Numerical Magnetohydrodynamics

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Outline

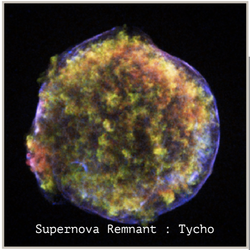
Computational Astrophysics : Area of Application.

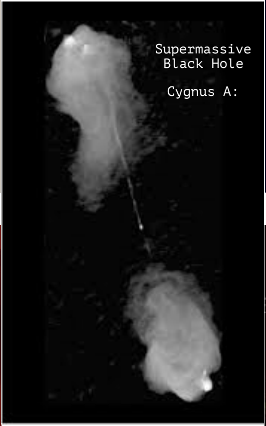
Computational Astrophysics opens a new window to perceive the heavens. Provides a string platform to describe the interiors of stars and planets, exterior phenomena such as discs, winds and jets, and also the interstellar medium, the intergalactic medium and cosmology etc. In continuum sense of treating astrophysical sources as fluids and further applying conservative equations.

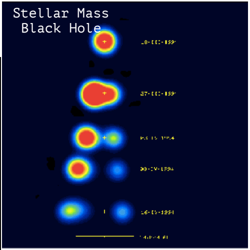
A fluid description is not applicable -

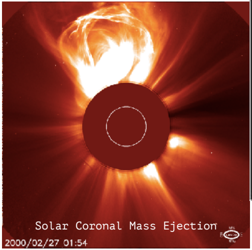
- in regions that are solidified, such as the rocky or icy cores of giant planets (under certain conditions)
- the crusts of neutron stars
- in very tenuous regions where the medium is not sufficiently collisional.

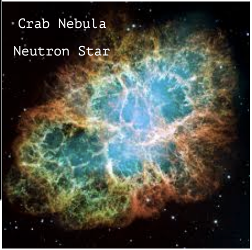
Astrophysical Flows











Computational Astrophysics : Area of Application.

Important Areas of applications include -

- Instabilities in astrophysical fluids
- Convection in stars
- Differential rotation and meridional flows in stars
- Stellar oscillations
- Astrophysical dynamos
- Magnetospheres of stars, planets and black holes
- Interacting binary stars and Roche-lobe overflow
- Tidal disruption and stellar collisions
- Supernovae
- Planetary Nebulae
- Jets and winds from stars and discs
- Star formation and the physics of the interstellar medium
- Astrophysical discs
- Other accretion flows (Bondi, Bondi–Hoyle, etc.)
- Processes related to planet formation and planet–disc interactions
- Planetary atmospheric dynamics
- Galaxy clusters and the physics of the intergalactic medium
- Cosmology and structure formation

Flavors of Astrophysical Fluid Dynamics

- *Basic Model* - A compressible, inviscid fluid in Newtonian (*non-relativistic*) framework → **Hydrodynamics**
- *Thermodynamics* - Treating fluids as either isothermal, adiabatic or including radiative process in various levels of details → **Equation of state**
- *Magnetic fields* - Including the dynamical effects of magnetic fields assuming infinite conductivity → **Ideal Magneto-hydrodynamics**
- *Dissipation Effects* - Include non-ideal effects due to viscosity, magnetic resistivity , Hall effect, Ambipolar diffusion etc.
- *Relativity* - Extend the basic model to incorporate effects due to special and general relativity.

Flavors of Astrophysical Fluid Dynamics

- **HD** - Hydrodynamics (Ideal and Non-Ideal)
- **RaHD** - Radiation Hydrodynamics *(RHD)*
- **SRHD** - Special Relativistic Hydrodynamics *(RHD)*
- **GRHD** - General Relativistic Hydrodynamics

- **MHD** - Magneto-Hydrodynamics (Ideal and Non-Ideal)
- **RaMHD** - Radiation Magneto-Hydrodynamics *(RMHD)*
- **SRMHD** - Special Relativistic Magneto-Hydrodynamics *(RMHD)*
- **GRMHD** - General Relativistic Magneto-Hydrodynamics

- **GRRaMHD** - General Relativistic Radiation Magneto-hydrodynamics.

Concept of *Fluid Element*

Small size ...

The size of the fluid element, $l_{\rm fe}$, should be smaller than a scale length for change of any relevant fluid variable q -

$$l_{\rm fe} \ll \frac{q}{|\nabla q|} \tag{1}$$

... yet large enough ...

But at the same time it should be large enough to contain a sufficient number of particles so as to ignore noise due to finite number of particles (*discreteness noise*). Thus for a system with n as the number of particles per unit volume, we should have

$$n l_{\rm fe}^3 \gg 1 \tag{2}$$

... to be *collisional*!

The size of fluid element should be large enough so that the constituent particles *know* about local conditions through collisions -

$$l_{\rm fe} \gg \lambda = \frac{1}{n\sigma} \tag{3}$$

Validity of Fluid Approach

Collisions and Fluid Approach

The equations that govern the dynamics of fluids are essentially derived from micro-physical considerations. The essential idea is that if particles inside a fluid element interact with each other (not necessarily via physical collisions), then they will attain a distribution of particle speed that maximizes the entropy of the system at that temperature. This allows us to define fluid quantities like density, pressure and derive a relation between then in form of Equation of state.
In some cases, in-spite of frequent collisions (i.e., $t_{\rm coll} \ll T_{\rm scale}$), small deviations to the distribution function of particles can arise. These small deviations can be well accounted for by including appropriate non-ideal effects like viscosity, heat conduction, resistivity etc.

