Producto Interno

Función: $\mathbb{V} \times \mathbb{V} \to \mathbb{K}$. Cumple

I- $\forall \lambda \in \mathbb{K},\ x,y,z \in \mathbb{V}$

- $(x+y,z) = (x,z) + (y,z)$
- $(\lambda x,y) = \lambda(x,y) \to (x,\lambda y) = \overline{(\lambda y,x)} = \overline{\lambda(y,x)} = \bar{\lambda}\,\overline{(y,x)} = \bar{\lambda}(x,y)$.

II- $(x,y) = \overline{(y,x)}$

III- $(x,x) > 0$ si $x \neq 0_{\mathbb{V}}$

- Cuando $\mathbb{K} = \mathbb{R}$: espacio euclídeo real.
- Cuando $\mathbb{K} = \mathbb{C}$: espacio euclídeo complejo

Matriz

$\mathbb{K}$-espacio vectorial $\mathbb{V}$ de dim. finita n y $B = \{v_1, v_2, \dots, v_n\}$ la matriz de Gram (G_B) tq.

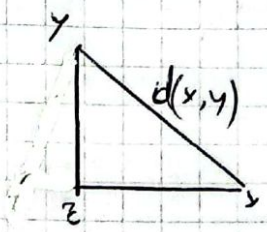
$$G_B = \begin{bmatrix} (v_1,v_1) & (v_1,v_2) & \dots & (v_1,v_n) \\ (v_2,v_1) & (v_2,v_2) & \dots & (v_2,v_n) \\ \vdots & \vdots & \ddots & \vdots \\ (v_n,v_1) & (v_n,v_2) & \dots & (v_n,v_n) \end{bmatrix} \Rightarrow \text{matriz de p.i. en la base } B.$$

- G es hermítica: $G_{ij} = \overline{G}_{ij}$
- $(u,v) = u^T G \bar{v}$ def. p.i. en $\mathbb{K}^n$ sii G es hermítica y def. positiva, $\forall u \in \mathbb{K}^n: u^T G \bar{u} > 0$, si $u \neq 0_{\mathbb{K}^n}$.

Norma

$\|\cdot\|: \mathbb{V} \to \mathbb{R}_0^+ : \|x\| = \sqrt{(x,x)},\ \forall x \in \mathbb{V}$.

- $\|\alpha x\| = |\alpha| \cdot \|x\|$.
- $\|x\| \geq 0 \wedge \|x\| = 0 \Leftrightarrow x = 0_{\mathbb{V}}$.
- Desigualdad de Cauchy-Schwarz: $|(x,y)| \leq \|x\| \|y\|$
- Desigualdad triangular: $\|x+y\| \leq \|x\| + \|y\|$.



- $d(x,y) = 0$ sii $x = y$.
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y)$.
- $d(\lambda x, \lambda y) = |\lambda| \cdot d(x,y)$

Identidad del Paralelogramo

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

↳ si se cumple $\Rightarrow (u,u) = \|u\|^2$

$\mathbb{R} \Rightarrow (u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right)$

$\mathbb{C} \Rightarrow (u,v) = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2\right)$

Ángulo

$\cos(\theta) = \frac{(x,y)}{\|x\|\|y\|} \qquad 0 \leq \theta \leq \pi$

Ortogonalidad

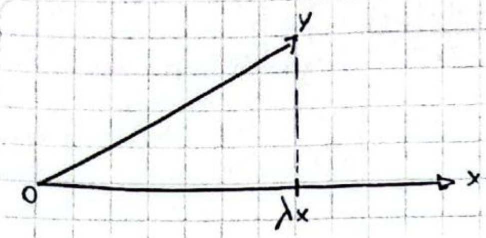
→ Si p.i $= 0$.

- Si $x \perp y \wedge x \neq 0_V \wedge y \neq 0_V \Rightarrow \theta = \frac{\pi}{2}$.
- Si x, y es L.D en $\mathbb{V} \Rightarrow \theta = 0 \vee \theta = \pi$
- $(x,y) = \|x\|\|y\|\cos(\theta)$

Pitágoras: $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ $\Rightarrow \|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle =$

$= \langle x, x\rangle + \underbrace{\langle x,y\rangle + \langle y,x\rangle}_{2\,\mathrm{Real}(\langle x,y\rangle)} + \langle y,y\rangle =$ $\Rightarrow x \cdot y \overset{P.I.}{=} 0$

$= \langle x,x\rangle + \langle y,y\rangle = \|x\|^2 + \|y\|^2$.



Quiero hallar λ tq. $(y-\lambda x) \perp x$

$\Rightarrow \quad y-\lambda x \perp x$ sii $\langle (y-\lambda x), x \rangle \overset{P.I.}{=} 0$

$$\langle y,x\rangle - \langle \lambda x, x\rangle = 0$$

$$\langle y,x\rangle - \lambda \langle x, x\rangle = 0$$

$$\langle y,x\rangle - \lambda \|x\|^2 = 0.$$

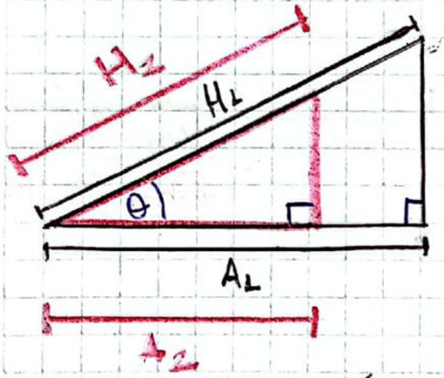
$$\langle y,x\rangle = \lambda \|x\|^2$$

$$\frac{\langle y,x\rangle}{\|x\|^2} = \lambda.$$

Si eso lo multiplicamos por x, tenemos proy. ortogonal de y sobre x:

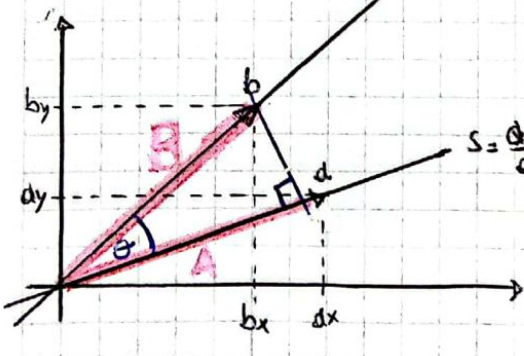
$$P_{gen\{x\}}(y) := \frac{\langle y,x\rangle}{\|x\|^2} . x = \boxed{\lambda . x}$$

TALES DE MILETO



$$\frac{A_1}{H_1} = \frac{A_2}{H_2}$$

$$\cos\theta = \frac{A_1}{H_1} = \frac{A_2}{H_2}$$



$s = \frac{dy}{dx} . t \rightarrow$ recta $\perp s$: $m = \overset{\text{inverso y opuesto}}{-\frac{dx}{dy}}$ que pasa por (bx, by)

$$\Rightarrow s - by = -\frac{dx}{dy}(t - bx)$$

$$\frac{dy}{dx} . t - by = -\frac{dx}{dy} t + \frac{dx . bx}{dy}$$

$$\frac{dy}{dx} . t + \frac{dx}{dy} t = \frac{dx\, bx}{dy} + by$$

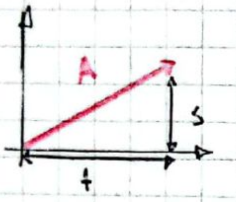
$$\frac{(dy)^2 + (dx)^2}{dx . dy} . t = \frac{dx\, bx + by . dy}{dy}$$

$$t = \frac{dx\, bx + by\, dy}{dy} . \frac{dx\, dy}{(dy)^2 + (dx)^2}$$

$$t = \frac{(d_x b_x + d_y b_y)\cdot d_x}{d_y^2 + d_x^2}$$

$$\Rightarrow s = \frac{d_y}{d_x}\cdot t = \frac{d_y}{\cancel{d_x}} \cdot \frac{(d_x b_x + d_y \cdot b_y)\cdot \cancel{d_x}}{d_y^2 + d_x^2} = d_y \cdot \frac{(d_x b_x + d_y b_y)}{d_y^2 + d_x^2}$$

Long. A:



$\text{long}(A) = \sqrt{t^2 + s^2}$ x pitágoras.

Log. B: $\sqrt{b_x^2 + b_y^2}$

$$\Rightarrow \cos(\theta) = \frac{\text{long } A}{\text{long } B} = \sqrt{\frac{t^2+s^2}{b_x^2+b_y^2}} = \sqrt{\frac{\left[\frac{(d_x b_x + d_y b_y)\cdot d_x}{d_x^2+d_y^2}\right]^2 + \left[\frac{d_y (d_x b_x + d_y b_y)}{d_x^2+d_y^2}\right]^2}{b_x^2 + b_y^2}} =$$

$$= \sqrt{\frac{(d_x^2+d_y^2)\cdot\left[\frac{d_x b_x + d_y b_y}{d_x^2+d_y^2}\right]^2}{b_x^2+b_y^2}} = \sqrt{\frac{\cancel{(d_x^2+d_y^2)}\cdot(d_x b_x + d_y b_y)^2}{(b_x^2+b_y^2)\cdot(d_x^2+d_y^2)^{\cancel{2}}}} =$$

$$= \sqrt{\frac{(d_x b_x + d_y b_y)^2}{(b_x^2+b_y^2)(d_x^2+d_y^2)}}\,\Bigg)^{>0} = \frac{\overbrace{[d_x b_x + d_y b_y]}^{\langle d,b\rangle}}{\sqrt{\underbrace{[b_x^2+b_y^2]}_{\langle b,b\rangle}\underbrace{[d_x^2+d_y^2]}_{\langle d,d\rangle}}} = \frac{\langle d,b\rangle}{\sqrt{(\langle b,b\rangle)(\langle d,d\rangle)}} = \frac{\langle d,b\rangle}{\underbrace{\sqrt{\langle b,b\rangle}}_{\sqrt{b\cdot b} = \|b\|^2}\cdot\underbrace{\sqrt{\langle d,d\rangle}}_{\|d\|^2}} =$$

$$= \frac{\langle d,b\rangle}{\|b\|^2 \cdot \|d\|^2}.$$

$$\overset{M^{E}_{B}}{}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \qquad B = \{v_1, v_2\}, \quad v_1 = \begin{bmatrix} d_{11} \\ d_{21} \end{bmatrix}, \quad v_2 = \begin{bmatrix} d_{12} \\ d_{22} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \cdot \begin{bmatrix} d_{11} & d_{21} \\ d_{12} & d_{22} \end{bmatrix} \cdot \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} \overbrace{d_{11}^2 + d_{21}^2}^{\langle v_1, v_1 \rangle} & \overbrace{d_{11} \cdot d_{12} + d_{21} \cdot d_{22}}^{\langle v_1, v_2 \rangle} \\ \underbrace{d_{12} \cdot d_{11} + d_{22} \cdot d_{21}}_{\langle v_1, v_2 \rangle} & \underbrace{d_{12}^2 + d_{22}^2}_{\langle v_2, v_2 \rangle} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$= \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

Y $\det\left(\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_1 \cdot v_2 & v_2 \cdot v_2 \end{bmatrix} \right) = \|v_1\|^2 \cdot \|v_2\|^2 - (v_1 \cdot v_2)^2 \geq 0.$ pq. $|v_1 \cdot v_2| \leq \|v_1\| \cdot \|v_2\|$ desigualdad Cauchy-Schwartz.

Defino $x \cdot y$: $\left\langle \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \right\rangle = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$ donde $\alpha > 0, \gamma > 0,$ $\alpha\gamma - \beta^2 \geq 0$

Producto Interno

- PI canónico en $\mathbb{R}^n$: $\left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n . y_n$.

- PI canónico en $\mathbb{C}^n$: $\left\langle \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = x_1 \overline{y_1} + x_2 \overline{y_2} + \ldots + x_n \overline{y_n}$.

Ej: $\left\langle \begin{bmatrix} 2+i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1-i \end{bmatrix} \right\rangle = (2+i).\overline{i} + 1.(\overline{1-i}) = (2+i).(-i) + 1.(1+i) = -2i - i^2 + 1 + i =$

$= -2i - (-1) + 1 + i = -i + 2 = 2 - i$.

Ejercicios:

- Tablas de multiplicar de $E = \{e_1, e_2\}$ (canónicos $\mathbb{R}^2$) y $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right\}$ en $\mathbb{R}^2$ con:
  - i. el p.i. canónico.
  - ii. el p.i. $\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ dado por $\langle x, y \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$.

i. $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

$$G_E = \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$G_B = \begin{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 5 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 26 \end{pmatrix}.$$

ii. $\langle x, y \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$.

$$G_E = \begin{pmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle \end{pmatrix} = \begin{pmatrix} 2.1.1 - 1.0 - 0.1 + 0.0 & 2.1.0 - 1.1 - 0.0 + 0.1 \\ 2.0.1 - 1.1 - 0.0 + 1.0 & 2.0.0 - 0.1 - 1.0 + 1.1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$G_B = \begin{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 5 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 5 \\ -1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} 2.1.1 - 1.2 - 2.1 + 2.2 & 2.1.5 - 1(-1) - 2.5 + 2(-1) \\ 2.5.1 - (-1).1 - 5.2 + (-1)2 & 2.5.5 - 5(-1) - (-1).5 + (-1).(-1) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 61 \end{pmatrix}.$$

Obs.:

- $x_1y_1+x_2y_2 = [x_1\ x_2]\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\begin{bmatrix}y_1\\y_2\end{bmatrix}$ · $2x_1y_1-x_1y_2-x_2y_1+x_2y_2=[x_1\ x_2]\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$.

$\Rightarrow G_B=\begin{pmatrix}\langle v,v\rangle & \langle v,w\rangle\\ \langle w,v\rangle & \langle w,w\rangle\end{pmatrix}$ "cómo se arma".

$\langle x,y\rangle=(x)^{B^T}\cdot G_B\cdot(\bar{y})^B$ "cómo se usa".

$=(\bar{y})^{B^T}\cdot G_B^T\cdot(x)^B \rightarrow$ así está en la guía.

$\rightarrow$ ¿Cómo sé si una matriz es la matriz de un p.i. en $\mathbb{R}^2$?

I. $\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle$ "lineal en la 1ra componente"

$\langle \lambda x,y\rangle=\lambda\langle x,y\rangle$.

$\rightarrow \langle x+\lambda y,z\rangle=z^T\cdot\begin{bmatrix}2&-1\\-1&1\end{bmatrix}[x+\lambda y]=[z_1,z_2]\cdot\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}x_1+\lambda y_1\\x_2+\lambda y_2\end{bmatrix}$.

$=\langle x,z\rangle+\lambda\langle y,z\rangle$.

II. $\langle x,y\rangle=\langle y,x\rangle$ "simétrico" $\rightarrow 2x_1y_1+x_2y_2-x_1y_2-x_2y_1=2y_1x_1+y_2x_2-y_1x_2-y_2x_1$ ✓

III. $\langle x,x\rangle>0$ "def. positiva".

$\hookrightarrow 2x_1^2+x_2^2-2x_1x_2=2(x_1^2-x_1x_2)+x_2^2=2\left(\left(x_1-\frac{1}{2}x_2\right)^2-\frac{1}{4}x_2^2\right)+x_2^2=2\underbrace{\left(x_1-\frac{1}{2}x_2\right)^2}_{\geq 0}+\frac{1}{2}\underbrace{x_2^2}_{\geq 0}\geq 0.$

$\langle x,x\rangle=0 \Leftrightarrow (x_2=0 \wedge x_1=0) \Leftrightarrow x=0_{\mathbb{R}^2}$.

Generalizo

¿$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathbb{R}^{2\times 2}$ es la matriz de un p.i.?

$\langle x,y\rangle=[y_1\ y_2]\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

I. Linealidad (siempre) ✓

II. Simetría

En part. debe suceder: $\langle e_1,e_2\rangle=\langle e_2,e_1\rangle \Rightarrow [0\ 1]\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=[1\ 0]\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

$\Leftrightarrow [0\ 1]\begin{bmatrix}a\\c\end{bmatrix}=[1\ 0]\begin{bmatrix}b\\d\end{bmatrix}$

$\Leftrightarrow c=b$

$\Rightarrow A=\begin{bmatrix}a&b\\b&d\end{bmatrix}$.

¿Alcanza con q. la matriz sea simétrica? Verificamos:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} d & b \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} d & b \\ b & d \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} dx_1 + bx_2 \\ bx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} dy_1 + by_2 \\ by_1 + dy_2 \end{bmatrix}$$

$$y_1(dx_1 + bx_2) + y_2(bx_1 + dx_2) = x_1(dy_1 + by_2) + x_2(by_1 + dy_2).$$

$$dy_1 y_1 + bx_2 y_1 + bx_1 y_2 + dx_2 y_2 = dx_1 y_1 + bx_1 y_2 + bx_2 y_1 + dx_2 y_2 \checkmark$$

III- En part., $\langle e_1, e_1 \rangle > 0$ si $x \neq 0_{\mathbb{R}^2}$

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} d & b \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} d \\ b \end{bmatrix} = d \Rightarrow d \text{ debe ser positivo.}$$

General: $\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} d & b \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = dx_1x_1 + 2bx_1x_2 + dx_2x_2 = dx_1^2 + 2bx_1x_2 + dx_2^2 =$

$$= d\left(x_1^2 + \frac{2bx_1x_2}{d}\right) + dx_2^2 = d.\left[\left(x_1 + \frac{2b}{2d}x_2\right)^2 - \frac{b^2}{d^2}x_2^2\right] + dx_2^2 =$$

$$= d.\left(x_1 + \frac{b}{d}x_2\right)^2 - \frac{\cancel{d}b^2}{d^{\cancel{2}}}x_2^2 + dx_2^2 = \underbrace{d}_{\geq 0}\underbrace{\left(x_1 + \frac{b}{d}x_2\right)^2}_{\geq 0} + \underbrace{x_2^2}_{\geq 0}\left(d - \frac{b^2}{d}\right) ?$$

si $d - \frac{b^2}{d} > 0 \Rightarrow \langle x, x \rangle > 0$ si $x \neq [0\,0]^T$

$\Rightarrow d.d - b^2 > 0 \longrightarrow$ ¡det > 0!

