

Abstract

Many scientific disciplines rely on observational data alone. For example, there is no current technology that can control the interaction between the solar wind and the Earth's magnetic field, so space weather studies rely on data collected without performing controlled experiments (i.e. without interventions to the system dynamics). Unfortunately, causal inference with such observational data sets is difficult. It is well known that correlation does not imply causation, which has lead to the development of several different time series causality tools. In this article, we introduce new tools for exploratory causal inference of a pair of time series data sets that are believed to be causally related. The tools, called penchants and leanings, do not depend on any assumed models for the times series, do not rely on any embedding procedures, are computationally straightforward to apply, and follow directly from assumptions of probabilistic causality, which may provide a clearer interpretation of the results than other existing time series causality tools.

Non-parametric causal inference in bivariate time series data

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1 Introduction

Casual inference in time series analysis centers on finding “driving” relationships between different time series data sets. Showing the existence, rather than the exact nature, of the “driving” relationship between the sets is often the primary goal. Thus, words like “driving”, “causality”, and related terms usually do not have straightforward analogs to the same terms used in other fields [?, ?, ?], e.g. , theoretical quantum or classical mechanics. The development and study of such causal inference techniques is often called *time series causality*. Most of these techniques fall into four broad categories, those related to transfer entropy [?], those related to Granger causality [?], those related to state space reconstruction (SSR) techniques [?], and those related to lagged cross-correlation [?, ?]. These techniques have found application in a wide range of fields including neuroscience (e.g. [?]), economics (e.g. [?, ?]), climatology (e.g. [?]), and others.

In this article, we introduce another time series causality technique derived directly from the definition of probabilistic causality [?]. The new technique is then applied to synthetic and empirical time series data sets with known, or intuitive, causal structure. We will discuss the strengths and weaknesses of the technique and try to frame how the technique may be useful for causal inference with empirical data. Formal relationships to the other well known time series causality techniques, such as Granger

causality or transfer entropy, will be the subject of future work.

2 Causal Pendant

Define the *causal pendant* as

$$\rho_{EC} := P(E|C) - P(E|\bar{C}) \quad . \quad (1)$$

The motivation for this expression is in the interpretation of ρ_{EC} as a causal indicator; i.e. if C causes (or *drives*) E , then $\rho_{EC} > 0$, and if $\rho_{EC} \leq 0$, then the direction of causal influence in the system is undetermined. If the effect E is assumed to be recorded in one time series and the cause C is assumed to be recorded in a different time series, then the direction of causal influence in the system can be determined by comparing various pendants calculated using both time series.

The pendant definition includes the probability of an assumed effect occurring given a lack of the assumed cause, i.e. $P(E|\bar{C})$. It may be argued that the lack of an assumed cause is unobservable, and thus, the pendant definition is not practically useful. Pearl, and others, argue that causality depends on intervention, which implies the occurrence probability of the assumed effect should be conditioned on performing or not performing an action rather than on the presence or lack of an assumed cause [?]. Bunge defines a causal relation as “a relation among events” [?], again implying the lack of an assumed cause cannot be used to

find causal relations. These issues have been a part of probabilistic definitions of causality at least since the 1960s [?], and we will not attempt to solve them in this article. Our goal is the define a practical tool for causal inference. As such, we will circumvent these philosophical issues in the definition of the penchant by deriving an expression that removes any conditioning on a lack of the assumed cause.

Eqn. 1 can be rewritten using the law of total probability, i.e

$$P(E) = P(E|C)P(C) + P(E|\bar{C})P(\bar{C}) \quad , \quad (2)$$

or Bayes theorem, i.e.

$$P(E|\bar{C}) = P(\bar{C}|E) \frac{P(E)}{P(\bar{C})} \quad . \quad (3)$$

The definition of probability complements implies $P(\bar{C}) = 1 - P(C)$ and $P(\bar{C}|E) = 1 - P(C|E)$. Applying Eqn. 3 leads to

$$\begin{aligned} P(\bar{C}|E) &= 1 - P(C|E) \\ &= 1 - P(E|C) \frac{P(C)}{P(E)} \end{aligned}$$

Thus,

$$\begin{aligned} P(E|\bar{C}) &= P(\bar{C}|E) \frac{P(E)}{P(\bar{C})} \\ &= \left(1 - P(E|C) \frac{P(C)}{P(E)} \right) \frac{P(E)}{1 - P(C)} \quad , \end{aligned}$$

and the second term in Eqn. 1 has been written in terms of the of the first term. This expression implies a penchant calculation containing only the conditional probability that is directly observed from the data, i.e.

$$\rho_{EC} = P(E|C) \left(1 + \frac{P(C)}{1 - P(C)} \right) - \frac{P(E)}{1 - P(C)} \quad (4)$$

This same expression can be derived without Eqn. 3 by using Eqn. 2 to make the appropriate substitution for $P(E|\bar{C})$ into Eqn. 1. This penchant calculation requires only a single conditional probability be estimated from the data.