Fluid Approach Fails

Cases where the mean flight time of microscopic particles, $< \tau >$ is comparable to characteristic time scale i.e., $T_{\rm scale}$, the fluid approach is no longer valid. Alternatively, in astrophysical systems where the *mean free path*, $\lambda = \frac{1}{n\sigma}$ is comparable to characteristic length scale, $L_{\rm scale}$ of the system, the fluid equations can not be applied.

Validity of Fluid Approach : Exercise

Astrophysical System	$\rho,\ n$	T	L_{scale}	$\lambda^{\dagger}$
Core of Sun-like star	$10^{2}\mathrm{gcm}^{-3}$	$10^{7}\ K$	$\approx 0.05 R_{\odot}$	2×10^{-8} c
Solar Corona	$10^{-15}\mathrm{gcm}^{-3}$	$10^{6}\ K$	$\sim 10 Mm$	?
ISM-Molecular clouds	$10^{3}\mathrm{cm}^{-3}$	$10\ K$	80 pc	?
ISM-Ionized Medium	$10^{-3}\mathrm{cm}^{-3}$	$10^{6}\ K$	1000-3000 pc	~ 3 pc

† NOTE : The *Columb cross section* for collisions, $\sigma \approx 10^{-4}(T/K)^{-2}$ cm^2 and mean free path $\lambda = \frac{1}{n\sigma}$.

Multi-fluid, Hybrid, Kinetic Approach

In cases where the basic single-fluid approach fails, we can adopt more complicated *multi-fluid* or hybrid models which allows us to treat constituent particles separately. For example, in solar corona we can treat ions and electrons separately and study their dynamics along with interactions among them.
The most consistent approach is the Kinetic approach, which really solves the *Boltzmann Equation* from first principle, however they can not be applied to study very large systems.

Fluid Variables and Derivatives

Symbols and Meanings

Cartesian co-ordinate

$\mathbf{x} = x\hat{i} + y\hat{j} + z\hat{k}$ and time t.

Fluid Variable	Symbol
Velocity	$\mathbf{v}(\mathbf{x}, t)$
Density	$\rho(\mathbf{x}, t)$
Pressure	$P(\mathbf{x}, t)$
Magnetic Fields	$\mathbf{B}(\mathbf{x}, t)$
Specific Volume	$1/\rho$
Temperature	$\propto P/\rho$
Current Density	$\nabla \times \mathbf{B}$

Lagrangian v/s Eulerian

Eulerian viewpoint - Consider the variation of properties of the fluid at a fixed point in space. (i.e., attached to the inertial co-ordinate system), time derivative and any quantity $\mathcal{Q}$ is given by -

$$\frac{\partial \mathcal{Q}}{\partial t}$$

Lagrangian viewpoint - Consider the variation of properties of the fluid at a point that moves with the fluid at velocity $\mathbf{v}(\mathbf{x}, t)$, Lagranian time derivative of quantity $\mathcal{Q}$ is given by -

$$\frac{D\mathcal{Q}}{Dt} = \frac{\partial \mathcal{Q}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{Q}$$

Vector Calculus

For any variable denoted by $\mathcal{Q}(\mathbf{x}, t) \equiv \mathcal{Q}(x, y, z, t)$, its partial derivatives are written as -

$$\mathcal{Q}_t \equiv \frac{\partial \mathcal{Q}}{\partial t}, \mathcal{Q}_x \equiv \frac{\partial \mathcal{Q}}{\partial x}, \mathcal{Q}_y \equiv \frac{\partial \mathcal{Q}}{\partial y}, \mathcal{Q}_z \equiv \frac{\partial \mathcal{Q}}{\partial z}$$

Given a scalar quantity ϕ that depends on spatial co-ordinates x, y and z, the gradient operator ∇ as applied to scalar ϕ is a vector given by -

$$\mathrm{grad}\phi \equiv \nabla\phi \equiv (\phi_{\mathrm{x}}, \phi_{\mathrm{y}}, \phi_{\mathrm{z}}) \equiv \left(\frac{\partial \phi}{\partial \mathrm{x}}, \frac{\partial \phi}{\partial \mathrm{y}}, \frac{\partial \phi}{\partial \mathrm{z}}\right)$$

The *divergence operator* applies to any vector **A** results in a scalar quantity -

$$\mathrm{div}\,\mathbf{A} \equiv \nabla \cdot \mathbf{A} \equiv \frac{\partial \mathrm{a}_1}{\partial \mathrm{x}} + \frac{\partial \mathrm{a}_2}{\partial \mathrm{y}} + \frac{\partial \mathrm{a}_3}{\partial \mathrm{z}}$$

Vector Products

The *dot product* of two vectors $\mathbf{A} = (a_1, a_2, a_3)$ and $\mathbf{B} = (b_1, b_2, b_3)$ is given by -

$$\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

One can also define *outer* product of the above two vectors as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

and in terms of index notation the above equation can be written as -

$$(\mathbf{A} \otimes \mathbf{B})_{ij} = a_i b_j$$

Hydrodynamics + Maxwell Equations

Conservative Form Ideal HD

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + P\mathbf{I}) &= 0 \\
\frac{\partial E}{\partial t} + \nabla \cdot (E + P)\mathbf{v} &= 0
\end{aligned}$$

where, E is total energy $\rightarrow$ sum of kinetic and internal energies and **I** is the identity matrix and P is a scalar pressure.

