Physics 305 – Computational Physics, Fall 2020
Homework Set 4
Due Date: Monday, October 5

1. **Higher-order ODEs [9 points]**

   Convert the following higher-order ODEs to systems to first-order ODEs:

   - $\frac{d^2y}{dt^2} - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{y} = 0$
   - $\frac{1}{t^2}\frac{d}{dt}\left(t^2\frac{dy}{dt}\right) + y = 0$
   - $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - t = 0$

   You don't need to write code for this problem, but explain your work.

2. **Self-convergence [15 points]**

   Not all problems have an exact solution to compare with. In these cases *self-convergence* must be demonstrated. The concept is simple: find the solution at multiple numerical resolutions, use the highest numerical resolution as the "reference" solution and demonstrate that the lower-resolution solutions converge to this reference solution. This can be done as follows: Let us consider that we have a solution that converges at order n, i.e.,

   $$y_{\mathrm{num}}(h) = y_{\mathrm{exact}} + A * h^n, \tag{1}$$

   where $y_{\mathrm{num}}(h)$ is the numerical solution at resolution h, y_{exact} the exact solution, and A some constant. Consider that we have the solution at different resolutions $h = h_1, h_2, h_3$, with $h_1 > h_2 > h_3$. The absolute error of the numerical solution then is

   $$y_{\mathrm{num}}(h_1) - y_{\mathrm{exact}} = A * h_1^n \tag{2}$$

   $$y_{\mathrm{num}}(h_2) - y_{\mathrm{exact}} = A * h_2^n \tag{3}$$

   $$y_{\mathrm{num}}(h_3) - y_{\mathrm{exact}} = A * h_3^n \tag{4}$$

   If we subtract Eq. (**??**) from Eqs. (**??**) and (**??**) by Eq. (**??**) we obtain

   $$y_{\mathrm{num}}(h_1) - y_{\mathrm{num}}(h_3) = A * (h_1^n - h_3^n), \tag{5}$$

   $$y_{\mathrm{num}}(h_2) - y_{\mathrm{num}}(h_3) = A * (h_2^n - h_3^n). \tag{6}$$

   If we divide Eqs. (**??**) and Eq. (**??**) we finally arrive at

   $$\frac{y_{\mathrm{num}}(h_1) - y_{\mathrm{num}}(h_3)}{y_{\mathrm{num}}(h_2) - y_{\mathrm{num}}(h_3)} = \frac{h_1^n - h_3^n}{h_2^n - h_3^n} \tag{7}$$

   In addition, let's consider that $h_2 = 2h_3$, and that $h_1 = h$.

   $$\frac{y_{\mathrm{num}}(h) - y_{\mathrm{num}}(h_3)}{y_{\mathrm{num}}(h_2) - y_{\mathrm{num}}(h_3)} = \frac{(h/h_3)^n - 1}{2^n - 1}. \tag{8}$$

   In this problem you'll use Eq.**??** to show self-convergence:

- Solve the following initial value problem

$$\frac{dy}{dt} = -2ty, 0 \leq t \leq 3, \; y(0) = 1. \tag{9}$$

using the RK3 and RK4 implementations developed in class. [5 points]

- Now set h_2 to the step size of the solution corresponding to $N = 800$, and h_3 to the step size corresponding to $N = 1600$. Then consider the step size h corresponding to $N = 100, 200, 300, 400, 500, 600$, and on a plot show the curves plot $\frac{y_{\text{num}}(h) - y_{\text{num}}(h_3)}{y_{\text{num}}(h_2) - y_{\text{num}}(h_3)}$ vs h and $\frac{(h/h_3)^n - 1}{2^n - 1}$ vs h. Use this plot to argue that your code is self-convergent. [10 points]

3. **The damped, driven pendulum [25 points]** We now extend the example of the damped pendulum from Lecture 14 to a damped, driven pendulum by applying a periodic driving torque of amplitude A:

$$\ddot{\theta} = -\sin\theta - \frac{1}{Q}\dot{\theta} - A\cos\omega_D t\,, \tag{10}$$

note that the definition of Q here is inverse to Lecture 14 - apologies!

I posted a note on D2L (with this assignment) that describes the behavior of the damped, driven pendulum. In this exercise, you will write code to study some of its properties.

- Adapt the code developed in Lecture 14 to simulate the damped, driven pendulum and produce a phase plot of the angular velocity ω vs. the pendulum angle θ for initial conditions $\theta_0 = 0$, $\omega_0 = 1$, and parameter values $A = 1.5$, and $Q = 1.2$ (assume $\omega_D = 2/3$).

  In the following steps, use a driving frequency of $\omega_D = 2/3$ and a driving amplitude $A = 1.5$, and vary the quality factor Q.

- Plot the angle of the pendulum and its angular velocity against time for a value of Q which leads to a simple, period-1 trajectory.
- Plot the phase diagram of this orbit with the Poincaré section superposed (i.e., use small dots for the trajectory and plot a larger dot of a different color once per drive period $2\pi/\omega_D$). Integrate the system for a long enough time before plotting so that the initial conditions are not seen in the phase plot.
- Repeat the same plot for a value of Q where spatial symmetry-breaking takes place, but the trajectory is still period-1.
- Increase Q to a value where the period of the pendulum has doubled, and plot both the time series and the phase plot/Poincaré section.
- For $Q = 2.1$, plot only the Poincaré section. Plot points for at least 1000 orbits so that the fractal nature of the section becomes apparent.