2 Medium Questions

5. (15 points) An alien spaceship from the planet Kepler 62f is in search of a rocky planet for a remote base. They're attracted to Earth because of a fortunate coincidence: its axis of rotation points directly at their home planet. That means they can have uninterrupted communication with home by planting fixed transmitters on The North Pole. But first, they need to find out if Earth's axis will always point in the same direction or if it undergoes precession. They can't know without years of observation, so they hope that we, its now-extinct intelligence, have left behind the answer.

   While orbiting Earth, they see a few remarkable structures, including the Hoover Dam in Nevada. Zooming in on the dam, a colorful plaza with peculiar markings on its floor catches their attention. Descending on the plaza, they realize the markings are a map of the sky when the dam was built, left to indicate the date to posterity. Figure 1 is an overhead architectural map of this plaza. The center-point depicts the north ecliptic pole, and the large circle represents the path of the Earth's axis throughout its counter-clockwise procession. As they interpret the map, they're dismayed to realize that their star has not been and will not be Earth's north star for very long.

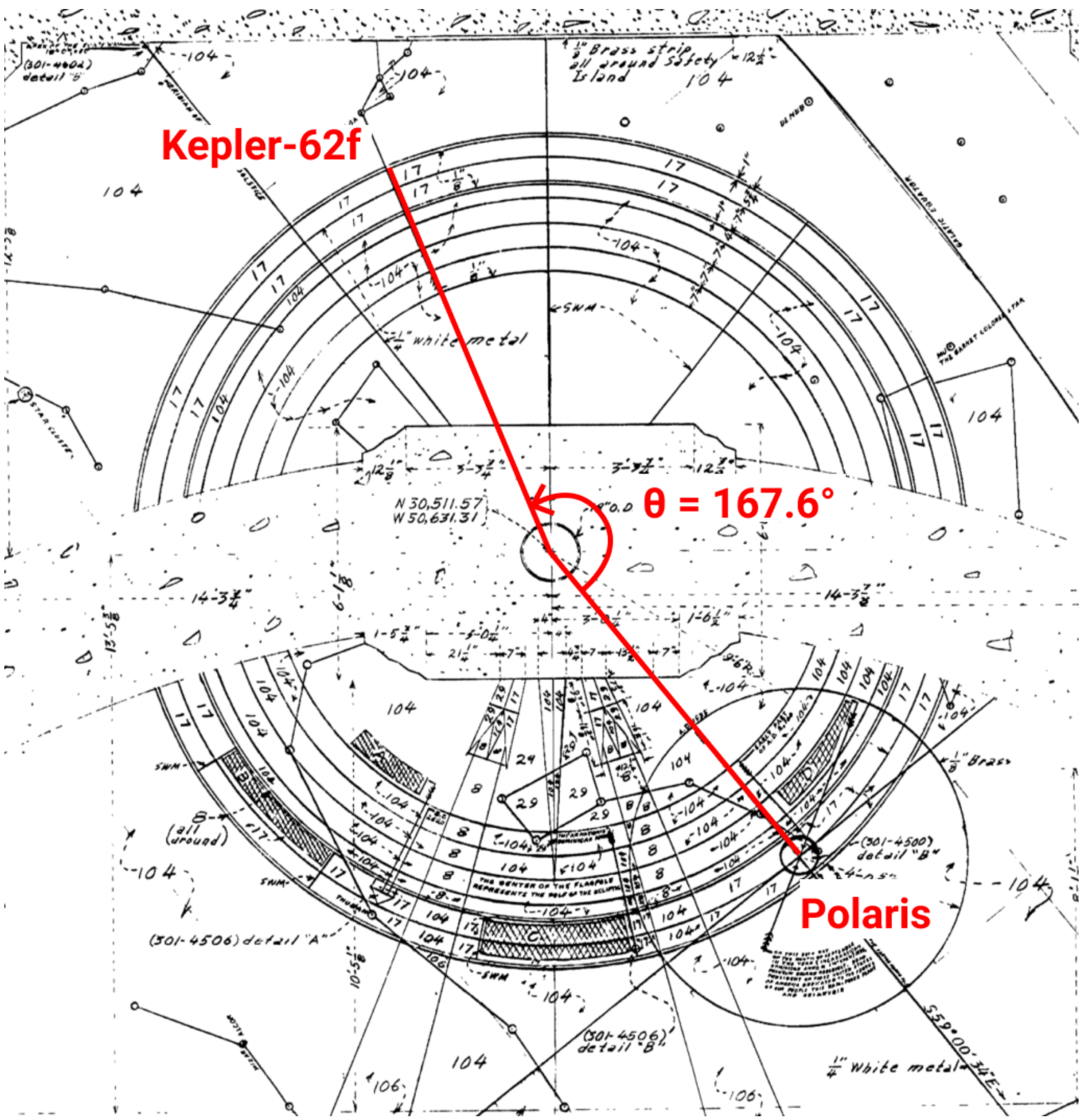

Figure 1: Overhead architectural plan of the Hoover Dam plaza depicting Polaris as north star

