Transformaciones Lineales

Sean V_K y W_K dos espacios vectoriales definidos sobre el mismo cuerpo de escalares K.
La función $T: V_K \to W_K$ es una transformación lineal sii cumple:

I.- $T(x+y) = T(x) + T(y), \quad \forall x,y \in V_K$

II.- $T(\alpha x) = \alpha . T(x), \quad \forall \alpha \in K, \forall x \in V_K$

Transformaciones Lineales Especiales

- **Transformación Identidad**

$I_{V_K}: V_K \to V_K : I_{V_K}(x) = x, \quad \forall x \in V_K$

- **Transformación Nula**

$O_L : V_K \to W_K : O_L(x) = O_{W_K}, \quad \forall x \in V_K$

- **Transformaciones Matriciales**

Dada matriz $A \in K^{m \times n}$, la aplicación $T: K^n \to K^m : T(x) = Ax$ es una T.L.

Función Lineal

Sea V_K un espacio vectorial def. sobre los escalares K.
Se llama función lineal sobre V_K a una función $f: V_K \to K$ q' cumple:

I.- $f(x+y) = f(x) + f(y), \quad \forall x,y \in V_K$.

II.- $f(\alpha x) = \alpha f(x), \quad \forall \alpha \in K, \forall x \in V_K$

Función de Coordenadas

De este modo, $\phi: V_K \to K^n$ es la T.L. que asigna a c/vector de V_K su vector de coordenadas en la base B

Propiedades de las Transformaciones Lineales

I. $T_{V_k}: V_k \to W_k$ es una TL, entonces $T(O_{V_k}) = O_{W_k}$

II. Sea $\{v_1, \dots, v_k\}$ un conj de vectores de V_k. Entonces $T\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i T(v_i)$; $\alpha_i \in k$

III. Sean f y $g: V_k \to W_k$ dos TL sobre mismo k. Las funciones $f \pm g: V_k \to W_k$ son TL.

IV. Sea $f: V_k \to W_k$ y $\alpha \in k$. Entonces la función $\alpha . f: V_k \to W_k$ es una transformación lineal.

V. La composición de TL es tmb una TL.

Imagen

La imagen de una T.L. $T: V_k \to W_k$ es el subconjunto de W_k, denotado $Im(T)$ tq $Im(T) = \{w \in W_k : w = T(x), x \in V\}$

↳ si es subespacio: $T(S) = \{w \in W_k : w = T(x), x \in S\}$

→ La imagen de una TL es un subespacio de W_k

Preimagen de un subespacio de W_k

La preimagen de un subconjunto de W_k a través de una TL $T: V_k \to W_k$ consta de todos los vectores de V_k q' se transforman en vectores de U a través de la TL T, y se denota $T^{-1}(U)$

* no es la inversa

Obtener imagen de una TL

$Im(T) = gen\{T(v_1), \dots, T(v_n)\}$

Clasificación de Transformaciones Lineales

I. T es inyectiva (o monomorfismo) si $T(x) = T(y) \Rightarrow x = y$, $\forall x, y \in V_k$ → cosas distintas a lugares distintos

II. T es sobreyectiva (o epimorfismo) si $\forall y \in W_k$ $\exists x \in V_k : y = T(x)$ → ningún elemento del codominio está libre

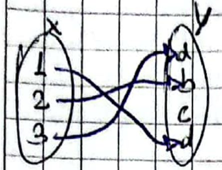
III. T es biyectiva (o isomorfismo) si es inyectiva y sobreyectiva

Núcleo

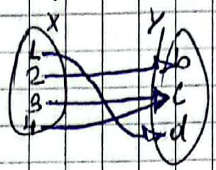
$Nu(T) = \{x \in V : T(x) = O_W\} = T^{-1}(\{O_W\}) \subseteq V$

$\dim(Nu(T)) + \dim(Im(T)) = \dim(V)$

Inyectividad



Suryectividad



- T es monomorfismo $\Leftrightarrow Nu(T) = \{0_V\}$
- T es epimorfismo $\Leftrightarrow Im(T) = W \Leftrightarrow \dim(Im(T)) = \dim(W)$
- Si $\dim(V) = n$ y $\dim(W) = m$:
  - Si T es monomorfismo $\Rightarrow \dim(V) \leq \dim(W)$
  - Si T es epimorfismo $\Rightarrow \dim(V) \geq \dim(W)$
  - Si T es isomorfismo $\Rightarrow \dim(W) = \dim(V)$

- Si $Nu(T) = \{0_V\} \rightarrow$ monofirmo
- Si $\dim[Im(T)] = \dim(W) \rightarrow$ epimorfismo

$\}\rightarrow$ Ambas: isomorfismo.

- Si $T: V \rightarrow W$ es un isomorfismo, entonces existe función inversa.

$T(T^{-1}(w)) = w \quad \wedge \quad T^{-1}(T(v)) = v$

$\forall w \in W \qquad\qquad \forall v \in V$

- $T(v) = w_0 \begin{cases} \text{si } w_0 \notin Im(T) \rightarrow \text{sist. incompatible} \\ \text{si } w_0 \in Im(T) \text{ y monomorfismo} \rightarrow \text{S.C.D.} \\ \text{si } w_0 \in Im(T) \text{ y no monomorfismo} \rightarrow \text{S.C.I.} \\ \text{soluciones son de la forma } x_p + x_N;\ x_N \in Nu(T) \text{ y } x_p \text{ sol. part.} \end{cases}$

Matriz

$x = \alpha_1 . v_1 + \dots + \alpha_n . v_n \quad \in \mathbb{V}$

$T(x) = T(\alpha_1 . v_1 + \dots + \alpha_n . v_n) \quad \in \mathbb{W}$

$[T(x)]^{B'} = [\alpha_1 . T(v_1) + \dots + \alpha_n . T(v_n)]^{B'} \in \mathbb{K}^m$

B → base $\mathbb{V}$

B' → base $\mathbb{W}$

$[T(x)]^{B'} = \alpha_1 . [T(v_1)]^{B'} + \dots + \alpha_n . [T(v_n)]^{B'}$

$[T(x)]^{B'} = \left[[T(v_1)]^{B'} | \dots | [T(v_n)]^{B'} \right] . \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$[T(x)]^{B'} = \left[[T(v_1)]^{B'} | \dots | [T(v_n)]^{B'} \right] . [x]^{B}$

$\Rightarrow [T(x)]^{B'} = [T]_{B}^{B'} . [x]^{B}$

- $\dim[Im(T)] = rg([T]_{B}^{B'})$
- $x \in N_u(T) \Leftrightarrow [T]_{B}^{B'} [x]_{B} = 0_{\mathbb{K}^m}$

EJERCICIOS

2.1. Verificar que las sigs. aplicaciones son TL.

a. $T_1 : \mathbb{R}^3 \to \mathbb{R}$ def. por $T_1([x_1 \ x_2 \ x_3]^T) := -3x_2 + 2x_3$

I. $T_1(x+y) = T(x) + T(y), \ \forall x,y \in \mathbb{R}^3$

$T_1([x_1+y_1 \ \ x_2+y_2 \ \ x_3+y_3]^T) = T([x_1 \ x_2 \ x_3]^T) + T([y_1 \ y_2 \ y_3]^T)$

$-3(x_2+y_2) + 2(x_3+y_3) = -3x_2 + 2x_3 - 3y_2 + 2y_3$ ✓

II. $T_1(\alpha x) = \alpha T_1(x), \ \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^3$

$T_1(\alpha [x_1 \ x_2 \ x_3]^T) = \alpha . T_1([x_1 \ x_2 \ x_3]^T)$

$T_1([\alpha x_1 \ \alpha x_2 \ \alpha x_3]^T) = \alpha . (-3x_2 + 2x_3)$

$-3\alpha x_2 + 2\alpha x_3 = -3\alpha x_2 + 2\alpha x_3$ ✓

b. $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ definida por $T_2([x_1 \ x_2 \ x_3]^T) := [-3x_2 + 2x_3 \ \ 3x_1 - x_3]^T$

I. $T_2(x+y) = T_2(x) + T_2(y), \ \forall x,y \in \mathbb{R}^3$

$T_2([x_1+y_1 \ \ x_2+y_2 \ \ x_3+y_3]^T) = T_2([x_1 \ x_2 \ x_3]^T) + T_2([y_1 \ y_2 \ y_3]^T)$

$[-3(x_2+y_2) + 2(x_3+y_3) \ \ \ 3(x_1+y_1) - (x_3+y_3)]^T = [-3x_2 + 2x_3 \ \ 3x_1 - x_3]^T + [-3y_2 + 2y_3 \ \ 3y_1 - y_3]^T$ ✓

II. $T_2(\alpha x) = \alpha . T_2(x), \ \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^3$

$T_2([\alpha x_1 \ \alpha x_2 \ \alpha x_3]^T) = \alpha . T_2([x_1 \ x_2 \ x_3]^T)$

$[-3\alpha x_2 + 2\alpha x_3 \ \ \ 3\alpha x_1 - \alpha x_3]^T = \alpha . [-3x_2 + 2x_3 \ \ 3x_1 - x_3]^T$ ✓

c. $T_3 : \mathbb{R}^3 \to \mathbb{R}^3$ def. por $T_3([x_1 \ x_2 \ x_3]^T) := [-3x_2 + 2x_3 \ \ 3x_1 - x_3 \ \ -2x_1 + x_2]^T$

I. $T_3(x+y) = T_3(x) + T_3(y), \ \forall x,y \in \mathbb{R}^3$

$[-3(x_2+y_2) + 2(x_3+y_3) \ \ 3(x_1+y_1) - (x_3+y_3) \ \ -2(x_1+y_1) + (x_2+y_2)]^T = [-3x_2 + 2x_3 \ \ 3x_1 - x_3 \ \ -2x_1 + x_2]^T + [-3y_2 + 2y_3 \ \ 3y_1 - y_3 \ \ -2y_1 + y_2]^T$ ✓

II. $T_3(\alpha x) = \alpha T_3(x)$

$[-3\alpha x_2 + 2\alpha x_3 \ \ 3\alpha x_1 - \alpha x_3 \ \ -2\alpha x_1 + \alpha x_2]^T = \alpha [-3x_2 + 2x_3 \ \ 3x_1 - x_3 \ \ -2x_1 + x_2]^T$ ✓

2.2. Usando q' toda T.L. $T: \mathbb{K}^n \to \mathbb{K}^m$ verifica q' $T(\alpha_1 v_1 + \dots + \alpha_k v_k) = \alpha_1 T(v_1) + \dots + \alpha_k T(v_k)$ para cualquier cantidad k de vectores $v_1, \dots, v_k \in \mathbb{K}^n$ y escalares $\alpha_1, \dots, \alpha_k \in \mathbb{K}$, comprobar que:

$T([x_1 \dots x_n]^T) = [T(e_1) \dots T(e_n)][x_1 \dots x_n]^T$.

Concluir que todas las TL de $\mathbb{K}^n$ en $\mathbb{K}^m$ son de la forma $T(x) = A_T x$, donde $A_T \in \mathbb{K}^{m \times n}$ es la matriz def. por: $A_T := [T(e_1) \dots T(e_n)]$ ↳ canónicos

$\cdot [x_1\ x_2 \dots x_n]^T = \sum_{i=1}^{n} x_i e_i$

$[x_1 \dots x_n]^T = x_1 . e_1 + \dots + x_n . e_n$

$T[x_1 \dots x_n]^T = T(x_1 e_1 + \dots + x_n e_n)$

$T[x_1 \dots x_n]^T = T(x_1 e_1) + \dots + T(x_n e_n)$

$T[x_1 \dots x_n]^T = x_1 T(e_1) + \dots + y_2 . T(e_n)$.

$T[x_1 \dots x_n]^T = \sum_{i=1}^{n} x_i . T(e_i)$.

$T[x_1 \dots x_n]^T = \sum_{i=1}^{n} y_i . \sum_{i=1}^{n} T(e_i)$.

$T[x_1 \dots x_n]^T = [T(e_1) \dots T(e_n)] . [x_1 \dots x_n]$.

I. $T_1: \mathbb{R}^3 \to \mathbb{R}$, $T([x_1 x_2 x_3]^T) = -3x_2 + 2x_3$

$\left.\begin{array}{l} \cdot T([1\ 0\ 0]) = 0 \\ \cdot T([0\ 1\ 0]^T) = -3 \\ \cdot T([0\ 0\ 1]^T) = 2 \end{array}\right\} A_{T_1} x = (0\ -3\ 2)(x_1\ x_2\ y_3)^T$

II. $T_2: \mathbb{R}^3 \to \mathbb{R}^2$, $T([x_1\ x_2\ x_3]^T) = [-3x_2 + 2x_3 \quad 3x_1 - x_3]^T$

$\left.\begin{array}{l} \cdot T([1\ 0\ 0]^T) = [0\ \ 3]^T \\ \cdot T([0\ 1\ 0]^T) = [-3\ \ 0]^T \\ \cdot T([0\ 0\ 1]^T) . [2\ \ -1]^T \end{array}\right\} A_{T_2} . x = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \end{pmatrix} . (x_1\ \ x_2\ x_3)^T$

III. $T_3: \mathbb{R}^3 \to \mathbb{R}^3$, $T([x_1\ x_2\ x_3]^T) = [-3x_2 + 2x_3 \quad 3x_1 - x_3 \quad -2x_1 + x_2]^T$.

$\left.\begin{array}{l} \cdot T([1\ 0\ 0]^T) = [0\ 3\ -2]^T. \\ \cdot T([0\ 1\ 0]^T) = [-3\ 0\ 1]^T \\ \cdot T([0\ 0\ 1]^T) = [2\ -1\ 0]^T \end{array}\right\} A_{T_3} x = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

EJERCICIOS

2.3. Sea $T: \mathbb{R}^5 \to \mathbb{R}^5$ la T.L. def. por $T([x_1\ x_2\ x_3\ x_4\ x_5]^T) = \begin{bmatrix} -x_1 + x_2 + x_3 - 2x_4 + x_5 \\ -x_1 + 3x_3 - 4x_4 + 2x_5 \\ -x_1 + 3x_3 - 5x_4 + 3x_5 \\ -x_1 + 3x_3 - 6x_4 + 4x_5 \\ -x_1 + 3x_3 - 6x_4 + 4x_5 \end{bmatrix}$

a. Hallar una base del núcleo de T.

$$\begin{bmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{bmatrix}$$

$\hookrightarrow$ Núcleo de $T = \{x \in \mathbb{R}^5 : T(x) = 0_{\mathbb{R}^5}\}$ $\Rightarrow$ $\left[\begin{array}{ccccc|c} -1 & 1 & 1 & -2 & 1 & 0 \\ -1 & 0 & 3 & -4 & 2 & 0 \\ -1 & 0 & 3 & -5 & 3 & 0 \\ -1 & 0 & 3 & -6 & 4 & 0 \\ -1 & 0 & 3 & -6 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} -1 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$-x_4 + x_5 = 0$
$\boxed{x_5 = x_4}$

$-x_2 + 2x_3 - 2x_4 + x_5 = 0$
$-x_2 + 2x_3 - 2x_5 + x_5 = 0.$
$-x_2 + 2x_3 - x_5 = 0.$
$\boxed{2x_3 - x_5 = x_2}$

$-x_1 + x_2 + x_3 - 2x_4 + x_5 = 0.$
$-x_1 + 2x_3 - x_5 + x_3 - 2x_5 + x_5 = 0.$
$-x_1 + 3x_3 - 2x_5 = 0.$
$\boxed{3x_3 - 2x_5 = x_1}$

$[x_1\ x_2\ x_3\ x_4\ x_5]^T = [3x_3 - 2x_5\ \ 2x_3 - x_5\ \ x_3\ \ x_5\ \ x_5]^T = x_3[3\ 2\ 1\ 0\ 0]^T + x_5[-2\ -1\ 0\ 1\ 1]^T$

$$\boxed{Nu(T) = ([3\ 2\ 1\ 0\ 0]^T, [-2\ -1\ 0\ 1\ 1]^T)}$$

b. Hallar una base de la imagen de T

$E_{\mathbb{R}^5} = \{[1\ 0\ 0\ 0\ 0]^T, [0\ 1\ 0\ 0\ 0]^T, [0\ 0\ 1\ 0\ 0]^T, [0\ 0\ 0\ 1\ 0]^T, [0\ 0\ 0\ 0\ 1]^T\}$

$Im(T) = gen\{T([1\ 0\ 0\ 0\ 0]^T), T([0\ 1\ 0\ 0\ 0]^T), T([0\ 0\ 1\ 0\ 0]^T), T([0\ 0\ 0\ 1\ 0]^T), T([0\ 0\ 0\ 0\ 1]^T)\}$

~~$Im(T) = gen$~~

$\boxed{y \in Im(T) \longleftrightarrow y \in Col(A)}$ $\therefore B(Im(T)) = B(Col(A))$

$A(x) = [A_1\ A_2\ A_3\ A_4\ A_5][x_1\ x_2\ x_3\ x_4\ x_5]^T = x_1.A_1 + x_2.A_2 + x_3.A_3 + x_4.A_4 + x_5.A_5$

$\to Col(A) = \{y \in \mathbb{R}^5 : y = A(x), x \in \mathbb{R}^5\}$
$\hookrightarrow$ C.L. de las columnas de A.