$\Rightarrow d > 0 \wedge \det\begin{bmatrix} d & b \\ b & d \end{bmatrix} > 0.$ ✓

Ejemplo: $G = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$, ¿es la matriz de un p.i.? $G^t = G$ ✓

$4 > 0$ ✓

$\det(G) = 3 > 0$ ✓

$<x,y> = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 4x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = 4x_1y_1 + x_2y_1 + x_1y_2 + x_2y_2$. es un p.i. en $\mathbb{R}^2$ ✓

¿Cómo afecta a la geometría este p.i.?

Calculo: $\left\|\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|$, $\left\|\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\|$ y $\theta = \measuredangle(e_1, e_2)$.

- $\|e_1\|^2 = <e_1, e_1> = <\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}> = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 4.$

  $\Rightarrow \|e_1\| = \sqrt{\|e_1\|^2} = \sqrt{4} = 2.$

- $\|e_2\|^2 = <\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}> = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$

  $\Rightarrow \|e_2\| = \sqrt{\|e_2\|^2} = \sqrt{1} = 1.$

- $<e_1, e_2> = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1.$ está en Gram

- $\theta = \arccos\left[\dfrac{<x,y>}{\|x\| \cdot \|y\|}\right]$ con $\theta \in [0, \pi]$.

  ↳ $\theta = \arccos\left[\dfrac{<e_1, e_2>}{\|e_1\| \cdot \|e_2\|}\right] = \arccos\left[\dfrac{1}{2 \cdot 1}\right] = \arccos\left[\dfrac{1}{2}\right] = \dfrac{\pi}{3}.$

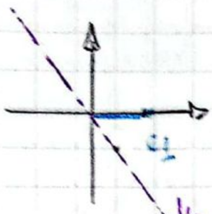
Otro ejemplo: $G_E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $G_B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$.

- Área $= \dfrac{\sqrt{\det(G_E)}}{2} = \dfrac{1}{2}$

- Área $= \dfrac{\sqrt{\det(G_B)}}{2} = \dfrac{\sqrt{3}}{2}.$

Ejercicio: Hallar un vector $v \in \mathbb{R}^2$ tq. el triángulo det. por $0, e_1, v$ sea rectángulo en O e isósceles con el p.i. dado por $G = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$.

rectángulo $\Leftrightarrow \theta = 90°$.



• Quiero que $\langle e_1, v \rangle = 0 \Leftrightarrow \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$

$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 0$

$4v_1 + v_2 = 0 \rightarrow$ ¡es una recta!

$y_2 = -4y_1$

$y_2 = -4y_1 \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\perp} = gen\left\{\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right\}; v = \alpha \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$

• Como tmb quiero q. sea isósceles $\Rightarrow \|e_1\| = \|v\| \Rightarrow 2 = \|v\|$.

$$\|v\|^2 = \langle v, v \rangle = \left\langle \alpha \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \alpha \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\rangle = \alpha^2 \left\langle \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\rangle = \alpha^2 \left(\begin{bmatrix} 1 & -4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right) =$$

$$= \alpha^2 \left(\begin{bmatrix} 1 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right) = \alpha^2 \cdot 12 \doteq 2^2.$$

$\Rightarrow \alpha^2 \cdot 12 = 4 \Rightarrow \alpha^2 = \frac{1}{3} \Rightarrow \alpha = \pm\sqrt{\frac{1}{3}}$

$v = \sqrt{\frac{1}{3}} \cdot \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{3}} & -4\sqrt{\frac{1}{3}} \end{bmatrix}^T$

$v = -\sqrt{\frac{1}{3}} \cdot \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{1}{3}} & 4\sqrt{\frac{1}{3}} \end{bmatrix}^T.$

Guía 3:

3.2. Verificar que $\langle x,y\rangle := y^T G x$ def. un p.i. en $\mathbb{R}^2$.

d- $G \in \mathcal{G}_4 = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a>0,\ \det\begin{bmatrix} a & b \\ b & c \end{bmatrix} > 0 \right\}$.

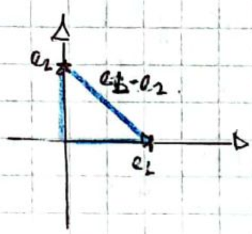
- $G^T = G$ ✓
- $a > 0$ ✓ $\Rightarrow G$ def. un p.i. en $\mathbb{R}^2$.
- $\det(G) > 0$ ✓

3.3. Hallar todos los p.i. q. convierten el triángulo $0, e_1, e_2$ en equilátero. Calcular, con alguno, α entre $v_1 = [1\ 1]^T$ y $v_2 = [-1\ 1]^T$. ¿Cuál es el área del triángulo $0, v_1, v_2$?

• equilátero $\Rightarrow \|e_1\| = \|e_2\| \Rightarrow \sqrt{\langle e_1, e_1\rangle} = \sqrt{\langle e_2, e_2\rangle}$.

$$\sqrt{[1\ 0]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \sqrt{[0\ 1]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\sqrt{a} = \sqrt{c}.$$



$\|e_1 - e_2\|^2 = \|(1,-1)\|^2$

$= [1\ -1]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} =$

$= [1\ -1]\begin{bmatrix} a-b \\ b-c \end{bmatrix} =$

$= a - b - b + c =$

$= a - 2b + c.$

$l_1 = \sqrt{a}$, $l_2 = \sqrt{c}$, $l_3 = \sqrt{a - 2b + c}$

$\Rightarrow a = c = a - 2b + c \Rightarrow a = a - 2b + c.$

$2b = c.$

$b = \frac{c}{2}$

$$\Rightarrow G = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} c & c/2 \\ c/2 & c \end{bmatrix} \rightsquigarrow \boxed{\begin{bmatrix} L & L/2 \\ L/2 & L \end{bmatrix}}$$

Calculo ángulo: v_1 y v_2.

$$\theta = \arccos\left(\frac{\langle v_1, v_2\rangle}{\|v_1\|\cdot\|v_2\|}\right) = \arccos\left(\frac{[-1\ 1]\begin{bmatrix} L & L/2 \\ L/2 & L \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{[1\ 1]\begin{bmatrix} L & L/2 \\ L/2 & L \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}\cdot\sqrt{[-1\ 1]\begin{bmatrix} L & L/2 \\ L/2 & L \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}}\right) = \arccos\left(\frac{[-1\ 1]\begin{bmatrix} \frac{3}{2}L \\ \frac{3}{2}L \end{bmatrix}}{\sqrt{L+\frac{L}{2}+\frac{L}{2}+L}\cdot\sqrt{[-1\ 1]\begin{bmatrix} -L+L/2 \\ -L/2+L \end{bmatrix}}}\right) =$$

$$= \arccos(0) = \boxed{90^\circ}$$

$$\text{Área} = \frac{\sqrt{\det(G)}}{2} = \sqrt{L^2 - \frac{L^2}{4}}\ .\ 2 = \sqrt{\frac{3}{4}L^2}\cdot\frac{1}{2} = \frac{1}{2}\sqrt{\frac{3}{4}}\cdot L = \boxed{\sqrt{\frac{3}{4}}\,L}$$

Escaneado con CamScanner

3.4. Sea $(V, \langle \cdot,\cdot \rangle)$ un $\mathbb{R}$-espacio euclídeo de dimensión 3 y sea $B = \{u_i : i \in I_3\} \subset \{u \in V : \|u\| = 1\}$ una base de V t.q. $\|u_i + u_j\|^2 = 2 + \sqrt{3}$ y $\|u_i - u_j\|^2 = 2 - \sqrt{3}$ para $i \neq j$.

• Hallar la matriz del prod. int. respecto de B.

Quiero hallar:

$$G_B = \begin{bmatrix} \langle u_1,u_1 \rangle & \langle u_1,u_2 \rangle & \langle u_1,u_3 \rangle \\ \langle u_2,u_1 \rangle & \langle u_2,u_2 \rangle & \langle u_2,u_3 \rangle \\ \langle u_3,u_1 \rangle & \langle u_3,u_2 \rangle & \langle u_3,u_3 \rangle \end{bmatrix}$$

$\|u_i\| = 1 \Rightarrow$
- $\|u_1\| = \sqrt{\langle u_1,u_1 \rangle} = 1 \Rightarrow \langle u_1,u_1 \rangle = 1$.
- $\|u_2\| = \sqrt{\langle u_2,u_2 \rangle} = 1 \Rightarrow \langle u_2,u_2 \rangle = 1$
- $\|u_3\| = \sqrt{\langle u_3,u_3 \rangle} = 1 \Rightarrow \langle u_3,u_3 \rangle = 1$.

Para el resto, uso la forma polar $\langle u_i,u_j \rangle = \frac{1}{4}\left(\|u_i+u_j\|^2 - \|u_i-u_j\|^2\right)$

- $\langle u_1,u_2 \rangle = \frac{1}{4}\left(\|u_i+u_j\|^2 - \|u_i-u_j\|^2\right) = \frac{1}{4}\cdot\left(2+\sqrt{3}-2+\sqrt{3}\right) = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$.
- $\langle u_1,u_3 \rangle = \frac{1}{4}\left(\|u_i+u_j\|^2 - \|u_i-u_j\|^2\right) = \frac{1}{4}\cdot\left(2+\sqrt{3}-2+\sqrt{3}\right) = \frac{\sqrt{3}}{2}$
- $\langle u_2,u_3 \rangle = \frac{1}{4}\left(\|u_i+u_j\|^2 - \|u_i-u_j\|^2\right) = \frac{1}{4}\left(2+\sqrt{3}-2+\sqrt{3}\right) = \frac{\sqrt{3}}{2}$.

Recuerdo, $\langle u_1,u_2 \rangle = \langle u_2,u_1 \rangle$, $\langle u_1,u_3 \rangle = \langle u_3,u_1 \rangle$, $\langle u_2,u_3 \rangle = \langle u_3,u_2 \rangle$

$$G_B = \begin{bmatrix} 1 & \sqrt{3}/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & \sqrt{3}/2 & 1 \end{bmatrix}$$

• Determinar la matriz $\Theta = [\arccos(\langle u_i,u_j \rangle)]_{i \in I_3, j \in I_3}$.

$$\theta = \arccos\left(\frac{\langle u_i,u_j \rangle}{\underbrace{\|u_i\|\cdot\|u_j\|}_{=1}}\right) = \arccos(\langle u_i,u_j \rangle).$$

Si $i = j \Rightarrow \theta = \arccos(1) = 0$

$i \neq j \Rightarrow \theta = \arccos\left(\frac{\sqrt{3}}{2}\right) = 30°$.

$$\Rightarrow \Theta = \begin{bmatrix} 0° & 30° & 30° \\ 30° & 0° & 30° \\ 30° & 30° & 0° \end{bmatrix}$$

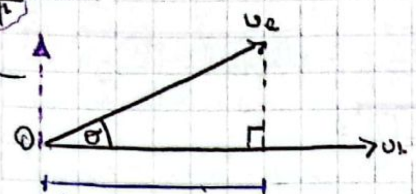
Guía

- Construir $\triangle$ vértices $0, u_1, u_2 - \lambda u_1$. ¿Es único?

Tiene q. cumplir $\langle u_2 - \lambda u_1, u_1 \rangle = 0 \Rightarrow \langle u_2, u_1 \rangle - \lambda \langle u_1, u_1 \rangle = 0 \Rightarrow \langle u_2, u_1 \rangle = \lambda \langle u_1, u_1 \rangle$

$$\Rightarrow \frac{\langle u_2, u_1 \rangle}{\langle u_1, u_1 \rangle} = \lambda. \Rightarrow \frac{\langle u_2, u_1 \rangle}{\|u_1\|^2} = \lambda.$$

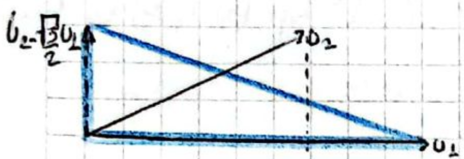
$u_2 - \underbrace{\frac{\langle u_2, u_1 \rangle}{\|u_1\|^2}}_{\lambda} . u_1$



- Hallamos proy. u_2 sobre u_1.

$\|u_2\| \cos(\theta) = \frac{\langle u_2, u_1 \rangle}{\|u_1\|} \to$ proy. genérica $_{u_1}(u_2) = \frac{\langle u_2, u_1 \rangle}{\|u_1\|^2} . u_1$.

$\lambda = \frac{\sqrt{3}/2}{1^2} = \frac{\sqrt{3}}{2} \Rightarrow$ vértices $0, u_1, u_2 - \frac{\sqrt{3}}{2} u_1$.



- Calcular área triángulo $0, u_1, u_2$.

Tres formas:

I_ $\frac{b.h}{2} \Rightarrow$ base $= \|u_1\|$, altura $= \|u_2 - \lambda u_1\| = \sqrt{\langle u_2 - \lambda u_1, u_2 - \lambda u_1 \rangle} = \sqrt{\langle u_2, u_2 - \lambda u_1 \rangle - \lambda \langle u_1, u_2 - \lambda u_1 \rangle}$

$$= \sqrt{\langle u_2, u_2 \rangle + \langle u_2, -\lambda u_1 \rangle - \lambda \langle u_1, u_2 \rangle - \lambda \langle u_1, -\lambda u_1 \rangle} =$$

$$= \sqrt{\langle u_2, u_2 \rangle - \lambda \langle u_2, u_1 \rangle - \lambda \langle u_1, u_2 \rangle + \lambda^2 \langle u_1, u_1 \rangle} =$$

$$= \sqrt{\|u_2\|^2 - 2\lambda \langle u_2, u_1 \rangle + \lambda^2 \|u_1\|^2} =$$

$$= \sqrt{1 - 2\lambda \frac{\sqrt{3}}{2} + \lambda^2 . 1} = \sqrt{1 - \lambda \sqrt{3} + \lambda^2} \quad \text{con } \lambda = \frac{\sqrt{3}}{2}.$$

$$= \sqrt{1 - \frac{\sqrt{3}}{2} . \sqrt{3} + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1 - 3/2 + 3/4} = \frac{1}{2}.$$

Área $= \frac{1 . 1/2}{2} = \frac{1}{4}$.

II_ Área $= \frac{1}{2} \|u_1\| \|u_2\| . \operatorname{sen} \theta = \frac{1}{2} . 1 . 1 . \operatorname{sen} 30° = \frac{1}{4}$.

III_ Área $= \frac{\sqrt{\det(G)}}{2} = \frac{1}{2} . \sqrt{\|u_1\|^2 . \|u_2\|^2 - (\langle u_1, u_2 \rangle)^2} = \frac{1}{2} . \sqrt{1 . 1 - \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{2} . \sqrt{\frac{1}{4}} = \frac{\sqrt{3}}{4}$

- Calcular área triángulo vértices u_1, u_2, u_3.

Traslado al origen: $0, u_2-u_1, u_3-u_1$

Como $\|u_i-u_j\|^2 = 2-\sqrt{3}$ para $i \neq j$, vale que: $\|u_2-u_1\|^2 = 2-\sqrt{3} = \|u_3-u_1\|^2 = \|u_3-u_2\|^2$.

$\Rightarrow \|u_2-u_1\| = \|u_3-u_1\| = \|u_3-u_2\| = \sqrt{2-\sqrt{3}}$.

Áng (triángulo equilátero) = $60°$.

$\Rightarrow$ Área $= \frac{1}{2} \cdot \|u_2-u_1\| \|u_3-u_1\| \cdot \operatorname{sen} 60° = \frac{1}{2} \cdot \sqrt{2-\sqrt{3}} \cdot \sqrt{2-\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{4} \cdot (2-\sqrt{3}) \sqrt{3} = \frac{1}{4}(2\sqrt{3}-3) =$

$= \boxed{\frac{1}{2}\sqrt{3} - \frac{3}{4}}$

3.6. En $\mathbb{R}^3$ con el p.i. canónico se considera el subespacio $\mathbb{S} = gen\{[1\,0\,{-1}]^T, [0\,1\,{-1}]^T\}$.

a. Escribir c/vector $x \in \mathbb{R}^3$ como $x = x_{\mathbb{S}} + x_{\mathbb{S}^\perp}$

$x = P_S(x) + P_{S^\perp}(x) \Rightarrow dim(\mathbb{R}^3) = dim(\mathbb{S}) + dim(\mathbb{S}^\perp) \Rightarrow 3 = 2 + dim(\mathbb{S}^\perp)$

$\Rightarrow dim(\mathbb{S}^\perp) = 1 \Rightarrow$ busco 1 generador de $\mathbb{S}^\perp$ tq. $(a\,b\,c)^T$ es $\perp$ a $\mathbb{S}$.

- $0 = \langle (a\,b\,c)^T, (1\,0\,{-1})^T \rangle$

$0 = [1\,0\,{-1}] \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = [1\,0\,{-1}] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = a - c$

$c = a$.

- $0 = \langle (a, b\,c)^T, (0\,1\,{-1})^T \rangle$

$0 = [0\,1\,{-1}] \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = [0\,1\,{-1}] \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = b - c$

$c = b$

$\hookrightarrow a = b = c \rightarrow (a\,a\,a) = a \cdot (1\,1\,1) \Rightarrow gen(\mathbb{S}^\perp) = \{[1\,1\,1]\}$

Ya tengo los generadores. Ahora busco expresar un $x = (x_1\,x_2\,x_3)$ como suma de las proyecciones de x sobre los subespacios $\mathbb{S}$ y $\mathbb{S}^\perp$: $x = P_{\mathbb{S}}(x) + P_{\mathbb{S}^\perp}(x)$.

Recuerdo que $P_{\mathbb{S}} + P_{\mathbb{S}^\perp} = I \Rightarrow$ Con saber uno, tengo el otro. Así q. averiguo el + fácil ($P_{\mathbb{S}^\perp}$ ya q. $\mathbb{S}^\perp$ gen por 1 vector).

