L} = \dot{\mathbf{H}} = \frac{d}{dt}([I]\boldsymbol{\omega}) = [I]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times ([I]\boldsymbol{\omega})$$

## Rigid Body Dynamics

Let the moment be taken either about the center of mass or the inertial coordinate frame origin, see [Angular Momentum in lecture note 02](#). In either case Euler's equation reduces to

$$\dot{\mathbf{H}} = \mathbf{L} \quad (4.1)$$



In the inertial frame  $\mathcal{N}$ ,  
vector  $\mathbf{R}$  is the inertial position vector of an infinitesimal mass element  $dm$ ;  
 $\mathbf{R}_c$  is the center of mass;  
 $\mathbf{r} = \mathbf{R} - \mathbf{R}_c$  is the relative position vector.

The angular momentum relative to  $O$  is then given by

$$\begin{aligned}
\mathbf{H}_O &= \int_B \mathbf{R} \times \dot{\mathbf{R}} dm \\
&= \int_B (\mathbf{R}_c + \mathbf{r}) \times (\dot{\mathbf{R}}_c + \dot{\mathbf{r}}) dm \\
&= \int_B \mathbf{R}_c \times \dot{\mathbf{R}}_c dm + \int_B \mathbf{r} \times \dot{\mathbf{R}}_c dm + \mathbf{R}_c \times \int_B \dot{\mathbf{r}} dm + \int_B \mathbf{r} \times \dot{\mathbf{r}} dm \\
&= \left( \int_B dm \right) \mathbf{R}_c \times \dot{\mathbf{R}}_c + \left( \int_B \mathbf{r} dm \right) \times \dot{\mathbf{R}}_c + \mathbf{R}_c \times \left( \int_B \dot{\mathbf{r}} dm \right) + \int_B \mathbf{r} \times \dot{\mathbf{r}} dm \\
&= \left( \int_B dm \right) \mathbf{R}_c \times \dot{\mathbf{R}}_c + \left( \int_B \mathbf{r} dm \right) \times \dot{\mathbf{R}}_c + \mathbf{R}_c \times \left( \int_B \dot{\mathbf{r}} dm \right) + \int_B \mathbf{r} \times \dot{\mathbf{r}} dm \\
&= \mathbf{R}_c \times M \dot{\mathbf{R}}_c + \int_B \mathbf{r} \times \dot{\mathbf{r}} dm
\end{aligned} \tag{4.5}$$

- The first term is the angular momentum of the mass center about the origin, and its behavior was studied when discussing the dynamics of a single particle (the superparticle).
- The second term is the angular momentum vector of the rigid body  $\mathcal{B}$  about its mass center  $\mathbf{R}_c$ .

We define the angular momentum vector  $\mathbf{H}_c$  of the rigid body  $\mathcal{B}$  about its mass center  $\mathbf{R}_c$

$$\mathbf{H}_c = \int_B \mathbf{r} \times \dot{\mathbf{r}} dm \tag{4.6}$$

⚠ From here on  $\mathbf{H}_c$  will be studied.

$$\begin{aligned}
\dot{\mathbf{r}} &= \frac{d}{dt} \mathbf{r} = \frac{d}{dt} \mathbf{r} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r} & (\text{transport theorem}) \\
&= \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{r} & (\text{rigid body})
\end{aligned}$$

So,

$$\mathbf{H}_c = \int_B \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) dm = \int_B -[\tilde{\mathbf{r}}][\tilde{\mathbf{r}}] \boldsymbol{\omega} dm = \left( \int_B -[\tilde{\mathbf{r}}][\tilde{\mathbf{r}}] dm \right) \boldsymbol{\omega}$$

Express all the coordinates in the body frame  $\mathcal{B}$ ,

$$\begin{aligned}
\mathbf{r} &= r_1 \hat{\mathbf{b}}_1 + r_2 \hat{\mathbf{b}}_2 + r_3 \hat{\mathbf{b}}_3 \\
\boldsymbol{\omega} &= \omega_1 \hat{\mathbf{b}}_1 + \omega_2 \hat{\mathbf{b}}_2 + \omega_3 \hat{\mathbf{b}}_3 \\
\mathbf{H}_c &= H_{c1} \hat{\mathbf{b}}_1 + H_{c2} \hat{\mathbf{b}}_2 + H_{c3} \hat{\mathbf{b}}_3
\end{aligned}$$

The component form of  $\mathbf{H}_c$  is obtained as:

$$\mathbf{H}_c = \begin{bmatrix} H_{c1} \\ H_{c2} \\ H_{c3} \end{bmatrix} = \left( \int_B \begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} dm \right) \cdot \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Define the inertia matrix at the center of mass  ${}^B[I_c]$  as

$${}^B[I_c] = \int_B \begin{bmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ -r_1 r_2 & r_1^2 + r_3^2 & -r_2 r_3 \\ -r_1 r_3 & -r_2 r_3 & r_1^2 + r_2^2 \end{bmatrix} dm \tag{4.14}$$

The inertia matrix is **coordinate system-dependent**. If a different coordinate system were assigned to the rigid body, the corresponding inertia matrix would be different.

It will be always assumed that the vectors and inertia matrices are written in the  $\mathcal{B}$  frame and the superscript is dropped. The angular momentum vector of a rigid body about its center of mass can then be written in its simplest form as

$$\mathbf{H} = [I_c] \boldsymbol{\omega} \tag{4.15}$$

**Transformation from center of mass to a different point:** (Parallel axis theorem) Given the moment of inertia matrix  $[I_c]$  of a rigid body  $\mathcal{B}$  about its center of mass and the position vector  $\mathbf{R}_c$  of this center of mass relative to some fixed point  $O$ , then the inertia matrix  $\mathcal{B}$  about  $O$  is given through the transformation

$$[I_O] = [I_c] + M [\tilde{\mathbf{R}}_c] [\tilde{\mathbf{R}}_c]^T \tag{4.19}$$

All the components must be expressed in the same coordinate system/frame.

**Transformation from one body frame to another:** Let the reference frames  $\mathcal{B}$  and  $\mathcal{F}$  both be proper body-fixed coordinate systems. The inertia matrix written in the  $\mathcal{B}$  frame is rewritten into the  $\mathcal{F}$  frame through the similarity transformation

$$\mathcal{F}[I] = [FB] \cdot {}^B[I] \cdot [FB]^T \quad (4.24)$$

If the above  $[FB]^T$  happens to be the matrix of eigenvectors of  ${}^B[I]$ , the resulting  $\mathcal{F}[I]$  will be a diagonal matrix with eigenvalues at the diagonal, which is called **Principal inertias**:

$${}^{\mathcal{F}} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = [\boldsymbol{\nu}_1 \quad \boldsymbol{\nu}_2 \quad \boldsymbol{\nu}_3]^T \cdot {}^B[I] \cdot [\boldsymbol{\nu}_1 \quad \boldsymbol{\nu}_2 \quad \boldsymbol{\nu}_3],$$

where  $\boldsymbol{\nu}_i$  and  $I_i$  are the eigenvector and eigenvalue satisfying  ${}^B[I]\boldsymbol{\nu}_i = I_i \boldsymbol{\nu}_i$ . This new set of body-fixed coordinate basis  $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3\}$  are called the **principal axes**.