• Por el teorema de la dimensión: $\dim[Col(A)] = \dim(\mathbb{R}^5) - \underbrace{\dim[Nul(A)]}_{\dim[Nu(A)]} \Rightarrow \dim[Col(A)] = 5 - 2 = 3.$

$\hookrightarrow$ elijo tres columnas LI.

$$\begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 3 & 3 \\ -2 & -4 & -5 & -6 & -6 \\ 1 & 2 & 3 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ → se anulan

$\Rightarrow$ $\boxed{B(Im(T)) = \{[-1\ -1\ -1\ -1\ -1]^T, [1\ 0\ 0\ 0\ 0]^T, [-2\ -4\ -5\ -6\ -6]^T\}}$

c. Comprobar vector $b = [1\ 2\ 2\ 2\ 2]^T \in Im(T)$ y resolver ec. $T(x) = b$.

$b = a.v_1 + b.v_2 + c.v_3$

$[1\ 2\ 2\ 2\ 2]^T = a.[-1\ -1\ -1\ -1\ -1]^T + b.[1\ 0\ 0\ 0\ 0]^T + c.[-2\ -4\ -5\ -6\ -6]^T$

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & 1 \\ -1 & 0 & -4 & 2 \\ -1 & 0 & -5 & 2 \\ -1 & 0 & -6 & 2 \\ -1 & 0 & -6 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -2 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c = 0 \rightarrow -b = 1$

$b = -1 \rightarrow -a - 1 = 1$

$-a = 2$

$a = -2.$

$$\boxed{T(x).\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}}$$

$3\times1.\quad 5\times1.$

2.4. Sea $T: \mathbb{R}_3[x] \rightarrow \mathbb{R}^3$ la aplicación def. por $T(p) = [p(0)\ p(1)\ p(2)]^T$.

a. Explicar porqué T es una T.L.

$p = a_0 + a_1x + a_2x^2 + a_3x^3$

$T(p) = [a_0\quad a_0 + a_1 + a_2 + a_3\quad a_0 + 2a_1 + 4a_2 + 8a_3]^T$

I. $T(p+q) = T(p) + T(q)$

$\hookrightarrow [p_0 + q_0\quad p_0 + q_0 + p_1 + q_1 + p_2 + q_2 + p_3 + q_3\quad p_0 + q_0 + 2(p_1 + q_1) + 4(p_2 + q_2) + 8(p_3 + q_3)]$

$= [p_0\quad p_0 + p_1 + p_2 + p_3\quad p_0 + 2p_1 + 4p_2 + 8p_3]^T + [q_0\quad q_0 + q_1 + q_2 + q_3\quad q_0 + 2q_1 + 4q_2 + 8q_3]^T$ ✓

II. $T(\alpha p) = \alpha . T(p)$

$\hookrightarrow [\alpha p_0\quad \alpha p_0 + \alpha p_1 + \alpha p_2 + \alpha p_3\quad \alpha a_0 + 2\alpha a_1 + 4\alpha a_2 + 8\alpha a_3]^T = \alpha.[p_0\quad p_0 + p_1 + p_2 + p_3\quad p_0 + 2p_1 + 4p_2 + 8p_3]^T$ ✓

b. Hallar una base del Nu(T)

$\rightarrow Nu(T) = \{p \in \mathbb{R}_3[x] : T(p) = 0_{\mathbb{R}^3}\}$

$\rightarrow p \in \mathbb{R}_3[x] : p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \rightarrow T(p) = [a_0\quad a_0 + a_1 + a_2 + a_3\quad a_0 + 2a_1 + 4a_2 + 8a_3]$

¿Cuándo $T(p) = 0$? $\rightarrow$ $a_0 = 0$, $a_0 + a_1 + a_2 + a_3 = 0$, $a_0 + 2a_1 + 4a_2 + 8a_3 = 0$ $\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 6 & 0 \end{array}\right]$

$\hookrightarrow 2a_2 + 6a_3 = 0$ $\quad \boxed{a_2 = -3a_3}$ $\quad a_1 + a_2 + a_3 = 0$ $\quad \boxed{a_1 = +2a_3}$ $\quad \boxed{a_0 = 0}$

$p(x) = 2a_3x - 3a_3x^2 + a_3x^3 = a_3(2x - 3x^2 + x^3)$

$B(Nu(T)) = \{(2x - 3x^2 + x^3)\} \rightarrow dim[Nu(T)] = 1$

EJERCICIOS

e. Mostrar q' para c/ $j \in I_3$, la ec. $T(p) = e_j$ admite sol. y hallar todas las soluciones de la misma

$$\begin{cases} j=1: & T(p) = e_1 = (1\ 0\ 0)^T \\ j=2: & T(p) = e_2 = (0\ 1\ 0)^T \\ j=3: & T(p) = e_3 = (0\ 0\ 1)^T \end{cases}$$

I. $\boxed{j=1} \Rightarrow T(p) = [1\ 0\ 0]^T = [d_0 \quad d_0+d_1+d_2+d_3 \quad d_0+2d_1+4d_2+8d_3]^T$

$$\begin{cases} d_0 = 1 \\ d_0+d_1+d_2+d_3 = 0 \\ d_0+2d_1+4d_2+8d_3 = 0 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 1&0&0&0&1 \\ 1&1&1&1&0 \\ 1&2&4&8&0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1&0&0&0&1 \\ 0&1&1&1&-1 \\ 0&0&2&6&1 \end{array}\right]$$

$2d_2 + 6d_3 = 1$

$\boxed{d_2 = \frac{1}{2} - 3d_3}$

$d_1 + d_2 + d_3 = -1$

$d_1 + \frac{1}{2} - 3d_3 + d_3 = -1$

$\boxed{d_1 = -\frac{3}{2} + 2d_3}$

$\boxed{d_0 = 1}$

$p(x) = 1 + \left(-\frac{3}{2} + 2d_3\right)x + \left(\frac{1}{2} - 3d_3\right)x^2 + d_3x^3$

$p(x) = 1 - \frac{3}{2}x + 2d_3x + \frac{1}{2}x^2 - 3d_3x^2 + d_3x^3$

$p(x) = \left(1 - \frac{3}{2}x + \frac{1}{2}x^2\right) + d_3\left(2x - 3x^2 + x^3\right)$

↳ sol. particular

$\therefore \boxed{T\left(1 - \frac{3}{2}x + \frac{1}{2}x^2\right) = (1\ 0\ 0)^T}$

↳ $\in Nu(T)$

$\therefore T(2x - 3x^2 + x^3) = (0\ 0\ 0)^T$

II. $\boxed{j=2} \Rightarrow T(p) = [0\ 1\ 0]^T = [\dots]^T$

$$\begin{cases} d_0 = 0 \\ d_0+d_1+d_2+d_3 = 1 \\ d_0+2d_1+4d_2+8d_3 = 0 \end{cases} \rightarrow \left[\begin{array}{cccc|c} 1&0&0&0&0 \\ 1&1&1&1&1 \\ 1&2&4&8&0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1&0&0&0&0 \\ 0&1&1&1&1 \\ 0&0&2&6&-2 \end{array}\right]$$

$2d_2 + 6d_3 = -2$

$\boxed{d_2 = -1 - 3d_3}$

$d_1 + d_2 + d_3 = 1$

$d_1 - 1 - 3d_3 + d_3 = 1$

$\boxed{d_1 = 2 + 2d_3}$

$\boxed{d_0 = 0}$

$p(x) = (2 + 2d_3)x + (-1 - 3d_3)x^2 + d_3x^3 = 2x + 2d_3x - x^2 - 3d_3x^2 + d_3x^3$

$p(x) = (2x - x^2) + (2x - 3x^2 + x^3)d_3$

↳ sol. part.

$\therefore \boxed{T(2x - x^2) = (0\ 1\ 0)^T}$

↳ $\in Nu(T)$

$\therefore T(2x - 3x^2 + x^3) = 0_{\mathbb{R}^3}$

$\boxed{III) = 3} \to T(p) = [0\ 0\ 1]^T = [\ldots]^T$

$$\begin{cases} d_0 = 0 \\ d_0 + d_1 + d_2 + d_3 = 0 \\ d_0 + 2d_1 + 4d_2 + 8d_3 = 1 \end{cases} \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 8 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 6 & 1 \end{array}\right]$$

$2d_2 + 6d_3 = 1$

$\boxed{d_2 = \frac{1}{2} - 3d_3}$

$d_1 + d_2 + d_3 = 0$

$d_1 + \frac{1}{2} - 3d_3 + d_3 = 0$

$d_1 + \frac{1}{2} - 2d_3 = 0$

$\boxed{d_1 = -\frac{1}{2} + 2d_3}$

$\boxed{d_0 = 0}$

$p(x) = \left(-\frac{1}{2} + 2d_3\right)x + \left(\frac{1}{2} - 3d_3\right)x^2 + d_3x^3 = -\frac{1}{2}x + 2d_3x + \frac{1}{2}x^2 - 3d_3x^2 + d_3x^3$

$p(x) = \left(-\frac{1}{2}x + \frac{1}{2}x^2\right) + \left(2x - 3x^2 + x^3\right)d_3$

$\hookrightarrow$ sol. part. $\qquad \hookrightarrow \in Nu(T)$

$\therefore T(2x - 3x^2 + x^3) = [0\ 0\ 0]^T$

$\therefore \boxed{T\left(-\frac{1}{2}x + \frac{1}{2}x^2\right) = [0\ 0\ 1]^T}$

d. Resolver la ec: $T(p) = [3\ 6\ 36]^T$

$T(p) = [d_0\quad d_0 + d_1 + d_2 + d_3\quad d_0 + 2d_1 + 4d_2 + 8d_3]^T = [3\ 6\ 36]^T$

$$\hookrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 36 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 2 & 6 & 27 \end{array}\right]$$

$2d_2 + 6d_3 = 27$

$\boxed{d_2 = \frac{27}{2} - 3d_3}$

$d_1 + d_2 + d_3 = 3$

$d_1 + \frac{27}{2} - 3d_3 + d_3 = 3$

$\boxed{d_1 = -\frac{21}{2} + 2d_3}$

$\boxed{d_0 = 3}$

$p(x) = 3 + \left(-\frac{21}{2} + 2d_3\right)x + \left(\frac{27}{2} - 3d_3\right)x^2 + d_3x^3 = 3 - \frac{21}{2}x + 2xd_3 + \frac{27}{2}x^2 - 3x^2d_3 + d_3x^3$

$p(x) = \left(3 - \frac{21}{2}x + \frac{27}{2}x^2\right) + d_3\left(2x - 3x^2 + x^3\right)$

$\hookrightarrow$ s.p. $\qquad \hookrightarrow \in Nu(T)$

Escaneado con CamScanner

EJERCICIOS

2.5. Sea $T: \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ la aplicación def. por: $T(p) = p + (1-x)p'$.

a. Explicar por qué T está bien def. y es una T.L.

$\dim(\mathbb{R}_3[x]) = 4 \to B = \{1, x, x^2, x^3\}$ $\qquad p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

↳ base conj. de partida

- $T(1) = 1 + (1-x).0 = 1$
- $T(x) = x + (1-x).1 = 1$
- $T(x^2) = x^2 + (1-x).2x = x^2 + 2x - 2x^2 = -x^2 + 2x$
- $T(x^3) = x^3 + (1-x).3x^2 = x^3 + 3x^2 - 3x^3 = -2x^3 + 3x^2$

} la imagen de c/elemento de una base de salida $(\mathbb{R}_3[x])$ se convierte en un único elemento del conj. de llegada $(\mathbb{R}_3[x])$ ✓

- Verifico sea T.L.

I. $T(p+q) = T(p) + T(q)$

$(p+q) + (1-x)(p+q)' = p + (1-x).p' + q + (1-x)q'$ ✓

II. $T(\alpha p) = \alpha.T(p)$

$\alpha p + (1-x).(\alpha p)' = \alpha\left[p + (1-x)p'\right]$ ✓

b. Hallar una base del núcleo de T.

$\to Nu(T) = \{p \in \mathbb{R}_3[x] : T(p) = 0_{\mathbb{R}_3[x]}\}$

$p + (1-x)p' = 0. \to p(x) + (1-x).p'(x) = 0 \to$ EDO de primer orden de variables separables

$p = -(1-x)\frac{dp}{dx} \to \frac{dx}{-(1-x)} = \frac{dp}{p} \to \int \frac{1}{(x-1)}dx = \int \frac{1}{p}dp \to \ln|x-1| + C_1 = \ln|p| \to (x-1).k = |p|$

$\to p = (x-1).K.$

$B[Nu(T)] = \{x-1\}, \dim(Nu(T)) = 1.$

c. Hallar una base de la imagen de T

$\underbrace{\dim[Nu(T)]}_{1} + \dim[Im(T)] = \underbrace{\dim(\mathbb{R}_3[x])}_{4} \to \dim[Im(T)] = 3.$

$Im(T) = gen\{T(1), T(x), T(x^2), T(x^3)\} = gen\{1, 1, -x^2+2x, -2x^3+3x^2\} \to$ son LI?

$\alpha + \beta(-x^2+2x) + \gamma.(-2x^3+3x^2) = 0.$

$\alpha - \beta x^2 + 2\beta x + (-2\gamma)x^3 + 3\gamma x^2 = 0.$

$\alpha + x(2\beta) + x^2(-\beta+3\gamma) + x^3(-2\gamma) = 0.$

$$\begin{cases} \alpha & = 0 \\ 2\beta & = 0 \\ -\beta + 3\gamma & = 0 \\ -2\gamma & = 0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \rightarrow \alpha = \beta = \gamma = 0$$

$\Downarrow$ son LI

$B[Im(T)] = \{1, -x^2 + 2x, -2x^3 + 3x^2\}$

↳ $\dim[Im(T)] = 3$ ✓

d- Comprobar que el polinomio $q = 1 + x + x^2 - x^3 \in Im(T)$ y resolver $T(p) = q$.

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$B[Im(T)] = \{1, -x^2 + 2x, -2x^3 + 3x^2\}$

↳ $\alpha . 1 + \beta . (-x^2 + 2x) + \gamma . (-2x^3 + 3x^2) = 1 + x + x^2 - x^3$

$\alpha + (-\beta) x^2 + 2\beta x + (-2\gamma) x^3 + 3\gamma x^2 = 1 + 1 . x + 1 x^2 + (-1) . x^3$

$\alpha + x . (2\beta) + x^2 . (-\beta + 3\gamma) + x^3 (-2\gamma) = 1 + x . 1 + x^2 . 1 + x^3 . (-1)$

↳ $\begin{cases} \alpha & = 1 \\ 2\beta & = 1 \\ -\beta + 3\gamma & = 1 \\ -2\gamma & = -1 \end{cases} \rightarrow \boxed{\alpha = 1} \quad \boxed{\beta = \frac{1}{2}}$

$-\frac{1}{2} + 3\gamma = 1$

$3\gamma = \frac{3}{2}$

$\boxed{\gamma = \frac{1}{2}}$

$p(x) = 1 . 1 + \frac{1}{2}(-x^2 + 2x) + \frac{1}{2}(-2x^3 + 3x^2) = 1 + x + x^2 - x^3$ ✓

$\Downarrow$ $\alpha . T(1) + \beta . T(x^2) + \gamma . T(x^3) = q$

$T(\alpha + \beta x^2 + \gamma x^3) = q$

↳ $\boxed{T^{-1}(q) = \alpha + \beta x^2 + \gamma x^3 = 1 + \frac{1}{2} x^2 + \frac{1}{2} x^3}$

Ejercicios

26. Sea $T: \mathbb{R}^3 \to \mathbb{R}^3$ la T.L. def. por $T([x_1\ x_2\ x_3]^T) = [bx_3 - x_2 \quad x_1 - ax_3 \quad ax_2 - bx_1]^T$, donde $a, b \in \mathbb{R}$ tal q: $Im(T) = gen\{[0\ 1\ -1]^T, [-1\ 0\ 1]^T\}$
Comprobar que $y = [2\ 2\ -4]^T \in Im(T)$ y resolver $T(x) = y$.