The use of either Eqn. 2 or Eqn. 3 may eliminate the concern that $P(E|\bar{C})$ is fundamentally unobservable. It may also, however, introduce new philosophical concerns in the definition of the penchant. For example, there has been no discussion of confounding. Assuming, e.g. $P(\bar{C}) = 1 - P(C)$ where $P(C)$ can be estimated from the data alone may be seen as an oversimplification of the dynamics generating the time series data; i.e. it may be seen as an assumption that the assumed effect is only caused by the assumed cause. Thus, it may be argued that the penchant is an indication of predictability rather than causality (similar to arguments made regarding Granger causality [?]). These concerns will not be addressed directly in this article. We emphasize, however, that we use terms like cause, effect, causal inference, and related terms to specifically refer to the expressions presented in this article and not to the general, or intuitive, meanings these terms are given elsewhere.

Assuming $P(E)$ and $P(C)$ can be estimated from the data alone is a key part of the penchant definition. The penchant is non-parametric in the sense that there is no assumed functional relationship between the times series being investigated. It will be shown below that the penchant is, essentially, a structured method for counting the data and using the those counts, along with notions from probabilistic causality, to infer which time series in a given pair might be seen as “driving” the other. Motivation for the penchant is the need for a times series causality tool that relies only on the data. It is often impossible to design experiments, or control interventions into the dynamics as discussed above, for many systems of interest, and it may be difficult to determine functional relationships between the time series data sets.

It follows from Eqn. 1 that

$$\rho_{EC} \in [1, -1] \quad , \quad (5)$$

but, more importantly for the calculations in the following sections, the penchant is not defined if $P(C)$ or $P(\bar{C})$ are zero (because the conditionals in Eqn. 1 would be undefined). Thus, the penchant is not defined if $P(C) = 0$ or if $P(C) = 1$. The former condition is interpreted intuitively as an inability to determine causal influence between two time series using points that do not appear in one of the series, and the latter condition is interpreted intuitively as an inability to determine causal influence between two time series if one of the data series is constant. The use of Bayes theorem in the derivation of Eqn. 4 implies that same conditions apply to $P(E)$. It will be seen below that there is no *a priori* assignments of "cause" or "effect" to a given time series when using penchants for causal inference. So, operationally, these conditions of $P(C)$ and $P(E)$ only mean that the penchant is undefined between pairs of time series where one series is constant.

The philosophical concerns are perhaps not as important as an answer to the straightforward question of whether or not the penchant is a useful tool for time series causality. The rest of this article will focus on answering that question.

3 Causal Leaning

Given a pair of times series $\{\mathbf{X}, \mathbf{Y}\}$, it is difficult to use the penchant directly for causal inference between the pair. Consider the assignment of \mathbf{X} as the cause, C , and \mathbf{Y} as the effect, E , i.e. $\{C, E\} = \{\mathbf{X}, \mathbf{Y}\}$. If $\rho_{EC} > 0$, then the probability that \mathbf{X} drives \mathbf{Y} is higher than the probability that it does not, which is stated more sufficiently as \mathbf{X} has a penchant to drive \mathbf{Y} or $\mathbf{X} \xrightarrow{pen} \mathbf{Y}$. It is possible, however, that the same penchant could be positive with the opposite cause-effect assignment, i.e. $\rho_{EC} > 0 \mid \{C, E\} = \{\mathbf{Y}, \mathbf{X}\} \Rightarrow \mathbf{Y} \xrightarrow{pen} \mathbf{X}$.

Even though it is possible that $\mathbf{X} \xrightarrow{pen} \mathbf{Y}$ and $\mathbf{Y} \xrightarrow{pen} \mathbf{X}$ are both true, such information does not provide information about the causal relationship within the pair $\{\mathbf{X}, \mathbf{Y}\}$.

The *leaning* is meant to address this problem and is defined as

$$\lambda_{EC} := \rho_{EC} - \rho_{CE} . \quad (6)$$

A positive leaning implies the cause C drives the effect E more than the effect drives the cause, a negative leaning implies the effect E drives the cause C more than the cause drives the effect, and a null leaning (i.e. $\lambda_{EC} = 0$) yields no causal information for the cause-effect pair $\{C, E\}$.