For the purpose of this question, assume that the Earth's axial tilt is a constant $i = 23.5°$ and its axis precesses at a constant rate.

a) Using the values on the map, and knowing that the aliens used carbon-aging to determine that the dam is 12,000 years old, find all possible values for the period of Earth's axial precession.

b) Using the most optimistic answer (longest period) from part (a), calculate how many arcseconds the Earth's axis precesses each day. Use the period you calculate here in the next two sections.

c) If they hadn't been lucky enough to come across the star map and decided to build a radio interferometer to observe the movement of the celestial pole over the course of 30 days instead, how many kilometers would the baseline of their telescope array have to be, assuming it operated at a 20cm wavelength?

d) As a last resort, to keep Earth's axis fixed, the aliens decide to counter the forces that cause the Earth's precession by building giant nuclear thrusters on the Earth's surface. Assume Earth's pre-

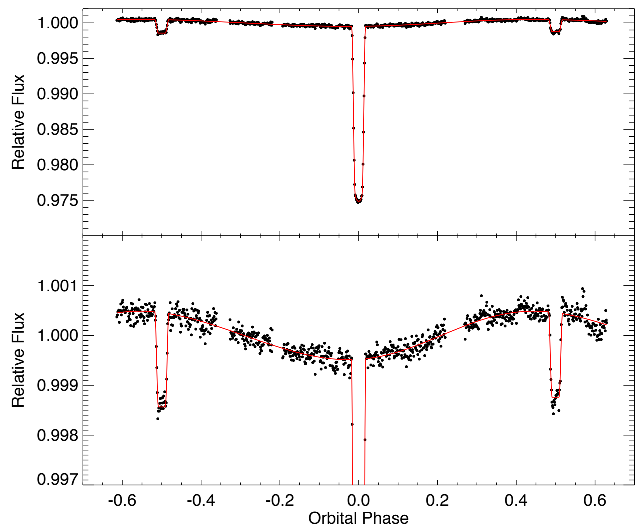

Figure 2: Full-phase light curve of HD 189733b. From Knutson et al. (2012).

cession is caused by external forces alone and calculate the average force (in kN) that a strategically positioned thruster would have to exert to counter them.

Solution:

(a) We know that $P_p = \frac{360^\circ}{\omega} = \frac{360t}{\theta_{total}}$ where $t = 12,000\,years$ and $\theta_{total} = 167.6^\circ$ or $360^\circ + 167.6^\circ$ or ... which gives: $P_p = 25775.6\,years$ or $P_p = 8188\,years$ or $P_p = 4867\,years$ etc.

(b) The axis is in the direction of the angular momentum vector, which traverses a path on a small circle of the celestial sphere, traveling a total of $2\pi \sin i$ radians in each period. So the axis moves $\frac{2\pi \sin i}{25775.6\times 365} \approx 0.05''$ per day.

(c) $R = \frac{\lambda}{D}$ so $D = \frac{0.2\times 3600\times 180}{0.05\times 30\times \pi} \approx 25$ km

(d) $\vec{\tau} = \vec{r} \times \vec{F}$ so $\left\|\vec{F}\right\| = \frac{\|\vec{\tau}\|}{R} = \frac{\left\|\frac{d\vec{L}}{dt}\right\|}{R} = \frac{\left\|\vec{L}\right\| \frac{d\theta}{dt}}{R} = \frac{I\|\vec{\omega}\| \frac{d\theta}{dt}}{R} = \frac{8.01\times 10^{37} \times \frac{2\pi}{24\times 3600} \times \frac{0.055''}{3600} \times \frac{1}{24\times 3600} \frac{\pi}{180}}{6.37\times 10^{6}} \approx$ 2.82×10^{12} kN

