AE_544_LecNote03__Rigid_Body_Kinematics__Ch03

> ## Disclaimers
>
> This repository contains markdown-based lecture notes for the Spring 2025 semester. These materials are intended solely for students enrolled in the course during this term. Mistakes may exist and will be corrected as they are identified, but the official textbook remains the ultimate reference for accuracy.

Fun facts about rigid body attitude coordinates from authors:

1. A minimum of three coordinates is required to describe the relative angular displacement between two reference frames.
2. Any minimal set of three attitude coordinates will contain at least one geometrical orientation where the coordinates are singular, namely at least two coordinates are undefined or not unique.
3. At or near such a geometric singularity, the corresponding kinematic differential equations are also singular.
4. The geometric singularities and associated numerical difficulties can be avoided altogether through a regularization. Redundant sets of four or more coordinates exist that are universally determined and contain no geometric singularities.
5. Two rigid body (or coordinate frame) orientations can differ at most by a 180 deg rotation.

Direction Cosine Matrix (DCM) Basics

The vectrix notation for two frames $\mathcal{N}$ and $\mathcal{B}$:

$$\{\hat{\boldsymbol{n}}\} \equiv \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix} \qquad \{\hat{\boldsymbol{b}}\} \equiv \begin{Bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{Bmatrix}$$

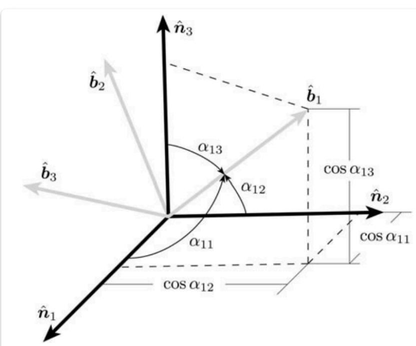


Fig. 3.1 Direction cosines.

The direction cosine matrix (DCM) $[C]$

$$\{\hat{\boldsymbol{b}}\} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \{\hat{\boldsymbol{n}}\} = [C]\{\hat{\boldsymbol{n}}\} \tag{3.5}$$

$$\begin{Bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{31} & \cos\alpha_{32} & \cos\alpha_{33} \end{bmatrix} \cdot \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix}$$

$[C]$ is an orthogonal matrix with all those good and convenient features, for example, $[C]^{-1} = [C]^T$.

The most powerful feature of the direction cosine is the ability to directly project (or transform) an arbitrary vector, with components written in one reference frame, into a vector with components written in another reference frame.

$$\boldsymbol{v} = v_{b_1}\hat{\boldsymbol{b}}_1 + v_{b_2}\hat{\boldsymbol{b}}_2 + v_{b_3}\hat{\boldsymbol{b}}_3$$

$$\boldsymbol{v} = v_{n_1}\hat{\boldsymbol{n}}_1 + v_{n_2}\hat{\boldsymbol{n}}_2 + v_{n_3}\hat{\boldsymbol{n}}_3$$

$$\boldsymbol{v}_b = [C]\boldsymbol{v}_n \tag{3.17}$$

The vectrix of another frame $\mathcal{R}$

$$\{\hat{\boldsymbol{r}}\} \equiv \begin{Bmatrix} \hat{\boldsymbol{r}}_1 \\ \hat{\boldsymbol{r}}_2 \\ \hat{\boldsymbol{r}}_3 \end{Bmatrix}$$

Composition of reference frame changes: $\mathcal{N} \to \mathcal{B} \to \mathcal{R}$

$$\{\hat{\boldsymbol{r}}\} = [C'] \{\hat{\boldsymbol{b}}\} = \underbrace{[C'] [C]}_{\text{composed}} \{\hat{\boldsymbol{n}}\} = \underbrace{[C'']}_{\text{as one}} \{\hat{\boldsymbol{n}}\} \tag{Used 3.5}$$

The direct transformation matrix from the first to the last cascading reference frame is clearly found by successive matrix-multiplications of each relative transformation matrix in reverse order as just shown.

> **Info**
>
> The direction cosine matrix is the most fundamental, but highly redundant, method of describing a relative orientation. The biggest asset of the direction cosine matrix is the ability to easily transform vectors from one reference frame to another.

Kinematics using DCMs

The instantaneous angular velocity vector $\boldsymbol{\omega}$ of the $\mathcal{B}$ frame relative to the $\mathcal{N}$ frame, expressed in $\mathcal{B}$ frame orthogonal components as

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega \hat{\boldsymbol{b}}_3$$

> **Expressed the body angular velocity in the body frame.**

Then the derivative of the basis of a frame $\mathcal{B}$ taken in another frame $\mathcal{N}$ can be easily expressed as

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t} \hat{\boldsymbol{b}}_i = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t} \hat{\boldsymbol{b}}_i + \boldsymbol{\omega} \times \hat{\boldsymbol{b}}_i \tag{3.22}$$

Define skew-symmetric tilde matrix operator such that for any given vector

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

a matrix $[\tilde{\boldsymbol{x}}]$ is defined as following,

$$[\tilde{\boldsymbol{x}}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

such that $[\tilde{\boldsymbol{x}}] = -[\tilde{\boldsymbol{x}}]^T$.

For example, applying this title matrix operation to Eq. (3.22) for $\hat{\boldsymbol{b}}_1$ in $\mathcal{B}$ gives

$$(\boldsymbol{\omega})_{\mathcal{B}} \times (\hat{\boldsymbol{b}}_i)_{\mathcal{B}} = \left(\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)_{\mathcal{B}} = \begin{vmatrix} \hat{\boldsymbol{b}}_1 & \hat{\boldsymbol{b}}_2 & \hat{\boldsymbol{b}}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix} = \omega_3 \hat{\boldsymbol{b}}_2 - \omega_2 \hat{\boldsymbol{b}}_3 = -\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{b}}_1 \\ \hat{\boldsymbol{b}}_2 \\ \hat{\boldsymbol{b}}_3 \end{Bmatrix}$$

> **This operator is independent of coordinate transformations (Cartesian) and holds for abstract vectors.**
>
> We want to show that the tilde operator doesn't change under a coordinate transformation from Cartesian frame $\mathcal{B}$ to Cartesian frame $\mathcal{F}$, where the orientation of $\mathcal{F}$ is given by the DCM $[FB]$, which can be expressed as
>
> $$([FB](\boldsymbol{\omega})_{\mathcal{B}}) \times ([FB](\hat{\boldsymbol{b}}_i)_{\mathcal{B}}) = [FB] \left((\boldsymbol{\omega})_{\mathcal{B}} \times (\hat{\boldsymbol{b}}_i)_{\mathcal{B}} \right) \tag{to prove}$$
>
> First, let's prove a general rule for the tilde operator. For two generic vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, after a transformation of DCM $[C]$, the new coordinates can be obtained as
>
> $$\boldsymbol{u}' = [C] \cdot \boldsymbol{u} \qquad \boldsymbol{v}' = [C] \cdot \boldsymbol{v}$$
>
> Using the definition of tilde operators, and plugging in the above transformed vectors, we have the first expression:

$$\boldsymbol{u} \times \boldsymbol{v} = -[\tilde{\boldsymbol{u}}] \cdot \boldsymbol{v}$$
$$\boldsymbol{u}' \times \boldsymbol{v}' = -[\tilde{\boldsymbol{u}}'] \cdot \boldsymbol{v}' = -[\tilde{\boldsymbol{u}}'] \cdot [C] \cdot \boldsymbol{v}$$

