3 Medium Questions

1. (**15 points**) An astronomer used his f/5 telescope with a diameter of 130 mm to observe a binary system. He is using an eyepiece with a field of view of 45^{o} and a focal length of 25 mm. In this system, star A has a mass of 18.9 solar masses, and an apparent magnitude in the V filter of 9.14. Star B has a mass of 16.2 solar masses, and an apparent magnitude in the V filter of 9.60. The period of the system is 108

days, and the distance between the binary stars and the Solar System is 2.29 kpc. The binary system has an edge-on orbit relative to the Solar System.

(a) What is the field of view of the telescope?
(b) What is the limiting magnitude of the telescope?
(c) What is the angular resolution of the telescope?
(d) What is the angular separation between the stars?
(e) Is the astronomer able to observe both stars as distinct points in the telescope? Answer as YES or NO.

The limiting magnitude for the human eye is 6.0, and the diameter of the pupil is equal to 7.0 mm. Also consider that visible light has a wavelength of 550 nm.

Solution:

(a) It is possible to divide the field of view of the eyepiece by the magnification of the telescope in order to obtain an approximation for the field of view of the telescope:

$$\begin{aligned}
FOV_{telescope} &= \frac{FOV_{eyepiece}}{m} \\
FOV_{telescope} &= \frac{FOV_{eyepiece}}{f_{telescope}/f_{eyepiece}} \\
FOV_{telescope} &= \frac{45^\circ}{(130 \times 5)/(25)} \\
FOV_{telescope} &= 1.7^\circ
\end{aligned}$$

(b)

$$\begin{aligned}
m_{eye} - m_{telescope} &= 2.5 \log \frac{D_{eye}^2}{D_{telescope}^2} \\
6.0 - m_{telescope} &= 2.5 \log \frac{7^2}{130^2} \\
m_{telescope} &= 12.3
\end{aligned}$$

Since both of stars are brighter than the limiting magnitude, they both can be detected by the telescope.

(c) It is possible to use the following expression to calculate the angular resolution of the telescope:

$$\begin{aligned}
\theta &= \frac{1.22\lambda}{D_{telescope}} \\
\theta &= \frac{1.22 \times 550 \times 10^{-9}}{130 \times 10^{-3}} \\
\theta &= 5.16 \times 10^{-6}\text{rad}
\end{aligned}$$

(d) Using Kepler's Third Law, it is possible to determine the separation between star A and star B:

$$\begin{aligned}
T^2/a^3 &= 1/M \\
(108/365.25 years)^2/a^3 &= 1/((18.9 + 16.2)) \\
a &= 1.45\, AU
\end{aligned}$$

Dividing the separation between the stars by the distance to the binary system, it is possible to obtain the angular separation between the stars:

$$\alpha = a/d$$
$$\alpha = (1.45AU)/(2.29 \times 10^3 \times 206265AU)$$
$$\alpha = 3.08 \times 10^{-9}\text{rad}$$

(e) Since the angular separation between the stars is smaller than the angular resolution, the astronomer will observe both stars as a single point in his telescope. Therefore, the correct answer is NO.

2. (**20 points**) The rotation curve of a particular spiral galaxy is modeled by an exponential function of the form $V(r) = V_0(1 - e^{-r/R})$, where $V_0 = 250$ km/s, $R = 7.5$ kpc, and r is measured radially from the center of the galaxy. Throughout parts (a)-(d), you may assume the galaxy is disk-shaped. Further, we'll assume that the distribution of mass in the galaxy depends only on the radial coordinate r (and is thus radially symmetric).

   (a) Find the period of rotation (in years) of a particle 10 kpc from the center of the galaxy. Also, find the mass enclosed within the (circular) orbit in solar masses, i.e. the mass within $r = 10$ kpc from the center of the galaxy.

   (b) Find the angular velocity of the galaxy very close to the center ($r \ll R$). Hint: $e^x \approx 1 + x$ for $|x| \ll 1$.

