Self-Gravity in Hydrodynamical Simiulations

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Abstract

In this lecture, we learn (1) how to evaluate the self-gravity and (2) how to include it in hydroyanical simulations. A successful method should guarantee the conservation of momentum and energy as well as that of mass. We learn such a scheme in this lecture. We learn two methods for solving the Poisson equation: Fast Fourier Transform (FFT) and multi grid iteration. The former is easy to implement and the students are expected to solve the homework using it.

1 Basic Equations

1.1 Gravity

N-body system:

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = -m_i \sum_{j \neq i} \frac{G m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) = m_i \boldsymbol{g}_i. \tag{1}$$

Continuous density distribution:

$$\boldsymbol{g}(\boldsymbol{r}) = -\int \frac{G\rho(\boldsymbol{x})}{|\boldsymbol{r} - \boldsymbol{x}|^3} (\boldsymbol{r} - \boldsymbol{x}) \, d\boldsymbol{x}. \tag{2}$$

Although Equation (2) is correct, we do not use it in our hydrodynamical equations because

1. It takes too much computation time. When the computation domain consists of N^3 cells, the computation time increases in proportion to N^6.

2. The solution tends to contain a large round off error. A contribution of a cell is often cancelled with that of a counter cell.

Use the Poisson equation,

$$\boldsymbol{\nabla}^2 \phi = 4\pi G \rho, \tag{3}$$

to obtain the gravity,

$$\boldsymbol{g} = -\boldsymbol{\nabla} \phi. \tag{4}$$

Equation (3) is analogous to

$$\nabla^2 \phi_{\rm e} = -\nabla \cdot \boldsymbol{E} = -\frac{\rho_{\rm e}}{\varepsilon_0}, \tag{5}$$

where $\phi_{\rm e}$ and $\rho_{\rm e}$ denote the elctrostatic potential and charge density, respectively. We can derive Eqution (3) by replacing the dielectric constant, $-1/\varepsilon_0$, with $-4\pi G$. This analogy works since the Newton's gravity and the Coulomb's law

$$\boldsymbol{g} = -GM\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}, \tag{6}$$

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}, \tag{7}$$

$$\tag{8}$$

are the same except for the multiplication factors.

1.2 Hydrodynamical Equations

The self-gravity does not alter the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{9}$$

where $\boldsymbol{v}$ denotes the velocity. The equation of motion is expressed as

$$\frac{D\boldsymbol{v}}{dt} = -\frac{1}{\rho}\nabla P + \boldsymbol{g}, \tag{10}$$

where D/Dt denotes the Lagrangian derivative. Combining Equations (9) and (10) we obtain the conservation of linear momentum,

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \nabla \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P\boldsymbol{I}) = \rho \boldsymbol{g}. \tag{11}$$

Equation (11) is the same as that for the external gravity. However, we have a constraint,

$$\int \rho \boldsymbol{g} dV = 0. \tag{12}$$

The self-gravity cannot accelerate the center of mass.

We use slightly different symbols for some variables. See Table 1 for the differences.

In the following we ignore the heating and cooling, i.e.,

$$T\frac{Ds}{Dt} = \frac{D\varepsilon}{Dt} + P\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{D}{Dt}\left(\varepsilon + \frac{P}{\rho}\right) - \frac{1}{\rho}\frac{DP}{dt} = 0, \tag{13}$$

	Prof. Kley July 27th	Hanawa July 29th
density	ρ	ρ
velocity	$\overrightarrow{u}$	$\boldsymbol{v}$
pressire	p	P
specific energy	ϵ	ε
specific energy plus kinetic energy	-	$E = \frac{\boldsymbol{v}^2}{2} + \epsilon$
external force (gravity)	$\overrightarrow{k}$	$\boldsymbol{g}$
direct product	$\overrightarrow{u} \otimes \overrightarrow{u}$	$\boldsymbol{vv}$

Table 1: Comparison of the symbols used in two lectures.

