Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. DARK MATTER HALOS POTENTIALS - GALACTIC ROTATION CURVES

This project is aimed at studying the properties of dark matter halo potentials and galactic rotation curves. Observations of visible stars and 21 cm observations of galaxies show that the rotational velocity profiles become almost flat one moves away significantly from the galactic center. This is completely counter-intuitive because if the radius is large enough that the entire baryonic content of the galaxy is enclosed, then one would expect the gravitational acceleration to simply be $GM_{\rm gal}/r^2$, where $M_{\rm gal}$ is the galaxy mass. Then for circular orbits we would expect $GM_{\rm gal}/r^2 = v^2/r$ which implies that the velocities profiles should decay as

$$v = \sqrt{\frac{GM_{\rm gal}}{r}}. \tag{1}$$

But, this is not observed. This led to the postulation that there must exist some invisible form of matter past the luminous parts of galaxies that accounts for the fact that the rotational velocity profiles become flat. You will be exploring a few different profiles that yield profiles that look similar to those of galaxies.

Based on Newtonian gravity the Poisson equation governs the accelation that objects experience, i.e., $\nabla^2\phi = -4\pi G\rho$, which in spherical symmetry becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi G\rho. \tag{2}$$

The gravitational accelaration is given by

$$g = -\frac{d\phi}{dr}. \tag{3}$$

Once g is obtained it must equal the centrifugal acceleration $g = v^2/r$, from which the rotational velocity can be obtained

$$v = \sqrt{gr} \tag{4}$$

To integrate numerically Eq. (2) it must first be cast to first-order form. Do so, and show your work in the term paper. You will also need a boundary condition at $r = 0$. The density profiles we will be working will have a finite density ρ at $r = 0$, where it will be maximum. This, implies that $\frac{d\phi}{dr}\big|_{r=0} = 0$, and $\phi(r = 0) = A$, where A is a constant. We are free to choose this constant and will use $A = 1$ here. Strictly speaking the constant should be determined by going far away and finding the total enclosed mass M and making sure that $\phi = M/r$. So, one can start with any initial value for ϕ and then add a constant everywhere so that $\phi = M/r$ at very large r. However, we are only interested in the acceleration here which is given by Eq. (3) and involves a derivative of ϕ. Hence, adding any constant to ϕ will give the same g. To sum the boundary (or initial) conditions are $\frac{d\phi}{dr}\big|_{r=0} = 0$, and $\phi(r = 0) = 1$. You will be treating r as a "time" parameter starting at $r = 0$ and integrating outward.

Now, when casting Eq. (2) to first order form some terms on the right-hand-sides of the equations will involve terms that are multiplied by $1/r$. Thus, at $r = 0$ these terms would become $1/0$ and the code would crash. To start the integration at $r = 0$ you will need the form of the solution near $r = 0$. To find that we can Taylor expand ϕ near $r = 0$ keeiping the initial conditions in mind

$$\phi = 1 + \frac{1}{2}\frac{d^2\phi}{dr^2}|_{r=0} r^2. \tag{5}$$

This implies, $r^2 d\phi/dr = \frac{d^2\phi}{dr^2}|_{r=0} r^3$. Plugging this equation in Eq. (2) for $\rho = \rho_c$ we obtain

$$3\frac{d^2\phi}{dr^2}|_{r=0} = -4\pi G\rho_c \Rightarrow \frac{d^2\phi}{dr^2}|_{r=0} = -\frac{4\pi G\rho_c}{3}. \tag{6}$$

Thus, the solution near $r = 0$ satisfies

$$\phi = 1 - \frac{4\pi G\rho_c}{6} r^2. \tag{7}$$

and

$$\frac{d\phi}{dr} = -\frac{4\pi G\rho_c}{3} r. \tag{8}$$

Using these last two equation is crucial to start the integration at $r = 0$ and avoid blow ups.

1. Implement RK4, to integrate numerically the first-order version of the system of ODEs (2).

2. You will be experimenting with a couple of density profiles. The first one results in analytic solution, and corresponds to a galaxy with a core with a density profile

$$4\pi G\rho = \frac{v_H^2}{r^2 + a_H^2}. \tag{9}$$

   The exact solution of (2) has velocity profiles given by

$$\frac{v(r)}{v_H} = \sqrt{1 - \frac{a_H}{r}\arctan\left(\frac{r}{a_H}\right)}. \tag{10}$$

   Introduce proper dimensionless quantities ($\tilde{r}$, $\tilde{v}$, and $\tilde{g}$) to solve Eq. (2) for any value of v_H and a_H. You will need to integrate sufficiently far enough in dimensionless radius (e.g., $\tilde{r} = 100$). Does this profile match (qualitatively) what is observed in galaxies?

   Make sure your code reproduces the above exact solution for the velocity profile by plotting the numerical solution for all r and overlaying the exact solution in the dimensionless quantities.

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes until you find a range of step sizes for which v at a particular large value of $\tilde{r}$ does not change appreciably.

3. Large scale numerical simulations of dark matter halos by Navarro, Frenk and White (1997) showed that dark matter halos have a universal profile irrespectively of the halo mass which is given by

$$\frac{\rho(r)}{\rho_c} = \frac{\delta_c}{\frac{r}{r_s}\left(1 + \frac{r}{r_s}\right)^2} \tag{11}$$

where the parameters ρ_c, δ_c and r_s are calculated if one knows the redshift where the halo forms. For the next question you will be using a slightly modified version of the NFW halo profile (which has a small core) given by

$$\frac{\rho(r)}{\rho_c} = \frac{\delta_c}{\left(\frac{r}{r_s} + 0.01\right)\left(1 + \frac{r}{r_s}\right)^2} \tag{12}$$

Introduce proper dimensionless quantities ($\tilde{r}$, $\tilde{v}$, and $\tilde{g}$) to solve Eq. (2) for any value of ρ_c, δ_c and r_s. You will need to integrate sufficiently far enough in dimensionless radius (e.g., $\tilde{r} = 100$).

Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes until you find a range of step sizes for which v at a particular large value of $\tilde{r}$ does not change appreciably.

4. Make a plot containing the curves $\tilde{v}$ vs $\tilde{r}$ (y axis log scale, x axis log scale). Does the NFW profile match (qualitatively) what is observed in galaxies? How does it differ from the previous profile? Do you think the NFW profile provides a more physical description of what happens in a dark matter halo?

5. **Convergence**: Use the exact solution we have, pick a particular value of $\tilde{r} = r_1$, and using a range of step sizes show that your numerical solution $\phi(r_1)$ converges to the exact solution. Does the order of self-convergence match your expectation? If not try to explain why this is case.

6. **Self-convergence**: Use the NFW solution, pick a particular value of $\tilde{r} = r_1$, and using a range of step sizes show that your numerical solution $\phi(r_1)$ self-converges to the exact solution. Does the order of self-convergence match your expectation? If not try to explain why this is case.

7. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for the solution for $\phi(r_1)$ at a time of your choosing.