Consider again the assignment of $\{C, E\} = \{\mathbf{X}, \mathbf{Y}\}$. If $\lambda_{EC} > 0$, then \mathbf{X} has a larger penchant to drive \mathbf{Y} than \mathbf{Y} does to drive \mathbf{X} . More verbosely, $\lambda_{EC} > 0$ implies the difference between the probability that \mathbf{X} drives \mathbf{Y} and the probability that it does not is higher than the difference between the probability that \mathbf{Y} drives \mathbf{X} and the probability that it does not. For convenience, this language is boiled down to $\mathbf{X} \xrightarrow{lean} \mathbf{Y}$, as in $\lambda_{EC} > 0 \mid \{C, E\} = \{\mathbf{X}, \mathbf{Y}\} \Rightarrow \mathbf{X} \xrightarrow{lean} \mathbf{Y}$, $\lambda_{EC} < 0 \mid \{C, E\} = \{\mathbf{X}, \mathbf{Y}\} \Rightarrow \mathbf{Y} \xrightarrow{lean} \mathbf{X}$, and $\lambda_{EC} = 0 \mid \{C, E\} = \{\mathbf{X}, \mathbf{Y}\} \Rightarrow$ no conclusion.

It follows from Eqn. 6 and the bound for the penchant that $\lambda_{EC} \in [-2, 2]$. The leaning is a function of four probabilities, $P(C)$, $P(E)$, $P(C|E)$ and $P(E|C)$. The usefulness of the leaning for causal inference will depend on an effective method for estimating these probabilities from times series data and a more careful definition of the cause-effect assignment within the time series pair. These topics will be discussed with a motivating toy model of a dynamical system for which the penchant and leaning calculations are simple enough to perform without any computational aid.

4 Motivating Example

Consider a time series pair $\bar{\mathbf{T}} = \{\mathbf{X}, \mathbf{Y}\}$ with

$$\begin{aligned}\mathbf{X} &= \{x_t \mid t \in [0, 9]\} \\ &= \{0, 0, 1, 0, 0, 1, 0, 0, 1, 0\} \\ \mathbf{Y} &= \{y_t \mid t \in [0, 9]\} \\ &= \{0, 0, 0, 1, 0, 0, 1, 0, 0, 1\}.\end{aligned}$$

It seems intuitive to say that \mathbf{X} drives \mathbf{Y} because $y_t = x_{t-1}$. However, to show this result using a leaning calculation requires specification of the cause-effect assignment $\{C, E\} = \{\mathbf{X}, \mathbf{Y}\}$. A cause must precede an effect in the cause-effect assignment for consistency with the intuitive definition of causality. It follows that a natural assignment may be $\{C, E\} = \{x_{t-l}, y_t\}$ where $l \in [1, 9]$. This cause-effect assignment will be referred to as the l -standard assignment.

4.1 Defining the pendants

Given $\bar{\mathbf{T}}$, one possible pendant that can be defined using the 1-standard assignment is

$$\rho_{y_t=1, x_{t-1}=1} = \kappa \left(1 + \frac{P(x_{t-1}=1)}{1 - P(x_{t-1}=1)} \right) - \frac{P(y_t=1)}{1 - P(x_{t-1}=1)},$$

with $\kappa = P(y_t=1|x_{t-1}=1)$. Another pendant defined using this assignment would be the corresponding term with $\kappa = P(y_t=0|x_{t-1}=0)$. These two pendants are called the *observed* pendants because κ can be found directly from the time series data.

Equations for the unobserved pendants corresponding to $\kappa = P(y_t=0|x_{t-1}=1)$ and $\kappa = P(y_t=1|x_{t-1}=0)$ can be written down. These pendants are defined, but in both cases $\kappa = 0 \Rightarrow \rho_{y_t x_{t-1}} < 0$. Thus unobserved pendants imply the effect, $y_t = 0$ or 1 (for this toy model) is most likely not caused by the postulated cause, $x_{t-1} = 1$ or 0 , respectively. Using

these unobserved pendants to define leanings becomes a comparison of how unlikely postulated causes are to cause given effects. Such comparisons are not as easily interpreted in the intuitive framework of causality, and as such, are not explored as tools for causal inference in this article.

4.2 Finding the pendants from the data

The probabilities in the pendant calculations can be estimated from the time series data with frequency counts, e.g.

$$P(y_t=1|x_{t-1}=1) = \frac{n_{EC}}{n_C} = \frac{3}{3} = 1,$$

where n_{EC} is the number of times $y_t = 1$ and $x_{t-1} = 1$ appears in $\bar{\mathbf{T}}$, and n_C is related to the number of times the assumed cause, $x_{t-1} = 1$, has appeared in $\bar{\mathbf{T}}$ and is defined in more detail below.

