Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. SPHERICAL STARS IN NEWTONIAN GRAVITY - THE LANE EMDEN EQUATION

This project is aimed to study the properties of **equilibrium** (stationary) spherical stars in Newtonian gravity that are described as polytropes. The stellar matter is treated as a perfect fluid (non-ideal effects such as heat conduction, viscosity etc. are neglected). Polytropes are fluids that are governed by the following equation of state

$$P = K\rho^{1+\frac{1}{n}}, \tag{1}$$

where K is called the polytropic constant and n is called the polytropic index.

Based on Newtonian gravity and hydrostatic equilibrium, the equation describing structure of spherical polytropic fluid configurations is referred to as the Lane-Emden equation. The Lane-Emden equation is written in dimensionless form as follows

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0. \tag{2}$$

where ξ is a dimensionless radius defined by

$$\xi = \frac{r}{\alpha}, \quad \alpha = \sqrt{\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}}, \tag{3}$$

with G Newton's constant, and ρ_c the central rest-mass density of the star. The function θ is related to the rest-mass density of the star through

$$\rho = \rho_c\theta^n. \tag{4}$$

When $n = 3/2$, equation (2) can described the structure of non-relativistic white dwarves, and some main sequence stars like the Sun. Moreover, for $n = 4$ we can model radiation-pressure dominated stars or convection dominated stars, and relativistic white dwarves.

In particular, for low-density white dwarves (up to $\rho_c = 10^6 \mathrm{g/cm}^3$ we have $n = 3/2$ and $K = 5.3802 \times 10^9$ in cgs units. For high-density white dwarves (up to $10^6 < \rho_c < 10^{11} \mathrm{g/cm}^3$ we have $n = 3$ and $K = 1.2223 \times 10^{15}$ in cgs units.

To integrate the Lane-Emden equation you need "initial" or better stated boundary conditions at $xi = 0$. These conditions are $\theta(0) = 1$ and $d\theta/d\xi = 0$. To solve numerically the Lane-Emden equation you will first need to cast it to first order form. Show your work in the term project report. Once you have that the steps for integrating are the following

Step 1. Set $\theta(0) = 1$ and $d\theta/d\xi = 0$.

Step 2. Integrate the Lane-Emden equation from the center of the star outward treating the dimensionless radius ξ as a "time" parameter marching forward.

Step 3. The value $\xi = \xi_1$ at which $|\theta(\xi) - \theta(0)|/\theta(0) \leq 10^{-20}$ is where we encounter the surface of the star and the integration must be terminated.

In solving the equation numerically you will have terms at the right-hand-side that are multiplied by $1/\xi$, and hence are singular at $\xi = 0$. Thus, if you were to start evaluating the right-hand-side at $\xi = 0$, the code would crash because it would encounter terms that are $1/0$. To avoid this you can Taylor expand the solution near $\xi = 0$, i.e., $\theta(\xi) = 1 + (d\theta/d\xi)_{\xi=0}\xi + 0.5(d^2\theta/d\xi^2)_{\xi=0}\xi^2 + \ldots$. But, the function θ is maximum at $\xi = 0$, thus $(d\theta/d\xi)_{\xi=0} = 0$. Therefore, near $\xi = 0$ we have $\theta(\xi) = 1 + 0.5(d^2\theta/d\xi^2)_{\xi=0}\xi^2$. This implies $\xi^2 d\theta/d\xi = (d^2\theta/d\xi^2)_{\xi=0}\xi^3$. Then, we have $d/d\xi(\xi^2 d\theta/d\xi) = 3(d^2\theta/d\xi^2)_{\xi=0}\xi^2$. Plugging this solution in Eq. (2) implies

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 3\left.\frac{d^2\theta}{d\xi^2}\right|_{\xi=0} = -1 \Rightarrow \left.\frac{d^2\theta}{d\xi^2}\right|_{\xi=0} = -\frac{1}{3}. \tag{5}$$

Therefore, the solution near $\xi = 0$, is given by

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2, \tag{6}$$

which implies

$$\frac{d\theta}{d\xi} = -\frac{1}{3}\xi. \tag{7}$$

Using the last two equations is crucial to start the integration.

Through the numerical integration we then find ξ_1, and then we can find the radius of the star as

$$R = \sqrt{\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}}\xi_1. \tag{8}$$

If we further know $\left|\frac{d\theta}{d\xi}\Big|_{\xi_1}\right|$, it can be shown that we can compute the mass of the star as follows

$$M = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{\frac{3-n}{2n}}\xi_1^2\left|\frac{d\theta}{d\xi}\Big|_{\xi_1}\right| \tag{9}$$

1. Implement RK4, to integrate numerically the system of ODEs (2).
2. The Lane-Emden equation has a few exact solutions that you can use to test your solver.
   - For $n = 0$, the exact solution is given by $\theta(\xi) = 1 - \frac{1}{6}\xi^2$
   - For $n = 1$, the exact solution is given by $\theta(\xi) = \frac{\sin(\xi)}{\xi}$.
   - For $n = 5$, the exact solution is given by $\theta(\xi) = \frac{1}{\sqrt{1+\xi^2/3}}$.

   Make sure your code reproduces all three solutions by plotting the numerical solution for all ξ, i.e., $\theta(\xi)$ and overlaying the exact solution.

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes until you find a range of step sizes for which ξ_1 and $\frac{d\theta}{d\xi}\Big|_{\xi_1}$, do not change appreciably.

3. Use $n = 3/2$ and $n = 3$ and integrate the Lane-Emden equation. Then use the values for K and n for white dwarves, and a range of central densities from $\rho_c = 10^3 - 10^7 \mathrm{gr/cm}^3$ (for $n = 3/2$) and $\rho_c = 10^6 - 10^{11} \mathrm{g/cm}^3$ for $n = 3$, and determine M, and R as functions of the central rest-mass density ρ_c.

4. Make one plot containing the curves $M/M_\odot$ vs ρ_c (y axis linear, x axis log scale) for both $n = 3/2$ and $n = 3$. Here $M_\odot = 2 \times 10^{33}\mathrm{g}$ is the mass of the Sun. What do you notice about mass M as the density increases? Does it keep increasing?

5. Make a plot of $M/M_\odot$ vs R (in units of km), again use y axis linear, x axis log scale. What is the mass-radius relationship at small masses? Do you see an expressiong like $MR^m = \text{const.}$? What is the value of m. What happens to the radius as the mass keeps increasing?

6. **Convergence**: For each of the 3 exact solutions, pick the value of $\xi = \xi_1/2$, and using a range of step sized show that your numerical solution $\theta(\xi_1/2)$ converges to the exact solution. Does the order of convergence match your expectation? If not try to explain why this is case.

7. **Self-convergence**: Pick the value $n = 3/2$. Use a number of step sizes and make a plot to demonstrate that the code solution for $\theta(\xi = 3)$ self-converges. Does the order of self-convergence match your expectation? If not try to explain why this is case.

8. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for $\theta(\xi_1/2)$ when $n = 3/2$.