2 Long Questions

1. In 2008, while observing WASP-14, a main sequence star of mass 1.211 $M_\odot$ and radius 1.306 $R_\odot$, an exoplanet called WASP-14b was discovered via the transit method. Photometry as well as radial velocity data are shown in the figures. Transits occur once every 2.243753 days. The radial velocity of the center of mass of WASP-14 and its planet is -4.99 km/s. Fitting of the radial velocity curve indicates that theargument of periastron of the orbit of WASP-14b is 254.9°.

   (a) Determine the length of the semimajor axis of the orbit of WASP-14b.

Solution: Answer: $a_p = 5.348 \times 10^9$ m = 0.0357 AU.
From Kepler's third law:

$$(a_* + a_p)^3 = \frac{G(M_* + M_p)}{4\pi^2} P^2$$

Assuming that the mass of the planet is negligible to the mass of the host star, and that the semimajor axis of the host star is negligible compared to the semimajor axis of the planet gives

$$a_p^3 = \frac{GM_*}{4\pi^2} P^2$$

Solving for a_p, we get

$$a_p = \left(\frac{GM_*}{4\pi^2} P^2\right)^{1/3} = 5.348 \times 10^9 \text{ m} = 0.0357 \text{ AU}$$

   (b) Determine the density of WASP-14b.

Solution: Answer: $\rho = 4843.7$ kg/m^3.

To find the mass, we assume the orbit is circular. Since the planet is transiting, its inclination must be approximately 90°. Therefore, the velocity of the host star's movement is the amplitude of the radial velocity curve.

$$v_* = \frac{(-4.02) - (-6.02)}{2} = 0.995 \text{ km/s}$$

The velocity of the planet can be calculated from the semimajor axis from part (a):

$$v_p = \frac{2\pi a_p}{P}$$

Finally, the two velocities v_* and v_p are related by the conservation of momentum: $M_* v_* = M_p v_p$. Solving for M_p gives

$$M_p = \frac{M_* v_*}{v_p} = 1.383 \times 10^{28} \text{ kg}$$

Note that given the eccentricity from part (c), we can find a more accurate mass of the planet; however, assuming circular orbits gives a close approximation.

To find the radius, we compare the magnitude difference during transit. A difference in magnitude Δm relates to flux by

$$\Delta m = -2.5 \log \frac{F_{transit}}{F}$$

The normalized drop in flux during transit (i.e. relative to the normal flux) is therefore

$$\Delta F_{norm} = \frac{F - F_{transit}}{F} = 1 - \frac{F_{transit}}{F} = 1 - 10^{-\Delta m/2.5} = 0.0093$$

The normalized drop in flux relates to the radii of the host star and planet by

$$\Delta F_{norm} = \left(\frac{R_p}{R_*}\right)^2$$

The radius of the planet can then be found: $R_p = R_* \sqrt{\Delta F_{norm}} = 8.786 \times 10^7$ m.

Density is simply

$$\rho = \frac{M_p}{V_p} = \frac{3M_p}{4\pi R_p^3} = 4843.7 \text{ kg/m}^3$$

(c) Determine the eccentricity of the orbit of WASP-14b.

Solution: Answer: $e = 0.0964$.

Because the planet is transiting, we know that the inclination must be 90°. Let ω be the argument of periastron, which is the angle between the periapsis and the plane of the sky. Let ϑ be the true anomaly, which is the angle between the planet and the periapsis. Therefore, $\vartheta + \omega$ is the angle between the planet and the plane of the sky.

Let z be the position of the planet along the axis perpendicular to the plane of the sky (i.e. toward and away from the observer). Let r be the distance from the planet to the host star. $z = r\sin(\vartheta + \omega)$. The radial velocity of the star is simply the time-derivative of z, which is $\dot{z} = \dot{r}\sin(\vartheta + \omega) + r\dot{\vartheta}\sin(\vartheta + \omega)$.

The orbit equation gives the distance from the host star to the planet as a function of true anomaly:

$$r = \frac{a(1 - e^2)}{1 + e\cos\vartheta}$$

Taking the time-derivative of r gives:

$$\dot{r} = \frac{a(1 - e^2)}{(1 + e\cos\vartheta)^2} \cdot e\sin\vartheta = \frac{r\dot{\vartheta} \cdot e\sin\vartheta}{1 + e\cos\vartheta}$$

$\dot{r}$ can then be substituted into the expression for radial velocity:

$$\dot{z} = r\dot{\vartheta}\left[\frac{e\sin\vartheta}{1 + e\cos\vartheta}\sin(\vartheta + \omega) + \cos(\vartheta + \omega)\right]$$

From Kepler's second law:

$$\dot{A} = \frac{1}{2}r^2\dot{\vartheta} = \frac{A}{P} = \frac{\pi a^2\sqrt{1 - e^2}}{P}$$

$$r\dot{\vartheta} = \frac{2\pi a^2\sqrt{1 - e^2}}{rP}$$

$$r\dot{\vartheta} = \frac{2\pi a^2\sqrt{1 - e^2}}{P} \cdot \frac{1 + e\cos\vartheta}{a(1 - e^2)} = \frac{2\pi a(1 + e\cos\vartheta)}{P\sqrt{1 - e^2}}$$