6. (15 points) Figure 2 shows a full-phase light curve ("phase curve") of the exoplanet HD 189733b taken by the Spitzer space telescope. Use this figure to answer the following questions. The star HD 189733 has an effective temperature of 4785 K and a radius of 0.805 Solar radii.

a) Use the depth of the planet's transit to estimate the radius of HD 189733b, in Jupiter radii.

b) Use the depth of the eclipse of the planet by the host star to estimate the ratio of the flux of the planet HD 189733b to that of the host star HD 189733.

c) HD 189733b is so close-in to its host star that it is expected to be tidally locked. Use the phase curve to estimate the ratio of the dayside flux emitted by the planet to the nightside flux emitted by the planet.

d) This phase curve also noticeably has a phase curve offset, that is, the maximum in planet and star flux does not occur exactly at secondary eclipse. What process that occurs in a planetary atmosphere could cause such a phase curve offset?

Solution: a) The transit depth (reading off top plot) is ≈ 0.025. Know that transit depth $T_d \propto (R_p/R_s)^2 \rightarrow R_p = R_s\sqrt{T_d} = 0.805 * 6.95 \times 10^8$ m/Solar radius$\sqrt{0.025}/6.91 \times 10^7$ m/Jupiter radius = 1.28 Jupiter radii.
b) Eclipse depth is ≈ 0.002 in normalized flux. The bottom of the eclipse occurs at a normalized flux of 0.9985. $F_p/F_s \approx 0.002/0.9985 \approx 2 \times 10^{-3}$.
c) Max normalized flux $\approx 1.0005 \rightarrow$ max normalized planet flux $\approx 1.0005 - 0.9985 = 2 \times 10^{-3}$. Min normalized flux $\approx 0.9996 \rightarrow$ min normalized planet flux $\approx 0.9996 - 0.9985 = 1.1 \times 10^{-3}$. Ratio of max to min planet flux= $2 \times 10^{-3}/1.1 \times 10^{-3} = 1.8$.
d) Atmospheric circulation that is driven by the large dayside-to-nightside pressure gradient due to tidal locking.

7. (15 points)

a) **Mass-Radius Relation** Stellar physics often involves guessing the equation of state for stars, which is typically a relation between the pressure P and the density ρ. A family of such guesses are known as polytopes and go as follows-

$$P = K \rho^{\gamma} \tag{1}$$

where K is a constant and the exponent γ is fixed to match a certain pressure and core temperature of a star. Given this, show that one can obtain a crude power-law scaling between the mass M of a polytopic star and its radius R of the form $M \propto R^{\alpha}$. Find the exponent α for polytopic stars (justify all steps in your argument). Also, indicate the exponent γ for which the mass is independent of the radius R. Bonus: Why is this case interesting?

b) **Black Holes as Blackbodies** The mass radius relation for ideal non-rotating, uncharged black holes is known from relativity to be

$$R = \frac{2GM}{c^2} \tag{2}$$

Moreover, Stephen Hawking showed that a black hole behaves like a blackbody, where its temperature (known as the Hawking temperature) is given by

$$T = \frac{\hbar c^3}{8\pi k_B GM} \tag{3}$$

Given this information, show that the lifetime of a black hole (justify this phrase!) t^* scales with its mass M as

$$t^* \propto M^{\beta} \tag{4}$$

where you should find the exponent β

c) **Minimal Black Holes** Using the information of the previous part, and Wien's displacement law, estimate the smallest possible mass of a black hole. State any possible flaws with this estimate.

Solution:

(a) To maintain hydrostatic equilibrium in a star, we need the pressure to balance the gravitational pull. This translates to the radial force balance equation

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \tag{5}$$

(Consider the forces on a thin shell of gas in the stellar interior to obtain this). Now to obtain a crude estimate, we do the following

$$\frac{dP}{dr} \approx \frac{\Delta P}{\Delta R} = \frac{-P}{R} \approx -\frac{GM\rho}{R^2} \tag{6}$$

where P is the pressure in the core of the star and R is the radius of the star. Thus, we obtain the scaling

$$P \sim \frac{M\rho}{R} \sim \rho^2 R^2 \tag{7}$$

Using (1), we obtain then

$$\rho \sim R^{\frac{2}{\gamma-2}} \tag{8}$$

Thus, we finally arrive at

$$M \sim R^{\frac{3\gamma-4}{\gamma-2}} \tag{9}$$

$$\implies \alpha = \frac{3\gamma - 4}{\gamma - 2} \tag{10}$$

Thus, if $\gamma = 4/3$, the mass of the polytopic star is independent of the radius R. This is an interesting case, since this has connections to the equation of state for a relativistic degenerate fermion gas, which is used to model white dwarves. This independence of the mass and radius can be thought of as a crude way to understand the Chandrashekar limit.

