USAAAO 2018 Second Round Solutions

March-April 2018

1 Short Questions

1. In order to detect an Earth-twin, we need significant advances in the precision of spectrographs to detect the periodic Doppler shift of nearby stars. Estimate the radial velocity semi-amplitude, in m/s, that a planet with the mass, radius, and semi-major axis of Earth would cause in the motion of a star with the mass of the Sun. Assume that the Earth-twin has zero eccentricity. Note that the mass of Earth is $5.97 \cdot 10^{24}$ kg, and the distance from the Earth to the Sun is $1.5 \cdot 10^8$ km.

Solution: Since $v_p = \frac{2\pi a}{T} = 29.9\,km/s$, and from 2^{nd} Kepler's law $\frac{v_s}{v_p} = \frac{m_p}{m_s}$. Calculating it ends up being

$$v_s = v_p \cdot \frac{m_p}{m_s} = 29.9 km/s \cdot \frac{M_e}{M_s},$$

where M_e is the mass of the Earth and M_s is the mass of the Sun. Plugging in the numbers we get $v_s = 0.090 m/s$. Thus the radial velocity semi-amplitude of $\sim 0.1 m/s$.

2. Planet nine is a hypothesized planet in the outer Solar System that may explain the clustering of orbital elements of distant trans-Neptunian objects. The hypothesized periapse of planet nine is $200AU$, and the apoapse is expected to be at approximately $1200AU$. What would the eccentricity of planet nine be? How does this eccentricity compare to that of the 8 major planets in the Solar System?

Solution: $e = \frac{r_a - r_p}{(r_a + r_p)} = .714$, much larger than any planet in the Solar System.

3. What is the main-sequence lifetime of a star with a mass of 0.1 Solar masses, and a star with a mass of 10 Solar masses? Assume that stellar luminosity, $L \propto M^{3.5}$, where M is stellar mass, and that the main-sequence lifetime of the Sun is 10 billion years.

Solution: If $L \propto M3.5$, then age $t \propto M^{-2.5}$. Plugging in for $M = 10$ and 0.1 solar masses and $t_0 = 10 Gyr$, we get:

$t = 31.6$ million years for $M = 10$ solar masses

$t = 3160$ billion years for $M = 0.1$ solar masses.

4. The Very Large Array radio interferometer ($\lambda = 1m$) has maximum baseline of $D = 36.4km$. How large will an optical telescope have to be to achieve a similar angular resolution in visible light ($\lambda = 5,500\mathring{A}$)?

Solution: Answer: $20mm$

$$\Theta[rad] = 1.22\frac{\lambda}{Diameter} = 1.22\frac{1}{36.4 \cdot 10^3} = 0.0000335rad$$

$$Diameter = 1.22\frac{\lambda}{\Theta} = 1.22\frac{550 \cdot 10^{-9}}{0.0000335} = 0.020m = 20mm$$

5. An amateur astronomer observes the Moon with $20cm$ telescope, and accomplishes 160x magnification with an eyepiece with focal length $10mm$. What is the f-number of the telescope?

Solution: Answer: 8

$Magnification = {}^{F}/_{f}$ where F is telescope focal length, f is eyepiece focal length

$$F = Magnification \cdot f = 160 \cdot 10 = 1600mm$$

$$F - number = \frac{F}{D} = \frac{1600}{200} = 8$$

6. The average person has $1.4m^2$ of skin. What is the energy per second radiated by the average person in the form of blackbody radiation? What is the peak wavelength of emitted radiation? Why can't we see it with our eyes?

Solution: Answer: $L = A \cdot \sigma \cdot T^4 = 1.4 \cdot 5.6710^{-8} \cdot (273 + 37)^4 = 733W$

Note that the answer may vary slightly ($700 - 730W$) on the assumed temperature of the human body. In the above example, we have chosen $37C$.

$\Lambda[A] = 2.9 \cdot \frac{10^7}{T[K]} = 93548\mathring{A} = 9.4microns \implies$ IR (human eye does not see in IR)

7. On March 21^{st} at true noon, length of the shadow of a vertical rod was equal to its height. On which geographic latitude did this happen?

Solution: Answer: $\varphi = \pm 45$.

21^{st} of March is the day of spring equinox. Thus the Sun can have a height of $h = 90 - \varphi$. Since the length of the shadow is equal to its height it means that height of the Sun was 45°. Thus geographic latitude of this place was $\varphi = \pm 45^\circ$(both north and south count).

8. In stars like the Sun, helium nuclei are formed by fusing hydrogen nuclei together in a process known as the proton-proton chain. One step of the proton-proton chain consists of a deuterium nucleus (m_d = 2.01410 u) fusing together with a hydrogen nucleus (m_H = 1.00783 u) to form a helium-3 nucleus (m_{He} = 3.01603 u), where $u = 1.6605 \times 10^{-27}kg$. How much energy is released during this fusion reaction?

