Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. THE LOTKA-VOLTERRA EQUATIONS: PREDATOR-PREY POPULATION EVOLUTION

You will be examining the properties of the Lotka-Volterra equations that describe the evolution of the populations between a predator and prey species. The equations are a pair of first-order non-linear ordinary differential equations given by

$$\begin{aligned}\frac{dx}{dt} &= bx - pxy,\\ \frac{dy}{dt} &= rxy - dy,\end{aligned} \tag{1}$$

where x is the number of prey, e.g., antelopes, and y is the number of a predator, e.g, lion. The constants b, p, r, d are positive real parameters describing the interaction of the two species. One of the useful properties of Eq. (1) is that it admits the following integral of the motion which must be constant

$$C = b\ln(y) - py - rx + d\ln(x). \tag{2}$$

The value of the constant C is determined by the initial conditions (think of this constant as the analogue of energy conservation). The existence of this integral of the motion will help you assess the validity of the numerical solutions of Eq. (1).

The equation has two equilibrium points ($dx/dt = 0 = dy/dt$). These are

$$(x = 0, y = 0), \text{ and } (x = d/r, y = b/p). \tag{3}$$

The first point implies extinction. The second point implies co-existence. You will be testing the stability of these points and also understanding the classic prey-predator cycles observed in ecological systems.

The equations of motion (1) have multiple time scales involved, and hence non-dimensionalizing the system completely is not possible. Instead you will be working with $b = d = p = r = 1$.

1. Use RK4 to integrate numerically the system of ODEs (1) from $t = 0$ and for sufficiently long times. You will need to be plotting x, y vs t to determine when to stop the integration. You will need initial conditions to integrate the equations.

   - First test the stability of the extinction point. You will be studying initial perturbations near the extinction point. Choose $x = y = 0.5$ at $t = 0$. Does the solution converge or stay close to the origin $x = y = 0$ or does it move away from it? Show one plot of x vs t and y vs t demonstrating your results. Discuss the plot. Choose $x = y = 0.01$ at $t = 0$, i.e., even closer to the extinction point. Does the solution converge or stay close to the origin $x = y = 0$ or does it move away from it? Show another plot of x vs t and y vs t demonstrating your result. What does this imply about the extinction point? Is it stable?
   - Now test the stability of the co-existence point. You will consider small perturbations about the $x = 1 = y$ point. Start with initial conditions $x = 1.5$, $y = 1.5$. Does the solution converge or stay close to the co-existence point $x = y = 1$ or does it move away from it? Show one plot of x vs t and y vs t demonstrating your results. Discuss the plot. Choose $x = y = 1.01$ at $t = 0$, i.e., even closer to the extinction point. Does the solution converge or stay close to the origin $x = y = 0$ or does it move away from it? Show another plot of x vs t and y vs t demonstrating your result. For both equilibrium points you considered perturbation with smaller amplitude. What happens to the amplitude of the initial perturbations as the amplitude decreases? How is the solution of the co-existence point different than the extinction point? What does this imply about the co-existence point? Is it stable?
   - Next you will run your code for multiple initial conditions to make the phase space plot, showing the classic prey-predator cycles observed in ecological systems. You will consider initial conditions $x = 1.25$, $y = 1.25$, $x = 1.5$, $y = 1.5$, $x = 1.75$, $y = 1.75$, $x = 2.25$, $y = 1.25$, $x = 2.5$, $y = 2.5$, $x = 2.75$, $y = 2.75$. Run your code until population cycles on the x vs t and y vs t planes are completed. Then using your 6 data sets from all these initial conditions make one plot of y vs x (label the axes predator and prey, respectively). Label the 6 curves corresponding to different initial conditions based on the value of the constant of the motion in Eq. (2) has for each curve.

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes and use the constant of the motion Eq. (2) to determine how small the step size has to be. If $|C(t) - C(0)|/|C(0)| < 10^{-3}$, you have decent accuracy.

2. **Convergence**: The quantity $|C(t) - C(0)|/|C(0)|$ must be zero at all times. Choose one set of initial conditions integrate to some final time of your choise t_f, solve the Lotka-Volterra system using a number of step sizes and make a plot to demonstrate that $|C(t_f) - C(0)|/|C(0)|$ at the final time converges. Does the order of convergence match your expectation? If not, try to explain why.

3. **Self-convergence**: Use a number of step sizes and make a plot to demonstrate that the code solution for x, y at the final time self-converges. Does the order of convergence match your expectation? If not, try to explain why.

4. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for $C(t)$ at a time t of your choosing.