This can then be substituted into the expression for radial velocity:

$$\dot{z} = \frac{2\pi a}{P\sqrt{1 - e^2}}\left[e\sin\vartheta \cdot \sin(\vartheta + \omega) + (1 + e\cos\vartheta) \cdot \cos(\vartheta + \omega)\right]$$

$$\dot{z} = \frac{2\pi a}{P\sqrt{1 - e^2}}\left[e\cos\omega + \cos(\vartheta + \omega)\right]$$

The term $\cos(\vartheta + \omega)$ determines the radial velocity over time, since all other variables are constants. Radial velocity is maximized when $\cos(\vartheta + \omega) = 1$ and is minimized when $\cos(\vartheta + \omega) = -1$. Let $k = \frac{2\pi a}{P\sqrt{1-e^2}}$.

$$\begin{cases} k(e\cos\omega + 1) = \dot{z}_{max} \\ k(e\cos\omega - 1) = \dot{z}_{min} \end{cases}$$

Subtracting the two equations gives $2k = \dot{z}_{max} - \dot{z}_{min}$, or $k = \frac{1}{2}(\dot{z}_{max} - \dot{z}_{min})$. Adding the two equations gives $2ke\cos\omega = \dot{z}_{max} + \dot{z}_{min}$. So,

$$e\cos\omega = \frac{\dot{z}_{max} + \dot{z}_{min}}{\dot{z}_{max} - \dot{z}_{min}}$$

$$e = \frac{\dot{z}_{max} + \dot{z}_{min}}{\cos\omega(\dot{z}_{max} - \dot{z}_{min})} = 0.0964$$

2. The star Sualocin (RA: 20^h 39.6^m, Dec: 15° 54.7′, absolute magnitude: -0.4) is about 78 pc away from our solar system, and the star Rotanev (RA: 20^h 37.5^m, Dec: 14°35.7′, absolute magnitude: 1.6) is about 31 pc away. An alien astronomer is on a planet with Earth's mass and radius orbiting Rotanev. The planet has a uniform albedo of 0.3.
   (a) What is the angular distance between Sualocin and Rotanev?
   (b) What is the distance between these stars in parsecs?
   (c) On the alien's planet, what is the angular separation in the sky between Sualocin and our Sun?
   (d) How much greater is the flux received by the planet from Sualocin than that received from our Sun?

3. Let's suppose that at some point in the recent past, all the hydrogen and helium in the universe had been instantly fused into iron in stars, and that the energy released was thermalized into black body radiation. Take the baryon density to be $\rho_b = 4.2 \times 10^{-31}$ g/cm^3. This is about 75% hydrogen (1 baryon) and 25% helium (4 baryons) by mass. The binding energy per nucleon of $^{56}_{26}$Fe is 8.8 MeV and that of $^{4}_{2}$He is 7.1 MeV.
   (a) What is the current temperature of this black body radiation?

Solution: Answer: $T = 4.40K$.
Without considering intermediary products, we have these two reactions:

$$56\ ^{1}_{1}\text{H} \rightarrow\ ^{56}_{26}\text{Fe}$$

$$14\ ^{4}_{2}\text{He} \rightarrow\ ^{56}_{26}\text{Fe}$$

We can calculate the energy released in each of these reactions by computing the binding energy on each side. There is no binding energy for a hydrogen atom, so the energy released in the first reaction is 56×8.8 MeV = 492.8 MeV per Fe. For the second reaction, the binding energy will by (56×8.8 MeV) - ($14 \times 4 \times 7.1$ MeV) = 95.2 MeV per Fe.
To find how much iron is actually produced, we need to determine the number densities of hydrogen and helium using the mass fraction given in the question:

$$n_H = \frac{0.75\rho_b}{m_H}$$

$$n_{He} = \frac{0.25\rho_b}{m_{He}}$$

where m_H and m_{He} are the masses of hydrogen and helium respectively. Then , the total energy density released is:

$$u = \frac{492.8MeV \times n_H}{56} + \frac{95.2MeV \times n_{He}}{14}$$

To get the temperature, we use:

$$u = aT^4$$

where a is the radiation constant. This gives us a temperature of 4.40 K.

   (b) Determine what wavelength the blackbody spectrum would peak at. What region of the electro-magnetic spectrum would this be in?

Solution: Answer: $\lambda = 6.59 \times 10^{-2}$ cm, microwave.
Using Wien's displacement law:

$$\lambda = \frac{b}{T}$$

where $b = 2.898 \times 10^{-3}$ m K, we get a wavelength of 6.59×10^{-2} cm, which is in the microwave region of the electomagnetic spectrum.

(c) How long would it take stars to fuse all the hydrogen and helium in the universe, given that the mean bolometric luminosity per unit volume emitted by stars today is about $3 \times 10^8\ L_\odot/\mathrm{Mpc}^3$? Compare this to the present age of the universe.

Solution: Answer: 7.11×10^{19} seconds (2250 Gyr)
The time it would take is given by the energy density (found in part (a)), divided by the rate of fusion, i.e. the luminosity given in the problem:

$$t = \frac{u}{L}$$

This gives us 7.11×10^{19} seconds, or about 2250 Gyr. This is much longer than the current age of the universe (13.7 Gyr).