Since $\boldsymbol{u}' \times \boldsymbol{v}'$ and $\boldsymbol{u} \times \boldsymbol{v}$ themselves are also a vector, so the following relation holds as our second expression:

$$\boldsymbol{u}' \times \boldsymbol{v}' = [C] \cdot (\boldsymbol{u} \times \boldsymbol{v}) = -[C] \cdot [\tilde{\boldsymbol{u}}] \cdot \boldsymbol{v}$$

Equate the above two expressions and we have:

$$-[\tilde{\boldsymbol{u}}'] \cdot [C] = -[C] \cdot [\tilde{\boldsymbol{u}}] \implies [\tilde{\boldsymbol{u}}'] = [C] \cdot [\tilde{\boldsymbol{u}}] \cdot [C]^T$$

Then let's apply this to our problem. After the coordinate transformation $[FB]$ ($\mathcal{B} \to \mathcal{F}$), the following derivation becomes apparent:

$$\begin{aligned}
([FB](\boldsymbol{\omega})_{\mathcal{B}}) \times ([FB](\hat{\boldsymbol{b}}_i)_{\mathcal{B}}) &= -[FB] \cdot ([\tilde{\boldsymbol{\omega}}])_{\mathcal{B}} \cdot [FB]^T \cdot [FB] \cdot (\hat{\boldsymbol{b}}_i)_{\mathcal{B}} \\
&= -[FB] \cdot ([\tilde{\boldsymbol{\omega}}])_{\mathcal{B}} \cdot \left([FB]^T \cdot [FB] \cdot (\hat{\boldsymbol{b}}_i)_{\mathcal{B}}\right) \\
&= -[FB] \cdot ([\tilde{\boldsymbol{\omega}}])_{\mathcal{B}} \cdot (\hat{\boldsymbol{b}}_i)_{\mathcal{B}} \\
&= [FB]\left((\boldsymbol{\omega})_{\mathcal{B}} \times (\hat{\boldsymbol{b}}_i)_{\mathcal{B}}\right)
\end{aligned}$$

Applying this tilde operator to Eq. (3.22), and we get a vectrix equation for the rate of the basis

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{b}}_i = \frac{{}^{\mathcal{B}}\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{b}}_i + \boldsymbol{\omega} \times \hat{\boldsymbol{b}}_i \tag{3.22 copied}$$

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} \tag{3.24}$$

Take time derivate to Eq. (3.5)

$$\{\hat{\boldsymbol{b}}\} = [C]\{\hat{\boldsymbol{n}}\} \tag{3.5 copied}$$

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}([C]\{\hat{\boldsymbol{n}}\}) = \left(\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}[C]\right)\{\hat{\boldsymbol{n}}\} + \cancelto{0}{[C]\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{n}}\}} \tag{3.25}$$

Shorten the notation using $\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}[C] = [\dot{C}]$:

$$\frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}\{\hat{\boldsymbol{b}}\} = \frac{{}^{\mathcal{N}}\mathrm{d}}{\mathrm{d}t}([C]\{\hat{\boldsymbol{n}}\}) = [\dot{C}]\{\hat{\boldsymbol{n}}\} \tag{3.25 continued}$$

ⓘ $[C]$ is $\hat{\boldsymbol{b}}_i$ expressed in $\mathcal{N}$ by $\hat{\boldsymbol{n}}_i$, so $[C]$ is expressed in $\mathcal{N}$.

Equate Eq. (3.25) and Eq. (3.24), we have

$$[\dot{C}]\{\hat{\boldsymbol{n}}\} = -[\tilde{\boldsymbol{\omega}}]\{\hat{\boldsymbol{b}}\} = -[\tilde{\boldsymbol{\omega}}]\,[C]\{\hat{\boldsymbol{n}}\} \tag{used Eq. 3.5}$$

Since this equation shall hold for the basis vectrix $\{\hat{\boldsymbol{n}}\}$ of any arbitrary frame $\mathcal{N}$, so there must be

$$[\dot{C}] = -[\tilde{\boldsymbol{\omega}}]\,[C] \tag{3.27}$$

It seems everything are expressed in the body frame $\mathcal{B}$ now, but we still need the reference frame $\mathcal{N}$ for calculation. But $\mathcal{N}$ doesn't need to be inertial. Using the explicit frame labelling notation by the authors, Eq. (3.27) can be expressed in a more explicit way as

$$[\dot{BN}] = -\left[\tilde{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{N}}\right][BN] \tag{3.28}$$

Now, it is clear that a reference frame $\mathcal{N}$ is always needed to describe attitude/orientation.

⚠ to correct.

But $\mathcal{N}$ is arbitrary, so if $\mathcal{B}(t)$ is time-varying and rotating, $\mathcal{B}(t)$ at $t = t_0$ can be chosen as $\mathcal{N}$.
In this way, $\mathcal{N}$ is $\mathcal{B}(t_0)$, and $[BN](t_0) = \boldsymbol{I}_3$.

A major advantage of the kinematic differential equation for $[C]$ is that it is linear and universally applicable. There are no geometric singularities present in the attitude description or its kinematic differential equations.

At the cost of having a highly redundant formulation.

⚠ **During this discussion/derivation of kinematic equations, no vectors have been moved in space, only their coordinates are being transformed (or mapped as the term the authors used) from one frame to another.**

In other words, everything happened at a fixed instant of t_0, it's only when you start propagate the ODE in Eq. (3.28) from t_0 to $t > 0$, the vector starts to change.

Explicit Frame Labeling (a textbook convention for notations) (Example 3.1)

Denote $[FB]$ as the DCM of frame $\mathcal{F}$ relative to $\mathcal{B}$.

Under this notation convention/system/definition, $[FB]$ ~~maps~~(change coordinate of vectors) written in $\mathcal{B}$ into (coordinates of) vectors written in $\mathcal{F}$.

As summary:

$$\{\hat{\boldsymbol{f}}\} = [FB]\{\hat{\boldsymbol{b}}\}$$

$$\begin{bmatrix} r_{f_1} \\ r_{f_2} \\ r_{f_3} \end{bmatrix} = (\boldsymbol{r})_{\mathcal{F}} = [FB](\boldsymbol{r})_{\mathcal{B}} = [FB] \begin{bmatrix} r_{b_1} \\ r_{b_2} \\ r_{b_3} \end{bmatrix}$$

Euler Angles $(\theta_1, \theta_2, \theta_3)$

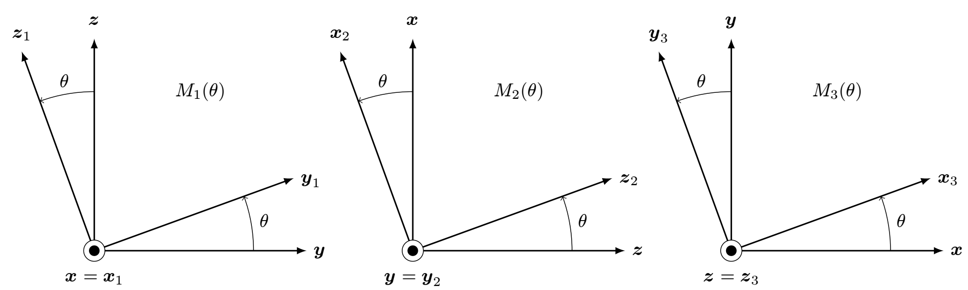
ⓘ **The popularity of Euler angles stems from the fact that the relative attitude is easy to visualize for small angles.**

The position of $\{\hat{\boldsymbol{b}}\}$ relative to $\{\hat{\boldsymbol{n}}\}$ is described by a sequence of three rigid rotations about prescribed body-fixed axes.

