Problem 3

a) $\tilde{I} = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) + c_4 f(x_4)$

x_1, x_2, x_3, x_4 are roots of $P_4(x)$.

$$P_4(x) = x_4 - \frac{6x^2}{7} + \frac{3}{35} = 0 \rightarrow \text{WOLFRAM-ALPHA}$$

This gives $x = \pm\sqrt{\frac{3}{7} \pm \frac{2\sqrt{6/5}}{7}}$

$\therefore\ x_1 \approx 0.86$, $x_2 \approx 0.34$

$x_3 \approx -0.86$, $x_4 \approx -0.34$

Now, we can calculate c_1, c_2, c_3, c_4:

$$c_1 = \int_{-1}^{1} \left(\frac{x-0.34}{0.86-0.34}\right)\left(\frac{x+0.86}{0.86+0.86}\right)\left(\frac{x+0.34}{0.86+0.34}\right) dx = \frac{1}{36}(18-\sqrt{30})$$

$$c_2 = \int_{-1}^{1} \left(\frac{x-0.86}{0.34-0.86}\right)\left(\frac{x+0.86}{0.34+0.86}\right)\left(\frac{x+0.34}{0.34+0.34}\right) dx = \frac{1}{36}(18+\sqrt{30})$$

$$c_3 = \int_{-1}^{1} \left(\frac{x-0.86}{-0.86-0.86}\right)\left(\frac{x-0.34}{-0.86-0.34}\right)\left(\frac{x+0.34}{-0.86+0.34}\right) dx = \frac{1}{36}(18-\sqrt{30})$$

$$c_4 = \int_{-1}^{1} \left(\frac{x-0.86}{-0.34-0.86}\right)\left(\frac{x-0.34}{-0.86-0.34}\right)\left(\frac{x+0.86}{-0.34+0.86}\right) dx = \frac{1}{36}(18+\sqrt{30})$$

$$\tilde{I} = \frac{1}{36}(18-\sqrt{30})f(0.86) + \frac{1}{36}(18+\sqrt{30})f(0.34)$$
$$+ \frac{1}{36}(18-\sqrt{30})f(-0.86) + \frac{1}{36}(18+\sqrt{30})f(-0.34)$$

This is exact for a polynomial of degree:

$2n-1 = 2(4)-1 = \boxed{7}$

$f(0.86) = -1129.49 \Rightarrow \frac{1}{36}(18-\sqrt{30})f(0.86) = -392.90$

$f(0.34) = 292.72 \Rightarrow \frac{1}{36}(18+\sqrt{30})f(0.34) = 190.90$

$f(-0.86) = -2828.50 \Rightarrow \frac{1}{36}(18-\sqrt{30})f(-0.86) = -983.91$

$f(-0.34) = -387.14 \Rightarrow \frac{1}{36}(18+\sqrt{30})f(-0.34) = -252.47$

$\boxed{\tilde{I} = \int_{-1}^{1} p(x)\,dx = -1438.38} \rightarrow$ WOLFRAM ALPHA.

$\tilde{I} = -392.90 + 190.90 - 983.91 - 252.47$

$\therefore \boxed{\tilde{I} = -1438.38}$

$\therefore \int_{-1}^{1} p(x)\,dx$ agrees with our $n=4$ Gaussian Quadrature expression.

b) $\tilde{I} = \int_0^{\pi} x\cos x \; dx$

Let $t = -\cos x$ [When, $x=0, t=-1$ & $x=\pi, t=1$].

$dt = \sin x \, dx \Rightarrow dx = \dfrac{dt}{\sin(\cos^{-1}(t))}$

$x = \cos^{-1}(t)$

$$\tilde{I} = \int_{-1}^{1} \cos^{-1}(t) \cdot \frac{(-t)\,dt}{\sin(\cos^{-1}(t))}$$

$$= \int_{-1}^{1} \frac{-t\cos^{-1}(t)\,dt}{\sqrt{1-t^2}}$$

$\boxed{\tilde{I} = -2}$

Now, applying Gaussian Quadrature,

For $n = 2$:

$$\tilde{I}_2 = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$
$$= 0.6755 - 1.5459$$
$$\boxed{\tilde{I}_2 = -0.8704}$$

$$\varepsilon_2 = \left|\frac{\tilde{I}_2 - \tilde{I}}{\hat{I}}\right|$$
$$= \left|\frac{-0.8704 + 2}{-2}\right|$$
$$\boxed{\varepsilon_2 = 0.5648}$$

For $n = 3$:

$$\tilde{I}_3 = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$
$$= 0.4659 + 0 - 1.6717$$
$$\boxed{\tilde{I}_3 = -1.2058}$$

$$\varepsilon_3 = \left|\frac{\tilde{I}_3 - \hat{I}}{\hat{\tilde{I}}}\right|$$
$$= \left|\frac{-1.2058 + 2}{-2}\right|$$
$$\boxed{\varepsilon_3 = 0.3971}$$

For $n = 4$:

$$\tilde{I}_4 = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) + c_4 f(x_4)$$
$$= c_1(-4.4182) + c_2(-0.6933) + c_3(0.9034) + c_4(0.4425)$$
$$= -1.5369 - 0.4521 + 0.3143 + 0.2886$$
$$\boxed{\tilde{I}_4 = -1.3862}$$

$$\varepsilon_4 = \left|\frac{\hat{I}_4 - \tilde{I}}{\tilde{I}}\right|$$
$$= \left|\frac{-1.3862 + 2}{-2}\right|$$
$$\boxed{\varepsilon_4 = 0.3069}$$

$\therefore$ Relative Error decreases as n increases!