Maxwell Equation in Vacuum [Gaussian Units]

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi \mathbf{Q} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \frac{1}{c}\left(4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)
\end{aligned}$$

where, **J** is current density and **Q** is charge density.

MHD Approximation - I

- From the first principle, the impact of electric field **E** and magnetic field **B** on a collection of charged particles is governed by *Boltzmann Equation with Lorentz Force*.
- We assume collisional limit and single fluid approximation (quasi-neutral, no distinction between electrons and ions) for deriving the standard MHD equations.
- Assume the system in plasma rest frame has infinite conductivity with finite current density (Ideal approximation) : $\mathbf{J}' = \sigma \mathbf{E}' \rightarrow \mathbf{E}' \approx 0$
- Transforming to Lab frame we have,

$$\begin{aligned} \mathbf{E}' &= \Gamma\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right) = 0 \\ \mathbf{E} &= -\frac{1}{c}\mathbf{v} \times \mathbf{B} \end{aligned}$$

- Under this MHD approximation, the ideal induction equation is given as -

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = \nabla \cdot (\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B})$$

MHD Approximation - II

Consequences of infinite conductivity –

- Neglect Electric component of Maxwell stress as its $\mathcal{O}(v^2/c^2)$ smaller than its magnetic counterpart
- Ratio of Displacement current to curl of magnetic field is also $\mathcal{O}(vV/c^2)$ so can be neglected :

$$\mathbf{J} = \frac{c}{4\pi}(\nabla \times \mathbf{B})$$

- The Lorentz force that needs to be incorporated into the conservation of momentum Equation becomes

$$\mathbf{F}_L = \frac{1}{c}\mathbf{J} \times \mathbf{B} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} = \left(\frac{\mathbf{B} \cdot \nabla\mathbf{B}}{4\pi} - \nabla\left(\frac{B^2}{8\pi}\right)\right)$$

- For infinite conductivity, the plasma is *frozen* on the magnetic fields i.e., as plasma moves, magnetic fields are dragged along and Magnetic flux $\Phi_B = \oint \mathbf{B} \cdot \hat{n} dA$ is always conserved.

Set of MHD Equations.

For completeness, here is the list of equations that one should solve to study fluid dynamical behavior in presence of magnetic fields. These are set of **Ideal** (infinite conductivity) MHD equations combines hydrodynamics with Maxwell Equations in vacuum ($\mu_0 = \epsilon_0 = c = 1; \mathbf{B} \rightarrow \mathbf{B}'/\sqrt{4\pi}$)

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \otimes \mathbf{v} - \mathbf{B} \otimes \mathbf{B} + \left(P + \frac{\mathbf{B} \cdot \mathbf{B}}{2} \right) \mathbf{I} \right) &= 0 \\
\frac{\partial E}{\partial t} + \nabla \cdot \left[(E + P + \frac{B^2}{2}) \mathbf{v} - \mathbf{B}(\mathbf{v} \cdot \mathbf{B}) \right] &= 0 \\
\frac{\partial B}{\partial t} + \nabla \cdot [\mathbf{v} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{v}] &= 0 \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}
$$

where total energy $E = \rho e + \frac{\rho \mathbf{v} \cdot \mathbf{v}}{2} + \frac{\mathbf{B} \cdot \mathbf{B}}{2}$ and the induction equation is given as -

$$\frac{\partial B}{\partial t} + \nabla \times \mathcal{E} = 0$$

where the electric field $\mathcal{E} = -\mathbf{v} \times \mathbf{B}$ also we assume no free charge density i.e., $\nabla \cdot \mathcal{E} = 0$. The current density $\mathbf{J} = \nabla \times \mathbf{B}$ (no displacement current – very negligible)

Equation of state (EoS)

Types of EoS

This is the *closure* relation to set of conservation laws. A system in thermodynamic equilibrium can be completely described by the basic thermodynamic variables pressure P and density ρ.

Ideal Gases

- **Thermally Ideal gas** - $P = \rho R T$, where R is the *gas constant*. The ratio of specific heats defined as $\gamma = c_p/c_v$ can in general be a function of temperature, i.e., $\gamma(T)$ as the specific heats can in general be a function of temperature particularly for poly-atomic gases.
- **Calorically Ideal gas** - More restrictive - $e = \frac{P}{(\gamma - 1)\rho}$, where c_v, c_p are constants and therefore also the adiabatic index γ is independent of temperature.

