ÁLGEBRA II (61.08 – 81.02)

Pionono — Segundo cuatrimestre – 2023

Duración: 3 horas. — 6/XII/23 – 7:00 hs.

Apellido y Nombres:

Legajo:

Curso:

1. Usando la técnica de mínimos cuadrados, ajustar los siguientes datos

x	-2	-1	0	1	2
y	5	2	0	1	4

mediante una parábola $y = ax^2 + bx + c$.

2. Hallar la matriz de rotación de ángulo $\frac{\pi}{3}$ alrededor del eje generado por el vector $\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$

3. Sea $A \in \mathbb{R}^{3\times 3}$ la matriz simétrica tal que $\operatorname{nul}\left(A - \frac{1}{2}I\right) = \left\{x \in \mathbb{R}^3 : 2x_1 + 2x_2 - x_3 = 0\right\}$ y $\operatorname{traza}(A) = 2$. Hallar $\lim\limits_{k\to\infty} A^k \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

4. Sea $A \in \mathbb{R}^{3\times 3}$ la matriz de rango 2 tal que $\begin{bmatrix} 2 & -6 & 3 \end{bmatrix}^T \in \operatorname{nul}(A)$ y

$$A \begin{bmatrix} 6 & -3 \\ 3 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{9} \end{bmatrix}.$$

Hallar todas las soluciones por cuadrados mínimos de la ecuación $Ax = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$ y determinar la de norma mínima.

5. Hallar una matriz $A \in \mathbb{R}^{2\times 3}$ tal que $\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}^T \in \operatorname{nul}(A)$, $A\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 9 & 12 \end{bmatrix}^T$ y $\max\limits_{\|x\|=1} \|Ax\| = 10$.

6. Sea $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ la proyección sobre el plano $\left\{x \in \mathbb{R}^3 : x_3 = 0\right\}$ en la dirección de la recta $\operatorname{gen}\left\{\begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T\right\}$. Hallar y graficar la imagen por Π de la esfera unitaria de $\mathbb{R}^3$.

2- $S = \text{gen}\{(2\ \ 2\ \ 1)^T\}$

$S^{\perp} = \text{gen}\{(5\ \ -4\ \ -2)^T, (0\ \ -1\ \ 2)^T\}$

$B_{ON_{\mathbb{R}^3}} = \text{gen}\left\{\left(2/3\ \ 2/3\ \ 1/3\right), \left(5/\sqrt{5}\ \ -4/\sqrt{5}\ \ -2/\sqrt{5}\right)\left(0\ \ -1/\sqrt{5}\ \ 2/\sqrt{5}\right)\right\}$?

$[R]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\text{sen}(\theta) \\ 0 & \text{sen}(\theta) & \cos(\theta) \end{pmatrix} \to$ si $\theta = \pi/3$

$$\begin{tikzcd} B \arrow[r, "{[R]_B}"] & B \arrow[d, "P"] \\ \mathcal{E}_{\mathbb{R}^3} \arrow[u, "P^T"] \arrow[r, "{[R]_{\mathcal{E}_{\mathbb{R}^3}}}"'] & \mathcal{E}_{\mathbb{R}^3} \end{tikzcd}$$

$[R]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix}$

$$A = \begin{pmatrix} 2/3 & 5/\sqrt{5} & -0/\sqrt{5} \\ 2/3 & -4/\sqrt{5} & -1/\sqrt{5} \\ 1/3 & -2/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2/3 & 2/3 & 1/3 \\ 5/\sqrt{5} & -4/\sqrt{5} & -2/\sqrt{5} \\ 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

3- $A \in \mathbb{R}^{3\times 3}$ matriz simétrica / $\text{nul}(A - \frac{1}{2} I)$

$= \{x \in \mathbb{R}^3 : 2x_1 + 2x_2 - x_3 = 0\}$ $\quad tr(A) = 2$

Hallar $\lim_{k\to\infty} A^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 1/2$