The three single-axis rotation matrices $[M_i(\theta)]$ are given by:

$$[M_1(\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad [M_2(\theta)] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad [M_3(\theta)] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

> tikz_elementary_rotation_matrix



(TikZ plots for three elementary rotation matrix.)

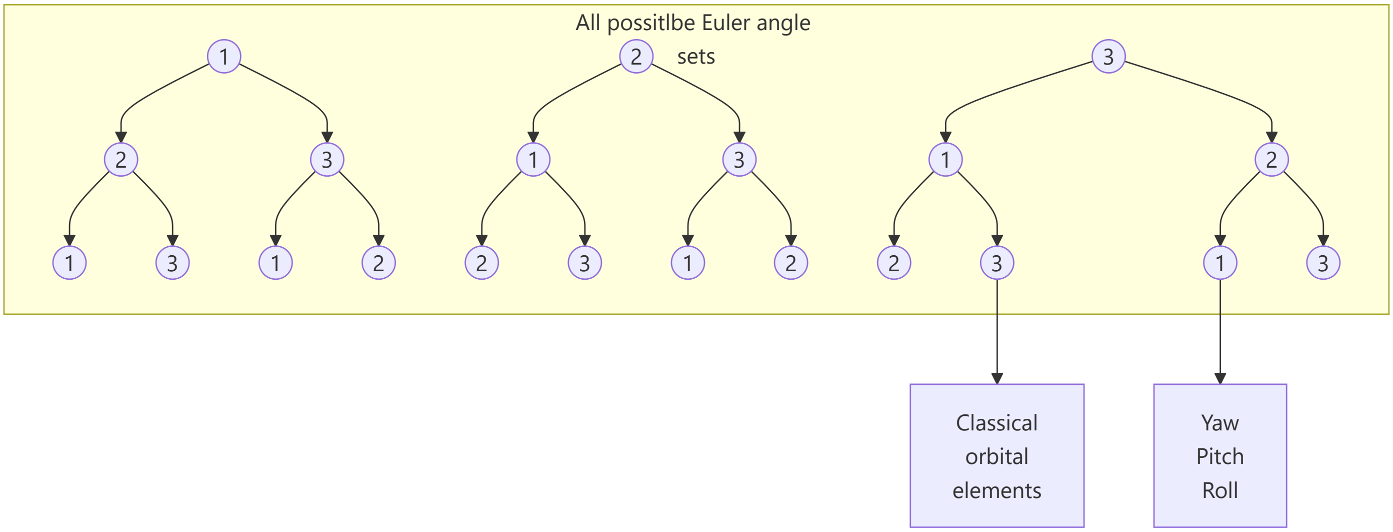
✓ **This $[M_i(\theta)]$ is the DCM of the rotated frame relative to the original frame.**

Let the (α, β, γ) ($\alpha, \beta, \gamma \in \{1, 2, 3\}$ are indices) Euler angle sequence be $(\theta_1, \theta_2, \theta_3)$ (θ_i are angles in radian by default), the direction cosine matrix of the rotated frame relative to the original frame is given as

$$[C(\theta_1, \theta_2, \theta_3)] = [M_\gamma(\theta_3)] \cdot [M_\beta(\theta_2)] \cdot [M_\alpha(\theta_1)] \tag{3.33}$$

⚠ **Here, using the textbook convention, Euler angle order and DCM multiplication order is reversed.**

All possible Euler Angle sets (a permutation problem requiring different adjacent elements):



Two types:

- Asymmetric Euler angle set: 1st and 3rd rotations around different body frame axis
- Symmetric Euler angle set: 1st and 3rd rotations around the same body frame axis

⚠ The term "symmetric" doesn't mean the order can be changed!

Asymmetric (3-2-1) Euler angles

In particular, for (3-2-1) Euler angle (θ_1, θ_2, θ_3), the full DCM can be resolved as:

$$[C]_{\text{3-2-1}} = \begin{bmatrix} c\theta_2 c\theta_1 & c\theta_2 s\theta_1 & -s\theta_2 \\ s\theta_3 s\theta_2 c\theta_1 - c\theta_3 s\theta_1 & s\theta_3 s\theta_2 s\theta_1 + c\theta_3 c\theta_1 & s\theta_3 c\theta_2 \\ c\theta_3 s\theta_2 c\theta_1 + s\theta_3 s\theta_1 & c\theta_3 s\theta_2 s\theta_1 - s\theta_3 c\theta_1 & c\theta_3 c\theta_2 \end{bmatrix} \tag{3.34, Didn't verify.}$$

where $c \equiv \cos$ and $s \equiv \sin$. This is most useful to convert from DCM to Euler angles, which leads to

$$\begin{aligned} (\theta_1)_{\text{3-2-1}} &= \tan^{-1}\left(\frac{C_{12}}{C_{11}}\right) \\ (\theta_2)_{\text{3-2-1}} &= -\sin^{-1}(C_{13}) \\ (\theta_3)_{\text{3-2-1}} &= \tan^{-1}\left(\frac{C_{23}}{C_{33}}\right) \end{aligned} \tag{Find through observations of elements.}$$

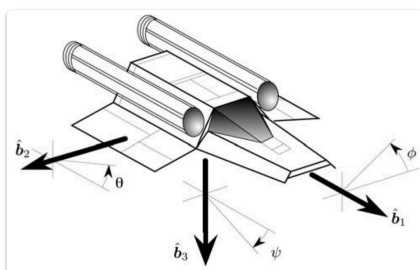


Fig. 3.2 Yaw, pitch, and roll Euler angles.