Sound Speed

$$c_0 = \sqrt{\left(\frac{\gamma P}{\rho}\right)}$$

MHD waves

When we perturb the Equations of MHD using linear approximation we obtain various waves collectively called as MHD waves :

- **Alfvén Wave** : The are transverse and non-compressive waves with wave speed (group velocity) given as

$$c_A = \frac{B}{\sqrt{4\pi\rho}}$$

  and its direction along the magnetic field **B**.
- **Magneto-Acoustic waves** : They are of two types : slow and fast and are compressive in nature $\rightarrow$ potential to form shocks. Their speeds are given as

$$\begin{aligned} c_{ms} &= \frac{1}{2}(c_0^2 + c_A^2) - \frac{1}{2}\sqrt{(c_0^2 + c_A^2)^2 - 4c_0^2 c_A^2 \cos^2(\Psi)} \\ c_{mf} &= \frac{1}{2}(c_0^2 + c_A^2) + \frac{1}{2}\sqrt{(c_0^2 + c_A^2)^2 - 4c_0^2 c_A^2 \cos^2(\Psi)} \end{aligned}$$

  where, Ψ is the angle between the direction of propagation and magnetic field **B**.

Ohm's Law Including Resistive and Hall MHD terms

$$\mathbf{cE} = -\mathbf{v} \times \mathbf{B} + \frac{\eta}{c}\mathbf{J} + \left(\frac{\mathbf{J}}{ne} \times \mathbf{B}\right) \quad (\mathbf{J} = c\nabla \times \mathbf{B})$$

Terms on the RHS :

- Convective Term
- Resistive Term (η denotes the magnetic resistivity)
- Hall Term

Ideal MHD only has the first term.

Including Source Terms

Viscous, Resistive Magneto-HydroDynamics with Gravity, Thermal Conduction & Chemical species as advective scalars.

$$\boxed{\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}}$$

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ E \\ \mathbf{B} \\ \rho \mathcal{X}_i \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 0 \\ \rho \mathbf{g} + \nabla \cdot \Pi \\ \rho \mathbf{v} \cdot \mathbf{g} - \nabla \cdot [(\eta \cdot \mathbf{J}) \times \mathbf{B}] + \nabla \cdot (\mathbf{v} \cdot \Pi) + \nabla \cdot \mathcal{F}_c \\ -\nabla \times (\eta \cdot \mathbf{J}) \\ \mathcal{R}_{f,i} - \mathcal{R}_{d,i} \end{pmatrix}$$

$$\mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v}\mathbf{v} + I p_g + I\mathbf{B}^2/2 - \mathbf{B}\mathbf{B} \\ (E + p_g + \mathbf{B}^2/2)\mathbf{v} - \mathbf{B}(\mathbf{v} \cdot \mathbf{B}) \\ \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v} \\ \rho \mathbf{v} \mathcal{X}_i \end{pmatrix}^T$$

$$E = \rho\epsilon + \frac{1}{2}\rho \mathbf{v}^2 + B^2 \qquad \rho\epsilon = \frac{p_g}{\gamma - 1}$$

Euler Equations - I

A set of non-linear hyperbolic conservation laws that govern the dynamics of a compressible material, such as gases or liquids at high pressure, for which the effects of body forces, viscous stress and heat flux are neglected.

Two types of variables -

- *Primitive Variables* - ρ, P, $\mathbf{v} \equiv u\hat{i} + v\hat{j} + w\hat{k}$.
- *Conservative Variables* - ρ, ρu, ρv, ρw, E

$$
\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y + (\rho w)_z &= 0 \\
(\rho u)_t + (\rho u^2 + P)_x + (\rho uv)_y + (\rho uw)_z &= 0 \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + P)_y + (\rho vw)_z &= 0 \\
(\rho w)_t + (\rho uw)_x + (\rho vw)_y + (\rho w^2 + P)_z &= 0 \\
E_t + [u(E+P)]_x + [v(E+P)]_y + (w[E+P])_z &= 0
\end{aligned}
$$

The total energy E per unit volume is sum of *kinetic* and *internal energy* ρe.

$$
E = \rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} + e\right) \equiv= \rho\left(\frac{1}{2}(u^2+v^2+w^2) + e\right)
$$

Euler Equations - II

The above conservation laws can be written in a very compact from by defining a column vector $\mathbf{U}$ of conservative variables and flux vectors along three directions as a function of conserved variable vector $\mathbf{U}$: $\mathbf{F}(\mathbf{U})$, $\mathbf{G}(\mathbf{U})$ and $\mathbf{H}(\mathbf{U})$ respectively -

$$\boxed{\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x + (\mathbf{G}(\mathbf{U}))_y + (\mathbf{H}(\mathbf{U}))_z = 0} \tag{4}$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}; \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ \rho uw \\ u(E+P) \end{bmatrix}; \mathbf{G}(\mathbf{U}) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + P \\ \rho vw \\ v(E+P) \end{bmatrix}; \mathbf{H}(\mathbf{U}) = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + P \\ w(E+P) \end{bmatrix}$$

Any set of partial differential equations written in form of Eq. 4 is called a system of conservation laws