where T, s, and ε denote the temperature, the specific entropy, and the specific internal energy, respectively. When the gas is a perfect one having the specific heat ratio, γ, the internal energy is expressed as

$$\varepsilon = \frac{P}{(\gamma - 1)\rho}. \tag{14}$$

Then we obtain Equation of the energy conservation[1] expressed as

$$\frac{\partial}{\partial t}(\rho E) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} H) = \rho \boldsymbol{v} \cdot \boldsymbol{g}, \tag{15}$$

$$E = \frac{\boldsymbol{v}^2}{2} + \varepsilon = \frac{\boldsymbol{v}^2}{2} + \frac{P}{(\gamma - 1)\rho}, \tag{16}$$

$$H = \frac{\boldsymbol{v}^2}{2} + \frac{\gamma P}{(\gamma - 1)\rho}. \tag{17}$$

Here, the last term in the right hand side of Equation (15) denotes the gravitational energy release, which should be equated with the change in the gravitational energy. We can evaluate the self-gravity by

$$E_{\mathrm{g}} = \frac{1}{2} \int \phi \rho dV \tag{18}$$

$$= \frac{1}{8\pi G} \int \phi \boldsymbol{\nabla}^2 \phi dV \tag{19}$$

$$= -\frac{1}{8\pi G} \int (\boldsymbol{\nabla} \phi)^2 \, dV + \frac{1}{8\pi G} \int \phi \boldsymbol{\nabla} \phi d\boldsymbol{S}. \tag{20}$$

We can ignore the last surface integral either if the outer boundary is very far from the gravitating body or if the system is periodic and does not have an

[1] We can derive Equation (15) from $\frac{\partial}{\partial t}(\rho\varepsilon) + \boldsymbol{\nabla} \cdot (\rho \varepsilon \boldsymbol{u}) = - P \boldsymbol{\nabla} \cdot \boldsymbol{v}$ (Eq. 16 in the lecture of Prof. Kley). Use $P\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot (P\boldsymbol{v}) - \boldsymbol{v} \cdot \boldsymbol{\nabla P}$, mass conservation, and momentum conservation.

outer boundary. We can evaluate the change in the gravitional energy by

$$\frac{\partial E_{\mathrm{g}}}{\partial t} = -\frac{1}{4\pi G}\int \boldsymbol{g}\cdot\frac{\partial \boldsymbol{g}}{\partial t}dV + \frac{1}{8\pi G}\int \frac{\partial}{\partial t}(\phi\boldsymbol{g})\cdot d\boldsymbol{S}. \tag{21}$$

In the following we prove that the gravitational energy release is equated with the change in the gravitational energy. The difference is

$$\begin{aligned}\int \left[\rho\boldsymbol{v}\cdot\boldsymbol{g} \right. &+ \left. \frac{\partial}{\partial t}\left(\frac{\boldsymbol{g}\cdot\boldsymbol{g}}{8\pi G}\right)\right] dV = \int \boldsymbol{g}\cdot\left(\rho\boldsymbol{v} + \frac{1}{4\pi G}\frac{\partial \boldsymbol{g}}{\partial t}\right) dV \qquad (22)\\ &= \int \phi\boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{v} + \frac{1}{4\pi G}\frac{\partial \boldsymbol{g}}{\partial t}\right) dV - \int \phi\left(\rho\boldsymbol{v} + \frac{1}{4\pi G}\frac{\partial \boldsymbol{g}}{\partial t}\right)\cdot d\boldsymbol{S}.\end{aligned}$$

The volume integral vanishes since Equation (9) and another equation,

$$\boldsymbol{\nabla}\cdot\frac{\partial \boldsymbol{g}}{\partial t} = -4\pi G\frac{\partial \rho}{\partial t}, \tag{23}$$

hold.

When we have no external heating or cooling, the total energy is conserved. This is the second constraint to construct a successful scheme to simulate a self-gravitating gas.