Otra forma

Una forma

Guía

$$P_{S^\perp}(x) = \frac{\langle x, gen(S^\perp)\rangle}{\|gen(S^\perp)\|^2} \cdot gen(S^\perp)$$

$\hookrightarrow P_{S^\perp}(x) = \frac{\langle x, [1\,1\,1]\rangle}{\|[1\,1\,1]\|^2} \cdot [1\,1\,1]$

$P_{S^\perp}(x) = [1\,1\,1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \cdot \frac{1}{[1\,1\,1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}} \cdot [1\,1\,1]$

$P_{S^\perp}(x) = (x_1+x_2+x_3) \cdot \frac{1}{3} \cdot [1\,1\,1]$

$P_{S^\perp}(x) = \left[\frac{x_1+x_2+x_3}{3} \quad \frac{x_1+x_2+x_3}{3} \quad \frac{x_1+x_2+x_3}{3}\right]$

I- $P_{S^\perp}(x) \in S^\perp$

II- $x - P_{S^\perp}(x) \perp S^\perp$

Cumplo los requisitos:

I- $P_{S^\perp}(x) = k \cdot gen(S^\perp)$

$P_{S^\perp}(x) = k \cdot [1\,1\,1]$ ①

II- $\langle x - P_{S^\perp}(x), gen(S^\perp)\rangle = 0$

$\langle x - P_{S^\perp}(x), [1\,1\,1]\rangle = 0$ ①

$\langle x - k[1\,1\,1], [1\,1\,1]\rangle = 0.$

$\langle [x_1\ x_2\ x_3] - k[1\,1\,1], [1\,1\,1]\rangle = 0.$

$\langle [x_1-k\ \ x_2-k\ \ x_3-k], [1\,1\,1]\rangle = 0.$

$x_1-k+x_2-k+x_3-k = 0.$

$x_1+x_2+x_3-3k = 0.$

$\frac{x_1+x_2+x_3}{3} = k$

$=$

$\Rightarrow P_{S^\perp}(x) = \left[\frac{x_1+x_2+x_3}{3} \quad \frac{x_1+x_2+x_3}{3} \quad \frac{x_1+x_2+x_3}{3}\right].$

Quiero un $(x_1\ x_2\ x_3) = \left[\frac{x_1+x_2+x_3}{3} \quad \frac{x_1+x_2+x_3}{3} \quad \frac{x_1+x_2+x_3}{3}\right] + P_S(x)$

$\Rightarrow (x_1\ x_2\ x_3) - \frac{x_1+x_2+x_3}{3}(1\,1\,1) = P_S(x).$

b. Hallar la matriz con resp. a la base canónica de la proy. ortogonal de $\mathbb{R}^3$ sobre S.

$E = \{[1\,0\,0][0\,1\,0][0\,0\,1]\}$

$P_S : \mathbb{R}^3 \to \mathbb{R}^3$

dim = 3

$$|P_S|_E^E = \left[\, [P_S(1\,0\,0)]^E \ [P_S(0\,1\,0)] \ [P_S(0\,0\,1)] \,\right]$$

- $P_s(1\,0\,0) = [1\,0\,0] - P_{S^\perp}[1\,0\,0] = [1\,0\,0] - \frac{(1+0+0)}{3}\cdot[1\,1\,1] = [1\,0\,0] - [\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}] = [\frac{2}{3}\,-\frac{1}{3}\,-\frac{1}{3}]$.

  $\cancel{P_S(1\,0\,0) = [1\,0\,0] - [\frac{2}{3}\,-\frac{1}{3}\,-\frac{1}{3}] = [\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}]}$

- $P_s(0\,1\,0) = [0\,1\,0] - P_{S^\perp}[0\,1\,0] = [0\,1\,0] - \frac{(0+1+0)}{3}\cdot[1\,1\,1] = [0\,1\,0] - \frac{1}{3}[1\,1\,1] = [-\frac{1}{3}\,\frac{2}{3}\,-\frac{1}{3}]$.

  $\cancel{P_S(0\,1\,0) = [0\,1\,0] - [-\frac{1}{3}\,\frac{2}{3}\,-\frac{1}{3}] = [\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}]}$

- $P_s(0\,0\,1) = [0\,0\,1] - P_{S^\perp}[0\,0\,1] = [0\,0\,1] - \frac{1}{3}[1\,1\,1] = [-\frac{1}{3}\,-\frac{1}{3}\,\frac{2}{3}]$

  $\cancel{P_S(0\,0\,1) = [0\,0\,1] - [-\frac{1}{3}\,-\frac{1}{3}\,\frac{2}{3}] = [\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}]}$

$$\Rightarrow |P_s|_E^F = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

c- Calcular la dist a S de c/ uno de los elementos base canónica $\mathbb{R}^3 \Rightarrow E = \{e_1, e_2, e_3\}$

$$\text{dist}\{(1\,0\,0), S\} = \|A - P_S(A)\| = \|P_{S^\perp}(A)\|$$

$$= \|P_{S^\perp}(1\,0\,0)\|$$

$$= \left\|\frac{1+0+0}{3}[1\,1\,1]\right\| = \left\|[\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}]\right\| = \frac{1}{3}\|[1\,1\,1]\| =$$

$$= \frac{1}{3}\sqrt{\langle[1\,1\,1],[1\,1\,1]\rangle} = \frac{1}{3}\sqrt{[1\,1\,1]\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}} =$$

$$= \boxed{\frac{1}{3}\sqrt{3}} \Rightarrow \text{la misma para todos.}$$

d- Hallar todos los $x \in \mathbb{R}^3$ talq. $P_S(x) = [1\,1\,-2]^T$ cuya dist a S sea igual a $\sqrt{3}$

- $P_S(x) = [1\,1\,-2]^T \Rightarrow \left[x_1 - \frac{x_1+x_2+x_3}{3},\ x_2 - \frac{x_1+x_2+x_3}{3},\ x_3 - \frac{x_1+x_2+x_3}{3}\right] = [1\,1\,-2]$

$$\begin{cases} \frac{2}{3}x_1 - \frac{1}{3}x_2 - \frac{1}{3}x_3 = 1 \\ -\frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 = 1 \\ -\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{2}{3}x_3 = -2 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2/3 & -1/3 & -1/3 & 1 \\ -1/3 & 2/3 & -1/3 & 1 \\ -1/3 & -1/3 & 2/3 & -2 \end{array}\right] \Rightarrow$$

- $\frac{1}{2}x_2 - \frac{1}{2}x_3 = \frac{3}{2}$

  $x_2 = (\frac{3}{2} + \frac{1}{2}x_3)\cdot 2$

  $x_2 = 3 + x_3$.

- $\frac{2}{3}x_1 - \frac{1}{3}x_2 - \frac{1}{3}x_3 = 1$

  $\frac{2}{3}x_1 - 1 - \frac{x_3}{3} - \frac{1}{3}x_3 = 1$.

  $\frac{2}{3}x_1 - \frac{2}{3}x_3 = \cancel{2}\,1$

  $\frac{1}{3}x_1 = 1 + \frac{x_3}{3}$.

  $x_1 = 3 + x_3$.

$$\Rightarrow (3+x_3,\ 3+x_3,\ x_3) = (3, 3, 0) + x_3(1, 1, 1)$$

Guía

$\bullet\ dist\{(x_1\ x_2\ x_3),\ S\} = \sqrt{3}$.

$dist\{(3+x_3,\ 3+x_3,\ x_3),\ S\} = \sqrt{3}$

$\| P_{S^\perp}(3+x_3,\ 3+x_3,\ x_3)\| = \sqrt{3}$

$\left\| \frac{3+x_3+3+x_3+x_3}{3} [1\ 1\ 1] \right\| = \sqrt{3}$

$\left|\frac{6+3x_3}{3}\right| \cdot \cancel{\sqrt{3}} = \cancel{\sqrt{3}}$

$|2+1x_3| = 1$ → $x_3 = -1$

$2 + x_3 = \pm 1$ → $x_3 = -3$.

$\Rightarrow \boxed{(2\ \ 2\ -1)} \ \vee\ \boxed{(0\ \ 0\ -3)}$

Compruebo

$[2\ 2\ -1]$ $\begin{cases} \cdot\ P_S(2\ 2\ -1) = [2\ 2\ -1] - \frac{2+2-1}{3}.[1\ 1\ 1] = [2\ 2\ -1] - [1\ 1\ 1] = [1\ 1\ -2] \checkmark \\ \cdot\ dist\{(2\ 2\ -1), S\} = \sqrt{3} \checkmark \end{cases}$

$[0\ 0\ -3]$ $\begin{cases} \cdot\ P_S(0\ 0\ -3) = [0\ 0\ -3] - \frac{-3}{3}[1\ 1\ 1] = [0\ 0\ -3] + [1\ 1\ 1] = [1\ 1\ -2] \checkmark \\ \cdot\ dist\{(0\ 0\ -3), S\} = \sqrt{3} \checkmark \end{cases}$

e- Hallar el conj. de los $v \in \mathbb{R}^3$ equidistantes a $[0\ 0\ 0]^T$ y $[6\ 6\ 6]^T$.

$dist\{(x_1\ x_2\ x_3), (0\ 0\ 0)\} = dist\{(x_1\ x_2\ x_3), (6\ 6\ 6)\}$

$\|(x_1\ x_2\ x_3)\| = \|(x_1-6\ \ x_2-6\ \ x_3-6)\|$.

$\sqrt{x_1^2+x_2^2+x_3^2} = \sqrt{(x_1-6)^2+(x_2-6)^2+(x_3-6)^2}$

$\cancel{x_1^2}+\cancel{x_2^2}+\cancel{x_3^2} = \cancel{x_1^2}-12x_1+36+\cancel{x_2^2}-12x_2+36+\cancel{x_3^2}-12x_3+36$

$0 = -12x_1+36-12x_2+36-12x_3+36$.

$x_1 = 3-x_2+3-x_3+3$.

$\to (x_1\ x_2\ x_3) = (3-x_2+3-x_3+3\ \ x_2\ \ x_3) = (9\ 0\ 0) + x_2(-1\ 1\ 0) + x_3(-1\ 0\ 1)$

$= (9\ 0\ 0) + x_2(-1\ 1\ 0) + x_3(-1\ 0\ 1)$

$gen\{(9\ 0\ 0), (-1\ 1\ 0), (-1\ 0\ 1)\}$

3.7. En $\mathbb{R}^3$ con p.i. def por $\langle x, y\rangle = y^T \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} x$ se consideran

$S_1 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$, $S_2 = \{x \in \mathbb{R}^3 : x_1 - x_3 = 0\}$

gen$\{(-1\ 1\ 0), (-1\ 0\ 1)\}$ gen$\{(0\ 1\ 0), (1\ 0\ 1)\}$.

a. Hallar las matrices con resp. a base canónica de las proy. ortogonales de $\mathbb{R}^3$ sobre $S_1^\perp$ y $S_2^\perp$.

$\dim(\mathbb{R}^3) = \dim(S_1) + \dim(S_1^\perp) \Rightarrow 3 = 2 + \dim(S_1^\perp) \Rightarrow 1 = \dim(S_1^\perp)$.

...

$\dim(S_2^\perp) = 1$.

• Hallo generador de $S_1^\perp$:

$* \, 0 = \langle (a\,b\,c)^T, (-1\ 1\ 0)^T \rangle$

$0 = (-1\ 1\ 0) \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = (-1\ 1\ 0) \begin{pmatrix} 2a - 2b \\ -2a + 5b + 4c \\ 4b + 6c \end{pmatrix}$

$0 = -2a + 2b - 2a + 5b + 4c$

$0 = -4a + 7b + 4c$

$* \, 0 = \langle (a\,b\,c)^T, (-1\ 0\ 1)^T \rangle$

$0 = [-1\ 0\ 1] \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = [-1\ 0\ 1] \begin{bmatrix} 2a - 2b \\ -2a + 5b + 4c \\ 4b + 6c \end{bmatrix}$

$0 = -2a + 2b + 4b + 6c$.

$0 = -2a + 6b + 6c$.

$\begin{cases} -4a + 7b + 4c = 0 \\ -2a + 6b + 6c = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} -4 & 7 & 4 & 0 \\ -2 & 6 & 6 & 0 \end{array}\right] \Rightarrow$ • $\frac{5}{2}b = -4c$.

$b = -4c \cdot \frac{2}{5}$

$b = -\frac{8}{5}c$.

• $-4a + 7b + 4c = 0$

$-4a + 7\left(-\frac{8}{5}c\right) + 4c = 0$

$-4a - \frac{56}{5}c + 4c = 0$.

$-4a - \frac{36}{5}c = 0$.

$a = -\frac{36}{5}c \cdot \frac{1}{4}$

$a = -\frac{36}{20}c$.

$\Rightarrow c\left(-\frac{\cancel{36}^{\,9}}{\frac{\cancel{20}}{5}}, -\frac{8}{5}, 1\right) = c\left(-\frac{9}{5}, -\frac{8}{5}, 1\right)$.

$\boxed{\text{gen}(S_1^\perp) = \left\{\left(-\frac{9}{5}\ -\frac{8}{5}\ 1\right)\right\}}$

Luz

- Hallo gen. de $S_2^{\perp}$

$* \; 0 = \langle (a\,b\,c), (0\,1\,0) \rangle$

$0 = [0\,1\,0] \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$0 = [0\,1\,0] \begin{bmatrix} 2a-2b \\ -2a+5b+4c \\ 4b+6c \end{bmatrix}$

$0 = -2a+5b+4c$

$* \; 0 = \langle (a\,b\,c), (1\,0\,1) \rangle$

$0 = [1\,0\,1] \begin{bmatrix} 2a-2b \\ -2a+5b+4c \\ 4b+6c \end{bmatrix}$.

$0 = 2a-2b+4b+6c$.

$0 = 2a+2b+6c$.

$0 = a+b+3c$.

$\begin{cases} -2a+5b+4c=0 \\ a+b+3c=0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} -2 & 5 & 4 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right]$

- $\frac{7}{2}b = -5c$

$b = -\frac{10}{7}c$

- $-2a + 5.\left(-\frac{10}{7}c\right) + 4c = 0$.

$-2a - \frac{50}{7}c + 4c = 0$.

$-2a = \frac{22}{7}c$.

$a = -\frac{22}{14}c$.

$(a\,b\,c) = \left(-\frac{11}{7}c \;\; -\frac{10}{7}c \;\; c\right) = c\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right)$

$\boxed{gen(S_2^{\perp}) = \left\{\left[-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right]\right\}}$

- $|P_{S_1^{\perp}}|_E^E = \left[[P_{S_1^{\perp}}(1\,0\,0)]^E \;\; [P_{S_1^{\perp}}(0\,1\,0)]^E \;\; P_{S_1^{\perp}}(0\,0\,1) \right]$

¿Y cómo es $P_{S_1^{\perp}}$? $\quad P_{S_1^{\perp}}(x) = \dfrac{\langle x, gen(S_1^{\perp}) \rangle}{\|gen(S_1^{\perp})\|^2} . gen(S_1^{\perp})$

$P_{S_1^{\perp}}(x) = \dfrac{\langle (x_1\,x_2\,x_3), \left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right) \rangle}{\left\|\left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right)\right\|^2} . \left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right)$

$= \left(-\frac{9}{5} \;\; -\frac{8}{5} \;\; 1\right) \begin{bmatrix} 2x_1-2x_2 \\ -2x_1+5x_2+4x_3 \\ 4x_2+6x_3 \end{bmatrix} . \dfrac{1}{\left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right) . \begin{bmatrix} 2\left(-\frac{9}{5}\right)+\frac{16}{5} \\ 2.\left(\frac{9}{5}\right)+5\left(-\frac{8}{5}\right)+4.1 \\ 4\left(-\frac{8}{5}\right)+6.1 \end{bmatrix}} . \left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right) =$

$= \left(-\frac{18}{5}x_1 + \frac{18}{5}x_2 + \frac{16}{5}x_1 - 8x_2 + \left(-\frac{8}{5}\right).4x_3 + 4x_2 + 6x_3\right) . \dfrac{1}{\left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right)\begin{pmatrix} -2/5 \\ -2/5 \\ -2/5 \end{pmatrix}} . \left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right) =$

$= \left(-\frac{2}{5}x_1 - \frac{2}{5}x_2 - \frac{2}{5}x_3\right) \dfrac{1}{\left(\frac{9}{5}.\frac{2}{5} + \frac{8}{5}.\frac{2}{5} - \frac{2}{5}\right)} . \left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right) = \dfrac{\left(-\frac{2}{5}x_1 - \frac{2}{5}x_2 - \frac{2}{5}x_3\right)}{24/25} . \left(-\frac{9}{5}\; -\frac{8}{5}\; 1\right)$

$$P_{S_1^\perp} = \frac{25}{24}\cdot\left(-\frac{2}{5}x_1 \;\; -\frac{2}{5}x_2 \;\; -\frac{2}{5}x_3\right)\cdot\left(-\frac{9}{5} \;\; -\frac{8}{5} \;\; 1\right) = \left(x_1 + x_2 + x_3\right)\cdot\left(-\frac{2}{5}\right)\left(\frac{25}{24}\right)\cdot\left(-\frac{9}{5} \;\; -\frac{8}{5} \;\; 1\right) =$$

$$= \left(x_1 + x_2 + x_3\right)\left(\frac{-5}{12}\right)\cdot\left(-\frac{9}{5} \;\; -\frac{8}{5} \;\; 1\right) = \boxed{\left(x_1 + x_2 + x_3\right)\cdot\left(\frac{9}{12} \;\; \frac{8}{12} \;\; \frac{-5}{12}\right)}$$

- $P_{S_1^\perp}(1\,0\,0) = 1\cdot\left(\frac{9}{12} \;\; \frac{8}{12} \;\; \frac{-5}{12}\right) = \left(\frac{9}{12} \;\; \frac{8}{12} \;\; -\frac{5}{12}\right)$
- $P_{S_1^\perp}(0\,1\,0) = \left(\frac{9}{12} \;\; \frac{8}{12} \;\; \frac{-5}{12}\right)$
- $P_{S_1^\perp}(0\,0\,1) = \left(\frac{9}{12} \;\; \frac{8}{12} \;\; \frac{-5}{12}\right)$

$$\boxed{\left|P_{S_1^\perp}\right|_E^E = \begin{bmatrix} 9/12 & 9/12 & 9/12 \\ 8/12 & 8/12 & 8/12 \\ -5/12 & -5/12 & -5/12 \end{bmatrix}}$$

