Unseen companions to astrometric and spectroscopic binaries:

Some math

From binary star orbits, we have a relation between the orbital separation, a, the orbital period, $P_{\rm orb}$, and the two stellar component masses, M_1 and M_2:

$$\left(\frac{2\pi}{P_{\rm orb}}\right)^2 = \frac{\mathcal{G}(M_1+M_2)}{a^3}. \tag{1}$$

For a binary with an eccentricity e, the separation of the two components as a function of the true anomaly, f, is:

$$r = \frac{1-e^2}{1+e\cos f}a. \tag{2}$$

If we want to split this separation into two separations corresponding to the distance of each component to the binary's center of mass, we multiply by a mass factor:

$$\begin{aligned} r_1 &= \frac{1-e^2}{1+e\cos f}\frac{M_2}{M_1+M_2}a \\ r_2 &= \frac{1-e^2}{1+e\cos f}\frac{M_1}{M_1+M_2}a. \end{aligned} \tag{3}$$

Under observation, all binaries suffer from perspective effects depending on the orientation of the binary relative to an observer. These are summarized by three angles: the inclination angle, I, the argument of periapse, ω, and the longitude of the ascending node, Ω. Dividing by the distance to the binary, d, provides an equation for the angular position of stellar component of a binary relative to its center of mass. Adding the center of mass coordinate gives an equation for the absolute angular position of a star in a binary as a function of f:

$$\begin{aligned} \alpha_1 &= \alpha + \frac{r_1}{d}\left[\cos\Omega\cos(\omega+f) - \sin\Omega\sin(\omega+f)\cos I\right] \\ \delta_1 &= \delta + \frac{r_1}{d}\left[\sin\Omega\cos(\omega+f) + \cos\Omega\sin(\omega+f)\cos I\right]. \end{aligned} \tag{4}$$

An analogous equation expresses the position of the secondary star, or it can be expressed as a function of (α_1, δ_1):

$$\begin{aligned} \alpha_2 &= -\frac{M_1}{M_2}(\alpha_1 - \alpha) \\ \delta_2 &= -\frac{M_1}{M_2}(\delta_1 - \delta). \end{aligned} \tag{5}$$

Velocity suffers from similar perspective effects as position. It can be shown that the radial velocity of a stellar component in a binary can be expressed as a function of f:

$$\mathrm{RV}_1 = \gamma + \frac{M_2}{M_1 + M_2}\frac{2\pi}{P_{\rm orb}}a\frac{1}{\sqrt{1-e^2}}\left[\cos(\omega + f)\sin I + e\cos\omega\sin I\right]. \tag{6}$$

Subtracting the minimum radial velocity from the maximum radial velocity and dividing by two yields K:

$$K = \frac{M_2}{(M_1 + M_2)^{2/3}}\left(\frac{\mathcal{G}2\pi}{P_{\rm orb}}\right)^{1/3}\frac{1}{\sqrt{1-e^2}}\sin I. \tag{7}$$

Therefore, for a known (observed) K, e, $P_{\rm orb}$, and I, one obtains the function:

$$M_1 = \left(\frac{M_2}{C_0}\right)^{2/3} - M_2, \tag{8}$$

where:

$$C_0 = \sqrt{1-e^2}\left(\frac{P_{\rm orb}}{\mathcal{G}2\pi}\right)^{1/3}\frac{K}{\sin I}. \tag{9}$$

Good orbital coverage with radial velocities can provide, $P_{\rm orb}$, K, and e. I can be determined separately if the absolute scale of the orbit can be observed with precise astrometry combined with a parallax distance.