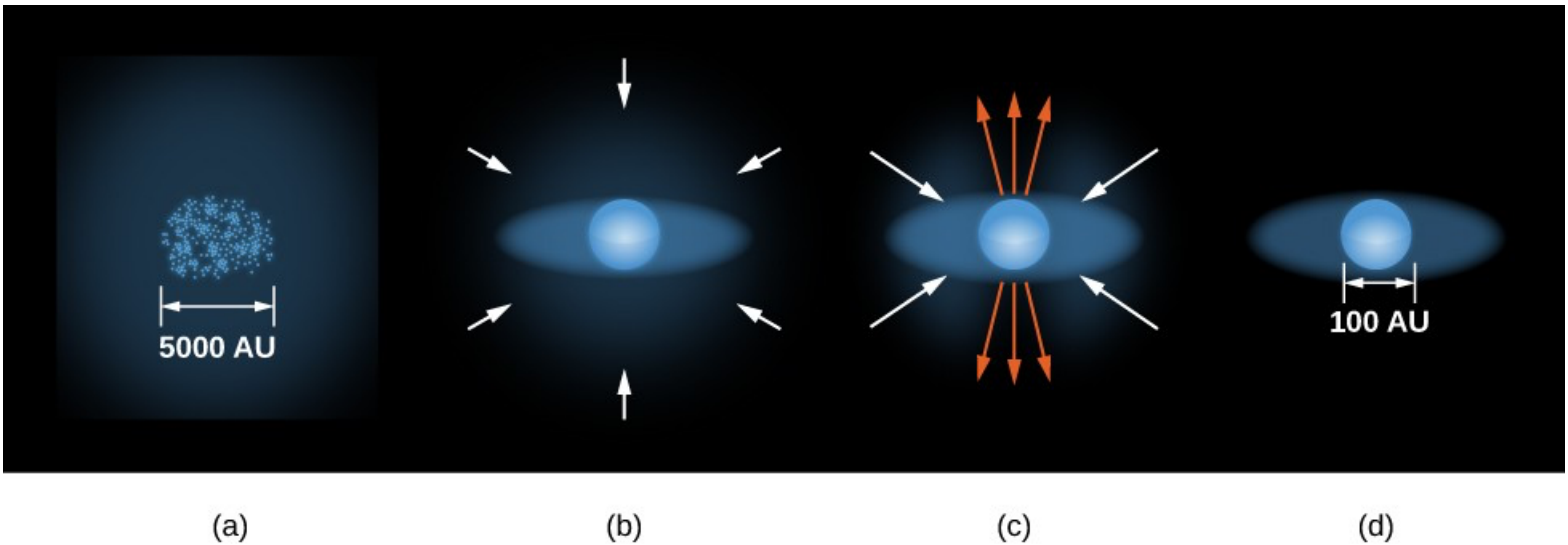
Self-gravity in hydrodynamical simulation

Contents:

Tomoyuki Hanawa (Chiba Univ.)

0. Needs for the self-gravity

- Star formation in the present day and in the early universe
- Large scale structure formation in the universe
- Supernova explosion



https://courses.lumenlearning.com/astronomy/chapter/star-formation/

Finite volume method and Conservation Law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

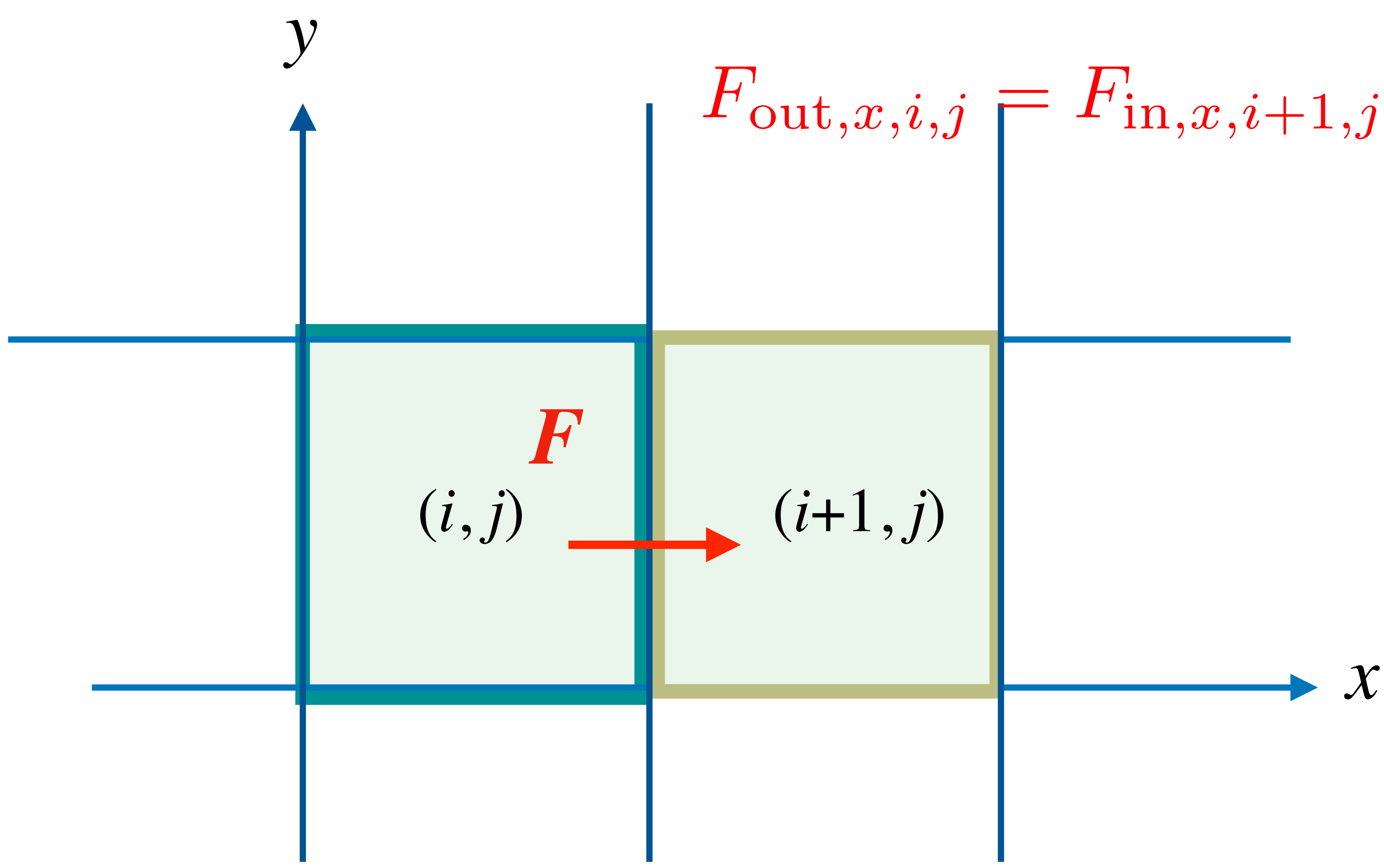
$$\frac{\partial}{\partial t} \int \rho dV + \int \boxed{\rho \boldsymbol{v}} d\boldsymbol{S} = 0$$