## Euler's Rotational Equations of Motion

Using transport theorem

$$\begin{aligned} \mathbf{L}_c &= \dot{\mathbf{H}}_c = \frac{{}^B d}{dt} \mathbf{H}_c + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times \mathbf{H}_c \\ &= \left( \frac{{}^B d}{dt} [I] \right) \boldsymbol{\omega} + [I] \frac{{}^B d}{dt} \boldsymbol{\omega} + \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} \times ([I] \boldsymbol{\omega}) && \text{(used } \mathbf{H}_c = [I] \boldsymbol{\omega} \text{ in body frame)} \\ &= [I] \frac{{}^B d}{dt} \boldsymbol{\omega} + \boldsymbol{\omega} \times ([I] \boldsymbol{\omega}) && ({}^B[I] \text{ is constant)} \\ &= [I] \frac{{}^B d}{dt} \boldsymbol{\omega} + [\tilde{\boldsymbol{\omega}}] [I] \boldsymbol{\omega} && ({}^B[I] \text{ is constant)} \\ &= [I] \left( \frac{{}^{\mathcal{N}} d}{dt} \boldsymbol{\omega} + \boldsymbol{\omega}_{\mathcal{N}/\mathcal{B}} \times \boldsymbol{\omega} \right) + [\tilde{\boldsymbol{\omega}}] [I] \boldsymbol{\omega} \\ &= [I] \dot{\boldsymbol{\omega}} + [\tilde{\boldsymbol{\omega}}] [I] \boldsymbol{\omega} && (4.32) \\ &= [I] \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times ([I] \boldsymbol{\omega}) && (4.32 \text{ alternative}) \end{aligned}$$

where we have used the fact that  $\boldsymbol{\omega} = \boldsymbol{\omega}_{\mathcal{B}/\mathcal{N}} = -\boldsymbol{\omega}_{\mathcal{N}/\mathcal{B}}$

In the principal body frame, the Euler's equation reduces to

$$\begin{aligned} I_{11} \dot{\omega}_1 &= -(I_{33} - I_{22}) \omega_2 \omega_3 + L_1 \\ I_{22} \dot{\omega}_2 &= -(I_{11} - I_{33}) \omega_3 \omega_1 + L_2 \\ I_{33} \dot{\omega}_3 &= -(I_{22} - I_{11}) \omega_1 \omega_2 + L_3 \end{aligned} \quad (4.33)$$

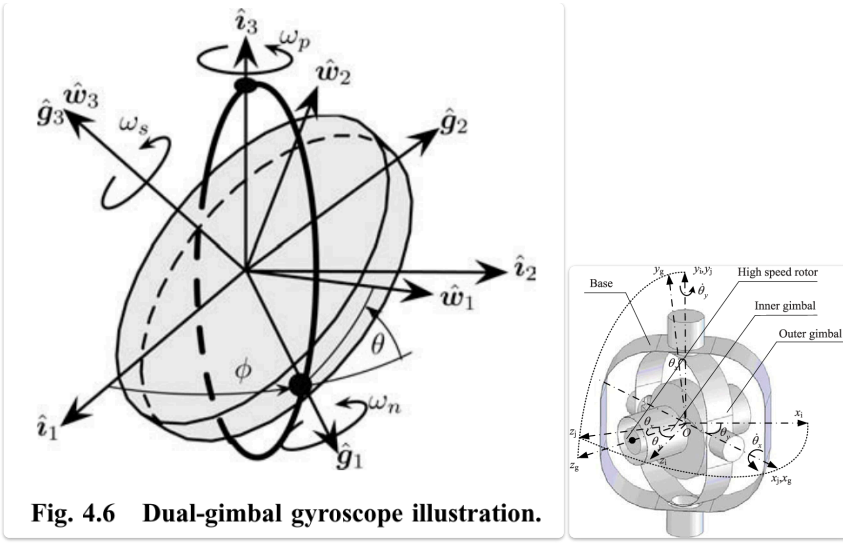
For an axially symmetric body with transverse inertia  $I_T = I_{11} = I_{22}$ , this can be further simplified to

$$\begin{aligned} I_T \dot{\omega}_1 &= -(I_{33} - I_T) \omega_2 \omega_3 \\ I_T \dot{\omega}_2 &= (I_{33} - I_T) \omega_3 \omega_1 \\ I_{33} \dot{\omega}_3 &= 0 \end{aligned} \quad (4.34)$$

which can be solved analytically. Then we can analyze the different precession behaviors of prolate and oblate bodies. (Refer to textbook Sec. 4.2.3)

Later, the same problem will be discussed in a more geometric perspective using Euler angles (refer to textbook Sec. 4.3.3, or lecture note section [Special Discussions About Axisymmetric Rigid Body](#) later).

## Example 4.4: dual-gimbal gyroscope



**Fig. 4.6 Dual-gimbal gyroscope illustration.**

Inertial frame  $\mathcal{I} : \{\hat{i}_1, \hat{i}_2, \hat{i}_3\}$

Gimbal frame  $\mathcal{G} : \{\hat{g}_1, \hat{g}_2, \hat{g}_3\}$

Wheel/disk frame  $\mathcal{W} : \{\hat{w}_1, \hat{w}_2, \hat{w}_3\}$

Angular velocities:

$$\omega_{\mathcal{W}/\mathcal{G}} = \omega_s \hat{w}_3 = \omega_s \hat{g}_3$$

$$\omega_{\mathcal{G}/\mathcal{I}} = \dot{\phi} \hat{i}_3 + \dot{\theta} \hat{g}_1 = \omega_p \hat{i}_3 + \omega_n \hat{g}_1$$

Since we want to expand  $\hat{i}_3$  using  $\mathcal{G}$  basis, we need the DCM from  $\mathcal{I}$  to  $\mathcal{G}$ , which is obtained by first a rotation of  $\phi$  around  $\hat{i}_1$  and then a rotation around of  $\theta$  around  $\hat{g}_1$ :

$$\begin{aligned} [GI] &= [M_1(\theta)] [M_3(\phi)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\cos \theta \sin \phi & \cos \theta \cos \phi & \sin \theta \\ \sin \phi \sin \theta & -\sin \theta \cos \phi & \cos \theta \end{bmatrix} \end{aligned}$$

Therefore we have

$$\hat{i}_3 = \sin \theta \hat{g}_2 + \cos \theta \hat{g}_3$$

and the gimbal angular velocity is

$$\omega_{\mathcal{G}/\mathcal{I}} = \omega_n \hat{g}_1 + \omega_p \sin \theta \hat{g}_2 + \omega_p \cos \theta \hat{g}_3$$

and the wheel angular velocity is

$$\omega_{\mathcal{W}/\mathcal{I}} = \omega_{\mathcal{W}/\mathcal{G}} + \omega_{\mathcal{G}/\mathcal{I}} = \omega_n \hat{g}_1 + \omega_p \sin \theta \hat{g}_2 + (\omega_s + \omega_p \cos \theta) \hat{g}_3$$

The wheel inertia in the gimbal frame  $\mathcal{G}$  is diagonal,

$$\dot{\mathbf{H}} = [I_W] \frac{^{\mathcal{G}}d}{dt} \omega_{\mathcal{W}/\mathcal{I}} + \omega_{\mathcal{G}/\mathcal{I}} \times ([I_W] \omega_{\mathcal{W}/\mathcal{I}})$$

The first term without  $[I_W]$  is:

$$\begin{aligned} \frac{^{\mathcal{G}}d}{dt} \omega_{\mathcal{W}/\mathcal{I}} &= \frac{^{\mathcal{G}}d}{dt} (\omega_n \hat{g}_1 + \omega_p \sin \theta \hat{g}_2 + (\omega_s + \omega_p \cos \theta) \hat{g}_3) \\ &= \dot{\omega}_n \hat{g}_1 + \frac{^{\mathcal{G}}d}{dt} (\omega_p \sin \theta) \hat{g}_2 + \frac{^{\mathcal{G}}d}{dt} (\omega_s + \omega_p \cos \theta) \hat{g}_3 \\ &= \dot{\omega}_n \hat{g}_1 + (\dot{\omega}_p \sin \theta + \omega_p \dot{\theta} \cos \theta) \hat{g}_2 + (\dot{\omega}_s + \dot{\omega}_p \cos \theta - \omega_p \dot{\theta} \sin \theta) \hat{g}_3 \\ &= \dot{\omega}_n \hat{g}_1 + (\dot{\omega}_p \sin \theta + \omega_p \omega_n \cos \theta) \hat{g}_2 + (\dot{\omega}_s + \dot{\omega}_p \cos \theta - \omega_p \omega_n \sin \theta) \hat{g}_3 \end{aligned}$$

The second term is:

$$\begin{aligned}
\omega_{G/I} \times ([I_W]\omega_{W/I}) &= (\omega_n \hat{g}_1 + \omega_p \sin \theta \hat{g}_2 + \omega_p \cos \theta \hat{g}_3) \times (I_1 \omega_n \hat{g}_1 + I_2 \omega_p \sin \theta \hat{g}_2 + I_3 (\omega_s + \omega_p \cos \theta) \hat{g}_3) \\
&= I_2 \omega_n \omega_p \sin \theta \hat{g}_3 - I_3 \omega_n (\omega_s + \omega_p \cos \theta) \hat{g}_2 \\
&\quad - I_1 \omega_p \omega_n \sin \theta \hat{g}_3 + I_3 \omega_p (\omega_s + \omega_p \cos \theta) \sin \theta \hat{g}_1 \\
&\quad + I_1 \omega_p \omega_n \cos \theta \hat{g}_2 - I_2 \omega_p^2 \cos \theta \sin \theta \hat{g}_1
\end{aligned}$$