When we solve the hydrodynamical equations in the conserved form, we obtain $(\rho, \rho\boldsymbol{v}, \rho E)$. We can obtain the primitive variables by

$$\boldsymbol{v} = \frac{(\rho\boldsymbol{v})}{\rho}, \tag{24}$$

$$P = \frac{2\rho E - (\rho\boldsymbol{v})\cdot\boldsymbol{v}}{2(\gamma-1)}. \tag{25}$$

2 How to Solve the Poisson Equation (Part 1)

$$\phi_{i,j,k} \equiv \phi(x_i,\, y_j,\, z_k), \tag{26}$$

$$\rho_{i,j,k} \equiv \rho(x_i,\, y_j,\, z_k), \tag{27}$$

$$x_i = hi, \tag{28}$$

$$y_j = hj, \tag{29}$$

$$z_k = hk. \tag{30}$$

h: cell width

Discretized Poisson Equation

$$\begin{aligned}\phi_{i+1,j,k} + \phi_{i-1,j,k} + \phi_{i,j+1,k} + \phi_{i,j-1,k} &+ \phi_{i,j,k+1} + \phi_{i,j,k-1}\\ &= 6\phi_{i,j,k} + 4\pi G h^2 \rho_{i,j,k}. \qquad (31)\end{aligned}$$

$$
\begin{aligned}
g_{x,i+1/2,j,k} &= -\frac{\phi_{i+1,j,k}-\phi_{i,j,k}}{h}, &(32)\\
g_{y,i,j+1/2,k} &= -\frac{\phi_{i,j+1,k}-\phi_{i,j,k}}{h}, &(33)\\
g_{z,i,j,k+1/2} &= -\frac{\phi_{i,j,k+1}-\phi_{i,j,k}}{h}. &(34)
\end{aligned}
$$

Popular methods

1. Multigrid Method [1, 2]
    - Efficient
    - Wide applicability
    - Difficult to master
2. Fast Fourier Transform (FFT)
    - Easy to use
    - Difficult to handle outer boundary except for the periodic one.

3 How to Solve the Hydrodynamical Equations

We assume that we can solve Equation (31).

Then the gravitational acceleration is expressed as

$$
\begin{aligned}
(\rho g_x)_{i,j,k} &= \frac{\rho_{i,j,k}\left(g_{x,i+1/2,j,k}+g_{x,i-1/2,j,k}\right)}{2} &(35)\\
&= -\frac{\rho_{i,j,k}\left(\phi_{i+1,j,k}-\phi_{i-1,j,k}\right)}{2h}, &(36)\\
(\rho g_y)_{i,j,k} &= \frac{\rho_{i,j,k}\left(g_{y,i,j+1/2,k}+g_{y,i,j-1/2,k}\right)}{2} &(37)\\
&= -\frac{\rho_{i,j,k}\left(\phi_{i,j+1,k}-\phi_{i,j-1,k}\right)}{2h}, &(38)\\
(\rho g_z)_{i,j,k} &= \frac{\rho_{i,j,k}\left(g_{z,i,j,k+1/2}+g_{z,i,j,k-1/2}\right)}{2} &(39)\\
&= -\frac{\rho_{i,j,k}\left(\phi_{i,j,k+1}-\phi_{i,j,k-1}\right)}{2h}, &(40)
\end{aligned}
$$

The center of mass continues to move at a constant speed if we use these acceleration.

The gravitational energy release is expressed as

$$
\begin{aligned}
(\rho \boldsymbol{v}\cdot\boldsymbol{g})_{i,j,k} &= \frac{1}{2}\Big[\left(\overline{\rho v_x}\right)_{i+1/2,j,k}\overline{g_{x,i+1/2,j,k}}+\left(\overline{\rho v_x}\right)_{i-1/2,j,k}\overline{g_{x,i-1/2,j,k}}\\
&\quad+\left(\overline{\rho v_y}\right)_{i,j+1/2,k}\overline{g_{y,i,j+1/2,k}}+\left(\overline{\rho v_y}\right)_{i,j-1/2,k}\overline{g_{y,i,j-1/2,k}}\\
&\quad+\left(\overline{\rho v_z}\right)_{i,j,k+1/2}\overline{g_{z,i,j,k+1/2}}+\left(\overline{\rho v_z}\right)_{i,j,k-1/2}\overline{g_{z,i,j,k-1/2}}\Big] \quad (41)
\end{aligned}
$$

Here, the over line denotes the time average.

$$\begin{aligned}\frac{h\left[\rho(t+\Delta t)-\rho(t)\right]}{\Delta t} &= -(\overline{\rho v_x})_{i+1/2,j,k}+(\overline{\rho v_x})_{i-1/2,j,k} \\ &\quad -(\overline{\rho v_y})_{i,j+1/2,k}+(\overline{\rho v_y})_{i,j-1/2,k} \\ &\quad -(\overline{\rho v_x})_{i-1/2,j,k}+(\overline{\rho v_z})_{i+1/2,j,k-1/2}, \end{aligned} \tag{42}$$

$$\overline{\boldsymbol{g}} = \frac{\boldsymbol{g}(t)+\boldsymbol{g}(t+\Delta t)}{2}. \tag{43}$$

This source term guarantees the conservation of total energy including the gravity.