   (c) Determine how the (gravitational) mass per unit area must vary with distance from the center of the galaxy in order to yield the given rotation curve. Find the expressions only for regions very far from the galactic center.

   (d) An astronomer measures the absolute bolometric magnitude of the galaxy to be -21.2. For comparison, the bolometric magnitude of the sun is 4.75. Assume that the luminous mass per unit area follows a profile given by $\sigma_L = \frac{k}{r}$ for $k = 2.55 \times 10^8$ M_{Sun}/kpc and that all of the luminous mass is in the form of Sun-like stars. Approximate the percentage of the galaxy's mass that is dark matter, out to the maximum distance (radius) that is still visibly defined.

Solution:

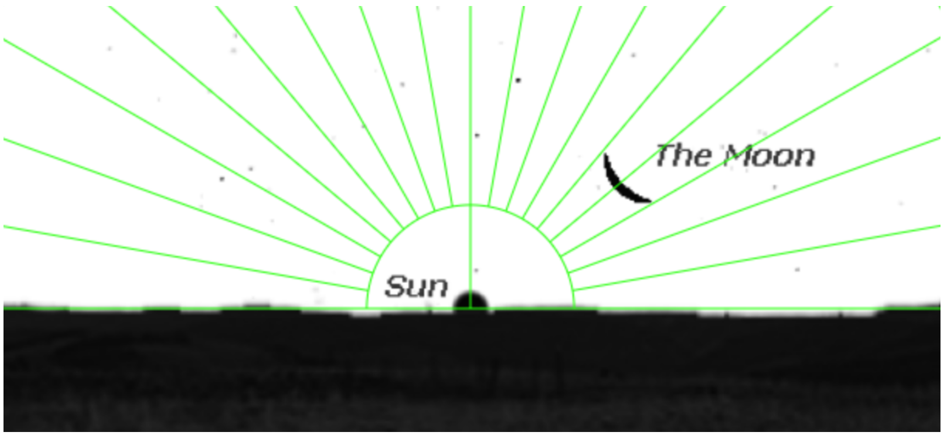
(a) The period is $T = \frac{2\pi r}{V}$, so we obtain $T = 334$ million years. Equating the centripetal force with the gravitational force (as is the condition for a circular orbit), $M = \frac{V^2 r}{G}$, so we obtain $M = 7.88 \times 10^{10}$ solar masses.

(b) $\omega = \frac{V}{r}$ is the angular velocity. Very close to the center, $e^{-r/R} \approx 1 - r/R$, so $\omega \approx V_0/R = 1.08 \times 10^{-15}$ rad/s.

(c) Recall $M = \frac{V^2 r}{G}$. Consider an annulus at radius r with small radial thickness δr. Letting σ be the desired mass per unit area, we have that $\sigma 2\pi r \delta r = \delta M$. But note that $\delta M \approx \frac{V^2 \delta r}{G}$ very far away from the center (as V is essentially constant). Thus we obtain $\sigma = \frac{V_0^2}{2\pi r G}$ very far away from the center ($r \gg R$).

(d) $M_{bol} = 4.75 - 2.5\log(L/L_{Sun})$, so $L/L_{Sun} = 2.40 \times 10^{10}$. The luminous mass enclosed by an annulus at radius r with small radial thickness δr is $2\pi r \delta r \sigma_L = 2\pi k \delta r$, which means we seek r_{max} such that $2\pi k r_{max} = 2.40 \times 10^{10}$. This gives $r_{max} = 14.97$ kpc as the maximum distance from the center of the galaxy for which the galaxy is still visibly defined. The total gravitational mass enclosed at this radius is $M = \frac{V^2 r}{G} = 1.63 \times 10^{11}$ solar masses. The fraction of dark matter is approximately $1 - \frac{2.40\times10^{10}}{1.63\times10^{11}} = 85\%$.