Solution: It is necessary to calculate the mass defect. The missing mass has been converted into energy. From conservation of mass,

$$m_d + m_H = m_{He} + \Delta m.$$

Rearranging,

$$\Delta m = m_d + m_H - m_{He},$$

and plugging in the given masses,

$$\Delta m = (2.01410 + 1.00783 - 3.01603)\ \text{u},$$

$$\Delta m = 0.0059\ \text{u},$$

$$\Delta m = (0.0059\ \text{u}) \left(1.6605 \times 10^{-27}\ \frac{\text{kg}}{\text{u}}\right),$$

$$\Delta m = 9.797 \times 10^{-30}\ \text{kg}.$$

Then to find the energy,

$$E = mc^2,$$

$$E = (9.797 \times 10^{-30}\ \text{kg}) \left(3.0 \times 10^{8}\ \frac{\text{m}}{\text{s}}\right)^2,$$

$$E = 8.817 \times 10^{-13}\ \text{J}.$$

9. Solar wind consists of protons that fly with the speed of 300 *km/s* and they fill the space of interplanetary matter around Earth with 10 particles/*cm*3. With what force is this "wind" hitting the Moon? Recall that mass of a proton is $m_p = 1.6 \cdot 10^{24}\ g$. Radius of the Moon is $R_m = 1737\ km$.

Solution: Answer: $F = 1.4$ tons.

From 2^{nd} Newton's Law $F = a \cdot m = \frac{\Delta V}{\Delta t} m = \frac{\Delta(V\,m)}{\Delta t}$, i.e. equals change of impulse per unit of time. We assume that the protons reaching to the Moon give its impuls without changing its mass. Let V be the wind speed and ρ the density of particles. Then ρV particles hit per unit time per unit area of the Moon, giving impulse $m_p V \rho V$. Over the whole surface of the Moon we get $F = \pi R_m^2 \rho m_p V^2 = 1.4$ tons.

10. Mars orbits the Sun at an average distance of 2.28×10^{11} m and has a radius of 3.39×10^{6} m. The Sun has a luminosity of 3.828×10^{26} W. How much solar energy falls on the surface of Mars each second? Ignore any effects of Mars' thin atmosphere.

Solution: At the distance of Mars' orbit, the Sun's energy output is spread over a sphere with an area of

$$A = 4\pi r^2 = 4\pi(2.28 \times 10^{11}\ \text{m})^2 = 6.53 \times 10^{23}\ \text{m}^2.$$

Dividing the luminosity of the Sun by this area gives,

$$\frac{L_\odot}{A} = \frac{3.828 \times 10^{26}\ \text{W}}{6.53 \times 10^{23}\ \text{m}^2} = 586\ \frac{\text{W}}{\text{m}^2}.$$

Mars presents a circular area to the Sun of

$$A_{\text{Mars}} = \pi r^2_{\text{Mars}} = \pi(3.39 \times 10^6\ \text{m})^2 = 3.61 \times 10^{13}\ \text{m}^2.$$

Therefore, the total energy that falls on the surface of Mars will be

$$\frac{L_\odot}{A} A_{\text{Mars}} = 586\ \frac{\text{W}}{\text{m}^2}\ (3.61 \times 10^{13}\ \text{m}^2) = 2.12 \times 10^{16}\ \text{W}.$$

11. When a gravitationally bound system (such as a galaxy) forms, it transitions from a just bound state ($E_{kin} = E_{pot}$) to a virialized state ($E_{kin} = 0.5\ E_{pot}$) and the excess binding energy has to be radiated away. Consider an idealized disk galaxy with an exactly flat rotation curve with a rotation speed of $v_{circ} = 220 km/s$ (you can neglect the kinetic energy in random motions). Its density profile cuts off abruptly at a radius of $R_{max} = 50 kpc$. Assume that it took 500 million years for this galaxy to collapse to its present state. What was its mean luminosity (in units of solar luminosity) due to the release of the binding energy during that period?

Solution: Since the final kinetic energy is only half of the potential energy, the amount radiated away must be also equal to the current kinetic energy.

First we need to estimate the total mass of the galaxy. The easiest way to do this is to look at the centripetal force acting on a particle at the outermost radius:

$$\frac{mv^2_{circ}}{R_{max}} = \frac{GmM}{R^2_{max}} \rightarrow M = \frac{v^2_{circ}}{GR^2_{max}}.$$

The outer radius is 50 kpc and v_{circ} = 220 km/s, so $M = 1.12 \times 10^{45}\ g = 5.63 \times 10^{11} M_\odot$. The kinetic energy is

$$\frac{1}{2} M v^2_{circ} = 2.71 \times 10^{59}\ erg.$$

The mean luminosity is just

$$L = \frac{E_{kin}}{t} = 1.715 \times 10^{43}\ ergs^{-1} = 4.48 \times 10^9\ L_\odot$$