This corresponds to the standard yaw-ψ, pitch-θ, roll-ϕ set, which is a (3-2-1) Euler angle set. To transform components of a vector in the $\mathcal{N}$ frame into the $\mathcal{B}$ frame. So, (yaw-ψ, pitch-θ, roll-ϕ) transforms another frame $\mathcal{N}$ to to the body frame $\mathcal{B}$, i.e., it gives the attitude of $\mathcal{N}$ relative to $\mathcal{B}$:

$$\{\hat{\boldsymbol{b}}\} = [BN]\{\hat{\boldsymbol{n}}\} = [C]_{\text{3-2-1}}\{\hat{\boldsymbol{n}}\}$$

Similar for the coordinate transformation:

$${}^{\mathcal{B}}\boldsymbol{v} = [BN] \cdot {}^{\mathcal{N}}\boldsymbol{v} = [C]_{\text{3-2-1}} \cdot {}^{\mathcal{N}}\boldsymbol{v}$$

Symmetric (3-1-3) Euler angles

In particular, for (3-1-3) Euler angle (θ_1, θ_2, θ_3), the full DCM can be resolved as:

$$
[C]_{3\text{-}1\text{-}3} = \begin{bmatrix} c\theta_3 c\theta_1 - s\theta_3 c\theta_2 s\theta_1 & c\theta_3 s\theta_1 + s\theta_3 c\theta_2 c\theta_1 & s\theta_3 s\theta_2 \\ -s\theta_3 c\theta_1 - c\theta_3 c\theta_2 s\theta_1 & -s\theta_3 s\theta_1 + c\theta_3 c\theta_2 c\theta_1 & c\theta_3 s\theta_2 \\ s\theta_2 s\theta_1 & -s\theta_2 c\theta_1 & c\theta_2 \end{bmatrix} \tag{3.36, Didn't verify.}
$$

where $c \equiv \cos$ and $s \equiv \sin$. This is most useful to convert from DCM to Euler angles, which leads to

$$
\begin{aligned}
(\theta_1)_{3\text{-}1\text{-}3} &= \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right) \\
(\theta_2)_{3\text{-}1\text{-}3} &= \cos^{-1}(C_{33}) \\
(\theta_3)_{3\text{-}1\text{-}3} &= \tan^{-1}\left(\frac{C_{12}}{C_{23}}\right)
\end{aligned} \tag{Find through observations of elements.}
$$

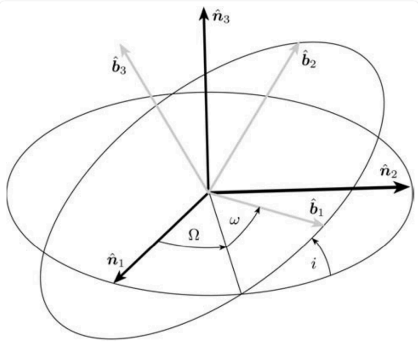


Fig. 3.4 (3-1-3) Euler angle illustration.

The ascending node-Ω, inclination-i, and argument of the periapsis ω is a (3-1-3) Euler angle set. Similarly, (Ω, i, ω) transforms $\mathcal{N}$ to $\mathcal{B}$, i.e., it gives the attitude of $\mathcal{N}$ relative to $\mathcal{B}$:

$$
\{\hat{\boldsymbol{b}}\} = [BN]\{\hat{\boldsymbol{n}}\} = [M_3(\omega)]\,[M_1(i)]\,[M_3(\Omega)]\{\hat{\boldsymbol{n}}\}
$$

$$
\begin{aligned}
\{\hat{\boldsymbol{f}}_{after-\Omega}\} &= [M_3(\Omega)]\{\hat{\boldsymbol{n}}\} \\
\{\hat{\boldsymbol{f}}_{after-i}\} &= [M_1(i)]\{\hat{\boldsymbol{f}}_{after-\Omega}\} \\
\{\hat{\boldsymbol{f}}_{after-\omega}\} &= [M_3(\omega)]\{\hat{\boldsymbol{f}}_{after-i}\} \\
\{\hat{\boldsymbol{b}}\} &= \{\hat{\boldsymbol{f}}_{after-\omega}\} \\
&= [M_3(\omega)]\{\hat{\boldsymbol{f}}_{after-i}\} \\
&= [M_3(\omega)]\,[M_1(i)]\{\hat{\boldsymbol{f}}_{after-\Omega}\} \\
&= [M_3(\omega)]\,[M_1(i)]\,[M_3(\Omega)]\{\hat{\boldsymbol{n}}\}
\end{aligned}
$$

ⓘ **When specifying Euler angle sets, it must be clear which frame is the reference. There must be two frames involved: one serves as the reference, the other is the one gets rotated.**

The vectrix only unifies directions of change of basis/frame and coordinate transformation in mathematical expression.

Physically, the "directions" of change of basis/frame and coordinate transformation are always "opposite", regardless of the mathematical conventions and notations.

Note that each of the 12 possible sets of Euler angles has a geometric singularity where two angles are not uniquely defined.

? **The singularity of Euler angles: an explanation without touching kinematics yet**

"Singularity" usually means something doesn't work, abnormal, ill-defined, unexpected, or giving infinity.
DCM has no singularity, if one cannot resolve a set of Euler angles from DCM, then it's a singularity of that particular set of Euler angles.

Go back and check Eqs. (3.34) and (3.36).

Textbook problem 3.7

3.7 The reference frames $\mathcal{N}$: $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ and $\mathcal{B}$: $\{\hat{\boldsymbol{b}}_L, \hat{\boldsymbol{b}}_\theta, \hat{\boldsymbol{b}}_r\}$ are shown in Fig. P3.7.
(a) Find the direction cosine matrix $[BN]$ in terms of the angle ϕ.
(b) Given the vector ${}^{\mathcal{B}}v = 1\hat{\boldsymbol{b}}_r + 1\hat{\boldsymbol{b}}_\theta + 2\hat{\boldsymbol{b}}_L$, find the vector ${}^{\mathcal{N}}v$.

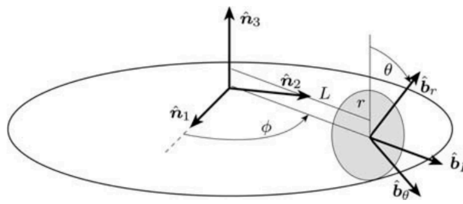


Fig. P3.7 Disk rolling on circular ring.

⚠ Notice the order of basis for $\mathcal{B}$.

(Similar to the grind in the problem 1.14 in lecture note 01).

(a) DCM transforms $\mathcal{N}$ to $\mathcal{B}$

Assumed no slip between the disk and the ground, leading to a constraint between θ and ϕ: the speed of the contact point should be the same

$$\dot{\phi} \cdot L = \dot{\theta} \cdot r$$

Given θ_0 and ϕ_0, a simple definite integration gives

$$\phi \cdot L + \phi_0 \cdot L = \theta \cdot r + \theta_0 \cdot r$$

Get DCM from two consecutive rotations:

$$\begin{aligned}
\begin{Bmatrix} \hat{\boldsymbol{b}}_L \\ \hat{\boldsymbol{b}}_\theta \\ \hat{\boldsymbol{b}}_r \end{Bmatrix} &= [M_1(-\theta)] \left([M_3(\phi)] \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix} \right) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\theta) & \sin(-\theta) \\ 0 & -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix} \\
&= [BN] \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix}
\end{aligned}$$

(b) coordinate transformation

The given vector $\boldsymbol{v}$ (not necessarily to be a velocity vector)

$${}^{\mathcal{B}}\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \text{(notice the order)}$$

Apply DCM

$${}^{\mathcal{N}}\boldsymbol{v} = [NB] \cdot {}^{\mathcal{B}}\boldsymbol{v} \implies {}^{\mathcal{N}}\boldsymbol{v} = [BN]^T \cdot {}^{\mathcal{B}}\boldsymbol{v}$$

Kinematics using Euler angles

To avoid having to integrate the direction cosine matrix directly given an $\boldsymbol{\omega}$ time history, the Euler angle kinematic differential equations are needed.

✓ The following derivation is the same for any set of Euler angles.

