4 Long Questions

1. (**30 points**) M15 is a globular cluster in the constellation Pegasus. The Hertzsprung–Russell diagram (apparent visual magnitude versus color index) of the cluster is shown in fig. 1. Considering that the mass (M)– luminosity (L) relation for main sequence stars is given by $\frac{L}{M^3}$ =constant, answer the following questions. In this problem, ignore the interstellar reddening and dust extinction effects.

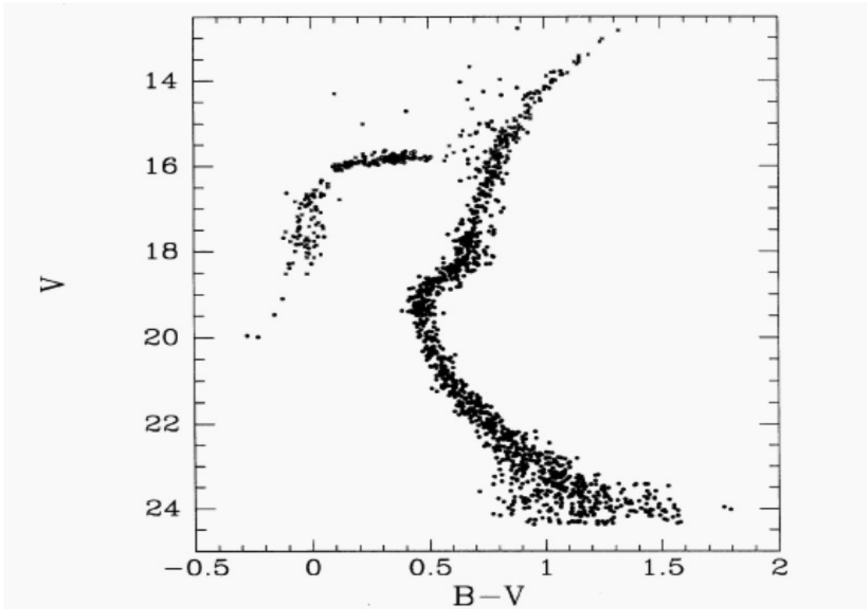


Figure 1: HR diagram for M15

(a) Given that all the stars are formed at the same time, estimate the age of the globular cluster. The color index of the sun ($(B-V)_\odot$) is 0.65 and its life time on the main sequence is 10 billion years.

(b) Estimate the distance of this globular cluster from the Earth. Give the answer in parsec. The absolute visual magnitude of the Sun is 4.83.

(c) Given that stars spend about 10% of their main-sequence life time in the post main sequence phase, find the mass of the most massive star in the post main sequence stage.

(d) The number of stars in the mass range of (M_1, M_2) can be written as:

$$N(M_1 \leqslant M \leqslant M_2) = A(M_1^{-1.35} - M_2^{-1.35}) \tag{1}$$

where A is a constant, M_1 and M_2 are in units of solar masses. Assuming that the number of stars in the post main sequence phase is 515, calculate the value of constant A in equation 1.

(e) M15 is one of the most densely packed globular clusters such that in a visual band ($\lambda \sim 5500\mathring{A}$) image of M15 taken by a telescope with diameter of 10 cm, the stars at the center of cluster cannot be resolved. Estimate the minimum number of stars in this cluster. The angular diameter of M15 is 12.3 arc minutes. Assume that the number density of stars is constant within the cluster.

(f) Use your answers from parts (d) and (e) to estimate the mass of the lowest possible mass star in this cluster. For this part, assume that the mass of the most massive star in the cluster is $20M_\odot$.

Solution:

(a) Measuring from the H-R diagram, a solar-like star with ($(B-V)_\odot$=0.65) has apparent visual magnitude $V_\odot = 21.4$. Also, this value is $V_t = 19.3$ for a star at the turning point of the main sequence. Thus, $V_\odot - V_t = 2.5\log(L_t/L_\odot)$ which gives $L_t/L_\odot \sim 7$. It allows us to estimate the life time of the star at the turning point which is the age of the cluster. Life time in the main sequence $\propto \frac{M}{L} \propto \frac{L^{1/3}}{L} \propto L^{-2/3} \Rightarrow$ Age of the cluster $= 10(L_t/L_\odot)^{-2/3}$ billion years=2.7 billion years. M15 is a very metal-poor globular cluster, so its actual age should be larger than this number (should be about the age of the Universe). The main reason for this discrepancy can be explained by a large uncertainty in B-V measurements.

