Problem 1 : Higher-Order ODEs

- $$\frac{d^2y}{dt^2} - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{y} = 0$$

$\rightarrow$ $\ddot{y} - \dot{y}^2/y - 1/y = 0$

$$\ddot{y} = \frac{\dot{y}^2}{y} + \frac{1}{y} \quad - ①$$

Assume, $u = \dot{y}$

Substituting in ①,

$$\dot{u} = \frac{u^2}{y} + \frac{1}{y}$$

Therefore, the given DE can be composed into a system of first-order ODEs as follows:

$$\boxed{\begin{aligned} \frac{dy}{dt} &= u \\ \frac{du}{dt} &= \frac{u^2}{y} + \frac{1}{y} \end{aligned}}$$

$$\cdot\ \frac{1}{t^2}\frac{d}{dt}\left(t^2\frac{dy}{dt}\right) + y = 0$$

$$\rightarrow \frac{d}{dt}\left(t^2\frac{dy}{dt}\right) + yt^2 = 0$$

$$2t\frac{dy}{dt} + \frac{d^2y}{dt^2}t^2 + yt^2 = 0.$$

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + y = 0$$

$$\ddot{y} = -\frac{2}{t}\dot{y} + y \quad - \quad ①$$

Assume, $\dot{y} = u$.

Substituting in ①,

$$\dot{u} = -\frac{2}{t}u + y$$

Therefore, the given DE can be composed into a system of first-order ODEs as follows:

$$\boxed{\begin{aligned} \frac{dy}{dt} &= u \\ \frac{du}{dt} &= -\frac{2}{t}u + y. \end{aligned}}$$

- $\frac{d^3y}{dt^3} + \frac{2d^2y}{dt^2} - \frac{dy}{dt} - t = 0$

$\dddot{y} = -2\ddot{y} + \dot{y} + t \quad - ①$

Assume, $\dot{y} = u$. Substituting in ①,

$\ddot{u} = -2\dot{u} + u + t \quad - ②$

Now, Assume $\dot{u} = v$. Substituting in ②,

$\dot{v} = -2v + u + t$

Therefore, the given DE can be composed into a system of first-order ODEs as follows:

$$\boxed{\begin{aligned} \frac{dy}{dt} &= u \\ \frac{du}{dt} &= v \\ \frac{dv}{dt} &= -2v + u + t \end{aligned}}$$