3. (**15 points**) An astro-photographer has taken the photo of the moon close to a new moon day shown below right before the sunset on December 21 (Winter Solstice) in a wide open area.



(a) In which hemisphere (Northern or Southern) is the photographer located?

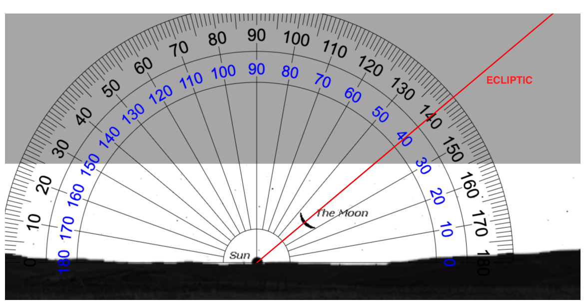
(b) Find the latitude of the photographer. Ignore the orbital inclination of the Moon and the ellipticity of the Earth's orbit. Hint: The green equiangular lines are added to the image to help you out in measuring any relevant angle.

(c) Calculate the sidereal time when the photo was taken.

Solution:

(a) By looking at the position of the moon with respect to the sun, we know that the photo has been taken from the Southern hemisphere.

(b) First off, we measure the angle between the horizon and ecliptic. It is $\sim 40°$.



Drawing the celestial spheres, let S be the celestial south pole, Z be the zenith, and K' be the South ecliptic pole. $\widehat{SZ} = 90 - \phi$; $\widehat{S(Sun)} = 90 - 23.5 = 66.5°$; $\widehat{ZK'} = 40°$; $\widehat{SK'} = 23.5°$; $\widehat{Z(Sun)} = 90°$; $\widehat{K'(Sun)} = 90°$ and $\angle K'S(Sun) = 180°$. So, Sun is the pole of a great circle crossing both K' and Z points, $\Rightarrow \angle S(Sun)Z = 40°$. Using the spherical law of cosines, $\sin\phi = \cos 23.5° \cos 40° \Rightarrow \phi = 44.6°$. Since the astrophotographer is in the Southern hemisphere, $\phi = 44.6° S$.

(c) Sidereal time$=H_\odot + RA_\odot$. Using the spherical law of cosines, $\cos H_\odot = -\tan 23.5° \tan 44.6° \Rightarrow H_\odot = 7^h 42^m$. $RA_\odot$(Winter Solstice)$= 18^h \Rightarrow ST = 7^h 42^m + 18^h = 1^h 42^m$.

4. (**20 points**) In general relativity, the orbit of satellites around a massive object (like a black hole) are known as geodesics and do not obey all of Kepler's laws for orbits. However, for objects that are moving at non-relativistic speeds, we can analyze the orbit using classical mechanics, with a corrective term added to Newton's Law of gravity. In this case, the potential energy of an object in orbit around a black hole is:

$$V_s(r) = -\frac{GMm}{r} - \frac{GML^2}{c^2 m r^3}$$

where M and m are the masses of the black hole and the object respectively, c is the speed of light, L is the angular momentum of the object in orbit and r is the distance of that object from the black hole. Likewise, the gravitational force from a black hole has magnitude:

$$F_s(r) = \frac{GMm}{r^2} + \frac{3GML^2}{c^2 m r^4}$$

You may assume that both conservation of energy and conservation of angular momentum hold in this regime.

(a) Argue which of Kepler's laws are still true.

(b) Calculate the radius of a stable circular orbit with an angular momentum L (you will get two solutions, the stable orbit generates the classical result under the proper limits)

(c) What is the radius, R_{ISCO} of the innermost stable circular orbit (the smallest stable circular obit) for a black hole of mass M? What is the numerical value of R_{ISCO} for Sagittarius A*, which has mass 3.6×10^6 solar masses?

(d) Suppose we discovered a new star orbiting Sagittarius A*, S99, that has a periapsis of $10R_{ISCO}$ and an apoapsis of 16 AU. Find the magnitude of velocity of S99 at both periapsis and apoapsis.