Estimating the other two probabilities in this pendant calculation using frequency counts from $\bar{\mathbf{T}}$ is slightly more subtle. The underlying assumption that the assumed cause must precede the assumed effect must be considered when defining the frequency counts. This concern is addressed by shifting \mathbf{X} and \mathbf{Y} into $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ such that, for any given t , $\tilde{\mathbf{X}}_t$ precedes $\tilde{\mathbf{Y}}_t$, and defining

$$P(y_t=1) = \frac{n_E}{L} = \frac{3}{9} \quad (7)$$

and

$$P(x_{t-1}=1) = \frac{n_C}{L} = \frac{3}{9}, \quad (8)$$

where n_C is the number of times $\tilde{x}_t = 1$, n_E is the number of times $\tilde{y}_t = 1$, and L is the library length of $\tilde{\mathbf{X}}$ and $\tilde{\mathbf{Y}}$ (which are assumed to be the same length). For this example, those subsets are

$$\begin{aligned}\tilde{\mathbf{X}} &= \{0, 0, 1, 0, 0, 1, 0, 0, 1\} \\ \tilde{\mathbf{Y}} &= \{0, 0, 1, 0, 0, 1, 0, 0, 1\}\end{aligned}$$

which are both shorter than there counterparts above by a single value because the penchants are being calculated using the 1-standard cause-effect assignment. It follows that $\tilde{x}_t = x_{t-1}$ and $\tilde{y}_t = y_t$.

4.3 Mean observed leaning for \bar{T}

The two observed penchants in this example that assume \mathbf{X} causes \mathbf{Y} (i.e. using the 1-standard assignment) are found from the data to be

$$\rho_{y_t=1, x_{t-1}=1} = 1 \quad (9)$$

and

$$\rho_{y_t=0, x_{t-1}=0} = 1 \quad (10)$$

The complements of these observed penchants are found using the complementary 1-standard assignment of $\{C, E\} = \{y_{t-1}, x_t\}$ and are found from the data to be

$$\rho_{x_t=1, y_{t-1}=0} = \frac{3}{7} \quad (11)$$

$$\rho_{x_t=0, y_{t-1}=1} = \frac{3}{7} \quad (12)$$

and

$$\rho_{x_t=0, y_{t-1}=0} = -\frac{3}{7} \quad (13)$$

The *mean observed penchant* is the algebraic mean of the observed penchants, i.e.

$$\begin{aligned} \langle \rho_{y_t, x_{t-1}} \rangle &= \frac{1}{2} (\rho_{y_t=1, x_{t-1}=1} + \rho_{y_t=0, x_{t-1}=0}) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \langle \rho_{x_t, y_{t-1}} \rangle &= \frac{1}{3} (\rho_{x_t=1, y_{t-1}=0} \\ &\quad + \rho_{x_t=0, y_{t-1}=1} + \rho_{x_t=0, y_{t-1}=0}) \\ &= \frac{1}{7} \end{aligned}$$

The *mean observed leaning* follows from the definition of the mean observed penchants as

$$\langle \lambda_{y_t, x_{t-1}} \rangle = \langle \rho_{y_t, x_{t-1}} \rangle - \langle \rho_{x_t, y_{t-1}} \rangle \quad (14)$$

$$= \frac{6}{7} \quad (15)$$

The positive leaning implies the probability that x_{t-1} drives y_t is higher than the probability that y_{t-1} drives x_t ; i.e. $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ given the 1-standard cause-effect assignment. This result is expected and agrees with the intuitive definition of causality in this example.

4.4 Unobserved penchants

The *unobserved* penchants (using the 1-standard assignment from the beginning of the subsection) for this example are

$$\rho_{y_t=1, x_{t-1}=0} = -1 \quad (16)$$

$$\rho_{y_t=0, x_{t-1}=1} = -1 \quad (17)$$

and their complements are

$$\rho_{x_t=1, y_{t-1}=1} = -\frac{3}{7} \quad (18)$$

These values can be incorporated into the averaging calculation to yield a *mean total penchant*; i.e.

$$\begin{aligned} \langle \langle \rho_{y_t, x_{t-1}} \rangle \rangle &= \frac{1}{4} (\rho_{y_t=1, x_{t-1}=1} + \rho_{y_t=0, x_{t-1}=0} \\ &\quad + \rho_{y_t=1, x_{t-1}=0} + \rho_{y_t=0, x_{t-1}=1}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle \langle \rho_{x_t, y_{t-1}} \rangle \rangle &= \frac{1}{4} (\rho_{x_t=1, y_{t-1}=1} + \rho_{x_t=0, y_{t-1}=0} \\ &\quad + \rho_{x_t=1, y_{t-1}=0} + \rho_{x_t=0, y_{t-1}=1}) \\ &= 0 \end{aligned}$$

Thus, the *mean total leaning* (defined analogous to Eqn. 14) would be $\langle \langle \lambda_{y_t, x_{t-1}} \rangle \rangle = \langle \langle \rho_{y_t, x_{t-1}} \rangle \rangle - \langle \langle \rho_{x_t, y_{t-1}} \rangle \rangle = 0$ and would not be useful for casual inference in this example.