Starting from the expression of the angular velocity $\boldsymbol{\omega}$ in the body frame $\mathcal{B}$:

$$\boldsymbol{\omega} = \omega_1 \hat{\boldsymbol{b}}_1 + \omega_2 \hat{\boldsymbol{b}}_2 + \omega_3 \hat{\boldsymbol{b}}_3 \tag{3.52}$$

Here the (3-2-1) Euler angle set (yaw-θ_1-ψ, pitch-θ_2-θ, roll-θ_3-ϕ) is used as an example.

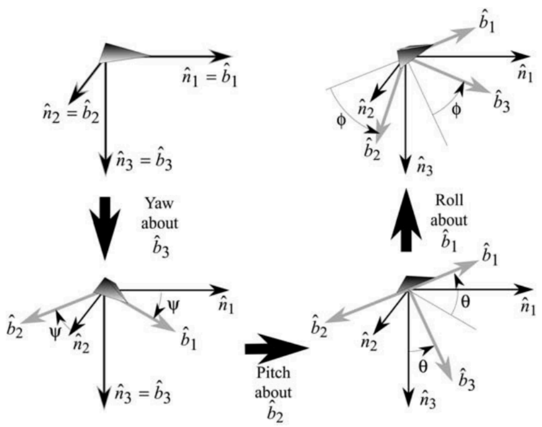


Fig. 3.3 Successive yaw, pitch, and roll rotations.

Another expression of $\boldsymbol{\omega}$ using Fig. 3.3 can be obtained as:

$$\boldsymbol{\omega} = \dot{\psi} \hat{\boldsymbol{n}}_3 + \dot{\theta} \hat{\boldsymbol{b}}_2' + \dot{\phi} \hat{\boldsymbol{b}}_1 \tag{3.53}$$

where $\hat{\boldsymbol{b}}_2'$ above is the $\hat{\boldsymbol{b}}_2$ (after pitch, before roll) on the third subplot in the order given by gigantic arrows, and $\hat{\boldsymbol{n}}_3$ is apparent.

> ⓘ The reason Eq. (3.53) is "apparent" in the textbook. >
>
> This equation is rigorous and we are not using any approximation here.
> Why it's apparent is because this is simply a composition, or an integration, of different sources of angular velocities. We have three sources of angular velocity (try think of three motors along each axis), each of them is along a well-defined axis. So, this is not about decomposition of a vector into a frame (like what Eq. 3.52 does), but to physically integrate all angular velocity vectors into a total velocity vector.
>
> If you treat it as a decomposition, then you will suffer from the dependencies among $\hat{\boldsymbol{n}}_3$, $\hat{\boldsymbol{b}}_2'$, and $\hat{\boldsymbol{b}}_1$ because they are non-orthogonal. But if you think of them as physical source of angular rotations, they are independent sources.

Next is to find expressions for $\hat{\boldsymbol{b}}_2'$ and $\hat{\boldsymbol{n}}_3$ in the body frame $\mathcal{B}$.

$$\hat{\boldsymbol{b}}_2' = \cos\phi \hat{\boldsymbol{b}}_2 - \sin\phi \hat{\boldsymbol{b}}_3 \tag{From the fourth plot}$$

$$\hat{\boldsymbol{n}}_3 = -\sin\theta \hat{\boldsymbol{b}}_1 + \sin\phi \hat{\boldsymbol{b}}_2 + \cos\phi\cos\theta \hat{\boldsymbol{b}}_3 \tag{Get from Eq. (3.34)}$$

Simply plugging in and get:

$$\begin{aligned}
\boldsymbol{\omega} &= \dot{\psi}(-\sin\theta \hat{\boldsymbol{b}}_1 + \sin\phi \hat{\boldsymbol{b}}_2 + \cos\phi\cos\theta \hat{\boldsymbol{b}}_3) + \dot{\theta}(\cos\phi \hat{\boldsymbol{b}}_2 - \sin\phi \hat{\boldsymbol{b}}_3) + \dot{\phi}\hat{\boldsymbol{b}}_1 \\
&= (-\dot{\psi}\sin\theta + \dot{\phi})\hat{\boldsymbol{b}}_1 + (\dot{\psi}\sin\phi + \dot{\theta}\cos\phi)\hat{\boldsymbol{b}}_2 + (\cos\phi\cos\theta - \dot{\theta}\sin\phi)\hat{\boldsymbol{b}}_3 \\
&= \qquad \omega_1 \hat{\boldsymbol{b}}_1 \qquad + \omega_2 \hat{\boldsymbol{b}}_2 \qquad + \omega_3 \hat{\boldsymbol{b}}_3 \qquad \text{(compare with 3.53)}
\end{aligned}$$

or

$$^{\mathcal{B}}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \sin\phi\cos\theta & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Finally, <u>the kinematic ODE of (3-2-1) Euler angle set</u> (yaw-θ_1-ψ, pitch-θ_2-θ, roll-θ_3-ϕ) is obtained by inversing the above equation, which gives:

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\phi\cos\theta & -\sin\phi\cos\theta \\ \cos\theta & \sin\phi\sin\theta & \cos\phi\sin\theta \end{bmatrix} \cdot {}^{\mathcal{B}}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{3.57}$$

Similarly, the ODE of (3-1-3) Euler angle set can be obtained as

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix} = \frac{1}{\sin\theta_2}\begin{bmatrix}\sin\theta_3 & \cos\theta_3 & 0\\ \cos\theta_3\sin\theta_2 & -\sin\theta_3\sin\theta_2 & 0\\ -\sin\theta_3\cos\theta_2 & -\cos\theta_3\cos\theta_2 & \sin\theta_2\end{bmatrix}\cdot {}^{\mathcal{B}}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{3.59}$$

Revisit singularity of Euler angle sets from the perspective of kinematics equations above.

- For (3-2-1), or general asymmetric sets, $\cos\theta = 0$ (2nd angle) gives a singularity in Eq. (3.57).
- For (3-1-3), or general symmetric sets, $\sin\theta_2 = 0$ (2nd angle) gives a singularity in Eq. (3.59).

Drawbacks of Euler angle set:

- A rigid body or reference frame is never further than a 90-deg rotation away from a singular orientation.
- Their kinematic differential equations are fairly nonlinear, containing computationally intensive trigonometric functions.
- The linearized Euler angle kinematic differential equations are valid only for a relatively small domain of rotations.

Principal Rotation Vector $(\hat{e}, \Phi)$

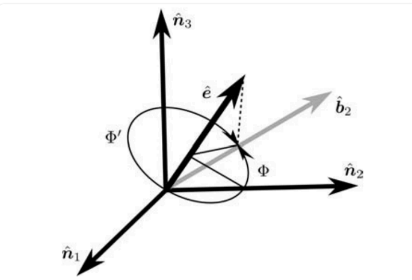


Fig. 3.8 Illustration of both principal rotation angles.

Theorem 3.1 (Euler's Principal Rotation)

A rigid body or coordinate reference frame can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single rigid rotation through a principal angle Φ about the principal axis $\hat{e}$; the principal axis is a judicious axis fixed in both the initial and final orientation.

Before proving it mathematically, just think of one 2D plane rotating in the 3D space: at two difference moments, two planes have a unique intersecting line, and the dihedral angle can be calculated.