- Hallo $\left|P_{S_2^\perp}\right|_E^E$

$$P_{S_2^\perp}(x) = \frac{\langle x, gen(S_2^\perp)\rangle}{\|gen(S_2^\perp)\|^2}\cdot gen(S_2^\perp) = \frac{\langle (x_1\, x_2\, x_3), \left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right)\rangle}{\langle \left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right), \left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right)\rangle}\cdot\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right) =$$

$$= \left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right)\begin{pmatrix} 2a-2b \\ -2a+5b+4c \\ 4b+6c \end{pmatrix}\cdot\frac{1}{\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right)\begin{pmatrix} 2\cdot\left(-\frac{11}{7}\right) - 2\cdot\left(-\frac{10}{7}\right) \\ -2\cdot\left(-\frac{11}{7}\right) + 5\cdot\left(-\frac{10}{7}\right) + 4\cdot 1 \\ 4\cdot\left(-\frac{10}{7}\right) + 6\cdot 1 \end{pmatrix}}\cdot\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right) =$$

$$= \left(-\frac{22}{7}a + \frac{22}{7}b + \frac{20}{7}a - \frac{50}{7}b - \frac{40}{7}c + 4b + 6c\right)\cdot\frac{1}{\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right)\begin{pmatrix} -2/7 \\ 0 \\ 2/7 \end{pmatrix}}\cdot\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right) =$$

$$= \left(-\frac{2}{7}a + \frac{2}{7}c\right)\frac{1}{\left(\frac{22}{49} + \frac{2}{7}\right)}\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right) = \frac{\cancel{2}}{\cancel{7}}(-a+c)\,\frac{\cancel{49}\,7}{\cancel{36}\,18}\cdot\left(-\frac{11}{7} \;\; -\frac{10}{7} \;\; 1\right) =$$

$$= \boxed{(-a+c)\left(\frac{-11}{18} \;\; \frac{-10}{18} \;\; \frac{7}{18}\right)}$$

Guía

- $P_{S_2^\perp}(1\,0\,0) = -1 \cdot \left(\frac{-11}{18}\ \frac{-10}{18}\ \frac{7}{18}\right) = \left(\frac{11}{18}\ \frac{10}{18}\ \frac{-7}{18}\right)$

- $P_{S_2^\perp}(0\,1\,0) = 0_{\mathbb{R}^3}$

- $P_{S_2^\perp}(0\,0\,1) = 1 \cdot \left(\frac{-11}{18}\ \frac{-10}{18}\ \frac{7}{18}\right) = \left(\frac{-11}{18}\ \frac{-10}{18}\ \frac{7}{18}\right)$

$$\boxed{\left|P_{S_2^\perp}\right|_E^E = \begin{bmatrix} 11/18 & 0 & -11/18 \\ 10/18 & 0 & -10/18 \\ -7/18 & 0 & 7/18 \end{bmatrix}}$$

b- Sea $b = [1\,1\,2]^T$. Hallar la dist. de b a $S_1^\perp$ y de b a $S_2^\perp$.

$$\text{dist.}\left\{[1\,1\,2], S_1^\perp\right\} = \left\|P_{S_1^\perp}(1\,1\,2)\right\| = \left\|4 \cdot \left(\frac{9}{12}\ \frac{8}{12}\ \frac{-5}{12}\right)\right\| = \left\|\left(\frac{9}{3}\ \frac{8}{3}\ \frac{-5}{3}\right)\right\| = \frac{1}{3}\sqrt{\langle(9\,8\,-5),(9\,8\,-5)\rangle}$$

$$= \frac{1}{3}\sqrt{[9\,8\,-5]\begin{bmatrix} 2.9-2.8 \\ -2.9+5.8+4(-5) \\ 4.8+6(-5) \end{bmatrix}} = \frac{1}{3}\sqrt{[9\,8\,-5]\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}} = \frac{1}{3}\sqrt{24} = \boxed{\frac{2}{3}\sqrt{6}}$$

$$\text{dist.}\left\{[1\,1\,2], S_2^\perp\right\} = \left\|P_{S_2^\perp}(1\,1\,2)\right\| = \left\|1 \cdot \left(\frac{-11}{18}\ \frac{-10}{18}\ \frac{7}{18}\right)\right\| = \frac{1}{18}\left\|(-11\,-10\,7)\right\| =$$

$$= \frac{1}{18}\sqrt{\langle(-11\,-10\,7),(-11\,-10\,7)\rangle} = \frac{1}{18}\sqrt{(-11\,-10\,7)\begin{pmatrix} 2(-11)-2(-10) \\ -2(-11)+5(-10)+4.7 \\ 4(-10)+6.7 \end{pmatrix}} =$$

$$= \frac{1}{18}\sqrt{(-11\,-10\,7)\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}} = \frac{1}{18}\sqrt{22+14} = \frac{6}{18} = \boxed{\frac{1}{3}}$$

e- Hallar todos los x a la misma dist. de $S_1^\perp$ que de $S_2^\perp$.

- $\text{dist}\left\{(x_1\,x_2\,x_3), S_1^\perp\right\} = \left\|P_{S_1^\perp}(x_1\,x_2\,x_3)\right\| = \left\|(x_1+x_2+x_3) \cdot \left(\frac{9}{12}\ \frac{8}{12}\ \frac{-5}{12}\right)\right\| = \left(\frac{x_1+x_2+x_3}{12}\right) \cdot \left\|(9\,8\,-5)\right\| =$

$$= \left(\frac{x_1+x_2+x_3}{12}\right) \cdot \sqrt{\langle(9\,8\,-5),(9\,8\,-5)\rangle} = \left(\frac{x_1+x_2+x_3}{12}\right)\sqrt{(9\,8\,-5)\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}} = \left(\frac{x_1+x_2+x_3}{12}\right)\sqrt{24}$$

- $\text{dist}\left\{(x_1\,x_2\,x_3), S_2^\perp\right\} = \left\|P_{S_2^\perp}(x_1\,x_2\,x_3)\right\| = \left\|(-x_1+x_3)\left(\frac{-11}{18}\ \frac{-10}{18}\ \frac{7}{18}\right)\right\| = (-x_1+x_3) \cdot \frac{1}{18}\left\|(-11\,-10\,7)\right\| =$

$$= \left(\frac{-x_1+x_3}{18}\right)\sqrt{(-11\,10\,7)\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}} = \frac{1}{3}(-x_1+x_3)$$

$$\Rightarrow \left(\frac{x_1+x_2+x_3}{12}\right)\sqrt{24} = \frac{1}{3}(-x_1+x_3) \Rightarrow \frac{3}{12}\sqrt{24}(x_1+x_2+x_3) = -x_1+x_3 \Rightarrow \frac{\sqrt{6}}{2}(x_1+x_2+x_3) = -x_1+x_3$$

$$\Rightarrow \frac{\sqrt{6}}{2}x_1 + x_1 + \frac{\sqrt{6}}{2}x_2 + \frac{\sqrt{6}}{2}x_3 - x_3 = 0 \Rightarrow \left(\frac{2+\sqrt{6}}{2}\right)x_1 + \frac{\sqrt{6}}{2}x_2 + \left(\frac{-2+\sqrt{6}}{2}\right)x_3 = 0$$

$$\frac{\sqrt{6}}{2}x_2 = -\left(\frac{2+\sqrt{6}}{2}\right)x_1 - \left(\frac{-2+\sqrt{6}}{2}\right)x_3$$

$$\frac{\sqrt{6}}{2}x_2 = -\left(\frac{2+\sqrt{6}}{2}\right)x_1 + \left(\frac{2-\sqrt{6}}{2}\right)x_3$$

$$x_2 = -\left(\frac{2+\sqrt{6}}{2}\cdot\frac{2}{\sqrt{6}}\right)x_1 + \left(\frac{2-\sqrt{6}}{2}\right)\left(\frac{2}{\sqrt{6}}\right)x_3.$$

$$x_2 = -\left(\frac{2+\sqrt{6}}{\sqrt{6}}\right)x_1 + \left(\frac{2-\sqrt{6}}{\sqrt{6}}\right)x_3.$$

$$x_2 = -\left(\frac{3+\sqrt{6}}{3}\right)x_1 + \left(\frac{-3+\sqrt{6}}{3}\right)x_3$$

$$x_2 = \left(\frac{-3-\sqrt{6}}{3}\right)x_1 + \left(\frac{-3+\sqrt{6}}{3}\right)x_3$$

$$(x_1 \; x_2 \; x_3) = \boxed{\left(x_1 \;\; \left(\frac{-3-\sqrt{6}}{3}\right)x_1 + \left(\frac{-3+\sqrt{6}}{3}\right)x_3 \;\; x_3\right)}$$

3.8. Sea $(\mathbb{V}, \langle \cdot,\cdot \rangle)$ un $\mathbb{R}$-espacio euclídeo de dim 3 y sea $B = \{v_1, v_2, v_3\}$ una base de $\mathbb{V}$ cuya matriz de Gram es $G_B = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$.

a. Hallar la matriz con resp. B de proy. ortogonal sobre $S = gen\{v_1, v_2\}$

$$|P_S|_B^E = \left[[P_S(v_1)]^E \; [P_S(v_2)]^E \; [P_S(v_3)]^E \right]$$

Hallo $P_S(x)$. Primero Obtengo $P_{S^\perp}(x)$. (ya que tiene dim = 1)

$$P_{S^\perp}(x) = \frac{\langle x, gen(S^\perp)\rangle}{\|gen\, S^\perp\|^2} \cdot gen(S^\perp) \to \, ?$$

G

- $0 = \langle (a\ b\ c)^T, v_1 \rangle$

$$0 = v_1 \cdot \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$0 = v_1 \cdot \begin{bmatrix} a + b/2 + c/3 \\ a/2 + b/3 + c/4 \\ a/3 + b/4 + c/5 \end{bmatrix}$$

$$0 = v_{1x} \cdot a + \frac{v_{1x}}{2} b + \frac{v_{1x}}{3} \cdot c + \frac{v_{1y}}{2} a + \frac{v_{1y}}{3} b + \frac{v_{1y}}{4} c + \frac{v_{1z}}{3} a + \frac{v_{1z}}{4} b + \frac{v_{1z}}{5} c$$

$$0 = \frac{11}{6} v_{1x} \cdot a + \frac{13}{12} v_{1y} b + \frac{47}{60} v_{1z} c$$

- $0 = \langle (a\ b\ c)^T, v_2 \rangle$

$$0 = v_2 \cdot \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$0 = \frac{11}{6} v_{2x} a + \frac{13}{12} v_{2y} b + \frac{47}{60} v_{2z} c.$$

Guía

$$\begin{cases} \frac{11}{6} v_{1x} \cdot a + \frac{13}{12} v_{1y} \cdot b + \frac{47}{60} v_{1z} \cdot c = 0 \\ \frac{11}{6} v_{2x} \cdot a + \frac{13}{12} v_{2y} \cdot b + \frac{47}{60} v_{2z} \cdot c = 0 \end{cases} \to \cdot \frac{11}{6} v_{1x} \cdot a + \frac{13}{12} v_{1y} \cdot b + \frac{47}{60} v_{1z} \cdot c = 0$$

$$\cdot \left(\frac{-13 . v_{1y} . v_{2x} + 13 v_{1x} . v_{2y}}{12 . v_{1x}}\right) b + \left(\frac{47 v_{1z} v_{2x} + 47 v_{1x} v_{2z}}{60 . v_{1x}}\right) \cdot c$$

$$\left(\frac{-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y}}{12 v_{1x}}\right) b = \left(\frac{47 v_{1z} v_{2x} - 47 v_{1x} v_{2z}}{60 v_{1x}}\right) c$$

$$\left(\frac{-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y}}{\cancel{12 v_{1x}}}\right) \frac{\overset{5}{\cancel{60 v_{1x}}}}{(47 v_{1z} v_{2x} - 47 v_{1x} v_{2z})} b = c$$

$$5 . \frac{(-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y})}{(47 v_{1z} v_{2x} - 47 v_{1x} v_{2z})} b = c .$$

$$\to \frac{11}{6} v_{1x} \cdot a + \frac{13}{12} v_{1y} b + \frac{47}{\underset{12}{\cancel{60}}} v_{1z} . \cancel{5} . \frac{(-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y})}{(47 v_{1z} v_{2x} - 47 v_{1x} v_{2z})} b = 0 .$$

$$\left[\frac{13}{12} v_{1y} + \frac{47}{12} v_{1z} . \left(\frac{-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y}}{47 v_{1z} v_{2x} - 47 v_{1x} v_{2z}}\right)\right] b = -\frac{11}{6} v_{1x} . a$$

$$\frac{-\cancel{6}}{11 v_{1x}} . \frac{1}{\underset{2}{\cancel{12}}} \left[13 v_{1y} + 47 v_{1z} \left(\frac{-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y}}{47 v_{1z} v_{2x} - 47 v_{1x} v_{2z}}\right)\right] b = a .$$

$$gen\left\{(a \ b \ c)\right\} = gen\left\{\left(\frac{-1}{22 v_{1x}} . \left[13 v_{1y} + 47 v_{1z} \left(\frac{-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y}}{47 v_{1z} v_{2x} - 47 v_{1x} v_{2z}}\right)\right] \quad 1 \quad 5 \left(\frac{-13 v_{1y} v_{2x} + 13 v_{1x} v_{2y}}{47 v_{1z} v_{2x} - 47 v_{1x} v_{2z}}\right)\right)\right\}$$

¿Qué haría?

- Hallo $P_{S^\perp}(x)$.
- Hallo $P_S(x)$ tq $x = P_S(x) + P_{S^\perp}(x)$.
- Armo $|P_S|_B^E$.

b- Proy. ortogonal de v_3 sobre S.

- Con $P_S(x)$ hago $P_S(v_3)$.

c- Calcular la dist. de v_3 a S

- Hago $dist(v_3, S) = \|P_S(v_3)\|$.

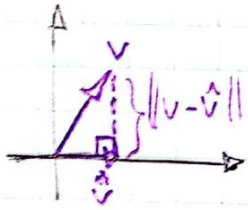
d- Conj. todos los $v \in V$ cuya dist. a v_1 = dist a v_2.

$$\|P_S(v_1)\| = \|P_S(v_2)\|$$

Ortogonalidad

- $(w)^{\perp} = \{v \in \mathbb{V} : \langle v, w \rangle = 0\}$
- $(w)^{\perp} = Nu(\phi_w)$. con $\phi_w(v) = \langle v, w \rangle$.
- $\{0_V\}^{\perp} = \mathbb{V}$
- $S^{\perp} = \{v \in \mathbb{V} : \langle v, w \rangle = 0 \quad \forall w \in S\} = \bigcap_{w \in S} \{w\}^{\perp}$
- $S \cap S^{\perp} = \{0\}$

$d(v, S) = \|v - \hat{v}\| = \min d(v, w)$

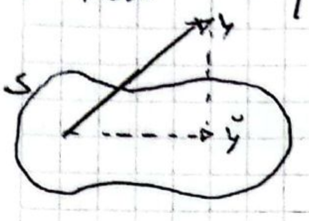


Ejemplo 2.

Se considera el espacio euclídeo $\left(\mathcal{C}([-1,1]), \langle \cdot,\cdot \rangle\right)$ con p.i: $\langle f,g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

a. Hallar proy ortogonal de $y = \operatorname{sen}(\pi x)$ sobre $\mathbb{R}_3[x]$.

$S = \mathbb{R}_3[x] \Rightarrow B = \{1, x, x^2, x^3\}$.



$$\text{Pido:} \begin{cases} \check{y} \in S \Rightarrow \check{y} = \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 x^2 + \alpha_4 x^3 \\ (y - \check{y}) \perp \forall \in B \Rightarrow y - \check{y} \perp \begin{cases} 1 \\ x \\ x^2 \\ x^3 \end{cases} \end{cases} \Rightarrow G_{rB} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \check{y}.$$

$$G_{rB} = \begin{bmatrix} \langle 1,1 \rangle & \dots & \langle 1,x^3 \rangle \\ \vdots & \vdots & \vdots \\ \langle x^3,1 \rangle & \dots & \langle x^3,x^3 \rangle \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{bmatrix}$$

$$\check{y} = \begin{bmatrix} \langle y,1 \rangle \\ \langle y,x \rangle \\ \langle y,x^2 \rangle \\ \langle y,x^3 \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\pi \\ 0 \\ \frac{2(\pi^2-6)}{\pi^3} \end{bmatrix}$$

$$G_{rB} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \check{y} \Rightarrow \cancel{G_{rB}} \cdot \frac{1}{\cancel{G_{rB}}} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \frac{1}{G_{rB}} \cdot \check{y} \Rightarrow I \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = (G_{rB})^{-1} \cdot \check{y}$$

$$\Rightarrow I \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-15\pi^2+315}{2\pi^3} \\ 0 \\ \frac{35\pi^2-525}{2\pi^3} \end{bmatrix} \Rightarrow \boxed{\check{y} = \frac{-15\pi^2+315}{2\pi^3} \cdot x + \frac{35\pi^2-525}{2\pi^3} x^3.}$$

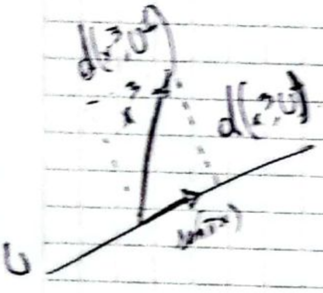
$$d(y, \mathbb{R}_3[x])^2 = \|y - \check{y}\|^2 = \|\operatorname{sen}(\pi x) - \check{y}\|^2 = \langle \operatorname{sen}(\pi x) - \check{y}, \operatorname{sen}(\pi x) - \check{y} \rangle = (0{,}094)^2$$

b. Calcular $\min\limits_{p\in\mathbb{R}_3[x]} \underbrace{\int_{-1}^{1} [\operatorname{sen}(\pi x) - p(x)]^2 \, dx}_{d(\operatorname{sen}(\pi x),\, p)^2}$

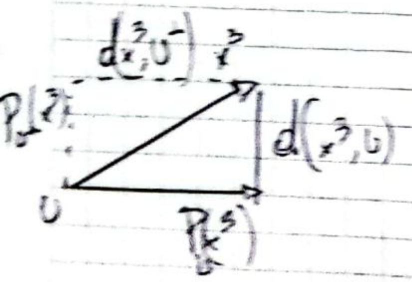
Tengo q: $d(y, \mathbb{R}_3[x])^2 = d(y, \hat{y})^2 = \min\limits_{p\in\mathbb{R}_3[x]} d(y, p)^2 = 0.094$

c. Calcular la distancia de $y = x^3$ al complemento ortogonal del subespacio gen(sen(πx))

$d(x^3, U^\perp)$ con $U = \operatorname{gen}\{\operatorname{sen}(\pi x)\}$.