$$\boldsymbol{F} = \rho \boldsymbol{v}$$

Gauss's theorem

$$\frac{\Delta \rho}{\Delta t} dV + \sum_j \boldsymbol{F}_j \cdot d\boldsymbol{S}_j = 0$$

$$F_{\mathrm{out},x,i,j} = F_{\mathrm{in},x,i+1,j} = F^*_{x,i+1/2,j}$$



Gravity

N-body

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = -m_i \sum_{j \neq i} \frac{G m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) = m_i \boldsymbol{g}_i.$$

Gas

$$\boldsymbol{g}(\boldsymbol{r}) = -\int \frac{G\rho(\boldsymbol{x})}{|\boldsymbol{r} - \boldsymbol{x}|^3} (\boldsymbol{r} - \boldsymbol{x})\, d\boldsymbol{x}.$$

Direct integral (summation) does not work! High cost and low accuracy.

The Poisson equation $\boldsymbol{\nabla}^2 \phi = 4\pi G \rho,$

$$\boldsymbol{g} = -\boldsymbol{\nabla}\phi. \qquad \boldsymbol{g} = -GM \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3},$$

$$\boldsymbol{\nabla}^2 \phi_{\mathrm{e}} = -\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\frac{\rho_{\mathrm{e}}}{\varepsilon_0}, \qquad \boldsymbol{E} = \frac{Q}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{|\boldsymbol{r}|^3},$$

$$4\pi G \longleftrightarrow -\frac{1}{\varepsilon_0}$$

1. Basics: Hydrodynamic Equations

The Euler-Equations in conservative form read

Prof. Kley

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad (1)$$

$\rho \boldsymbol{g}$

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) = -\nabla p + \rho \vec{k} \qquad (2)$$

$$\boxed{\frac{\partial (\rho \epsilon)}{\partial t} + \nabla \cdot (\rho \epsilon \vec{u}) = -p \nabla \cdot \vec{u}} \qquad (3)$$

cf. pyro https://pyro2.readthedocs.io/en/latest/

gravitational energy release

$$\frac{\partial}{\partial t}(\rho E) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} H) = \boxed{\rho \boldsymbol{v} \cdot \boldsymbol{g},} \qquad (15)$$

When $TDs/dt = 0$

(no heating/cooling)

$$E = \frac{\boldsymbol{v}^2}{2} + \varepsilon = \frac{\boldsymbol{v}^2}{2} + \frac{P}{(\gamma - 1)\rho}, \qquad (16)$$

$$H = \frac{\boldsymbol{v}^2}{2} + \frac{\gamma P}{(\gamma - 1)\rho}. \qquad (17)$$

enthalpy

Gravity and gravitational energy release are source terms.

No acceleration of the center of mass

$$\int \rho \boldsymbol{g} dV \quad = \quad 0.$$

action and counteraction

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = -m_i \sum_{j \neq i} \frac{G m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} (\boldsymbol{r}_i - \boldsymbol{r}_j) = m_i \boldsymbol{g}_i.$$

$$\rho \boldsymbol{g} \quad = \quad -\boldsymbol{\nabla} \cdot \boldsymbol{T}_g \;=\; \frac{(\boldsymbol{\nabla} \times \boldsymbol{g}) \times \boldsymbol{g}}{4\pi G} - \frac{(\boldsymbol{\nabla} \cdot \boldsymbol{g})}{4\pi G} \boldsymbol{g}$$

Gravitational energy is conservative.

$$E_{\mathrm{g}} \quad = \quad \frac{1}{2} \int \phi \rho dV \quad = \quad -\frac{1}{8\pi G} \int (\boldsymbol{\nabla} \phi)^2 dV + \frac{1}{8\pi G} \int \phi \boldsymbol{\nabla} \phi d\boldsymbol{S}.$$

$$\frac{\partial E_{\mathrm{g}}}{\partial t} \quad = \quad -\frac{1}{4\pi G} \int \boldsymbol{g} \cdot \frac{\partial \boldsymbol{g}}{\partial t} dV + \frac{1}{8\pi G} \int \frac{\partial}{\partial t} (\phi \boldsymbol{g}) \cdot d\boldsymbol{S}$$

$$\int \left[\rho \boldsymbol{v} \cdot \boldsymbol{g} \quad + \quad \frac{\partial}{\partial t} \left(\frac{\boldsymbol{g} \cdot \boldsymbol{g}}{8\pi G} \right) \right] dV = \; - \int \phi \left(\rho \boldsymbol{v} + \frac{1}{4\pi G} \frac{\partial \boldsymbol{g}}{\partial t} \right) \cdot d\boldsymbol{S}$$

gravitational energy release

$$\boxed{\text{if} \quad \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{g}}{\partial t} \quad = \quad -4\pi G \frac{\partial \rho}{\partial t},}$$

Another Form of Hydrodynamical Equations

no source term

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = \boxed{0,}$$

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P\boldsymbol{I} + \boldsymbol{T}_g) = \boxed{0,}$$

$$\frac{\partial}{\partial t}\Bigg(\rho E - \underbrace{\boxed{\frac{\boldsymbol{g}^2}{8\pi G}}}_{E_{\mathrm{g}}}\Bigg) + \boldsymbol{\nabla} \cdot \left\{\rho \boldsymbol{v} H + \phi \underbrace{\boxed{\left[\rho \boldsymbol{v} - \frac{\partial}{\partial t}\left(\frac{\boldsymbol{g}}{4\pi G}\right)\right]}}_{\text{divergence free}}\right\} = \boxed{0,}$$

$$\boldsymbol{T}_g = \frac{1}{4\pi G}\left[\boldsymbol{\nabla}\phi\boldsymbol{\nabla}\phi - \frac{1}{2}(\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla})\,\phi\boldsymbol{I}\right] = \frac{1}{4\pi G}\left[\boldsymbol{g}\boldsymbol{g} - \frac{1}{2}(\boldsymbol{g}\cdot\boldsymbol{g})\,\boldsymbol{I}\right],$$

