Physics 305 – Computational Physics, Fall 2020
Term Project
Full project submission Due Date: Tuesday December 15, 5pm
Presentation Phase: November 30 - December 11

The program in your term project can be either submitted as a python program or ipython notebook, where the latter is preferred. The program, an explanation of what the program does, along with answers to **all** questions asked should be uploaded to d2l.

You are expected to write a term paper (in word or Latex) on your project that discusses the problem you are trying to solve, the basic equations that govern the problem, includes plots that show the solutions, and describes the solution and the numerical method involved. In addition, you must demonstrate that your solution is correct by showing that the code converges at the expected order. If your code does not converge at the expected order you should try to identify potential reasons for why this is the case. You are expected to work on your term project by yourself..

Your term project will receive full credit **only** if: (a) the program runs successfully without errors using python 3, (b) the programs have explanatory comments and variable names that identify with the problem equations you are trying to solve, (c) give the correct output, and (d) demonstrate the validity of the solution through convergence plots. No credit will be given to late term projects.

The term paper is as important as the code (50% of the term project credit will go to the code and the other 50% to the paper). Answers to the questions and analysis requested below should be elaborated in the report. Plots should be clearly labeled and be properly described in the report, and not just shown. You will need to explain what each and every plot demonstrates. A polished paper written in word or LaTex (preferred, e.g. please try overleaf) is expected to get full credit.

Note: Before you present results from numerical integrations that answer the questions in the project, it is critical to *first* perform the convergence tests for one case, and to estimate errors. This will tell you how small a step size is necessary for accurate solutions. Only after errors are estimated, does it make sense to run your code for producing results that help you learn more about the system you study.

I. COSMOLOGICAL SLOW ROLL INFLATION

This project is aimed to find numerical solutions to the Friedman equations supplemented with a slowly rolling scalar field that drives an exponential expansion of the Universe. This exponential expansion of the Universe early in its course of evolution is called inflation and is a proposed solution to 3 major cosmological puzzles: the Horizon, the flatness and relic problems.

The gist of the horizon problem has to do with the isotropy of the cosmic microwave background (CMB) radiation that we observe. The CMB is remarkably uniform all across the sky but the different regions are causally disconnected; So, how is it possible for the CMB to be so isotropic if it is composed by causally disconnected regions?

The flatness problem is about the radius of curvature of the Universe. Any triangle in flat space has angles that sum up to 180 degrees. This appears to be the case in our Universe. Which implies that either the entire Universe is flat, or that the radius of curvature of the Universe is larger than the observable Universe.

The relic problem has to do with the absence of magnetic monopoles.

Inflation solves all these problems by invoking the presence of some form of energy early in the Universe that drives exponential expansion of the Universe. This exponential expansion dilutes the density of relics, makes the current observable Universe smaller than the radius of curvature of the Universe, and increase the horizon early on so that the observable CMB consists of patches that were in causal contact in the past. We will not elaborate further on these as a proper course in cosmology is necessary. Here we will discuss and solve the equations that govern inflation.

The Friedman equations dictate the evolution of the so-called scale factor (a – the "size" of the Universe) and the evolution of the energy density (ρ) content in the Universe. The equations are given by (setting Newton's constant

$G = 1$ and the speed of light $c = 1$)

$$\begin{aligned} \frac{d\rho}{dt} =& -3H\left(\rho + P\right) \\ H^2 =& \left(\frac{d\ln a}{dt}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \\ \frac{1}{a}\frac{d^2a}{dt^2} =& -\frac{4\pi}{3}(\rho + 3P), \end{aligned} \tag{1}$$

The k term in the second equation is called the curvature term and is responsible for the curvature of universe. The last equation describes the acceleration of the Universe, and the Universe is accelerating if $\rho + 3P < 0$. For matter with an equation of state $P = w\rho$, acceleration occurs when $(1 + 3w)\rho < 0$, since $\rho > 0$, this implies $w < -1/3$ results in acceleration, while $w > -1/3$ results in deceleration. Note that dark matter has $w = 0$ and radiation has $w = 1/3$. Thus, in a universe with "normal" matter and radiation the Universe is decelerating. To solve the horizon, flatness and relics problems we need the Universe to accelerate so that the Horizon of the Universe expands rapidly early on. The horizon of the Universe is given by

$$\eta = \int d\ln a \frac{1}{Ha} \tag{2}$$

Here η is the conformal time, but in units where $c = 1$ it also equals the (particle) horizon size. For $P = w\rho$ the Hubble length – the distance between two areas that move away from each other at the speed of light due to the expansion of the Universe) – scales as $\frac{1}{Ha} \sim \alpha^{(1+3w)/2}$. Thus, in a decelerating Universe the $\frac{1}{Ha}$ increases as a increases, thus Horizon gets most of its contribution late in the evolution, but then the CMB has already exited the horizon and hence its different parts are not in causal contact. Instead, in an accelerating Universe $\frac{1}{Ha}$ is decreasing as a increases, and the horizon then gets most of contribution from early in the evolution of the Universe when the CMB is inside the horizon and hence in causal contact. Later on the CMB will exit the horizon but within the observable Universe the patch of the CMB arises from an early phase when it was in causal contact.