$\text{avts}_{\lambda_1 = 1/2} = [1\ \ 1\ \ 5]^T$

$[1\ \ 0\ \ 2]^T$

$\sigma(A) = \{1/2, 1\}$

(doble

$\text{avt}_{\lambda_2 = 1} = [2\ \ 2\ \ -1]^T$

$$A^k \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\lambda_1^k}_{\to 0} \cdot \alpha \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + \underbrace{\lambda_1^k}_{\to 0} \cdot \beta \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \underbrace{\lambda_2^k}_{\to 1} \cdot \gamma \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \gamma \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\1\\4\end{bmatrix} + \beta\begin{bmatrix}1\\0\\2\end{bmatrix} + \gamma\begin{bmatrix}2\\2\\-1\end{bmatrix}$$

$$1 = \alpha + \beta + 2\gamma$$
$$1 = \alpha + 2\gamma$$
$$1 = 4\alpha + 2\beta - \gamma$$

$$\beta = 0$$
$$1-\alpha = 2\gamma$$
$$\frac{1-\alpha}{2} = \gamma$$
$$1 = 4\alpha - \frac{1-\alpha}{2}$$
$$\frac{3}{2} = \frac{9}{2}\alpha$$
$$\frac{3}{2}\cdot\frac{2}{9} = \alpha = \frac{1}{3}$$
$$\gamma = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

$$\lim_{k\to\infty} A^k \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}\cdot\begin{bmatrix}2\\2\\-1\end{bmatrix}$$

Hallar A:

HACER UNA $B_{OG} \rightarrow$ $W_1 = V_1$ $\quad W_2 = V_2 - \frac{\langle V_2, W_1\rangle}{\langle W_1, W_1\rangle} V_1$

$$W_2 = [1\ 0\ 2]^T - \frac{9}{18}\cdot[1\ 1\ 4]^T$$
$$W_2 = [1\ 0\ 2]^T - \frac{1}{2}\cdot[1\ 1\ 4]^T$$
$$W_2 = [\tfrac{1}{2}\ \ -\tfrac{1}{2}\ \ 0] \rightarrow W_2 = [1\ \ -1\ \ 0]$$
$$B_{OG} = \text{gen}\{(1\ 1\ 4)^T, (1\ -1\ 0)^T\}$$
$$B_{ON} = \text{gen}\{(1/\sqrt{18}\ \ 1/\sqrt{18}\ \ 4/\sqrt{18})^T, (1/\sqrt{2}\ \ -1/\sqrt{2}\ \ 0)^T\}$$

$$A = \begin{bmatrix} 1/\sqrt{18} & 1/\sqrt{2} & 2/3 \\ 1/\sqrt{18} & -1/\sqrt{2} & 2/3 \\ 4/\sqrt{18} & 0 & -1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{18} & 1/\sqrt{18} & 4/\sqrt{18} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$$

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4. $A \in \mathbb{R}^{3\times 3}$ $\quad rg(A) = 2 \;/\; [2 \;\; -6 \;\; 3]^T \in nul(A)$

$$A \begin{bmatrix} 6 & -3 \\ 3 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & 0 \\ 0 & 2/9 \end{bmatrix}$$

Hallar sol. por cuad. mín. de la ec:

$Ax = [1 \;\; -1 \;\; 1]$ y determinar la de

norma mínima

$$\underbrace{A \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = 2/3 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \quad A \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix} = 2/9 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}_{\text{genera } Col(A)}$$

3×3 $\quad$ 1×2

$nul(A) = gen\{[2 \;\; -6 \;\; 3]^T\}$

$Col(A) = gen\{[1 \;\; 2 \;\; -2]^T, [2 \;\; 2 \;\; 1]^T\}$

$\hat{x} = A^+ b$

VALORES SINGULARES

$$A \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \quad A \begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$A \begin{bmatrix} 6/7 \\ 3/7 \\ 2/7 \end{bmatrix} = \frac{2}{7} \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \quad A \begin{bmatrix} -3/7 \\ 2/7 \\ 6/7 \end{bmatrix} = \frac{2}{21} \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$A = \overbrace{\begin{bmatrix} 1/3 & 2/3 \\ -2/3 & 2/3 \\ 2/3 & 1/3 \end{bmatrix}}^{Col(A)}\overbrace{\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}}^{Col^{\perp}(A)} \underbrace{\begin{bmatrix} 2/7 & 0 & 0 \\ 0 & 2/21 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 6/7 & 3/7 & 2/7 \\ -3/7 & 2/7 & 6/7 \\ 2/7 & -6/7 & 3/7 \end{bmatrix}}_{P^{-1}} \begin{array}{l} \} Fil(A) \\ \\ ] \; nul(A) \end{array}$$