Once the proper gravitational acceleration is given, we can obtain a solution of second order accuracy in space and time by the following procedure.

1. Compute the gravity, $\boldsymbol{g}\,(t_0)$ at $t=t_0$ from the density, $\rho(t_0)$. This step can be omitted from the second cycle, since the gravity has been computed in the previous cycle.
2. Compute the density, $\rho^*\,(t_0+\Delta t)$ and momentum density $(\rho\boldsymbol{v})^*\,(t_0+\Delta t)$ by solving the equation of continuity and store the numerical flux, $[\rho\boldsymbol{v}]\,(t_0)$.
3. Evaluate the gravity at $t=t_0+\Delta t$, $\boldsymbol{g}^*\,(t_0+\Delta t)$, from the density, $\rho^*\,(t_0+\Delta t)$.
4. Compute the internal energy and pressure, $\varepsilon^*_{i,j,k}\,(t_0+\Delta t)$ and $P^*\,(t_0+\Delta t)$ by solving Equation (15) with the gravitational energy release given by Equation (41). Here the gravity is the average of those obtained at steps 1 and 3. The numerical flux at step 2 are used in the evaluation of Equation (42) so that the total energy is conserved.
5. Update the density, $\rho\,(t_0+\Delta t)$, by using the average of numerical fluxes at $t=t_0$ and $t=t_0+\Delta t$.
6. Compute the gravity, $\boldsymbol{g}\,(t_0+\Delta t)$ at $t=t_0$ from the density $\rho\,(t_0+\Delta t)$ and store it for later use.
7. Update the momentum density, $(\rho\boldsymbol{v})\,(t_0+\Delta t)$ by using the average of numerical fluxes at $t=t_0$ and $t_0+\Delta t$ and the average of $\boldsymbol{g}\,(t_0)$ and $\boldsymbol{g}\,(t_0+\Delta t)$.
8. Update the internal energy and pressure from $t=t_0$ to $t=t_0+t_0+\Delta t$ by using the average numerical flux and gravitational energy release given by Equation (41). Here the average gravitational energy release from each cell boundary is defined as the product of average mass flux and average gravity.
9. Go back to step 1 and repeat.

Here, the symbols with asterisk denote the values of the first order accuracy in time. This scheme can be applied to the system in which the gravity is external and stationary. Step 3 can be omitted although the total energy is not conserved at step 4. The total energy conservation is guaranteed by step 8.

4 How to Solve the Poisson Equation (Part 2)

4.1 Fast Fourier Transform (FFT)

When the density distribution is expressed as

$$\rho(x, y, z) = \rho_0 \cos(k_x x) \cos(k_y y) \cos(k_z z), \tag{44}$$

the corresponding gravitational potential is expressed as

$$\phi(x, y, z) = -\frac{4\pi G \rho_0}{\bar{k}^2} \cos(k_x x) \cos(k_y y) \cos(k_z z), \tag{45}$$

$$\bar{k}^2 = {k_x}^2 + {k_y}^2 + {k_z}^2. \tag{46}$$

When the density distribution is expressed as

$$\rho_{i,j,k} = \rho_0 \cos(k_x hi) \cos(k_y hj) \cos(k_z hk), \tag{47}$$

the corresponding potential is given by

$$\phi_{i,j,k} = -\frac{4\pi G \rho_0}{\tilde{k}^2} \cos(k_x hi) \cos(k_y hj) \cos(k_z hk), \tag{48}$$

$$\tilde{k}^2 = \frac{4}{h^2} \left(\sin^2 \frac{k_x h}{2} + \sin^2 \frac{k_y h}{2} + \sin^2 \frac{k_z h}{2} \right). \tag{49}$$