The torque acting on the rotor is

$$\mathbf{L} = L_1 \hat{g}_1 + L_2 \hat{g}_2 + L_3 \hat{g}_3$$

Equating everything gives:

$$\begin{aligned}
L_1 &= I_1 \dot{\omega}_n + I_3 \omega_p (\omega_s + \omega_p \cos \theta) \sin \theta - I_2 \omega_p^2 \cos \theta \sin \theta \\
L_2 &= I_2 (\dot{\omega}_p \sin \theta + \omega_p \omega_n \cos \theta) - I_3 \omega_n (\omega_s + \omega_p \cos \theta) + I_1 \omega_p \omega_n \cos \theta \\
L_3 &= I_3 (\dot{\omega}_s + \dot{\omega}_p \cos \theta - \omega_p \omega_n \sin \theta) + I_2 \omega_n \omega_p \sin \theta - I_1 \omega_p \omega_n \sin \theta
\end{aligned}$$

Next, solve first-order ODEs numerically.

## Textbook problem 4.10 (cylinder in gyroscope)

**4.10** A solid cylinder of mass  $m$ , radius  $a$ , and length  $l$  is pivoted about a transverse axis  $(B-B')$  through its center of mass as shown in Fig. P4.10. The axis  $(A-A')$  rotates with a constant angular velocity  $\Omega$ . Assume  $l > \sqrt{3}a$ .

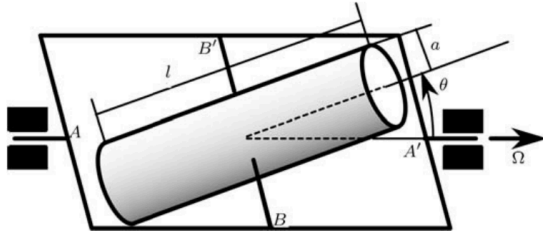
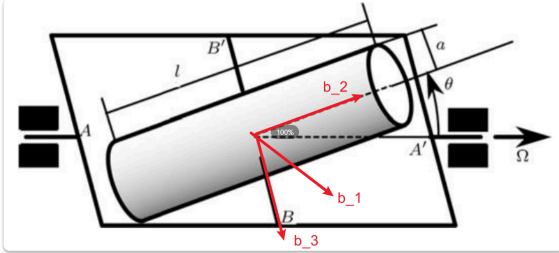


Fig. P4.10 Solid cylinder in a two-hinge gyroscope.

- Find the frequency  $\omega_n$  of small oscillations about  $\theta = \pi/2$ .
- What is the angular velocity  $\dot{\theta}^*$  when  $\theta = \pi/2$ , if the cylinder is released from  $\theta = 0$  with a very small positive value of  $\theta_0$ ? Determine  $\dot{\theta}^*$  as a function of  $m$ ,  $a$ ,  $l$ , and  $\Omega$ .

Defining the body frame  $\{\hat{b}\}$  as shown in the figure:



### (a) natural frequency around $\theta = \pi/2$

The cylinder inertias are given by

$$\begin{aligned}
I_t &= \frac{m}{12}(3a^2 + l^2) \\
I_a &= \frac{m}{2}a^2
\end{aligned}$$

Angular velocity vector due to the prescribed gimbal motion and the rotation around  $B-B'$  axle:

$$\begin{aligned}
\boldsymbol{\omega} &= \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3 \\
&= \sin \theta \Omega \hat{b}_1 + \cos \theta \Omega \hat{b}_2 + \dot{\theta} \hat{b}_3
\end{aligned}$$

Using the Euler equation derived before

$$\begin{aligned}
I_{11} \dot{\omega}_1 &= -(I_{33} - I_{22}) \omega_2 \omega_3 + L_1 \\
I_{22} \dot{\omega}_2 &= -(I_{11} - I_{33}) \omega_3 \omega_1 + L_2 \\
I_{33} \dot{\omega}_3 &= -(I_{22} - I_{11}) \omega_1 \omega_2 + L_3
\end{aligned} \tag{4.33}$$

we have,

$$I_t \dot{\omega}_3 = -(I_a - I_t) \omega_1 \omega_2 + 0$$

$$\ddot{\theta} = \frac{I_t - I_a}{I_t} \sin \theta \cos \theta \Omega^2 = \frac{I_t - I_a}{2I_t} \Omega^2 \sin(2\theta)$$

The frequency of small oscillations at  $\theta = \pi/2$  means the linear frequency around the equilibrium  $\theta = \pi/2$ , so we can do a variationally analysis by plugging in  $\theta + \delta\theta$  and expand using Taylor series around  $\theta = \pi/2$ , which leads to:

$$\ddot{\delta\theta} = 0 + \frac{I_t - I_a}{2I_t} \Omega^2 2 \cos(2\theta) \Big|_{\pi/2} \delta\theta + O(\delta\theta^2)$$

Upon checking that  $I_t - I_a = \frac{m}{12}(3a^2 + l^2) - \frac{m}{2}a^2 = \frac{m}{12}(l^2 - 3a^2) < 0$ , we know that the solution to the first-order approximation will be a trigonometric (harmonic) function with a natural frequency  $\omega_n$  of

$$\omega_n = \sqrt{\frac{I_t - I_a}{I_t} \Omega^2} = \Omega \sqrt{\frac{\frac{m}{12}(3a^2 + l^2) - \frac{m}{2}a^2}{\frac{m}{12}(3a^2 + l^2)}} = \Omega \sqrt{\frac{l^2 - 3a^2}{l^2 + 3a^2}}$$

## (b) final velocity

Multiple  $\dot{\theta}$  at both size and do a definite integral:

$$\int_0^t \ddot{\theta} \dot{\theta} dt = \int_0^t d\left(\frac{\dot{\theta}^2}{2}\right) = \int_0^t \frac{k}{2} \sin(2\theta) \dot{\theta} dt = \int_0^{\theta(t)} \frac{k}{2} \sin(2\theta) d\theta$$

where  $k = \omega_n^2 = \frac{l^2 - 3a^2}{l^2 + 3a^2} \Omega^2$ .

So,

$$\frac{\dot{\theta}^2}{2} \Big|_0^t = \frac{\dot{\theta}^2(t)}{2} = \frac{k}{4} \cos(2\theta) \Big|_0^{\theta(t)} = \frac{k}{4} (1 - \cos(2\theta(t)))$$

$$\dot{\theta}(t)^2 = \frac{k}{2} (1 - \cos(2\theta(t)))$$

At  $\theta_f = \theta(t_f) = \pi/2$ ,

$$\dot{\theta}_f = \dot{\theta}(t_f) = \sqrt{\frac{k}{2}} (1 - (-1)) = \sqrt{k} = \Omega \sqrt{\frac{l^2 - 3a^2}{l^2 + 3a^2}}$$

## Rotational Kinetic Energy

The rigid body rotational kinetic energy expression as

$$T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{H}_c = \frac{1}{2} \boldsymbol{\omega}^T [I] \boldsymbol{\omega} \quad (4.52)$$

The work done by the torque  $\mathbf{L}_c$  onto a rigid body  $\mathcal{B}$

$$\dot{T}_{\text{rot}} = \dot{\boldsymbol{\omega}}^T [I] \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \mathbf{L}_c$$

where  $\mathbf{L}_c = [I] \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times ([I] \boldsymbol{\omega})$ , as given in Eq. (4.32 alternative).

## Torque-Free Rigid Body Rotation (a geometric approach)

If no external torques are acting on a system, the total angular momentum vector  $\mathbf{H}$  is constant in the inertial frame, and the total rotational kinetic energy is also a constant.

When expressed in body frame  $\mathcal{B}$ , the angular momentum vector  $\mathbf{H}$  will generally not appear to be constant but rotating. However, the angular momentum magnitude  $H$  is still a constant.

**Principal axes assumption:** We always assume that the body-fixed coordinate axes are all aligned with principal inertia axes.

