Problem Set 3

Physics 305, Fall 2020
Due Sept 21, at 5.00pm

Problem 1. Romberg Integration [15 Points] Write a Python function `romberg(func, a, b, eps, *P)` which computes the integral of the given function between the given limits, using Romberg integration to obtain a solution to within an error of approximately `eps`. Use this function to solve the following problem.

The flux leaving a square centimeter of a blackbody radiator at temperature T is $F = \sigma T^4$ (where σ is the Stephan-Boltzmann constant). This is the result of integrating the flux from a Plank spectrum over all frequencies

$$\sigma T^4 = \pi \int_0^\infty B_\nu(T)\, d\nu$$

where the Planck function is

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \left\{ e^{h\nu/kt} - 1 \right\}^{-1}$$

Start by reducing the integral to

$$\int_0^\infty \frac{x^3\, dx}{e^x - 1}$$

by a change of variable. Use your Romberg integration function to obtain the value of this integral and evaluate σ in the units of your choice. Check your answer against the correct value.

Note that this and the next problem present integrals with several mathematical difficulties. The integral above is singular at the origin and both this and the next problem are integrals with an infinite range.

HINT: Note that both integrands go rapidly to zero outside a quite narrow range in x. How large a range of integration does one need, in either case, to achieve an approximation to a given precision?

Problem 2. Gaussian Integral [10 Points] It is reported that Lord Kelvin (of thermodynamic fame) wrote the following equality on the board

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \pi^{1/2}$$

and then quipped "a mathematician is one to whom *that* is as obvious as that twice two makes four is to you."

Use the same Romberg integrator you implemented for the previous problem to evaluate this integral. Once again, evaluate your answer for accuracy. How does the result change with changes in the limits of integration? Is there a symmetry in the integral which you can exploit?

Problem 3. Gaussian integration [15 Points]:

(a) Derive the formula for $n = 4$ Gaussian quadrature using the recipe based on the Legendre polynomials. Up to what degree polynomial is the $n = 4$ Gaussian quadrature exact for? Use the polynomial $p(x) = 36x^7 + x^6 - 58x^5 - 3600x^4 + 5x^3 - x^2 + 10^3x + 1$ to test your expression. What is $\int_{-1}^{1} p(x)dx$ equal to and does it agree with your $n = 4$ Gaussian quadrature expression? [8 Points]

(b) Use the $n = 2, 3, \text{ and } 4$ Gaussian quadrature methods to compute the integral $\mathbf{I} = \int_{\mathbf{0}}^{\pi} \mathbf{x}\cos(\mathbf{x})\mathbf{dx}$. The exact value of the integral is $\mathbf{I} = -\mathbf{2}$. What is the relative error of the $\mathbf{n} = 4$ Gaussian quadrature? How does it compare to the relative error of the $\mathbf{n} = 2$ and $\mathbf{n} = 3$ Gaussian quadrature methods? Does the error converge to 0 as n increases? [7 Points]

Note: You don't need to write a code for this problem. Show your work.