last time $\Omega_{DM} \approx 0.24$
DM timeline

Ryden Ch 12

consider $\rho_{DM} \propto a^{-3}$ always

$\rho \equiv \rho_{DM}$

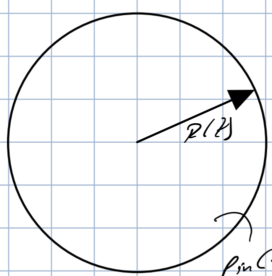
$\delta \equiv$ "Density Contrast" $= \frac{\rho - \bar{\rho}}{\bar{\rho}}$

f^n of $\vec{x}, t$

avg DM mass density

only talk about linear term w/ small perturbation $|\delta| \ll 1$

$\longrightarrow O(\delta^2) \to 0$




$M = \frac{4}{3}\pi \bar{\rho} R_0^3 = \text{constant}$

$\rho_{in}(t) = \bar{\rho}(1 + \delta(t))$

Newton's 2nd law for R(t)

$$\ddot{R}(t) = -\frac{G}{R^2}(\Delta M) = -\frac{G}{R^2}\left(\frac{4\pi}{3}\bar{\rho}\cdot\delta\right)R^3$$

ex 1: no expansion, $\bar{\rho} = \text{const}$

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3} G\bar{\rho}\delta$$

mass conservation: $M = \frac{4\pi}{3}\rho(t) R(t)^3 = \text{const}$

$$\longrightarrow R(t)^3 = \frac{K}{\rho(t)}$$

$$R(t) = \frac{K^{1/3}}{\bar{\rho}^{1/3}(1+\delta(t))^{1/3}}$$

$$= \left(\frac{K}{\bar{\rho}}\right)^{1/3}(1+\delta(t))^{1/3}$$

$$= \left(\frac{K}{\bar{\rho}}\right)^{1/3}\left((1 - \tfrac{1}{3}\delta(t)) + O(\delta^2) + \dots\right)$$

$$R(t) = R_0(1 - \tfrac{1}{3}\delta(t))$$

$$\ddot{R}(t) = -\tfrac{1}{3}R_0\ddot{\delta}(t)$$

$$\ddot{R}(t) = -\frac{1}{3} R_0 \ddot{\delta}(t)$$

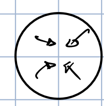
$$\approx -\frac{1}{3} R \ddot{\delta}(t) + O(\delta^2) + \ldots$$

$$\frac{\ddot{R}(t)}{R(t)} = -\frac{1}{3} \ddot{\delta}(t)$$

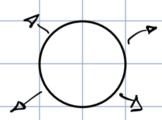
$$-\frac{1}{3} \ddot{\delta}(t) = -\frac{4\pi}{3} G \bar{\rho}\, \delta(t)$$

$$\ddot{\delta}(t) = 4\pi G \bar{\rho}\, \delta(t)$$

if $\delta < 0$ $\rightarrow$ collapse
$\dot{R} < 0$



if $\delta > 0$ $\rightarrow$ expand/dissipate
$\dot{R} > 0$



general sol$^n$ for $\delta(t)$?

$$\ddot{\delta} = 4\pi G \bar{\rho}\, \delta$$

$$= \left(\frac{1}{t_{dyn}^2}\right) \delta \qquad t_{dyn} = \left(\frac{1}{4\pi G \bar{\rho}}\right)^{1/2}$$

$$\longrightarrow \delta(t) = \underbrace{A e^{t/t_{dyn}}}_{\text{grow}} + \underbrace{B e^{-t/t_{dyn}}}_{\text{decay}}$$

matter has pressure when collapsing, keeping air molecules from clumping together

Jeans Scale $\lambda_J$

pressure : $P = f(\rho)$ — energy density

$\rightarrow$ equation of state

sound speed : $c_s^2 = \frac{dP}{d\rho}$ (material specific)

When is pressure impt?