Discrete Fourier Transform

$$A_k = \sum_{n=0}^{N-1} a_n \left[\cos\left(\frac{2\pi kn}{N}\right) - \sqrt{-1} \cos\left(\frac{2\pi kn}{N}\right) \right] \tag{50}$$

$$= \sum_{n=0}^{N-1} a_n \exp\left[-\sqrt{-1} \left(\frac{2\pi kn}{N}\right) \right], \tag{51}$$

$$a_n = \frac{1}{N} \sum_{k=0}^{N-1} A_k \left[\cos\left(\frac{2\pi kn}{N}\right) + \sqrt{-1} \cos\left(\frac{2\pi kn}{N}\right) \right] \tag{52}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} A_k \exp\left[\sqrt{-1} \left(\frac{2\pi kn}{N}\right) \right]. \tag{53}$$

These are analogous to the Fourier Transform,

$$A(x) = \int_{-\infty}^{\infty} \tilde{A}(k) \exp\left(\sqrt{-1} kx\right) dk. \tag{54}$$

Fast Fourier Transform algorithm enables us to compute the right hand sides of Equations (51) and (53) by an order of not N^2 but $N \log_2 N$ operations when $N = 2^\nu$.

$$\tilde{\rho}_{\ell,m,n} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \rho_{i,j,k} \exp\left[-2\pi\sqrt{-1}\frac{(i\ell + jm + kn)}{N}\right], \tag{55}$$

$$\tilde{\phi}_{\ell,m,n} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \phi_{i,j,k} \exp\left[-2\pi\sqrt{-1}\frac{(i\ell + jm + kn)}{N}\right]. \tag{56}$$

$$\tilde{\phi}_{\ell,m,n} = \begin{cases} 0 & (\ell = m = n = 0) \\ -\dfrac{4\pi G\tilde{\rho}_{i,j,k}}{\left(\tilde{k}_{\ell,m,n}\right)^2} & \text{(otherwise)} \end{cases}, \tag{57}$$

$$\left(\tilde{k}_{\ell,m,n}\right)^2 = \frac{4}{h^2}\left[\sin^2\left(\frac{\pi\ell}{N}\right) + \sin^2\left(\frac{\pi m}{N}\right) + \sin^2\left(\frac{\pi n}{N}\right)\right]. \tag{58}$$

$$\phi_{i,j,k} = \frac{1}{N^3} \sum_{\ell=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \tilde{\phi}_{\ell,m,n} \exp\left[2\pi\sqrt{-1}\frac{(i\ell + jm + kn)}{N}\right]. \tag{59}$$

You can find various packages offering FFT routines online[5]. LOC has prepared one of them at `https://github.com/astrofum/na2020`.

4.2 Multigrid Method

Jacobi iteration (Gauss Seidel) iteration:

$$\begin{aligned} \phi_{i,j,k}^{(n+1)} &= \frac{1}{6}\left(\phi_{i+1,j,k}^{(n)} + \phi_{i-1,j,k}^{(n)} + \phi_{i,j+1,k}^{(n)} + \phi_{i,j-1,k}^{(n)} + \phi_{i,j,k+1}^{(n)} + \phi_{i,j,k-1}^{(n)}\right) \\ &\quad -\frac{2}{3}\pi G h^2 \rho_{i,j,k}. \end{aligned} \tag{60}$$

We can speed up the iteration by factor 2 if we compute only $\phi_{i,j,k}^{(n+1)}$ of which $i + j + k + n$ is even. This improvement is known as red-black ordering.

When the right hand side is the solution of Equation (31), the right hand is the solution. When the residual at the n-th iteration is given by

$$\begin{aligned} \delta\phi_{i,j,k}^{(n)} &= \phi_{i,j,k}^{(n)} - \phi_{i,j,k}^{(\mathrm{exact})} &(61) \\ &= A_n \cos(k_x hi)\cos(k_y hj)\cos(k_z hk), &(62) \end{aligned}$$

each iteration decreases the amplitude decreases by a factor of

$$\left|\frac{A_{n+1}}{A_n}\right| = \frac{1}{3}\left|\cos hk_x + \cos hk_y + \cos hk_z\right| \le 1. \tag{63}$$