- $T([1 0 0]^T) = [0\ 1\ -b]^T$
- $T([0 1 0]^T) = [-1\ 0\ a]^T$
- $T([0 0 1]^T) = [b\ -a\ 0]^T$

$Im(T) = gen\{T(e_1), T(e_2), T(e_3)\} = gen\{[0\ 1\ -b]^T, [-1\ 0\ a]^T, [b\ -a\ 0]^T\}$
$\quad = gen\{[0\ 1\ -1]^T, [-1\ 0\ 1]^T\}$

$\Rightarrow x_1 [0\ 1\ -b]^T + x_2 [-1\ 0\ a]^T + x_3 [b\ -a\ 0]^T = x_4 [0\ 1\ -1]^T + x_5 [-1\ 0\ 1]^T$

$$\begin{cases} -1 x_2 + bx_3 + x_5 = 0 \\ 1x_1 - ax_3 - x_4 = 0 \\ -bx_1 + ax_2 + x_4 - x_5 = 0 \end{cases} \to \left[\begin{array}{ccccc|c} 0 & -1 & b & 0 & 1 & 0 \\ 1 & 0 & -a & -1 & 0 & 0 \\ -b & a & 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & -a & -1 & 0 & 0 \\ 0 & -1 & b & 0 & 1 & 0 \\ 0 & 0 & 0 & -b+1 & a-1 & 0 \end{array}\right]$$

- $x_1 - ax_3 - x_4 = 0$
  $x_1 - x_4 = ax_3$
  $\frac{x_1 - x_4}{x_3} = a$
- $-x_2 + bx_3 + x_5 = 0$
  $bx_3 = x_2 - x_5$
  $b = \frac{x_2 - x_5}{x_3}$
- $(-b+1)x_4 = (-a+1)x_5$
  $\left[-\left(\frac{x_2 - x_5}{x_3}\right) + 1\right] x_4 = \left[-\left(\frac{x_1 - x_4}{x_3}\right) + 1\right] x_5$

$$\alpha \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = T(e_i) \overset{\text{por ej}}{=} T(e_3) = T([0\ 0\ 1]^T) = [b\ -a\ 0]^T$$

$$\left[\begin{array}{cc|c} 0 & -1 & b \\ 1 & 0 & -a \\ -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -a \\ 0 & -1 & b \\ 0 & 0 & -a+b \end{array}\right] \to \boxed{\begin{matrix} a = -1 \\ b = -1 \\ a = b \end{matrix}}$$

$y \in Im(T)$ si y es CL. de gen$(Im(T))$
$\hookrightarrow \alpha [0\ 1\ -1]^T + \beta [-1\ 0\ 1]^T + \gamma [1\ -1\ 0]^T = [2\ 2\ -4]^T$

$$\left[\begin{array}{ccc|c} 0 & -1 & 1 & 2 \\ 1 & 0 & -1 & 2 \\ -1 & 1 & 0 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-\beta + \gamma = 2$
$\gamma = 2 + \beta$

$\alpha - \gamma = 2$
$\alpha - 2 - \beta = 2$
$\alpha = 4 + \beta$

$[\alpha\ \beta\ \gamma] = [4+\beta\ \ \beta\ \ 2+\beta]$
$= \underbrace{\beta[1\ 1\ 1]}_{\in Nu(T)} + \underbrace{[4\ 0\ 2]}_{\text{sol. part}}$

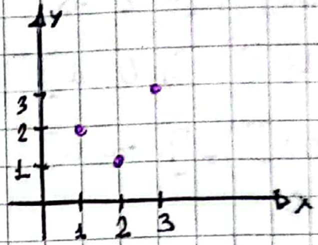
2.8. Sea $T: \mathbb{R}^2 \to \mathbb{R}^2$ la T.L. def. por $T(x) = Ax$, donde $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ y sean e_1, e_2 los vectores de la base canónica $\mathbb{R}^2$: $e_1 = [1\ 0]^T$; $e_2 = [0\ 1]^T$. Hallar y graficar la imagen por T del conj $R \subseteq \mathbb{R}^2$ def. por

a. $R = \{e_1, e_2, e_1 + e_2\} = \{[1\ 0]^T, [0\ 1]^T, [1\ 1]^T\}$

$T([1\ 0]^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [2\ 1]^T$

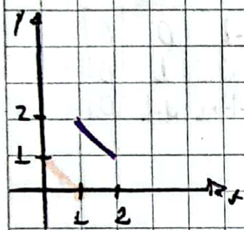
$T([0\ 1]^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [1\ 2]^T$

$T([1\ 1]^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [3\ 3]^T$



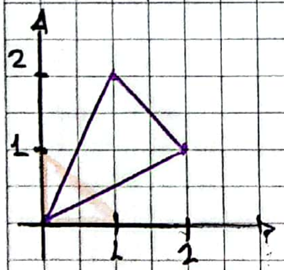
b. R es el segmento de recta que une a los ptos e_1 y e_2, es decir, $R = \mathcal{C}(\{e_1, e_2\})$

$T(x) = T(\lambda_1 e_1 + \lambda_2 e_2) = \lambda_1 . T(e_1) + \lambda_2 . T(e_2) = \lambda_1 . [2\ 1]^T + \lambda_2 [1\ 2]^T$ con $\lambda_1 + \lambda_2 = 1$



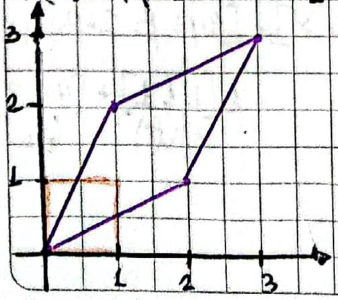
c. R es el Δ de vértices $0, e_1, e_2$, es decir, $R = \mathcal{C}(\{0, e_1, e_2\})$

$T(x) = T(\lambda_1 0 + \lambda_2 e_1 + \lambda_3 e_2) = \lambda_1 . [0\ 0]^T + \lambda_2 . [2\ 1]^T + \lambda_3 [1\ 2]^T$



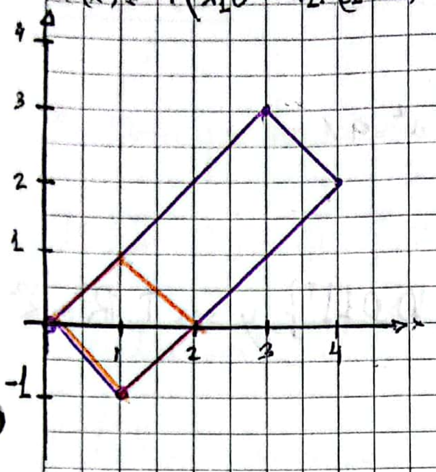
d. R es el cuadrado de vértices $0, e_1, e_2, e_1 + e_2$, es decir, $R = \mathcal{C}(\{0, e_1, e_2, e_1 + e_2\})$

$T(x) = T(\lambda_1 . 0 + \lambda_2 . e_1 + \lambda_3 . e_2 + \lambda_4 . (e_1 + e_2)) = \lambda_1 . [0\ 0]^T + \lambda_2 . [2\ 1]^T + \lambda_3 [1\ 2]^T + \lambda_4 [3\ 3]^T$



e- R es el paralelogramo de vértices $0, e_1+e_2, e_1-e_2, 2e_1$, es decir, $R = \mathcal{C}(\{0, e_1+e_2, e_1-e_2, 2e_1\})$

$T(x) = T(\lambda_1 0 + \lambda_2 (e_1+e_2) + \lambda_3 (e_1-e_2) + \lambda_4 (2e_1)) = \lambda_1 [0\ 0]^T + \lambda_2 [3\ 3]^T + \lambda_3 [1\ -1]^T + \lambda_4 [4\ 2]^T$



2.9. Hallar todos los $a \in \mathbb{R}$ para los cuales $\exists$ t.l. $T: \mathbb{R}^3 \to \mathbb{R}^3$ tq

$T([1\ 1\ 1]^T) = [1\ a\ 1]^T$

$T([1\ 0\ -1]^T) = [1\ 0\ 1]^T$

$T([-1\ -1\ 0]^T) = [1\ 2\ 3]^T$

$T([1\ -1\ -1]^T) = [5\ 1\ a^2]^T$

$\Rightarrow$ Podría decir que una base de $\mathbb{R}^3$ ($\dim(\mathbb{R}^3) = 3$) está generada por $B_{\mathbb{R}^3} = \text{gen}\{[1\ 1\ 1]^T, [1\ 0\ -1]^T, [-1\ -1\ 0]^T, [1\ -1\ -1]^T\}$ $\hookrightarrow$ son 4.

$\hookrightarrow$ veo si son L.I. los 3 primeros

$a \cdot [1\ 1\ 1]^T + b \cdot [1\ 0\ -1]^T + c \cdot [-1\ -1\ 0]^T$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow a = b = c = 0 \checkmark$$

$B_{\mathbb{R}^3} = \{[1\ 1\ 1]^T, [1\ 0\ -1]^T, [-1\ -1\ 0]^T\}$

$\Rightarrow [1\ -1\ -1]$ es c.l. de los otros tres $\Rightarrow [1\ -1\ -1] = \alpha [1\ 1\ 1]^T + \beta [1\ 0\ -1]^T + \gamma [-1\ -1\ 0]^T$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \gamma = 2,\ \beta = 2,\ \alpha = 1.$$

$[1\ {-1}\ {-1}]^T = [1\ 1\ 1]^T + 2.[1\ 0\ {-1}]^T + 2.[-1\ {-1}\ 0]^T$

$T[1\ {-1}\ {-1}]^T = T[1\ 1\ 1]^T + 2.T[1\ 0\ {-1}]^T + 2.T[-1\ {-1}\ 0]^T$

$[5\ 1\ a^2]^T = [1\ a\ 1]^T + 2.[1\ 0\ 1]^T + 2.[1\ 2\ 3]^T$

$[5\ 1\ a^2]^T = [5\ \ a+4\ \ 9]^T \Rightarrow 5=5 \checkmark \wedge 1 = a+4 \wedge a^2 = 9 \checkmark$

$-3 = a \checkmark$

$a = -3$

2.10. Sea B la base de $\mathbb{R}^3$ def. por $B = \{[1\ 1\ 0]^T, [1\ {-1}\ 0]^T, [0\ 0\ 1]^T\}$, y sea $T: \mathbb{R}^3 \to \mathbb{R}^3$ una T.L. q' actúa sobre la base B de la siguiente manera

$T([1\ 1\ 0]^T) = [1\ {-3/2}\ 2]^T$,

$T([1\ {-1}\ 0]^T) = [-3\ 9/2\ {-6}]^T$,

$T([0\ 0\ 1]^T) = [2\ {-3}\ 4]^T$

a. Hallar una base del núcleo de T y describirlo geométricamente.

$Nu(T) = \{(x_1\ x_2\ x_3)^T \in \mathbb{R}^3 : T([x_1\ x_2\ x_3]^T) = [0\ 0\ 0]^T\}$

~~b. Hallar una base de la imagen de T~~

$a.T(e_1) + b.T(e_2) + c.T(e_3) = 0_{\mathbb{R}^3}$

$a.[1\ {-3/2}\ 2]^T + b.[-3\ 9/2\ {-6}]^T + c[2\ {-3}\ 4]^T = [0\ 0\ 0]^T$

$$\begin{cases} a - 3b + 2c = 0 \\ -\frac{3}{2}a + \frac{9}{2}b - 3c = 0 \\ 2a - 6b + 4c = 0 \end{cases} \to \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ -\frac{3}{2} & 9/2 & -3 & 0 \\ 2 & -6 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$a - 3b + 2c = 0$

$a = 3b - 2c$

$(3b-2c).T([1\ 1\ 0]^T) + b.T([1\ {-1}\ 0]^T) + c.T([0\ 0\ 1]^T) = [0\ 0\ 0]^T$

$T([3b-2c\ \ 3b-2c\ \ 0]^T) + T([b\ {-b}\ 0]^T) + T([0\ 0\ c]^T) = [0\ 0\ 0]^T$

$T([4b-2c\ \ 2b-2c\ \ c]^T) = [0\ 0\ 0]^T$

$T(\ b.[4\ 2\ 0]^T + c[-2\ {-2}\ 1]^T) = [0\ 0\ 0]^T$

$\hookrightarrow \boxed{B[Nu(T)] = \{[4\ 2\ 0]^T, [-2\ {-2}\ 1]^T\}}$ PLANO!

EJERCICIOS

b. Hallar una base de la imagen de T.

$\underbrace{\dim[Nu(T)]}_{2} + \dim[Im(T)] = \underbrace{\dim(\mathbb{R}^3)}_{3} \Rightarrow \dim[Im(T)] = 1.$

$\Rightarrow$ Noto que las imágenes de la base son múltiplos $\rightarrow$ elijo una:

$B[Im(T)] = \{[1 \ -\frac{3}{2} \ 2]^T\}$

$\hookrightarrow \dim[Im(T)] = 1$ ✓

c. Hallar $T([x_1 \ x_2 \ x_3]^T)$ y calcular $T([2\,3\,5]^T)$

$a.[1\,1\,0]^T + b.[1\,-1\,0]^T + c.[0\,0\,1]^T = [x_1 \ x_2 \ x_3]^T$

$[a \ a \ 0]^T + [b \ -b \ 0]^T + [0\,0\,c]^T = [x_1 \ x_2 \ x_3]^T$

$[a+b \ \ a-b \ \ c] = [x_1 \ x_2 \ x_3] \rightarrow \begin{matrix} x_1 = a+b \\ x_2 = a-b \\ x_3 = c \end{matrix} \Rightarrow a = \frac{x_2+x_1}{2}, \ b = \frac{x_1-x_2}{2}, \ c = x_3$

$a.T[1\,1\,0]^T + b.T[1\,-1\,0]^T + c.T[0\,0\,1]^T = T([x_1 \ x_2 \ x_3])^T$

$a.[1 \ -\frac{3}{2} \ 2]^T + b.[-3 \ \frac{9}{2} \ -6]^T + c.[2\,-3\,4]^T = T([x_1 \ x_2 \ x_3]^T)$

$\left(\frac{x_2+x_1}{2}\right).[1 \ -\frac{3}{2} \ 2]^T + \left(\frac{x_1-x_2}{2}\right)[-3 \ \frac{9}{2} \ -6]^T + x_3[2 \ -3 \ 4]^T = T([x_1 \ x_2 \ x_3]^T)$

$\left[\frac{x_2}{2} + \frac{x_1}{2} - \frac{3}{2}x_1 + \frac{3}{2}x_2 + 2x_3 \quad -\frac{3}{4}x_2 - \frac{3}{4}x_1 + \frac{9}{4}x_1 - \frac{9}{4}x_2 - 3x_3 \quad x_2 + x_1 - 3x_1 + 3x_2 + 4x_3\right]^T = T([\ldots]^T)$

$\left[-x_1 + 2x_2 + 2x_3 \quad \frac{3}{2}x_1 - 3x_2 - 3x_3 \quad -2x_1 + 4x_2 + 4x_3\right]^T = T([x_1 \ x_2 \ x_3]^T)$

$\Rightarrow T([2\,3\,5]^T) = [-2+2.3+2.5 \quad 3-9-15 \quad -4+12+20]^T = [14 \ -21 \ 28]^T$

2.11. Sea B la base de $\mathbb{R}^3$ def. por $B = \{[1\,0\,0]^T, [0\,1\,1]^T, [0\,1\,-1]^T\}$, y sea $T: \mathbb{R}^3 \rightarrow \mathbb{R}_2[x]$ una T.L. q' actúa sobre la base B de la sig. manera:

$T([1\,0\,0]^T) = 1-x$

$T([0\,1\,1]^T) = 1+x^2$

$T([0\,1\,-1]^T) = x+x^2$

Comprobar q' el polinomio $p = 2+x-3x^2 \in Im(T)$ y det. $T^{-1}(p) = T^{-1}(\{p\})$

$\alpha . T(e_1) + \beta . T(e_2) + \gamma\ T(e_3) = p$

$\alpha . (1-x) + \beta . (1+x^2) + \gamma . (x+x^2) = 2+x+3x^2$

$\alpha - \alpha x + \beta + \beta x^2 + \gamma x + \gamma x^2 = 2+x+3x^2$

$(\alpha+\beta) + x(-\alpha+\gamma) + x^2(\beta+\gamma) = 2+x+3x^2$

$\begin{cases} \alpha+\beta = 2 \\ -\alpha+\gamma = 1 \\ \beta+\gamma = 3 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ $\alpha+\beta = 2$. $\beta = 2-\alpha$ $\quad \beta+\gamma = 3$. $2-\alpha+\gamma = 3$. $\gamma = 1+\alpha$.