Como $\dim(U) = 1 \Rightarrow P_U(x^3) = \frac{\langle x^3, \operatorname{gen}(U)\rangle}{\|\operatorname{gen}(U)\|^2} \cdot \operatorname{gen}(U) = \frac{\langle x^3, \operatorname{sen}(\pi x)\rangle}{\langle \operatorname{sen}(\pi x), \operatorname{sen}(\pi x)\rangle} \operatorname{sen}(\pi x) =$



$= \frac{2(\pi^2 - 6)}{\pi^3} \cdot \frac{1}{1} \cdot \operatorname{sen}(\pi x) = \frac{2(\pi^2-6)\operatorname{sen}(\pi x)}{\pi^3}$

$\Rightarrow x^\perp$ :: de geometría (pitágoras) : $d(x^3, U^\perp)^2 + \Big[\underbrace{P_{U^\perp}(x^3)}_{d(x^3, U)}\Big]^2 = \|x^3\|^2$

$d(x^3, U^\perp)^2 = \|P_U(x^3)\|^2$ $\qquad$ $d(x^3, U^\perp)^2 = \|x^3\|^2 - d(x^3, U)^2$

$= \langle P_U(x^3), P_U(x^3)\rangle$

$= \frac{4(\pi^2-6)^2}{\pi^6} \simeq 0.062$

- espacio nulo de $A \perp$ (espacio fila A = espacio col A^T).
- espacio nulo de $A^T \perp$ (espacio col A = espacio fil A^T).

<u>Resolver ec. por mín. cuadrados</u> significa def.: $\underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \|b - Ax\| = \{x \in \mathbb{R}^n : \|b - Ax\| \leq \|b - A\xi\| \; \forall \xi \in \mathbb{R}^n\}$

↳ el conj. de todos los $x \in \mathbb{R}^n$ cuyas imágenes por A minimizan la dist. al vector B.

↗ p.i. canónico siempre

Ej. Hallar sol por mín. cuadrados de: $Ax = b \rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \\ 2 \end{bmatrix}}_{b}$ y calcular el error cometido.

ya q. si lo calculo normal me da sist. incompatible.

Planteo $A^T \cdot (Ax - b) = 0 \rightarrow A^T A x - A^T b = 0 \rightarrow A^T A x = A^T b$

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 3 & 1 \\ 2 & 2 & 4 & 2 & 0 \\ 2 & 3 & 5 & 4 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 12 & 14 & 19 \\ 12 & 24 & 24 & 36 \\ 14 & 24 & 28 & 38 \\ 19 & 36 & 38 & 55 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 18 \\ 34 \\ 36 \\ 52 \end{bmatrix} \xrightarrow{\times b_1} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 11/12 \\ 0 \\ 0 \end{bmatrix}$$

- $x_1 + 2x_3 + x_4 = 1$
  $x_1 = 1 - 2x_3 - x_4$
- $x_2 + x_4 = 11/12$
  $x_2 = 11/12 - x_4$

$\rightarrow [x_1 \; x_2 \; x_3 \; x_4] = [1 - 2x_3 - x_4 \quad \frac{11}{12} - x_4 \quad x_3 \quad x_4]$.
$= [1 \; 11/12 \; 0 \; 0] + x_3 [-2 \; 0 \; 1 \; 0] + x_4 [-1 \; -1 \; 0 \; 1]$

$$\underset{x \in \mathbb{R}^4}{\operatorname{argmin}} \|b - Ax\| = \underbrace{[1 \; \tfrac{11}{12} \; 0 \; 0]^T}_{x_p} + \underbrace{\operatorname{gen}\{[-2 \; 0 \; 1 \; 0]^T, [-1 \; -1 \; 0 \; 1]^T\}}_{\operatorname{nul}(A)}$$

El error cuadrático que se comete es $\min_{x \in \mathbb{R}^4} \|b - Ax\|^2 = \|b - Ax_p\|^2$

$$b - A \cdot x_p = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 11/12 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 23/12 \\ 17/6 \\ 19/4 \\ 15/4 \\ 11/12 \end{bmatrix} = \begin{bmatrix} 1/12 \\ 1/6 \\ 1/4 \\ -3/4 \\ 13/12 \end{bmatrix} = \frac{1}{12} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ -9 \\ 13 \end{bmatrix}$$

$\|b - Ax_p\|^2 = 11/6$

Ej Hallar la sol. por mín. cuadrados de norma mín. de $Ax=b$.

• Teníamos: $\begin{bmatrix} 1&0&2&1\\0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$ Con algoritmo espacio-filas: $B_1=\{x_1, x_2\}$ con $x_1=[1\,0\,2\,1]^T$, $x_2=[0\,1\,0\,1]^T$

→ base Col (A^T)

• Planteamos $[A x_1 \ A x_2][\check{x}]^{B_1} = P_{col(A)}(b)$. → $A.[x_1\ x_2][\check{x}]^{B_1} = \underbrace{P_{col(A)}(b)}_{A.x_p}$

$$\rightarrow \begin{bmatrix} 1&1&2&2\\1&2&2&3\\2&3&4&5\\1&3&2&4\\0&1&0&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\2&0\\1&1\end{bmatrix}.\begin{bmatrix}\varphi_1\\\varphi_2\end{bmatrix} = \frac{1}{12}.\begin{bmatrix}23\\34\\57\\45\\11\end{bmatrix}$$

$$\begin{bmatrix}7&3\\8&5\\15&8\\9&7\\1&2\end{bmatrix}.\begin{bmatrix}\varphi_1\\\varphi_2\end{bmatrix} = \frac{1}{12}\begin{bmatrix}23\\34\\57\\45\\11\end{bmatrix} \rightarrow \left[\begin{array}{cc|c}7&3&23/12\\0&11/7&9/14\end{array}\right]$$

$\cdot\ 7\varphi_1+3\varphi_2=\frac{23}{12}$ $\cdot\ \frac{11}{7}\varphi_2=\frac{9}{14_2}$

$7\varphi_1+\frac{27}{22}=\frac{23}{12}$ $\varphi_2=\frac{9}{22}$

$7\varphi_1=\frac{91}{132}$

$\varphi_1=\frac{13}{132}$

$$\begin{bmatrix}\varphi_1\\\varphi_2\end{bmatrix} = \frac{1}{132}\begin{bmatrix}13\\54\end{bmatrix}$$

• Construimos $\check{x}$

$$[\check{x}]^{B_1} = \frac{1}{132}\begin{bmatrix}13\\54\end{bmatrix} \Leftrightarrow \check{x} = \underbrace{\begin{bmatrix}1&0\\0&1\\2&0\\1&1\end{bmatrix}}_{[x_1\ x_2]}.\frac{1}{132}\begin{bmatrix}13\\54\end{bmatrix} \rightarrow \check{x}=\frac{1}{132}.\begin{bmatrix}13\\54\\26\\67\end{bmatrix}$$

$$\boxed{\check{y}=\left[\frac{13}{132}\ \frac{54}{132}\ \frac{26}{132}\ \frac{67}{132}\right]^T}$$

$$\boxed{\|\check{x}\| = \sqrt{\frac{169+2916+676+4489}{17424}} = \sqrt{\frac{125}{264}} \approx 0{,}69}$$

$$\check{x} = \frac{13}{132}.\begin{bmatrix}1\\0\\2\\1\end{bmatrix}+\frac{54}{132}.\begin{bmatrix}0\\1\\0\\1\end{bmatrix} = \begin{bmatrix}13/132\\54/132\\26/132\\67/132\end{bmatrix} ✓$$

Ej. Hallar la sol. por mín. cuadrados de $\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}}_{b}$

Como A es de rango máx (col. son LI.) $\Rightarrow$ sol $= \check{x} = A^+ b$

$$A^+ = (A^T \cdot A)^{-1} \cdot A^T = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 5/6 & 2/6 & -1/6 \\ -1/2 & 0 & 1/2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{bmatrix}$$

$$\check{x} = A^+ \cdot b = \frac{1}{6} \cdot \begin{bmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 30 \\ -18 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \Rightarrow \boxed{\check{x} = \begin{bmatrix} 5 & -3 \end{bmatrix}^T}$$

- $A^+ \cdot A = I$
- $A \cdot A^+ = [P_{col(A)}]$

Ej. Hallar la matriz de la proy. ortogonal de $\mathbb{R}^3$ sobre el subespacio $S = gen\{[1 \; 1 \; 1]^T, [0 \; 1 \; 2]^T\}$

$\dim(S) = 2 \Rightarrow A \in \mathbb{R}^{3\times 2}$ de rng máx $\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow S = col(A) \wedge A A^+ = P_{col(A)}$.

$$[P_S] = A A^+ = A \cdot (A^T \cdot A)^{-1} \cdot A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} =$$

$$= \boxed{\frac{1}{6} \cdot \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}}$$

Ej. Recolecta info:

Temperatura: T	1	1,2	1,4	1,6	1,8
Proporción: p	0,45	0,54	0,62	0,75	0,92

a. Hallar la recta $p = a + bT$ que mejor se ajusta a los datos.

$$a + b\cdot T_i = p_i \Rightarrow 1\cdot a + T_i\cdot b = p_i \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1,2 \\ 1 & 1,4 \\ 1 & 1,6 \\ 1 & 1,8 \end{bmatrix}}_{V_2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0,45 \\ 0,54 \\ 0,62 \\ 0,75 \\ 0,92 \end{bmatrix}$$

rango máx.

$$V_2^+ = (V_2^T V_2)^{-1} \cdot V_2^T = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1,2 & 1,4 & 1,6 & 1,8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1,2 \\ 1 & 1,4 \\ 1 & 1,6 \\ 1 & 1,8 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1,2 & 1,4 & 1,6 & 1,8 \end{bmatrix} = \begin{pmatrix} 5 & 7 \\ 7 & 10,2 \end{pmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1,2 & 1,4 & 1,6 & 1,8 \end{bmatrix}$$

$$= \begin{bmatrix} 51/10 & -7/2 \\ -7/2 & 5/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1,2 & 1,4 & 1,6 & 1,8 \end{bmatrix} = \frac{1}{10} \cdot \begin{bmatrix} 16 & 9 & 2 & -5 & -12 \\ -10 & -5 & 0 & 5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16 & 9 & 2 & -5 & -12 \\ -10 & -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 0,45 \\ 0,54 \\ 0,62 \\ 0,75 \\ 0,92 \end{bmatrix} = \frac{1}{10} \cdot \frac{1}{100} \cdot \begin{bmatrix} -149 \\ 575 \end{bmatrix} = \frac{1}{1000} \cdot \begin{bmatrix} -149 \\ 575 \end{bmatrix}$$

$$\Rightarrow \boxed{p = -\frac{149}{1000} + \frac{575}{1000} \cdot T}$$

b. Hallar la parábola $p = a + bT + cT^2$ q. mejor se ajusta a los datos.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 1,2 & 1,2^2 \\ 1 & 1,4 & 1,4^2 \\ 1 & 1,6 & 1,6^2 \\ 1 & 1,8 & 1,8^2 \end{bmatrix}}_{V_3} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0,45 \\ 0,54 \\ 0,62 \\ 0,75 \\ 0,92 \end{bmatrix}$$

$$V_3^+ = \frac{1}{70} \begin{bmatrix} 582 & -172 & -456 & -270 & 386 \\ -710 & 315 & 700 & 385 & -630 \\ 250 & -125 & -250 & -125 & 250 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = V_3^+ \cdot p = \frac{1}{7000} \begin{bmatrix} 3892 \\ -3325 \\ 2625 \end{bmatrix} \Rightarrow \boxed{p = \frac{3892}{7000} + \frac{(-3325)}{7000} \cdot T + \frac{2625}{1000} \cdot T^2}$$

c- Det. q. modelo se adapta mejor

$$\varepsilon = p - V \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\varepsilon_1 = \begin{bmatrix} 0.45 \\ 0.54 \\ 0.62 \\ 0.75 \\ 0.92 \end{bmatrix} - \begin{bmatrix} 1 & 1.0 \\ 1 & 1.2 \\ 1 & 1.4 \\ 1 & 1.6 \\ 1 & 1.8 \end{bmatrix} \cdot \frac{1}{1000} \cdot \begin{bmatrix} -149 \\ 575 \end{bmatrix} = \frac{1}{1000} \cdot \begin{bmatrix} 24 \\ -1 \\ -36 \\ -21 \\ 34 \end{bmatrix} \rightarrow \|\varepsilon_1\|^2 = \frac{3410}{10^6}.$$

$$\varepsilon_2 = \begin{bmatrix} 0.45 \\ 0.54 \\ 0.62 \\ 0.75 \\ 0.92 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 1.2 & 1.2^2 \\ 1 & 1.4 & 1.4^2 \\ 1 & 1.6 & 1.6^2 \\ 1 & 1.8 & 1.8^2 \end{bmatrix} \cdot \frac{1}{1000} \begin{bmatrix} 3892 \\ -3325 \\ 2625 \end{bmatrix} = \frac{1}{1000} \begin{bmatrix} -6 \\ 14 \\ -6 \\ -6 \\ 4 \end{bmatrix} \rightarrow \|\varepsilon_2\|^2 = \frac{320}{10^6}.$$

$$\|\varepsilon_2\|^2 < \|\varepsilon_1\|^2$$

$\hookrightarrow$ mejor cuadrático.

Ejercicios ♡ - P.I.2

Dada $A \in \mathbb{R}^{4\times 3}$, $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}$, y $b = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}$ hallar la sol. por mín. cuadrados de $Ax=b$, det. la de norma mín y calcular error cuadrático $\min_{x \in \mathbb{R}^3} \|b-Ax\|^2$.

¿$rg(A)$ es máx? No, xq las columnas (A) no son LI (3^{ra} col $= 1^{ra} + 2^{da}$).

⇒ **Sol**: $A^T A x = A^T b$

$$\begin{bmatrix} 1&1&1&1 \\ 1&2&3&4 \\ 2&3&4&5 \end{bmatrix} \cdot \begin{bmatrix} 1&1&2 \\ 1&2&3 \\ 1&3&4 \\ 1&4&5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1&1&1&1 \\ 1&2&3&4 \\ 2&3&4&5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 4&10&14 \\ 10&30&40 \\ 14&40&54 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 46 \\ 62 \end{bmatrix} \to \left[\begin{array}{ccc|c} 4&10&14&16 \\ 0&5&5&6 \\ 0&0&0&0 \end{array}\right] \to$$

- $4x_1 + 10x_2 + 14x_3 = 16$
- $5x_2 + 5x_3 = 6$. $x_2 = \frac{6}{5} - x_3$.

$4x_1 + \frac{60}{5} - 10x_3 + 14x_3 = 16$.

$4x_1 + 4x_3 = 4$.

$x_1 = 1 - x_3$.

$$[x_1 \; x_2 \; x_3]^T = [1-x_3 \;\; \tfrac{6}{5}-x_3 \;\; x_3]^T = [1 \;\; \tfrac{6}{5} \;\; 0]^T + x_3[-1 \; -1 \; 1]^T.$$

$$\underset{x \in \mathbb{R}^3}{\operatorname{argmin}} \|b-Ax\| = \underbrace{[1 \;\; \tfrac{6}{5} \;\; 0]^T}_{x_p} + \underbrace{gen\{[-1 \; -1 \; 1]^T\}}_{nul(A)}$$

Error cuadrático: $\min_{x \in \mathbb{R}^3} \|b-Ax\|^2 = \|b-Ax_p\|^2$.

$$b - Ax_p = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 1&1&2 \\ 1&2&3 \\ 1&3&4 \\ 1&4&5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 11/5 \\ 17/5 \\ 23/5 \\ 29/5 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 3/5 \\ -3/5 \\ 1/5 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$$\|b-Ax_p\|^2 = \sqrt{\frac{1+9+9+1}{25}}^{\,2} = \frac{\cancel{20}^{\,4}}{\cancel{25}_{\,5}} = \boxed{\frac{4}{5}}$$

<u>x norma mín</u>: $\overset{\vee}{x}$.

① Triangulo A: $\begin{bmatrix} 1&1&2 \\ 0&1&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix} \to B_1 = \{v_1, v_2\}$ con $v_1 = [1\;1\;2]^T$, $v_2 = [0\;1\;1]^T$.

↳ $B_{col(A^T)}$

② Planteo: $A.[v_1 \; v_2][\overset{\vee}{x}]^{B_1} = A.x_p$.

$$\begin{bmatrix}1&1&2\\1&2&3\\1&3&4\\1&4&5\end{bmatrix}\cdot\begin{bmatrix}1&0\\1&1\\2&1\end{bmatrix}\cdot\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix}=\begin{bmatrix}1&1&2\\1&2&3\\1&3&4\\1&4&5\end{bmatrix}\begin{bmatrix}1\\6/5\\0\end{bmatrix}$$

$$\begin{bmatrix}6&3\\9&5\\12&7\\15&9\end{bmatrix}\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix}=\begin{bmatrix}11/5\\17/5\\23/5\\29/5\end{bmatrix}\rightarrow\left[\begin{array}{cc|c}6&3&11/5\\0&1/2&1/10\\0&0&0\\0&0&0\end{array}\right]$$

$\cdot\ \frac{1}{2}\delta_2=\frac{1}{10}$ $\quad\boxed{\delta_2=\frac{1}{5}}$

$\cdot\ 6\delta_1+\frac{3}{5}=\frac{11}{5}$

$6\delta_1=\frac{8}{5}$

$\boxed{\delta_1=\frac{\cancel{8}^{\,4}}{\cancel{30}_{15}}}$

$$\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix}=\begin{bmatrix}4/15\\1/5\end{bmatrix}$$

$$\check{x}=\frac{4}{15}\cdot\begin{bmatrix}1\\1\\2\end{bmatrix}+\frac{1}{5}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}4/15\\7/15\\11/15\end{bmatrix}=\boxed{\frac{1}{15}\cdot\begin{bmatrix}4\\7\\11\end{bmatrix}}$$

$$\|\check{x}\|=\sqrt{\frac{16+49+121}{225}}=\sqrt{\frac{186}{225}}=\boxed{\frac{\sqrt{186}}{15}\simeq 0.91}$$

Ejercicio 2: Dada $A\in\mathbb{R}^{3\times 2}$, $A=\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}$, hallar sol. por mín. cuadrados de $Ax=b$ con $b=\begin{bmatrix}2&4&7\end{bmatrix}^T$ y calcular error cuadrático $\|b-A\check{x}\|^2$.