(b) The distance module for the cluster is $V_{\odot} - M_v(\odot)$ where $M_v(\odot)$ is the absolute visual magnitude of the sun which is given, so distance module= 21.4-4.83=16.57. distance module= $5\log(d(pc)/10) \Rightarrow$ d$\sim$ 20 kpc. The actual distance of this globular cluster is $\sim$ 10 kpc. Our overestimation in the distance originates from the dust extinction effect that we ignored in this problem.

(c) The most massive star in the post main sequence phase spent $\frac{2.7}{110\%} = 2.45$ billion years on the main sequence. Life time in the main sequence $\propto \frac{M}{L} \propto \frac{M}{M^3} \propto M^{-2} \Rightarrow M_{max}$(post main sequence)$= (\frac{2.45}{10})^{-0.5} M_{\odot} = 2.02\ M_{\odot}$

(d) The mass of the star at the turning point is $(L_t/L_{\odot})^{(1/3)} = 7^{(1/3)} = 1.91 M_{\odot}$. Thus, $A = \frac{515}{1.91^{-1.35} - 2.02^{-1.35}} \sim 16945$. This number can be changes based on your normalization for masses (M_1 and M_2). Here we consider that they are given in solar masses.

(e) Consider a cylinder with a height of 2R (diameter of the cluster) and diameter of the telescope resolution at the center of the cluster. Within this cylinder, we should have at least two stars to observe the stars at the center of the cluster image unresolved. Telescope resolution: $D = 1.22 \frac{\lambda = 550nm}{D = 0.1m} = 1.38\ arcsec \Rightarrow \frac{2}{2R \times \pi \frac{D^2}{4}} = \frac{N_{min}}{\frac{4}{3}\pi R^3} \Rightarrow N_{min} = \frac{16}{3}(\frac{R}{D})^2 = \frac{16}{3}(\frac{12.3/2\ arcmin}{1.38\ arcsec})^2 \sim 3.8 \times 10^5$.

(f) Using the results from part d and e: $M_{min} = (\frac{N_{min}}{A} + M_{max}^{-1.35})^{\frac{1}{-1.35}} = (\frac{3.8 \times 10^5}{16987} + 20^{-1.35})^{\frac{1}{-1.35}} \sim 0.1\ M_{\odot}$.