(b) We have that

$$R = \frac{2GM}{c^2} \tag{11}$$

$$T = \frac{\hbar c^3}{8\pi k_B GM} \tag{12}$$

Using the assumption that a black hole is a blackbody, we apply Stefan's law to find the power of radiation emitted by a black hole

$$\frac{dE}{dt} = 4\pi R^2 \sigma T^4 = \frac{\hbar^4 c^8 \sigma}{256\pi^3 G^2 k_B^4} \frac{1}{M^2} \tag{13}$$

Since this energy has to arise from somewhere, we take this to emerge from the mass-energy of a black hole, giving us

$$\frac{dE}{dt} = -c^2 \frac{dM}{dt} \propto \frac{1}{M^2} \tag{14}$$

Thus, we get

$$M^2 dM = -K dt \tag{15}$$

where K is a constant of proportionality involving the fundamental constants and σ. Integrating the above equation, we see that the blackhole loses mass and its lifetime scales with its mass as follows

$$t^* \propto M^3 \tag{16}$$

$$\implies \beta = 3 \tag{17}$$

(c) Under the assumption that a black hole is a blackbody, we are justified in thinking that its spectrum has the characteristic Planckian spectrum with the Wien's law peak for the emitted photon being given by:

$$\lambda = \frac{b}{T}, \qquad b \;=\; 2.9 \times 10^{-3}\text{m K} \tag{18}$$

If we estimate that one photon (or any $\mathcal{O}(1)$ photons, as required to preserve momentum) having this wavelength carries away the entire mass energy of the black hole, we estimate that

$$Mc^2 \approx \frac{2\pi\hbar c}{\lambda} \tag{19}$$

$$\implies M^2 \approx \left(\frac{\hbar c}{2\sqrt{Gk_B b}}\right)^2 \tag{20}$$

$$M \approx \left(\frac{\hbar c}{2\sqrt{Gk_B b}}\right) \approx 9.68 \times 10^{-9} \text{ kg} \tag{21}$$

$$M \approx 4.84 \times 10^{-39} M_{\odot} \tag{22}$$

There are several problems with this method of estimation. Firstly radiation is a fully quantum process, so our assumption of radiating a single photon is not really correct. Moreover, we use Wien's law in its wavelength form, to obtain energy but we really need it in the frequency form to actually obtain the peak of the energy of the photon spectrum. This is an order of magnitude estimation, so any reasonably justified argument should be given full credit.

8. (15 points) In a rather weird universe, the gravitational constant G varies as a function of the scale factor $a(t)$.

$$G = G_0 f(a) \tag{23}$$

Consider the model $f(a) = e^{b(a-1)}$ where $b = 2.09$.

a) Assuming that the universe is flat, dark energy is absent, and the only constituent is matter, estimate the present age of this weird universe according to this model. Assume that the Friedmann equation:

$$H(a)^2 = H_0^2 \;(\Omega_m + \Omega_r + \Omega_k + \Omega_\Lambda) \tag{24}$$

still holds in this setting.

b) What is the behaviour of the age of the universe t as the scale factor $a(t) \to \infty$?

Note that all parameters with subscript $_0$ indicate their present value. Take the value of Hubble's constant as $H_0 = 67.8$ kms^{-1}Mpc^{-1}

Hint: You might need the following integrals

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \qquad\qquad \int_0^1 x^2 e^{-x^2} dx \approx 0.189471 \tag{25}$$

Solution:

(a) We have from the Friedmann equations that in such a matter-only universe,

$$H(a)^2 = H_0^2 \;\Omega_m \tag{26}$$

where

$$\Omega_m = \frac{\rho_m}{\rho_c} \tag{27}$$

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{28}$$

Using eqs. (26) to (28) and the relation $\rho_m = \rho_{m_0} a^{-3}$ we infer that

$$\Omega_m = \Omega_{m_0} f(a) a^{-3} = f(a) a^{-3} \tag{29}$$

where for a matter-only universe $\Omega_{m_0} = 1$. Using now the relation $H(t) = \frac{\dot{a}}{a}$, we obtain that

$$t = \int_0^t dt' = \int_0^{a(t)} \frac{dt'}{da'} da' = \int_0^{a(t)} \frac{a'}{\dot{a}'a'} da' = \int_0^{a(t)} \frac{da'}{H(a')a'} \tag{30}$$

$$t = \frac{1}{H_0} \int_0^{a(t)} \frac{da'}{a'\sqrt{f(a')a'^{(-3)}}} \tag{31}$$

Using $f(a) = e^{b(a-1)}$, and the substitution $x = \sqrt{a'b/2}$ simplifies this to

$$t = \frac{4\sqrt{2}e^{b/2}}{b^{3/2}H_0} \int_0^{\sqrt{a(t)b/2}} dx\ x^2 e^{-x^2} \tag{32}$$

Using the given value of H_0 and the integrals in the hint, we arrive at the present age ($a(t) = 1$) of such an obscure universe to be

$$t \approx 15.0 \text{ billion years} \tag{33}$$

which is surprisingly close to the current estimate of our universe's age.

(b) What is even more surprising in this model, is that if one were to set $a(t) = \infty$ one would find that the integral above is finite, as shown in the hint.

$$t = \frac{\sqrt{2\pi}e^{b/2}b^{-3/2}}{H_0} \tag{34}$$

The value $b = 2.09$ gives us $t \approx 34.1$ billion years. Thus, the scale factor blows up in a finite amount of time, which is a weird feature of this model given this simplistic treatment.

(The above model appeared in the paper *Varying-G Cosmology with Type Ia Supernovae* by Rutger Dungan and Harrison B. Prosper (arXiv:0909.5416v2) where the authors showed that the type Ia supernovae data alone wasn't sufficient to conclude the existence of dark energy. The value $b = 2.09$ was obtained by fitting the experimental luminosity distance to the theoretical luminosity distance obtained in the model.)

9. (15 points)

a) Find the shortest distance from Boston ($42.3601^\circ\ N,\ 71.0589^\circ\ W$) to Beijing ($39.9042^\circ\ N,\ 116.4074^\circ\ E$)traveling along the Earth's surface. Assume that the Earth is a uniform sphere of radius 6371 km.

b) What fraction of the path lies within the Arctic circle (north of $66.5608^\circ\ N$)?

Solution:

(a) We choose a coordinate system with Boston lying on the x-axis and the axis of the Earth's rotation corresponding to the z-axis. The two cities are separated by $360 - (116.4074 + 71.0589) = 172.5337^\circ$ of longitude. Now, in spherical coordinates, we get that (θ, ϕ) is given by $(90 - 42.3601, 0) = (47.6399^\circ, 0^\circ)$ for Boston and by $(90 - 39.9042, 172.5337) = (50.0958^\circ, 172.5337^\circ)$ for Beijing. Converting to cartesian coordinates on the unit sphere (we multiply by the radius later to get the actual distance), we get that $(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ are given by $\vec{r_1} = (0.738925, 0, 0.673788)$ and $\vec{r_2} = (-0.760614, 0.0996817, 0.641506)$. The cosine of the angle θ between them is given by $\cos(\theta) = \vec{r_1} \cdot \vec{r_2} = -0.129798 \implies \theta = 97.458^\circ = 1.701$. Noting that the distance is given by $D = R_{Earth}\theta$ when θ is measured in radians, we get $D = 10840$ km.

(b) The equation of the plane passing through the origin and the two cities is given by $(\vec{r_1} \times \vec{r_2}) \cdot \vec{x} = 0$. Now, solving the equations $-0.0671643x - 0.986517y + 0.0736573z = 0$. Setting $z = \sin(66.5608^\circ) = 0.917483$, we get $0.0671643x + 0.986517y = 0.0675793$ and $x^2 + y^2 = 0.158225$. Writing y in terms of x, we get $x^2 + \frac{(0.0675793 - 0.0671643x)^2}{0.986517^2} - 0.158225 = 0$. Thus, the product of x values that satisfy this equation is -0.152824. Doing the same with the variables switched, we get $y^2 + \frac{(0.0675793 - 0.986517y)^2}{0.0671643^2} - 0.158225 = 0$. The product of y-values is thus given by 0.00394098. Thus, the dot product of the points of contact with the Arctic circles is given by 0.692892. This gives an angle of 46.141°. Therefore, the final ratio is $\frac{46.14}{97.458} = 0.47$.