4.5 Cause-effect assignment independence

It may be argued that the causal inference above was a little disingenuous in that the assumed cause-effect relationship was known to

be correct. It can be shown, however, that causal inference is independent of the assumed cause-effect relationship. For example, consider the 1-standard cause-effect assignment $\{C, E\} = \{y_{t-l}, x_t\}$. The mean observed leaning would be

$$\begin{aligned} \langle \lambda_{x_t, y_{t-1}} \rangle &= \langle \rho_{x_t, y_{t-1}} \rangle - \langle \rho_{y_t, x_{t-1}} \rangle \quad (19) \\ &= -\frac{6}{7}, \quad (20) \end{aligned}$$

which implies $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$, as expected for this example.

In general, $\lambda_{AB} := \rho_{AB} - \rho_{BA} \Rightarrow -\lambda_{AB} = \rho_{BA} - \rho_{AB} := \lambda_{BA}$. Thus, the causal inference is independent of which times series is initially assumed to be the cause (or effect).

4.6 Weighted Mean Observed Leaning

The *weighted mean observed penchant* is defined similarly to the mean observed penchant but each term is weighted by the number of times that penchant appears in the data; e.g.

$$\begin{aligned} \langle \rho_{y_t, x_{t-1}} \rangle_w &= \frac{1}{L-l} (n_{y_t=1, x_{t-1}=1} \rho_{y_t=1, x_{t-1}=1} \\ &\quad + n_{y_t=0, x_{t-1}=0} \rho_{y_t=0, x_{t-1}=0}) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \langle \rho_{x_t, y_{t-1}} \rangle_w &= \frac{1}{L-l} (n_{x_t=1, y_{t-1}=0} \rho_{x_t=1, y_{t-1}=0} \\ &\quad + n_{x_t=0, y_{t-1}=1} \rho_{x_t=0, y_{t-1}=1} \\ &\quad + n_{x_t=0, y_{t-1}=0} \rho_{x_t=0, y_{t-1}=0}) \\ &= \frac{3}{63}, \end{aligned}$$

where n_{ab} is the number of times the assumed cause a appears with the assumed effect b in the data, L is the library length of the times series data, and l is the lag used in the l -standard cause-effect assignment under which these penchants are being calculated.

The *weighted mean observed leaning* follows naturally as

$$\begin{aligned} \langle \lambda_{y_t, x_{t-1}} \rangle_w &= \langle \rho_{y_t, x_{t-1}} \rangle_w - \langle \rho_{x_t, y_{t-1}} \rangle_w \\ &= \frac{60}{63}. \end{aligned}$$

For this example, $\langle \lambda_{y_t, x_{t-1}} \rangle_w \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ as expected.

Conceptually, the weighted mean observed penchant is preferred to the mean penchant because it accounts for the frequency of observed cause-effect pairs within the data, which is assumed to be a predictor of causal influence. For example, given some pair $\{\mathbf{A}, \mathbf{B}\}$, if it is known that a_{t-1} causes b_t and both $b_t = 0 | a_{t-1} = 0$ and $b_t = 0 | a_{t-1} = 1$ are observed in the data, then comparison of the frequencies with which the pair occur would be used to determine which of the two pairs represents the true cause-effect relationship and which pair represents, e.g., the effects of noise in the system.

For this example, the weighted mean observed leaning provides the same causal inference as the mean observed leaning. The weighted mean calculation will be used in the examples of the following sections.

4.7 Tolerance Domains

If the example time series was the expected time series but the data points were measured in the presence of noise, then the result may be the noisy time series pair $\bar{\mathbf{T}}' = \{\mathbf{X}', \mathbf{Y}'\}$ with

$$\begin{aligned} \mathbf{X}' &= \{x'_t \mid t \in [0, 9]\} \\ &= \{0, 0, 1.1, 0, 0, 1, -0.1, 0, 0.9, 0\} \\ \mathbf{Y}' &= \{y'_t \mid t \in [0, 9]\} \\ &= \{0, -0.2, 0.1, 1.2, 0, 0.1, 0.9, -0.1, 0, 1\}. \end{aligned}$$

The previous time series pair, $\bar{\mathbf{T}}$ had only five observed penchants, but $\bar{\mathbf{T}}'$ has more due to the noise. It can be seen in the time series

definitions that $x'_t = x_t \pm 0.1 := x_t \pm \delta_x$ and $x'_t = x_t \pm 0.2 := x_t \pm \delta_y$. The weighted mean observed leaning for $\bar{\mathbf{T}}'$ is $\langle \lambda_{y'_t, x'_{t-1}} \rangle_w \approx 0.19$.