Solution:

(a) Only Kepler's law of areas still holds from the fact that we have a radial force.

(b) Using centripetal force, we get that

$$\frac{L^2}{mr^3} = \frac{GMm}{r^2} + \frac{3GML^2}{c^2 m r^4}$$

A little algebra later, we get the quadratic equation,

$$r^2 - \frac{L^2}{GMm^2} r + \frac{3L^2}{m^2 c^2} = 0$$

Using the quadratic equation, we get:

$$r = \frac{L^2}{2GMm^2}\left(1 \pm \sqrt{1 - 12\frac{G^2M^2m^2}{L^2c^2}}\right)$$

Taking the limit that $c \to \infty$ we see that the plus solution gives us the classical result. Thus the stable orbit is

$$r = \frac{L^2}{2GMm^2}\left(1 + \sqrt{1 - 12\frac{G^2M^2m^2}{L^2c^2}}\right)$$

and the unstable orbit is

$$r = \frac{L^2}{2GMm^2}\left(1 - \sqrt{1 - 12\frac{G^2M^2m^2}{L^2c^2}}\right)$$

(c) The ISCO occurs with the smallest allowed value of L which is $L^2 = \frac{12G^2M^2m^2}{c^2}$. This gives a value of $R_{ISCO} = \frac{6GM}{c^2}$ and calculating numerically we get $R_{ISCO} = 3.20 \times 10^{10} \approx 0.21$ AU.

(d) One way to solve this problem is to first find the angular momentum of the orbit; to do this we use conservation of energy:

$$\frac{L^2}{2mr_1^2} - \frac{GMm}{r_1} - \frac{GML^2}{c^2mr_1^3} = \frac{L^2}{2mr_2^2} + -\frac{GMm}{r_2} - \frac{GML^2}{c^2mr_2^3}$$

Substituting $l = \frac{L}{m}$ and solving, we see that:

$$l^2 = \frac{2GM(1/r_1 - 1/r_2)}{(1/r_1^2 - 1/r_2^2) - \frac{2GM}{c^2}(1/r_1^3 - 1/r_2^3)} = \frac{R_s(1/r_1 - 1/r_2)}{(1/r_1^2 - 1/r_2^2) - R_s(1/r_1^3 - 1/r_2^3)}c^2$$

Where $R_s = 0.07$ AU is the Schwarchild radius of Sagittarius A*, $r_1 = 2.1$ AU and $r_2 = 16$ AU. Doing the calculations we get that $l = 0.36c\times$ AU. Thus the velocities are $v_1 = l/r_1 = 0.171c = 5.19 \times 10^7$ m/s and $v_2 = l/r_2 = 0.022c = 6.82 \times 10^6$ m/s.

5. (**15 points**) The following table gives the numerical values for some physical properties of four stars. The quantities that are affected by, i.e. include the effects of, interstellar extinction are marked with a star (*). You may consider that all stars are black bodies. The temperature of a star can be calculated directly from its B-V index, by using Ballesteros' formula:

$$T_{eff} = f(B-V) = 4600\left(\frac{1}{0.92(B-V)+1.7} + \frac{1}{0.92(B-V)+0.62}\right)K.$$

Determine the numerical values of all the other physical characteristics presented in the given table. For full credit, show your full work by writing all the mathematical expressions used in the calculation.