A rng máx $\Rightarrow$ $\check{x}=A^{+}\cdot b=(A^T\cdot A)^{-1}\cdot A^T\cdot b=\left(\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\cdot\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\right)^{-1}\cdot\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\cdot\begin{bmatrix}2\\4\\7\end{bmatrix}=$

$$=\begin{bmatrix}14&6\\6&3\end{bmatrix}^{-1}\cdot\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}2\\4\\7\end{bmatrix}=\frac{1}{6}\begin{bmatrix}3&-6\\-6&14\end{bmatrix}\cdot\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\cdot\begin{bmatrix}2\\4\\7\end{bmatrix}=$$

$$=\frac{1}{6}\begin{bmatrix}-3&0&3\\8&2&-4\end{bmatrix}\begin{bmatrix}2\\4\\7\end{bmatrix}=\frac{1}{6}\begin{bmatrix}15\\-4\end{bmatrix}=\begin{bmatrix}\frac{5}{2}&-\frac{2}{3}\end{bmatrix}^T.$$

$$\|b-A\check{x}\|^2=\left\|\begin{bmatrix}2\\4\\7\end{bmatrix}-\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\cdot\begin{bmatrix}5/2\\-2/3\end{bmatrix}\right\|^2=\left\|\begin{bmatrix}2\\4\\7\end{bmatrix}-\frac{1}{6}\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\cdot\begin{bmatrix}15\\-4\end{bmatrix}\right\|^2=\left\|\begin{bmatrix}2\\4\\7\end{bmatrix}-\frac{1}{6}\cdot\begin{bmatrix}11\\26\\41\end{bmatrix}\right\|^2=$$

$$=\left\|\begin{bmatrix}1/6\\-2/6\\1/6\end{bmatrix}\right\|^2=\frac{1+4+1}{36}=\frac{6}{36}=\boxed{\frac{1}{6}}$$

Ejercicios ♡ - P.S.2.

Ej. 3. Ajustar datos con recta $p_1(x) = a_0 + a_1 . x$, con cuadrática $p_2(x) = a_0 + a_1x + a_2x^2$ y cúbica $p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

x	-4	-2	0	2	4
y	9	12	14	13	8

Lineal

$$1 . a_0 + x . a_1 = p_1 \rightarrow \underbrace{\begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}}_{V_1} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Incompatible

$$V_1^+ = (V_1^T . V_1)^{-1} . V_1^T = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 0 & 2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \right)^{-1} . \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 0 & 2 & 1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathring{x} = V_1^+ . p_1 = \frac{1}{20} \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} . \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix} = \begin{bmatrix} 56/5 \\ -1/20 \end{bmatrix}$$

$$\Rightarrow \boxed{p_1(x) = \frac{56}{3} - \frac{1}{20} x}$$

Cuadrática

$$1 a_0 + x a_1 + x^2 a_2 = p_2(x) \rightarrow \underbrace{\begin{bmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix}}_{V_2} . \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix}$$

$$V_2^+ = (V_2^T . V_2)^{-1} . V_2^T = \begin{bmatrix} -3/35 & 12/35 & 17/35 & 12/35 & -3/35 \\ -1/10 & -1/20 & 0 & 1/20 & 1/10 \\ 1/28 & -1/56 & -1/28 & -1/56 & 1/28 \end{bmatrix}$$

$$\mathring{x} = V_2^+ . p_2 = \begin{bmatrix} 487/35 \\ -1/20 \\ -19/56 \end{bmatrix}$$

$$\Rightarrow \boxed{p_2(x) = \frac{487}{35} - \frac{1}{20} x - \frac{19}{56} x^2}$$

Cúbica:

$$1d_0 + x d_1 + x^2 d_2 + x^3 d_3 = p_3(x) \Rightarrow \underbrace{\begin{bmatrix} 1 & -4 & 16 & -64 \\ 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \end{bmatrix}}_{V_3} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix}$$

$$V_3^+ = (V_3^T \cdot V_3)^{-1} \cdot V_3^T = \begin{bmatrix} -3/35 & 12/35 & 17/35 & 12/35 & -3/35 \\ 1/84 & -1/3 & 0 & 1/3 & -1/84 \\ 1/28 & -1/56 & -1/28 & -1/56 & 1/28 \\ -1/96 & 1/48 & 0 & -1/48 & 1/96 \end{bmatrix}$$

$$\check{y} = V_3^+ \cdot p_3 = \begin{bmatrix} 487/35 \\ 3/8 \\ -19/56 \\ -1/32 \end{bmatrix}$$

$$\Rightarrow \boxed{p_3(x) = \frac{487}{35} + \frac{3}{8}x - \frac{19}{56}x^2 - \frac{1}{32}x^3.}$$

$$\varepsilon_1 = p - V_1 \cdot \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 56/5 \\ -1/20 \end{bmatrix} = \begin{bmatrix} -12/5 \\ -7/10 \\ 14/5 \\ 19/10 \\ -3 \end{bmatrix}$$

$$\hookrightarrow \|\varepsilon_1\|^2 = \frac{144}{25} + \frac{49}{100} + \frac{196}{25} + \frac{361}{100} + 9 = \boxed{\frac{267}{10}} \simeq 26,7$$

$$\varepsilon_2 = p - V_2 \begin{bmatrix} d_0 \\ d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 16 \\ 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} 487/35 \\ -1/20 \\ -19/56 \end{bmatrix} = \begin{bmatrix} 11/35 \\ -23/35 \\ 3/35 \\ 19/35 \\ -10/35 \end{bmatrix}$$

$$\hookrightarrow \|\varepsilon_2\|^2 = \frac{1120}{1225} = \boxed{\frac{32}{35}} \simeq 0,91$$

$$\varepsilon_3 = p - V_3 \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 14 \\ 13 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 16 & -64 \\ 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \end{bmatrix} \cdot \begin{bmatrix} 487/35 \\ 3/8 \\ -19/56 \\ -1/32 \end{bmatrix} = \begin{bmatrix} 1/70 \\ -2/35 \\ 3/35 \\ -2/35 \\ 1/70 \end{bmatrix}$$

$$\hookrightarrow \|\varepsilon_3\|^2 = \frac{1+16+36+16+1}{4900} = \frac{70}{4900} = \boxed{\frac{1}{70}} \simeq 0,01$$

↓ Mejor.

Ejercicios

Ej. 4. Ecuación: $V_R(t) = V_0 \cdot e^{-t/\tau_0}$. Utilizando mín. cuadrados, ¿cuáles son los valores de $V_0 \in \mathbb{R}^+$ y $\tau_0 \in \mathbb{R}^+$ ($\tau_0 = R\cdot C$) q mejor se ajustan a los datos?

t	1	2	3	4	5
V_R	0,4	0,25	0,17	0,07	0,05

$$\ln(V_R) = \ln\left(V_0 e^{-t/\tau_0}\right) \to \ln(V_R) = \ln(V_0) + \ln\left(e^{-t/\tau_0}\right) \to \ln(V_R) = \ln(V_0) - \frac{t}{\tau_0}$$

$$\Rightarrow \ln(V_R) = \underbrace{\ln(V_0)}_{a} - \underbrace{\frac{1}{\tau_0}}_{b} \cdot t \to 1 \cdot \ln(V_0) + \left(\frac{-1}{\tau_0}\right) \cdot t = \ln(V_R).$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}}_{A} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \ln 0{,}4 \\ \ln 0{,}25 \\ \ln 0{,}17 \\ \ln 0{,}07 \\ \ln 0{,}05 \end{bmatrix}$$

$$A^+ = (A^T \cdot A)^{-1} \cdot A^T = \begin{bmatrix} 4/5 & 1/2 & 1/5 & -1/10 & -2/5 \\ -1/5 & -1/10 & 0 & 1/10 & 1/5 \end{bmatrix}$$

$$x^* = A^+ \cdot \begin{bmatrix} \ln 0{,}4 \\ \ln 0{,}25 \\ \ln 0{,}17 \\ \ln 0{,}07 \\ \ln 0{,}05 \end{bmatrix} = \begin{bmatrix} -0{,}32 \\ -0{,}54 \end{bmatrix}$$

$$\Rightarrow \ln(V_R) = -0{,}32 - 0{,}54 \cdot t \longrightarrow \begin{cases} \ln(V_0) = -0{,}32 \Rightarrow V_0 = 0{,}73 \\ \frac{1}{\tau_0} = 0{,}54 \Rightarrow \tau_0 = 1{,}84 \end{cases}$$

$$\boxed{V_R(t) = 0{,}73 \cdot e^{(-t/1{,}84)}}$$

Isometría: c/vector conserva su long.

Ejercicio 1: Construir base ortonormal del subespacio $S = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ en el espacio euclídeo $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ con el p.i. def. por:

$$\langle x, y \rangle = y^T \cdot \underbrace{\begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix}}_{G} x.$$

Elijo $B_S = \{v_1, v_2\}$, por ej. $v_1 = [1\ {-1}\ 0]^T$ y $v_2 = [1\ 0\ {-1}]^T$. Def. $w_1 = v_1$ y calculo w_2: $w_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$

- $\|v_1\|^2 = \langle v_1, v_1 \rangle = v_1^T \cdot \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} v_1 = [1\ {-1}\ 0] \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} =$

$= [4\ {-7}\ {-4}] \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 11$

- $\langle v_2, v_1 \rangle = v_1^T \cdot \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} \cdot v_2 = [1\ {-1}\ 0] \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = [4\ {-7}\ {-4}] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 8.$

$\Rightarrow w_2 = [1\ 0\ {-1}]^T - \frac{8}{11} \cdot [1\ {-1}\ 0]^T = [1\ 0\ {-1}]^T + [-\tfrac{8}{11}\ \tfrac{8}{11}\ 0]^T = [\tfrac{3}{11}\ \tfrac{8}{11}\ {-1}]^T =$

$= \frac{1}{11} \cdot [3\ 8\ {-11}]^T.$

$\|w_2\|^2 = w_2^T \cdot G \cdot w_2 = \frac{1}{11} \cdot [3\ 8\ {-11}] \cdot \begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 0 & 6 \end{bmatrix} \cdot \frac{1}{11} \cdot [3\ 8\ {-11}]^T = \frac{264}{121}.$

Quiero base ortonormal: $U = \{u_1, u_2\}$

- $u_1 = \frac{w_1}{\|w_1\|} = \frac{[1\ {-1}\ 0]^T}{\sqrt{11}} = \frac{1}{\sqrt{11}} \cdot [1\ {-1}\ 0]^T$
- $u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{11} [3\ 8\ {-11}]^T \cdot \frac{1}{\sqrt{\frac{264}{121}}} = \frac{1}{\sqrt{264}} \cdot [3\ 8\ {-11}]^T.$

Ejercicios

Ejercicio 1. Se considera el $\mathbb{R}$-espacio euclídeo $(\mathbb{R}_2[x], <\cdot,\cdot>)$ con pi:
(Tipo 3.18) $<p,q> := p(1)q(1) + p(3)\cdot q(3) + p(5)\cdot q(5)$.
Comprobar que $B = \{\underbrace{\frac{1}{8}(x-3)(x-5)}_{p_1}, \underbrace{\frac{1}{4}(x-1)(x-5)}_{p_2}, \underbrace{\frac{1}{8}(x-1)(x-3)}_{p_3}\}$ es una base ortonormal de $(\mathbb{R}_2[x], <\cdot,\cdot>)$ y hallar el vector de coordenadas del polinomio $p = x^2 - x + 3$ en base B.

Compruebo que B es ortonormal: long.=1 y perpendiculares entre sí.

$p_1(1) = 1$
$p_1(3) = 0$
$p_1(5) = 0$

$$\|p_1\|^2 = <p_1,p_1> = p_1(1)^2 + p_1(3)^2 + p_1(5)^2 = \left[\frac{1}{8}(1-3)(1-5)\right]^2 + \left[\frac{1}{8}(3-3)(3-5)\right]^2 + \left[\frac{1}{8}(5-3)(5-5)\right]^2$$
$$= \left[\frac{1}{8}(-2)(-4)\right]^2 + 0 + 0 = \left[\frac{1}{8}\cdot 8\right]^2 = 1. \checkmark$$

$p_2(1) = 0$
$p_2(3) = 1$
$p_2(5) = 0$

$$\|p_2\|^2 = <p_2,p_2> = p_2(1)^2 + p_2(3)^2 + p_2(5)^2 = \left[\frac{1}{4}(1-1)(1-5)\right]^2 + \left[\frac{1}{4}(3-1)(3-5)\right]^2 + \left[\frac{1}{4}(5-1)(5-5)\right]^2$$
$$= 0 + \left[\frac{1}{4}(2)(-2)\right]^2 + 0 = 1. \checkmark$$

$p_3(1) = 0$
$p_3(3) = 0$
$p_3(5) = 1$

$$\|p_3\|^2 = <p_3,p_3> = p_3(1)^2 + p_3(3)^2 + p_3(5)^2 = \left[\frac{1}{8}(5-1)(5-3)\right]^2 = \left(\frac{1}{8}\cdot 4\cdot 2\right)^2 = 1. \checkmark$$

$<p_1,p_2> = p_1(1)\cdot p_2(1) + p_1(3)\cdot p_2(3) + p_1(5)\cdot p_2(5) = 1.0 + 0.1 + 0.0 = 0 \checkmark$
$<p_1,p_3> = p_1(1)\cdot p_3(1) + p_1(3)\cdot p_3(3) + p_1(5)\cdot p_3(5) = 1.0 + 0.0 + 0.1 = 0 \checkmark$
$<p_2,p_1> = 0 \checkmark$
$<p_2,p_3> = p_2(1)\cdot p_3(1) + p_2(3)\cdot p_3(3) + p_2(5)\cdot p_3(5) = 0.0 + 1.0 + 0.1 = 0 \checkmark$
$<p_3,p_1> = 0. \checkmark$
$<p_3,p_2> = 0. \checkmark$

$\Rightarrow$ B es ortonormal $\Rightarrow$ B L.i. $\Rightarrow$ card(B) = dim($\mathbb{R}_2[x]$) = 3 $\Rightarrow$ B es base

Tengo $p = x^2 - x + 3$.

$p = \alpha p_1 + \beta p_2 + \gamma p_3$

Utilizo lo que obtuve recién

- $\langle p, p_1 \rangle = \langle \alpha p_1 + \beta p_2 + \gamma p_3 , p_1 \rangle = \alpha \underbrace{\langle p_1, p_1 \rangle}_{1} + \beta \underbrace{\langle p_2, p_1 \rangle}_{0} + \gamma \underbrace{\langle p_3, p_1 \rangle}_{0} = \alpha.$

$\Rightarrow \langle p, p_1 \rangle = \alpha \Rightarrow \langle x^2 - x + 3, \frac{1}{8}(x-3)(x-5) \rangle = \alpha$

$$\left.\begin{array}{l} p(1) = 3,\ p(3) = 9,\ p(5) = 23 \\ p_1(1) = 1,\ p_1(3) = 0,\ p_1(5) = 0 \end{array}\right\} p(1) \cdot p_1(1) + p(3) \cdot p_1(3) + p(5) \cdot p_1(5) = \alpha.$$

$\boxed{3 + 0 + 0 = \alpha}$

- $\langle p, p_2 \rangle = \langle \alpha p_1 + \beta p_2 + \gamma p_3 , p_2 \rangle = \alpha \underbrace{\langle p_1, p_2 \rangle}_{0} + \beta \underbrace{\langle p_2, p_2 \rangle}_{1} + \gamma \underbrace{\langle p_3, p_2 \rangle}_{0} = \beta.$

$p(1) \underbrace{p_2(1)}_{0} + p(3) \underbrace{p_2(3)}_{1} + p(5) \underbrace{p_2(5)}_{0} = p(3) = 9 \Rightarrow \boxed{\beta = 9}$

- $\langle p, p_3 \rangle = \alpha \underbrace{\langle p_1, p_3 \rangle}_{0} + \beta \underbrace{\langle p_2, p_3 \rangle}_{0} + \gamma \underbrace{\langle p_3, p_3 \rangle}_{1} = \gamma.$

$p(1) \underbrace{p_3(1)}_{0} + p(3) \underbrace{p_3(3)}_{0} + p(5) \underbrace{p_3(5)}_{1} = p(5) = \boxed{23 = \gamma.}$

$\rightarrow$ El vector es $[3 \ 9 \ 23]$

Ejercicios

Ejercicio 2. Se considera $\mathbb{R}^2$ con ρ: $\langle x,y\rangle = y^T \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} x$
(Tipo 3.20)

a. Describir significado geométrico del conj. $\Sigma = \{x\in\mathbb{R}^2 : 3x_1^2 + 4x_1x_2 + 3x_2^2 = 9\}$.

$\hookrightarrow \Sigma = \{x\in\mathbb{R}^2 : \underbrace{\|x\|^2 = 9}_{\text{Circunferencia } r=3}\}$

b. Hallar dos vectores $v_1, v_2 \in \mathbb{R}^2$ tales q. $\Sigma = \{y_1 \cdot v_1 + y_2 \cdot v_2 : y_1, y_2 \in \mathbb{R},\ y_1^2 + y_2^2 = 9\}$

$= \{3\cos(\theta)\, v_1 + 3\,\text{sen}(\theta)\, v_2 : \theta\in[0,2\pi)\}$.

$v_1 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\hookrightarrow 3\cos\theta \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\,\text{sen}\,\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [3\cos\theta,\ 3\,\text{sen}\,\theta]^T \to$ circ. $r=3$.

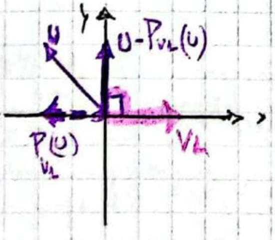
Busco $\{v_1, v_2\}$ ortonormal

$x\in\mathbb{R}^2 \Rightarrow x = y_1v_1 + y_2v_2$

$\|x\|^2 = \langle x,x\rangle = \langle y_1v_1 + y_2v_2,\ y_1v_1 + y_2v_2\rangle = y_1^2 \underbrace{\langle v_1,v_1\rangle}_{1} + y_2^2 \underbrace{\langle v_2,v_2\rangle}_{1} = y_1^2 + y_2^2$.