The effects of noise on the leaning calculations can be addressed by using the tolerances δ_x and δ_y in the probability estimations from the data. For example, the penchant calculation in Eqn. 9 relied on estimating $P(y_t = 1 | x_{t-1} = 1)$ from the data, but if, instead, the data is known to be noisy, then the relevant probability estimate may be $P(y_t \in [1 - \delta_y, 1 + \delta_y] | x_{t-1} \in [1 - \delta_x, 1 + \delta_x])$.

If the tolerances, δ_x and δ_y , are made large enough, then the noisy system (i.e. $\bar{\mathbf{T}}'$) weighted mean observed leaning, $\langle \lambda_{y'_t \pm \delta_y, x'_{t-1} \pm \delta_x} \rangle_w$, can, at least in the simple examples considered here, be made equal to the noiseless system (i.e. $\bar{\mathbf{T}}$) weighted mean observed leaning, i.e. $\langle \lambda_{y'_t \pm \delta_y, x'_{t-1} \pm \delta_x} \rangle_w = \langle \lambda_{y_t, x_{t-1}} \rangle_w$.

The noise in this example did not affect the causal inference, i.e. $\langle \lambda_{y'_t, x'_{t-1}} \rangle_w \Rightarrow \mathbf{X}' \xrightarrow{\text{lean}} \mathbf{Y}'$ as expected. However, consider the time series pair $\bar{\mathbf{T}}'' = \{\mathbf{X}, \mathbf{Y}'\}$, which is a clean impulse \mathbf{X} with a noisy response \mathbf{Y}' . The 1-standard assignment mean observed penchants for $\bar{\mathbf{T}}''$ are $\langle \rho_{y'_t=a, x_{t-1}=b} \rangle$ with $\{a, b\} = \{0, 0\}, \{0, -0.2\}, \{0, -0.1\}, \{0, -0.1\}, \{1, 1.2\}, \{1, 0.9\},$ and $\{1, 1\}$, and $\langle \rho_{x_t=c, y'_{t-1}=d} \rangle$ with $\{c, d\} = \{0, 0\}, \{-0.2, 1\}, \{0.1, 0\}, \{1.2, 0\}, \{0, 1\}, \{0.9, 0\},$ and $\{0, 0\}$, which lead to a weighted mean observed leaning of $\langle \lambda_{y'_t, x_{t-1}} \rangle_w \approx -0.05$. This value implies $\mathbf{Y}' \xrightarrow{\text{lean}} \mathbf{X}$, which disagrees with intuition. The signal $x_{t-1} = 1$ is known to cause the response $y_t = 1$, but in the noisy response $x_{t-1} = 1$ is observed to precede three different possible effects (i.e. $y_t = 1.2, 0.9,$ or 1). This effect of the noise in \mathbf{Y}' makes it more difficult to correctly infer how x_t might be related to y_t , which is illustrated sharply by the counterintuitive leaning calculation in this example.

It is not always true that a clean impulse

and noisy response lead to leanings that disagree with intuition (e.g. the pair $\{\mathbf{X}, \mathbf{Y}'' = \{0, 0, 0, 1.2, 0, 0, 0.9, -0.1, 0, 1\}\}$ has a weighted mean observed leaning of $\langle \lambda_{y''_t, x_{t-1}} \rangle_w \approx 0.43 \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}''$ as expected). However, tolerance domains are an important part of using leaning calculations for causal inference.

Tolerance domains, however, can be set too large. If the tolerance domain is large enough to encompass every point in the time series, then the probability of the assumed cause becomes one, which leads to undefined penchants. For example, given the symmetric definition of the tolerance domain used in this section, $\delta_x = 2$ implies $P(x_{t-1} = 1 \pm \delta_x) = 1$, which implies $\langle \lambda_{y'_t, x_{t-1}} \rangle_w$ is undefined.

It follows that the use of leaning calculations for causal inference depends on an understanding on the noise in the data, which can be troublesome if very little is known about the data sources. One strategy is to calculate the leanings with several different tolerances, increasing the size of the tolerance domains to the point where the penchants become undefined, and finding the tolerance domains for which the leaning changes sign. The sizes of these domains can then be compared to suspected noise levels. This strategy, and others, will be discussed in more detail in the example data sections below. If the noise level is known, then the task becomes much simpler and the tolerances should just be set to the known (or estimated) noise levels for the individual time series.