Important Relation

$$\boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v} - \frac{1}{4\pi G}\frac{\partial}{\partial t}\boldsymbol{g}\right) = \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v} + \frac{1}{4\pi G}\frac{\partial}{\partial t}\boldsymbol{\nabla}\phi\right) = \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) + \frac{\partial \rho}{\partial t} = 0$$

Properly Evaluated Gravity: $-\boldsymbol{\nabla}\cdot\mathbf{T_g} = \rho_{i,jk}\boldsymbol{g}_{i,j,k}$

$$g_{x,i,j,k} \equiv \frac{g_{x,i+1/2,j,k} + g_{x,i-1/2,j,k}}{2} = \frac{\phi_{i+1,j,k} - \phi_{i-1,j,k}}{2\Delta x}$$

$$g_{x,i,j+1/2,k} + g_{x,i,j-1/2,k} = g_{x,i,j,k+1/2} + g_{x,i,j,k-1/2} = g_{x,i,j,k}$$

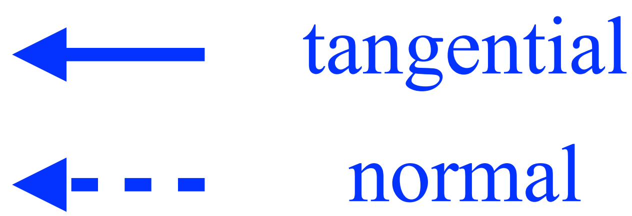
$$g_{x,i,j-1/2,k}\Delta x + g_{y,i+1/2,j,k}\Delta y - g_{x,i,j+1/2,k}\Delta x - g_{y,i-1/2,j,k}\Delta y = 0$$

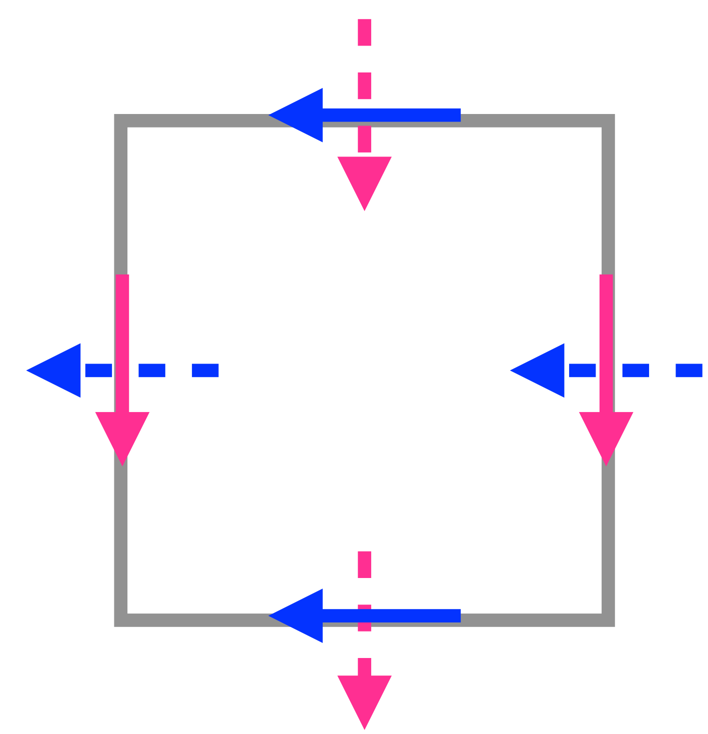
requirement

$$\rho\boldsymbol{g} = -\boldsymbol{\nabla}\cdot\boldsymbol{T}_g = -\frac{(\boldsymbol{\nabla}\boldsymbol{g})\times\boldsymbol{g}}{4\pi G} + \frac{(\boldsymbol{\nabla}\cdot\boldsymbol{g})}{4\pi G}\boldsymbol{g}$$

$$\boldsymbol{T}_g = \frac{1}{4\pi G}\left[\boldsymbol{\nabla}\phi\boldsymbol{\nabla}\phi - \frac{1}{2}(\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla})\,\phi\boldsymbol{I}\right]$$

$$\boldsymbol{\nabla}\times\boldsymbol{g} = 0 \Longleftrightarrow \oint\boldsymbol{g}\cdot d\boldsymbol{s} = 0$$





Total Energy is conserved if we solve the Hydrodynamical Equations without source terms.

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) &= 0, \\
\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v} + P\boldsymbol{I} + \boldsymbol{T}_g) &= 0, \\
\frac{\partial}{\partial t}\left(\rho E + \frac{\rho\phi}{2}\right) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} H + \boldsymbol{F}_g) &= 0,
\end{aligned}
$$

cf. Jiang+13

similar to MHD

$$
\begin{aligned}
\rho \boldsymbol{g} &= -\boldsymbol{\nabla} \cdot \boldsymbol{T}_g = \frac{(\boldsymbol{\nabla} \times \boldsymbol{g}) \times \boldsymbol{g}}{4\pi G} - \frac{(\boldsymbol{\nabla} \cdot \boldsymbol{g})}{4\pi G}\boldsymbol{g} \\
\boldsymbol{T}_g &= \frac{1}{4\pi G}\left[\boldsymbol{g}\boldsymbol{g} - \frac{1}{2}\boldsymbol{g} \cdot \boldsymbol{g}\boldsymbol{I}\right] \\
\boldsymbol{F}_g &= \frac{1}{8\pi G}\left(\phi \boldsymbol{\nabla}\dot{\phi} - \dot{\phi}\boldsymbol{\nabla}\phi\right) + \rho \boldsymbol{v}\phi \\
\boldsymbol{\nabla}^2 \phi &= 4\pi G \rho.
\end{aligned}
$$

$$\boldsymbol{g} \leftrightarrow \boldsymbol{B}$$

Cons: new unknons, $\dot{\phi}, \boldsymbol{\nabla}\dot{\phi}$

Poisson Equation in its Discrete Form

$$\simeq 2\phi_{i,j,k} + \frac{\partial^2 \phi}{\partial x^2} h^2 \qquad \simeq 2\phi_{i,j,k} + \frac{\partial^2 \phi}{\partial y^2} h^2 \qquad \simeq 2\phi_{i,j,k} + \frac{\partial^2 \phi}{\partial z^2} h^2$$