Using coordinates of the angular momentum  $\mathbf{H}$  and angular velocity  $\boldsymbol{\omega}$  in body frame  $\mathcal{B}$ , we have the angular momentum ellipsoid as

$$H^2 = H_1^2 + H_2^2 + H_3^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 \quad (4.64 \text{ and } 4.62)$$

and the energy ellipsoid as

$$T = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \quad (4.63)$$

The geometric interpretation of this is that  $\omega(t)$  must lie on the intersection of the momentum and energy ellipsoid surfaces.

To simplify one ellipsoid to a sphere, using  $H_i$  as independent coordinates, so we have the momentum sphere

$$H^2 = H_1^2 + H_2^2 + H_3^2 \quad (4.64)$$

and the energy ellipsoid (substituting  $\omega_i$  with  $H_i/I_i$ )

$$T = \frac{H_1^2}{2I_1} + \frac{H_2^2}{2I_2} + \frac{H_3^2}{2I_3} \quad (4.65 \text{ altered})$$

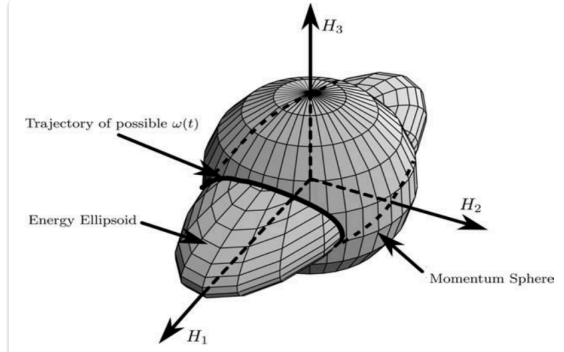


Fig. 4.8 General intersection of the momentum sphere and the energy ellipsoid.

Further requiring

$$I_1 \geq I_2 \geq I_3$$

the largest ellipsoid semi-axis occurs along  $\hat{b}_1$  and the smallest along  $\hat{b}_3$ . The overall shape and aspect ratio of the ellipsoid will remain the same for each choice in the energy  $T$ , and varying  $T$  will only uniformly scale the ellipsoid.

For a general rigid body motion as shown in Fig. 4.8, once the initial kinetic energy  $T$  and angular momentum vector  $\mathbf{H}$  are established, the angular velocity vector  $\mathbf{x}$  will theoretically trace out a particular intersection curve forever.

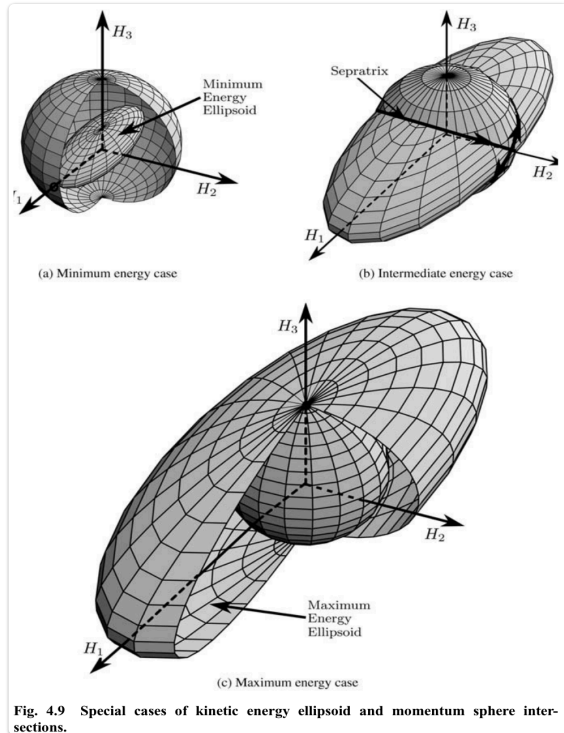


Fig. 4.9 Special cases of kinetic energy ellipsoid and momentum sphere intersections.

- For this minimum kinetic energy case, the rigid body B is spinning purely about its axis of maximum inertia  $\mathbf{b}_1$ , and the corresponding kinetic energy is

$$T_{\min} = \frac{H^2}{2I_1}$$

- For the intermediate kinetic energy case, the kinetic energy along the separatrix is

$$T_{\text{intermediate}} = \frac{H^2}{2I_2}$$

- For the maximum kinetic energy case, the rotation is along  $\hat{b}_3$ , and

$$T_{\text{max}} = \frac{H^2}{2I_3}$$

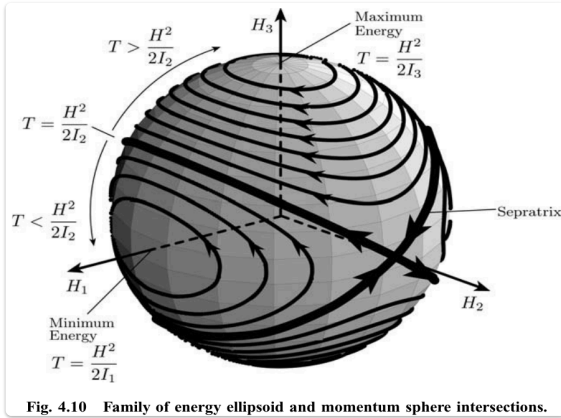


Fig. 4.10 Family of energy ellipsoid and momentum sphere intersections.

All feasible  $\omega(t)$  paths form closed trajectories. The energy is not conserved in reality, so the motion will wobble along  $\hat{b}_3$  when the energy is higher than that on the separatrix, then wobble along  $\hat{b}_2$  after crossing the separatrix, and eventually stabilized and slows down along  $\hat{b}_1$ .

## Special Discussions on Axisymmetric Rigid Body

Without loss of generality, assume that  $I_2 = I_3$ . Then the yaw, pitch, and roll angle rates are given as

$$\begin{aligned}\dot{\psi} &= -\frac{H}{I_2} \\ \dot{\theta} &= 0 \\ \dot{\phi} &= H \left( \frac{I_2 - I_1}{I_1 I_2} \right) \sin \theta\end{aligned}\tag{4.77}$$

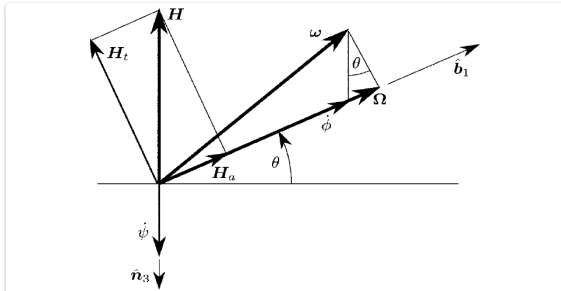


Fig. 4.12 Angular velocity and momentum vector relationship for the case  $I_2 > I_1$ .

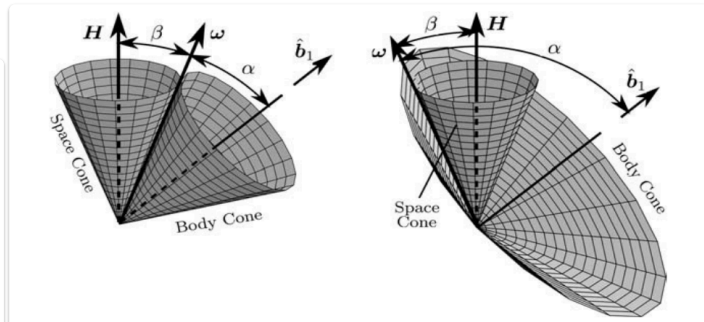
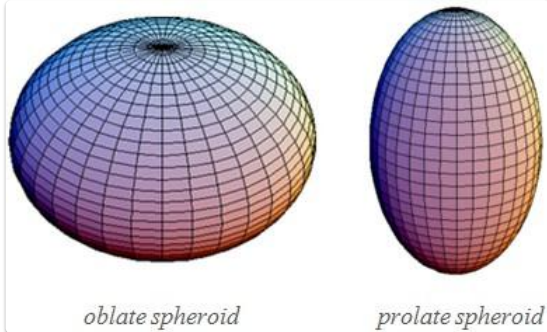


Fig. 4.13 Conic illustration of direct and retrograde precession of a freely rotating rigid body.

Using a (3-

2-1) Euler angle set, i.e. yaw-pitch-roll set, the precession motion of oblate (disk) or prolate (stick) bodies.



Details omitted, please refer to the textbook section 4.3.3.

