3 Long Questions

10. (25 points) In this problem, we will try to understand the relationship between magnetic moments and angular momenta, first for charged particles and how this can be extended to planetary objects.

    (a) (5 points) Consider a charge e and mass m moving in circular orbit of radius r with constant speed v. Write down the angular momentum L of the charge and magnetic moment μ of the effective current loop. Recall that the magnetic moment of a current loop with current I and radius r is given as $\mu = IA$ where A is the area of the loop.

    (b) (3 points) Use the above results to find a relationship between the magnetic moment μ and angular momentum L in terms of intrinsic properties of the particle (charge,mass).

    (c) (2 points) The relationship from part (b) can be expressed as $\mu = \gamma L$. γ is usually referred to as the *classical* gyromagnetic ratio of a particle. Evaluate the classical gyromagnetic ratio for an electron and for a neutron in SI units.

    (d) (7 points) For extended objects such as planets, the magnetic dipole moment is not directly accessible whereas the surface magnetic field can be measured. Assuming a magnetic dipole of magnetic moment μ located at the center of a sphere of radius r, write down the expression for the surface magnetic field B_{surf} and the surface magnetic moment defined as $\mathcal{M}_{surf} = B_{surf} r^3$. You may use the value of the angular dependence at the magnetic equator for the following parts.

    (e) (3 points) Assuming a gyromagnetic relationship exists between magnetic moment μ and angular momentum L of an extended object, write down the relationship between the surface magnetic moment $\mathcal{M}_{surf}$ and angular momentum L as $\mathcal{M}_{surf} = \kappa L$. You will observe that κ depends only on fundamental constants and intrinsic properties of the extended object.

    (f) (3 points) The surface magnetic moments for Mercury and Sun are 5×10^{12} T m^3 and 3×10^{23} T m^3 respectively. Assuming the bodies are perfect spheres, evaluate the constant κ for Mercury and the Sun. Comment on values obtained and if they fit into the model developed in parts (c) and (d).

(g) (5 points) The surface magnetic moments $\mathcal{M}_{surf}$ and angular momenta L of various solar system bodies are plotted in the figure 3. Justify that the data implies $\mathcal{M}_{surf} \sim L^{\alpha}$ and calculate the constant α. What is the expected value of α from the model developed in parts (c) and (d)?

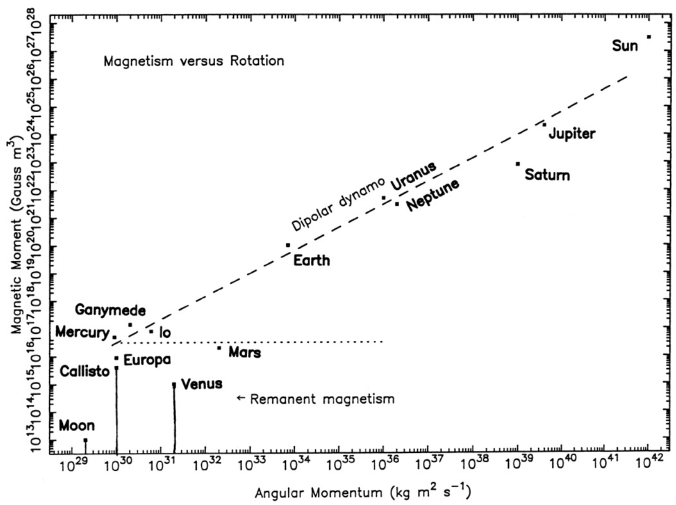

Figure 3: Surface magnetic moment vs angular momentum for solar system objects. Figure taken from Vallée, Fundamentals of Cosmic Physics, Vol. 19, pp 319-422, 1998.

(h) (2 points) Certain bodies such as Venus, Mars and the Moon are remarkably separated from the trend observed for other bodies. What can you say about magnetism in these bodies when compared to the others?

Solution:

(a) $L = mvr$ and $\mu = IA = \frac{ev}{2\pi r}(\pi r^2) = \frac{evr}{2}$

(b) $\mu = \frac{e}{2m}L$

(c) $\gamma_e = 8.794 \times 10^{10}$ C/kg and $\gamma = 0$ for a neutron, since the charge is zero.

(d) Use the magnetic field of a dipole μ as $B_{surf} = \frac{\mu_0}{4\pi}\frac{\mu}{r^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})$. The r^3 dependence is crucial here which follows for any dipole type distribution; partial credit if that is shown. $\mathcal{M}_{surf} = \frac{\mu_0}{4\pi}\mu(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})$

(e) Neglecting the angular dependence, $\mathcal{M}_{surf} = \left(\frac{\mu_0}{4\pi}\gamma\right) L = \kappa L$

(f) For Mercury, angular momentum $L = 10^{30}$ kg m^2/s, so $\kappa = 5 \times 10^{-18}$ T s^2/(kg m). For Sun,angular momentum $L = 10^{42}$ kg m^2/s, so $\kappa = 3 \times 10^{-19}$ T s^2/(kg m). These two values are approximately close to within an order of magnitude. We therefore expect that the mechanisms for magnetic dipole generation are similar in Mercury and the Sun.

