Fourier Transform $\xrightarrow[\text{limit}]{\text{continuous}}$ $f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{-ikx}$

inverse transform: $\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{ikx}$

Dirac Delta f^{n}

δ_D

$$\int_{-\infty}^{\infty} dx\, \delta_D(x) = \int_{-\infty}^{\infty} dx\, f(x)\, \delta(x-a) = f(x-a)$$

FT: $\int_{-\infty}^{\infty} dx\, e^{ikx} \delta_D(x) = e^{ik\cdot 0} = \underbrace{1 = f(x)}$

define δ_D as inverse FT: $\delta_D(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \cdot f(x) = \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx}}_{*}$ $\quad k = \frac{2\pi}{\lambda}$

recall correlation f^{n}: $C\!R(\vec{R}) = \left\langle \sum_i \delta_i(\vec{r}_i)\, \delta_i(\vec{r}_i + \vec{R}) \right\rangle$

take continuous limit: $C(\vec{R}) = \int d^3r\, \delta(r)\, \delta(r+R)$

inverse F.T. of $\delta(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{r}} \underbrace{\tilde{\delta}(\vec{r},t)}_{\to FT}$ .. mode f^{n} in k-space

for each r_x, r_y, r_z

$$C(R) = \int d^3r \left(\int \frac{d^3k}{(2\pi)^3} \cdot e^{-i\vec{k}\vec{r}} \delta_k \right)\left(\int \frac{d^3k'}{(2\pi)^3} e^{-i\vec{k}'\cdot(\vec{r}+\vec{R})} \delta_{k'} \right)$$

$$= \int \frac{d^3k}{(2\pi)^3} \cdot \int \frac{d^3k'}{(2\pi)^3} \cdot \underbrace{\int d^3r\, e^{\overline{-i(\vec{k}+\vec{k}')\cdot\vec{r}}}}_{\text{only } \vec{r} \text{ dependence}} \underbrace{\delta_k \delta_{k'} \cdot e^{\overline{-ik'R}}}_{\text{indpt of } \vec{r}}$$

3-D δ_D $\to$ $\overbrace{\int d^3r\, e^{-i(\vec{k}+\vec{k}')\vec{r}}}\ \underbrace{=}_{*} (2\pi)^3 \delta_D^3(\vec{k}+\vec{k}')$

kill $\frac{d^3k'}{(2\pi)^3}$ integral with $\delta_D^3(\vec{k}+\vec{k}')$ so $\vec{k}' \to -\vec{k}$

$k+k'=0$

$$C(\vec{R}) = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{d^3k'}{(2\pi)^3} \cdot (2\pi)^3 \delta_D^3(\vec{k}+\vec{k}')\ \delta_k \cdot \delta_{k'}\ e^{-i\vec{k}'\cdot\vec{R}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \underbrace{\delta_k\, \delta_{-k}}_{\to |\delta_k|^2} e^{i\vec{k}\vec{R}}$$

$*$ — $\delta(\vec{r},t) \in \mathbb{R}$

for any real $f(x)$

$\tilde{f}_k = \int_{-\infty}^{\infty} dk\, e^{ikx} f(x)$

$\longrightarrow f(-k) = f^*(k)$

$$C(\vec{R}) = \int \frac{d^3k}{(2\pi)^3} |\delta_k|^2 e^{i\vec{k}\vec{R}}$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{R}} |\delta_k|^2$$

$P(k) = |\delta_k|^2$ power spectrum

Derivatives of FT

$$f(x) = \int \frac{dk}{(2\pi)} e^{-ikx} \tilde{f}_k$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int \frac{dk}{(2\pi)} e^{-ikx} \tilde{f}_k$$

$$= \int \frac{dk}{2\pi} e^{-ikx} (-ik \tilde{f}_k)$$

$$\frac{\partial^2 f}{\partial x^2} = \int \frac{dk}{2\pi} e^{-ikx} (-k^2 \tilde{f}_k)$$

Full Equation of Motion for $\delta(\vec{r},t)$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2} \nabla^2 \delta \underbrace{- \frac{3}{2} \Omega_m H^2 \delta}_{-4\pi G \bar{\rho}_m \delta} = 0$$

simplify: no expansion, drop $+2H\dot{\delta}$, $a = 1$

$$\ddot{\delta} - c_s^2 \nabla^2 \delta - 4\pi G \bar{\rho}_m \delta = 0$$

$$\delta = \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m}$$

$$\rho_m = \rho_b + \underbrace{\rho_{DM}}_{\sim 5 \rho_b}$$

write $\delta(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}} \delta_k(t)$

$$\ddot{\delta} = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{r}} \cdot \ddot{\delta}_k$$

$$\nabla^2 \delta = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{r}} (-k^2) \delta_k$$

$$= \int \frac{d^3k}{(2\pi)^3} \Big[\underbrace{\ddot{\delta}_k + c_s^2 k^2 \delta_k - 4\pi G \bar{\rho}_m \delta_k}_{\to \text{ need } = 0} \Big]$$

factorize modes $\to$ indpt

$$\ddot{\delta}_k + \delta_k (c_s^2 k^2 - 4\pi G \bar{\rho}_m) = 0$$

seek solutions $\delta_k(t) = A e^{\pm i \omega k t}$

$$-\omega^2 (A e^{\pm i\omega k t}) + (A e^{\pm i\omega k t})(c_s^2 k^2 - 4\pi G \bar{\rho}_m) = 0$$

$$\rightarrow \omega^2 = c_s^2 k^2 - 4\pi G \bar{\rho}_m$$

earlier: $t_{collapse} \sim (G\bar{\rho}_m)^{-1/2}$ grav-only

$c_s^2 = \frac{dP}{d\rho}$ time for sound wave $t_{press} \sim R/c_s$

$t_{coll} \sim t_{press}$

$R = \lambda_J$ (Jeans length)

$$\frac{\lambda_J}{c_s} = \frac{1}{(G\bar{\rho}_m)^{1/2}} \rightarrow \lambda_J \sim \frac{c_s}{(G\bar{\rho}_m)^{1/2}}$$

$$\delta_k(t) = A\, e^{\pm i\left(\sqrt{c_s^2 k^2 - 4\pi G\bar{\rho}_m}\right) t}$$

$\omega(k_J) = 0$

$\hookrightarrow \lambda_J = c_s\left(\frac{\pi}{G\bar{\rho}_m}\right)^2$

if $c_s^2 k^2 \gg 4\pi G \rho_m$

$\rightarrow \delta_k \sim e^{\pm i c_s k t}$ sin/cos wave

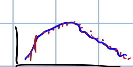
if $c_s^2 k^2 \ll 4\pi G \rho_m$

$\rightarrow \delta_k \sim e^{\pm (4\pi G \rho_m)^{1/2} t}$ decay/growth

$\sim e^{\pm t/t_{collapse}}$

$$C(R) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\vec{R}} \underbrace{|\delta_k|^2}_{\equiv P(k)}$$ power spectrum

data power spectrum agrees if $\rho_{DM} \cong 5\rho_m$

small physical & large k → non-linear, need numerical methods 

No DM: dominated by pressure