Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. FRIEDMAN EQUATION

This project is aimed to find numerical solutions to the Friedman equations that described a isotropic and homogeneous universe. The Friedman equations possess analytic solutions that will serve as diagnostics that your numerical implementation for solving the equations is correct.

The Friedman equations dictate the evolution of the so-called scale factor (a – the "size" of the Universe) and the evolution of the energy density (ρ) content in the Universe. The equations are given by (setting Newton's constant $G = 1$ and the speed of light $c = 1$)

$$\begin{aligned} \frac{d\rho}{dt} &= -3H\left(\rho + P\right) \\ H^2 &= \left(\frac{d\ln a}{dt}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \\ \frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi}{3}(\rho + 3P), \end{aligned} \tag{1}$$

The k term in the second equation is called the curvature term and since cosmological observations suggest that $k = 0$ we will be setting it equal to 0 throughout this project. Thus, the eqs. become

$$\begin{aligned} \frac{d\rho}{dt} &= -3H\left(\rho + P\right) \\ H^2 &= \left(\frac{d\ln a}{dt}\right)^2 = \frac{8\pi}{3}\rho \\ \frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi}{3}(\rho + 3P), \end{aligned} \tag{2}$$

To close the system of ODEs, we must use an equation of state, relating the Pressure P to the energy density ρ. We will use the standard form $P = w\rho$. For dark matter $w_M = 0$, for radiation $w_R = 1/3$ for dark energy $w_\Lambda = -1$. The first equation above can then be integrated directly because for this equation of state it is possible to separate variables, yielding

$$\frac{d\ln\rho}{dt} = -3\frac{d\ln a}{dt}\Big(1+w\Big) \tag{3}$$

whose solution is

$$\ln\rho = -3(1+w)\ln a + C \Rightarrow \rho = \rho_0 a^{-3-3w}, \tag{4}$$

where ρ_0 is the energy density at $a = 1$. Note, that for radiation we have $\rho_R = \rho_{0,R}a^{-4}$, for matter we get $\rho_M = \rho_{0,M}a^{-3}$, and for dark energy $\rho_\Lambda = \rho_{0,\Lambda}$. Thus, once the scale factor a is determined we can immediately find how the energy densities evolve.

The second equation applies to any time and so it applies today, i.e, $t = t_0$ for which

$$H_0^2 = \frac{8\pi}{3}\rho_0 \tag{5}$$

If we divide the second equation in Eq. (2) by H_0 we obtain

$$\frac{H^2}{H_0^2} = \frac{8\pi\rho}{3H_0} \tag{6}$$

In all previous equations the density ρ equals the sum of the dark matter and radiation energy density content of the Universe plus the "vacuum" energy density which corresponds to the so-called dark energy. The same applies to the pressure. In other words:

$$\begin{aligned}\rho &= \rho_{\text{matter}} + \rho_{\text{rad}} - \frac{\Lambda}{8\pi}\\ p &= p_{\text{matter}} + p_{\text{rad}} - \frac{\Lambda}{8\pi}\end{aligned} \tag{7}$$

The ratio of $\Omega = \frac{8\pi\rho}{3H^2}$ is called the density parameter. Since from Eq. (4) we know how the densities evolve we can straightforwardly find that: a) the matter density parameter behaves as $\Omega_M = \Omega_{0,M}/a^3$, b) similarly the radiation density parameter behaves as $\Omega_R = \Omega_{0,R}/a^4$, and c) the radiation density parameter is unchanged $\Omega_\Lambda = \Omega_{0,\Lambda}$. Thus, the independent Friedman equation becomes

$$\begin{aligned}\frac{1}{H_0}\frac{da}{dt} &= \pm\sqrt{\Omega_{0,M}a^{-1} + \Omega_{0,R}/a^{-2} + \Omega_{0,\Lambda}a^2}\\ \frac{d\eta}{dt} &= \frac{1}{a}.\end{aligned} \tag{8}$$

where $a = 1$ corresponds to today. The positive (+) solution corresponds to an expanding Universe, the negative (−) solution corresponds to a contracting Universe. Since, our Universe is expanding we will choose the expanding solution. The nice aspect of this equation is that it is almost dimensionless where the matter content is cast in terms of a fraction of the total matter content of the universe today, such that $\Omega_{0,M}+\Omega_{0,R}+\Omega_{0,\Lambda} = 1$. The second Equation in (8) is auxiliary and is for computing the quantity η which is called the conformal time, and in units where $c = 1$, it is also the (particle) horizon. Introduce a new dimensionless time (call it $\tilde{t}$) and conformal time (call it $\tilde{\eta}$), such that the above equations become completely dimensionless and independent of H_0. Show your work in the term paper.