$$\boxed{\phi_{i+1,j,k} + \phi_{i-1,j,k}} + \boxed{\phi_{i,j+1,k} + \phi_{i,j-1,k}} + \boxed{\phi_{i,j,k+1} + \phi_{i,j,k-1}}$$
$$= 6\phi_{i,j,k} + 4\pi G h^2 \rho_{i,j,k}.$$

$$\Delta\rho = 4\pi G\rho$$

$$\begin{aligned}
\phi_{i,j,k} &\equiv \phi(x_i,\, y_j,\, z_k), \\
\rho_{i,j,k} &\equiv \rho(x_i,\, y_j,\, z_k), \\
x_i &= hi, \\
y_j &= hj, \\
z_k &= hk.
\end{aligned}$$

unknown

given

h: cell width

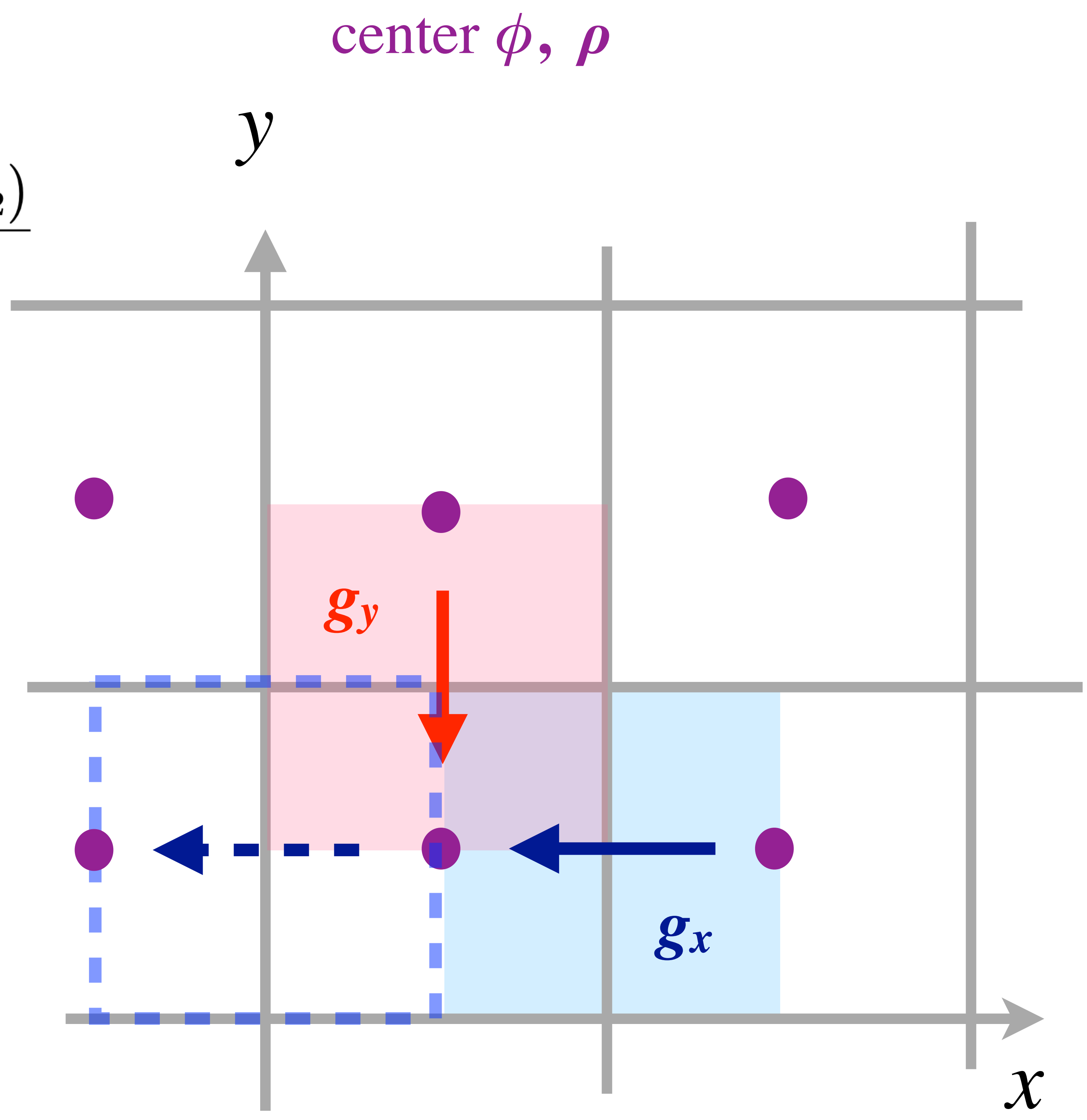
i, j, k: indexes

Gravity on the cell surface

$$\begin{aligned}
g_{x,i+1/2,j,k} &= -\frac{\phi_{i+1,j,k} - \phi_{i,j,k}}{h}, \\
g_{y,i,j+1/2,k} &= -\frac{\phi_{i,j+1,k} - \phi_{i,j,k}}{h}, \\
g_{z,i,j,k+1/2} &= -\frac{\phi_{i,j,k+1} - \phi_{i,j,k}}{h}.
\end{aligned}$$

$$
\begin{aligned}
(\rho g_x)_{i,j,k} &= \frac{\rho_{i,j,k}\left(g_{x,i+1/2,j,k} + g_{x,i-1/2,j,k}\right)}{2} \\
&= -\frac{\rho_{i,j,k}\left(\phi_{i+1,j,k} - \phi_{i-1,j,k}\right)}{2h}, \\
(\rho g_y)_{i,j,k} &= \frac{\rho_{i,j,k}\left(g_{y,i,j+1/2,k} + g_{y,i,j-1/2,k}\right)}{2} \\
&= -\frac{\rho_{i,j,k}\left(\phi_{i,j+1,k} - \phi_{i,j-1,k}\right)}{2h}, \\
(\rho g_z)_{i,j,k} &= \frac{\rho_{i,j,k}\left(g_{z,i,j,k+1/2} + g_{z,i,j,k-1/2}\right)}{2} \\
&= -\frac{\rho_{i,j,k}\left(\phi_{i,j,k+1} - \phi_{i,j,k-1}\right)}{2h},
\end{aligned}
$$

Gravity and mass flux are defined on the boundary.