Let the principal axis unit vector $\hat{e}$ be written in $\mathcal{B}$ and $\mathcal{N}$ frame components as

$$\begin{aligned}\hat{\boldsymbol{e}} &= e_1\hat{\boldsymbol{b}}_1 + e_2\hat{\boldsymbol{b}}_2 + e_3\hat{\boldsymbol{b}}_3\\ \hat{\boldsymbol{e}} &= e_1\hat{\boldsymbol{n}}_1 + e_2\hat{\boldsymbol{n}}_2 + e_3\hat{\boldsymbol{n}}_3\end{aligned}$$

so, they will have the same vector components in the two frames, which leads to

$${}^{\mathcal{B}}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix} = [C]\cdot {}^{\mathcal{N}}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix}$$

the principal axis unit vector $\hat{\boldsymbol{b}}$ is the unit eigenvector of Φ corresponding to the eigenvalue $+1$. Thus the proof of the principal rotation theorem reduces to proving the Φ has an eigenvalue of $+1$.

Logic to prove $[C]$ has only one eigenvalue $+1$.

Because $\det([C]) = \lambda_1\lambda_2\lambda_3 = 1$ and all λ_i are on the unit circle:

- if two of them are $+1$ (let's say $\lambda_1 = \lambda_2 = 1$), there must be $\lambda_3 = 1$, which leads to a zero rotation.

- if none of them are $+1$, let's say $\lambda_1 = \bar{\lambda}_2$ are conjugate to each other, then λ_3 must be real and thus ± 1. Since we also assumed a right-handled frame, $\lambda_3 = +1$, which is contradicting.

The eigenvector corresponding to $+1$ is unique to within a sign of $\hat{\boldsymbol{e}}$ and Φ, except for the case of a zero rotation. The sets $(\hat{\boldsymbol{e}}, \Phi)$ and $(-\hat{\boldsymbol{e}}, -\Phi)$ both describe the same orientation, so the lack of sign uniqueness will not cause any practical problems.

In most cases the magnitude of Φ is simply chosen to be less than or equal to 180 deg.

Conversions between principal axis and DCM

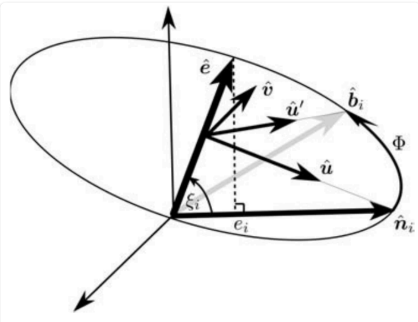


Fig. 3.9 Mapping $\hat{\boldsymbol{n}}_i$ into $\hat{\boldsymbol{b}}_i$ base vectors.

Our goal for the following discussion is: Rotate $\{\hat{n}\}$ to $\{\hat{b}\}$, in other words, to find basis vector $\hat{b}_i$ in terms of $(\hat{e}, \Phi)$ and $(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3)$.

Define the intermediate frame $\{\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}', \hat{\boldsymbol{e}}\}$ using geometric relationship in the figure:

$$\hat{\boldsymbol{v}} = \frac{\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i}{\|\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i\|} = \frac{1}{\sin \xi_i}(\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) \tag{3.66}$$

$$\begin{aligned}
\hat{\boldsymbol{u}} &= \hat{\boldsymbol{v}} \times \hat{\boldsymbol{e}} = \frac{1}{\sin \xi_i}(\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) \times \hat{\boldsymbol{e}} && (3.67)\\
&= \frac{-1}{\sin \xi_i}\hat{\boldsymbol{e}} \times (\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) \\
&= \frac{-1}{\sin \xi_i}(\hat{\boldsymbol{e}}(\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{n}}_i) - \sin \xi_i \hat{\boldsymbol{n}}_i(\hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{e}})) && \text{(bac-cab rule)}\\
&= \frac{-1}{\sin \xi_i}(\cos \xi_i \hat{\boldsymbol{e}} - \hat{\boldsymbol{n}}_i) \\
&= \frac{1}{\sin \xi_i}(\hat{\boldsymbol{n}}_i - e_i \hat{\boldsymbol{e}}) && (3.69)
\end{aligned}$$

where we have used the following identity to simplify notations

$$e_i = \cos \xi_i = \hat{\boldsymbol{e}} \cdot \hat{\boldsymbol{n}}_i.$$

After a rotation of Φ, $\hat{\boldsymbol{u}}$ is rotated to a new direction,

$$\hat{\boldsymbol{u}}' = \cos \Phi \hat{\boldsymbol{u}} + \sin \Phi \hat{\boldsymbol{v}}.$$

In this intermediate frame, $\hat{\boldsymbol{n}}_i$ is expressed as

$$\hat{\boldsymbol{n}}_i = \cos \xi_i \hat{\boldsymbol{e}} + \sin \xi_i \hat{\boldsymbol{u}}.$$

After the rotation, since $\hat{\boldsymbol{b}}_i$ and $\hat{\boldsymbol{n}}_i$ are in the same plane that is normal to $\hat{\boldsymbol{e}}$, only the transversal component of $\hat{\boldsymbol{n}}_i$ gets changed:

$$\begin{aligned}
\hat{\boldsymbol{b}}_i &= \cos \xi_i \hat{\boldsymbol{e}} + \sin \xi_i \hat{\boldsymbol{u}}' \\
&= \cos \xi_i \hat{\boldsymbol{e}} + \sin \xi_i (\cos \Phi \hat{\boldsymbol{u}} + \sin \Phi \hat{\boldsymbol{v}}) \\
&= e_i \hat{\boldsymbol{e}} + \sin \xi_i \cos \Phi \left(\frac{1}{\sin \xi_i}(\hat{\boldsymbol{n}}_i - e_i \hat{\boldsymbol{e}}) + \sin \Phi \frac{1}{\sin \xi_i}(\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) \right) \\
&= e_i \hat{\boldsymbol{e}} + \cos \Phi (\hat{\boldsymbol{n}}_i - e_i \hat{\boldsymbol{e}}) + \sin \Phi (\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) \\
&= \cos \Phi \hat{\boldsymbol{n}}_i + (1 - \cos \Phi) e_i \hat{\boldsymbol{e}} + \sin \Phi (\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) \\
&= \cos \Phi \hat{\boldsymbol{n}}_i + (1 - \cos \Phi)(^{\mathcal{N}}\hat{\boldsymbol{n}}_i)\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T\{\hat{\boldsymbol{n}}\} + \sin \Phi (\hat{\boldsymbol{e}} \times \hat{\boldsymbol{n}}_i) && \text{(3.70 altered)}
\end{aligned}$$

ⓘ **Derive $e_i \hat{e}$ explicitly.**

$$e_i\hat{\boldsymbol{e}} = e_i(e_1\hat{\boldsymbol{n}}_1 + e_2\hat{\boldsymbol{n}}_2 + e_3\hat{\boldsymbol{n}}_3) = ({}^{\mathcal{N}}\hat{\boldsymbol{n}}_i)^T \cdot \begin{bmatrix} e_1e_1 & e_1e_2 & e_1e_3 \\ e_2e_1 & e_2e_2 & e_2e_3 \\ e_3e_1 & e_3e_2 & e_3e_3 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix} = ({}^{\mathcal{N}}\hat{\boldsymbol{n}}_i)\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T \begin{Bmatrix} \hat{\boldsymbol{n}}_1 \\ \hat{\boldsymbol{n}}_2 \\ \hat{\boldsymbol{n}}_3 \end{Bmatrix}$$

