Electric Field / Potential

$\vec{\nabla}\times\vec{\nabla}f = 0$

$\vec{\nabla}\times\vec{E} = 0$

$\vec{E}$ is special

$\vec{E} = -\vec{\nabla}V$

$\vec{\nabla}\times E = \vec{\nabla}\times(-\vec{\nabla}V) = 0$

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

$\vec{E} = -\vec{\nabla}V$ can add any value to V b/c $\vec{\nabla}(c) = 0$

$V \rightarrow V + C$

$-\vec{\nabla}(V+c) = -\vec{\nabla}V - \vec{\nabla}C = -\vec{\nabla}V$

ambiguity in V w/o changing $\vec{E}$

Magnetic Field / Potential

$\vec{\nabla}\cdot(\vec{\nabla}\times f) = 0$

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{B}$ is special

$\vec{B} = \vec{\nabla}\times\vec{A}$

$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

$\vec{B} = \vec{\nabla}\times\vec{A}$ can add any value to $\vec{A}$ b/c $\vec{\nabla}\times(\vec{\lambda}) = 0$

$\vec{A} \rightarrow \vec{A} + \vec{\lambda}$

$\vec{\nabla}\times(\vec{A}+\vec{\lambda}) = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\lambda} = \vec{\nabla}\times\vec{A}$

ambiguity in $\vec{A}$ w/o changing $\vec{B}$

V & $\vec{A}$ have gauge freedom w/o changing $\vec{E}$ & $\vec{B}$

can fix V with $V(r=\infty) = 0$

many ways to fix $\vec{A}$

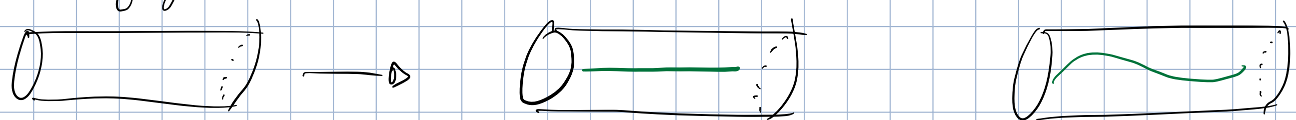
coulumb gauge: $\vec{\nabla}\cdot\vec{A} = 0$

can add $\vec{\lambda} = \vec{\nabla}f$ to satisfy coulumb gauge

$\vec{A}' = \vec{A} + \vec{\lambda} \rightarrow \vec{\nabla}\cdot(\vec{A}+\vec{\lambda}) = \vec{\nabla}\vec{A} + \underbrace{\nabla^2 f}_{\text{poisson eq}^{n}} = \vec{\nabla}\cdot\vec{A} - \vec{\nabla}\vec{A} = 0$

$\vec{\nabla}\vec{A}' = 0$

to fix a gauge for cylindrical rod, draw line



Ampere's Law: $\vec\nabla \times \vec B = \mu_0 \vec J$

$$\vec\nabla \times \vec B = \vec\nabla \times (\vec\nabla \times \vec A) = \vec\nabla (\underbrace{\vec\nabla \cdot \vec A}_{=0}) - \nabla^2 \vec A = -\nabla^2 \vec A$$

$$\nabla^2 \vec A = -\mu_0 \vec J$$

poisson eq^n $\left(\nabla^2 V = -\frac{\rho}{\epsilon_0}\right)$

$$\epsilon_0 \to \frac{1}{\mu_0} \qquad \longrightarrow V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho(\vec r')}{\mathscr{r}}\, d\tau'$$

$$V \to A_i$$

$$\rho \to J_i$$

$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x(\vec r')}{\mathscr{r}}\, d\tau'$$

$$\vec B = \vec\nabla \times \vec A = \vec\nabla \times \left(\frac{\mu_0}{4\pi}\int_{V'} \frac{\vec J(\vec r')}{\mathscr{r}}\, d\tau'\right)$$

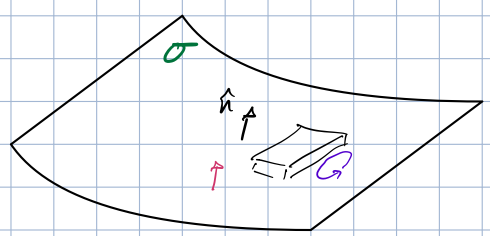
$$= \frac{\mu_0}{4\pi}\int_{V'} \vec\nabla \times \frac{\vec J(\vec r')}{\mathscr{r}}\, d\tau')$$

$$-\vec J(\vec r') \times \vec\nabla \frac{1}{\mathscr{r}}$$

$$\vec J(\vec r') \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\vec B = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec J(\vec r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Biot-Savart!