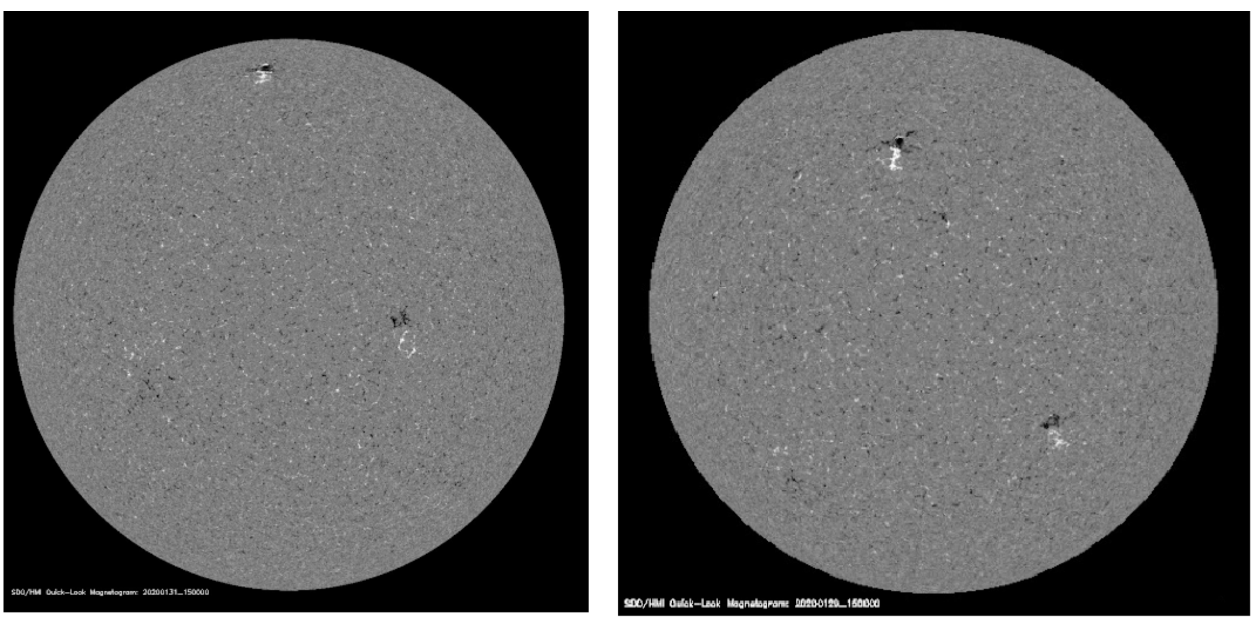
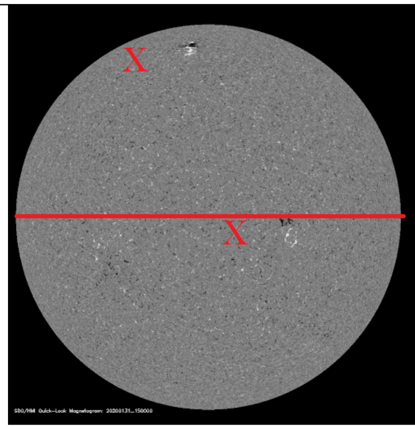
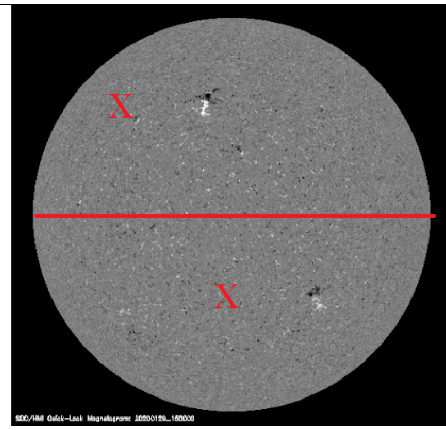
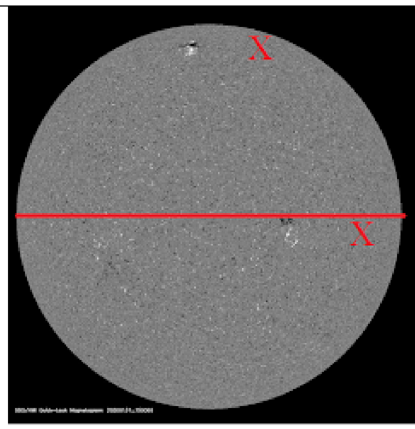
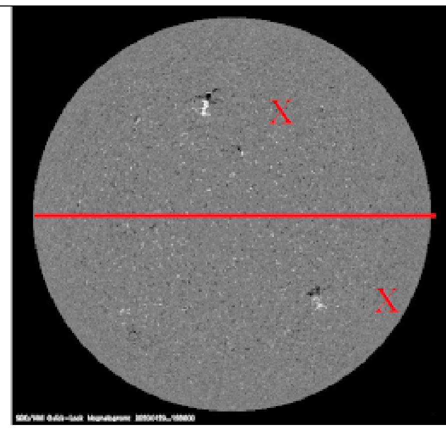
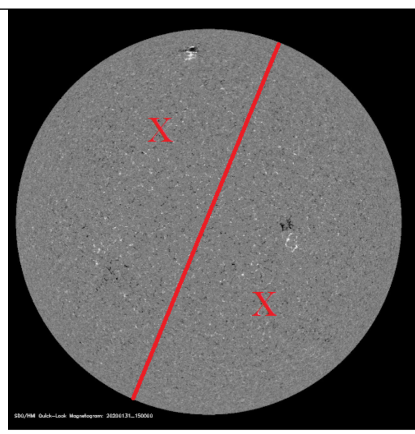