$[\alpha\ \beta\ \gamma] = \{\underbrace{\alpha[1\ -1\ 1]^T}_{\text{b}\in Nu(T)} + \underbrace{[0\ 2\ 1]^T}_{\text{b sol. part.}}\}$

2.12. Sea B la base de R_3 def. por $B = \{[2\ 2\ 1]^T, [-2\ 1\ 2]^T, [1\ -2\ 2]^T\}$, y sea $T: R^3 \rightarrow R^3$ una T.L. q' actúa sobre la base B de la sig manera:

$T([2\ 2\ 1]^T) = [2\ -1\ -1]^T$;

$T([-2\ 1\ 2]^T) = [-1\ 2\ -1]^T$;

$T([1\ -2\ 2]^T) = [-1\ -1\ 2]^T$

a- Hallar la imagen por T del subespacio gen $\{[1\ 0\ 1]^T, [1\ 1\ 5]^T\}$

- $[1\ 0\ 1]^T = a.[2\ 2\ 1] + b.[-2\ 1\ 2]^T + c[1\ -2\ 2]^T$

$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 1 \\ 2 & 1 & -2 & 0 \\ 1 & 2 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 1 & 1 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 9/2 & 3/2 \end{array}\right]$ $\quad \frac{9}{2}c = \frac{3}{2}$, $c = \frac{1}{3}$ $\quad 3b - 1 = -1$. $b = 0$. $\quad 2a + \frac{1}{3} = 1$. $a = \frac{1}{3}$

$\rightarrow [1\ 0\ 1]^T = \frac{1}{3}[2\ 2\ 1]^T + \frac{1}{3}[1\ -2\ 2]^T$.

$T[1\ 0\ 1]^T = \frac{1}{3}.[2\ -1\ -1]^T + \frac{1}{3}.[-1\ -1\ 2]^T$.

$T[1\ 0\ 1]^T = \left[\frac{1}{3}\ -\frac{2}{3}\ \frac{1}{3}\right]^T$

- $[1\ 1\ 5]^T = a.[2\ 2\ 1]^T + b.[-2\ 1\ 2]^T + c[1\ -2\ 2]^T$.

... $\frac{9}{2}c = \frac{9}{2}$, $c = 1$ $\quad 3b - 3 = 0$. $3b = 3$, $b = 1$ $\quad 2a - 2 + 1 = 1$. $2a = 2$. $a = 1$.

$\rightarrow$ ... $T[1\ 1\ 5]^T = 1.[2\ -1\ -1]^T + 1.[-1\ 2\ -1]^T + 1.[-1\ -1\ 2]^T$.

$T[1\ 1\ 5]^T = [0\ 0\ 0]^T$. $\Rightarrow Im(T) = gen\{[1\ -2\ 1]^T, \underbrace{[0\ 0\ 0]^T}_{\text{b ya está por def.}}\}$

b. Hallar la preimagen por T del subespacio $\{y \in \mathbb{R}^3 : y_1 - y_3 = 0 ,\ y_1 + y_2 + y_3 = 0\}$

$\cdot\ y_1 = y_3 \qquad 2y_1 + y_3 = 0$
$\qquad\qquad\qquad y_3 = -2y_1$

$S = \begin{bmatrix} y_1 \\ -2y_1 \\ y_1 \end{bmatrix} \rightarrow$ Una base de S es: $B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

$T^{-1}(S) = \{ x \in \mathbb{R}^3 / T(x) = S \}$

$T(x) = \lambda\, [1\ -2\ 1]^T$

$\alpha . T(e_1) + \beta . T(e_2) + \gamma . T(e_3) = \lambda [1\ -2\ 1]^T$

$\alpha . [2\ -1\ -1]^T + \beta . [-1\ 2\ -1]^T + \gamma . [-1\ -1\ 2]^T = \lambda [1\ -2\ 1]^T$

$\hookrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 1 \\ -1 & 2 & -1 & -2 \\ -1 & -1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 1 \\ 0 & 3/2 & -3/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\hookrightarrow \frac{3}{2}\beta - \frac{3}{2}\gamma = -\frac{3}{2}\lambda \qquad 2\alpha - \beta - \gamma = \lambda$
$\qquad\qquad\qquad\qquad\qquad\quad 2\alpha - \lambda - \gamma - \gamma = \lambda$
$\beta - \gamma = \lambda \qquad\qquad\quad 2\alpha - 2\lambda - 2\gamma = 0 \qquad \Rightarrow \alpha = \beta = \lambda + \gamma$
$\beta \quad = \lambda + \gamma \qquad\quad 2\alpha = 2\lambda + 2\gamma$
$\qquad\qquad\qquad\qquad\quad \alpha = \lambda + \gamma$

$\Rightarrow x \in T^{-1}(S) \Leftrightarrow x = \alpha . [2\ 2\ 1]^T + \beta . [-2\ 1\ 2]^T + \gamma . [1\ -2\ 2]^T$

$x = \ldots$

* A ver sin λ ...

...

$\hookrightarrow \frac{3}{2}\beta - \frac{3}{2}\gamma = -\frac{3}{2} \qquad 2\alpha - \beta - \gamma = 1$
$\beta - \gamma = -1 \qquad\qquad 2\alpha + 1 - \gamma - \gamma = 1$
$\beta = -1 + \gamma \qquad\qquad 2\alpha = 2\gamma$
$\qquad\qquad\qquad\qquad\quad \alpha = \gamma$

$\Rightarrow x = \alpha . [2\ 2\ 1]^T + \beta . [-2\ 1\ 2]^T + \gamma . [1\ -2\ 2]^T$
$x = \gamma [2\ 2\ 1]^T + (-1+\gamma) [-2\ 1\ 2]^T + \gamma [1\ -2\ 2]^T$
$x = \gamma . \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

$\hookrightarrow \boxed{x = \gamma . [1\ 1\ 5]^T + [2\ -1\ -2]^T \in T^{-1}(S)}$

2.13. Sea $T \in L(V, W)$ con V y W de dimensión finita. Sea $[T]_B^C$ la matriz de T con respecto a las bases B de V y C de W. Verificar:

~~a. T es monomorfismo sii $nul([T]_B^C) = \{0\}$~~

representación matricial: $[T]_B^C$, $B = \{v_1, \dots, v_n\}$, $C = \{w_1, \dots, w_m\}$

$v = \sum_{j=1}^{n} \alpha_j v_j \Rightarrow [v]^B = [\alpha_1 \dots \alpha_n]$

$T(v) = T\left(\sum_{j=1}^{n} \alpha_j v_j\right) = \sum_{j=1}^{n} \alpha_j T(v_j)$

$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$

$\Rightarrow T(v) = \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{m} \alpha_{ij} w_i = \underbrace{\sum_{j=1}^{n} \sum_{i=1}^{m} \alpha_j \alpha_{ij}}_{B_i} w_i = \boxed{\sum_{i=1}^{m} B_i w_i} \dots$

a. T es monomorfismo sii $nul([T]_B^C) = \{0\}$

$\hookrightarrow Nu(T) = \{0\}$

T es mono $\Leftrightarrow (T(v) = 0_W \Leftrightarrow v = 0_W) \Leftrightarrow Nu(T) = \{0_W\}$

Si $T(v) = 0_W \Leftrightarrow [T(v)]^C = 0_{K^m} \Leftrightarrow [T]_B^C [v]^B = 0_{K^m}$

...

2.14. Sea $T \in L(V, W)$ donde V y W son $\mathbb{R}^n$, $\mathbb{R}^{m \times n}$, $\mathbb{R}_n[x]$. Hallar para c/caso, la A_T con respecto a las bases canónicas de V y W, y det. las prop. de T.

a. $T: \mathbb{R}^2 \to \mathbb{R}^3$ es la T.L. def. por $T(x) = Ax$, donde $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$\underbrace{B_{\mathbb{R}^2}}_{B} = \{\underbrace{[1\ 0]^T}_{v_1}, \underbrace{[0\ 1]^T}_{v_2}\}$ y $\underbrace{B_{\mathbb{R}^3}}_{C} = \{\underbrace{[1\ 0\ 0]^T}_{w_1}, \underbrace{[0\ 1\ 0]^T}_{w_2}, \underbrace{[0\ 0\ 1]^T}_{w_3}\}$.

¿$[T]_B^C$? $[T(v)]^C = [T]_B^C [v]^B$

$\cdot T([1\ 0]^T) = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}_{[T(v_1)]^C} \to T([1\ 0]^T) = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}_{3\times 1} \cdot [w_1\ w_2\ w_3] \to T([1\ 0]^T) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \cdot [[1 0 0]^T [0 1 0]^T [0 0 1]^T]$

$\to T([1\ 0]^T) = [1\ 0\ 0]^T + [0\ 3\ 0]^T + [0\ 0\ 5]^T = [1\ 3\ 5]^T$

$\cdot T([0\ 1]^T) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [2\ 4\ 6]^T \to T([0\ 1]^T) = 2 \cdot [1\ 0\ 0]^T + 4 \cdot [0\ 1\ 0]^T + 6 [0\ 0\ 1]^T = [2\ 4\ 6]^T$

Cuando hago matriz canónica$_V$ = coordenadas $\to T(v) =$ coord. canónico$_W$

~~Así también~~ Luego $[T]_B^C = [T(v_1)\ T(v_2)\ T(v_3) \dots]$

$\Rightarrow [T]^C_B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$\hookrightarrow \dim\left[Col\left([T]^C_B\right)\right] = 2.$

• T es epimorfismo $\Leftrightarrow \dim(\mathbb{R}^3) = \dim\left[Col\left([T]^C_B\right)\right]$ $\Rightarrow$ no es epi ni mono.

$3 \neq 2$ ⊗

• T es isomorfismo si $\dim(\mathbb{R}^2) = \dim(\mathbb{R}^3)$

$2 \neq 3$ ⊗

b. $T: \mathbb{R}^3 \to \mathbb{R}^2$ es la T.L. def por $T(x) = Ax$, donde $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

• $T([1\,0\,0]^T) = \underbrace{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}}_{2\times 3} \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{3\times 1} = [1\;2]^T \rightarrow T([1\,0\,0]^T) = 1\cdot[1\,0]^T + 2\cdot[0\,1]^T = [1\;2]^T.$

• $T([0\,1\,0]^T) = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [3\;4]^T \rightarrow T([0\,1\,0]^T) = 3\cdot[1\,0]^T + 4\cdot[0\,1]^T = [3\;4]^T$

• $T([0\,0\,1]^T) = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [5\;6]^T \rightarrow T([0\,0\,1]) = 5\cdot[1\,0]^T + 6\cdot[0\,1]^T = [5\;6]^T$

$\Rightarrow [T]^C_B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$\hookrightarrow \dim\left[Col\left([T]^C_B\right)\right] = 3$

• No es epi ya que $\dim(V) \neq \dim(W)$ $\Rightarrow$ no es isomorfismo

$3 \neq 2$

• $\dim\left[Nu\left([T]^C_B\right)\right] \neq 0 \Rightarrow$ no es mono

c. $T: \mathbb{R}_3[x] \to \mathbb{R}^4$ es la TL def por $T(p) := [p(0)\;\; p(1)\;\; p(10)\;\; p(100)]^T$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \rightarrow p(0) = a_0$ $\quad p(10) = a_0 + 10a_1 + 100a_2 + 1000a_3.$

$p(1) = a_0 + a_1 + a_2 + a_3$ $\quad p(100) = a_0 + 100a_1 + 10000a_2 + 1000000a_3.$

$B_{\mathbb{R}_3[x]} = \{1, x, x^2, x^3\}$, $B_{\mathbb{R}^4} = \{[1\,0\,0\,0]^T, [0\,1\,0\,0]^T, [0\,0\,1\,0]^T, [0\,0\,0\,1]^T\}$

$T(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 1000 & 1000000 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [1\;1\;1\;1]^T$

$T(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 1000 & 1000000 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = [0\;1\;10\;100]^T$

$\Rightarrow \boxed{[T]^C_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 1000 & 1000000 \end{bmatrix}}$

$T(x^2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 1000 & 1000000 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = [0\;1\;100\;10000]^T$

$\hookrightarrow \dim[Nu(T)] = \{0\}.$

$\hookrightarrow$ T es mono y además epi

$\Rightarrow$ T es iso

$T(x^3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 10 & 100 & 1000 \\ 1 & 100 & 10000 & 1000000 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = [0\;1\;1000\;1000000]^T$

d. $T: \mathbb{R}_2[x] \to \mathbb{R}^{2\times2}$ es la T.L. def por $T(p) = \begin{bmatrix} p(0) & p(1) \\ p'(0) & p'(1) \end{bmatrix}$.

$p(x) = d_0 + d_1 x + d_2 x^2$ $\quad$ $p'(x) = d_1 + 2d_2 x$

- $p(0) = d_0 \to [1\,0\,0]^T$ $\quad$ • $p(1) = d_0 + d_1 + d_2 \to [1\,1\,1]^T$
- $p'(0) = d_1 \to [0\,1\,0]^T$ $\quad$ • $p'(1) = d_1 + 2d_2 \to [0\,1\,2]^T$

$B_{\mathbb{R}_2[x]} = \{1, x, x^2\}$, $B_{\mathbb{R}^{2\times2}} = \left\{ \begin{bmatrix} 1&0\\0&0 \end{bmatrix}, \begin{bmatrix} 0&1\\0&0 \end{bmatrix}, \begin{bmatrix} 0&0\\1&0 \end{bmatrix}, \begin{bmatrix} 0&0\\0&1 \end{bmatrix} \right\}$.

Si $p = 1 \to T(1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Si $p = x \to T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \Rightarrow [T]_B^C = \begin{bmatrix} 1&0&0\\1&1&1\\0&1&0\\0&1&2 \end{bmatrix}$

Si $p = x^2 \to T(x^2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

$\to$ es monomorfismo

$\to \dim[C_0([T]_B^C)] \neq \dim(\mathbb{R}^{2\times2}) \to$ es epi
$\quad 3 \neq 4$.

* Para q' sea mono $\to$ busco homogéneo.
  " epi $\to \dim[Col([T]_B^C)] = \dim(W)$.

2.15. Sea $T \in L(\mathbb{R}_2[x], \mathbb{R}^3)$ la T.L. def. por $[T]_B^C = \begin{bmatrix} 1&0&1\\0&1&1\\1&0&1 \end{bmatrix}$, donde B y C son las bases de $\mathbb{R}_2[x]$ y $\mathbb{R}^3$ def. por $B = \{1+x^2, 1+x, x+x^2\}$, $C = \{[1\,1\,0]^T, [1\,0\,1]^T, [0\,1\,1]^T\}$

a. Analizar prop. T.

- Calculo $Nu(T) = \{x \in \mathbb{R}_2[x] / T(x) = 0_{\mathbb{R}^3}\}$

$\underset{3\times3}{\begin{bmatrix} 1&0&1\\0&1&1\\1&0&1 \end{bmatrix}} \cdot \underset{3\times1}{\begin{bmatrix} a\\b\\c \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 0\\0\\0 \end{bmatrix}} \to \left[\begin{array}{ccc|c} 1&0&1&0\\0&1&1&0\\1&0&1&0 \end{array}\right] \to$ $b+c=0$, $-b=c$ $\quad$ $a+c=0$, $a-b=0$, $a=b$

$x = a.(1+x^2) + b(1+x) + c.(x+x^2) = b + bx^2 + b + bx - bx - bx^2 = 2b$.

$T(v_1) = T(1+x^2) = \alpha.[1\,1\,0]^T + \beta.[1\,0\,1]^T + \gamma[0\,1\,1]^T$.

$\begin{bmatrix} 1&1&0\\1&0&1\\0&1&1 \end{bmatrix}$

Como $\dim(\mathbb{R}_2[x]) = \dim(\mathbb{R}^3) = 3 \Rightarrow$ con probar que T es mono, ya pruebo que es epi: (y viceversa)

$\hookrightarrow$ T es mono $\Leftrightarrow \mathrm{Nul}([T]_B^C) = \{0_{\mathbb{R}^3}\}$

$\mathrm{rg}([T]_B^C) = 2$

$$\underbrace{\dim[\mathrm{rg}([T]_B^C)]}_{2} + \dim[\mathrm{Nul}([T]_B^C)] = \underbrace{\dim(\mathbb{R}_2[x])}_{3} \Rightarrow \dim[\mathrm{Nul}([T]_B^C)] = 1$$

Como T no es mono, entonces no es epi. Y por ende, no es iso.

b_ Hallar $T^{-1}([1\ 0\ 1]^T)$

Como T no es iso, no existe.