$$\vec\nabla \cdot \vec E = \frac{\rho}{\epsilon_0}$$

$$\oint \vec E \cdot d\vec a = \frac{\sigma A}{\epsilon_0}$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$

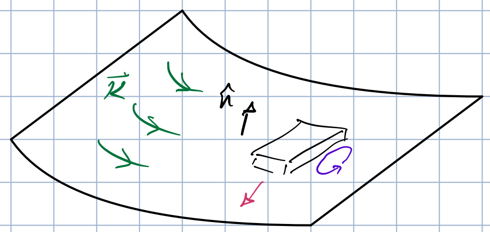
G

$$\vec\nabla \times \vec E = 0$$

$$\oint \vec E \cdot d\vec l = 0$$

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

$$\vec E_{above} - \vec E_{below} = \frac{\sigma}{\epsilon_0}\hat n$$



$$\vec{\nabla}\cdot\vec{B}=0$$

$$\oint \vec{B}\cdot d\vec{a}=0$$

$$B^{\perp}_{above}-B^{\perp}_{below}=0$$

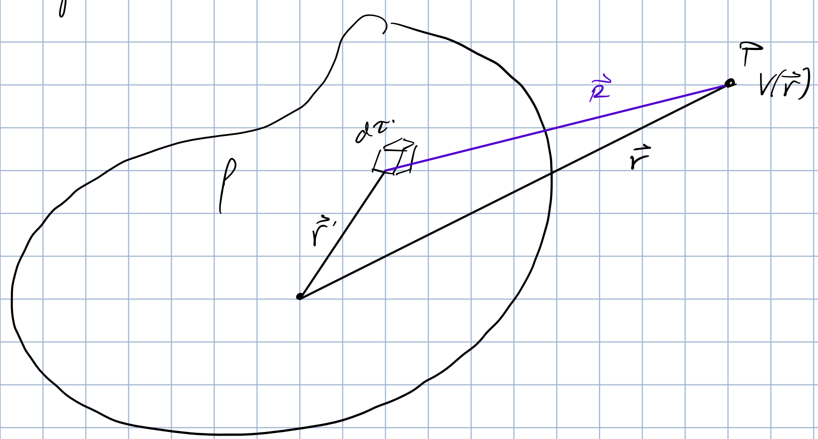
G $$\vec{\nabla}\times\vec{B}=0$$

$$\oint \vec{B}\cdot d\vec{l}=\mu_0 K\cdot l$$

$$B^{\parallel}_{above}-B^{\parallel}_{below}=\mu_0 K$$

$$\vec{B}_{above}-\vec{B}_{below}=\mu_0\,\underline{\hat{K}\times\hat{n}}$$

Multipole Expansion

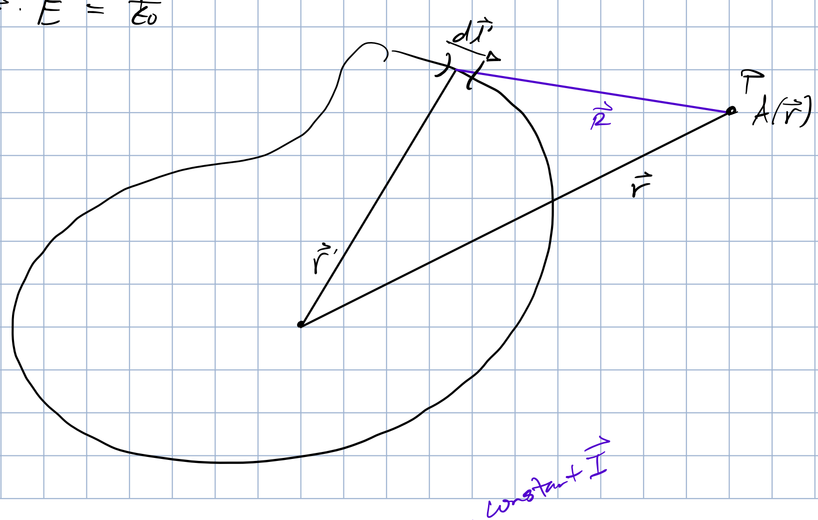


$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{R}\,d\tau'$$

$$\frac{1}{R}=\sum\left(\frac{r'}{r}\right)^n P_n(\cos\theta)$$

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')}{R}\,d\tau=\frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int\rho(\vec{r}')d\tau'+\frac{1}{r^2}\int\rho(\vec{r}')\cos\theta\, r'd\tau'+\dots\right]$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}$$