## Dynamics of Dual-Spin Spacecraft (or Reaction Wheels)



The dual-spin spacecraft is a simple system where passive attitude stability is achieved by adding a single fly-wheel to the rigid spacecraft.

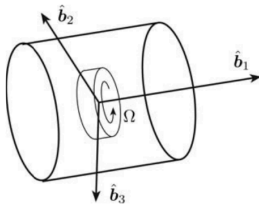


Fig. 4.15 Illustration of a dual-spin spacecraft with a rotating component aligned with the first spacecraft principal axis.

To develop the dual-spin system equations of motion, assume that the rotating fly-wheel is aligned with the first principal axis  $\hat{b}_1$  of the main spacecraft component as illustrated in Fig. 4.15.

Let  $\omega = \omega_{B/N}$  be the body angular velocity of the main craft, while  $\omega_{W/B} = \Omega \hat{b}_1$  is the angular velocity of the fly-wheel relative to the spacecraft.

The total angular momentum is then given by

$$\mathbf{H} = [I_s]\omega + [I_W](\Omega \hat{b}_1 + \omega) \quad (4.82)$$

where  $[I_s]$  is the inertia matrix of the main spacecraft system, while  $[I_W]$  is the inertia of the fly-wheel component.

⚠  $[I_s]$  doesn't include  $[I_W]$  in Eq. (4.82).

Using Euler's equation  $\dot{\mathbf{H}} = \mathbf{L}$ , we have

$$\dot{\mathbf{H}} = \mathbf{L} = ([I_s] + [I_W])\dot{\omega} + [\tilde{\omega}]([I_s] + [I_W])\omega + [I_W](\dot{\Omega} \hat{b}_1) + [\tilde{\omega}][I_W](\Omega \hat{b}_1) \quad (4.83)$$

Skew-symmetric tilde operator:

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Assuming no external torque is present, and the body frame is principal,

$$\mathbf{0} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_r} \end{bmatrix} \dot{\Omega} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_r} \end{bmatrix} \Omega \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Inspect each components and we have

$$\begin{aligned} I_1 \dot{\omega}_1 - \omega_3 \omega_2 I_2 + \omega_2 \omega_3 I_3 + I_{W_r} \dot{\Omega} &= 0 \\ I_2 \dot{\omega}_2 + \omega_3 \omega_1 I_1 - \omega_1 \omega_3 I_3 + I_{W_s} \omega_3 \Omega &= 0 \\ I_3 \dot{\omega}_3 - \omega_2 \omega_1 I_1 + \omega_1 \omega_2 I_2 - I_{W_s} \omega_2 \Omega &= 0 \end{aligned}$$

Move all the derivatives to the LHS and we have

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3 - I_{W_r} \dot{\Omega} \\ I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_1 \omega_3 - I_{W_s} \omega_3 \Omega \\ I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2 + I_{W_s} \omega_2 \Omega \end{aligned} \quad (4.85)$$

For the dual-spin spacecraft concept, the spin rate  $\Omega$  is typically held at a constant value, so we can eliminate  $\dot{\Omega}$  and get

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_2 - I_3) \omega_2 \omega_3 \\ I_2 \dot{\omega}_2 &= (I_3 - I_1) \omega_1 \omega_3 - I_{W_s} \omega_3 \Omega \\ I_3 \dot{\omega}_3 &= (I_1 - I_2) \omega_1 \omega_2 + I_{W_s} \omega_2 \Omega \end{aligned} \quad (4.86 \text{ dual-spin S/C})$$

It is apparent that the only equilibrium spin configuration (meaning  $\omega$  is time-invariant) is when both  $\omega_2$  and  $\omega_3$  are zero

$$\omega_e = \omega_{e_1} \hat{b}_1 \quad (4.87)$$

Linear stability analysis around the equilibrium point for a dual-spin S/C. Let the actual angular velocity be given by

$$\omega = \omega_e + \delta\omega = \begin{bmatrix} \omega_{e_1} + \delta\omega_1 \\ \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} \quad (4.88 \text{ expanded})$$

then substituting into Eq. (4.86) and dropping higher-order (nonlinear) terms to get

$$\dot{\delta\omega}_1 = 0 \quad (4.89a)$$

$$\dot{\delta\omega}_2 = \left( \frac{I_3 - I_1}{I_2} \omega_{e1} - \frac{I_{W_s}}{I_2} \Omega \right) \delta\omega_3 \quad (4.89b)$$

$$\dot{\delta\omega}_3 = \left( \frac{I_1 - I_2}{I_3} \omega_{e1} + \frac{I_{W_s}}{I_3} \Omega \right) \delta\omega_2$$

Eq. (4.89a) tells  $\delta\omega_1$  is constant, and thus  $\omega_{e1} + \delta\omega_1$  along  $\hat{\mathbf{b}}_1$  is also constant.

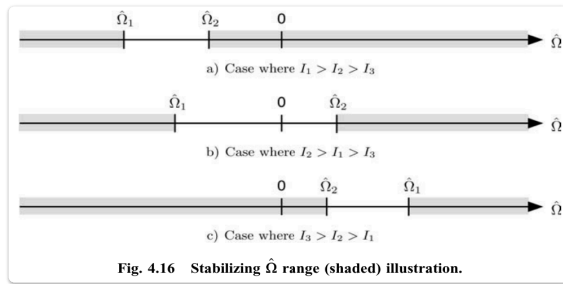
Rewrite the two coupled first-order differential equations into two uncoupled second-order differential equations, by taking derivative of Eq. (4.89b):

$$\begin{aligned} \ddot{\delta\omega}_2 &= \left( \frac{I_3 - I_1}{I_2} R \omega_{e1} - \frac{I_{W_s}}{I_2} \Omega \right) \dot{\delta\omega}_3 \\ &= - \left( \frac{I_1 - I_3}{R} I_2 \omega_{e1} + \frac{I_{W_s}}{I_2} \Omega \right) \left( \frac{I_1 - I_2}{I_3} \omega_{e1} + \frac{I_{W_s}}{I_3} \Omega \right) \delta\omega_2 \\ &= - \frac{\omega_{e1}^2}{I_2 I_3} \left( I_1 - I_3 + I_{W_s} \frac{\Omega}{\omega_{e1}} \right) \left( I_1 - I_2 + I_{W_s} \frac{\Omega}{\omega_{e1}} \right) \delta\omega_2 \\ &= - \frac{\omega_{e1}^2}{I_2 I_3} \left( I_1 - I_3 + I_{W_s} \hat{\Omega} \right) \left( I_1 - I_2 + I_{W_s} \hat{\Omega} \right) \delta\omega_2 \end{aligned} \quad (4.92--95 \text{ combined})$$

If and only if the factor is negative, the system is linearly stable (when the solution is trigonometric function, otherwise it's exponential function). Two critical wheel speeds are:

$$\hat{\Omega}_1 = \frac{I_3 - I_1}{I_{W_s}} \quad (\text{leads 1st term to 0})$$

$$\hat{\Omega}_2 = \frac{I_2 - I_1}{I_{W_s}} \quad (\text{leads 2nd term to 0})$$



A lot of interesting discussions can be conducted about this results. Basically, when the wheel is spinning fast enough and governs the ratio of  $H_1, H_2, H_3$ , it can change the intersection curve of the energy ellipsoid and momentum sphere. (Omitted here and refer to the textbook)

[Revisit Fig. 4.9](#) >

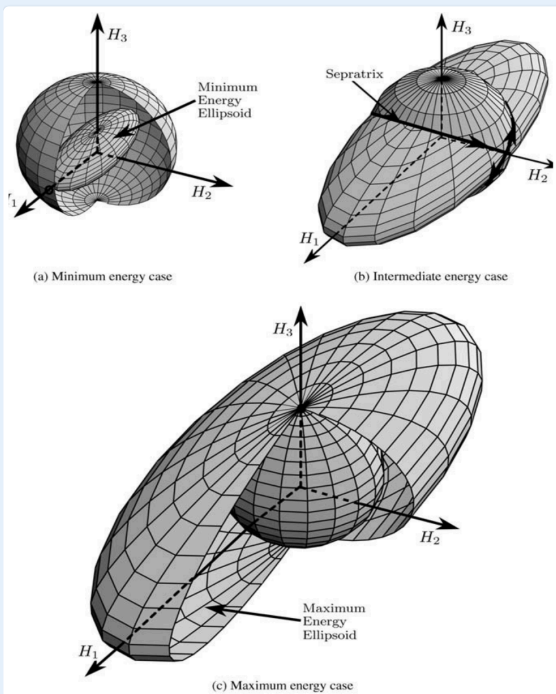


Fig. 4.9 Special cases of kinetic energy ellipsoid and momentum sphere intersections.