(g) Observe that the plot is on log-log scale. A linear relationship on this plot implies a power-law form. From the slope of linear trend-line on the graph, $\alpha \approx 0.8$. Full credit for realizing $\alpha < 1.0$; if $\alpha = 1.0$ is written, partial credit.

(h) Venus, Mars and Moon do not have a active dynamo and hence the magnetism present with them is of a different nature from the other bodies. These bodies had a magnetic dynamo in the past which has since died down. The magnetism is only remnant from this past dynamo.

11. (25 points) Cygnus X-1/HDE 226868 is a binary system consisting of a black hole Cygnus X-1 and blue supergiant HDE 226868. The mass of HDE 226868 is $30M_\odot$ and the period of the binary system is 5.6 days. Radial velocity data reveals that the orbital velocity of HDE 226868 is 116.68 km/s at apoapse and 123.03 km/s at periapse.

   (a) (5 points) Determine the eccentricity of the orbit of HDE 226868.

   (b) (5 points) Determine the length of the semimajor axis of the orbit of HDE 226868.

   (c) (5 points) Determine the mass of Cygnus X-1, to at least 3 significant figures.

   The peak blackbody temperature of an accretion disk occurs at a distance of r_{peak} and a temperature of T_{peak}. One can determine the peak blackbody temperature by assuming that it corresponds to the peak in the x-ray spectrum. Due to relativistic effects, the actual peak blackbody temperature T_{peak} is related to the peak color temperature T_{color} derived from observed spectral data by $T_{color} = f_{GR} f_{col} T_{peak}$, where $f_{GR} \approx 0.510$ and $f_{col} \approx 1.7$. Three x-ray spectra of Cygnus X-1 are shown in Figure 4.

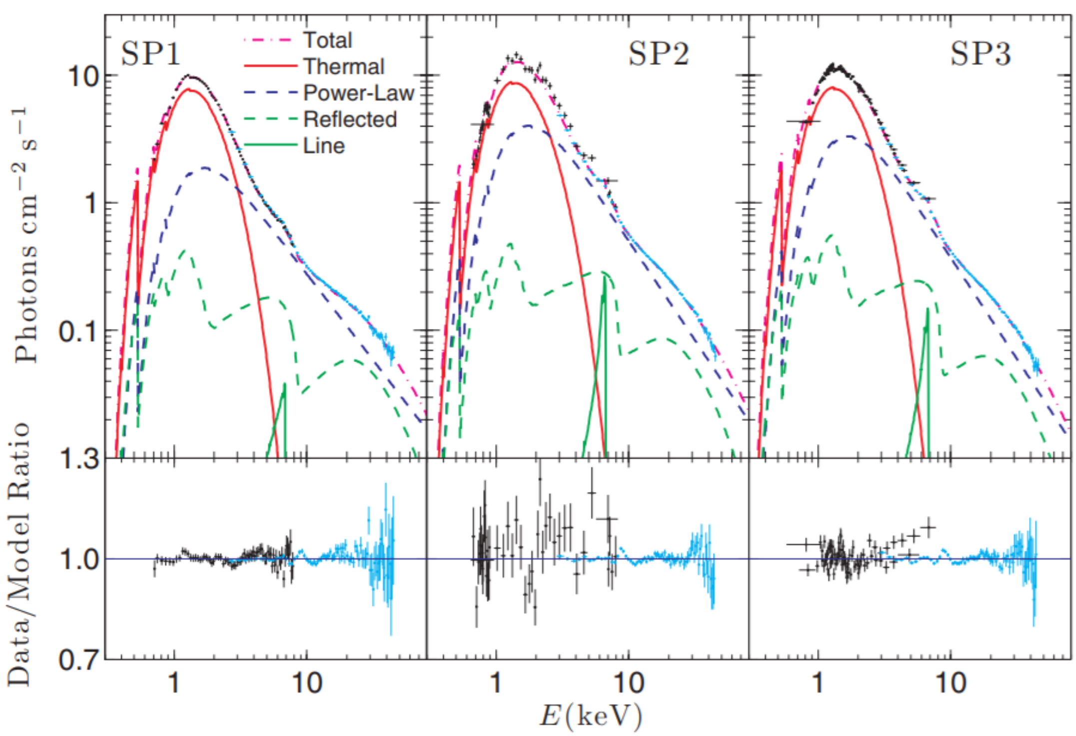

Figure 4: Three x-ray spectra from Cygnus X-1. From Gou et al. (2011).

   (d) (4 points) Using spectrum SP2, determine the peak blackbody temperature T_{peak} of the accretion disk around Cygnus X-1.