Hint: You might use the following empirical relation:

$$\frac{A_V}{E_{B-V}} = 3.2$$

Star	κ Velorum	β Tauri	Sirius A	Sun
Annual Parallax $p^*(10^{-3} arcsec)$	6.05	24.89	379.2	-
Distance to Sun $\Delta^*(pc)$				-
Interstellar Extinction in V Band A_V (mag)	0.20	0.08	Negligible	-
$10^{0.2\times A_V}$				-
Distance to Sun $\Delta(pc)$				4.85×10^{-6}
Annual Parallax $p(10^{-3}$ arcsec)				
Distance Modulus $\mu = m - M$				
Visual apparent magnitude $m^*(mag)$	2.86	1.68	-1.47	-26.73
Visual apparent magnitude m (mag)				
Visual absolute magnitude M_V (mag)				
Color Index $(B-V)^*$ (mag)	-0.14	-0.06	+0.01	+0.65
Extinction E_{B-V} (mag)				
Color Index $(B-V)$ (mag)				
Effective Temperature $T_{eff} = f(B-V)$ (K)				
λ_m (nm)				
Radius (Solar Radius, R_S)	9.10	4.60	1.71	1.00
Total Luminosity (Solar Luminosity,L_S				1.00
Absolute Bolometric Magnitude M_{bol} (mag)				4.64
Bolometric Correction BC for V band (mag)				-0.20

Solution:

It is known that:

$$\Delta^*_{pc} = \frac{1}{p^*_{arcsec}}$$

Also,

$$m = M + 5\log\Delta_{pc} - 5$$

without extinction, and

$$m^* = M + 5\log\Delta^*_{pc} - 5$$

with interstellar extinction. Because the luminous flux is dimmed by the interstellar extinction,

$$m^* = m + A_V$$

Thus,

$$\Delta_{pc} = \Delta_{pc}^{*} \times 10^{-0.2A_V}$$

Distance modulus is given by:

$$\mu = m - M = 5 \log \Delta_{pc} - 5$$

And thus, the absolute visual magnitude can be computed as:

$$M_V = m - \mu$$

For the Color Excess, we know that:

$$E_{B-V} = (B-V)^{*} - (B-V)$$

In order to find the temperature of a given star, we can use Ballesteros' formula, given in the question. Next, from Wien's formula:

$$\lambda \times T_{eff} = b$$

Moreover,

$$L_s = \sigma \times T^4 \times 4\pi \times R^2$$

So, we can show that:

$$M_{BOL} - M_{BOL_{Sun}} = -2.5 \log \frac{L}{L_{Sun}}$$

Lastly, the Bolometric Correction is:

$$B.C. = M_{BOL} - M_V$$

Star	κ Velorum	β Tauri	Sirius A	Sun
Annual Parallax $p^*(10^{-3}$ arcsec)	6.05	24.89	379.2	-
Distance to Sun Δ^* (pc)	165.28	40.17	2.63	-
Interstellar Extinction in V Band A_V (mag)	0.20	0.08	Negligible	-
$10^{0.2\times A_V}$	1.10	1.04	1	-
Distance to Sun $\Delta(pc)$	150.26	38.63	2.63	4.85×10^{-6}
Annual Parallax p $(10^{-3}$ arcsec)	6.655	25.880	379.200	286.05
Distance Modulus $\mu = m - M$	5.89	2.93	-2.89	-31.57
Visual apparent magnitude m^* (mag)	2.86	1.68	-1.47	-26.73
Visual apparent magnitude m (mag)	2.26	1.60	-1.47	-27.73
Visual absolute magnitude M_V (mag)	-3.63	-1.33	1.42	4.84
Color Index $(B-V)^*$ (mag)	-0.14	-0.06	+0.01	+0.65
Extinction E_{B-V} (mag)	0.0625	0.025	negligible	-
Color Index (B-V) (mag)	-0.2025	-0.085	+0.01	+0.65
Effective Temperature $T_{eff} = f(B-V)$ (K)	13616	11270	9936	5750
λ_m (nm)	220.00	266.19	301.93	521.73
Radius (Solar Radius, R_S)	9.100	4.600	1.711	1.000
Total Luminosity (Solar Luminosity,L_S	2350	277	23	1.00
Absolute Bolometric Magnitude M_{bol} (mag)	-3.79	-1.48	+1.21	4.64
Bolometric Correction BC for V band (mag)	-0.16	-0.15	-0.20	-0.20