Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. MOTION OF A SPINNING TOP IN A GRAVITATIONAL FIELD

A spinning top position is completely determined by three angles (the Euler angles) as shown in Fig. 1.

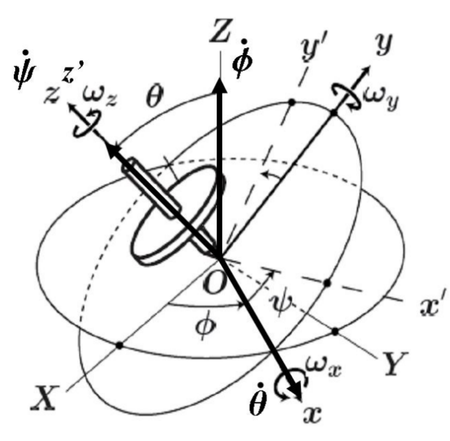

FIG. 1. Geometry of a spinning top. Any rigid body has 3 principal axes about which it can rotate. These are denoted by x, y, z in the figure. The Euler angles θ, ϕ and ψ that determine the position of the top are depicted. The angle ψ is measuring rotation about the axis of the top (the z axis) and is measured from the x principal axis of the top, which is chosen to always lies in the $X-Y$ plane of the coordinate system. The angle θ measures rotation about the x principal axis of top and is measured from the coordinate system Z axis (this angle is called nutation). The ϕ angle is the angle between the the X axis of the coordinate system and the principal x axis of the top.

If the length of the top is L and we know the Euler angles θ and ϕ we can always find the location of the tip of the top using the standard spherical polar coordinate expressions

$$x = L\sin\theta\cos\phi, \quad y = L\sin\theta\sin\phi, \quad z = L\cos\theta. \tag{1}$$

These expressions will be useful in generating plots of the motion of the top or plotting the orbit of the tip.

Using Newtonian/Lagrangian or Hamiltonian mechanics you can show that for a top spinning on the surface of a table with gravity pointing downwards, the equations of motion that determine the angles θ, ϕ, ψ are given by (when $\theta \neq 0$)

$$\begin{aligned}
\frac{d\phi}{dt} =& \frac{p_\phi - p_\psi\cos\theta}{I_0\sin^2\theta}\\
\frac{d\psi}{dt} =& \frac{p_\psi}{I} - \frac{p_\phi\cos\theta - p_\psi\cos^2\theta}{I_0\sin^2\theta}\\
\frac{d^2\theta}{dt^2} =& \frac{\big(2\cos(\theta)(p_\phi/I_0)^2 + 2\cos(\theta)(p_\psi/I_0)^2 - (\cos(2\theta)+3)(p_\phi/I_0)(p_\psi/I_0)\big)}{2\sin^3(\theta)} + \frac{gMz_G\sin(\theta)}{I_0}
\end{aligned} \tag{2}$$

where I is the moment of inertia about the axis of the top, I_0 the moment of inertia along the other principal axes of the top (which are normal to the symmetry axis of the top), M is the mass of the top, g is the gravitational acceleration, z_G the location of the center of mass of the top, and p_ϕ, p_ψ are constants that are integrals of the motion related to derivatives of the angles and the angles as follows

$$\begin{aligned}
p_\phi =& I\cos\theta\frac{d\psi}{dt} + (I_0\sin^2\theta + I\cos^2\theta)\frac{d\phi}{dt}\\
p_\psi =& I\frac{d\psi}{dt} + I\cos^2\theta\frac{d\phi}{dt}.
\end{aligned} \tag{3}$$

Thus, the entire evolution is determined once the evolution for θ is found.

When $\theta = 0$ and $d\theta/dt = 0$, i.e., the spinning top is spinning along the z-axis (where gravity points), then Eq. (3) becomes

$$\begin{aligned}
p_\phi =& I\frac{d\psi}{dt} + I\frac{d\phi}{dt}\\
p_\psi =& I\frac{d\psi}{dt} + I\frac{d\phi}{dt}.
\end{aligned} \tag{4}$$

Thus, $p_\phi = p_\psi = $ const., and since $\phi = $ const. for $\theta = 0$, we must have $\frac{d\psi}{dt} = p_\psi = $ const.. In other words the top is spinning with constant angular frequency.

We will be focusing on cases with $\theta \neq 0$ from now on. This system has an integral of the motion which is the total energy, i.e.,

$$E_{\rm tot} = \frac{1}{2}I_0\left[\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\phi}{dt}\right)^2\sin^2\theta\right] + \frac{1}{2}I(\frac{d\psi}{dt} + \frac{d\phi}{dt}\cos\theta)^2 + mgz_G\cos\theta. \tag{5}$$

The fact that $E_{\rm tot}$ is conserved will serve as a useful diagnostic to validate your numerical integration of this system.