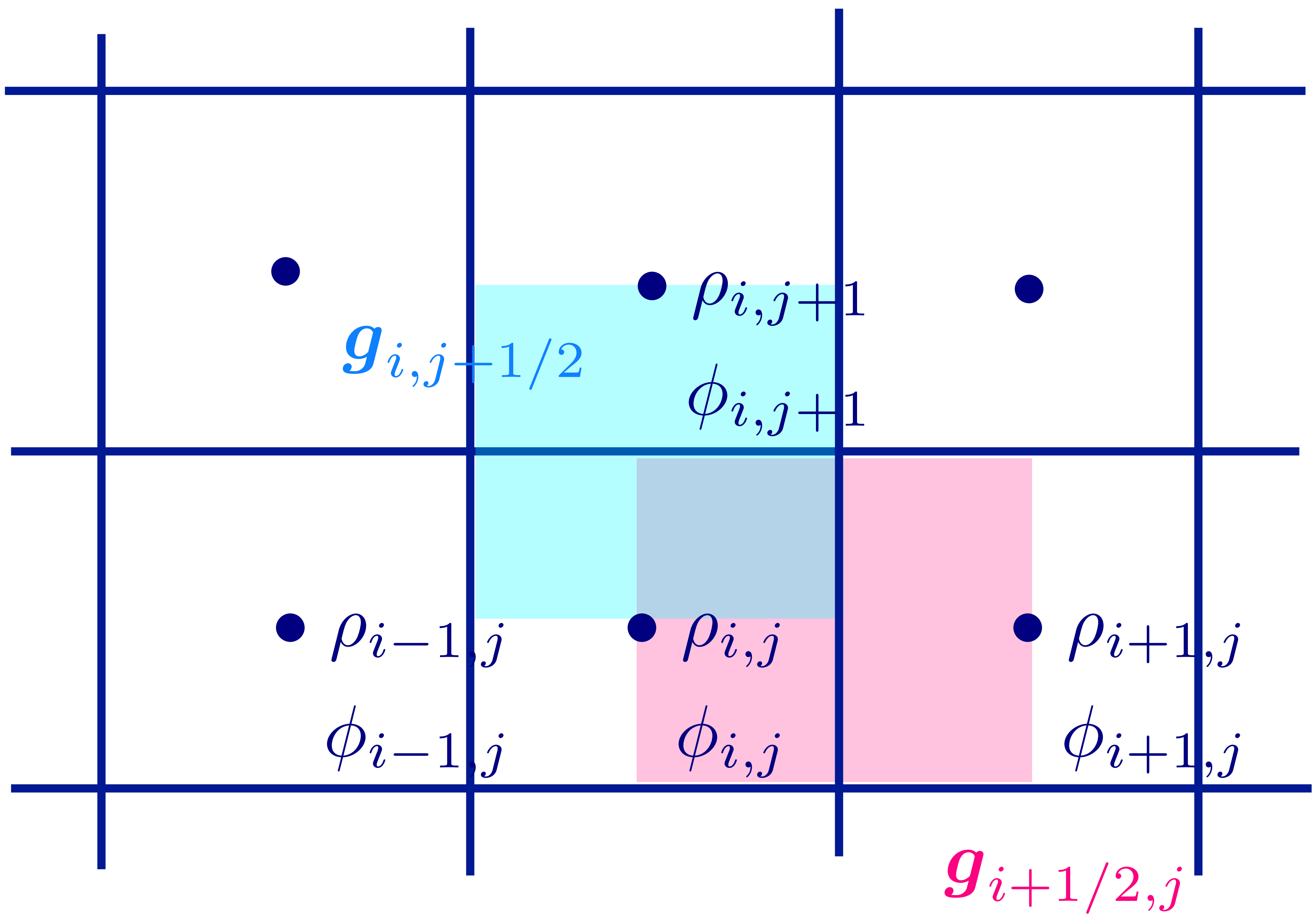


Two Forms of E_{g} are equivalent

$$\frac{1}{2}\iiint \rho\phi dV = -\iiint \frac{\boldsymbol{g}^2}{8\pi G} dV + \frac{1}{2}\iint_S \phi \boldsymbol{\nabla}\phi dS$$

$$-\frac{\boldsymbol{g}^2}{8\pi G} = \frac{\rho\phi}{2} + \boldsymbol{\nabla}\cdot\left[\frac{(\phi\boldsymbol{g})}{8\pi G}\right]$$

cf. Katz+16



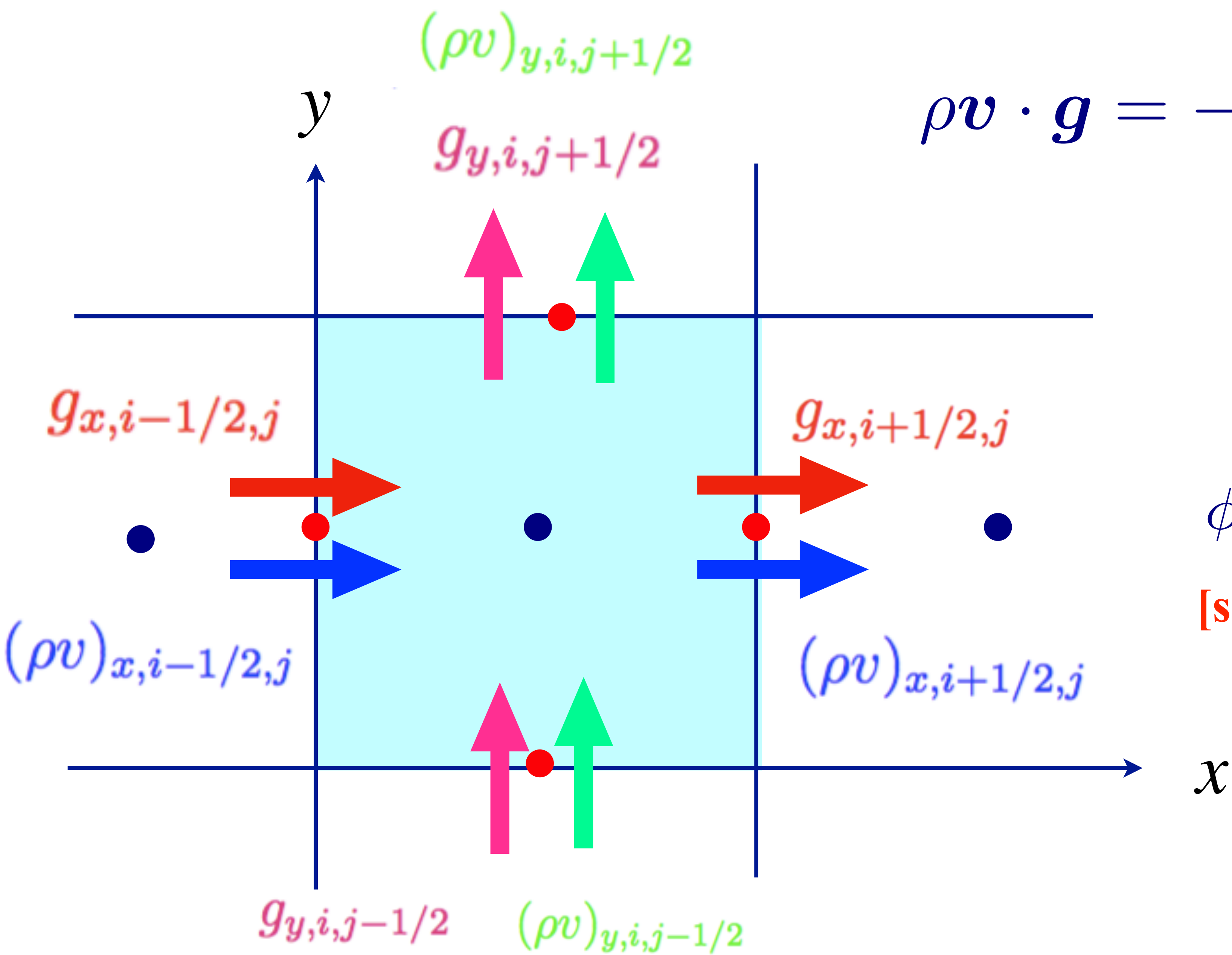
- Gravity, $\boldsymbol{g}$, should be evaluated on the cell surface.
- Only the normal component of $\boldsymbol{g}$ is taken into account.