Scalar Equation in 1D

We will for this course only deal with set of 1D equation for any general conservation law -

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \tag{5}$$

Lets assume for simplicity that $\mathbf{F}(\mathbf{U})$ is a linear function of $\mathbf{U}$ i.e., $\mathbf{F}(\mathbf{U}) = A\mathbf{U}$, where A is a constant and $\mathbf{U} = \{q_1, q_2, ...q_n\}^T$ are the conservative variables. So for say single variable q_1, we have the following equation,

$$\frac{\partial q_1}{\partial t} + a\frac{\partial q_1}{\partial x} = 0 \tag{6}$$

such an equation is called a scalar equation (or advection) equation moving with constant advection speed a. For example mass continuity equation with constant speed $\rightarrow q_1(x,t) = \rho(x,t); u(x,t) = a$

Solution Approach

Grid-based Approach

The numerical domain under consideration is divided into discrete cells on which the above equations are approximated and evolved using either FV, FD or FE methods.
Example Codes : PLUTO, Athena, FLASH, Enzo, RAMSES and many more.
Further, codes like AREPO uses a fully dynamic unstructured mesh based on Voronoi tessellations.

Particle-Based

Traditionally particle-based approach were used to develop N-body codes to understand the collapse in presence of gravity and associated motion.
Using particles for description of fluid quantities via the use of weighted kernel gave rise to the development of Smoothed Particle Hydro-dynamics (SPH).
Example Codes : GADGET, NDSPMHD code

Numerical framework for PDEs

1. **Finite Difference Method (FDM)** - Uses the differential form of Conservation Equations.
   PRO : Most simplest to code. It is based on Taylor expansion of differential equations.
   CON : Very tedious be apply in more complex geometries in multiple dimensions particularly in presence of shocks.
2. **Finite Volume Method (FVM)** - Uses the weak integral form of the Conservation Equation.
   PRO : Can handle discontinuities well and be easily extended to multiple dimensions.
   CON : Slightly more involved coding.
3. **Finite Element Method (FEM)** - Similar to FVM uses the integral form, the difference lies definition of the integral form particularly in way weights are defined.

Time Stepping

Consider a ODE -

$$\frac{dq(t)}{dt} = F(t, q(t))$$

1. **Explicit** -
   Conditionally stable, does not involve any matrix manipulation. The next time solution depends only on previous time values. (*Forward Euler*)

$$\frac{q_{n+1} - q_n}{\Delta t} = F(t_n, q_n)$$

2. **Implicit** -
   Unconditionally stable, however involves matrix manipulation. The solution at the next time depends on both the previous time and next time value. (*Backward Euler*)

$$\frac{q_{n+1} - q_n}{\Delta t} = F(t_{n+1}, q_{n+1})$$

Hyperbolic PDEs

In general $\mathbf{F}(\mathbf{U})$ may be a general function of $\mathbf{U}$.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x} = 0$$

where the *Jacobian* matrix $\mathbf{A} = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}$
1D Isothermal Hydrodynamics Equation

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \end{bmatrix} = \begin{bmatrix} \rho \\ m \end{bmatrix} ; \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + P \end{bmatrix} = \begin{bmatrix} m \\ \frac{m^2}{\rho} + \rho c_s^2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ c_s^2 - \frac{m^2}{\rho^2} & 2\frac{m}{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c_s^2 - u^2 & 2u \end{bmatrix} \tag{7}$$

Eigenvalues : $\lambda_1 = u - c_s,\ \lambda_2 = u + c_s$

Godunov based Methods - I

The scalar Equation in integral form can be written as -

$$\frac{d}{dt}\int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{U}(x,t)dx = \mathbf{F}(\mathbf{U}(x_{i-1/2},t)) - \mathbf{F}(\mathbf{U}(x_{i+1/2},t))$$

The above equation can be re-written as -

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x}(\mathbf{F}_{i-1/2} - \mathbf{F}_{i+1/2})$$

where,

$$\mathbf{F}_{i\pm1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}[\mathbf{U}(x_{i\pm1/2},t)]dt$$

and

$$\mathbf{U}_i^n = \frac{1}{\Delta x}\int_{x-1/2}^{x+1/2} \mathbf{U}(x,t^n)dt$$

Godunov based Methods - II

Standard Godunov scheme is first order accurate as it assumes piece-wise constant values of $\mathbf{U}_i$. For a simple advection problem $\mathbf{F} = \lambda\mathbf{U}$ with λ being a constant. It can be easily shown that the scheme now becomes

