Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. SPHERICAL STARS IN GENERAL RELATIVITY

This project is aimed to study the properties of **equilibrium** (stationary) spherical stars in general relativity. The stellar matter is treated as a perfect fluid (non-ideal effects such as heat conduction, viscosity etc. are neglected).

Based on Einstein's theory of general relativity, the equations describing stellar structure of spherical fluid configurations are referred to as the Tolman-Oppenheimer-Volkoff (TOV) equations after these three people. Adopting geometrized units in which Newton's constant and the speed of light are set to unity, $G = c = 1$) these equations are given by

$$\begin{aligned}
\frac{dm_0}{dr} &= 4\pi r^2 \sqrt{1-\frac{2m}{r}}\rho_0 \\
\frac{dm}{dr} &= 4\pi r^2 \rho \\
\frac{dP}{dr} &= -\frac{\rho m}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^3}{m}\right)\left(1-\frac{2m}{r}\right)^{-1}
\end{aligned} \tag{1}$$

where m_0 is the enclosed rest-mass-energy up to radius r (i.e., the sum of all the masses of the baryons), ρ_0 is the rest-mass (energy) density (i.e., the baryon rest-mass times the number density of baryons) at radius r, m is the enclosed **total** mass-energy up to radius r, ρ is the **total** energy density (sum of rest-mass density ρ_0 and the internal energy density ρ_i) at radius r, and P is the fluid pressure at radius r. Thus, all quantities m_0, m, P, ρ, ρ_0 are functions of the coordinate radius r. To close the system of coupled ODEs (1), you will need to supplement it with an equation of state, i.e., an expression that relates P and ρ to ρ_0. We will consider a piecewise polytropic equation

of state given by

$$P = \begin{cases} k_1 \rho_0^{\Gamma_1} & \rho_0 \leq \rho_1 \\ k_2 \rho_0^{\Gamma_2} & \rho_1 < \rho_0 \leq \rho_2 \\ k_3 \rho_0^{\Gamma_3} & \rho_2 < \rho_0. \end{cases} \tag{2}$$

You will also need an expression that determines ρ_0 from P. Use the last equation to find P_1 and P_2 at $\rho_0 = \rho_1$ and $\rho_0 = \rho_2$, respectively. Solve the last equation with respect to ρ_0 as a function of P and cast it in a piecewise form using the transition pressures P_1 and P_2. This will give you an equation $\rho_0 = \rho_0(P)$. Show your work in the report.

Using the first law of thermodynamics it can be shown shown that the internal energy density ρ_i is given by

$$\frac{\rho_i}{\rho_0} = \begin{cases} n_1 k_1 \rho_0^{1/n_1} & \rho_0 \leq \rho_1 \\ n_2 k_2 \rho_0^{1/n_2} + c_2 & \rho_1 < \rho_0 \leq \rho_2 \\ n_3 k_3 \rho_0^{1/n_3} + c_3 & \rho_2 < \rho_0, \end{cases} \tag{3}$$

where $c_2 = (n_1 - n_2) k_1 \rho_1^{1/n_1}$ and $c_3 = c_2 + (n_2 - n_3) k_2 \rho_2^{1/n_2}$. and $\Gamma_i = 1 + 1/n_i,\ i = 1, 2, 3$. Having the internal energy density ρ_i and the rest-mass energy density ρ_0 we can determine the total energy density as

$$\rho = \rho_i + \rho_0. \tag{4}$$

You will be using the following parameters to integrate the TOV equations using the above equation of state: $k_3 = 4458.0491$, $n_1 = 2$, $n_2 = 2.96736$, $n_3 = 0.716$, $\log(\rho_1/\rho_{\rm nuc}) = -6.39356$, $\log(\rho_2/\rho_{\rm nuc}) = 0.208502$, where $\rho_{\rm nuc} = 1.48494 \times 10^{-4} {\rm km}^{-2}$ is the nuclear saturation density (in geometrized units) and k_3 is in geometrized units with the unit of length being 1 km. The constants k_1, k_2 are determined by demanding continuity of the pressure, at the transition densities, i.e., continuity for $P(\rho_1)$ and $P(\rho_2)$, in other words

$$k_1 \rho_1^{\Gamma_1} = k_2 \rho_1^{\Gamma_2}, \text{ and } k_2 \rho_2^{\Gamma_2} = k_3 \rho_2^{\Gamma_3}. \tag{5}$$