$$\int \boldsymbol{g}\cdot d\boldsymbol{S} = -4\pi G \int \rho dV$$

Source Term Evaluation based on Mass Flux

cf. Mikami+08, Springel 10, Katz+16

$$\rho \boldsymbol{v} \cdot \boldsymbol{g} \quad = \quad (\text{mass flux}) \cdot (\text{gravity}) \;\neq\; (\text{density}) \times (\text{velocity}) \cdot (\text{gravity})$$



$$\rho \boldsymbol{v} \cdot \boldsymbol{g} = -\boldsymbol{\nabla} \cdot \boxed{(\phi \rho \boldsymbol{v})} + \boxed{\phi} \boldsymbol{\nabla} \cdot \boxed{(\rho \boldsymbol{v})}$$

cell surface

cell center

$$\phi_{i+1/2,j} = \phi_{i,j} - \frac{g_{x,i+1/2,j}}{2} \Delta x$$

[surface] [center]

Use time averaged mass flux and gravity.

$$(\rho \boldsymbol{v} \cdot \boldsymbol{g})_{i,j,k} = \frac{1}{2}\Big[(\overline{\rho v_x})_{i+1/2,j,k}\,\overline{g_{x,i+1/2,j,k}} + (\overline{\rho v_x})_{i-1/2,j,k}\,\overline{g_{x,i-1/2,j,k}}$$
$$+ (\overline{\rho v_y})_{i,j+1/2,k}\,\overline{g_{y,i,j+1/2,k}} + (\overline{\rho v_y})_{i,j-1/2,k}\,\overline{g_{y,i,j-1/2,k}}$$
$$+ (\overline{\rho v_z})_{i,j,k+1/2}\,\overline{g_{z,i,j,k+1/2}} + (\overline{\rho v_z})_{i,j,k-1/2}\,\overline{g_{z,i,j,k-1/2}}\Big]$$

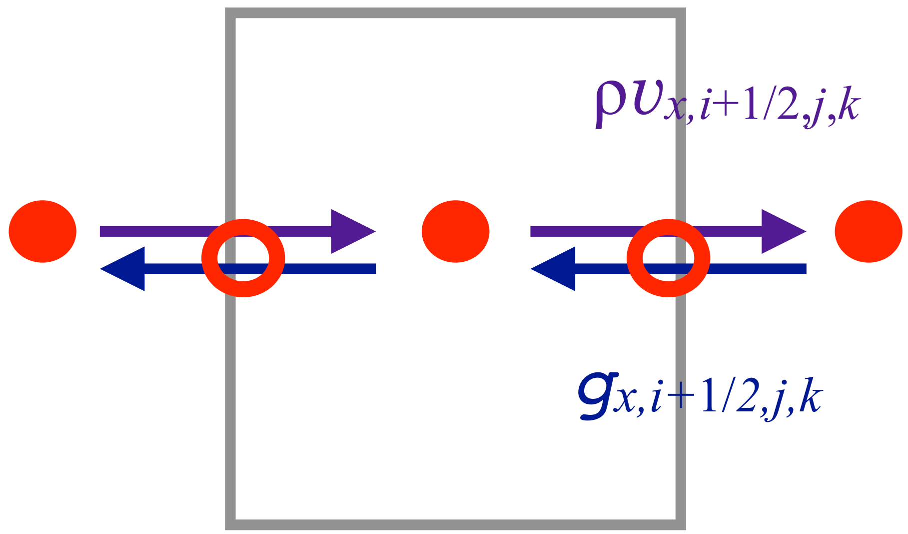
$$\frac{h\,[\rho(t+\Delta t) - \rho(t)]}{\Delta t} = -(\overline{\rho v_x})_{i+1/2,j,k} + (\overline{\rho v_x})_{i-1/2,j,k}$$
$$-(\overline{\rho v_y})_{i,j+1/2,k} + (\overline{\rho v_y})_{i,j-1/2,k}$$
$$-(\overline{\rho v_x})_{i-1/2,j,k} + (\overline{\rho v_z})_{i+1/2,j,k-1/2}$$

$$\overline{\boldsymbol{g}} = \frac{\boldsymbol{g}(t) + \boldsymbol{g}(t+\Delta t)}{2}$$

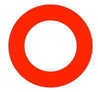
Gravitational Energy Release

$$\begin{aligned} \rho \boldsymbol{v} \cdot \boldsymbol{g} &= -\rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \phi \\ &= -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \phi) + [\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v})]\, \phi \end{aligned}$$

$$\begin{aligned} \phi_{i+1/2,j,k} &= \phi_{i,j,k} - \frac{g_{x,i+1/2,j,k} \Delta x}{2} \\ \phi_{i,j+1/2,k} &= \phi_{i,j,k} - \frac{g_{y,i,j+1/2,k} \Delta y}{2} \\ \phi_{i,j,k+1/2} &= \phi_{i,j,k} - \frac{g_{z,i,j,k+1/2} \Delta z}{2} \end{aligned}$$



$$\begin{aligned} [\rho v_x g_x]_{i,j,k} &= \frac{1}{2} (\rho v_x)^*_{i+1/2,j,k}\, g_{x,i+1/2,j,k} \\ &+ \frac{1}{2} (\rho v_x)^*_{i-1/2,j,k}\, g_{x,i-1/2,j,k} \end{aligned}$$



potential on the cell surface

cf Springel 10

How to Solve Poisson Equation.

