Comments on entropy

in an expanding universe, total E not conserved (Lagrange eq$^{\underline{n}}$)

in equilibrium, entropy is conserved

$\mathcal{S}$ = total entropy in box $V = a^3 \quad = \quad a^3 s(T)$

$$s(T) = \frac{2\pi^2}{45} g_{*s}(T)\, T^3 = \sum_{\text{particles}} s_i$$

$$\frac{d\mathcal{S}}{dt} = 0 \quad \longrightarrow \quad \frac{d(a^3 \cdot s)}{dt} = a^3 \cdot \frac{ds}{dt} + s\,\frac{da^3}{dt}$$

$$0 = a^3 \frac{dT}{dt} \cdot \frac{ds}{dT} + 3\dot{a}a^2 s$$

$$-3 \underbrace{\frac{\dot{a}}{a}}_{=H} s = \frac{dT}{dt}\frac{ds}{dT} = \left(\frac{2\pi^2}{45}\right)\frac{dT}{dt}\left[T^3 \frac{dg_{*s}}{dT} + 3T^2 g_{*s}\right]$$

$$-3H \cdot g_{*s} T^3 = \frac{dT}{dt}\left[3T^2 g_{*s} + T^3 \frac{dg_{*s}}{dT}\right]$$

$$-HT = \frac{dT}{dt}\left[1 + \frac{T}{3g_{*s}} \cdot \frac{dg_{*s}}{dT}\right]$$

$$\frac{dT}{dt} = -HT \cdot \left(1 + \frac{T}{3g_{*s}}\frac{dg_{*s}}{dT}\right)^{-1}$$

if g_{*s} constant: $\frac{dT}{dt} = -HT \quad \longrightarrow \quad \frac{dT}{T} = -H\,dt = -\frac{da}{dt}\frac{1}{a}dt = -\frac{da}{a}$

$\longrightarrow \log(T) = \log(a^{-1})$

$$(\mathcal{S}_{tot})_i = s(T_i)\,a_i^3 = (\mathcal{S}_{tot})_f = s(T_f)\,a_f^3$$

with known T_i & T_f, then can find volume ratio $\frac{a_f}{a_i}$ "how much volume has changed"

Dark Matter Freeze Out

last time: Boltzmann eq$^{\underline{n}}$: $\underbrace{\chi}_{DM}\bar{\chi} \longleftrightarrow \underbrace{f\bar{f}}_{\text{always in eq.}}$ starts in eq. $n_\chi(T \gg m_\chi) = (n_\chi)_{eq}$

assume $m_\chi = m \gg m_t \approx 175\text{ GeV}$

$\longrightarrow g_{*s} = 106.75 \quad @\ T = m$

$n_{\bar{\chi}} = n_\chi$, $\mu_\chi = 0$ $\qquad n_{DM} = n_\chi + n_{\bar{\chi}}$

$$3Hn_\chi + \frac{dn_\chi}{dt} = -\langle\sigma v\rangle\left(n_\chi n_{\bar{\chi}} - (n_\chi)_{eq}(n_{\bar{\chi}})_{eq}\right) = -\langle\sigma v\rangle\left(n_\chi^2 - (n_\chi^{eq})^2\right)$$

$$\frac{1}{a^3} \cdot \frac{d(a^3 n_\chi)}{dt}$$

$Y = \frac{n_\chi}{s}$ (comoving yield)

$$\frac{1}{a^3} \cdot \frac{d(a^3 \cdot s \cdot Y)}{dt} = \frac{a^3 s}{a^3}\frac{dY}{dt} = s\frac{dY}{dt}$$

const. entropy conservation

$$s\frac{dY}{dt} = -\langle\sigma v\rangle\left(s^2Y^2 - s^2Y_{eq}^2\right) \qquad Y_{eq} = \frac{n_{\chi,eq}}{s}$$

dimensionless time: $z \equiv \frac{m}{T}$ not redshift

$$\frac{dz}{dt} = -\frac{m}{T^2}\cdot\dot{T} = -\left(\frac{m}{T}\right)\cdot\left(\frac{1}{T}\frac{dT}{dt}\right)$$