$$t_{press} \sim \frac{R}{c_s}$$ — size of system

Jeans length $\lambda_J$ : $t_{press} = t_{dyn}$

$$\frac{\lambda_J}{c_s} = \frac{1}{(4\pi G \bar{\rho})^{1/2}}$$

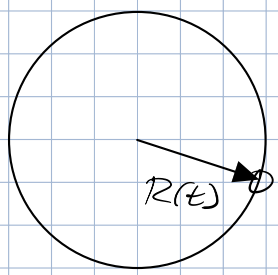
critical value for collapse or survival

$$\lambda_J = \frac{c_s}{(4\pi G \bar{\rho})^{1/2}}$$

if $R > \lambda_J \rightarrow$ collapse

if $R < \lambda_J \rightarrow$

ex 2: expansion! over/under dense sphere




what is acceleration of test mass?

$$\rho(t) = \underbrace{\bar{\rho}(t)}_{\propto a^{-3}} (1+\delta(t))$$

$$\ddot{R} = -\frac{G}{R^2} M = -\frac{G}{R^2}\left(\frac{4\pi}{3}\rho R^3\right)$$

$$= -G\,\frac{4\pi}{3}\rho(t)\, R(t)$$

$$\frac{\ddot{R}}{R} = -G\,\frac{4\pi}{3}\rho(t)$$

$$\star\ \frac{\ddot{R}}{R} = -G\,\frac{4\pi}{3}\bar{\rho}(t)\,(1+\delta(t))$$

mass conservation: $M = \frac{4\pi}{3}\rho(t) R^3 = \frac{4\pi}{3}\underbrace{\bar{\rho}(t)\,(1+\delta(t))\,R(t)^3}_{\to k} = \text{const}\cdot K$

$$R^3 = \frac{k}{\bar{\rho}(1+\delta)}$$

$$R = \frac{k^{1/3}}{\bar{\rho}^{1/3}(1+\delta)^{1/3}}$$

$\bar{\rho} \propto a^{-3}$

$\bar{\rho}^{1/3} \propto a^{-1}$ $\qquad \bar{\rho}^{1/3} = \alpha \cdot a(t)^{-1} \qquad k' \equiv \frac{k^{1/3}}{\alpha}$

$$R = \frac{k'}{a^{-1}(1+\delta)^{1/3}}$$

$$R(t) = k'\, a(t)\,(1+\delta(t))^{-1/3}$$

expand to $O(\delta)$

$$R(t) = k' a(1 - \tfrac{1}{3}\delta)$$

$$\dot{R}(t) = k'\left(\dot{a}(1-\tfrac{1}{3}\delta) + a(0 - \tfrac{1}{3}\dot{\delta})\right)$$

$$\ddot{R}(t) = k'\left(\ddot{a}(1-\tfrac{1}{3}\delta) + \dot{a}(0-\tfrac{1}{3}\dot{\delta}) + \dot{a}(-\tfrac{1}{3}\dot{\delta}) + a(-\tfrac{1}{3}\ddot{\delta})\right)$$

$$= k'\left(\ddot{a}(1-\tfrac{1}{3}\delta) - \tfrac{2}{3}\dot{a}\dot{\delta} - \tfrac{1}{3}a\ddot{\delta}\right)$$

$$\frac{\ddot{R}}{R} = \frac{k'\left(\ddot{a}(1-\tfrac{1}{3}\delta) - \tfrac{2}{3}\dot{a}\dot{\delta} - \tfrac{1}{3}a\ddot{\delta}\right)}{k'\, a(1-\tfrac{1}{3}\delta)}$$

$$= \frac{1}{a(1-\tfrac{1}{3}\delta)}\left(\ddot{a}(1-\tfrac{1}{3}\delta) - \tfrac{2}{3}\dot{a}\dot{\delta} - \tfrac{1}{3}a\ddot{\delta}\right)$$

$$\frac{1}{a}(1-\tfrac{1}{3}\delta)^{-1} \approx \frac{1}{a}(1+\tfrac{1}{3}\delta)$$

small $\delta$

$$\frac{\ddot{R}}{R} = \frac{(1+\frac{1}{3}\delta)}{a}\left(\ddot{a}(1-\tfrac{1}{3}\delta) - \tfrac{2}{3}\dot{a}\dot{\delta} - \tfrac{1}{3}a\ddot{\delta}\right)$$

drop all $\delta^2$, $\dot{\delta}^2$, $\delta\cdot\dot{\delta}$, ...