You will also need "initial" conditions to integrate the TOV equations. You can view the radius r as a "time" parameter starting at $r = 0$ and integrating forward to larger r from the center of the star. The "initial" conditions are: $\rho_0(0) = \rho_c$, $m_K(0) = 0$, $m(0) = 0$, $m_0(0) = 0$, $P(0) = P(\rho_c)$. In other words, you choose a central rest-mass density $\rho_0(0) = \rho_c$ from this you compute the central Pressure $P(0)$ and central **total** energy density $\rho(0)$ using the equation of state, and combine this with $m(0) = 0$, $m_0(0) = 0$ you get all the initial values at $r = 0$. Note, that the first, second and third equations of the TOV equations appear to be singular at $r = 0$ because of the m/r and m/r^2 terms. But, near $r = 0$, $m(r) = \rho(0) * (4/3) * \pi r^3$, thus $m(r)/r \to 0$ and $m(r)/r^2 \to 0$ as $r \to 0$. Thus, when evaluating the right-hand-sides of the TOV equations at $r = 0$, you will set $m/r = 0$ and $m/r^2 = 0$. The integration continues for large r until the pressure vanishes $P(r) = 0$. The radius at that location is the radius of the star R, i.e., $P(R) = 0$. In practise the integration is terminated at a radius r at which the ratio of the pressure $P(r)$ to the central pressure becomes smaller than a very small number, e.g. $P(r)/P(0) \leq 10^{-20}$.

Thus, the steps for integrating the TOV equations are as follows:

Step 1. Pick a value for the central rest-mass density at $r = 0$ ($\rho_0(0) = \rho_c$), and compute $P(0)$, $\rho(0)$.

Step 2. Integrate the TOV equations: at each new step that you have a new value for P use the expression $\rho_0 = \rho_0(P)$ you will derive to find ρ_0. Once you have ρ_0 you can compute the new total energy density ρ, and with the new P you can evaluate the right-hand-sides of the TOV equations to perform the next step in the integration.

Step 3. The value $r = R$ at which $P(r)/P(0) \leq 10^{-20}$ is the radius of the star, $m(R) = M$ is the **total** or **gravitational** mass of the star, while $m_0(R) = M_0$ is the total rest-mass of the star.

1. Implement RK4, to integrate numerically the system of ODEs (1).

2. Use a range of central densities from $\rho_c = 10^{-10}\rho_{\rm nuc}$ to $\rho_c = 10\rho_{\rm nuc}$, to integrate the TOV equations and determine M, M_0, R as functions of the central rest-mass density ρ_c.

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes until you find a range of step sizes for which the solutions you obtain for M, M_0 and R do not change appreciably.

3. Make one plot containing the curves $M/M_\odot$ vs $\rho_c/\rho_{\rm nuc}$ and $M_0/M_\odot$ vs $\rho_c/\rho_{\rm nuc}$ (y axis linear, x axis log scale). Where $M_\odot = 1.477$km is the mass of the Sun in geometrized units. What do you notice about the **gravitational** mass M and the rest-mass M_0? Is M greater/smaller than or equal to M_0? If there is a difference between these masses try to understand.

4. When $dM/d\rho_c > 0$ the stellar configurations are stable, while $dM/d\rho_c < 0$ implies unstable stars. How many stable and unstable branches can you identify on your $M/M_\odot$ vs $\rho_c/\rho_{\rm nuc}$. Do you see two local maxima in your plot? The smaller local maximum corresponds to the maximum possible mass that a white dwarf can have, while the larger local maximum corresponds to the maximum mass of neutron star can have with this equation of state.

5. Make a plot of $M/M_\odot$ vs R (in units of km), again use y axis linear, x axis log scale. What is the radius of the maximum-mass white dwarf configuration? What is the radius of the maximum-mass neutron star configuration?

6. **Convergence**: Pick one value of the central rest-mass density near the neutron star maximum mass. Use a number of step sizes and make a plot to demonstrate that the code solution for δM converges to zero. Does the order of convergence match your expectation? If not try to explain why this is case.

7. **Self-convergence**: Pick one value of the central rest-mass density near the neutron star maximum mass. Use a number of step sizes and make a plot to demonstrate that the code solution for $M(\rho_c)$ self-converges. Does the order of self-convergence match your expectation? If not try to explain why this is case.

8. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for the solution for $\rho(r)$ at a radius inside the star of your choosing.