Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. THE CATENARY PROBLEM: SHAPE OF A ROPE-BRIDGE FIXED AT TWO END POINTS IN A CONSTANT GRAVITATIONAL POTENTIAL

The catenary is the curved configuration $z = z(x)$ of a uniform inextensible rope with two fixed endpoints at rest in a constant gravitational This is the curve that minimizes the gravitational potential energy given by

$$W = \rho g \int_{x_1}^{x_2} z ds = \rho g \int_{x_1}^{x_2} z\sqrt{dx^2 + dz^2} = \rho g \int_{x_1}^{x_2} z\sqrt{1 + \left(\frac{dz}{dx}\right)^2} dx, \tag{1}$$

where ρ is the linear mass density of the rope, and g the gravitational acceleration. This is a classic problem for the calculus of variations. One important aspect of the problem is that the length of the rope is going to be fixed. Thus, we must use the method of Lagrange multipliers to add constraints to the system. If we call ℓ the length of the rope, the constraint here is $\int ds = \ell$. Then, the functional with the constraint becomes

$$(W/\rho g) - \lambda(\int ds - \ell) = \int_{x_1}^{x_2} (z-\lambda)ds + \lambda\ell = \int_{x_1}^{x_2} (z-\lambda)\sqrt{dx^2 + dz^2} + \lambda\ell = \int_{x_1}^{x_2} (z-\lambda)\sqrt{1 + \left(\frac{dz}{dx}\right)^2} dx + \lambda\ell, \tag{2}$$

where λ is the Lagrange multiplier. Since this is a bit more complicated problem we will drop the constraint and assume that the rope is "flexible". The equation governing the shape $z(x)$ is determined by the Euler-Lagrange equations:

$$\frac{\partial L}{\partial z} - \frac{d}{dx}\left(\frac{\partial L}{\partial \frac{dz}{dx}}\right) = 0 \tag{3}$$

with Lagrangian

$$L = z\sqrt{1 + \left(\frac{dz}{dx}\right)^2} = z\sqrt{1 + z'^2}, \tag{4}$$

where $z' = dz/dx$. If we plug the last equation into Eq. (3) we obtain

$$\sqrt{1 + z'^2} - \frac{d}{dx}\left(\frac{zz'}{(1 + z'^2)^{1/2}}\right) = \frac{1 + z'^2 - zz''}{(1 + z'^2)^{3/2}} = 0 \Rightarrow \tag{5}$$

$$\frac{d^2 z}{dx^2} = \frac{1}{z}\left(\frac{dz}{dx}\right)^2 + \frac{1}{z} \tag{6}$$

This second order non-linear ordinary differential equation is essentially a boundary value problem between $x = 0$ and $x = b$ with $z(0)$ and $z(b)$ known. Is it always possible to rescale the x coordinate, such that the upper bound $b = 1$? To solve Eq. (6) first it has to be reduced to a system of first order ODEs. Show your work in the term paper.

Equation (6) also has an integral of the motion because the Lagrangian does not depend on x. This is given by

$$C = \frac{z}{\sqrt{1 + z'^2}}. \tag{7}$$

Do the following:

1. Use RK4 to integrate numerically the dimensionless Eq. (6) from $x = 0$ to $x = 1$ with $z(0)$ and $z(1)$ as described below. Note, that although a boundary value problem you can cast it into an initial value problem by using the value $z(0)$, and using trial and error for the initial derivative $dz/dx|_{x=0}$ using the shooting method we discussed in class. Show plots of your solutions for the following cases.

   - Consider the end points at the same height: $z(0) = 1$, $z(1) = 1$ and separately $z(0) = 3$, $z(1) = 3$. Show a plot of containing both solutions. Is the shape in the larger height case the same as that in the lower height but simply shifted?
   - Consider the left point at larger height: $z(0) = 10$, $z(1) = 6$, and separately $z(0) = 5$, $z(1) = 1$. Show a plot of containing both solutions. Is the shape in the larger heights case the same as that in the lower heights but simply shifted?
   - Consider the right point at larger height: $z(0) = 6$, $z(1) = 10$, and compare it to the $z(0) = 10$, $z(1) = 6$ solution. Is the solution symmetric with respect to the $x = 0.5$ line? Show this in a plot. If yes, can you understand why based on the governing differential equation?

   It used to be thought that the shape would be a parabola. Does the shape look like a parabola in any of the cases above?

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes and use the solution z at the final x (for a fixed choice of $dz/dx|_{x=0}$) and choose the step size such that z at that x does not change appreciably. You can also use the integral of the motion if $\delta C = |(C(x) - C(0))/C(0)|$ is appreciably small, say within 10^{-3} or less, then you have decent accuracy.

2. **Convergence**: Use a number of step sizes and for a specific choice of initial and final z as well as $dz/dx|_{x=0}$, make a plot to demonstrate that δC at $x = 1$ converges to 0. Does the order of convergence match your expectation? If not, try to explain why.

3. **Self-convergence**: Use a number of step sizes and for a specific choice of initial and final z as well as $dz/dx|_{x=0}$, make a plot to demonstrate that the code solution for z at $x = 1$ self-converges. Does the order of convergence match your expectation? If not, try to explain why.

4. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for the solution for $z(x)$ at $x = 0.9$.