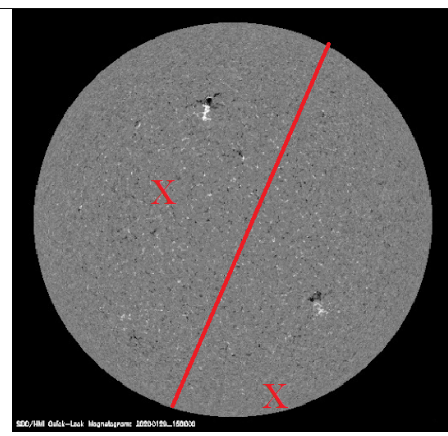
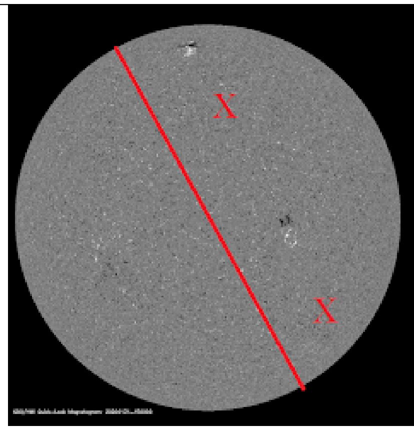
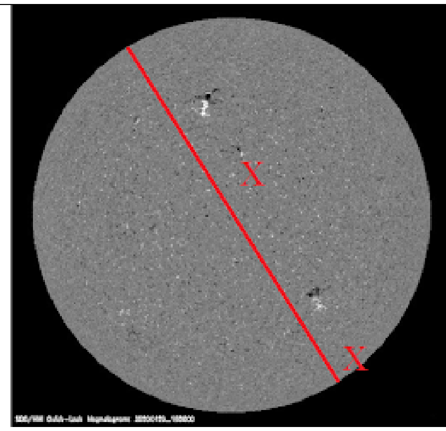
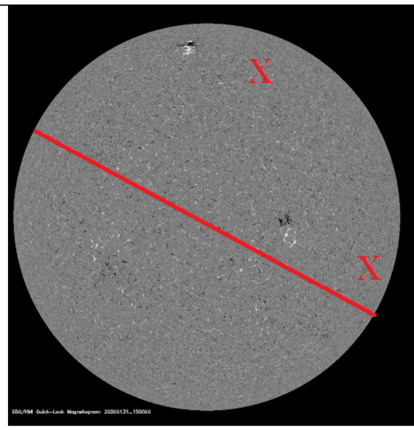
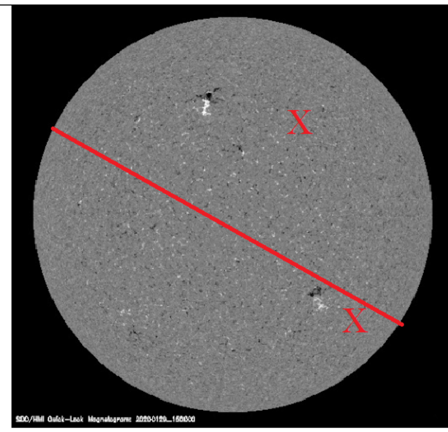