1. First let us focus on a system with $\theta = $ const. $\neq 0$, and $d\theta/dt = 0$. Then Eq. (2) becomes

$$\begin{aligned}
\frac{d\phi}{dt} =& \frac{p_\phi - p_\psi\cos\theta}{I_0\sin^2\theta}\\
\frac{d\psi}{dt} =& \frac{p_\psi}{I} - \frac{p_\phi\cos\theta - p_\psi\cos^2\theta}{I_0\sin^2\theta}
\end{aligned} \tag{6}$$

This system means that $d\phi/dt = $ const. and $d\psi/dt = $ const.. However, the two angular frequencies are not independent. The last of Eq. (2) implies

$$\frac{\Big(2\cos(\theta)p_\phi^2 + 2\cos(\theta)p_\psi^2 - (\cos(2\theta)+3)p_\phi p_\psi\Big)}{2\sin^3(\theta)I_0^2} + \frac{I_0 gMz_G\sin(\theta)}{I_0^2} = 0 \tag{7}$$

After, substituting Eq. (3) in the last equation we arrive at

$$(I_0 - I)\cos\theta \left(\frac{d\phi}{dt}\right)^2 - I\frac{d\phi}{dt}\frac{d\psi}{dt} + Mgz_G = 0 \tag{8}$$

This is a quadratic equation for $\frac{d\phi}{dt}$. While it is possible to solve we will make one approximation, that $d\psi/dt \gg d\phi/dt$ as is usually with tops and gyroscopes. This way we can drop the $\left(\frac{d\phi}{dt}\right)^2$ term, and the equation becomes

$$\frac{d\phi}{dt} = \frac{Mgz_G}{I\frac{d\psi}{dt}}. \tag{9}$$

Then, for this last equation introduce a dimensionless time (call it $\tilde{t}$), and integrate numerically the resulting dimensionless equations adopting the RK4 algorithm. This problem is trivial because the solution is analytic. Ensure your numerical solution reproduces the analytic solution. Using Eqs. (1) and setting $\theta = \pi/6$, $L = 1$ along with your solution for ϕ, make a 3D plot of the orbit of the tip of the top.

2. Next focus on nutating and precessing systems. This will require solution of the full system Eq. (2). Notice that gMz_G/I_0 has dimensions of $1/\text{time}^2$. Based on this introduce a dimensionless time (call it $\tilde{t}$), and corresponding dimensionless constants $\tilde{p}_\phi/I_0$ and $\tilde{p}_\psi/I_0$.

   To solve the resulting equation using RK4 you will first need to cast it to first-order form. You will then need initial conditions for ϕ, ψ, θ and $d\theta/dt$. Choose $\phi(0) = 0$ $\psi(0) = 0$, $\theta(0) = \pi/6$, $d\theta/d\tilde{t} = 0$. Notice that apart from initial conditions this system requires the we set the constants $\tilde{p}_\psi = \sqrt{\frac{I_0}{gMz_G}}\frac{p_\psi}{I_0}$, $\tilde{p}_\phi = \sqrt{\frac{I_0}{gMz_G}}\frac{p_\phi}{I_0}$. Experiment with $\tilde{p}_\psi = 5$, $\tilde{p}_\phi = 3.5$. You might want to experiment with other values, too. You will also need to relate I to to I_0. Set $I = I_0/2$.

   Cast the equation for the total energy (5) also to dimensionless form, using the new dimensionless time you will introduce, and also normalizing by mgz_G.

   Use RK4, to integrate numerically the non-dimensional system of ODEs for sufficiently long times. You will need to use the solution for ϕ and θ to plot the tip of the top assuming unit length $L = 1$ and determine if the orbit is closed. Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes. Use $\delta E = |(E_{\text{tot}} - E_{\text{tot}}(t = 0))/E_{\text{tot}}(t = 0)|$ to determine the accuracy. If δE is smaller than 10^{-3} for all integration times, then you have decent accuracy.

3. **Convergence:** Demonstrate that as you decrease the step size h, $\delta E(\tilde{t} = 10)$ converges to 0. Make a plot to determine the order of convergence of δE, and discuss why or why not this matches your expectation.

4. **Self-convergence**: Use a number of step sizes and make a plot to demonstrate that the code solution for ϕ and θ at $\tilde{t} = 10$ self-converges.

5. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for the solution for $\theta_1(100)$.