For any k_x, k_y and k_z, the residual decreases by each iteration except when $hk_x = hk_y = hk_z = \pi$.[2] However, the damping is very slow when all of $k_x h$, $k_y h$ and $k_z h$ are zero or close to zero, since

$$\left|\frac{A_{n+1}}{A_n}\right| \quad \simeq \quad 1 - \frac{h^2}{6}\left({k_x}^2 + {k_y}^2 + {k_z}^2\right). \tag{64}$$

When $k_x h = 2\pi/N$ and $k_y = k_z = 0$,

$$\frac{A_{n+1}}{A_n} \quad \simeq \quad 1 - \frac{2\pi^2}{3N^2}. \tag{65}$$

The residuals only a factor of 1.39×10^{-3} by N^2 times iteration. We need to accelerate the iteration. SOR (Successive Over Relaxation) accelerate the convergence only slightly.

Multigrid iteration combines coarse and fine grids to damp various residuals.

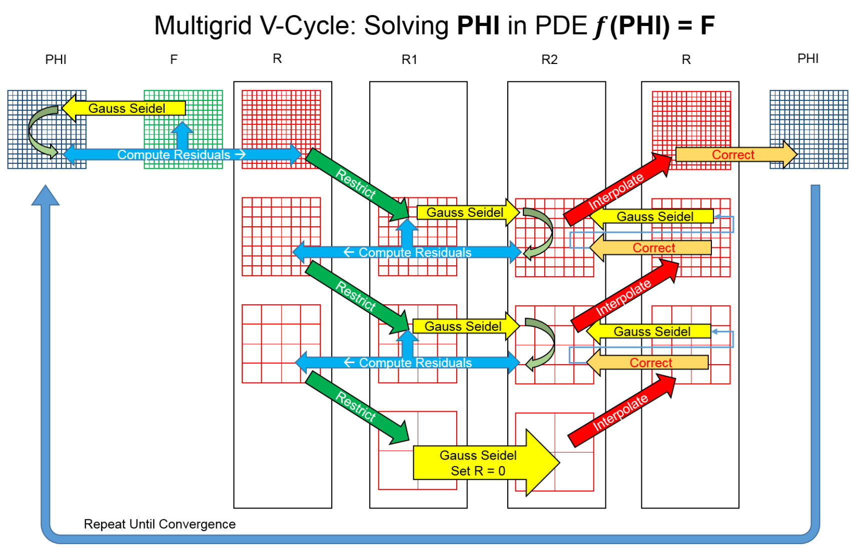

Figure 1: Taken from Wikipedia on Multigrid Method[4]

A code and samples are available at “pyro”[7].

4.3 Boundary Condition

Boundary conditions:

- Periodic: $\phi(x + L, y, z) = \phi(x, y, z)$.
- Dirichlet: ϕ on the boundary is given.

[2]The red-black ordering erases the mode of $hk_x = hk_y = hk_z = \pi$ at the first iteration.

- Neumann: $\nabla\phi$ on the boundary is given.

When the system is periodic, you can apply FFT directly. Otherwise, we separate the potential into two parts,

$$\phi = \phi' + \Phi, \tag{66}$$

$$\Delta\phi' = 4\pi G\rho - \Delta\Phi, \tag{67}$$

and assume that ϕ' is periodic. We can take the following approximation,

$$\Phi = -\frac{GM}{\sqrt{r^2 + a^2}}, \tag{68}$$

$$M = \iiint \rho(x, y, z)dxdydz, \tag{69}$$

where a is an arbitrary size parameter. When $\rho - \Delta\phi/4\pi G$ consists of small scale fluctuations, ϕ' and hence ϕ are good approximations.

When we use the multi grid iteration, we can set the potential outside the computation domain. The potential outside the boundary is given by

$$\phi_{N+1,j,k} = \begin{cases} \phi(x_{N+1}, y_j, z_k) & \text{for Dirichlet} \\ \phi_{N,j,k} - hg_x\left(x_{N+1/2}, y_j, z_k\right) & \text{for Neumann} \end{cases}. \tag{70}$$

We can regard the potential outside the boundary, $\phi_{N+1,j,k}$, as a source term.