P

$$\begin{vmatrix} i & j & k \\ 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{vmatrix} = \begin{bmatrix} -2/3 & 1/3 & 2/3 \end{bmatrix}^T$$

Filas en Col → columnas en fil

$$A^+ = V_r \cdot \Sigma^{-1} \cdot U_r^T$$

$$A^+ = \begin{bmatrix} 6/7 & -3/7 \\ 3/7 & 2/7 \\ 2/7 & 6/7 \end{bmatrix} \begin{bmatrix} 7/2 & 0 \\ 0 & 21/2 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$A^+ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \hat{x}$$

$$\begin{bmatrix} 3 & -9/2 \\ 3/2 & 3 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} = \cancel{\begin{bmatrix} -2 & -5 & 1/2 \\ 3/2 & 1 & 2 \\ 19/3 & & \end{bmatrix}}$$

$$\hat{x} = \overbrace{\underbrace{\begin{bmatrix} 7/2 \\ 7/2 \\ 19/3 \end{bmatrix}}_{\text{Norma mínima}} + \alpha \underbrace{\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}}_{nul}}^{general}$$

5- $A \in \mathbb{R}^{2\times 3}$ / $[1\ 2\ 2]^T \in Nul(A)$;

$A[2\ -1\ \ 2]^T = [9\ 12]^T$

$\max\limits_{\|x\|=1} \|Ax\| = 10 \longrightarrow \sigma_1 \to \lambda_1 = 100$

Hallar A

$\dim(Nul) > 0$

$\hookrightarrow$ hay un ava = 0 o más

$A\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}9\\12\end{pmatrix}$

$Nul(A) = gen\{(-1\ 2\ 2)^T\}$

$A \cdot 3\begin{bmatrix}2/3\\-1/3\\2/3\end{bmatrix} = \begin{bmatrix}9/15\\12/15\end{bmatrix} \cdot 15$

$A\begin{bmatrix}2/3\\-1/3\\2/3\end{bmatrix} = 5\begin{bmatrix}9/15\\12/15\end{bmatrix}$

$\longrightarrow \sigma_2 \to \lambda_2 = \sqrt{25}$

$\sigma_A = \{0, 25, 100\}$ $\quad \dim(Nul(A)) = 1$

$Fil(A) = gen\{(2\ -1\ 2)^T, (2\ 2\ -1)^T\}$ OG $\hookrightarrow$ OG a $Nul(A)$

$$A = \underbrace{\begin{bmatrix}-12/15 & 9/15\\ 9/15 & 12/15\end{bmatrix}}_{U_r}\underbrace{\begin{bmatrix}10 & 0\\ 0 & 5\end{bmatrix}}_{\Sigma_r}\underbrace{\begin{bmatrix}2/3 & 2/3 & -1/3\\ 2/3 & -1/3 & 2/3\end{bmatrix}}_{V_r^T}$$

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$$A = DVS = U_r \cdot \Sigma_r \cdot V_r^T$$

$$A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \end{bmatrix}$$

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6 — $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ proy sobre $\{x \in \mathbb{R}^3 : x_3 = 0\}$

en dirección de la recta gen$\{[-2\ 0\ 1]^T\}$

Hallar y graficar la img por π de la esfera unitaria

$$B = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$v_1 \quad v_2 \quad v_3$

$$[\pi]_B^B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\pi]_E^E = \begin{pmatrix} | & | & | \\ \pi(E_1) & \pi(E_2) & \pi(E_3) \\ | & | & | \end{pmatrix}$$

$\pi(1\,0\,0) = (1\,0\,0)$

$\pi(0\,1\,0) = (0\,1\,0)$

$\pi(-2\,0\,1) = (0\,0\,0)$

$\pi(0\,0\,1) = \alpha\pi(-2\,0\,1) + \beta\pi(1\,0\,0) + \gamma\pi(0\,1\,0)$

$\pi(0\,0\,1) = \pi(\alpha(-2\,0\,1) + \beta(1\,0\,0) + \gamma(0\,1\,0))$

$(0\,0\,1)^T = \alpha(-2\,0\,1) + \beta(1\,0\,0) + \gamma(0\,1\,0))$

$\alpha = 1 \quad \gamma = 0$

$\beta = 2$

$\pi(0\,0\,1) = 1 \cdot (0\,0\,0)^T + 2(1\,0\,0)^T + 0 \cdot (0\,1\,0)^T$

$\pi(0\,0\,1) = (2\,0\,0)^T$

$$A \qquad Nul(A) = (-2\ 0\ 1)$$

$$[\pi]_E^E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & [illegible] \\ 0 & 1 & 0 \\ 0 & 0 & [illegible] \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