When the density distribution is expressed as

$$\rho(x, y, z) = \rho_0 \cos(k_x x) \cos(k_y y) \cos(k_z z), \tag{44}$$

the corresponding gravitational potential is expressed as

$$\phi(x, y, z) = -\frac{4\pi G \rho_0}{\bar{k}^2} \cos(k_x x) \cos(k_y y) \cos(k_z z), \tag{45}$$

$$\bar{k}^2 = {k_x}^2 + {k_y}^2 + {k_z}^2 \tag{46}$$

When the density distribution is expressed as

$$\rho_{i,j,k} = \rho_0 \cos(k_x hi) \cos(k_y hj) \cos(k_z hk), \tag{47}$$

the corresponding potential is given by

$$\phi_{i,j,k} = -\frac{4\pi G \rho_0}{\tilde{k}^2} \cos(k_x hi) \cos(k_y hj) \cos(k_z hk), \tag{48}$$

$$\tilde{k}^2 = \frac{4}{h^2} \left(\sin^2 \frac{k_x h}{2} + \sin^2 \frac{k_y h}{2} + \sin^2 \frac{k_z h}{2} \right). \tag{49}$$

Use Principle of Superposition.

Discrete Fourier Transform

$$
\begin{aligned}
A_k &= \sum_{n=0}^{N-1} a_n \left[\cos\left(\frac{2\pi kn}{N}\right) - \sqrt{-1}\cos\left(\frac{2\pi kn}{N}\right)\right] && (50)\\
&= \sum_{n=0}^{N-1} a_n \exp\left[-\sqrt{-1}\left(\frac{2\pi kn}{N}\right)\right], && (51)\\
a_n &= \frac{1}{N}\sum_{k=0}^{N-1} A_k \left[\cos\left(\frac{2\pi kn}{N}\right) + \sqrt{-1}\cos\left(\frac{2\pi kn}{N}\right)\right] && (52)\\
&= \frac{1}{N}\sum_{k=0}^{N-1} A_k \exp\left[\sqrt{-1}\left(\frac{2\pi kn}{N}\right)\right]. && (53)
\end{aligned}
$$

These are analogous to the Fourier Transform,

$$
A(x) = \int_{-\infty}^{\infty} \tilde{A}(k)\exp\left(\sqrt{-1}kx\right)dk. \quad (54)
$$

We can evaluate the right hand sides of equations (51) - (53) by an order of $N \log_2 N$ times operations (Fast Fourier Transform, FFT).

$N = 2^{\nu}$

$N \log_2 N = 896$ but $N^2 = 16383$ for $N = 128$.

How to Solve the Poisson Equation with FFT

Step 1 (FFT)

$$\tilde{\rho}_{\ell,m,n} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \rho_{i,j,k} \exp\left[-\sqrt{-1}\frac{(i\ell + jm + kn)}{2\pi}\right]$$

Step 2

$$\tilde{\phi}_{\ell,m,n} = \begin{cases} 0 & (\ell = m = n = 0) \\ -\dfrac{4\pi G \tilde{\rho}_{i,j,k}}{\left(\tilde{k}_{\ell,m,n}\right)^2} & (\text{otherwise}) \end{cases},$$

$$\left(\tilde{k}_{\ell,m,n}\right)^2 = \frac{4}{h^2}\left[\sin^2\left(\frac{\pi\ell}{N}\right) + \sin^2\left(\frac{\pi m}{N}\right) + \sin^2\left(\frac{\pi n}{N}\right)\right]$$

Step 3 (FFT)

$$\phi_{i,j,k} = \frac{1}{N^3}\sum_{\ell=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \tilde{\phi}_{\ell,m,n} \exp\left[2\pi\sqrt{-1}\frac{(i\ell + jm + kn)}{N}\right]$$

Numerical Recipes

1. Compute the gravity, $\boldsymbol{g}\,(t_0)$ at $t = t_0$ from $\rho(t_0)$.
2. Compute $\rho^*\,(t_0 + \Delta t)$ and $(\rho \boldsymbol{v})^*\,(t_0 + \Delta t)$.
3. Evaluate $\boldsymbol{g}^*\,(t_0 + \Delta t)$ from $\rho^*\,(t_0 + \Delta t)$.
4. Compute $P^*\,(t_0 + \Delta t)$ from the energy conservation.
5. Update the density, $\rho\,(t_0 + \Delta t)$.
6. Compute $\boldsymbol{g}\,(t_0 + \Delta t)$.
7. Update the momentum density, $(\rho \boldsymbol{v})\,(t_0 + \Delta t)$
8. Update $P(t_0 + \Delta t)$.
9. Go back to step 1 and repeat.

toward multi grid iteration

Jacobi (Gauss-Seidel) iteration

Unknown

$$
\begin{aligned}
\phi_{i+1,j,k} + \phi_{i-1,j,k} + \phi_{i,j+1,k} + \phi_{i,j-1,k} & + \phi_{i,j,k+1} + \phi_{i,j,k-1} \\
& = \boxed{6\phi_{i,j,k}} + 4\pi G h^2 \rho_{i,j,k}.
\end{aligned}
$$

$$
\begin{aligned}
\phi^{(n+1)}_{i,j,k} &= \frac{1}{6}\left(\phi^{(n)}_{i+1,j,k} + \phi^{(n)}_{i-1,j,k} + \phi^{(n)}_{i,j+1,k} + \phi^{(n)}_{i,j-1,k} + \phi^{(n)}_{i,j,k+1} + \phi^{(n)}_{i,j,k-1}\right) \\
&\quad -\frac{2}{3}\pi G h^2 \rho_{i,j,k} \qquad (60)
\end{aligned}
$$

Red-black ordering: Compute $\phi^{(n+1)}_{i,j,k}$ only when $i+j+k+n$ is even.

