Rotation Curves of Galaxies Page Design

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1 Methods

1.1 Spherically Symmetric Components (Bulge)

I assume an Sérsic profile for the surface brightness (in physical units) of a component of the galaxy in a given filter, of the form

$$I(r) = I_0 e^{\left(-\left(\frac{r}{\mathcal{R}}\right)^{\frac{1}{n}}\right)}, \tag{1}$$

where I_0 is a central surface brightness (at $r = 0$), n is the Sérsic index, r is the distance from the center of the galaxy as projected on the sky, and $\mathcal{R}$ is a scale length over which the surface brightness decreases by one e-folding.

Assuming this form describes the underlying luminosity integrated over a line of sight in a spherical body, I define a luminosity density

$$\lambda(R) = \lambda_0 f(R) e^{\left(-\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}\right)}, \tag{2}$$

where λ_0 is the central luminosity density, R is the spatial distance from the center of the galaxy, and $f(R)$ is an (as-yet) unknown function.

The surface brightness at a distance r from the center of the galaxy the requires integrating over a column

$$I(r) = \int_{-\infty}^{\infty} \lambda(r^2 + z^2) dz, \tag{3}$$

where z describes height above the plane of the sky bisecting the galaxy. Since the galaxy taken to be radially symmetric, the integrands may be simplified to

$$I(r) = 2\int_{0}^{\infty} \lambda(r^2 + z^2) dz. \tag{4}$$

This equation can be simplified by taking $R^2 \equiv r^2 + z^2$, such that $z = \sqrt{R^2 - r^2}$ and $dz = \left(1 - \left(\frac{r}{R}\right)^2\right)^{-\frac{1}{2}} dR$; in this form the equation is

$$I(r) = 2\int_{r}^{\infty} \lambda_0 f(R) e^{\left(-\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}\right)} \left(1 - \left(\frac{r}{R}\right)^2\right)^{-\frac{1}{2}} dR. \tag{5}$$

Finally, define $x \equiv \left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}$, so that $R = \mathcal{R}x^n$ and $dR = n\mathcal{R}x^{n-1}$, leading to

$$I(r) = 2\int_{\left(\frac{r}{\mathcal{R}}\right)^{\frac{1}{n}}}^{\infty} \lambda_0 f(r,x) n\mathcal{R} e^{-x} \frac{x^{2n-1}}{\sqrt{x^{2n} - \left(\frac{r}{\mathcal{R}}\right)^2}} dx. \tag{6}$$

In order for this to simplify to Eq. 1 requires

$$f(x) = x^{1-2n}\sqrt{x^{2n} - \left(\frac{r}{\mathcal{R}}\right)^2}. \tag{7}$$

Eq. 6 thus becomes

$$I(r) = 2\int_{\left(\frac{r}{\mathcal{R}}\right)^{\frac{1}{n}}}^{\infty} n\mathcal{R}\lambda_0 e^{-x} dx, \tag{8}$$

leading to the desried result in Eq. 1. We also find from this the central surface brightness

$$I_0 = n\mathcal{R}\lambda_0. \tag{9}$$

The general luminosity density can be found using Eqs.2 and 7 and taking $r = 0$, resulting in

$$\lambda(x) = \lambda_0 x^{1-n} e^{-x}. \tag{10}$$

I maintain the form using x instead of R to simplify the following integrals.

The luminosity density is related to the mass density by the stellar mass-to-light ratio, $\Upsilon_\star$, according to

$$\rho(x) = \Upsilon_\star \lambda(x) = \rho_0 x^{1-n} e^{-x}, \tag{11}$$

where $\rho_0 \equiv \Upsilon_\star \lambda_0$.

Now having the density function, we can find the mass interior to a distance R (or r) from the center of the galaxy as

$$M(R) = \int_0^{\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}} 4\pi\mathcal{R}^2 x^{2n} \rho(x) dx, \tag{12}$$

where I again assume spherical symmetry and use the relation $R = \mathcal{R}x^n$.

This leads to

$$M(R) = \int_0^{\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}} 4\pi\mathcal{R}^2 \rho_0 x^{n+1} e^{-x} dx. \tag{13}$$

For $R = \infty$, this leads to

$$M(\infty) = 4\pi\mathcal{R}^2 \rho_0 \Gamma(n+2), \tag{14}$$

where Γ is the gamma function; this demonstrates that the mass of the galaxy is bounded for $n > 0$.

For finite R, we must solve the integral in Eq. 13; this cannot be done analytically, so I use the Taylor expansion for e:

$$e^x = \sum_i \frac{x^i(-1)^i}{i!}, \tag{15}$$

leading to

$$M(R) = \int_0^{\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}} 4\pi\mathcal{R}^2 \rho_0 \sum_i \frac{x^{1+n+i}(-1)^i}{i!} dx, \tag{16}$$

and resulting in

$$M(R) = 4\pi\mathcal{R}^2 \rho_0 \left(\sum_i \frac{x^{2+n+i}(-1)^i}{i!(2+n+i)} \right) \Bigg|_0^{\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n}}}. \tag{17}$$

The lower bound $x = 0$ results in 1; the upper bound must be found numerically. Defining

$$g(i, n, x) \equiv \frac{x^{2+n+i}(-1)^i}{i!(2+n+i)}, \tag{18}$$

and recognizing that the factorial component will increase in magnitude faster than the polynomial component, we sum until we find

$$\frac{g(I, n, x)}{\sum_{i=0}^{I} g(i, n, x)} < \varepsilon \tag{19}$$

where ε is a small value (e.g. 0.001).

Now that there is a solution for the mass as a function of radius, we can find the velocity profile

$$v(R) = \sqrt{\frac{GM(R)}{R}}, \tag{20}$$

where G is the Newtonian gravitational constant.

We can also find the surface brightness profile using the relation

$$\mu(R) = 21.572 + M_{\odot} + 2.5\log(I(R)), \tag{21}$$

where μ is the surface brightness (in mag. / sq. arc-sec) and $M_{\odot}$ is the absolute magnitude of the Sun in the desired filter.

1.2 Cylindrically Symmetric Components (Disk)

The disk has a Sérsic-like surface brightness profile of the form

$$I(R) = I_0 e^{\left(-\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n_r}}\right)} e^{\left(-\left(\frac{z}{\mathcal{H}}\right)^{\frac{1}{n_z}}\right)}, \tag{22}$$

where I_0 is a central surface brightness (at $r, z = 0$), n_r and n_z are the radial and vertical (respectively) Sérsic indices, R is the distance from the center of the galaxy as projected on the sky, z is height above the plane of the disk, $\mathcal{R}$ is a scale length in the radial direction over which the surface brightness decreases by one e-folding, and $\mathcal{H}$ is a scale height above and below the plane of the disk over which the surface brightness decreases by one e-folding.

I define a luminosity density for the disk

$$\lambda_{disk}(R, z) = \lambda_0 f(R, z) e^{\left(-\left(\frac{R}{\mathcal{R}}\right)^{\frac{1}{n_r}}\right)} e^{\left(-\left(\frac{z}{\mathcal{H}}\right)^{\frac{1}{n_z}}\right)}, \tag{23}$$

where $f(R, z)$ is an as-yet unknown function