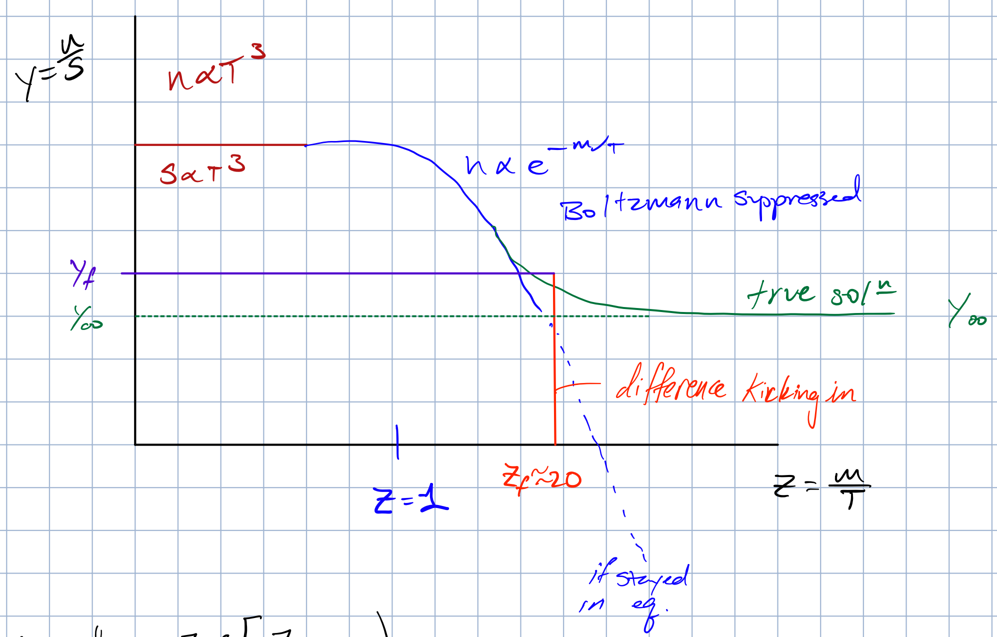
earlier discussion

g_{*s} = const

$$= -z\left(\frac{1}{T}\cdot -HT\right) = zH$$

$$\frac{dz}{dt} = Hz \to \frac{dz}{Hz} = dt$$

$$Hz\cdot\frac{dY}{dz} = -\langle\sigma v\rangle s\left(Y^2 - Y_{eq}^2\right)$$



want Y_∞

consider $z\in[z_f,\infty)$

$z > z_f$, neglect Y_{eq}^2

$$Hz\frac{dY}{dz} = -\langle\sigma v\rangle s Y^2$$

rewrite s & H in Y & z

now assume radiation domination $T \gg T_{eq}$

$$H = 1.66\sqrt{g_*}\frac{T^2}{m_{pl}} = 1.66\sqrt{g_*}\frac{m^2}{m_{pl}}\cdot\frac{1}{z^2} = H(T=m)\cdot\frac{1}{z^2} = H(m)\frac{1}{z^2}$$

$$s = \frac{2\pi^2}{45}g_{*s}T^3 = \frac{2\pi^2}{45}g_{*s}\frac{m^3}{z^3} = s(T=m)\frac{1}{z^3} = s(m)\frac{1}{z^3}$$

$$\frac{dY}{Y^2} = -\frac{\langle\sigma v\rangle}{Hz}s\,dz = -\frac{\langle\sigma v\rangle}{z}\cdot\left(\frac{s(m)}{z^3}\right)\left(\frac{z^2}{H(m)}\right)dz = -\underbrace{\frac{\langle\sigma v\rangle s(m)}{H(m)}}_{\equiv\lambda}\frac{1}{z^2}\cdot dz$$