$$\frac{\ddot{R}}{R} = \frac{1}{a}\left(\cancel{\tfrac{1}{3}\delta\ddot{a}} + \ddot{a}(1-\cancel{\tfrac{1}{3}\delta}) - \tfrac{2}{3}\dot{a}\dot{\delta} - \tfrac{1}{3}a\ddot{\delta}\right)$$

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{2}{3}\left(\frac{\dot{a}}{a}\right)\dot{\delta} - \frac{1}{3}\ddot{\delta}$$

$$-G\frac{4\pi}{3}\bar{\rho}(1+\delta) = \frac{\ddot{a}}{a} - \frac{2}{3}H\dot{\delta} - \frac{1}{3}\ddot{\delta}$$

In matter domination $\bar{\rho} = \rho_{tot} \propto a^{-3}$

Friedmann: $H = \frac{\dot{a}}{a} = \left(\frac{8\pi}{3}G\rho_{tot}\right)^{1/2}$ $\qquad \rho_{tot} = \rho_i a^{-3}$

$$\frac{\dot{a}}{a} = \left(\frac{8\pi}{3}G\rho_i\right)^{1/2}\cdot a^{-3/2}$$

$$\dot{a} = \left(\frac{8\pi}{3}G\rho_i\right)^{1/2} a^{-1/2}$$

$$\Big\downarrow \frac{d}{dt}$$

$$\ddot{a} = \left(\frac{8\pi}{3}G\rho_i\right)^{1/2}\left(-\tfrac{1}{2}\right)a^{-3/2}\dot{a}$$

$$= \left(\frac{8\pi}{3}G\rho_i a^{-3}\right)^{1/2}\left(-\tfrac{1}{2}\dot{a}\right)$$

$$= \left(\frac{8\pi}{3}G\rho_{tot}\right)^{1/2}\left(-\tfrac{1}{2}\dot{a}\right)$$

$$\frac{\ddot{a}}{a} = \left(\frac{8\pi}{3}G\rho_{tot}\right)^{1/2}\left(-\tfrac{1}{2}\right)\left(\frac{\dot{a}}{a}\right)$$

$$= H\left(-\tfrac{1}{2}\right)H$$

$$= -\tfrac{1}{2}H^2$$

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\frac{8\pi}{3}G\bar{\rho} = -\frac{4\pi}{3}G\bar{\rho}$$

sub in ① & get $\quad \ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta$

$$\frac{\ddot{a}}{a} - \frac{2}{3}H\dot{\delta} - \frac{1}{3}\ddot{\delta} = -G\frac{4\pi}{3}\bar{\rho}(1+\delta)$$

$$\cancel{-\frac{4\pi}{3}G\bar{\rho}} - \frac{2}{3}H\dot{\delta} - \frac{1}{3}\ddot{\delta} = \cancel{-G\frac{4\pi}{3}\bar{\rho}} - G\frac{4\pi}{3}\bar{\rho}\delta$$

$$\frac{2}{3}H\dot{\delta} + \frac{1}{3}\ddot{\delta} = G\frac{4\pi}{3}\bar{\rho}\delta$$

$\downarrow \cdot 3$

$$\ddot{\delta} + 2H\dot{\delta} = G\, 4\pi \underline{\bar{\rho}}\, \delta$$

General form: $\Omega_M = \bar{\rho}_M / \rho_{tot}$

$$\bar{\rho}_M = \Omega_M\, \rho_{tot}$$

$$H^2 = \frac{8\pi}{3} G \rho_{tot}$$

$$\rightarrow \rho_{tot} = \frac{3H^2}{8\pi G}$$

$$\bar{\rho}_M = \Omega_M \cdot \frac{3H^2}{8\pi G}$$

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G \left( \Omega_M \frac{3H^2}{8\pi G} \right) \delta$$

$$\ddot{\delta} + 2H\dot{\delta} = \frac{3}{2} G\, \Omega_M H^2 \delta$$