For a rigid body, its distribution of  $\mathbf{H}$  along three principal axes is determined by the inertia  $I_1, I_2, I_3$ . But for a dual-spin S/C, it can increase a particular  $H_i$  without affecting the other two components much. Therefore, even for the intermediate energy case in (b), the wheel can spin up fast enough to blow up the ellipsoid along  $\hat{\mathbf{b}}_2$  or  $H_2$  direction until it becomes the minimum energy case in (a). Note that when the ellipsoid is being blown up, the energy sphere is also increasing.

## Dynamics of Multiple Control Moment Gyroscopes (CMGs)

Instead of using thrusters to perform precise spacecraft attitude maneuvers, typically control moment gyroscopes (CMGs) or reaction wheels (RWs) are used. Either device can change its internal angular momentum vector and thus, through Euler's equation, produce an effective torque on the spacecraft.

A single-gimbal CMG contains a wheel spinning at a constant rate. To exert a torque onto the spacecraft, this wheel is gimballed or rotated about a fixed axis. A separate feedback control loop is used to spin up the rotor to the required spin rate and maintain it. The advantage of a CMG is that a relatively small gimbal torque input is required to produce a large effective torque output on the spacecraft. The drawback of the single-gimbal CMGs is that their control laws can be fairly complex and that such CMG systems encounter certain singular gimbal angle configurations. At these singular configurations the CMG cluster is unable to produce the required torque exactly, or any torque at all if the required torque is orthogonal to the plane of allowable torques. Double-gimbal CMGs have fewer problems with singularities. However, they are also much more costly and complicated devices than the single-gimbal CMGs.

Reaction wheels, on the other hand, have a wheel spinning about a body-fixed axis whose spin speed is variable. Torques are produced on the spacecraft by accelerating or decelerating the reaction wheels. RW systems do not have singular configurations and typically have simpler control laws than CMG clusters. Drawbacks to the reaction wheels include a relatively small effective torque being produced on the spacecraft and the possible reaction wheel saturation. To exert a given torque onto a spacecraft, reaction wheels typically require more energy than CMGs.

Variable speed control moment gyroscopes (VSCMGs) combine positive features of both the single-gimbal CMGs and the RWs.

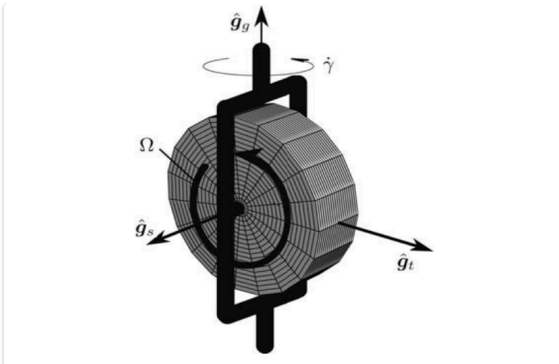


Fig. 4.17 Illustration of a variable speed control moment gyroscope.

The reference frame is the spacecraft body frame  $\mathcal{B}$  (usually the principal inertia frame).

Gimbal frame  $\mathcal{G} : \{\hat{b}_s, \hat{b}_t, \hat{g}_g\}$ , where the gimbal axis  $\hat{g}_g$  is fixed in  $\mathcal{B}$ , and the spin axis  $\hat{g}_s$  and the transverse axis  $\hat{g}_t$  are time varying in  $\mathcal{B}$  described by the gimbal angle  $\gamma(t)$ .  $\mathcal{G}$  is aligned with the gimbal principal inertia axes.

The wheel frame  $\mathcal{W} : \{\hat{w}_1, \hat{w}_2, \hat{g}_s\}$ , where the 3rd axis is aligned with  $\mathcal{G}$ .

The spinning angular velocity vector of the wheel is  $\omega_{\mathcal{W}/\mathcal{G}} = \Omega \hat{g}_g$ .

The angular velocity vector of the gimbal frame  $\mathcal{G}$  is  $\omega_{\mathcal{G}/\mathcal{B}} = \dot{\gamma} \hat{g}_g$ .

Given an initial gimbal angle  $\gamma_0$ , we have the evolution of spin and transverse axes as

$$\begin{aligned}\hat{g}_s(t) &= \cos(\gamma(t) - \gamma_0) \hat{g}_s(t_0) + \sin(\gamma(t) - \gamma_0) \hat{g}_t(t_0) \\ \hat{g}_t(t) &= -\sin(\gamma(t) - \gamma_0) \hat{g}_s(t_0) + \cos(\gamma(t) - \gamma_0) \hat{g}_t(t_0)\end{aligned}$$

which can be explained as simply a elementary rotation along the 3rd axis, indicating a matrix multiplication using  $M_3(\gamma(t) - \gamma_0)$ .

Gimbal frame inertial matrix is  ${}^{\mathcal{G}}[I_G] = \begin{bmatrix} I_{G_s} & 0 & 0 \\ 0 & I_{G_t} & 0 \\ 0 & 0 & I_{G_g} \end{bmatrix}$ .

The reaction wheel inertia about the same axes is  ${}^{\mathcal{G}}[I_W] = \begin{bmatrix} I_{W_s} & 0 & 0 \\ 0 & I_{W_t} & 0 \\ 0 & 0 & I_{W_l} \end{bmatrix}$ .

In this development the RW and gimbal frame inertias are not combined early on into one overall VSCMG inertia matrix; rather, they are retained as separate entities until later in the development. This will allow for a precise formulation of the actual physical motor torques that drive the RWs or the CMGs.

Assuming that the coordinate of each gimbal frame vector is given in  $\mathcal{B}$ , we have the DCM of  $\mathcal{G}$  relative to  $\mathcal{B}$  as

$$\{\hat{g}\} = [GB]\{\hat{b}\} = \begin{bmatrix} {}^{\mathcal{B}}(\hat{g}_s)^T \\ {}^{\mathcal{B}}(\hat{g}_t)^T \\ {}^{\mathcal{B}}(\hat{g}_g)^T \end{bmatrix} \{\hat{b}\}$$

which gives the reversed conversion, i.e., the DCM of  $\mathcal{B}$  relative to  $\mathcal{G}$ , as

$$[BG] = \begin{bmatrix} {}^{\mathcal{B}}(\hat{g}_s) & {}^{\mathcal{B}}(\hat{g}_t) & {}^{\mathcal{B}}(\hat{g}_g) \end{bmatrix}.$$

The inertial of the gimbal frame and the reaction wheel in the body frame can be found through

$$\begin{aligned}{}^{\mathcal{B}}[I_G] &= [BG] \cdot {}^{\mathcal{G}}[I_G] \cdot [BG]^T = I_{G_s} \hat{g}_s \hat{g}_s^T + I_{G_t} \hat{g}_t \hat{g}_t^T + I_{G_g} \hat{g}_g \hat{g}_g^T \\ {}^{\mathcal{B}}[I_W] &= [BG] \cdot {}^{\mathcal{G}}[I_W] \cdot [BG]^T = I_{W_s} \hat{g}_s \hat{g}_s^T + I_{W_t} \hat{g}_t \hat{g}_t^T + I_{W_g} \hat{g}_g \hat{g}_g^T\end{aligned} \quad (4.105 \text{ and } 4.106)$$

🔗 Now, find the total angular momentum as a summation of three parts: S/C with a static VSCMG (body), the gimbal frame of VSCMG (gimbal), and the wheel of VSCMG (wheel).