If the field driving the inflation (inflaton) is a scalar field, then it's energy density and pressure are give by

$$\rho_\phi = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V(\phi), \quad P_\phi = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V(\phi). \tag{3}$$

Thus, if the field is kinetic energy dominated $\left(\frac{d\phi}{dt}\right)^2 \gg V(\phi)$, then $w = P_\phi/\rho_\phi \sim 1$ and this field leads to deceleration. If the field is potential energy dominated $\left(\frac{d\phi}{dt}\right)^2 \ll V(\phi)$, then $w = P_\phi/\rho_\phi \sim -1$, and the field drives acceleration. Thus, we will have inflation whenever the potential energy of the scalar field dominates. This is done when the field rolls down the potential slowly. Hence the term slow-roll inflation. Typical potentials that are used to drive inflation include

- $V(\phi) = \lambda(\phi^2 - M^2)^2$ Higgs potential
- $V(\phi) = m^2\phi^2$ Massive scalar field

The equations describing inflation are

$$\begin{aligned} H = \frac{1}{a}\frac{da}{dt} =& \sqrt{\frac{8\pi}{3}\left[V(\phi) + \left(\frac{d\phi}{dt}\right)^2\right] - \frac{k}{a^2}} \\ \frac{d^2\phi}{dt^2} =& -3H\frac{d\phi}{dt} - \frac{dV}{d\phi}. \end{aligned} \tag{4}$$

For simplicity we will drop the curvature term, i.e., set $k = 0$, because this term will quickly become insignificant as a will increase rapidly.

Slow roll inflation means that we have negligible acceleration in the scalar field, and that the field is potential energy dominated. Then the slow-roll inflation equations of motion become

$$\begin{aligned} H = \frac{1}{a}\frac{da}{dt} =& \sqrt{\frac{8\pi}{3}V(\phi)} \\ 3H\frac{d\phi}{dt} =& -\frac{dV}{d\phi}. \end{aligned} \tag{5}$$

We introduce the slow-roll parameters

$$\epsilon(\phi) = \frac{1}{16\pi}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2, \quad \eta(\phi) = \frac{1}{8\pi}\frac{1}{V}\frac{d^2V}{d\phi^2}, \tag{6}$$

where the $\epsilon(\phi)$ measures the slope of the potential and $\eta(\phi)$ the curvature. The necessary conditions for the slow-roll appproximation to hold are

$$\epsilon(\phi) \ll 1, \quad |\eta(\phi)| \ll 1. \tag{7}$$

The amount of inflation is encoded by the logarithm of the amount of expansion, the number of e-foldings N, given by

$$N \equiv \frac{a(t_{\text{end}})}{a(t_{\text{start}})} = \int_{t_{\text{start}}}^{t_{\text{end}}} H dt. \tag{8}$$

To avoid dealing with integrals, we can rewrite the integral as its equivalent ODE

$$\frac{dN}{dt} = H. \tag{9}$$

We will now focus on the massive scalar field potential $V(\phi) = m^2\phi^2$. For this field $dV/d\phi = 2m\phi$ and $d^2V/d\phi^2 = 2m$. Thus, the slow roll parameters become

$$\epsilon(\phi) = \frac{1}{4\pi\phi^2} = \eta(\phi). \tag{10}$$

Therefore, slow roll requires $\phi \gg \phi_{\text{crit}} = 1/\sqrt{4\pi}$. The slow roll equations of motion for this potential become

$$\begin{aligned}
\frac{1}{a}\frac{da}{dt} &= \sqrt{\frac{8\pi}{3}m^2\phi^2} \\
\frac{d\phi}{dt} &= -\frac{2}{3}m\phi/\sqrt{\frac{8\pi}{3}\phi^2} \\
\frac{dN}{dt} &= \sqrt{\frac{8\pi}{3}m^2\phi^2} \\
\frac{d\eta}{dt} &= \frac{1}{a}.
\end{aligned} \tag{11}$$

First, introduce a proper dimensionless time (call it $\tilde{t}$) and dimensionless conformal time (call it $\tilde{\eta}$) to be able to solve the ODEs for any scalar field mass m. To solve this system of ODEs you need initial conditions. The initial conditions are $a(0) = 1$, $N = 1$, and you will consider $\phi(0) \approx 11.024\phi_{\text{crit}} = 3.11$.

Do the following and show your work in the term paper.

1. Implement RK4, to integrate numerically the dimensionless version of equation (11) for the 4 different initial conditions. You will integrate forward in time until the number of e-foldings becomes flat and does not change appreciably as the time moves forward.

   Show in four plots a vs t, ϕ vs t and N vs t η vs t for the each of the different initial conditions. In other words each plot will have 4 curves. Does a increase exponentially with time? How does the horizon size η evolve with time?

   Use your judgement as to how small a step size you need to solve this system accurately. If you cannot figure this out from pure thought, experiment with different step sizes and evolve to a given time. If the solution at that time does not change appreciably with step size, then you have found a decent step size.

2. Run your code also with $\phi(0) \approx 5\phi_{\text{crit}}$, $\phi(0) \approx 11.024\phi_{\text{crit}} = 3.11$, $\phi(0) \approx 15\phi_{\text{crit}}$, $\phi(0) \approx 20\phi_{\text{crit}}$. How does the final number of e-foldings depend on the $\phi(0)$? Is it linear, quadratic, cubic, logarithmic etc.?

3. **Self-convergence**: Use a number of step sizes for one of the initial conditions you used, and make a plot to demonstrate that the code solution for $\phi(t)$ self-converges at some time t of your choice. Does the order of self-convergence match your expectation? If not try to explain why this is case.

4. Using the order of convergence you determined, employ Richardson extrapolation to determine an error for the number of e-foldings for each case you ran.