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \mathcal{C}(\mathbf{U}_i - \mathbf{U}_{i-1}) \text{ if}\lambda > 0$$

and

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \mathcal{C}(\mathbf{U}_{i+1} - \mathbf{U}_i) \text{ if}\lambda < 0$$

where $\mathcal{C} = \frac{\lambda\Delta t}{\Delta x}$ is called the Courant factor.
To improve the accuracy, the values of $\mathbf{U}_i^n$ are interpolated via linear (second order) or parabolic (higher order) functions. To avoid spurious oscillations typically seen in higher order schemes, one should ensure Total Variation Diminishing formulation - $TV(\mathbf{U}^{n+1}) \leq TV(\mathbf{U}^n)$, where

$$TV(Q^n) = \Sigma_{i=-\infty}^{i=\infty}|Q_i^n - Q_{i-1}^n|$$

Under the hood I

Reconstruction from cell averages $\mathbf{U}_i$

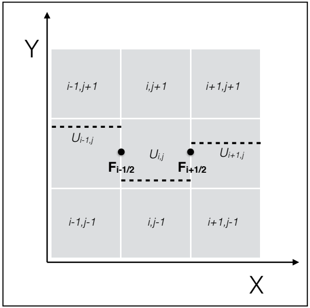
- Computing conservative states at the interface : $\mathbf{U}^{-}_{i+\frac{1}{2}}, \mathbf{U}^{+}_{i-\frac{1}{2}}$
- Options : Flat, Piecewise Linear (PLM), Piecewise Parabolic (PPM), *WENO, MP5*

Mathematical Form of the Conservative Equation :

$$\boxed{\mathbf{U_i}^{n+1} = \mathbf{U_i}^{n} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}\right)}$$

Interface states → Fluxes

- By solving the *Riemann problems* in an approximate manner : Lax-Friedrichs, HLL, HLLC, Roe, HLLD(for MHD module).
- Additionally use of *slope limiters* [i.e., to limit primitive states] in regions of shocks for TVD scheme.



Explicit Time Stepping

- Options : Euler, Runge-Kutta II, Runge-Kutta IV, Primitive Hancock.

Under the hood II

Ensuring $\nabla \cdot \mathbf{B} = 0$

- Physically no magnetic monopoles ... however treatment required for numerical schemes
- Options : Powell 8 wave method, Divergence cleaning , Constraint Transport

Dissipative Terms

- Parabolic Terms like Thermal Conduction, Magnetic Resistivity & Viscosity treated in conservative manner
- Handled explicitly using techniques like *Super Time Stepping, Runge Kutta Legendre*

Explicit Source Terms

- Source terms : Gravity, Chemistry & User-defined source functions.
- Coupling with conservative updates using the 2nd order accurate Strang Operator Split

Ideal MHD Equations in 1D

For a 1D (along $\hat{x}$) the ideal MHD equations can be represented as -

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \\ B_y \\ B_z \end{bmatrix} ; \mathbf{F(U)} = \begin{bmatrix} \rho u \\ \rho u^2 + P - B_x^2 \\ \rho uv - B_x B_y \\ \rho uw - B_x B_z \\ u(E+P) - B_x(B_x u + B_y v + B_z w) \\ B_y u - B_x v \\ B_z u - B_x w \end{bmatrix}$$

where $\mathbf{v} = \{u, v, w\}$ and $\mathbf{B} = \{B_x, B_y, B_z\}$ and total energy $E = \frac{P}{(\gamma-1)} + \frac{\rho \mathbf{v}\cdot\mathbf{v}}{2} + \frac{\mathbf{B}\cdot\mathbf{B}}{2}$, the solenoidal condition requires B_x is constant.

Eigenspeeds in Ideal MHD

Compute the Jacobian : $\frac{\partial \mathbf{F(U)}}{\partial \mathbf{U}}$.
It has 7 real eigenvalues, one for each wave :

- 2 fast magneto-sonic waves : $\lambda_1 = u - c_{mf}$, $\lambda_7 = u + c_{mf}$
- 2 Alfven waves : $\lambda_2 = u - c_A$, $\lambda_6 = u + c_A$
- 2 slow magneto-sonic waves : $\lambda_3 = u - c_{ms}$, $\lambda_5 = u + c_{ms}$
- 1 entropy wave: $\lambda_4 = u$

Fast magnetosonic waves are longitudinal waves with variations in pressure and density (correlated with magnetic field)
Slow magnetosonic waves are longitudinal waves with variations in pressure and density (anti-correlated with magnetic field)
On comparing,$\lambda_7 \geq \lambda_6 \geq \lambda_5 \geq \lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1$
In some cases, wavespeeds can be equal and therefore MHD system is not strictly hyperbolic and can admit *compound* waves.

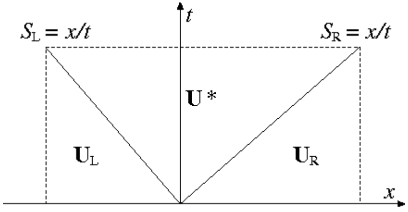
Lessons from Godunov Method

- Using interpolation methods, reconstruct conservative variables at the interfaces from cell averages $\mathbf{U}_i$.
- Get the estimate of Flux at the interface from the Left and right states using approximate Riemann solution.
- The solution of equations in conservative form allows to treat discontinuities/shocks using Rankine-Huoniot Jump conditions as $S[\mathbf{U}] = [\mathbf{F}(\mathbf{U})]$, where S denotes the speed of discontinuities and [.] denotes jumps across the discontinuity.
- In case of contact discontinuity (moving with speed λ_4), we have

$$
\begin{aligned}
[B_y] = [B_z] = [P] = [v] = [w] &= 0; B_x \neq 0 \\
\left[P + \frac{B_y^2 + B_z^2}{2}\right] &= 0; B_x = 0
\end{aligned}
$$