The total angular momentum  $\mathbf{H}$  of the spacecraft and the VSCMG (frame and wheel) about the spacecraft center of mass is given by

$$\mathbf{H} = \mathbf{H}_B + \mathbf{H}_G + \mathbf{H}_W \quad (4.107)$$

where the angular momentum component of spacecraft is

$$\mathbf{H}_B = [I_s] \boldsymbol{\omega}_{B/N} \quad (4.109)$$

the component of the gimbal frame is

$$\begin{aligned} \mathbf{H}_G &= [I_G] \boldsymbol{\omega}_{G/N} \\ &= (I_{G_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{G_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{G_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \cdot (\boldsymbol{\omega}_{G/B} + \boldsymbol{\omega}_{B/N}) \end{aligned} \quad \begin{array}{l} (4.109) \\ \text{(used 4.105)} \end{array}$$

$$\begin{aligned} &= (I_{G_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{G_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{G_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \cdot (\dot{\gamma} \hat{\mathbf{g}}_g + \boldsymbol{\omega}_{B/N}) \\ &= (I_{G_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{G_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{G_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \cdot \boldsymbol{\omega}_{B/N} + \dot{\gamma} I_{G_g} \hat{\mathbf{g}}_g \end{aligned} \quad (4.110)$$

$$\begin{aligned} &= \omega_s I_{G_s} \hat{\mathbf{g}}_s + \omega_t I_{G_t} \hat{\mathbf{g}}_t + \omega_g I_{G_g} \hat{\mathbf{g}}_g + \dot{\gamma} I_{G_g} \hat{\mathbf{g}}_g \\ &= \omega_s I_{G_s} \hat{\mathbf{g}}_s + \omega_t I_{G_t} \hat{\mathbf{g}}_t + (\omega_g + \dot{\gamma}) I_{G_g} \hat{\mathbf{g}}_g \end{aligned} \quad (4.112)$$

and the component of the reaction wheel (RW) is obtained similarly as

$$\begin{aligned} \mathbf{H}_W &= [I_W] \boldsymbol{\omega}_{W/N} \\ &= (I_{W_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{W_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{W_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \cdot (\boldsymbol{\omega}_{W/G} + \boldsymbol{\omega}_{G/B} + \boldsymbol{\omega}_{B/N}) \end{aligned} \quad \begin{array}{l} (4.113) \\ \text{(used 4.106)} \end{array}$$

$$\begin{aligned} &= (I_{W_s} \hat{\mathbf{g}}_s \hat{\mathbf{g}}_s^T + I_{W_t} \hat{\mathbf{g}}_t \hat{\mathbf{g}}_t^T + I_{W_g} \hat{\mathbf{g}}_g \hat{\mathbf{g}}_g^T) \cdot (\Omega \hat{\mathbf{g}}_s + \dot{\gamma} \hat{\mathbf{g}}_g + \boldsymbol{\omega}_{B/N}) \\ &= \Omega I_{W_s} \hat{\mathbf{g}}_s + \dot{\gamma} I_{W_g} \hat{\mathbf{g}}_g + \omega_s I_{W_s} \hat{\mathbf{g}}_s + \omega_t I_{W_t} \hat{\mathbf{g}}_t + \omega_g I_{W_g} \hat{\mathbf{g}}_g \\ &= (\Omega + \omega_s) I_{W_s} \hat{\mathbf{g}}_s + \omega_t I_{W_t} \hat{\mathbf{g}}_t + (\dot{\gamma} + \omega_g) I_{W_g} \hat{\mathbf{g}}_g \end{aligned} \quad (4.114)$$

In the above calculation, the angular velocity  $\boldsymbol{\omega}_{B/N}$  has been expressed in the gimbal frame  $\mathcal{B}$  as

$${}^G \boldsymbol{\omega} = \omega_s \hat{\mathbf{g}}_s + \omega_t \hat{\mathbf{g}}_t + \omega_g \hat{\mathbf{g}}_g. \quad (4.115)$$

🔗 Now, find inertial derivatives in order to apply Euler's equation of rigid body rotation.

The equations of motion of a system of rigid bodies follow from Euler's equation

$$\dot{\mathbf{H}} = \mathbf{L}$$

$$\mathbf{H} = \mathbf{H}_B + \mathbf{H}_G + \mathbf{H}_W \quad (4.107)$$

From Eq. 4.107 we know that in order to find  $\dot{\mathbf{H}}$ , we need to find the time derivatives of the basis vectors of  $\mathcal{G}$  first, and then the derivatives of  $\omega_s, \omega_t, \omega_g$ .

Using transport theorem, we have

$$\begin{aligned} \dot{\hat{\mathbf{g}}}_s &= \frac{{}^B d}{dt} \hat{\mathbf{g}}_s + \boldsymbol{\omega} \times \hat{\mathbf{g}}_s \\ &= \cancel{\frac{{}^G d}{dt} \hat{\mathbf{g}}_s} + \boldsymbol{\omega}_{G/B} \times \hat{\mathbf{g}}_s + \boldsymbol{\omega} \times \hat{\mathbf{g}}_s \\ &= \dot{\gamma} \hat{\mathbf{g}}_g \times \hat{\mathbf{g}}_s + (\omega_s \hat{\mathbf{g}}_s + \omega_t \hat{\mathbf{g}}_t + \omega_g \hat{\mathbf{g}}_g) \times \hat{\mathbf{g}}_s \\ &= (\dot{\gamma} + \omega_g) \hat{\mathbf{g}}_t - \omega_t \hat{\mathbf{g}}_g \end{aligned}$$

and similarly

$$\begin{aligned} \dot{\hat{\mathbf{g}}}_t &= -(\dot{\gamma} + \omega_g) \hat{\mathbf{g}}_s + \omega_s \hat{\mathbf{g}}_g \\ \dot{\hat{\mathbf{g}}}_g &= \omega_t \hat{\mathbf{g}}_s - \omega_s \hat{\mathbf{g}}_t \end{aligned} \quad (4.117)$$

Next, the time derivatives of the basis are obtained as

$$\begin{aligned} \dot{\boldsymbol{\omega}}_s &= \frac{{}^N d}{dt} (\boldsymbol{\omega} \cdot \hat{\mathbf{g}}_s) = \dot{\boldsymbol{\omega}} \cdot \hat{\mathbf{g}}_s + \boldsymbol{\omega} \cdot \frac{{}^N d}{dt} \hat{\mathbf{g}}_s \\ &= \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + (\omega_s \hat{\mathbf{g}}_s + \omega_t \hat{\mathbf{g}}_t + \omega_g \hat{\mathbf{g}}_g) \cdot ((\dot{\gamma} + \omega_g) \hat{\mathbf{g}}_t - \omega_t \hat{\mathbf{g}}_g) \\ &= \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + (\omega_s \hat{\mathbf{g}}_s + \omega_t \hat{\mathbf{g}}_t + \omega_g \hat{\mathbf{g}}_g) \cdot ((\dot{\gamma} + \omega_g) \hat{\mathbf{g}}_t - \omega_t \hat{\mathbf{g}}_g) \\ &= \hat{\mathbf{g}}_s^T \dot{\boldsymbol{\omega}} + \omega_t \dot{\gamma} \end{aligned}$$

and similarly

$$\begin{aligned} \dot{\omega}_t &= -\dot{\gamma} \omega_s + \hat{\mathbf{g}}_t^T \dot{\boldsymbol{\omega}} \\ \dot{\omega}_g &= \hat{\mathbf{g}}_g^T \dot{\boldsymbol{\omega}} \end{aligned} \quad (4.121)$$

Other derivatives are considered known or given, such as  $\dot{\Omega}, \dot{\gamma}, \ddot{\gamma}$  etc.

🔗 Now, isolating the dynamics of the RW.

Let  $\mathbf{L}_W$  be the torque the gimbal frame exerts on the RW, Euler's equation states that

$$\dot{\mathbf{H}}_W = \mathbf{L}_W$$

where, using the auxiliary results derived above, the inertial derivative of  $\mathbf{H}_W$  is

$$\begin{aligned}\dot{\mathbf{H}}_W &= \hat{\mathbf{g}}_s \left[ I_{W_s} (\dot{\Omega} + \hat{\mathbf{g}}_s^T \dot{\Omega} + \dot{\gamma} \omega_t) \right] \\ &\quad + \hat{\mathbf{g}}_t \left[ I_{W_s} (\dot{\gamma} (\omega_s + \Omega) + \Omega \omega_g) + I_W \hat{\mathbf{g}}_t^T \dot{\Omega} + (I_{W_s} - I_{W_t}) \omega_s \omega_g - 2I_W \omega_s \dot{\gamma} \right] \\ &\quad + \hat{\mathbf{g}}_g \left[ I_{W_s} (\hat{\mathbf{g}}_g^T \dot{\Omega} + \ddot{\gamma}) \right] + (I_{W_t} - I_{W_s}) \omega_s \omega_t - I_{W_s} \Omega \omega_t \\ &= u_s \hat{\mathbf{g}}_s + u_t \hat{\mathbf{g}}_t + u_g \hat{\mathbf{g}}_g\end{aligned}\tag{4.122}$$

In Eq. (4.122), the spin torque  $u_s$  is provided by the RW torque motor,

$$u_s = I_{W_s} (\dot{\Omega} + \hat{\mathbf{g}}_s^T \dot{\Omega} + \dot{\gamma} \omega_t)$$

while the other two torques are the reaction torque produced by the gimbal frame.