Write all three equations in the vectrix format:

$$\{\hat{\boldsymbol{b}}\} = (\cos\Phi[I_{3\times3}] + (1-\cos\Phi)\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T - \sin\Phi\,[\tilde{\boldsymbol{e}}])\{\hat{\boldsymbol{n}}\} \tag{3.71}$$

Finally, the transformation to DCM can be directly extracted from Eq. (3.71) as, notating $\Sigma = 1 - c\Phi$,

$$[\boldsymbol{C}] = \begin{bmatrix} e_1^2\Sigma + c\Phi & e_1e_2\Sigma + e_3s\Phi & e_1e_3\Sigma - e_2s\Phi \\ e_2e_1\Sigma - e_3s\Phi & e_2^2\Sigma + c\Phi & e_2e_3\Sigma + e_1s\Phi \\ e_3e_1\Sigma + e_2s\Phi & e_3e_2\Sigma - e_1s\Phi & e_3^2\Sigma + c\Phi \end{bmatrix}. \tag{3.72}$$

The inverse transformation from the direction cosine matrix $[C]$ to the principal rotation elements $(\hat{\boldsymbol{e}}, \Phi)$ is found to be

$$\cos\Phi = \frac{1}{2}(C_{11} + C_{22} + C_{33} - 1) \tag{3.73}$$

$$\hat{\boldsymbol{e}} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \frac{1}{2\sin\Phi}\begin{pmatrix} C_{23} - C_{32} \\ C_{31} - C_{13} \\ C_{12} - C_{21} \end{pmatrix} \tag{3.74}$$

The principal rotation vector $\boldsymbol{\gamma}$ is defined as

$$\boldsymbol{\gamma} = \Phi\hat{\boldsymbol{e}} \tag{3.77}$$

then

$$\boldsymbol{\omega} = \dot{\Phi}\hat{\boldsymbol{e}} \qquad \text{and} \qquad [\tilde{\boldsymbol{\omega}}] = \dot{\Phi}\,[\tilde{\boldsymbol{e}}] \tag{3.78 and 3.79}$$

$$[\dot{C}] = -\,[\tilde{\boldsymbol{\omega}}]\,[C] \tag{3.27 copied}$$

$$\begin{aligned} \frac{\mathrm{d}[C]}{\mathrm{d}t} &= -\dot{\Phi}\,[\tilde{\boldsymbol{e}}]\,[C] \\ \frac{\mathrm{d}[C]}{\mathrm{d}t} &= -\frac{d\Phi}{\mathrm{d}t}[\tilde{\boldsymbol{e}}]\,[C] \\ \frac{\mathrm{d}[C]}{d\Phi} &= -\,[\tilde{\boldsymbol{e}}]\,[C] \\ [C] &= e^{-\Phi[\tilde{\boldsymbol{e}}]} = e^{-[\tilde{\boldsymbol{\gamma}}]} \end{aligned} \tag{3.81}$$

Express the matrix power using infinite time series definition,

$$[C] = e^{-[\tilde{\boldsymbol{\gamma}}]} = \sum_{n=0}^{\infty}\frac{1}{n!}(-\,[\tilde{\boldsymbol{\gamma}}])^n$$

and we will get the same results to Eqs. (3.71) and (3.72):

$$[C] = \cos\Phi[I_{3\times3}] + (1-\cos\Phi)\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T - \sin\Phi\,[\tilde{\boldsymbol{e}}] \tag{3.71 copied}$$

The inverse transformation from $[C]$ to $\boldsymbol{\gamma}$ is also given my matrix logarithm as

$$[\tilde{\boldsymbol{\gamma}}] = -\ln[C] = \sum_{n=0}^{\infty}\frac{1}{n}(1 - [C])^n \tag{3.83}$$

This derivation of using principle rotation vector to find DCM also holds for higher dimension rotations.

Euler Parameters $\boldsymbol{\beta}$ (aka. Quaternions sometimes)

The Euler parameter vector $\boldsymbol{\beta}$ is defined in terms of the principal rotation elements as

$$\begin{aligned} \beta_0 &= \cos(\Phi/2) \\ \beta_1 &= e_1\sin(\Phi/2) \\ \beta_2 &= e_2\sin(\Phi/2) \\ \beta_3 &= e_3\sin(\Phi/2) \end{aligned}$$

There is one holonomic constraint

$$\beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2=1$$

which describes a 4D unit sphere. Any rotation described through the Euler parameters has a trajectory on the surface of this constraint sphere.

Given a certain attitude, there are actually two sets of Euler parameters that will describe the same orientation. This is due to the non-uniqueness of the principal rotation elements themselves. Try flip the sign of $\hat{e}$ and Φ in the above definition.

Because any point on the unit constraint sphere surface represents a specific orientation, the anti-pole to that point represents the exact same orientation.

> $$[C]=\begin{bmatrix} e_1^2\Sigma+c\Phi & e_1e_2\Sigma+e_3s\Phi & e_1e_3\Sigma-e_2s\Phi \\ e_2e_1\Sigma-e_3s\Phi & e_2^2\Sigma+c\Phi & e_2e_3\Sigma+e_1s\Phi \\ e_3e_1\Sigma+e_2s\Phi & e_3e_2\Sigma-e_1s\Phi & e_3^2\Sigma+c\Phi \end{bmatrix}. \qquad (3.72\text{ copied})$$
> $$\Sigma=1-c\Phi$$

Using Eq. (3.72) and half-angle identities,

$$\begin{aligned} \cos\Phi &= 2\cos^2(\Phi/2)-1=1-2\sin^2(\Phi/2) \\ \sin\Phi &= 2\sin(\Phi/2)\cos(\Phi/2) \end{aligned}$$

the DCM converted from principle rotation parameters, we can find the DCM converted from Euler parameters as

$$[C]=\begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \qquad (3.93)$$

The inverse transformation can be found as:

$$\begin{aligned} \beta_0 &= \pm\frac{1}{2}\sqrt{C_{11}+C_{22}+C_{33}+1} \\ \beta_1 &= \frac{C_{23}-C_{32}}{4\beta_0} \\ \beta_2 &= \frac{C_{31}-C_{13}}{4\beta_0} \\ \beta_3 &= \frac{C_{12}-C_{21}}{4\beta_0} \end{aligned}$$

Initially one simply picks an initial condition on one Euler parameter trajectory and then remains with it.

A very important composite rotation property of the Euler parameters is the manner in which they allow two sequential rotations to be combined into one overall composite rotation.

$$[FN(\boldsymbol{\beta})]=[FB(\boldsymbol{\beta}'')]\cdot[BN(\boldsymbol{\beta}')]$$

$$\begin{bmatrix}\beta_0\\ \beta_1\\ \beta_2\\ \beta_3\end{bmatrix}=\begin{bmatrix} \beta_0'' & -\beta_1'' & -\beta_2'' & -\beta_3'' \\ \beta_1'' & \beta_0'' & \beta_3'' & -\beta_2'' \\ \beta_2'' & -\beta_3'' & \beta_0'' & \beta_1'' \\ \beta_3'' & \beta_2'' & -\beta_1'' & \beta_0'' \end{bmatrix}\begin{bmatrix}\beta_0'\\ \beta_1'\\ \beta_2'\\ \beta_3'\end{bmatrix}$$

The composition only involved matrix multiplication.