4.8 Stationarity Dependence

Both \mathbf{X} and \mathbf{Y} are stationary in the original example time series pair $\bar{\mathbf{T}}$. Given a time series pair $\bar{\tau}_{10} = \{\mathbf{X}, \mathbf{R} = \{0, 0, 0, 1, 1, 1, 2, 2, 2, 3\}\}$ containing a non-stationary response signal \mathbf{R} , the weighted mean observed leaning calculated under the 1-standard assignment with no tolerance domains still leads to a causal inference

that agrees with intuition; i.e. $\langle \lambda_{r_t, x_{t-1}} \rangle_w \approx 0.11 \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{R}$ as expected. This result, however, depends on the library length of the data.

$\bar{\tau}_{10}$ is a specific instance of the following time series pair:

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{R}\} = \{\{x_t\}, \{r_t\}\} \quad (21)$$

where $t \in [0, L]$,

$$x_t = \begin{cases} 0 & \forall t \in \{t \mid t \bmod 3 \neq 0\} \\ 1 & \forall t \in \{t \mid t \bmod 3 = 0\} \end{cases} \quad (22)$$

and

$$r_t = x_{t-1} + r_{t-1} \quad (23)$$

with $r_0 = 0$. The weighted mean observed leaning, under the 1-standard assignment with no tolerance domains, for $\bar{\tau}_L$ depends on L . As L is increased, the leaning calculation will eventually lead to causal inferences that do not agree with intuition; e.g. $L = 20 \Rightarrow \langle \lambda_{r_t, x_{t-1}} \rangle_w \approx 1.8 \times 10^{-3} \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{R}$ and $L = 50 \Rightarrow \langle \lambda_{r_t, x_{t-1}} \rangle_w \approx -2.5 \times 10^{-3} \Rightarrow \mathbf{R} \xrightarrow{\text{lean}} \mathbf{X}$.

As the L is increased, the number of possible observed effects for a given observed cause increases. Thus, under the 1-standard assignment $\{C, E\} = \{x_{t-1}, r_t\}$, $x_{t-1} = 1$ precedes three different values, $r_t = 1, 2$, and 3, if $L = 10$, but it precedes fifteen different values if $L = 50$. The leaning calculations are methods for counting (in a specific way) the number of times (and ways in which) an observed cause-effect pair appears in the data. The causal inference becomes more difficult for non-stationary time series pairs because repeated cause-effect pairs in the data may be more rare than in the cyclic stationary examples. This effect is very similar to the effect seen when the impulse signal was clean but the response was noisy. Unfortunately, it cannot be remedied with tolerance domains for the non-stationary case. For example, for $\bar{\tau}_L$, the cardinality of the set $\{r_t \mid x_{t-1} = 1\} \rightarrow \infty$ as

$L \rightarrow \infty$, and penchants are not defined given a tolerance domain for \mathbf{R} of $\delta_r = \infty$.

These shortcomings of the weighted mean observed leaning when applied to non-stationary data, however, do not imply that causal inference of non-stationary data cannot be done using a different application of the observed penchants. For example, replacing the weighted mean calculation in the weighted mean observed leaning calculation with a median calculation leads to a *median observed leaning*, $[\lambda_{r_t, x_{t-1}}] \approx 5.3 \times 10^{-3} \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{R}$ for $L = 50$ as expected, where $[\cdot]$ is used to denote the median. Of course, even though the median leaning calculation agrees with intuition for a library length where the mean leaning calculation did not, there is no reason to believe the median leaning calculation will not also eventually provide counterintuitive causal inferences as L is increased.

A more basic strategy to deal with non-stationary data would be to define the observed penchant using a different cause-effect assignment. For example, the l -standard assignment (with $l = 1$) used above, i.e. $\{C, E\} = \{x_{t-1}, r_t\}$, might be replaced with an l -AR (autoregressive) assignment with $l = 1$ of $\{C, E\} = \{(x_{t-1}, r_{t-1}), r_t\}$. An observed penchant may be calculated with an assumed cause of $(x_{t-1} = 1, r_{t-1} = 0)$ and an assumed effect of $r_t = 1$. The algorithms to compute the observed penchants from the data become more complicated as the cause-effect assignment becomes more complicated, but the basic definition of the penchant provides a very general conceptual framework for causal inference. This work will only use the l -standard cause-effect assignment in the examples that follow.

5 Simple Example Systems

In this section, the leaning, specifically the weighted mean observed leaning using the l -

standard cause-effect assignment for various l , will be applied to dynamical systems and empirical data sets with known causal relationships. The usefulness of the leaning as a tool for causal inference is tested directly with empirical and synthetic time series data sets for which there is an intuitive understanding of the driving relationships within the system.