$v_1 = \dfrac{[1\ 0]^T}{\|[1\ 0]^T\|} = \boxed{\dfrac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{3}}}$

$\hookrightarrow \|[1\ 0]^T\| = \sqrt{\langle [1\ 0]^T, [1\ 0]^T\rangle} = \left([1\ 0]\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^{1/2} = \left([1\ 0]\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)^{1/2} = (3)^{1/2}$.



cualquiera q. no sea $[1\ 0]$.
$u = [0\ 1]$

$P_{v_1}(u) = \dfrac{\langle u, v_1\rangle}{\|v_1\|^2} \cdot v_1 = \left(\dfrac{1}{\sqrt{3}}[1\ 0]\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \cdot \dfrac{1}{\sqrt{3}}[1\ 0] =$

$= \dfrac{1}{\sqrt{3}}[1\ 0]\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \dfrac{1}{\sqrt{3}}[1\ 0] = \dfrac{2}{3}[1\ 0]$

$u - P_{v_1}(u) = [0\ 1] - \dfrac{2}{3}[1\ 0] = \left[-\dfrac{2}{3}\ \ 1\right] \to$ Misma dir: $[-2\ 3]$

$\|[-2\ 3]\| = (\langle[-2\ 3],[-2\ 3]\rangle)^{1/2} = \left([-2\ 3]\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)^{1/2} = \left([-2\ 3]\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right)^{1/2} = \sqrt{15}$.

$$V_2 = \frac{v - P_{v_1}(u)}{\|v_1 - P_{v_1}(v)\|} = \frac{\begin{bmatrix} -2 & 3 \end{bmatrix}^T}{\sqrt{15}} = \boxed{\frac{1}{\sqrt{15}} \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}}$$

Ejercicio 3. (Tipo 3.21.)

Sea $A = \begin{bmatrix} 4 & 6 & 5 & 6 & 1 \\ 3 & 3 & 3 & 4 & 1 \\ 1 & 3 & 2 & 2 & 0 \\ -2 & 0 & -1 & -2 & -1 \end{bmatrix}$ y p.i. $\langle x,y\rangle = y^T \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$

a. Comprobar que $b = [10\ 8\ 2\ -6]^T \in Col(A)$

• Triangulo $Ax=b$

$$\left[\begin{array}{ccccc|c} 4 & 6 & 5 & 6 & 1 & 10 \\ 3 & 3 & 3 & 4 & 1 & 8 \\ 1 & 3 & 2 & 2 & 0 & 2 \\ -2 & 0 & -1 & -2 & -1 & -6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 1/2 & 1 & 1/2 & 3 \\ 0 & 1 & 1/2 & 1/3 & -1/6 & -1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\to$ Quiere decir que.

$3 \cdot Col_1(A) + \left(-\frac{1}{3}\right) Col_2(A) = b$

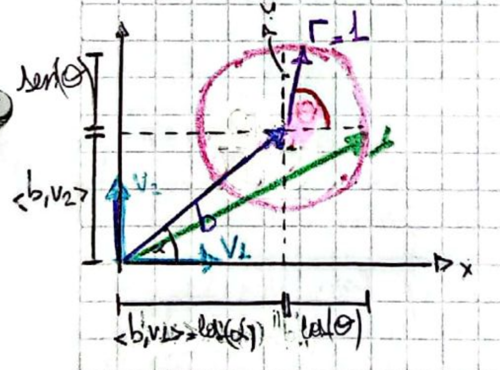
$$3 \begin{bmatrix} 4 \\ 3 \\ 1 \\ -2 \end{bmatrix} + \left(-\frac{1}{3}\right) \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 2 \\ -6 \end{bmatrix}.$$

$$\begin{bmatrix} 12 \\ 9 \\ 3 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 2 \\ -6 \end{bmatrix} ✓$$

$\Rightarrow b \in Col(A)$ ✓

b. Mostrar que existen $\alpha_1, \alpha_2 \in \mathbb{R}$ y $v_1, v_2 \in Col(A)$ tales q.

$$\{y \in Col(A) : d(b,y) = 1\} = \{(\alpha_1 + \cos\theta) v_1 + (\alpha_2 + \operatorname{sen}\theta) v_2 : \theta \in [0, 2\pi]\}$$



$y = (\operatorname{sen}\theta + \langle b, v_2\rangle) v_2 + (\cos\theta + \langle b, v_1\rangle) v_1.$

$d^2(b,y) = 1$

$y = \alpha_1 w_1 + \alpha_2 w_2$. $\quad B = \{w_1, w_2\}$ base $Col(A)$.

$\|b-y\|^2 = \|b\|^2 - 2\langle b,y\rangle + \|y\|^2.$

$= \|b\|^2 - 2\langle b, \alpha_1 w_1 + \alpha_2 w_2\rangle + \langle \alpha_1 w_1 + \alpha_2 w_2, \alpha_1 w_1 + \alpha_2 w_2\rangle$

$= \|b\|^2 - 2(\alpha_1 \langle b, w_1\rangle + \alpha_2 \langle b, w_2\rangle) + \alpha_1^2 \langle w_1, w_1\rangle + \alpha_1\alpha_2 \langle w_1, w_2\rangle + \alpha_2\alpha_1 \langle w_2, w_1\rangle + \alpha_2^2 \langle w_2, w_2\rangle$

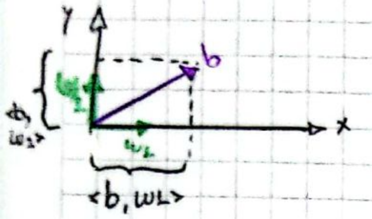
$\to$ S: B ortonormal

$= \|b\|^2 - 2\alpha_1 \langle b, w_1\rangle - 2\alpha_2 \langle b, w_2\rangle + \alpha_1^2 \cdot 1 + \alpha_1\alpha_2 \cdot 0 + \alpha_2 \alpha_1 \cdot 0 + \alpha_2^2 \cdot 1$

$= \|b\|^2 - 2\alpha_1 \langle b, w_1\rangle - 2\alpha_2 \langle b, w_2\rangle + \alpha_1^2 + \alpha_2^2.$

$$\cdot \|b\|^2 = \langle b,w_1\rangle^2 + \underbrace{\langle b,w_1\rangle^2 - 2\alpha_1\langle b,w_1\rangle + \alpha_1^2}_{(\langle b,w_1\rangle - \alpha_1)^2} + \langle b,w_2\rangle^2 + \underbrace{\langle b,w_2\rangle^2 - 2\alpha_2\langle b,w_2\rangle + \alpha_2^2}_{(\langle b,w_2\rangle - \alpha_2)^2}.$$

$$= \|b\|^2 - \langle b,w_1\rangle^2 + (\langle b,w_1\rangle - \alpha_1)^2 - \langle b,w_2\rangle^2 + (\langle b,w_2\rangle - \alpha_2)^2.$$



$\Rightarrow$ x' pitágoras: $\|b\|^2 = \langle b,w_1\rangle^2 + \langle b,w_2\rangle^2 \Rightarrow \|b\|^2 - \langle b,w_1\rangle^2 - \langle b,w_2\rangle^2 = 0.$

$$b = 0 + (\langle b,w_1\rangle - \alpha_1)^2 + (\langle b,w_2\rangle - \alpha_2)^2.$$

$$\Rightarrow \|b-y\|^2 = \underbrace{(\langle b,w_1\rangle - \alpha_1)}_{\cos\theta}^2 + \underbrace{(\langle b,w_2\rangle - \alpha_2)}_{\operatorname{sen}\theta}^2 \stackrel{\circ}{=} 1$$

$$\Rightarrow \begin{cases} \cos\theta = (\langle b,w_1\rangle - \alpha_1) = (\alpha_1 - \langle b,w_1\rangle) \\ \operatorname{sen}\theta = (\langle b,w_2\rangle - \alpha_2) = (\alpha_2 - \langle b,w_2\rangle) \end{cases} \Rightarrow \begin{matrix} \alpha_1 = \cos\theta + \langle b,w_1\rangle. \\ \alpha_2 = \operatorname{sen}\theta + \langle b,w_2\rangle. \end{matrix}$$

$$y = (\langle b,w_1\rangle + \cos\theta)w_1 + (\langle b,w_2\rangle + \operatorname{sen}\theta)w_2$$

Base Col A:

$\cdot\ \operatorname{Col}(A) = \operatorname{gen}\left\{\underbrace{\overset{1^{era}\text{ fila}}{[4\ 3\ 1\ {-2}]^T}}_{v_1}, \underbrace{\overset{2^{da}\text{ fila}}{[6\ 3\ 3\ 0]^T}}_{v_2}\right\}$ $\quad \cdot\ B = \{\check{v}_1, \check{v}_2\}$ con $\check{v}_1 = \dfrac{v_1}{\|v_1\|}$

$$\check{v}_2 = \frac{\left(v_2 - \frac{\langle v_2,v_1\rangle}{\|v_1\|^2} \cdot v_1\right)}{\left\|v_2 - \frac{\langle v_2,v_1\rangle}{\|v_1\|^2} v_1\right\|}$$

Producto Interno 3

Teórica

Ejercicio 1. En $(\mathbb{R}^3, \langle \cdot,\cdot \rangle)$ con p.i.: $\langle x,y\rangle = y^T \cdot \underbrace{\begin{bmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{bmatrix}}_{G} x$, consideramos la funcional

$\phi : \mathbb{R}^3 \to \mathbb{R}$ def. por $\phi(x) = x_1 + x_2 + x_3$. Hallar $v \in \mathbb{R}^3$ t.q. $\phi(x) = \langle x, v\rangle \ \forall x \in \mathbb{R}^3$

$$v = G^{-1} \cdot \begin{bmatrix} \phi(e_1) \\ \phi(e_2) \\ \phi(e_3) \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 7 & 6 & -4 \\ 6 & 6 & -4 \\ -4 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 9 \\ 8 \\ -5 \end{bmatrix}$$

$$\{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}^{\perp} = gen\{[9\ 8\ -5]^T\}.$$

Ejercicio 2. En $(\mathbb{R}_2[x], \langle \cdot,\cdot \rangle)$ con p.i. def por $\langle p, q\rangle := \int_0^1 p(x) q(x)\, dx$, consideramos la función lineal $\phi : \mathbb{R}_2[x] \to \mathbb{R}$ def. por $\phi(p) = p'(0)$ Hallar $q \in \mathbb{R}_2[x]$ t.q. $\phi(p) = \langle p, q\rangle \ \forall p \in \mathbb{R}_2[x]$.

Elijo base canónica: $B = \{1, x, x^2\}$, con $q = a_0 + a_1 x + a_2 x^2$

$$\Rightarrow q(x^i) = \langle x^i, q\rangle = \langle x^i, a_0 + a_1 x + a_2 x^2\rangle = \langle x^i, a_0\rangle + \langle x^i, a_1 x\rangle + \langle x^i, a_2 x^2\rangle =$$

$$= a_0 \langle x^i, 1\rangle + a_1 \langle x^i, x\rangle + a_2 \langle x^i, x^2\rangle.$$

$\hookrightarrow \phi(1) = 1' = 0, \quad \phi(x) = x' = 1, \quad \phi(x^2) = x^{2\prime}(0) = 2x(0) = 2.0 = 0, \quad \int_0^1 x^k dx = \left(\frac{x^{k+1}}{k+1}\right)\Big|_0^1 = \frac{1}{k+1}$

$$G = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \Rightarrow G \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi(1) \\ \phi(x) \\ \phi(x^2) \end{bmatrix} \to \simeq G \cdot a = \phi.$$

$$a = \phi \cdot G^{-1}.$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -36 \\ 192 \\ -180 \end{bmatrix}$$

Por lo tanto: $q = -36 + 192x - 180x^2$.

Ejercicio 3. En $\mathbb{R}^4$, con p.i. canónico, consideramos los vectores $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

Hallar sist. ortonormal.

$j=1$:

$$w_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \|w_1\|^2 = 2, \quad u_1 = \frac{w_1}{\|w_1\|} = \frac{[1\ 0\ 0\ {-1}]^T}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

$j=2$:

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1 = v_2 - \frac{2}{2} \cdot w_1 = v_2 - w_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix},$$

$$\|w_2\|^2 = 4, \quad u_2 = \frac{w_2}{\|w_2\|} = \frac{[0\ 2\ 0\ 0]^T}{2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$j=3$:

$$w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} \cdot w_2 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} \cdot w_1 = v_3 - \frac{2}{4} \cdot [0\,2\,0\,0]^T - \frac{4}{2} \cdot [1\,0\,0\,{-1}]^T$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} - [0\,1\,0\,0]^T - 2[1\,0\,0\,{-1}]^T = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\|w_3\|^2 = 3, \quad u_3 = \frac{w_3}{\|w_3\|} = \frac{[1\,0\,1\,1]^T}{\sqrt{3}}$$

$\Rightarrow$ Sist. ortonormal: $u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $u_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Poniendo $B = \{v_1, v_2, v_3\}$ y $S = \{w_1, w_2, w_3\}$ $\Rightarrow$ $M_B^S = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

$\hookrightarrow v_3 = 2w_1 + \frac{1}{2}w_2 + 1w_3.$

$\hookrightarrow v_2 = 1 \cdot w_1 + 1w_2 + 0w_3$

$\hookrightarrow v_1 = 1 \cdot w_1 + 0w_2 + 0w_3.$

Quiere decir que $[v_1\ v_2\ v_3] = [w_1\ w_2\ w_3] \cdot M_B^S$

Producto Interno 3

Teórica

Ejercicio 4. En espacio euclídeo $C([-1,1],\mathbb{R})$ (funciones continuas) con p.i.: $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

Consideramos $v_1 = 1$, $v_2 = x$, $v_3 = x^2$. Construir sist. ortonormal.

$j=1$:

- $w_1 = v_1 = 1.$
- $\|w_1\|^2 = \langle 1,1\rangle = \int_{-1}^{1} 1\,dx = 2.$
- $u_1 = \dfrac{w_1}{\|w_1\|} = \dfrac{1}{\sqrt{2}}$

$j=2$:

- $w_2 = v_2 - \dfrac{\langle v_2,w_1\rangle}{\|w_1\|^2}\cdot w_1 = x - \dfrac{\langle x,1\rangle}{2}\cdot 1 = x - \int_{-1}^{1} x\,dx\,\dfrac{1}{2} = x - \left(\dfrac{x^2}{2}\right)\Big|_{-1}^{1}\dfrac{1}{2} = x.$
- $\|w_2\|^2 = \langle w_2,w_2\rangle = \int_{-1}^{1} x^2\,dx = \left(\dfrac{x^3}{3}\right)\Big|_{-1}^{1} = \dfrac{2}{3}$
- $u_2 = \dfrac{w_2}{\|w_2\|} = \dfrac{x}{\sqrt{2/3}} = \sqrt{\dfrac{3}{2}}\cdot x.$

$j=3$:

- $w_3 = v_3 - \dfrac{\langle v_3,w_2\rangle}{\|w_2\|^2}\cdot w_2 - \dfrac{\langle v_3,w_1\rangle}{\|w_1\|^2}\cdot w_1 = v_3 - \dfrac{\langle x^2,x\rangle}{2/3}\cdot x - \dfrac{\langle x^2,1\rangle}{2}\cdot x^2 = x^2 - \dfrac{3}{2}x\cdot\langle x^2,x\rangle - \dfrac{x^2}{2}\langle x^2,1\rangle$

  $= x^2 - \dfrac{3}{2}x\left(\dfrac{x^4}{4}\right)\Big|_{-1}^{1} - \dfrac{x^2}{2}\cdot\left(\dfrac{x^3}{3}\right)\Big|_{-1}^{1} = x^2 - 0 - \dfrac{x^2}{2}\cdot\dfrac{2}{3} = x^2 - \dfrac{x}{3} = x^2 - \dfrac{1}{3}x.$
- $\|w_3\|^2 = \langle w_3,w_3\rangle = \int_{-1}^{1}\left(x^2 - \dfrac{1}{3}x\right)^2 dx = \dfrac{64}{135}$
- $u_3 = \dfrac{w_3}{\|w_3\|} = \dfrac{x^2 - x/3}{\sqrt{64/135}} = \sqrt{\dfrac{135}{64}}\cdot\left(x^2 - \dfrac{x}{3}\right) = \dfrac{\sqrt{135}}{8}\left(x^2 - \dfrac{x}{3}\right)$

Sist. ortonormal: u_1, u_2, u_3.

Producto Interno 3
Práctica

Ejercicio 1. Sea el p.i. $\langle \cdot , \cdot \rangle$ en $\mathbb{R}^3$ def por $\langle x, y\rangle = y^T \cdot G \cdot x$ con $G = \begin{bmatrix} 8 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$.
Considerando la funcional lineal $\phi: \mathbb{R}^3 \to \mathbb{R}$ dada por $\phi(x) = x_1 + 2x_2 - 3x_3$ hallar $v \in \mathbb{R}^3$ tq. $\phi(x) = \langle x, v\rangle \ \forall x \in \mathbb{R}^3$.

$$[v]^B = G^{-1} \cdot \begin{bmatrix} \phi(v_1) \\ \phi(v_2) \\ \phi(v_3) \end{bmatrix} \quad \text{con } B = \{v_1, v_2, v_3\}$$

$$\Rightarrow B_{\mathbb{R}^3} = \{[1\,0\,0]^T, [0\,1\,0]^T, [0\,0\,1]^T\}.$$

$$\Rightarrow [v]^B = G^{-1} \cdot \begin{bmatrix} \phi(e_1) \\ \phi(e_2) \\ \phi(e_3) \end{bmatrix} \Rightarrow [v]^B = \frac{1}{6} \cdot \begin{bmatrix} 2 & -2 & 4 \\ -2 & 4 & -6 \\ 4 & -6 & 13 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \frac{1}{6} \cdot \begin{bmatrix} -14 \\ 24 \\ -47 \end{bmatrix}$$

Ejercicio 2. En $\mathbb{R}_2[x]$ con p.i. $\langle p, q\rangle = \int_0^1 x\, p(x) q(x)\, dx$ hallar $q \in \mathbb{R}_2[x]$ tq. $\phi(p) = \langle p, q\rangle \ \forall p \in \mathbb{R}_2[x]$, donde $\phi(x) = \mathbb{R}_2[x] \to \mathbb{R}$ es la función lineal def. por $\phi(p) = \int_0^1 x\, p(x) \operatorname{sen}(\pi x)\, dx$

Defino una base de $\mathbb{R}_2[x]$: $E = \{1, x, x^2\}$.

$$\Rightarrow \begin{cases} \phi(1) = 1/\pi \\ \phi(x) = (\pi^2 - 4) : \pi^3 \\ \phi(x^2) = (\pi^2 - 6) : \pi^3 \end{cases}$$

• Hallo la matriz de Gram $\Rightarrow$ $G = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{bmatrix} = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \end{bmatrix}$

$$\Rightarrow [q]^E = G^{-1} \cdot \begin{bmatrix} \phi(1) \\ \phi(x) \\ \phi(x^2) \end{bmatrix} = \begin{bmatrix} (12\pi^2 - 120) : \pi^3 \\ (-60\pi^2 + 720) : \pi^3 \\ (60\pi^2 - 720) : \pi^3 \end{bmatrix}$$

$$\hookrightarrow \boxed{q = \frac{12\pi^2 - 120}{\pi^3} \cdot 1 + \frac{(-60\pi^2 + 720)}{\pi^3} x + \frac{60\pi^2 - 720}{\pi^3} x^2}$$ $\to$ representante de Riesz.