2.46_ Sean $\mathbb{V} = \{A \in \mathbb{R}^{2\times2} : A^T = A\}$ el $\mathbb{R}$-espacio vectorial de las matrices simétricas de $\mathbb{R}^{2\times2}$, y $T \in L(\mathbb{V}, \mathbb{R}^3)$ tq: $[T]_B^C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, donde es la matriz de T con respecto a las bases $B = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ de $\mathbb{V}$ y $C = \{[1\ 1\ 0]^T, [1\ 0\ 1]^T, [0\ 1\ 1]^T\}$ de $\mathbb{R}^3$.

Hallar el conjunto solución de la ecuación $T(A) = [1\ 0\ 1]^T$.

$\Rightarrow T(A) = [1\ 0\ 1]^T$

$\hookrightarrow$ c.l. de la base de C $\Rightarrow [1\ 0\ 1]^T = \alpha.[1\ 1\ 0]^T + \beta.[1\ 0\ 1]^T + \gamma.[0\ 1\ 1]^T$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{array}\right] \Rightarrow \gamma = 0 \Rightarrow \beta = 1 \Rightarrow \alpha = 0 \rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [A]^C$$

• $[T(A)]^C = [T]_B^C\,[A]^B = ([1\ 0\ 1]^T)^C$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = [0\ 1\ 0]^T$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{ll} \alpha_1 + \alpha_3 = 0 & \alpha_2 + \alpha_3 = 1 \\ \alpha_1 = -\alpha_3 & \alpha_2 = 1 - \alpha_3 \end{array}$$

$$\Rightarrow [A]^B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Entonces, me resta encontrar un A tq $T(A) = [1\ 0\ 1]^T$. Se que las coord son $\begin{bmatrix} -\alpha_3 \\ 1-\alpha_3 \\ \alpha_3 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha_1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = -\alpha_3 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + (1-\alpha_3) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha_3 & 1-\alpha_3 \\ 1-\alpha_3 & -\alpha_3 \end{bmatrix} = [A]$$

$$\hookrightarrow \underbrace{\alpha_3 \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}}_{\text{sol. homogénea}} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{sol. part.}}$$

2.17. Sea $T \in L(\mathbb{R}_2[x], \mathbb{R}^3)$ la T.L. def. por $[T]_B^C = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix}$ donde B y C son las bases de $\mathbb{R}_2[x]$ y $\mathbb{R}^3$ respectivamente, def. por: $B = \{\underbrace{\frac{1}{2}x(x-1)}_{V_1}, \underbrace{-x(x-2)}_{V_2}, \underbrace{\frac{1}{2}(x-1)(x-2)}_{V_3}\}$ y $C = \{[2\ 2\ 1]^T, [-2\ 1\ 2]^T, [1\ -2\ 2]^T\}$.

a- Analizar las prop. de T.

$\det([T]_B^C) \neq 0 \Rightarrow$ Es isomorfismo.

$\hookrightarrow \dim(\mathrm{Im}(T)) = 3$

$\dim(\mathrm{Nu}(T)) = 0$.

b- Hallar la matriz T con respecto a la base canónica de $\mathbb{R}_2[x]$ y la base C de $\mathbb{R}^3$.

- $[T]_E^C = [T]_B^C \cdot M_E^B$

- $\alpha \cdot x^2 + \beta \cdot x + \gamma \cdot 1 = V_1 = \frac{1}{2}x^2 - \frac{1}{2}x \Rightarrow [V_1]_E = [\frac{1}{2}\ -\frac{1}{2}\ 0]$

$= V_2 = -x^2 + 2x \Rightarrow [V_2]_E = [-1\ 2\ 0]$

$= V_3 = \frac{1}{2}x^2 \underbrace{-x - \frac{1}{2}x}_{-\frac{3}{2}x} + 1 \Rightarrow [V_3]_E = [\frac{1}{2}\ -\frac{3}{2}\ 1]$

$\rightarrow M_B^E = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 2 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$

Pero busco $M_E^B \rightarrow M_E^B = (M_B^E)^{-1} \Rightarrow M_E^B = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\hookrightarrow [T]_E^C = [T]_B^C \cdot M_E^B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 2 \\ -4 & -2 & 0 \\ -6 & -2 & 3 \end{bmatrix}}$

c- Hallar la matriz T con respecto a la base B de $\mathbb{R}_2[x]$ y la base canónica de $\mathbb{R}^3$.

$[T]_B^E = M_C^E \cdot [T]_B^C \rightarrow$ orden $M_{(b)}^{(a)} = M_C^{(a)} \cdot M_{(b)}^C \rightarrow 2.$

$\alpha \cdot [1\ 0\ 0]^T + \beta \cdot [0\ 1\ 0]^T + \gamma \cdot [0\ 0\ 1]^T = [2\ 2\ 1]^T \rightarrow [2\ 2\ 1]^T$

$= [-2\ 1\ 2]^T \rightarrow [-2\ 1\ 2]^T \rightarrow M_C^E = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$

$= [1\ -2\ 2]^T \rightarrow [1\ -2\ 2]^T$

$$\Rightarrow [T]_B^{E} = M_C^{E} \cdot [T]_B^{C} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ 3 & -2 & -3 \\ -6 & 5 & 9 \end{bmatrix}$$

d. Hallar la matriz de T con respecto a las bases canónicas de $\mathbb{R}_2[x]$ y $\mathbb{R}^3$

$E \to \mathbb{R}_2[x]$, $E' \to \mathbb{R}^3$

$$[T]_E^{E'} = M_C^{E'} \cdot [T]_B^{C} \cdot M_E^{B} = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 4 & 7 \\ 10 & 4 & -2 \\ -19 & -7 & 8 \end{bmatrix}}$$

e. Hallar la imagen por T del subespacio $\text{gen}\{\underbrace{2+3x+2x^2}_{\to v_1}, \underbrace{5+5x+4x^2}_{\to v_2}\}$

$\rightarrow$ Los polinomios no son múltiplos $\Rightarrow$ son base de S. $\dim(S) = 2$

• Hallo $[v_1]^B$ y $[v_2]^B$

• $[v_1]^B \rightarrow$ $2+3x+2x^2 = \alpha \cdot \frac{1}{2}x(x-1) + \beta(-x)(x-2) + \gamma \cdot \frac{1}{2}(x-1)(x-2)$

$2+3x+2x^2 = \alpha\left(\frac{1}{2}x^2 - \frac{1}{2}x\right) + \beta \cdot (-x^2+2x) + \gamma\left(\frac{1}{2}x^2 - x - \frac{1}{2}x + 1\right)$

$2+3x+2x^2 = \frac{\alpha}{2}x^2 - \frac{\alpha}{2}x - \beta x^2 + 2\beta x + \frac{\gamma}{2}x^2 - \frac{3}{2}\gamma x + \gamma$

$$\to \begin{cases} \boxed{2 = \gamma} \\ 3 = -\frac{\alpha}{2} + 2\beta - \frac{3}{2}\gamma \to 3 = -\frac{\alpha}{2} + 2\beta - \frac{3}{2}\cdot 2 \to 6 = -\frac{\alpha}{2} + 2\beta \\ 2 = \frac{\alpha}{2} - \beta + \frac{\gamma}{2} \to 2 = \frac{\alpha}{2} - \beta + 1 \to 1 = \frac{\alpha}{2} - \beta \end{cases}$$

$$\left[\begin{array}{cc|c} \alpha & \beta & z \\ -\frac{1}{2} & 2 & 6 \\ \frac{1}{2} & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} \alpha & \beta & z \\ -\frac{1}{2} & 2 & 6 \\ 0 & 1 & 7 \end{array}\right]$$

$\to \boxed{\beta = 7}$

$\to -\frac{1}{2}\alpha + 14 = 6$

$-\frac{1}{2}\alpha = -8$

$\boxed{\alpha = 16}$

$\Rightarrow [v_1]^B = [16\ 7\ 2]^T$

• $[v_2]^B \rightarrow$ $5+5x+4x^2 = \alpha \cdot \frac{1}{2}x(x-1) + \beta(-x)\cdot(x-2) + \gamma \cdot \frac{1}{2}(x-1)(x-2)$

...

$5+5x+4x^2 = \frac{\alpha}{2}x^2 - \frac{\alpha}{2}x - \beta x^2 - 2\beta x + \frac{\gamma}{2}x^2 - \frac{3}{2}\gamma x + \gamma$

$$\to \begin{cases} \boxed{5 = \gamma} \\ 5 = -\frac{\alpha}{2} + 2\beta - \frac{3}{2}\gamma \to 5 = -\frac{\alpha}{2} + 2\beta - \frac{15}{2} \to \frac{25}{2} = -\frac{\alpha}{2} + 2\beta \\ 4 = \frac{\alpha}{2} - \beta + \frac{\gamma}{2} \to 4 = \frac{\alpha}{2} - \beta + \frac{5}{2} \to \frac{3}{2} = \frac{\alpha}{2} - \beta \end{cases}$$

$$\left[\begin{array}{cc|c} -1/2 & 2 & 25/2 \\ 1/2 & -1 & 3/2 \end{array}\right] \to \left[\begin{array}{cc|c} -1/2 & 2 & 25/2 \\ 0 & 1 & 14 \end{array}\right]$$

$\to \boxed{\beta = 14}$

$\to -\frac{\alpha}{2} + 28 = \frac{25}{2}$

$-\frac{\alpha}{2} = -\frac{31}{2}$

$\Rightarrow [v_2]^B = [31\ 14\ 5]^T$

$$[T(v_1)]^C = [T]_B^C \cdot [v_1]^B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix} \cdot [16\ 7\ 2]^T = [9\ -14\ -12]^T$$

$$[T(v_2)]^C = [T]_B^C \cdot [v_2]^B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix} \cdot [31\ 14\ 5]^T = [19\ -26\ -19]^T$$

$[T(v)]^C$ son las coord. de $T(v)$ en C, entonces:

- $T(v_1) = 9 \cdot [2\ 2\ 1]^T + (-14) \cdot [-2\ 1\ 2]^T + (-12) \cdot [1\ -2\ 2]^T$

  $T(v_1) = [18\ 18\ 9]^T + [28\ -14\ -28]^T + [-12\ 24\ -24]^T$

  $\boxed{T(v_1) = [34\ 28\ -43]^T}$

- $T(v_2) = 19 \cdot [2\ 2\ 1]^T + (-26) \cdot [-2\ 1\ 2]^T + (-19) \cdot [1\ -2\ 2]^T$

  $T(v_2) = [38\ 38\ 19]^T + [52\ -26\ -52]^T + [-19\ 38\ -38]^T$

  $\boxed{T(v_2) = [71\ 50\ -71]^T}$

$$\boxed{T(S) = gen\left\{\begin{bmatrix} 34 \\ 28 \\ -43 \end{bmatrix}, \begin{bmatrix} 71 \\ 50 \\ -71 \end{bmatrix}\right\}} \rightarrow \text{gen. un plano}$$

2.18. Sea $T_1 \in L(\mathbb{R}^3)$ la T.L def. en el ej. 2.10, y sea $T_2 \in L(\mathbb{R}^3, \mathbb{R}_2[x])$ la T.L. def. por

$T_2([a\ b\ c]^T) = (a+b) + (a+c)x + (b+c)x^2$

$\rightarrow T_1([a\ b\ c]^T) = [-a + 2b + 2c \quad \frac{3}{2}a - 3b - 3c \quad -2a + 4b + 4c]^T$

a. Hallar las matrices de T_1, T_2 y T_2^{-1} con respecto a las bases canónicas que correspondan.

- Hallo $[T_1]_B^C$ con $B = C = B_{\mathbb{R}^3}$ $\rightarrow$ los transformados de B

$T_1([1\ 0\ 0]^T) = [-1\ \ 3/2\ \ -2]^T$

$T_1([0\ 1\ 0]^T) = [2\ \ -3\ \ 4]^T$

$T_1([0\ 0\ 1]^T) = [2\ \ -3\ \ 4]^T$

$$\rightarrow \begin{bmatrix} -1 & 2 & 2 \\ 3/2 & -3 & -3 \\ -2 & 4 & 4 \end{bmatrix} = [T_1]_B^C$$

• Hallo $[T_2]_B^C$ con $B = B_{\mathbb{R}^3}$ y $C = B_{\mathbb{R}_2[x]}$

$$T_2([1\ 0\ 0]^T) = 1 + x$$
$$T_2([0\ 1\ 0]^T) = 1 + x^2 \quad \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = [T_2]_B^C$$
$$T_2([0\ 0\ 1]^T) = x + x^2$$

• Hallo $\left([T_2]_B^C\right)^{-1} \Rightarrow$ la inversa de T_2

¿Es isomorfismo? Como $\dim(\mathbb{R}^3) = \dim(\mathbb{R}_2[x])$, con probar que es monomorfismo, pruebo que es ep:

ó $rg([T_2]_B^C) = 3$.

$$\hookrightarrow \underbrace{\dim[rg([T_2]_B^C)]}_{3} + \dim[Nu([T_2]_B^C)] = \underbrace{\dim(\mathbb{R}^3)}_{3} \Rightarrow \dim[Nu([T_2]_B^C)] = 0.$$

$\Rightarrow$ Es monomorfismo $\Rightarrow$ es epimorfismo $\Rightarrow$ es isomorfismo $\Rightarrow \exists (T_2)^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\Rightarrow \boxed{\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \left([T_2]_B^C\right)^{-1}}$$

b. Hallar la matriz de $\underbrace{T_1 \circ (T_2)^{-1}}_{T}$ con resp. a las mismas bases y utilizarla para hallar una base de $Nu(T_1 \circ T_2^{-1})$

$$\left.\begin{array}{l} T_1: \mathbb{R}^3 \to \underline{\mathbb{R}^3} \\ T_2: \mathbb{R}^3 \to \mathbb{R}_2[x] \\ \hookrightarrow (T_2)^{-1}: \underline{\mathbb{R}_2[x]} \to \mathbb{R}^3 \end{array}\right\} \to T_1 \circ (T_2)^{-1}: \underbrace{\mathbb{R}_2[x]}_{B} \to \underbrace{\mathbb{R}^3}_{C}$$

$$T(p) = (T_1 \circ T_2^{-1})(p) = T_1(T_2^{-1}(p)) = T_1(w) = 0$$

$$[T]_B^C = [(T_1 \circ T_2^{-1})]_B^C = [T_1]_C^C \cdot [T_2^{-1}]_B^C \to [T]_{\mathbb{R}_2[x]}^{\mathbb{R}^3} = [T_1]_{\mathbb{R}^3}^{\mathbb{R}^3} \cdot [T_2^{-1}]_{\mathbb{R}_2[x]}^{\mathbb{R}^3}$$

$\to$ ya lo tengo.

$$[T]_{\mathbb{R}_2[x]}^{\mathbb{R}^3} = \begin{bmatrix} -1 & 2 & 2 \\ 3/2 & -3 & -3 \\ -2 & 4 & 4 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} -1 & -1 & 5 \\ 3/2 & 3/2 & -15/2 \\ -2 & -2 & 10 \end{bmatrix}}$$

$\hookrightarrow$ ya lo tengo

Hallo $B_{Nu(T)}$: $\frac{1}{2}\begin{bmatrix} -1 & -1 & 5 \\ 3/2 & 3/2 & -15/2 \\ -2 & -2 & 10 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 & 5/2 \\ 3/4 & 3/4 & -15/4 \\ -1 & -1 & 5 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -1/2 & -1/2 & 5/2 & 0 \\ 3/4 & 3/4 & -15/4 & 0 \\ -1 & -1 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1/2 & -1/2 & 5/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$-\frac{1}{2}\alpha - \frac{1}{2}\beta + \frac{5}{2}\gamma = 0.$

$\frac{5}{2}\gamma = \frac{1}{2}(\alpha+\beta).$

$\gamma = \frac{1}{5}(\alpha+\beta)$

$Nu(T): [\alpha\ \beta\ \gamma]^T = [\alpha\ \beta\ \frac{1}{5}\alpha + \frac{1}{5}\beta]^T = \alpha[1\ 0\ 1/5]^T + \beta[0\ 1\ 1/5]^T \Rightarrow$ ∞ soluciones

$Nu(T) = gen\{[1\ 0\ 1/5]^T, [0\ 1\ 1/5]^T\}$

$\hookrightarrow \boxed{B(Nu(T)) = \{1 + \frac{1}{5}x^2,\ x + \frac{1}{5}x^2\}}$

2.20. Sean $\mathbb{V}$ un $\mathbb{R}$-espacio vectorial de dimensión 3, $B = \{v_1, v_2, v_3\}$ una base de $\mathbb{V}$, S_1 y S_2 los subespacios de $\mathbb{V}$ def. por $S_1 = gen\{v_1 - 2v_2, v_1 + v_3\}$, $S_2 = gen\{v_2 - v_3\}$.

a. Comprobar que $\mathbb{V} = S_1 \oplus S_2$

$(S_1 + S_2) = \{[1\ -2\ 0]^T, [1\ 0\ 1]^T, [0\ 1\ -1]^T\} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ ✓ son L.I.