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{V'}\frac{\vec{J}(\vec{r}')}{R}\,d\tau' = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}}{R}$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\cancelto{0}{\oint d\vec{l}} + \frac{1}{r^2}\oint r'\,\vec{J}(\vec{r}')\cdot\cos\theta\,d\tau' + \dots\right]$$

no magnetic monopole. if nonzero, dipole dominates

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\underbrace{\int r'\rho(\vec{r}')\cos\theta\,d\tau'}$$

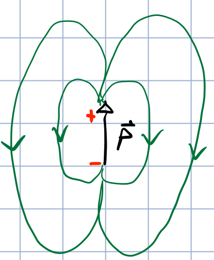
$$\int \vec{r}'\cdot\hat{r}\,\rho(\vec{r}')\,d\tau$$

$$\hat{r}\cdot\underbrace{\int \vec{r}'\rho(\vec{r}')\,d\tau}_{\equiv\vec{p}}$$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$\vec{E}_{dipole} = -\vec{\nabla}V_{dipole}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



$$A_{dipole} = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\underbrace{\oint r'\cos\theta\,d\vec{l}'}$$

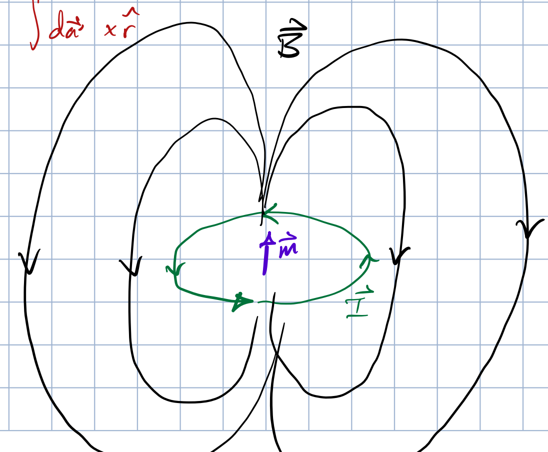
$$\oint \vec{r}'\cdot\hat{r}\,d\vec{l}' = \hat{r}\cdot\oint\vec{r}'\,d\vec{l} = -\hat{r}\times\int d\vec{a}' = \int d\vec{a}'\times\hat{r}$$

$$= \frac{\mu_0}{4\pi}\cdot\frac{1}{r^2}\cdot\underbrace{I\cdot\int d\vec{a}\times\hat{r}}$$

$$\equiv \vec{m} = I\cdot\int d\vec{a}' = I\cdot\vec{a}$$

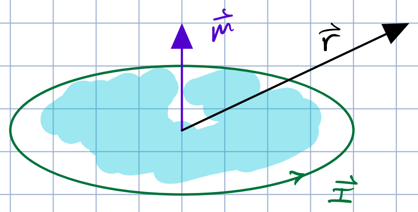
$$A_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$\vec{B}_{dipole} = \vec{\nabla}\times\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\cdot\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

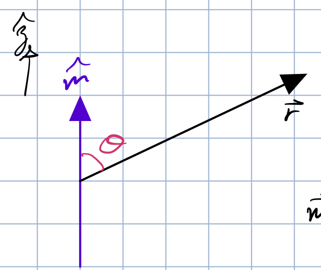


magnetic dipole moment

$$\vec{m} = I \cdot \oint da'$$



vector area: Right hand rule



$\otimes$ $\vec{I}$ $\hat{\phi}$ in page

$\vec{m} = m\hat{z}$

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$A_r = 0$

$A_\theta = 0$

$A_\phi = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}$

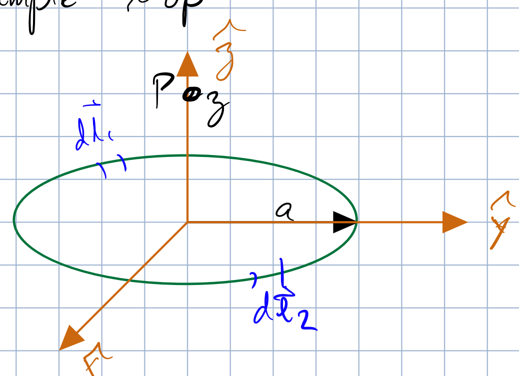
$$\vec{B}_{dp}(\vec{r}) = \vec{\nabla}\times\vec{A}_{dp} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\cdot A_\phi\right]\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\cdot A_\phi\right]\hat{\theta}$$

$$B_{dip,r} = \frac{1}{r\sin\theta}\left(\cos\theta\cdot\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2} + \sin\theta\cdot\frac{\mu_0}{4\pi}\frac{m\cos\theta}{r^2}\right) = \frac{\mu_0 m}{4\pi r^3}\cdot 2\cos\theta$$

