A Multiplication Game

- proof of the mathematical formula -

Stan and Piet choose a number between 2 and 9, and multiply it to the previous calculated number. The initial number the game starts with is 1.

Both players play the game perfectly, meaning that one tries to win the game in the shortest possible rounds.

Round	N	Winner	Range	Description
1	0	Stan	2-9	Stan is going to be the winner if the number falls in this range.
2	0	Piet	10-18	Now, the possible range of the numbers that might be chosen by Stan and Piet is: min = 2 * 2 = 4, max = 9 * 9 = 81. Thus, from 4 to 81. We already know that number 4 – 9 are the winning numbers for Stan. So, we can safely exclude them from the range. We are now left with: 10 – 81. When the number is 10, Stan won't be able to reach this number in the first round. However, no matter which number Stan chooses, Piet will be able to reach or exceed 10. Stan will do his best to minimise the maximum value of the range. For instance, if the number is 19, Stan can choose 2, and limit the maximum value that Piet can reach to 18 (2 * 9). Piet won't be able to reach this number, and no matter what number Piet chooses, Stan can win in the third round (e.g., 2 * 2 * 5 = 20). This maximum value of the range can be calculated by multiplying the maximum number with the number of turns of the player that can win (if the game is played perfectly), and multiplying it with the minimum value of the range with the number of turns of the player that cannot win. So we get, 2 * 9 = 18.
3	1	Stan	19-162	Possible total range: minimum = 8 (2^3), maximum = 729 (9^3). Same reasoning as above. Maximum value = 9^2 * 2 = 162.
4	1	Piet	163-324	Possible total range: minimum = 16 (2^4), maximum = 6561 (9^4). Same reasoning as above. Maximum value = 9^2 * 2^2 = 324.
5	2	Stan		
6	2	Piet		

7	3	Stan		
8	3	Piet		

The maximum value in the range can be calculated as follows:

Odd round (Stan wins): Piet will minimise the numbers while Stan will maximise the numbers.

So the formula is: 2^((k-1)/2) * 9^((k-1)/2) * 9. The numbers under the N column can be obtained by: (k- 1)/2. Replacing (k-1)/2 with n in the formula, gives us 2^n * 9^n * 9 = 2^n * 9^(n+1).

Even round (Piet wins): Now, Stan will minimise the numbers while Piet will maximise the numbers.

Thus the formula is: 2^(k/2) * 9^(k/2). The numbers under the N column can be obtained by: k/2 – 1. Hence, n + 1 = k/2. Replacing k/2 with n + 1 in the formula gives us: 2^(n+1) * 9^(n+1).

Having these two formulas, we can obtain the formulas of the min values.

Odd round (Stan wins):

Max formula of Piet wins + 1 gives us: 2^(n+1) * 9^(n+1) + 1. So when we take n to be 0, it will give us the min value of Stan for n = 1. Clearly, we get the min value of n+1 when we choose n. Hence, k = n + 1, n = k -1. After replacing the n in the formula with k -1, we get 2^(k-1+1) * 9^(k-1+1) + 1 = 2^k * 9^k + 1, equivalent to 2^n * 9^n + 1.

Even round (Piet wins):

Max formula of Stan wins + 1 gives us: 2^n * 9^(n+1) + 1.

Hence the formulas which indicate the number ranges are:

Winner	Min	Max
Stan	2^n * 9^n + 1	2^n * 9^(n+1)
Piet	2^n * 9^(n+1) + 1	2^(n+1) * 9^(n+1)