$\underbrace{\dim(S_1 + S_2)}_{3} = \underbrace{\dim(S_1)}_{2} + \underbrace{\dim(S_2)}_{1} + \dim(S_1 \cap S_2) \Rightarrow \dim(S_1 \cap S_2) = 0$ ✓

b. Hallar las matrices con resp. a la base B de las proyecciones y simetrías inducidas por la partición $\mathbb{V} = S_1 \oplus S_2$.

$[T]_B^B = [T]_{B'}^{B'} \cdot M_B^{B'}$

• $B = gen\{v_1, v_2, v_3\}$ $\qquad B' = gen\{\underbrace{v_1 - 2v_2, v_1 + v_3}_{S_1}, \underbrace{v_2 - v_3}_{S_2}\}$

• $M_B^{B'} = [[v_1]_{B'}\ [v_2]_{B'}\ [v_3]_{B'}]$

• $v_1 = \alpha.(v_1 - 2v_2) + \beta.(v_1 + v_3) + \gamma.(v_2 - v_3) = \alpha v_1 - 2\alpha . v_2 + \beta . v_1 + \beta . v_3 + \gamma . v_2 - \gamma . v_3$

$= v_1.(\alpha + \beta) + v_2(-2\alpha + \gamma) + v_3(\beta - \gamma)$

$\hookrightarrow \begin{cases} \alpha + \beta = 1 \\ -2\alpha + \gamma = 0 \\ \beta - \gamma = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & -3/2 & -1 \end{array}\right] \Rightarrow$

$-\frac{3}{2}\gamma = -1$ $\qquad 2\beta + \frac{2}{3} = 2$ $\qquad \alpha + \frac{2}{3} = 1$

$\frac{3}{2}\gamma = 1.$ $\qquad 2\beta = \frac{4}{3}$ $\qquad \boxed{\alpha = \frac{1}{3}}$

$\boxed{\gamma = \frac{2}{3}}$ $\qquad \boxed{\beta = \frac{2}{3}}$

$\Rightarrow [V_1]^{B'} = [1/3 \;\; 2/3 \;\; 2/3]^T$

- $V_2 = V_1(\alpha+\beta) + V_2(-2\alpha+\gamma) + V_3(\beta-\gamma)$

$\begin{cases} \alpha+\beta=0 \\ -2\alpha+\gamma=1 \\ \beta-\gamma=0 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -3/2 & -1/2 \end{array}\right] \Rightarrow$

$-\frac{3}{2}\gamma = -\frac{1}{2}$ $2\beta + \frac{1}{3} = 1$ $\alpha + \frac{1}{3} = 0$

$\frac{3}{2}\gamma = \frac{1}{2}$ $2\beta = 2/3$ $\boxed{\alpha = -1/3}$

$\boxed{\gamma = \frac{1}{3}}$ $\boxed{\beta = 1/3}$

$\Rightarrow [V_2]^{B'} = [-1/3 \;\; 1/3 \;\; 1/3]^T$

- $V_3 = V_1(\alpha+\beta) + V_2(-2\alpha+\gamma) + V_3(\beta-\gamma)$

$\begin{cases} \alpha+\beta=0 \\ -2\alpha+\gamma=0 \\ \beta-\gamma=1 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -3/2 & 1 \end{array}\right] \rightarrow$

$-\frac{3}{2}\gamma = 1$ $2\beta + \gamma = 0$ $\alpha + \frac{1}{3} = 0$

$\boxed{\gamma = -\frac{2}{3}}$ $2\beta - \frac{2}{3} = 0$ $\boxed{\alpha = -\frac{1}{3}}$

$2\beta = \frac{2}{3}$

$\boxed{\beta = \frac{1}{3}}$

$\Rightarrow [V_3]^{B'} = [-1/3 \;\; 1/3 \;\; -2/3]^T$

$\hookrightarrow M_B^{B'} = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ 2/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}$

2.22 - Sea $T \in L(\mathbb{R}^3)$ la T.L. def. por $T([x_1 \; x_2 \; x_3]^T) = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} [x_1 \; x_2 \; x_3]^T$.

Hallar la matriz con resp. a la base canónica de la proyección de $\mathbb{R}^3$ sobre $Im(T)$ en la dirección de $Nu(T)$

$\underbrace{Nu(T)}_{S_2} \oplus \underbrace{Im(T)}_{S_1} = V$, $Nu(T) = \{x \in \mathbb{R}^3 : T(x) = 0_{\mathbb{R}^3}\}$

$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ $x_1 + x_2 - x_3 = 0 \;\wedge\; x_2 = 0$

$\rightarrow x_1 - x_3 = 0$

$x_1 = x_3$

$\} \; [x_1 \; x_2 \; x_3]^T = x[1 \; 0 \; 1]^T$

- $Nu(T) = gen\{[1\;0\;1]^T\}$
- $Im(T) = gen\{[1\;-1\;1]^T, [1\;1\;3]^T\}$

$\hookrightarrow Col(A)$

$\rightarrow \underbrace{\dim[Nu(T)]}_{1} + \underbrace{\dim[Im(T)]}_{2} = \underbrace{\dim(V)}_{3}$ ✓

$B(\Pi_{S_1 S_2}) = \{[1\;0\;1]^T, [1\;-1\;1]^T, [1\;1\;3]^T\}$

Base de $\mathbb{R}^3$: $[1 0 0]^T, [0 1 0]^T, [0 0 1]^T$

- $[1 0 0]^T = a.[1 0 1]^T + b[1 -1 1]^T + c.[1 1 3]^T$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{array}\right]$$

$a+b+c=1 \wedge -b+c=0 \wedge c=\frac{-1}{2}$

$b=\frac{-1}{2}$

$a=2$

$\Rightarrow [a\, b\, c]^T = [2\ -\frac{1}{2}\ -\frac{1}{2}]^T$

$[1 0 0]^T = 2.[1 0 1]^T + \left(-\frac{1}{2}\right).[1 -1 1]^T + \left(-\frac{1}{2}\right).[1 1 3]^T$.

$T[1 0 0]^T = 2.T[1 0 1]^T + \left(-\frac{1}{2}\right).T[1 -1 1]^T + \left(-\frac{1}{2}\right).T[1 1 3]^T$.

$T[1 0 0]^T = 2.\underbrace{[0 0 0]^T}_{\text{nulo}} - \frac{1}{2}.[1 -1 1]^T - \frac{1}{2}.[1 1 3]^T$. $\Rightarrow$ T manda S_1 a sí mismos y S_2 al 0

$\boxed{T[1 0 0]^T = [-1\ 0\ -2]^T}$.

- $[0 1 0]^T = a.[1 0 1]^T + b.[1 -1 1]^T + c.[1 1 3]^T$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$a+b+c=0 \wedge -b+c=1 \wedge c=0$.

$a-1=0 \quad -b=1$

$a=1 \quad b=-1$

$\Rightarrow [a\, b\, c]^T = [1\ -1\ 0]^T$

$[0 1 0]^T = [1 0 1]^T - [1 -1 1]^T$

$T[0 1 0]^T = [0 0 0]^T - [1 -1 1]^T = [-1\ 1\ -1]^T \Rightarrow \boxed{T([0 1 0]^T) = [-1\ 1\ -1]^T}$

- $[0 0 1]^T = a.[1 0 1]^T + b.[1 -1 1]^T + c.[1 1 3]^T$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

$a+b+c=0 \wedge -b+c=0. \wedge 2c=1$

$a+1=0 \quad c=b \quad c=\frac{1}{2}$

$a=-1$

$\Rightarrow [a\, b\, c]^T = [-1\ \frac{1}{2}\ \frac{1}{2}]^T$.

$[0 0 1]^T = -[1 0 1]^T + \frac{1}{2}[1 -1 1]^T + \frac{1}{2}[1 1 3]^T$.

$T[0 0 1]^T = -T[1 0 1]^T + \frac{1}{2}T[1 -1 1]^T + \frac{1}{2}.T[1 1 3]^T$.

$\boxed{T[0 0 1]^T = [1 0 2]^T}$.

$$\Rightarrow \boxed{[\Pi_{S_1 S_2}]_E^E = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}}$$

2.23. Verificar las sigs. afirmaciones.

a. Si $T \in \mathcal{L}(V)$ es tq $T^2 = T$, entonces T es la proyección de V sobre $Im(T)$ en la dirección de $Nu(T)$.

$T: V \to V$ (endomorfismo). Se llama proyector si $T \circ T = T^2 = T$.

Como $V = Im(T) \oplus Nu(T)$ para cada $v \in V$ existen únicos $v_1 \in Nu(T)$ y $v_2 \in Im(T)$ tq $v = v_1 + v_2$

$\to$ Se tiene que $T(v) = T(v_1 + v_2) = T(v_1) + T(v_2) = 0_V + v_2 = v_2$

Es por def. $\Pi_{S_1 S_2}$ (con $S_1 = Im(T) \wedge S_2 = Nu(T)$).

b. Si $T \in \mathcal{L}(V)$ es tq. $T^2 = T$, entonces $S = I_V - 2T$ es tq. $S^2 = I_V$

Dado q' $\Pi_{S_1 S_2}(v) = I_V - \Pi_{S_2 S_1}(v) \to S = \Pi_{S_1 S_2}(v) - \Pi_{S_2 S_1}(v)$

$S = I_V - 2\Pi_{S_2 S_1}(v)$

$\to v = v_1 + v_2, \forall v \in V$ con $v_1 \in S_1 \wedge v_2 \in S_2$

$S(S(v_1+v_2)) = S(\underbrace{S(v_1)}_{v_1} + \underbrace{S(v_2)}_{-v_2}) = S(v_1 - v_2) = S(v_1) - S(v_2) = v_1 + v_2 = v, \forall v \in V$

...

2.24. Sean $T, S \in \mathcal{L}(\mathbb{R}^3)$ las T.L def. por: $T([x_1\ x_2\ x_3]^T) = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \wedge S([x_1\ x_2\ x_3]^T) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

a. Comprobar que T es proyección y hallar una base B de $\mathbb{R}^3$ tq. $[T]_B^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$B(T) = \{v_1, v_2, v_3\} \to$ base ordenada

• $T(v_1) = [1\ 0\ 0]^T$ $\quad$ • $T(v_2) = T(v_3) = [0\ 0\ 0]^T$

$\hookrightarrow \in Im(T)$ $\quad$ $\hookrightarrow \in Nu(T)$

• Si T es un proyector, $T^2 = T \to$ la composición de dos transformaciones lineales se traduce como la multiplicación de sus matrices

$$\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 2/4 & 0 & -2/4 \\ 0 & 0 & 0 \\ -2/4 & 0 & 2/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$ $\to$ Como $T^2 = T \Rightarrow T$ es proyección

• Dado q. v_2 y v_3 van al núcleo, busco dos vectores tq. $T(x) = 0_{\mathbb{R}^3}$

$$\left[\begin{array}{ccc|c} 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \frac{1}{2}x_1 - \frac{1}{2}x_3 = 0 \quad x_1 = x_3$$

$\Rightarrow [x_1\ x_2\ x_3]^T = [x_1\ x_2\ x_1]^T = x_1[1\ 0\ 1]^T + x_2[0\ 1\ 0]^T$

$\therefore Nu(T) = \{\underbrace{[1\ 0\ 1]^T}_{v_2}, \underbrace{[0\ 1\ 0]^T}_{v_3}\}$

Falta hallar v_1: $T(v_1)=v_1 \to T(v_1)-v_1=0 \Rightarrow (T-I)(v_1)=0_{\mathbb{R}^3}$.

$$\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} -1/2 & 0 & -1/2 & 0 \\ 0 & -1 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1/2 & 0 & -1/2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \nearrow -\frac{1}{2}x_1-\frac{1}{2}x_3=0 \wedge -x_2=0$$

$x_1=-x_3$.

$\hookrightarrow [x_1\ x_2\ x_3]^T=[x_1\ 0\ -x_1]^T=x_1[1\ 0\ -1]^T \Rightarrow Im(T)=gen\{[1\ 0\ -1]^T\}$

$\cdot V=Nu(T)\oplus Im(T) \to B(V)=B(Nu(T))\cup B(Im(T))$

$B=\{[1\ 0\ 1]^T, [0\ 1\ 0]^T, [1\ 0\ -1]^T\}$.

b. Comprobar que S es una simetría y hallar una base B de $\mathbb{R}^3$ tq $[S]_B^B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$S^2=I_V \to$ simetría

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$\cdot$ Hallo base B tq. $B(S)=\{v_1, v_2, v_3\} \wedge B(v_1)=[1\ 0\ 0]^T \wedge B(v_2)=[0\ -1\ 0]^T \wedge B(v_3)=[0\ 0\ -1]^T$

$\cdot S$: S es simetría $\Rightarrow V=Nu(S-I_V)\oplus Nu(S+I_V)$.

$\to Nu(S-I_V)$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \nearrow -x_1+x_3=0 \wedge x_2=0$$

$x_3=x_1$

$\Rightarrow [x_1\ x_2\ x_3]^T=[x_1\ 0\ x_1]^T = x_1[1\ 0\ 1]^T$.

$\to Nu(S+I_V)$.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \nearrow x_1+x_3=0.$$

$x_3=-x_1$.

$\Rightarrow [x_1\ x_2\ x_3]^T=x_1[1\ 0\ -1]^T+x_2[0\ 1\ 0]^T$

$B(V)=B(S)=\{[1\ 0\ 1]^T, [1\ 0\ -1]^T, [0\ 1\ 0]^T\}$

$\to$ Recuerdo. $S(v_1)-v_1=0_{\mathbb{R}^3} \Rightarrow S-I_V=0$.

$\left.\begin{array}{l} S(v_2)+v_2=0_{\mathbb{R}^3} \\ S(v_3)+v_3=0_{\mathbb{R}^3} \end{array}\right\} S+I_V=0$

$\cdot S([1\ 0\ 0]^T)=[1\ 0\ 0]^T$

$\cdot S([0\ 1\ 0]^T)=[0\ -1\ 0]^T$

$\cdot S([0\ 0\ 1]^T)=[0\ 0\ -1]^T$

$B=\{[1\ 0\ 0]^T, [0\ -1\ 0]^T, [0\ 0\ -1]^T\}$

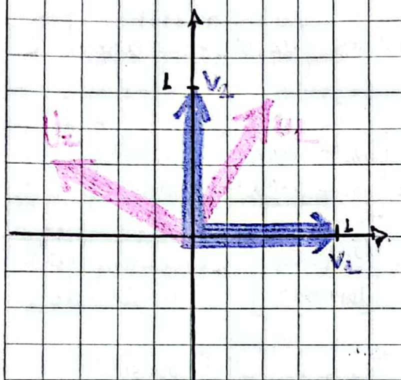
2.25. Sea $O(2,\mathbb{R}) := \{R_\theta, S_\theta : \theta \in \mathbb{R}\} \subset \mathcal{L}(\mathbb{R}^2)$ el conj. de todas las T.L. de $\mathbb{R}^2$ en $\mathbb{R}^2$ def. por $R_\theta([x_1\ x_2]^T) := \begin{bmatrix}\cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$ $\wedge$ $S_\theta([x_1\ x_2]^T) := \begin{bmatrix}\cos\theta & \operatorname{sen}\theta \\ \operatorname{sen}\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$

a. Hallar y graficar la imagen de la base canónica de $\mathbb{R}^2$ por $R_{\pi/3}$.