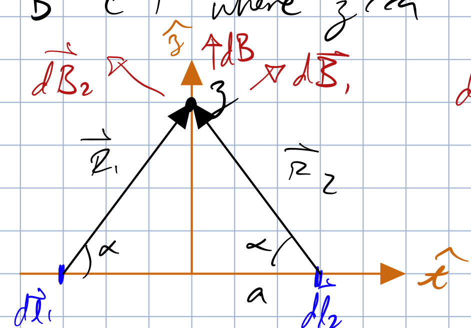
$$B_{dip,\theta} = -\frac{1}{r}\left(\frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^2} + r(-2)\frac{\mu_0 m}{4\pi}\frac{\sin\theta}{r^3}\right) = \frac{\mu_0 m}{4\pi r^3}\sin\theta$$

$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\cdot\hat{\theta}\right)$$

Example: loop



calculate B @ P where $z \gg a$



$$dB = d\vec{B}_1 + d\vec{B}_2$$

$$d\vec{B}_1 = \frac{\mu_0}{4\pi} I \frac{d\vec{l}_1 \times \hat{R}_1}{R_1^2}$$

$$d\vec{B}_2 = \frac{\mu_0}{4\pi} I \frac{d\vec{l}_2 \times \hat{R}_2}{R_2^2}$$

$$R_1 = R_2 \equiv R$$

$$d\vec{l}_1 \times \hat{R}_1 = dl(\cos\alpha\, \hat{z} + \sin\alpha\, \hat{t})$$

$$d\vec{l}_2 \times \hat{R}_2 = dl(\cos\alpha\, \hat{z} - \sin\alpha\, \hat{t})$$

$$d\vec{l}_1 \times \hat{R}_1 + d\vec{l}_2 \times \hat{R}_2 = 2\cos\alpha\, \hat{z}\, dl$$

$$\int d\vec{B} = \frac{\mu_0}{4\pi} I \cdot \int \frac{2\cos\alpha\, dl}{R^2} \hat{z}$$

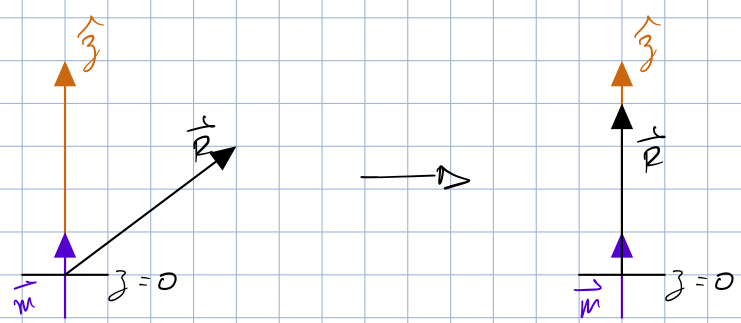
$$= \frac{\mu_0 I \cos\alpha}{2\pi R^2} \hat{z} \underbrace{\int dl}_{\text{half circle : } \pi R}$$

$$\vec{B} = \frac{\mu_0 I a^2}{2R^2} \hat{z} \underset{z \gg a}{\approx} \frac{\mu_0 I a^2}{2|z|^3} \hat{z}$$

$$\rightarrow z^2 + a^2 = z^2\left(1 + \frac{a^2}{z^2}\right) \approx z^2$$

OR just use multipole expansion

dominant will be dipole : $\vec{B} \cong \vec{B}_{dip}$



$$\vec{B}_{dip} = \frac{\mu_0 m}{4\pi r^3} \cdot (2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta})$$

$$m = I\vec{a} = I \cdot \pi a^2$$

our case, $r = R$ & $\theta = 0$

$$\rightarrow \vec{B}_{dip} = \frac{\mu_0 \cdot \pi a^2 I}{4\pi R^3} \cdot (2\hat{r} + 0\hat{\theta})$$

$$= \frac{\mu_0 a^2 I}{2R^3} \hat{r}$$