Ejercicio 3 (Tipo 3.14/25) - Sea $V = \mathbb{R}_2[x]$ el $\mathbb{R}$-esp. euclídeo dado por p.i. $<p,q> = \int_{-1}^{1} p(x)q(x)dx$ y sea $S = gen\{1+x\}$. Hallar $q \in S^\perp$ t.q. $\int_{-1}^{1} (q(x)-1-x^2)^2 dx$ tome el mejor valor posible.

Queremos minimizar: $\int_{-1}^{1} (q(x)-(1+x^2))^2 dx = \int_{-1}^{1} [q(x)-(1+x^2)] \cdot [q(x)-(1+x^2)] dx =$

$= <q(x)-(1+x^2), q(x)-(1+x^2)> = \| q(x)-(1+x^2) \|^2.$

Es decir, buscamos $q(x) \in S^\perp$ q. minimice $\| q(x)-(1+x^2)\| = dist(q, 1+x^2)$.

$q(x) = P_{S^\perp}(1+x^2) \wedge q(x) = P_S(q) + P_{S^\perp}(q) \rightarrow \mathbb{R}_2[x] = S \oplus S^\perp$

$$\begin{cases} P_{S^\perp}(f) = f & \text{si } f \in S^\perp \\ P_S(f) = 0 & \text{si } f \in S \end{cases}$$

Sé q. $\dim(\mathbb{R}_2[x]) = \dim(S) + \dim(S^\perp)$

$3 = \underset{\text{S gen. por 1 polinomio}}{1} + \dim(S^\perp) \Rightarrow \dim(S^\perp) = 2$.

$\Rightarrow p = P_S(p) + P_{S^\perp}(p) \Rightarrow (1+x^2) = \underbrace{P_S(1+x^2)}_{\dim 1} + \underbrace{P_{S^\perp}(1+x^2)}_{\dim 2} \Rightarrow$ Utilizo S (+ fácil)

$\Rightarrow P_{S^\perp}(1+x^2) = (1+x^2) - P_S(1+x^2)$.

- $P_S(1+x^2) = \dfrac{<p, gen(S)>}{\|gen(S)\|^2} \cdot gen(S) = \dfrac{<1+x^2, 1+x>}{\|1+x\|^2} \cdot (1+x)$.

  - $<1+x^2, 1+x> = \int_{-1}^{1} (1+x^2)(1+x)dx = 8/3$
  - $<1+x, 1+x> = \int_{-1}^{1} (1+x)^2 dx = 8/3$

  $\Rightarrow P_S(1+x^2) = 1+x$.

$\Rightarrow P_{S^\perp}(1+x^2) = (1+x^2)-(1+x) = 1-1+x^2-x = x^2-x$.

$\longrightarrow$ $\boxed{q(x) = x^2-x}$ es el polinomio en $S^\perp$ más cercano a $1+x^2$.

Y la dist. es: $dist(1+x^2, x^2-x) = \|(1+x^2)-(x^2-x)\| = \|1+x\| = \sqrt{\frac{8}{3}}$.

Producto Interno 3

Práctica

Ejercicio 4- Hallar descomposición QR de la matriz $A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & -2 \\ 2 & 1 & 2 \end{bmatrix}$

(Tipo 3_26/27)

¿rango(A)? $\det(A) \neq 0 \Rightarrow$ rango$(A) = 3$.

$[A] = \overset{\text{Base de } Col(A)}{[Q]} \cdot [R]$

$\hookrightarrow \in \mathbb{R}^{3\times 3} \to$ igual q. A. $\quad \hookrightarrow \in \mathbb{R}^{3\times 3} \to$ rango(A) . rango(A)

$Col(A) = gen\underbrace{\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \right\}}_{L.i.} \Rightarrow Col(A) = \mathbb{R}^3 \to$ No sirve cualquier BON

Si uso por ej. $E = \{e_1, e_2, e_3\}$.

$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . R = I.R = R$

$\therefore R = A \Rightarrow R$ no es triangular superior

Necesito BON tq. R es triangular sup. con diagonal de elementos positivos

$A = \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}}_{Q} . \underbrace{\begin{bmatrix} *_1 & \cdots & \cdots \\ 0 & *_2 & \vdots \\ 0 & 0 & \\ 0 & 0 & *_3 \end{bmatrix}}_{R}$

$\to$ Como 1^{ra} $Col(A) = Q \cdot 1^{ra}$ $Col(R)$

1^{ra} $Col(A) = *_1 . v_1 \Rightarrow v_1 = \dfrac{1^{ra} Col(A)}{*_1}$

2^{da} $Col(A) = Q . 2^{da}$ $Col(R)$

2^{da} $Col(A) = *_1 v_1 + *_2 . v_2 \Rightarrow v_2 = \dfrac{2^{da} Col(A) - *_1 v_1}{*_2}$

Además, por BON, $v_1 \perp v_2$.

... Estamos haciendo Gram-Schmidt!!!

i. Consigo conj. ortogonal $\{w_1, w_2, w_3\}$.

$j=1$: $w_1 = v_1 = [0\ 1\ 2]^T$.

$\|w_1\|^2 = 5$

$u_1 = \frac{1}{\sqrt{5}} . [0\ 1\ 2]^T$

$j=2$

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1 = [0\ 3\ 1]^T - \frac{5}{5}[0\ 1\ 2]^T = [0\ 2\ -1]^T$

$\|w_2\|^2 = 5$

$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{5}} \cdot [0\ 2\ -1]^T$

$j=3$

$w_3 = v_3 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} \cdot w_2 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} \cdot w_1 = [2\ -2\ 2]^T - \frac{(-6)}{5}[0\ 2\ -1]^T - \frac{2}{5}[0\ 1\ 2]^T$

$= [2\ -2\ 2]^T + \frac{6}{5}[0\ 2\ -1]^T - \frac{2}{5}[0\ 1\ 2]^T = [2\ 0\ 0]^T.$

$\|w_3\|^2 = 4$

$u_3 = \frac{1}{\sqrt{4}}[2\ 0\ 0]^T = [1\ 0\ 0]^T.$

$\Rightarrow$ Base ortonormal: $\mathcal{B} = \{w_1, w_2, w_3\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

$Q = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

$\hookrightarrow$ despejo R. $A = Q.R \Rightarrow Q^{-1}.A = R$

$Q^{-1}. \begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & -2 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1/5 & 2/5 \\ 0 & 2/5 & -1/5 \\ 1/2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & -2 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2/5 \\ 0 & 1 & -6/5 \\ 0 & 0 & 1 \end{bmatrix} = R.$

$\mathcal{B} = \left\{ \frac{1}{\sqrt{5}}[0\ 1\ 2]^T, \frac{1}{\sqrt{5}}[0\ 2\ -1]^T, [1\ 0\ 0]^T \right\}.$

Producto Interno 3

Práctica

Ejercicio 5. Dar la descomposición QR de $A = \begin{bmatrix} 4 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow A$ ya es triangular sup.

$Col(A) = gen\left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\underbrace{A}_{\in \mathbb{R}^{3\times 2}} = \overset{\in \mathbb{R}^{3\times 2}}{Q} \cdot \underbrace{R}_{\in \mathbb{R}^{2\times 2}}$

Busco BON

$w_1 = v_1 = [4\ 0\ 0]^T$

$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} \cdot w_1 = [2\ 1\ 0]^T - \frac{8}{16} \cdot [4\ 0\ 0]^T = [2\ 1\ 0]^T - [2\ 0\ 0]^T$

$= [0\ 1\ 0]^T.$

$\mathcal{G} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

sist. ortonormal

$U = \left\{ u_1, u_2 \right\} = \left\{ \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$A = [u_1\ u_2] \cdot R \rightarrow \begin{bmatrix} 4 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot R \rightarrow \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}$

Repaso Guía 2
Práctica.

1. Resolver una ecuación del tipo $T(x)=b$

$$\left.\begin{array}{l} \to T([1\,0\,0]^T) = 1-x \\ T([0\,1\,1]^T) = 1+x^2 \\ T([0\,1\,-1]^T) = x+x^2 \end{array}\right\} \underset{\in\mathbb{R}^{3\times1}}{T(v)} = \underset{\in\mathbb{R}_2[x]}{p} \to T\in[\mathbb{R}^3, \mathbb{R}_2[x]]$$

Me pide $T(v) = 2+x+3x^2$. → enunciado

Escribo las bases:

- $B=\{e_1,e_2,e_3\}=\{[1\,0\,0]^T,[0\,1\,0]^T,[0\,0\,1]^T\}$
- $\mathcal{E}=\{1,x,x^2\}$

$$[T]_B^{\mathcal{E}} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

→ escalares por los que al multiplicar los elementos de $\mathcal{E}$ me da los valores del enunciado ($1-x$, $1+x^2$, $x+x^2$)

Necesito $[T]_{\mathcal{E}}^{\mathcal{E}}$, tengo $[T]_B^{\mathcal{E}}$ → $[T]_{\mathcal{E}}^{\mathcal{E}} = [T]_B^{\mathcal{E}} \cdot M_{\mathcal{E}}^{B}$
$= [T]_B^{\mathcal{E}} \cdot (M_B^{\mathcal{E}})^{-1}$

→ $M_B^{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ ← vectores del enunciado

$$\Rightarrow [T]_{\mathcal{E}}^{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 2 & 1 & 1 \\ -2 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

Y tengo que $T(v) = p$ →

$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ -1 & 1/2 & -1/2 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 = 2$; $x_2 = 3$
$x_1 + \frac{3}{2} + \frac{1}{2}x_3 = 2$.
$x_1 + \frac{1}{2}x_3 = \frac{1}{2}$.
$x_1 = \frac{1}{2} - \frac{1}{2}x_3$

→ $[x_1\,x_2\,x_3] = \underbrace{[1/2\;\;3\;\;0]}_{sp} + x_3[-1/2\;\;0\;\;1]$

2. Matriz de una simetría.

$$\left[\Sigma_{\underbrace{Im(T)\ nul(T)}_{\text{en direcc. de}}}\right]_B^B \qquad B = \{v_1, v_2, v_3\}$$

$T(v_1) = v_1$

$T(v_2) = v_1 + v_2$ → enunciado.

$T(v_3) = -v_1$

datos del enunciado

$[T]_B^B$ → $nul([T]_B^B)$, $col([T]_B^B)$ ⇒ $$[T]_B^B = \begin{array}{ccc} v_1 & v_1 & -v_1 \\ \hline \end{array}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

• $Col([T]_B^B) = gen\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \cancel{\begin{bmatrix}-1\\0\\0\end{bmatrix}}\right\} = gen\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

puedo hacer el seg menos el primero

• $Nul([T]_B^B) = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left.\begin{array}{l} \bullet x_1 - x_3 = 0 \quad \bullet x_2 = 0 \\ x_1 = x_3 \end{array}\right\}$ $[x_1\ x_2\ x_3] = [x_1\ 0\ x_1]$
$= x_1[1\ 0\ 1]$

$= gen\{[1\ 0\ 1]^T\}$.

$$C = \Bigg\{\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}}_{Col}, \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{Nul}\Bigg\} \to [\Sigma]_C^C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

igual — opuesta

$$[\Sigma] = M_C^E . [\Sigma]_C^C . M_E^C = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Repaso Guía 2

Práctica

3. Ecuaciones diferenciales: $L[y]=x^k.e^{\lambda x}$

$L[y]=e^{-x}$ $\qquad x^k.e^{\lambda x} \Rightarrow A=(D-\lambda I)^{k+1} \Rightarrow A=(D-(-I))^{0+1}=(D+I)^1=D+I$

$L=(D-I)(D+I)$

$A=D+I$

$AL=(D-I)(D+I)(D+I)=(D-I)(D+I)^2$

$B_{AL}=\{e^x, e^{-x}, xe^{-x}\}$ $\qquad B_L=\{e^x, e^{-x}\}$

$B_{AL}-B_L=\{xe^{-x}\}$

Sol part. de forma $(xe^{-x}).a \Rightarrow y_p=a.x.e^{-x}$

$L[y_p]=(D-I).ae^{-x}=-a.e^{-x}-a.e^{-x}=-2ae^{-x}=e^{-x}$

$\Rightarrow a=-\frac{1}{2} \Rightarrow y_p=-\frac{1}{2}xe^{-x}$

Como $B_L=\{e^x, e^{-x}\} \Rightarrow y=-\frac{1}{2}xe^{-x}+ae^x+be^{-x}$

4. Sea $y\in C^\infty(\mathbb{R})$ la sol. de $y''+4y=sen(2t)$, $y(0)=1$, $y'(0)=0$

- Hallo homogénea: $y''+4y=0$ $\qquad \Rightarrow\{e^{at}sen(bt), e^{at}cos(bt)\}$

$L[y]=D^2+4I$ $\qquad Nu(L)=gen\{sen(2t), cos(2t)\}$

x^2+4

$x=a+bi$ raíz $x^2-2ax+(a^2+b^2)=0$

$A=D^2+4I$ $\qquad AoL=(D^2+4I).(D^2+4I)=(D^2+4I)^2$

$B_{Nu(A.L)}=\{cos(2t), sen(2t), tcos(2t), t\,sen(2t)\}$

$B_{Nu(AL)}-B_{Nu(L)}=\{tcos(2t), t\,sen(2t)\}$

$y_p=a.tcos(2t)+b.t\,sen(2t) \Rightarrow L[y_p]=sen(2t)$

$y_p''+4y_p=sen(2t) \Rightarrow [a.t.cos(2t)+bt\,sen(2t)]''+4(a.t.cos(2t)+bt\,sen(2t))=sen(2t)$

2. Matriz de una simetría.

$$\left[\Sigma_{Im(T)\, nul(T)}\right]_B^B \qquad B = \{v_1, v_2, v_3\}$$

en direcc. de

$T(v_1) = v_1$

$T(v_2) = v_1 + v_2$ — enunciado

$T(v_3) = -v_1$

$$[T]_B^B \begin{cases} nul\left([T]_B^B\right) \\ col\left([T]_B^B\right) \end{cases} \rightarrow [T]_B^B = \begin{array}{ccc} v_1 & v_1 & -v_1 \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} & \left.\begin{matrix} -1 \\ 0 \\ 0 \end{matrix}\right] \end{array}$$

datos del enunciado (v_2)

- $Col\left([T]_B^B\right) = gen\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \cancel{\begin{bmatrix}-1\\0\\0\end{bmatrix}}\right\} = gen\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

puedo hacer el seg menos el primero

- $Nul\left([T]_B^B\right) = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ $x_1 - x_3 = 0$, $x_2 = 0$. $x_1 = x_3$.

$[x_1\ x_2\ x_3] = [x_1\ 0\ x_1]$.

$= x_1 [1\ 0\ 1]$.

$= gen\{[1\ 0\ 1]^T\}$.

$$C = \underbrace{\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right.}_{Col}, \underbrace{\left.\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}}_{Nul} \rightarrow [\Sigma]_C^C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

igual — opuesta

$$[\Sigma] = M_C^E \cdot [\Sigma]_C^C \cdot M_E^C = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Repaso Guía 2
Práctica

3. Ecuaciones diferenciales: $L[y] = x^k . e^{\lambda x}$

$L[y] = e^{-x}$ $\qquad x^k . e^{\lambda x} \Rightarrow A = (D - \lambda I)^{k+1} \Rightarrow A = (D - (-I))^{0+1} = (D+I)^1 = D+I$

$L = (D-I)(D+I)$

$A = D+I$

$AL = (D-I)(D+I)(D+I) = (D-I)(D+I)^2$

$B_{AL} = \{e^x, e^{-x}, xe^{-x}\}$ $\qquad B_L = \{e^x, e^{-x}\}$

$B_{AL} - B_L = \{xe^{-x}\}$

Sol part. de forma $(xe^{-x}).\alpha \Rightarrow y_p = \alpha . x . e^{-x}$

$L[y_p] = (D-I).\alpha e^{-x} = -\alpha.e^{-x} - \alpha.e^{-x} = -2\alpha e^{-x} = e^{-x}$

$\Rightarrow \alpha = -\frac{1}{2} \Rightarrow y_p = -\frac{1}{2} x e^{-x}$

Como $B_L = \{e^x, e^{-x}\} \Rightarrow y = -\frac{1}{2} xe^{-x} + ae^x + be^{-x}$

4. Sea $y \in C^\infty(\mathbb{R})$ la sol. de $y'' + 4y = sen(2t)$, $y(0) = 1$, $y'(0) = 0$

• Hallo homogénea: $y'' + 4y = 0$ $\qquad \Rightarrow \{e^{at} sen(bt), e^{at} cos(bt)\}$

$L[y] = D^2 + 4I$ $\qquad Nu(L) = gen\{sen(2t), cos(2t)\}$
$x^2 + 4$

$x = a+bi$ raíz $x^2 - 2ax(a^2+b^2) = 0$

$A = D^2 + 4I$ $\qquad A \circ L = (D^2+4I).(D^2+4I) = (D^2+4I)^2$

$B_{Nu(A.L)} = \{cos(2t), sen(2t), t\,cos(2t), t\,sen(2t)\}$

$B_{Nu(AL)} - B_{Nu(L)} = \{t\,cos(2t), t\,sen(2t)\}$

$y_p = a . t\,cos(2t) + b . t\,sen(2t) \Rightarrow L[y_p] = sen(2t)$

$y_p'' + 4y_p = sen(2t) \Rightarrow [a.t.cos(2t) + bt\,sen(2t)]'' + 4(a.t.cos(2t) + bt\,sen(2t)) = sen(2t)$