Reimann Solver - I : HLL

Assumes an average intermediate state between the fastest (λ_7) and slowest (λ_1) waves.



$$
\begin{aligned}
\mathbf{U}^* &= \frac{S_R\mathbf{U}_R - S_L\mathbf{U}_L - \mathbf{F}_R + \mathbf{F}_L}{S_R - S_L} \\
\mathbf{F}^* &= \frac{S_R\mathbf{F}_L - S_L\mathbf{F}_R + S_R S_L(\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L}
\end{aligned}
$$

$$
\mathbf{F}_{HLL} = \begin{cases} \mathbf{F}_L, & \text{if } S_L > 0 \\ \mathbf{F}^*, & \text{if } S_L \le 0 \le S_R \\ \mathbf{F}_R, & \text{if } S_R > 0 \end{cases}
$$

The speeds of discontinuities on the left S_L and that on right interface S_R is obtained from various estimates e.g., Davis estimate -

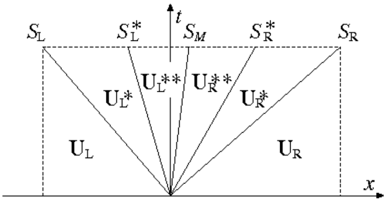
$$
\begin{aligned}
S_L &= \min\,[\lambda_1(\mathbf{U}_L), \lambda_1(\mathbf{U}_R)] \\
S_R &= \max\,[\lambda_7(\mathbf{U}_L), \lambda_7(\mathbf{U}_R)]
\end{aligned}
$$

Riemann Solver - II : HLLD

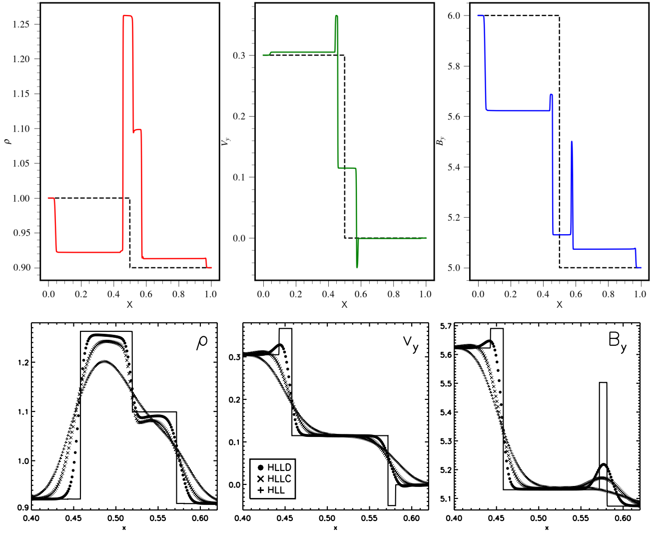
Standard HLLC solver used for Hydrodynamics have inconsistency with respect to jump conditions.
Single state HLL solver to be divided into 4 intermediate stages :
$\mathbf{U}_L^*, \mathbf{U}_L^{**}, \mathbf{U}_R^*, \mathbf{U}_R^{**}$ accounting for rotational discontinuities propagating with the Alfven waves and also the contact discontinuity propagating with entropy wave.
The wave speed S_M (from HLL average) and value of the total pressure P_T in the intermediate stages is given by

$$S_M = \frac{\rho u^*}{\rho^*} = \frac{(S_R - u_R)\rho_R u_R - (S_L - u_L)\rho_L u_L - P_{T,R} + P_{T,L}}{(S_R - u_R)\rho_R - (S_L - u_L)\rho_L}$$

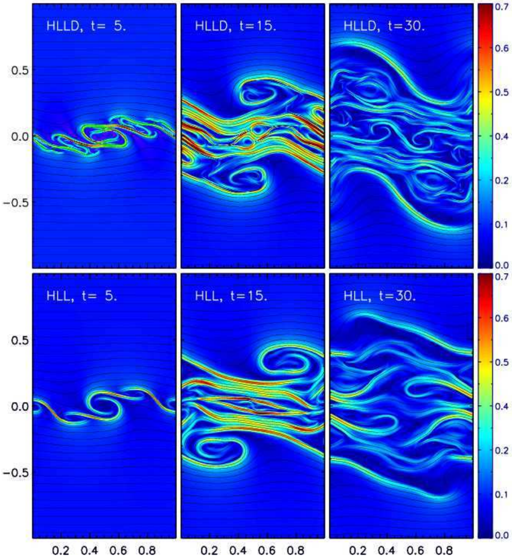
$$P_T^* = \frac{(S_R - u_R)\rho_R P_{T,L} - (S_L - u_L)\rho_L P_{T,R} + \rho_L\rho_R(S_R - u_R)(S_L - u_L)(u_R - u_L)}{(S_R - u_R)\rho_R - (S_L - u_L)\rho_L}$$



Miyoshi & Kusano; Journal of Computational Physics 208 (2005) 315-344



Mignone et. al. 2009, MNRAS



Mignone et. al. 2009, MNRAS

Divergence of B Control

The solenoidal constraint of magnetic fields from the Maxwell's Equations needs to be ensured numerically. This is unique to the MHD solvers and several techniques are available in the literature :

- Powell's Eight wave (Powell 1994)
- Hyperbolic Divergence Cleaning (Dredner 2002)
- Constraint Transport (Balsara & Spicer 1999)

Powell's Eight wave formalism

The magnetic fields have cell centred representation and the divergence of magnetic field is kept to 0 at the truncation level and **NOT** at machine accuracy.