Cosmological observations suggest that $\Omega_{0,\Lambda} = 0.6847$, $\Omega_{0,M} = 0.315$, leaving $\Omega_{0,R} = 0.0003$. You will be using these parameters and variations of them in order to solve Eq. (8). To solve it you will also need initial conditions which are $a = 1$. The integration of the equation will be performed backwards in time, i.e., $t = -ih, i = 0, 1, \ldots N$ with h the step size. Once solved you will be computing several diagnostics through numerical quadratures. In particular, we want to compute

- The comoving distance (this is the distance between two objects in the "frame" of the expanding universe, which in a spatially flat Universe is given by

$$d_C(z) = d_H\int_0^z \frac{dz'}{\sqrt{\Omega_{0,M}(1+z)^3 + \Omega_{0,R}(1+z)^4 + \Omega_{0,\Lambda}}}, \tag{9}$$

where $d_H = c/H_0$, and you will use $H_0 = 67.4$km/s/Mpc, $c = 3 \times 10^5$km/s. The quantity z is called the cosmological redshift (it equals the redshift of spectral lines of distant galaxies that are moving away from us due to the expansion of the Universe) and it is defined from the equation $1 + z = \frac{1}{a}$. Thus, today when $a = 1$, $z = 0$. For $a < 1$, i.e., in the past, then $z > 0$.

- The angular diameter distance

$$d_A = \frac{d_C(z)}{1+z} \tag{10}$$

  This is the distance based on an angle measured on the sky from an object whose physical length is known; An object of size x at redshift z that appears to have angular size $\delta\theta$ has the angular diameter distance of $d_A(z) = x/\delta\theta$. This is commonly used to observe so called standard rulers, for example in the context of baryon acoustic oscillations.

- The luminosity distance

$$d_L = d_C(z)(1+z) \tag{11}$$

  This is the distance based on a source whose luminosity is known. If the intrinsic luminosity L of a distant object is known, we can calculate its luminosity distance by measuring the flux S and determine $d_L(z) = \sqrt{L/4\pi S}$, which is equivalent to the expression above for $d_L(z)$. This quantity is important for measurements of standard candles like type Ia supernovae, which were first used to discover the acceleration of the expansion of the universe.

Do the following and show your work in the term paper.

1. Implement RK4, to integrate numerically the dimensionless version of the Friedman equations (8) with initial condition $a(\tilde{t} = 1) = 1$. You will integrate backwards in time until the moment of the Big Bang, when the scale factor becomes 0. In practice, the integration will terminate when $a < \epsilon$, choose $\epsilon = 10^{-20}$. The equation admits exact solutions for certain cases that you will reproduce.

   - For $\Omega_{0,\Lambda} = 0$, $\Omega_{0,M} = 1$, $\Omega_{0,R} = 0$, i.e., a matter dominated Universe, the solution is $a = \frac{(3t-1)^{2/3}}{2^{2/3}}$.
   - For $\Omega_{0,\Lambda} = 0$, $\Omega_{0,M} = 0$, $\Omega_{0,R} = 1$, i.e., a radiation pressure dominated Universe, the solution is $a = \sqrt{2t-1}$.
   - For $\Omega_{0,\Lambda} = 1$, $\Omega_{0,M} = 0$, $\Omega_{0,R} = 0$, i.e., a dark energy dominated Universe, the solution is $a = e^{t-1}$. This is known as the De Sitter solution.

   Show three plots of your numerical solution in each, and overlay the exact solutions to see how they compare.

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes and evolve to a given time. If the solution at that time does not change appreciably with step size, then you have found a decent step size.

2. Using $\Omega_{0,\Lambda} = 0.6847$, $\Omega_{0,M} = 0.315$, $\Omega_{0,R} = 0.0003$ (the Planck measurements) and separately $\Omega_{0,\Lambda} = 0.690$, $\Omega_{0,M} = 0.308$, $\Omega_{0,R} = 0.002$ (the Dark Energy survey measurements). Solve numerically the Friedman equation and show a plot of your two solutions. Are the two solutions distinguishable?

   What happens to the horizon η of the universe as a function of a from the Big Bang until now? Does it increase or decrease? Show a plot of η vs a. The (particle) horizon is the maximum distance from which light from particles could have traveled to the observer in the age of the universe, and hence come into causal contact.

   Using the two sets of measurements also compute d_A and d_L as a function of z for $0 < z < 2000$. Use the composite Simpson method for the numerical integrations. Ensure that your using dense enough points for the integration error to be small. Are d_A and d_L distinguishable for the two sets of measurements?

3. **Convergence**: Use a number of step sizes and make a plot to demonstrate that the code $a(t)$ solution for $\Omega_{0,\Lambda} = 0$, $\Omega_{0,M} = 0$, $\Omega_{0,R} = 1$ at $t = 3/4$ converges to the exact solution. Does the order of convergence match your expectation? If not try to explain why this is case.

4. **Self-convergence**: Use a number of step sizes and make a plot to demonstrate that the code $a(t)$ solution for $\Omega_{0,\Lambda} = 0.6847$, $\Omega_{0,M} = 0.315$, $\Omega_{0,R} = 0.0003$ self-converges at some time t of your choice before $a = 0$. Does the order of self-convergence match your expectation? If not try to explain why this is case.

5. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for the solution for a(t) at a time of your choosing.