$B_{\mathbb{R}} = \{\underbrace{[1\ 0]^T}_{v_1}, \underbrace{[0\ 1]^T}_{v_2}\}$

$\to R_{\pi/3}([1\ 0]^T) = \begin{bmatrix}\cos\pi/3 & -\operatorname{sen}\pi/3 \\ \operatorname{sen}\pi/3 & \cos\pi/3\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix}1/2 \\ \sqrt{3}/2\end{bmatrix} \to u_1$

$\to R_{\pi/3}([0\ 1]^T) = \begin{bmatrix}\cos\pi/3 & -\operatorname{sen}\pi/3 \\ \operatorname{sen}\pi/3 & \cos\pi/3\end{bmatrix}\begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/2 \\ 1/2\end{bmatrix} \to u_2$

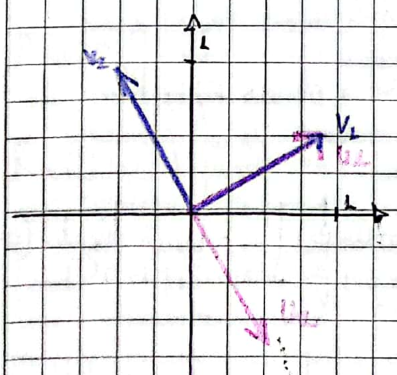


$\Rightarrow R_{\pi/3}$ genera rotación.

b. Hallar y graficar la imagen de la base $\{\overset{v_1}{\begin{bmatrix}\sqrt{3}/2 \\ 1/2\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}-1/2 \\ \sqrt{3}/2\end{bmatrix}}\}$ por $S_{\pi/3}$

$S_{\pi/3}([\sqrt{3}/2\ \ 1/2]^T) = \begin{bmatrix}\cos\pi/3 & \operatorname{sen}\pi/3 \\ \operatorname{sen}\pi/3 & -\cos\pi/3\end{bmatrix}\begin{bmatrix}\sqrt{3}/2 \\ 1/2\end{bmatrix} = \begin{bmatrix}\sqrt{3}/4 + \sqrt{3}/4 \\ 3/4 - 1/4\end{bmatrix} = \begin{bmatrix}\sqrt{3}/2 \\ 1/2\end{bmatrix} \to u_1$

$S_{\pi/3}([-1/2\ \ \sqrt{3}/2]^T) = \begin{bmatrix}\cos\pi/3 & \operatorname{sen}\pi/3 \\ \operatorname{sen}\pi/3 & -\cos\pi/3\end{bmatrix}\begin{bmatrix}-1/2 \\ \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}-1/4 + 3/4 \\ -\sqrt{3}/4 - \sqrt{3}/4\end{bmatrix} = \begin{bmatrix}1/2 \\ -\sqrt{3}/2\end{bmatrix} \to u_2$



$\Rightarrow S_{\pi/3}$ genera simetría

c. Hallar y graficar la imagen de la base canónica de $\mathbb{R}^2$ por R_θ.

$B(\mathbb{R}^2) = \{[1\ 0]^T, [0\ 1]^T\} \Rightarrow R_\theta([1\ 0]^T) = \begin{bmatrix}\cos\theta & -\text{sen}\,\theta\\ \text{sen}\,\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \text{sen}\,\theta\end{bmatrix}$

$\rightarrow R_\theta([0\ 1]^T) = \begin{bmatrix}\cos\theta & -\text{sen}\,\theta\\ \text{sen}\,\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\text{sen}\,\theta\\ \cos\theta\end{bmatrix}$

d. Comprobar que $\left\{\overset{v_1}{\begin{bmatrix}\cos\theta/2\\ \text{sen}\,\theta/2\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}-\text{sen}\,\theta/2\\ \cos\theta/2\end{bmatrix}}\right\}$ es una base de $\mathbb{R}^2$ y hallar su imagen por S_θ

$\dim(\mathbb{R}^2) = 2 \Rightarrow$ si demuestro que v_1 y v_2 son L.I. $\Rightarrow$ son base de $\mathbb{R}^2$.

$a.v_1 + b.v_2 = 0_{\mathbb{R}^2} \Rightarrow a[\cos\theta/2\ \ \text{sen}\,\theta/2]^T + b[-\text{sen}\,\theta/2\ \ \cos\theta/2]^T = [0\ 0]^T$

$M = \begin{bmatrix}\cos\theta/2 & -\text{sen}\,\theta/2\\ \text{sen}\,\theta/2 & \cos\theta/2\end{bmatrix} \Rightarrow \det(M) = \cos^2\theta/2 + \text{sen}^2\theta/2 \neq 0 \Rightarrow$ son L.I. $\Rightarrow \{v_1, v_2\}$ es base de $\mathbb{R}^2$.

$S_\theta([x_1\ x_2]^T) = \begin{bmatrix}\cos\theta & \text{sen}\,\theta\\ \text{sen}\,\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$\rightarrow S_\theta([\cos\theta/2\ \ \text{sen}\,\theta/2]^T) = \begin{bmatrix}\cos\theta & \text{sen}\,\theta\\ \text{sen}\,\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta/2\\ \text{sen}\,\theta/2\end{bmatrix}$

$= \begin{bmatrix}\cos\theta/2\\ \text{sen}\,\theta/2\end{bmatrix}$ → no cambia de sentido

$\rightarrow S_\theta([-\text{sen}\,\theta/2\ \ \cos\theta/2]^T) = \begin{bmatrix}\cos\theta & \text{sen}\,\theta\\ \text{sen}\,\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}-\text{sen}\,\theta/2\\ \cos\theta/2\end{bmatrix}$

$= \begin{bmatrix}\text{sen}\,\theta/2\\ -\cos\theta/2\end{bmatrix}$ → cambia sentido

$\text{Im} = \{[\cos\theta/2\ \ \text{sen}\,\theta/2]^T, [\text{sen}\,\theta/2\ \ -\cos\theta/2]^T\}$.

e. ¿Cuál es el significado geométrico de S_θ sobre vectores de $\mathbb{R}^2$?

→ Es una simetría resp. de la recta dirigida por $[\cos\theta/2\ \ \text{sen}\,\theta/2]^T$.

f. Dados $\alpha, \beta \in \mathbb{R}$, hallar las matrices respecto a la base canónica de las sigs. T.L.

I. $R_\alpha \circ R_\beta$ II. $S_\alpha \circ S_\beta$ III. $S_\alpha \circ R_\beta$ IV. $R_\beta \circ S_\alpha$.

I. $R_\alpha \circ R_\beta = \begin{bmatrix}\cos\alpha & -\text{sen}\,\alpha\\ \text{sen}\,\alpha & \cos\alpha\end{bmatrix}\cdot\begin{bmatrix}\cos\beta & -\text{sen}\,\beta\\ \text{sen}\,\beta & \cos\beta\end{bmatrix} = \begin{bmatrix}\cos\alpha+\beta & -\text{sen}(\alpha+\beta)\\ \text{sen}\,\alpha+\beta & \cos\alpha+\beta\end{bmatrix}$ → rotación ángulo $(\alpha+\beta)$

II. $S_\alpha \circ S_\beta = \begin{bmatrix}\cos\alpha & \text{sen}\,\alpha\\ \text{sen}\,\alpha & -\cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & \text{sen}\,\beta\\ \text{sen}\,\beta & -\cos\beta\end{bmatrix} = \begin{bmatrix}\cos\alpha-\beta & -\text{sen}\,\alpha-\beta\\ \text{sen}\,\alpha-\beta & \cos\alpha-\beta\end{bmatrix}$

III. $S_\alpha \circ R_\beta = \begin{bmatrix}\cos\alpha & \text{sen}\,\alpha\\ \text{sen}\,\alpha & -\cos\alpha\end{bmatrix}\cdot\begin{bmatrix}\cos\beta & -\text{sen}\,\beta\\ \text{sen}\,\beta & \cos\beta\end{bmatrix} = \begin{bmatrix}\cos\alpha-\beta & \text{sen}\,\alpha-\beta\\ \text{sen}\,\alpha-\beta & -\cos\alpha-\beta\end{bmatrix}$

IV. $R_\beta \circ S_\alpha = \begin{bmatrix}\cos\beta & -\text{sen}\,\beta\\ \text{sen}\,\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & \text{sen}\,\alpha\\ \text{sen}\,\alpha & -\cos\alpha\end{bmatrix} = \begin{bmatrix}\cos\alpha+\beta & \text{sen}\,\alpha+\beta\\ \text{sen}\,\alpha+\beta & -\cos\alpha+\beta\end{bmatrix}$

$\to A \cdot A^T = I$

matriz ortogonal $\angle$ col. ortogonales y normal 1.

g. Concluir que el conj. $O(2,\mathbb{R})$ es cerrado por composiciones

$\to$ Quiere decir q. siempre gira sobre lo mismo, se trata de rot. o simetrías. El result. de componer dos rot. o dos simetrías resulta en mismo mov., rot. o sim., resp.

h. Observar q. $R_0 = I_{\mathbb{R}^2}$.

$$R_0 = \begin{bmatrix} \cos 0 & -\operatorname{sen} 0 \\ \operatorname{sen} 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow R_0([x_1\ x_2]^T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

i. Comprobar que R_θ y S_θ son isomorfismos y hallar R_θ^{-1} y S_θ^{-1}.

· Si $\det(T) \neq 0 \Rightarrow$ isomorfismo $\Rightarrow \exists T^{-1}$.

$$\to |R_\theta| = \begin{vmatrix} \cos\theta & -\operatorname{sen}\theta \\ \operatorname{sen}\theta & \cos\theta \end{vmatrix} = (\cos^2\theta + \operatorname{sen}^2\theta) = 1 \neq 0 \Rightarrow \exists R_\theta^{-1}.$$

$$\to |S_\theta| = \begin{vmatrix} \cos\theta & \operatorname{sen}\theta \\ \operatorname{sen}\theta & -\cos\theta \end{vmatrix} = -(\cos^2\theta + \operatorname{sen}^2\theta) = -1 \neq 0 \Rightarrow \exists S_\theta^{-1}.$$

Como son matrices ortogonales, la inversa es la transpuesta.

$$R_\theta^{-1} = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{bmatrix}$$

$\to$ lo sé pq. producto int. da 0 $\to R_\theta \to \cos\cdot -\operatorname{sen} + \operatorname{sen}\cdot\cos = 0$.

$S_\theta \to \cos\cdot\operatorname{sen} + \operatorname{sen}(-\cos) = 0$.

$$S_\theta^{-1} = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta \\ \operatorname{sen}\theta & -\cos\theta \end{bmatrix}$$

2.26. Observar q. la T.L. $R: \mathbb{R}^3 \to \mathbb{R}^3$ def. por $R([1\,0\,0]^T) := [\cos\theta\ \operatorname{sen}\theta\ 0]^T$, $R([0\,1\,0]^T) := [-\operatorname{sen}\theta\ \cos\theta\ 0]^T$, $R([0\,0\,1]^T) := [0\ 0\ 1]^T$

$$R([x_1\,x_2\,x_3]^T) = \begin{bmatrix} \cos\theta & -\operatorname{sen}\theta & 0 \\ \operatorname{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

es la rot. del ángulo θ en sentido antihorario del plano xy alrededor del eje z.

a. Hallar y graficar la imagen de los sigs. vectores por la rot. de ángulo $\pi/4$ en sentido antihorario del plano xy alrededor del eje z: $v_1 = [1\,0\,0]^T$, $v_2 = [1\,1\,0]^T$, $v_3 = [1\,0\,1]^T$.

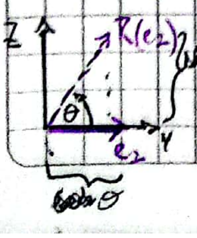
$$R_{\pi/4}([1\,0\,0]^T) = \begin{bmatrix} \cos\pi/4 & -\operatorname{sen}\pi/4 & 0 \\ \operatorname{sen}\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\pi/4 \\ \operatorname{sen}\pi/4 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}.$$

$$R_{\pi/4}([1\,1\,0]^T) = \begin{bmatrix} \cos\pi/4 & -\operatorname{sen}\pi/4 & 0 \\ \operatorname{sen}\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\pi/4 - \operatorname{sen}\pi/4 \\ \operatorname{sen}\pi/4 + \cos\pi/4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$R_{\pi/4}([1\,0\,1]^T) = \begin{bmatrix} \cos\pi/4 & -\operatorname{sen}\pi/4 & 0 \\ \operatorname{sen}\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\pi/4 \\ \operatorname{sen}\pi/4 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 1 \end{bmatrix}$$

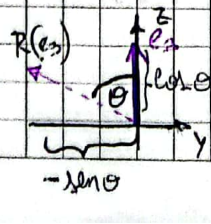
b. Hallar la matriz resp. base canónica de la rot. ángulo θ en sentido antihorario del plano yz alrededor eje x.

$e_1 = [1\,0\,0]^T \to$ queda fijo.



fijo

$\to [0\ \cos\theta\ \operatorname{sen}\theta]$

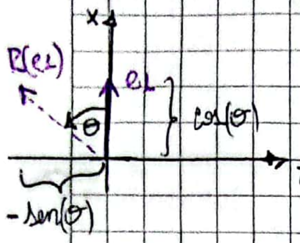


$\cos\theta \to [0\ -\operatorname{sen}\theta\ \cos\theta]$

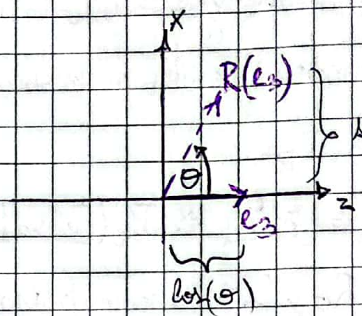
$$\Rightarrow R([x_1\,x_2\,x_3]^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\operatorname{sen}\theta \\ 0 & \operatorname{sen}\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c. Hallar la matriz resp. base canónica de la rot. del áng. θ en sentido antihorario zx alrededor del eje y.

$e_2 = [0\ 1\ 0]^T \Rightarrow$ queda fijo $\Rightarrow R(e_2) = [0\ 1\ 0]^T$.



$\Rightarrow R(e_1) = [-\text{sen}\,\theta\ \ 0\ \ \cos\theta]^T$.



$\Rightarrow R(e_3) = [\cos\theta\ \ 0\ \ \text{sen}\,\theta]^T$.

$$\Rightarrow R([x_1\ x_2\ x_3]^T) = \begin{bmatrix} -\text{sen}\,\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \text{sen}\,\theta \end{bmatrix}$$

2.27. Sea $D: C^{\infty}(\mathbb{R},\mathbb{C}) \to C^{\infty}(\mathbb{R},\mathbb{C})$ el operador derivación.

d. Sea $\lambda \in \mathbb{C}$. Verificar q' para todo $k \in \mathbb{N}$ vale q. $(D-\lambda I)^k[f(x)e^{\lambda x}] = f^{(k)}(x)e^{\lambda x}$ $\forall f \in C^{\infty}(\mathbb{R},\mathbb{C})$

I. Si $k=1$

$$(D-\lambda I)^1[f(x)\,e^{\lambda x}] \overset{\text{dist.}}{=} D[f(x)e^{\lambda x}] - \lambda I[f(x)e^{\lambda x}] = (f(x).e^{\lambda x})' - \lambda(f(x)e^{\lambda x}) \overset{\text{deriva}}{=} f'(x).e^{\lambda x} + \cancel{f(x).\lambda.e^{\lambda x}} - \cancel{\lambda f(x) e^{\lambda x}} =$$
$$= f'(x).e^{\lambda x}$$

II. Si vale para $k \geq 1 \Rightarrow$ vale para $k+1$

$$(D-\lambda I)^{k+1}[f(x)e^{\lambda x}] = (D-\lambda I)\underbrace{(D-\lambda I)^k[f(x)e^{\lambda x}]}_{f^k(x).e^{\lambda x}} = (D-\lambda I)[f^k(x)e^{\lambda x}] = D[f^k(x)e^{\lambda x}] - \lambda.f^k(x)e^{\lambda x} =$$
$$= f^{k+1}(x)e^{\lambda x} + \cancel{f^k(x)\lambda e^{\lambda x}} - \cancel{\lambda f^k(x).\lambda e^{\lambda x}} = f^{k+1}(x)e^{\lambda x}.$$

$\therefore$ se cumple $\forall k \in \mathbb{N}$ q.: $(D-\lambda I)^k[f(x)e^{\lambda x}] = f^k(x)e^{\lambda x}$.

2.28. Resolver:

a. $\frac{dy}{dx} - y = 0 \Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx \Rightarrow \int \frac{1}{y}dy = \int 1\,dx \Rightarrow \ln|y| = x + C \Rightarrow \boxed{y = e^x.K}$

b. $\frac{dy}{dx} - y = e^{2x} \Rightarrow$ Bernoulli $\rightarrow y = u.v \Rightarrow y' = u'.v + u.v'$.

$u'.v + u.v' - u.v = e^{2x} \rightarrow u'.v + u(v'-v) = e^{2x} \Rightarrow v'-v = 0. \Rightarrow v' = v \Rightarrow \frac{dv}{dx} = v \Rightarrow \ln|v| = x + C \Rightarrow v = e^x.K \Rightarrow v = e^x$.