Figure 2: Solar magnetograms taken at the end of Jan 2020

2. (**25 points**) Fig. 2 shows two magnetograms of the Sun taken with the Heliseismic and Magnetic Imager (HMI) at the Solar Dynamics Observatory (SDO) towards the end of January 2020. The picture on the left was taken three days after the image on the right.

(a) Select the pair of images (numbered from I to V) in which the lines are drawn at the Sun's Equator and the Xs correspond to the position of each sunspot 4 days before the pictures were taken.

I		
II		
III		

IV		
V		

(b) Estimate the absolute value of the latitude of both Sunspots in Figure 2.

(c) Fig. 3 is a magnetogram of the Sun in normal activity. It is possible to notice that the sunspots have different orientations in different hemispheres. In one of the hemispheres, each spot has the white part on the left and the black one on the right, and vice-versa. However, this is not the case for the images presented in fig. 2. Suggest an explanation for the anomaly on the images in fig. 2.

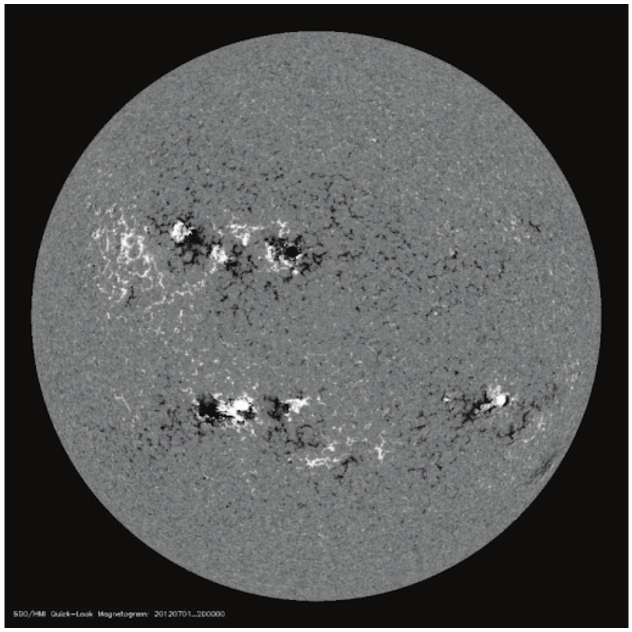


Figure 3: Sun in normal activity

(d) Assume for the sake of simplicity that a specific sunspot has a shape very similar to that of a spherical triangle. The sides of the triangle are equal to $0.176°$, $0.0981°$, and $0.201°$. Calculate the value of the three internal angles in degrees.

For the following parts, assume that this sunspot is centered at $7.89°$ South and $51.74°$ East of the center of the Solar disk for an observer on Earth.

(e) For an observer on Earth, what is the ratio between the area of the Solar disk and the observed area of the sunspot? Note that the required ratio is between the areas observed by someone on Earth, not the ratio between the actual areas. The area of a spherical triangle is equal to $\pi R^2 E/180°$, in which the spherical excess (E) in (deg) is equal to the sum of the internal angles minus $180°$ and R is the radius of the sphere on which the spherical triangle lies.

(f) If an observer on Earth uses a huge f/5 telescope with a focal length of 13 m to look at this sunspot, will it be possible to resolve it? Visible light is centered at 550 nm.

(g) The Sun generates its luminosity by converting Hydrogen into Helium in the proton-proton chain. In the most energetic branch of the chain, 4 protons fuse into a helium nucleus. Considering that only 10% of the solar mass can be converted into energy, calculate the time that the Sun spends in the Main Sequence.

Solution:

(a) The correct answer is III.

(b) The acceptable range will be 26^o to 29^o for full credit, and 25^o or 30^o to half of the punctuation.

It is useful to know that sunspots always come in pairs and have the same absolute value of latitude (one is positive and the other negative).

(c) Every 11 years (the period of the Schwabe Cycle), the Solar magnetosphere flips its orientation. Therefore, the orientation of the sunspots on each hemisphere is reversed. The pictures on item A were likely taken while the Sun was going through this transition, which explains their anomaly.

(d) A, B, and C are the internal angles of the spherical triangle.

Using the Law of the Cosines for spherical triangles: $\cos(0.176^o) = \cos(0.0981^o) \times \cos(0.201^o) + \sin(0.0981^o) \times \sin(0.201^o) \times \cos(A)$

$$\cos(A) = \frac{\cos(0.176^o) - \cos(0.0981^o) \times \cos(0.201^o)}{\sin(0.0981^o) \times \sin(0.201^o)}$$

$A = 61.11699^o$

Using the Law of the Sines for spherical triangles:

$\sin(61.11699^o)/\sin(0.176^o) = \sin(B)/\sin(0.201^o)$

$B = 89.67051^o$

Using the Law of the Sines for spherical triangles to calculate the last internal angle:

$\sin(61.11699^o)/\sin(0.176^o) = \sin(C)/\sin(0.0981^o)$

$C = 29.21265^o$

(e) First, it is necessary to calculate the spherical excess:

$E = 61.1^o + 89.7^o + 29.2^o - 180^o$

$E = (1.51 \times 10^{-4})^o$

Then, it is possible to calculate the area of the spherical triangle:

$A = \pi \times R_{Sun}^2 \times 1.51 \times 10^{-4}/180$

$A = 8.41 \times 10^{-7} \times \pi \times R_{Sun}^2$

Since the question asks about the observed area it is necessary to correct it with the inclination of the tangential plane.