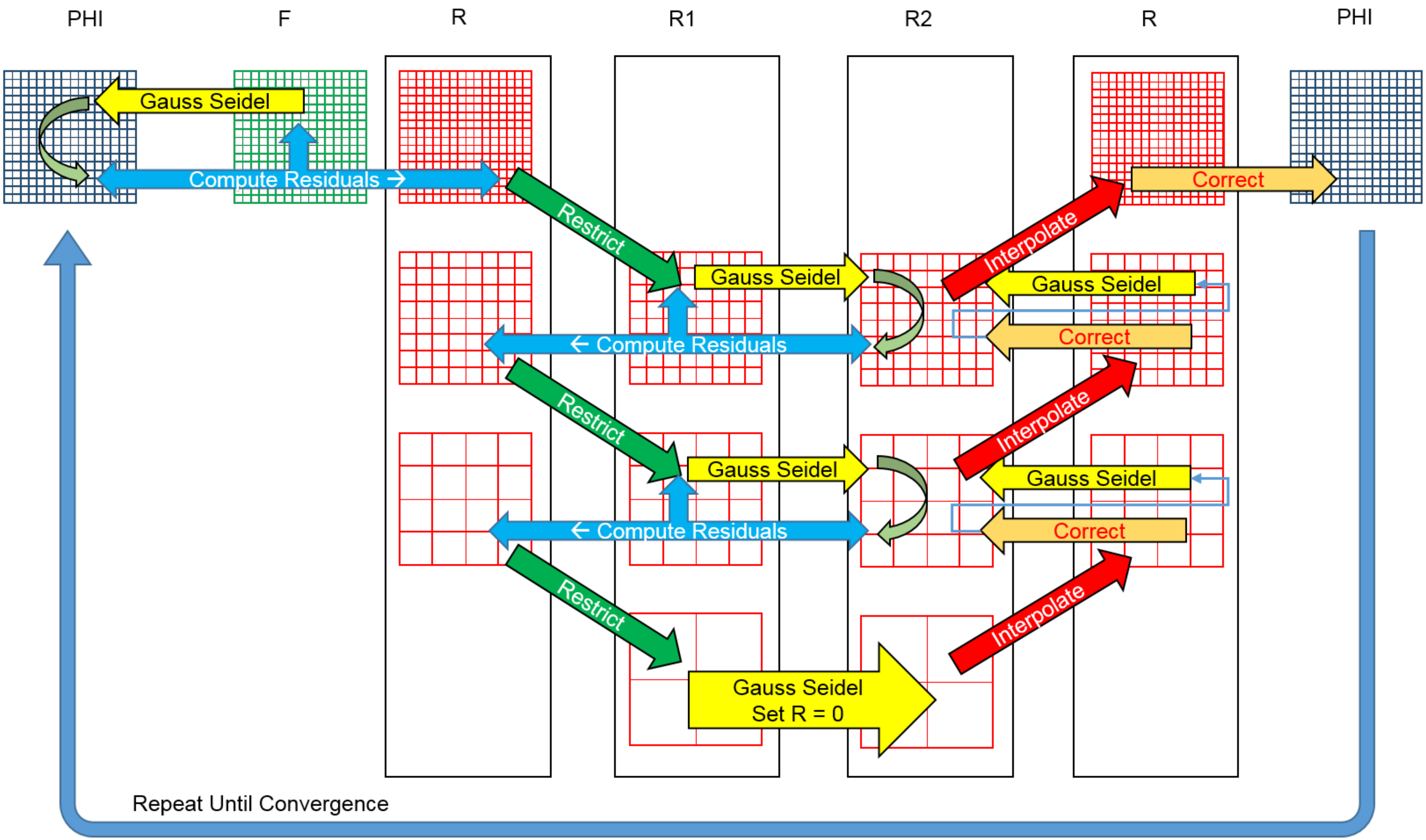
$$i+j+k = 2m \longleftrightarrow i+j+k = 2m+1$$

$$
\begin{aligned}
\delta\phi^{(n)}_{i,j,k} &= \phi^{(n)}_{i,j,k} - \phi^{(\mathrm{exact})}_{i,j,k} \\
&= A_n \cos(k_x h i)\cos(k_y h j)\cos(k_z h k),
\end{aligned}
$$

$$\left|\frac{A_{n+1}}{A_n}\right| = \frac{1}{3}\left|\cos hk_x + \cos hk_y + \cos hk_z\right| \leq 1.$$

The error decreases but only slowly for a small ***k***.

Multigrid V-Cycle: Solving **PHI** in PDE f **(PHI) = F**



Wikipedia

https://pyro2.readthedocs.io/en/latest/

Boundary Condition

(a) Periodic: $\phi(x+L) = \phi(x)$
(b) Dirichlet: $\phi(x)$ given on the boundary
(c) Neumann: $\nabla\phi$ given on the boundary

$$\phi_{N+1,j,k} = \begin{cases} \phi(x_{N+1}, y_j, z_k) & \text{for Dirichlet} \\ \phi_{N,j,k} - h g_x \left(x_{N+1/2}, y_j, z_k \right) & \text{for Neumann} \end{cases} .$$

When the system is periodic, you can apply FFT directly. Otherwise, we separate the potential into two parts,

$$\phi = \phi' + \Phi \tag{66}$$

$$\Delta\phi' = 4\pi G\rho - \Delta\Phi, \tag{67}$$

and assume that ϕ' is periodic. We can take the following approximation,

$$\Phi = -\frac{GM}{\sqrt{r^2 + a^2}}, \tag{68}$$

$$M = \iiint \rho(x,y,z)\,dx\,dy\,dz, \tag{69}$$

cf. $\Delta\phi = 4\pi G\left(\rho - \langle\rho\rangle\right)$ for periodic boundary

Homework

Obtain ϕ for

$$\rho(r) = \begin{cases} \dfrac{\rho_0 a}{r} \sin \dfrac{r}{a} & \left(r \leq \dfrac{\pi a}{2}\right) \\ 0 & \left(r > \dfrac{\pi a}{2}\right) \end{cases},$$

1. Confirm that the solution obtained with FFT satisfies the discrete Poisson equation.
2. Evaluate the numerical error of the solution. Please keep in mind that the numerical solution converges to the exact one in the limit of $h \to 0$ and $2^\nu h \to \infty$. You can choose a boundary condition as you like.
   (a) Evaluate the dependence on h for a given $2^\nu h$ (truncation error).
   (b) Evaluate the dependence on ν for a given h.
   (c) Propose an optimum choice of h for a given ν.

A challenging problem:

$$\phi_{\rm M}(R,z) = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}},$$

$$\rho_{\rm M}(R,z) = \left(\frac{b^2M}{4\pi}\right)\frac{aR^2+(a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{[R^2+(a+\sqrt{z^2+b^2})^2]^{5/2}(z^2+b^2)^{3/2}},$$

Miyamoto & Nagai (1975)