NO HACE FALTA SACAR
LA MATRIZ

$$A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \sigma_A = \{0,\ \overset{\text{doble}}{1}\}$$

$$A \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

6 − $\pi : \mathbb{R}^3 \to \mathbb{R}^3$ proy sobre $\{x \in \mathbb{R}^3 : x_3 = 0\}$ en dir. de la recta $\{[-2 \ 0 \ 1]^T\}$.

Hallar y graficar la imagen por π de la esfera unitaria de $\mathbb{R}^3$

$\pi((0\ 1\ 0)^T) = (0\ 1\ 0)^T$] Col

$\longmapsto$ Fil

$\pi((1\ 0\ 0)^T) = (1\ 0\ 0)^T$

TODOS OG

$\pi((-2\ 0\ 1)^T) = (0\ 0\ 0)^T$] Nul

$\hookrightarrow$ Nul

$$V^T = \begin{bmatrix} \text{— Fil —} \\ \text{— Fil —} \\ \text{— Nul —} \end{bmatrix}$$

$$A = U \ \Sigma \ V^T$$

Calculo $B_{ON} \to \{(0\ 1\ 0)^T, (-2/\sqrt{5}\ \ 0\ \ 1/\sqrt{5}), (1/\sqrt{5}\ \ 0\ \ 2/\sqrt{5})^T\}$

$$\pi(1\ 0\ 2) = \pi(\alpha(0\ 1\ 0)^T + \beta(1\ 0\ 0)^T + \gamma(-2\ 0\ 1)^T)$$

$$\pi(1\ 0\ 2) = \pi(0(0\ 1\ 0)^T + 4(1\ 0\ 0)^T + 2(-2\ 0\ 1)^T)$$

$$\pi(1\ 0\ 2) = (4\ 0\ 0)^T$$

$\sigma_1 = 0$

$A\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \begin{pmatrix}4\\0\\0\end{pmatrix}$ $\quad A\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}4\\0\\0\end{pmatrix}$

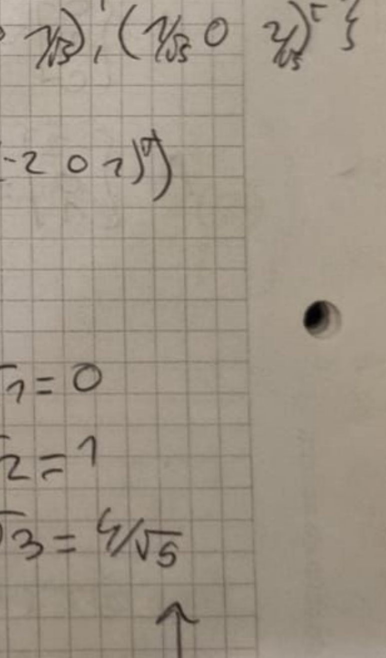
$\sigma_2 = 1$

$\sigma_3 = 4/\sqrt{5}$

$A\begin{pmatrix}0\\1\\0\end{pmatrix} = 1 \cdot \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad A\sqrt{5}\begin{pmatrix}1/\sqrt{5}\\0\\2/\sqrt{5}\end{pmatrix} = 4\begin{pmatrix}1\\0\\0\end{pmatrix}$

$A\begin{pmatrix}1/\sqrt{5}\\0\\2/\sqrt{5}\end{pmatrix} = \frac{4}{\sqrt{5}}\begin{pmatrix}1\\0\\0\end{pmatrix}$

VS



$A\begin{pmatrix}-2\\0\\1\end{pmatrix} = 0\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

los ejes de la imagen de la esfera unitaria son los autoespacios de A^TA

$$A = \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}4/\sqrt{5}&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}1/\sqrt{5}&0&2/\sqrt{5}\\0&1&0\\-2/\sqrt{5}&0&1/\sqrt{5}\end{bmatrix}}_{V^T}$$