Inclination of the tangential plane (Law of the Cosines with a right spherical triangle containing both coordinates):

$\cos(i) = \cos(7.89^o) \times \cos(51.74^o)$

$i = \arccos(0.613)$

$i = 52.2^o$

Now, it is possible to correct the observed area with the inclination:

$A_c = 8.41 \times 10^{-7} \times \pi \times R_{Sun}^2 \times \cos(52.2^o)$

$A_c = 5.16 \times 10^{-7} \times \pi \times R_{Sun}^2$

Considering that the area of the Solar disk is $\pi \times R_{Sun}^2$, it is possible to calculate the ratio between the areas observed:

$r = 5.16 \times 10^{-7} \times \pi \times R_{Sun}^2/(\pi \times R_{Sun}^2)$

$r = 5.16 \times 10^{-7}$

It is important to consider that small rounding errors will lead to huge differences in the spherical excess, so students won't be penalized if their answer is different than expected due to rounding errors. In fact, if students notice that the spherical excess is so small that the triangle is basically flat, they will also get full credit for this item. In this case, the calculations for the value of r would be the following (it is possible to use any pair of sides in the triangle and the angle in between them):

$A = 1/2 \times (2 \times \pi \times R_{Sun} \times 0.0981^o/360^o) \times (2 \times \pi \times R_{Sun} \times 0.201^o/360^o) \times \sin(61.1^o)$

$A = 8.37 \times 10^{-7} \times \pi \times R_{Sun}^2$

$A_c = 8.37 \times 10^{-7} \times \pi \times R_{Sun}^2 \times \cos(52.2^o)$

$A_c = 5.13 \times 10^{-7} \times \pi \times R_{Sun}^2$

$r = 5.13 \times 10^{-7} \times \pi \times R_{Sun}^2/(\pi \times R_{Sun}^2)$

$r = 5.13 \times 10^{-7}$

(f) First, it is necessary to calculate the diameter of the telescope:

$f/D = 5$

$13/D = 5$

$D = 2.6m$

Then, it is possible to calculate the angular resolution of the telescope:

$\theta = 1.22\lambda/D$

$\theta = 1.22 \times 5.5 \times 10^{-7}/2.6$

$\theta = 2.58 \times 10^{-7} rad$

It is possible to use the smallest side of the triangle to see if it will be resolved by the telescope. Length of the smallest size of the triangle:

$l = 2 \times \pi \times R_{Sun} \times 0.0981^o/360^o$

$l = 1.19 \times 10^6 m$

Correcting it for the inclination:

$l_c = 1.19 \times 10^6 m \times cos(52.2^o)$

$l_c = 7.31 \times 10^5 m$

Angle that corresponds to this length:

$\alpha = 7.31 \times 10^5/(1.496 \times 10^{11})$

$\alpha = 4.89 \times 10^{-6}\text{rad}$

Since $\alpha > \theta$, it is possible to resolve the sunspot.

(g) Variation in mass per reaction:

$$\Delta m = 4m_H - m_{He}$$
$$\Delta m = 4 \times 1.6725 \times 10^{-27} kg - 6.644 \times 10^{-27} kg$$
$$\Delta m = 4.600 \times 10^{-29} kg$$

Fraction of mass converted:

$$f = m/(4m_H)$$
$$f = 4.600 \times 10^{-29} kg/(4 \times 1.6725 \times 10^{-27} kg)$$
$$f = 6.876 \times 10^{-3}$$

Mass converted into energy while the Sun is in the Main Sequence:

$$\Delta M = M \times f \times 0.10$$
$$\Delta M = 1.989 \times 10^{30} \times 6.876 * 10^{-3} \times 0.10$$
$$\Delta M = 1.368 \times 10^{27} kg$$

Total energy converted:

$$E = \Delta M c^2$$
$$E = 1.368 \times 10^{27} kg \times (2.998 \times 10^8 m/s)^2$$
$$E = 1.229 \times 10^{44} J$$

Time in the main sequence:

$$\Delta t = E/L$$
$$\Delta t = 1.229 * 10^{44} J/(3.826 \times 10^{26} W)$$
$$\Delta t = 3.213 * 10^{17} s$$
$$\Delta t \approx \text{ 10 billion years}$$