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ E_t\\ \mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{v}\otimes\mathbf{v}+\left(P+\frac{B^2}{2}\right)\mathbf{I}-\mathbf{B}\otimes\mathbf{B}\\ \left(E+P+\frac{B^2}{2}\right)\mathbf{v}-\mathbf{B}(\mathbf{v}\cdot\mathbf{B})\\ \mathbf{v}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{v}\end{bmatrix}=-\nabla\cdot\mathbf{B}\begin{bmatrix}0\\ \mathbf{B}\\ \mathbf{v}\cdot\mathbf{B}\\ \mathbf{v}\end{bmatrix}$$

Derived the characteristics of the ideal MHD system with this source as indicated above. This results in addition wave to the already know 7 waves. The effect of the additional wave to advect away $\nabla\cdot\mathbf{B}/\rho$ with the flow.

Hyperbolic Divergence Cleaning

Divergence free constraint is enforced by solving a modified system of conservation laws, where the induction equation is coupled to generalized Lagrangian multiplier (GLM)

$$\begin{cases} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (c\mathbf{E}) + \nabla \psi = 0 \\ \frac{\partial \psi}{\partial t} + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_p^2} \psi \end{cases}$$

where, $c_h = CFL \times \Delta l_{\min}/\Delta t^n$ is the maximum speed compatible with the step size, $c_p = \sqrt{\Delta l_{\min} c_h/\alpha}$ and $\Delta l_{\min}$ is minimum length scale. The free parameter α controls the damping of monopole, it has an optimal range of $0.05 \leq \alpha \leq 0.3$.

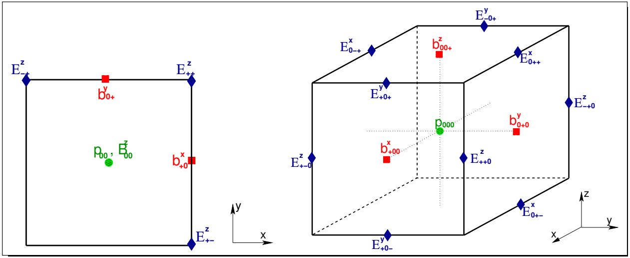
- Divergence Errors are transported to domain boundaries with maximum admissible speed and are damped at the same time.
- The magnetic field are defined at the cell centred position just as in Powell's Eight wave method, but the hyperbolic divergence has much better performance in constraining the solenoidal condition.

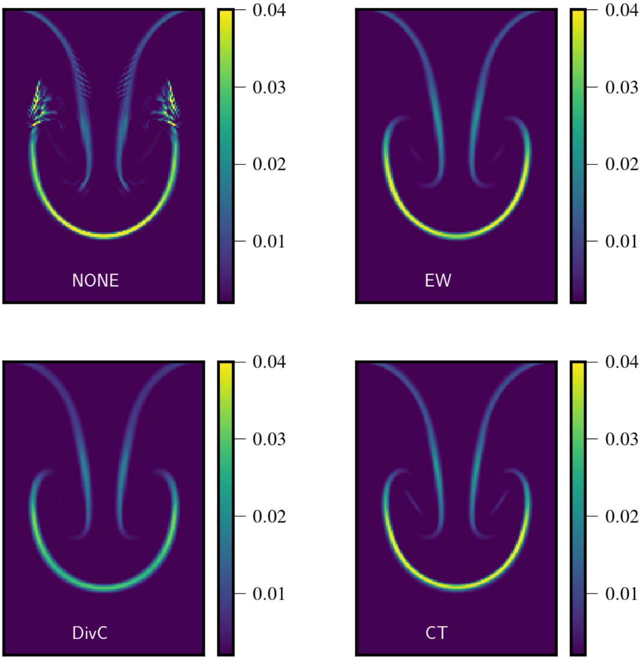
Constraint Transport

- Magnetic fields are defined at the Cell centre **B** and also at the face center **b**.
- Electric fields $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ are computed at cell edges from flux estimates of Riemann solver and appropriately averaged.
- Magnetic fields at the face centre are updated using *Stokes* theorem -

$$\int \left(\frac{\partial \mathbf{b}}{\partial t} + \nabla \times \mathbf{E} \right) \cdot d\mathbf{S}_d = 0$$

- Maintains $\nabla \cdot \mathbf{B} = 0$ upto machine accuracy.

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- *Lectures on Computational Astrophysics* by Roman Teyssier. Link : `https://www.ics.uzh.ch/~teyssier/comp_astro_lectures/`