5 Homework

Obtain the potential and gravity numerically on the Cartesian grid,

$$(x_i, y_j, z_k) = h\left(i - \frac{N+1}{2}, j - \frac{N+1}{2}, k - \frac{N+1}{2}\right), \tag{71}$$

where the indices, i, j, and k range from 1 to $N = 2^\nu$, for the density distribution,

$$\rho(r) = \begin{cases} \dfrac{\rho_0 a}{r}\sin\dfrac{r}{a} & \left(r \le \dfrac{\pi a}{2}\right) \\ 0 & \left(r > \dfrac{\pi a}{2}\right) \end{cases}, \tag{72}$$

The analytic solution is expressed as

$$\phi(r) = \begin{cases} -\dfrac{4\pi G\rho_0 a^3}{r}\sin\left(\dfrac{r}{a}\right) & \left(r \le \dfrac{\pi a}{2}\right) \\ -\dfrac{4\pi G\rho_0 a^3}{r} & \left(r > \dfrac{\pi a}{2}\right) \end{cases}, \tag{73}$$

$$g_r = \begin{cases} -\dfrac{4\pi G\rho_0 a^3}{r^2}\left[\sin\left(\dfrac{r}{a}\right) - \dfrac{r}{a}\cos\left(\dfrac{r}{a}\right)\right] & \left(r \le \dfrac{\pi a}{2}\right) \\ -\dfrac{4\pi G\rho_0 a^3}{r^2} & \left(r > \dfrac{\pi a}{2}\right) \end{cases}. \tag{74}$$

Problems:

1. Confirm that the solution obtained with FFT satisfies the discrete Poisson equation.

2. Evaluate the numerical error of the solution. Please keep in mind that the numerical solution converges to the exact one in the limit of $h \to 0$ and $2^\nu h \to \infty$. You can choose a boundary condition as you like.

   (a) Evaluate the dependence on h for a given $2^\nu h$ (truncation error).

   (b) Evaluate the dependence on ν for a given h.

   (c) Propose an optimum choice of h for a given ν.

A challenging problem: Compute the potential for Miyamoto & Nagai[6] model. This potential is designed to describe the potential of a disk galaxy,

$$\phi_{\mathrm{M}}(R,z) = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}, \tag{75}$$

$$\rho_{\mathrm{M}}(R,z) = \left(\frac{b^2 M}{4\pi}\right)\frac{aR^2+(a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2+(a+\sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}}, \tag{76}$$

where $R=\sqrt{x^2+y^2}$ denote the radial distance from the axis. You can use this potential as a test the boundary condition. The model parameters, M, a, and b denote the mass, disk radius, and disk thickness, respectively.

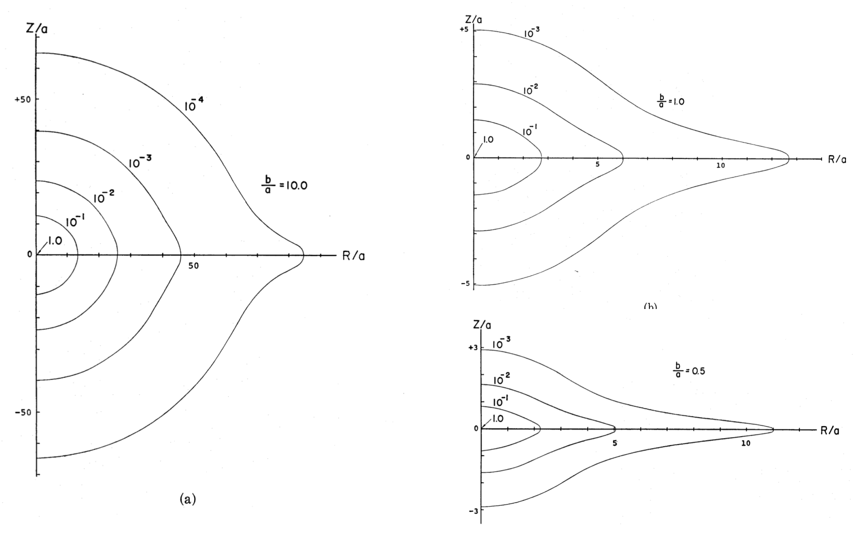

Figure 2: Each panel denotes the density $\rho_{\mathrm{M}}(R,z)/\rho_{\mathrm{M}}(0,0)$ for a given b/a. These panels are taken from Miyamoto & Nagai [6].

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