These transformations provide a simple, nonsingular, and bilinear method to combine two successive rotations described through Euler parameters. For other attitude parameters such as the Euler angles, this same composite transformation would yield a very complicated, transcendental expression. (refer to textbook for those equations)

Next, discuss the kinematics using Euler parameters. Take time derivative directly:

$$\dot{\beta}_0=\frac{\dot{C}_{11}+\dot{C}_{22}+\dot{C}_{33}}{8\beta_0} \qquad (3.101)$$

> $$[\dot{C}]=-[\tilde{\boldsymbol{\omega}}][C]=-\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}[C] \qquad (3.27\text{ copied})$$

Using Eq. 3.27 to find derivatives of C_{ij}, then plugging in to Eq. 3.101,

$$\dot{\beta}_0 = \frac{1}{2}\left(-\frac{C_{23}-C_{32}}{4\beta_0}\omega_1 - \frac{C_{31}-C_{13}}{4\beta_0}\omega_2 - \frac{C_{12}-C_{21}}{4\beta_0}\omega_3\right) \tag{3.102}$$

Then replacing all C_{ij} using the Euler parameters β_i in DCM, we get

$$\dot{\beta}_0 = \frac{1}{2}(-\beta_1\omega_1 - \beta_2\omega_2 - \beta_3\omega_3) \tag{3.103}$$

Similarly, we can get

$$\begin{aligned}
\dot{\beta}_1 &= \frac{1}{2}(\beta_0\omega_1 - \beta_3\omega_2 + \beta_2\omega_3) \\
\dot{\beta}_2 &= \frac{1}{2}(\beta_3\omega_2 + \beta_0\omega_2 - \beta_1\omega_3) \\
\dot{\beta}_3 &= \frac{1}{2}(-\beta_2\omega_1 + \beta_1\omega_2 + \beta_0\omega_3)
\end{aligned}$$

Wrap up everything in matrix format, and we have

$$\dot{\boldsymbol{\beta}} = \frac{1}{2}[B(\boldsymbol{\beta})] \cdot {}^{\mathcal{B}}\boldsymbol{\omega} \tag{3.106}$$

where

$$[B(\boldsymbol{\beta})] = \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}_{4\times 3} \tag{3.107}$$

Other detailed discussions are omitted here. The essentials are that Euler parameters have very good matrix expressions without relying on trigonometric functions or the original DCM. It is kind of "self-contained".

Quaternion is usually another equivalent way to express Euler parameters. But quaternion also has its own ambiguities in definition in different context. Relatively speaking, Euler parameters are more consistently defined throughout different textbooks.

Classical Rodrigues Parameters

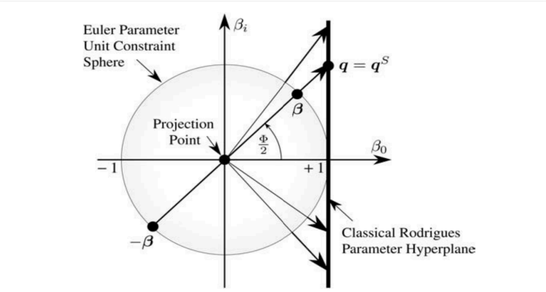


Fig. 3.10 Stereographic projection of Euler parameters to classical Rodrigues parameters.

This rigid body attitude coordinate set reduces the redundant Euler parameters to a minimal three-parameter set through the transformation

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1,2,3 \tag{3.114}$$

The inverse transformation

$$\begin{aligned}
\beta_0 &= \frac{1}{\sqrt{1+\boldsymbol{q}^T\boldsymbol{q}}} \\
\beta_i &= \frac{q_i}{\sqrt{1+\boldsymbol{q}^T\boldsymbol{q}}}
\end{aligned}$$

Classical Rodrigues parameters can also be defined using principal rotation parameters as

$$\boldsymbol{q} = \tan\frac{\Phi}{2}\hat{\boldsymbol{e}}$$

$$[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \tag{3.93 copied}$$

DCM can be get from DCM of Euler parameters (Eq. 3.93) above,

$$[C] = \frac{1}{1+\boldsymbol{q}^T\boldsymbol{q}} \begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3) & 2(q_1q_3-q_2) \\ 2(q_2q_1-q_3) & 1-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1) \\ 2(q_3q_1+q_2) & 2(q_3q_2-q_1) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

The equation of motion is found by taking derivatives to the definition and simply the results using relations to β_i and also the EOM expressed in Euler parameters,

$$\dot{\boldsymbol{q}} = \frac{1}{2} \begin{bmatrix} 1+q_1^2 & q_1q_2-q_3 & q_1q_3+q_2 \\ q_2q_1+q_3 & 1+q_2^2 & q_2q_3-q_1 \\ q_3q_1-q_2 & q_3q_2+q_1 & 1+q_3^2 \end{bmatrix} \cdot {}^{\mathcal{B}}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Modified Rodrigues Parameters

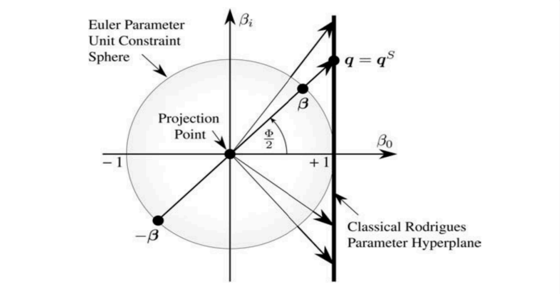


Fig. 3.10 Stereographic projection of Euler parameters to classical Rodrigues parameters.

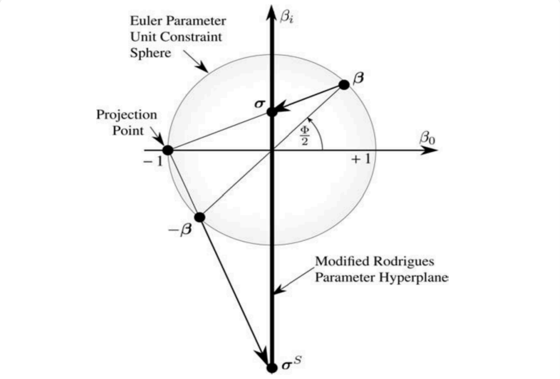


Fig. 3.11 Stereographic projection of Euler parameters to modified Rodrigues parameters.

The MRP vector s is defined in terms of the Euler parameters as the transformation

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1,2,3 \tag{3.137}$$

The inversion is given by

$$\beta_0 = \frac{1-\sigma^2}{1+\sigma^2} \quad \beta_i = \frac{2\sigma_i}{1+\sigma^2} \quad i = 1,2,3$$

✓ End of Lecture Note 04.

More useful and interesting contents are available in the textbook, but are omitted here.
Understanding DCM, Euler angles, quaternions, and Rodrigues parameters are adequate in many cases.