🔗 Now, isolating the RW and the gimbal frame as a whole

Let  $\mathbf{L}_G$  be the torque vector that spacecraft exerts on the VSCMG system, then Euler's equation states again that

$$\dot{\mathbf{H}}_G + \dot{\mathbf{H}}_W = \mathbf{L}_G$$

where  $\dot{\mathbf{H}}_W$  is given in Eq. (4.122) and  $\dot{\mathbf{H}}_G$  is calculated as

$$\begin{aligned}\dot{\mathbf{H}}_G &= \hat{\mathbf{g}}_s \left[ (I_{G_s} - I_{G_t} + I_{G_g}) \dot{\gamma} \omega_t + I_{G_s} \hat{\mathbf{g}}_s^T \dot{\Omega} + (I_{G_s} - I_{G_t}) \omega_t \omega_g \right] \\ &\quad + \hat{\mathbf{g}}_t \left[ (I_{G_s} - I_{G_t} - I_{G_g}) \dot{\gamma} \omega_s + I_{G_t} \hat{\mathbf{g}}_t^T \dot{\Omega} + (I_{G_s} - I_{G_g}) \omega_s \omega_g \right] \\ &\quad + \hat{\mathbf{g}}_g \left[ I_{G_g} (\hat{\mathbf{g}}_g^T \dot{\Omega} + \ddot{\gamma}) \right] + (I_{G_t} - I_{G_s}) \omega_s \omega_t\end{aligned}\tag{4.124}$$

The torque component of  $\mathbf{L}_G$  along  $\hat{\mathbf{g}}_g$  is produced by the gimbal torque motor, so, adding Eqs. (4.122) and (4.124) and equating the components along  $\hat{\mathbf{g}}_g$  gives

$$u_g = J_g (\hat{\mathbf{g}}_g^T \dot{\Omega} + \ddot{\gamma}) - (J_s - J_t) \omega_s \omega_t - I_{W_s} \Omega \omega_t$$

where we have combined the inertia matrices of RW and the gimbal frame into a single matrix  $[J]$

$$\mathcal{G}[J] = \mathcal{G}[I_G] + \mathcal{G}[I_W] = \begin{bmatrix} I_s + I_s & 0 & 0 \\ 0 & I_t + I_t & 0 \\ 0 & 0 & I_g + I_g \end{bmatrix} = \begin{bmatrix} J_s & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & J_g \end{bmatrix}.$$

🔗 Now, combine everything as a whole.

To further simplify the equations of motion, the total spacecraft inertia matrix  $[I]$  is defined as

$${}^B[I] = {}^B[I_s] + {}^B[J] = {}^B[I_s] + [BG] \cdot \mathcal{G}[J] \cdot [BG]^T\tag{4.128 altered}$$

The final equations of motion of a rigid spacecraft with a single VSCMG are

$$\begin{aligned}[I] \dot{\Omega} &= -\Omega \times [I] \Omega - \hat{\mathbf{g}}_s \left[ J_s \dot{\gamma} \omega_t + I_{W_s} \dot{\Omega} - (J_t - J_g) \omega_t \dot{\gamma} \right] \\ &\quad - \hat{\mathbf{g}}_t \left[ (J_s \omega_s + I_{W_s} \Omega) \dot{\gamma} - (J_t + J_g) \omega_s \dot{\gamma} + I_{W_s} \Omega \omega_g \right] \\ &\quad - \hat{\mathbf{g}}_g \left[ J_g \ddot{\gamma} - I_{W_s} \Omega \omega_t \right] \\ &\quad + \mathbf{L}\end{aligned}\tag{4.129}$$

🔗 A common assumption that the gimbal frame inertial  $I_{G_s}$  about the spin axis is negligible.

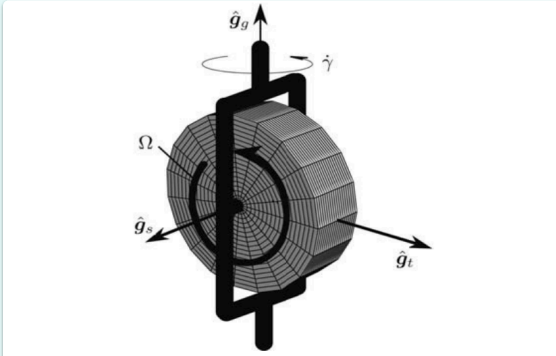


Fig. 4.17 Illustration of a variable speed control moment gyroscope.

Because the frame is usually designed as lightweight as possible to avoid interfering with the spinning wheel. The magnitude of difference can be 2 to 3 magnitudes.

With this assumption,  $J_s = I_{G_s} + I_{W_s} \approx I_{W_s}$  and we have the more common equations of motion as follows,

$$\begin{aligned}
 [I]\dot{\omega} = & -\omega \times [I]\omega - \hat{g}_s \left[ J_s(\dot{\gamma}\omega_t + \dot{\Omega}) - (J_t - J_g)\omega_t\dot{\gamma} \right] \\
 & - \hat{g}_t \left[ J_s(\omega_s + \Omega)\dot{\gamma} - (J_t + J_g)\omega_s\dot{\gamma} + J_s\Omega\omega_g \right] \\
 & - \hat{g}_g \left[ J_g\ddot{\gamma} - J_s\Omega\omega_t \right] \\
 & + \mathbf{L}
 \end{aligned} \tag{4.131}$$

If we set  $\dot{\Omega} = 0$ , it means the RW spin speed is constant, thus the equations of motion becomes those of a single-gimbal CMG (not varying-speed), which is

$$\begin{aligned}
 [I]\dot{\omega} = & -\omega \times [I]\omega - \hat{g}_s \left[ J_s(\dot{\gamma}\omega_t + \dot{\Omega}^0) - (J_t - J_g)\omega_t\dot{\gamma} \right] \\
 & - \hat{g}_t \left[ J_s(\omega_s + \Omega)\dot{\gamma} - (J_t + J_g)\omega_s\dot{\gamma} + J_s\Omega\omega_g \right] \\
 & - \hat{g}_g \left[ J_g\ddot{\gamma} - J_s\Omega\omega_t \right] \\
 & + \mathbf{L}
 \end{aligned} \tag{Example 4.8}$$

If we set the gimbal to be fixed, thus  $\dot{\gamma} = \ddot{\gamma} = 0$ , we get the equations of motion for RW as

$$\begin{aligned}
 [I]\dot{\omega} = & -\omega \times [I]\omega - \hat{g}_s \left[ J_s(\dot{\gamma}^0\omega_t + \dot{\Omega}) - (J_t - J_g)\omega_t\dot{\gamma}^0 \right] \\
 & - \hat{g}_t \left[ J_s(\omega_s + \Omega)\dot{\gamma}^0 - (J_t + J_g)\omega_s\dot{\gamma}^0 + J_s\Omega\omega_g \right] \\
 & - \hat{g}_g \left[ J_g\ddot{\gamma}^0 - J_s\Omega\omega_t \right] \\
 & + \mathbf{L} \\
 = & -\omega \times [I]\omega - \hat{g}_s J_s \dot{\Omega} - \hat{g}_t J_s \Omega \omega_g + \hat{g}_g J_s \Omega \omega_t + \mathbf{L} \\
 = & -\omega \times [I]\omega - \hat{g}_s J_s \dot{\Omega} - \omega \times J_s \Omega \hat{g}_s + \mathbf{L}
 \end{aligned} \tag{Example 4.8}$$

🔗 Now, extend to multiple VSCMGs.

The essential is to notice that all the above derivations are the same and will be repeated for each VSCMG introduced to the spacecraft, but the only difference is how the DCM  $[BG]_i$  is defined for the  $i$ -th VSCMG.

(Refer to Sec. 4.5.2 for details.)