$$\frac{dY}{Y^2} = -\frac{dz}{z^2}\,\lambda$$

integrate $z_f \to \infty$ $\qquad\qquad$ $f \to$ freeze out

$$\int_{Y_f}^{Y_\infty} \frac{dY}{Y^2} = -\lambda \int_{z_f}^{\infty} \frac{dz}{z^2}$$

$$\left(-\frac{1}{Y}\right)\Bigg|_{Y_f}^{Y_\infty} = -\lambda\left(-\frac{1}{z}\right)\Bigg|_{z_f}^{\infty}$$

$$\frac{1}{Y_f} - \frac{1}{Y_\infty} = -\lambda\left(\overset{0}{\cancel{\frac{-1}{\infty}}} + \frac{1}{z_f}\right)$$

$Y_f \gg Y_\infty$, so $\frac{1}{Y_f} - \frac{1}{Y_\infty} \approx -\frac{1}{Y_\infty}$

$$\frac{1}{Y_\infty} = \frac{\lambda}{z_f}$$

$$Y_\infty = \frac{z_f}{\lambda} = \frac{z_f}{\langle\sigma v\rangle}\cdot\frac{H(m)}{s(m)} \overset{\text{also}}{=} \frac{n_\infty}{s_\infty}$$

$$S_\infty = \frac{2\pi^2}{45}\,\underbrace{g_{*s}(T_0)}_{3.9}\,\underbrace{T_0}_{2.72K}{}^{3}$$

$$n_\infty = Y_\infty Z_\infty$$

now bring back in $\bar{\chi}$, all previous was for just χ. $\chi \to \underline{2\chi}$

$$\rho_{DM,0} = m(n_\infty)$$

$$= m\cdot(2Y_\infty)\left(\frac{2\pi^2}{45}\,g_*(T_0)\,T_0^3\right)$$

$$= m\cdot\frac{2z_f}{\langle\sigma v\rangle}\,\frac{\left(1.66\sqrt{g_*(m)}\,m^2/m_{pl}\right)}{s(m)}\;s(\infty)$$

$$= m\left(\frac{2z_f}{\langle\sigma v\rangle}\right)\left(\frac{\frac{1.66\sqrt{g_*(m)}\,m^2}{m_{pl}}}{\cancel{\frac{2\pi^2}{45}}\,g_{*s}(m)\,m^3}\right)\cancel{\frac{2\pi^2}{45}}\,g_{*s}(T_0)\,T_0^3$$

mass cancels

$g_*(m) = g_{*s}(m) = 106.75$

$g_{*s}(T_0) = 3.9$

$T_0 = 2.72K$

$$\rho_{DM,0} \simeq \left(\frac{3z_f}{\langle\sigma v\rangle}\right)\frac{\sqrt{g_*(m)}\;g_{*s}(T_0)}{g_{*s}(m)}\;\frac{T_0^3}{m_{pl}}$$

$$\Omega_{DM} = \frac{\rho_{DM,0}}{\rho_{tot}} \qquad \rho_{tot} = 4\cdot10^{-47}\,GeV^4 = \frac{3}{8\pi G}H_0^2$$

$$\Omega_{DM_0} = 0.24 \left(\frac{z_f}{20}\right)\left(\frac{10^{-9}\,GeV^{-2}}{\langle\sigma v\rangle}\right)$$

$z_f = 20$ $\quad \langle \sigma v \rangle = 10^{-9}\,GeV^{-2}$ for our measurements

true in any theory of DM freeze out
for any mass

$$\langle \sigma v \rangle \propto (G_F)^2 \cdot (\text{energy})^2$$

$$\left.\begin{array}{l} G_F^{\,2} \sim 10^{-10}\,GeV^{-4} \\ m \sim \text{few } GeV \end{array}\right\} \Rightarrow \langle\sigma v\rangle \approx 10^{-9}\,GeV^{-2}$$