$u'.e^x = e^{2x} \rightarrow u' = e^2 \rightarrow \frac{du}{dx} = e^2 \rightarrow u = e^2.x + C$

$\Rightarrow y = u.v = (e^2 x + C)\,e^x = \boxed{e^{2x}x + C.e^x}$

c- $y'-y=xe^{2x}$ $\rightarrow$ Bernoulli: $\Rightarrow y=u.v \Rightarrow y'=u'v+u.v'$

$u'v+uv'-u.v=xe^{2x} \rightarrow u'v+u(v'-v)=xe^{2x}$ $\quad v'-v=0 \rightarrow v'=v \Rightarrow \frac{dv}{dx}=v \Rightarrow v=e^{x}$ (k)

$\hookrightarrow u'e^{x}=xe^{2x} \rightarrow u'=xe^{x} \rightarrow \frac{du}{dx}=xe^{x} \rightarrow du=xe^{x}dx \rightarrow u=(x-1)e^{x}+C$

$\Rightarrow y=u.v=((x-1)e^{x}+C).e^{x}=(x-1)e^{2x}+e^{x}.C=\boxed{xe^{2x}-e^{2x}+e^{x}.C}$

d- $y'-y=(3+5x).e^{2x}$ $\rightarrow$ $\frac{dy}{dx}-y=(3+5x).e^{2x}$ $\rightarrow$ Bernoulli: $y=u.v \Rightarrow y'=\frac{dy}{dx}=u'v+uv'$

$u'v+u.v'-u.v=(3+5x).e^{2x} \rightarrow u'v+u(v'-v)=(3+5x).e^{2x}$ $\quad v'-v=0 \ldots v=e^{x}$ (k)

$u'(v)=(3+5x)e^{2x} \rightarrow u'.e^{x}=(3+5x)e^{2x} \rightarrow u'=(3+5x).e^{x}$

$\Rightarrow \frac{du}{dx}=3e^{x}+5xe^{x} \rightarrow du=(3e^{x}+5xe^{x})dx \rightarrow \int 1du=\int 3e^{x}dx+\int 5xe^{x}dx \Rightarrow u=3e^{x}+5(x-1)e^{x}+C$

$\Rightarrow y=u.v=(3e^{x}+5xe^{x}-5e^{x}+C)e^{x}=3e^{2x}+5xe^{2x}-5e^{2x}+C.e^{x}=\boxed{-2e^{2x}+5xe^{2x}+Ce^{x}}$

e- ~~$y''-2y'+y=(3+5x)e^{2x}=3e^{2x}+5xe^{2x}$~~

~~$p(r)=r^2-2r+1 \rightarrow r_1=1, r_2=1 \rightarrow y_h(t)=C_1e^{t}+C_2e^{t}=(C_1+C_2)e^{t}$~~

$(D-1I)\circ(D-1I)(y)=0$ ~~$\rightarrow$ sol. h $\{e^{1x}, xe^{1x}\} \Rightarrow y_h=C_1e^{1x}+C_2xe^{1x}=(C_1+C_2x)e^{t}$~~

$y''-2y'+y=(3+5x)e^{2x}$

$\hookrightarrow L=D^2-2D+I \rightarrow p(x)=x^2-2x+1=(x-1)^2$

$\hookrightarrow (D-I)^2[y]=(3+5x)e^{2x}$ donde $y=y_p+y_h$

$\cdot y_h \rightarrow N_u((D-I)^2)=gen\{e^{x}, e^{x}.x\} \Rightarrow y_h=\alpha e^{x}+\beta xe^{x}$.

$\cdot y_p=F(x)e^{x} \Rightarrow F''(x)=(3+5x).e^{2x}.e^{-x}=(3+5x)e^{x}$

$F'(x)=(5x-2)e^{x}$

$F(x)=(5x-7)e^{x} \Rightarrow y_p=5xe$

- $(D-\lambda I)^k [F(x)\, e^{\lambda x}] = F^{k}(x)\, e^{\lambda x}$
- $N_u((D-\lambda I)^k) = \text{gen}\{x^{i} e^{\lambda x}\}$
- $y_p = F(x)\, e^{\lambda x}$, $F^{k}(x) = g(x)\, e^{-\lambda x}$.
- $y = y_p + y_H$

Ejemplos

a. $y' - 3y = 0 \Rightarrow (D-3I)[y] = 0 \Rightarrow L = D - 3I \wedge p(x) = x - 3$

↳ raíz: 3.

Homogénea $\Rightarrow y = y_h \to N_u(D-3I) = \text{gen}\{x^0 e^{3x}\} = \text{gen}\{e^{3x}\}$.

$\Rightarrow \boxed{y = \alpha . e^{3x}}$, $\alpha \in \mathbb{R}$

b. $y' - 3y = (4x + 5x^2).e^{2x} \Rightarrow (D - 3I)[y] = g \Rightarrow g = (4x + 5x^2) e^{2x}$

$y = y_p + y_h$

- Homogénea $\to N_u(D-3I) = \text{gen}\{x^0 e^{3x}\} = \text{gen}\{e^{3x}\} \Rightarrow y_h = \alpha . e^{3x}$
- Particular $\to y_p = F(x).e^{3x}$ con $F'(x) = g(x).e^{-3x}$

$\Rightarrow F'(x) = (4x + 5x^2) e^{2x} . e^{-3x} = (4x + 5x^2).e^{-x}$

$F(x) = -(5x^2 + 14x + 14) e^{-x} \Rightarrow y_p = -(5x^2 + 14x + 14) e^{-x} . e^{3x} = -(5x^2 + 14x + 14).e^{2x}$.

$\Rightarrow y = y_p + y_h = \boxed{-e^{2x}(5x^2 + 14x + 14) + \alpha e^{3x}}$

c. $y''' + 6y'' + 12y' + 8y = 3x e^{5x} \Rightarrow (D^3 + 6D^2 + 12D + 8I)[y] = g \wedge g = 3x e^{5x}$

↳ $L = D^3 + 6D^2 + 12D + 8I \Rightarrow p(x) = x^3 + 6x^2 + 12x + 8$.

↳ raíces: $-2, -2, -2$ (triple)

- Homogénea $\Rightarrow N_u[(D+2I)^3] = \text{gen}\{e^{-2x}, x e^{-2x}, x^2 e^{-2x}\}$

$y_h = \alpha_1 . e^{-2x} + \alpha_2 e^{-2x} . x + \alpha_3 . e^{-2x} . x^2$.

- Particular $\to y_p = F(x).e^{-2x}$ con $F'''(x) = g(x) e^{2x} \wedge g(x) = 3x e^{5x} \Rightarrow F'''(x) = 3x e^{7x}$

$F''(x) = \frac{(21x - 3)}{49} e^{7x}$

$F'(x) = \frac{(21x - 6)}{343} e^{7x}$

$F(x) = \frac{(21x - 9)}{2401} e^{7x}$

$\Rightarrow y_p = \frac{(21x - 9)}{2401} . e^{7x} . e^{-2x} = \frac{(21x - 9)}{2401} . e^{5x}$

$\boxed{y = \alpha_1 . e^{-2x} + \alpha_2 . e^{-2x} x + \alpha_3 . e^{-2x} x^2 + \frac{(21x - 9)}{2401} e^{5x}}$

2.28.

a- $y' - y = 0 \rightarrow (D - I)[y] = 0$

$L = D - I \rightarrow p(x) = x - 1 \rightarrow \lambda = 1$

Homogénea $\Rightarrow y = y_h$ con $y_h \in Nu(L) = Nu[(D-I)] = gen\{e^x\}$

$\Rightarrow \boxed{y = \alpha e^x}$

b. $y' - y = e^{2x} \rightarrow (D-I)[y] = g(x) \wedge g(x) = e^{2x}$

$\rightarrow L = D - I \rightarrow p(x) = x - 1 \longrightarrow$ raíz: $1 \rightarrow \lambda$.

↳ pol. característico

$y = y_H + y_P$

- Homogénea $= y_H \rightarrow y_H \in Nu(L) = Nu[(D-I)] = gen\{e^x\} \Rightarrow y_H = \alpha . e^x$.
- Particular $= y_P \rightarrow y_P = f(x) . e^{\lambda x} \wedge f'(x) = g(x) . e^{-x} = e^{2x} . e^{-x} = e^x$.

↳ $f(x) = e^x$

$\Rightarrow y_P = e^x . e^x = e^{2x}$

$\Rightarrow y = y_H + y_P = \boxed{\alpha e^x + e^{2x}}$

c.- $y' - y = x e^{2x} \rightarrow (D-I)[y] = g(x) \wedge g(x) = x e^{2x}$

$\rightarrow L = D - I \rightarrow p(x) = x - 1 \rightarrow \lambda = 1$.

- Homogénea $= y_H \rightarrow y_H \in Nu(L) = Nu[(D-I)] = gen\{e^x\} \Rightarrow y_H = \alpha . e^x$.
- Particular $= y_P \rightarrow y_P = f(x) e^{\lambda x} \wedge f'(x) = g(x) . e^{-\lambda x} = x e^{2x} . e^{-x} = x . e^x$

$f(x) = (x-1) e^x$

$\Rightarrow y_P = (x-1) e^x . e^x = (x-1) e^{2x}$.

$\Rightarrow \boxed{y = \alpha e^x + (x-1) e^{2x}}$

d- $y' - y = (3 + 5x) e^{2x} \rightarrow (D - I)[y] = g(x) \wedge g(x) = (3 + 5x) e^{2x}$

$\rightarrow L = D - I \rightarrow p(x) = x - 1 \rightarrow \lambda = 1$.

- $y_H \in Nu(L) = Nu[(D-I)] = gen\{e^x\} \Rightarrow y_H = \alpha . e^x$
- $y_P = f(x) . e^{\lambda x} \wedge f'(x) = g(x) e^{-\lambda x} \Rightarrow f'(x) = (3 + 5x) e^{2x} . e^{-x} = (3 + 5x) e^x$

↳ $f(x) = (5x - 2) e^x$

$\Rightarrow y_P = (5x-2) e^x . e^x$

$\Rightarrow \boxed{y = \alpha e^x + e^{2x} (5x - 2)}$

a.- $y'' - 2y' + y = (3+5x)e^{2x} \Rightarrow D^2 - 2D + I = g(x) \wedge g(x) = (3+5x)e^{2x}$

$\Rightarrow L = D^2 - 2D + I \Rightarrow p(x) = x^2 - 2x + 1 \Rightarrow \lambda = 1$ (doble) $\Rightarrow p(x) = (x-1)^2 \Rightarrow L = (D-I)^2$.

$y = y_P + y_H$

- Homogenea: $y_H \in Nu(L) = Nu((D-I)^2) = gen\{e^x, e^x x\} \Rightarrow y_H = \alpha . e^x + \beta . x . e^x$
- Particular: $y_P = F(x).e^{\lambda x} \wedge F''(x) = g(x).e^{-\lambda x} \Rightarrow F''(x) = (3+5x)e^{2x}.e^{-x} = (3+5x).e^x$

$F'(x) = (5x-2)e^x$

$F(x) = (5x-7)e^x$

$\Rightarrow y_P = (5x-7)e^x.e^x = (5x-7)e^{2x}$.

$$\boxed{y = \alpha e^x + \beta x e^x + (5x-7)e^{2x}}$$

F.- $(D-I)^3[y] = (3+5x)e^{2x} \Rightarrow (D-I)^3[y] = g(x) \wedge g(x) = (3+5x)e^{2x}$

$\hookrightarrow (x-1)^3 = x^3 - 3x^2 + 3x - 1 \qquad \hookrightarrow \lambda = 1$ (triple)

- Homogenea: $y_H \in Nu(L) = Nu[(D-I)^3] = gen\{e^x, xe^x, x^2e^x\} \Rightarrow y_H = \alpha e^x + \beta . xe^x + \gamma . x^2 e^x$
- Particular: $y_P = F(x).e^{\lambda x} \wedge F'''(x) = g(x).e^{-\lambda x} \Rightarrow F'''(x) = (3+5x)e^{2x}.e^{-x} = (3+5x)e^x$.

$F''(x) = (5x-2)e^x$

$F'(x) = (5x-7)e^x$

$F(x) = (5x-12)e^x \Rightarrow y_P = (5x-12)e^x e^x$

$$\Rightarrow \boxed{y = \alpha e^x + \beta x e^x + \gamma x^2 e^x + (5x-12)e^{2x}}$$

Álgebra II - 2021

2.30

$L: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ def. por $L := (D-2)(D-4)(D+3)^2 \wedge L[y] = p(x)$ con $p(x) = 5x^3 e^{-3x}$

a. Hallar base B_L de $Nu(L)$.

$B_L = \{e^{2x}, e^{4x}, e^{-3x}, x.e^{-3x}\} \Rightarrow y_H = \alpha.e^{2x} + \beta.e^{4x} + \gamma.e^{-3x} + \delta.e^{-3x}.x$

b. Comprobar q. el operador $A = (D+3I)^4$ es un aniquilador de p: $A[p] = 0$.

$A[p] = 0 \Rightarrow A[5x^3 e^{-3x}] = 0 \Rightarrow (D+3I)^4[5x^3 e^{-3x}] = 0 \to$ se q.: $(D-\lambda I)^k[f(x)e^{\lambda x}] = f^{k}(x)e^{\lambda x}$

Con $\lambda = -3$, $k = 4$, $f(x) = 5x^3 \Rightarrow f^4(x)e^{-3x} = (5x^3)^{IV}.e^{-3x} = (15x^2)'''.e^{-3x} = (30x)''.e^{-3x} = (30)'.e^{-3x} = 0$ ✓

c. Hallar base B_{AL} de $Nu(A \circ L)$ que contenga a la base B_L.

$A[L[y]] = A[p] = 0 \to (D+3I)^4(D-2I)(D-4I)(D+3I)^2[y] = 0.$

$(D+3I)^6(D-2I)(D-4I)[y] = 0.$

$\Rightarrow Nu[(D+3I)^6(D-2I)(D-4I)] = gen\{e^{-3x}, e^{-3x}x, e^{-3x}x^2, e^{-3x}x^3, e^{-3x}x^4, e^{-3x}x^5, e^{2x}, e^{4x}\}$

$\Rightarrow B_{AL} = \{e^{-3x}, xe^{-3x}, x^2e^{-3x}, x^3e^{-3x}, x^4e^{-3x}, x^5e^{-3x}, e^{2x}, e^{4x}\}$

d. $\exists\, y_p$ de $L[y] = p \in gen(B_{AL}/B_L)$

$B_{AL}/B_L = \{x^2e^{-3x}, x^3e^{-3x}, x^4e^{-3x}, x^5e^{-3x}\} \Rightarrow y_p = \alpha.x^2e^{-3x} + \beta.x^3e^{-3x} + \gamma x^4e^{-3x} + \delta x^5 e^{-3x}.$

$L[y_p] = p. \to L[e^{-3x}(\alpha x^2 + \beta x^3 + \gamma x^4 + \delta x^5)] = 5x^3e^{-3x} \to (D-2I)(D-4I)(D+3I)^2[e^{-3x}(\alpha x^2 + \beta x^3 + \gamma x^4 + \delta x^5)] = 5x^3e^{-3x}$

$\to (D-2I)(D-4I)(D\underbrace{+3}_{-\lambda}I)^{\overset{k}{2}}[\underbrace{e^{-3x}}_{e^{\lambda x}}\underbrace{(\alpha x^2 + \beta x^3 + \gamma x^4 + \delta x^5)}_{f(x)}] = 5x^3e^{-3x}$

$\Rightarrow (D-2I)(D-4I)[(\alpha x^2 + \beta x^3 + \gamma x^4 + \delta x^5)''e^{-3x}] = (D-2I)(D-4I)[(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3)e^{-3x}]$

$= (D-2I)[D[(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3)e^{-3x}] - 4.[(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3)e^{-3x}]] =$

$= (D-2I)[(6\beta + 24\gamma x + 60\delta x^2)e^{-3x} + (2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3).(-3)e^{-3x} + (-4)e^{-3x}(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3)] =$

$= (D-2I)[(6\beta + 24\gamma x + 60\delta x^2)e^{-3x} - 7e^{-3x}(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3)].$

$= [(24\gamma + 120\delta x)e^{-3x} + (6\beta + 24\gamma x + 60\delta x^2)(-3)e^{-3x} + 21e^{-3x}(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3) - 7e^{-3x}(6\beta + 24\gamma x + 60\delta x^2)]$

$- 2I[\,]$

$= [(24\gamma + 120\delta x)e^{-3x} - 10(6\beta + 24\gamma x + 60\delta x^2)e^{-3x} + 21e^{-3x}(2\alpha + 6\beta x + 12\gamma x^2 + 20\delta x^3)] - 2[\,] \ldots = 5x^3e^{-3x}$

↳ resuelvo sist.