   The total luminosity of the blackbody component of the accretion disk can be estimated by $L_{disk} \approx 4\pi\sigma r_{peak}^2 T_{peak}^4$ (Makishima et al. 1986). The radius r_{last} of the innermost edge of the accretion disk is related to the radius r_{peak} of the peak blackbody temperature by $r_{peak} = \eta r_{last}$, where $\eta \approx 0.63$. In 1996, the blackbody luminosity of the accretion disk around Cygnus X-1 was estimated to be 2.2×10^{37} ergs/s.

   (e) (4 points) Determine the radius r_{last} of the innermost edge of the accretion disk around Cygnus X-1.

Assume that the innermost edge of the accretion disk is located at the innermost stable circular orbit (ISCO), whose radius r_{isco} is a function of the spin of the black hole. The relationship between r_{isco} and a_*, the spin parameter of the black hole, can be estimated by:

$$r_{isco} = \frac{GM}{c^2}\left(\sqrt{8.354 \cdot [(2-a_*)^2 - 1]} + 1\right)$$

(f) (2 points) Determine the spin parameter a_* of Cygnus X-1.

Solution:

(a) In the absence of external forces, the total angular momentum of the binary system is conserved. The angular momentum $l = mr^2\omega = mrv$ of each component must also be conserved. Therefore, $m_1 r_{1,a} v_{1,a} = m_1 r_{1,p} v_{1,p} \implies \frac{v_{1,p}}{v_{1,a}} = \frac{r_{1,a}}{r_{1,p}} = \frac{a(1+\epsilon)}{a(1-\epsilon)} \implies \epsilon = \frac{v_{1,p}-v_{1,a}}{v_{1,p}+v_{1,a}} = \frac{123.03-116.68}{123.03+116.68} = 0.0265.$

(b) The area swept by the vector $\mathbf{r_1}$ (from the center of mass to the primary) over time interval Δt is approximately a right triangle of base r_1 and height $v_1\Delta t$, and therefore area $\frac{1}{2}r_1 v_1 \Delta t$. Over a single orbit, the area swept is $A = \frac{1}{2}r_1 v_1 T = \pi a_1^2\sqrt{1-\epsilon^2}$, for (r_1, v_1) at any point in the orbit (by Kepler's second law, the rate of area swept is constant). So, $\pi a_1^2\sqrt{1-\epsilon^2} = \frac{1}{2}a_1(1+\epsilon)v_{1,a}T \implies a_1 = \sqrt{\frac{1+\epsilon}{1-\epsilon}} \cdot \frac{v_{1,a}T}{2\pi} = \sqrt{\frac{1+0.0265}{1-0.0265}} \cdot \frac{116680 \cdot 5.6 \cdot 24 \cdot 3600}{2\pi} = 9.23 \times 10^9$ m.

(c) Kepler's third law states that $(a_1 + a_2)^3 = \frac{G(m_1+m_2)}{4\pi^2}T^2$, so $a_2 = \left[\frac{G(m_1+m_2)}{4\pi^2}T^2\right]^{1/3} - a_1$. Additionally, the definition of center of mass is that $a_1 m_1 = a_2 m_2$, so $a_2 = \frac{a_1 m_1}{m_2}$. Putting these two equations together yields $\left[\frac{G(m_1+m_2)}{4\pi^2}T^2\right]^{1/3} - a_1 = \frac{a_1 m_1}{m_2}$, or $\left[\frac{G(m_1+m_2)}{4\pi^2}T^2\right]^{1/3} - a_1 - \frac{a_1 m_1}{m_2} = 0$. Solving by iteration gives $m_2 = 2.40 \times 10^{31}$ kg, or $m_2 = 12.1 M_\odot$.

(d) The spectrum peaks at about 1.5 keV, or 2.4×10^{-16} J. The energy of a photon is related to its wavelength by $E = \frac{hc}{\lambda}$, which is related to the peak blackbody temperature by $T_{color} = \frac{b}{\lambda}$, where b is Wein's constant. Therefore, $E = \frac{hcT_{color}}{b}$, or $T_{color} = \frac{Eb}{hc} = \frac{2.4\times10^{-16}\cdot 2.898\times10^{-3}}{6.626\times10^{-34}\cdot 3.00\times10^{8}} = 3.50 \times 10^6$ K. Finally, $T_{peak} = \frac{T_{color}}{f_{GR} f_{col}} = 4.04 \times 10^6$ K.

(e) $L_{disk} = 2.2 \times 10^{37}$ ergs/s, or 2.2×10^{30} W. $L_{disk} = 4\pi\sigma r_{peak}^2 T_{peak}^4$, so $r_{peak} = \frac{1}{T_{peak}^2}\sqrt{\frac{L_{disk}}{4\pi\sigma}} = \frac{1}{(4.04\times10^6)^2}\sqrt{\frac{2.2\times10^{30}}{4\pi\cdot 5.67\times10^{-8}}} = 107.66$ km. $r_{last} = \eta r_{peak} = 67.83$ km.