5.1 Impulse with Noisy Response Linear Example

Consider the linear example dynamical system of

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{Y}\} = \{\{x_t\}, \{y_t\}\} \quad (24)$$

where $t \in [0, L]$,

$$x_t = \begin{cases} 2 & t = 1 \\ 0 & \forall t \in \{t \mid t \neq 1 \text{ and } t \bmod 5 \neq 0\} \\ 2 & \forall t \in \{t \mid t \bmod 5 = 0\} \end{cases}$$

and

$$y_t = x_{t-1} + B\eta_t$$

with $y_0 = 0$, $B \in \mathbb{R} \geq 0$ and $\eta_t \sim \mathcal{N}(0, 1)$. Specifically, consider $B \in [0, 1]$. The driving system \mathbf{X} is a periodic impulse with a signal amplitude above the maximum noise level of the response system, and the response system \mathbf{Y} is a lagged version of the driving signal with standard Gaussian noise of amplitude B applied at each time step.

Figure 1 shows how the weighted mean observed leaning using the 1-standard cause-effect assignment, $\tilde{\lambda}$, changes as the noise amplitude B and tolerance δ_y in increments of 0.01. The synthetic data sets \mathbf{X} and \mathbf{Y} are constructed such that intuitively \mathbf{X} drives \mathbf{Y} . Thus, it is expected that $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ which implies $\tilde{\lambda} > 0$. Figure 1 shows that this expectation is met except when $\delta_y < B$ even for a short library length of $L = 10$. Examples of undefined penchants due to large tolerance domains, as discussed in section 4.7, are seen as δ_y is increased in the $L = 10$ example.

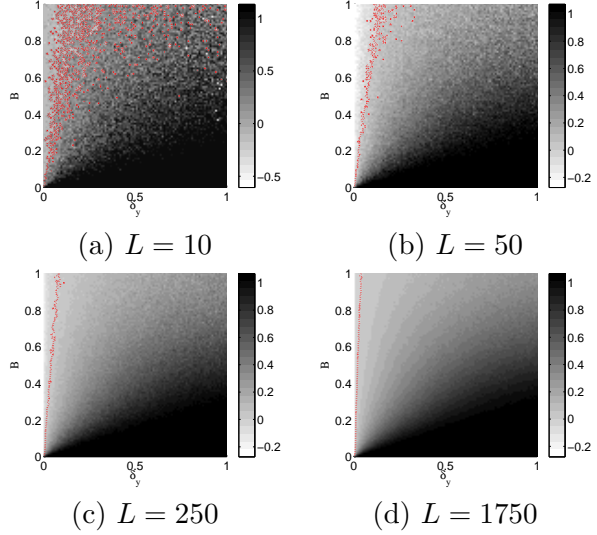


Figure 1: (Color available online.) The unitless leaning is a function of both the noise, the tolerance used for terms from \mathbf{Y} , and the library length of the signals. This synthetic data is used to explore the leaning, so \mathbf{X} and \mathbf{Y} do not have explicit units, from which it follows that both δ_y and B are unitless. The red dashed line is the zero contour. See the text for an explanation of the missing data for large δ_y .

Figure 1 shows using the strategy of $\delta_y = B$ always leads to causal inferences that agree with intuition for $L > 10$ in this example. However, as discussed in section 4.7, knowing B *a priori* may be unrealistic with empirical data sets. Consider the following three methods for estimating δ_y from the data:

1. *lagged linear response deviation* - The y-tolerance is set to the mean absolute deviation of y_t from x_{t-1} ; i.e. $\delta_y = \langle |y_t - x_{t-1}| \rangle$.
2. *normalized standard deviation* - The y-tolerance is set to the standard deviation of $\{|\mathbf{Y} - \langle \mathbf{Y} \rangle|\}$ where $\langle \mathbf{Y} \rangle$ is the mean of \mathbf{Y} ; i.e. $\delta_y = \sigma_{|y_t - \langle y_t \rangle|}$.
3. *n-bin mean standard deviation* - The y-tolerance is set to the mean standard deviation of y_t across n bins.

B	method 1	method 2	method 3
0.0	1.0	1.0	1.0
0.1	0.40	1.0	0.48
0.5	0.39	0.79	0.26
0.8	0.30	0.44	0.10

Table 1: $\tilde{\lambda}$ using three different estimation methods for δ_y .

ation of n bins of \mathbf{Y} ; i.e. $\delta_y = \langle \sigma_{B_i} \rangle$ where B_i is the i th bin of an n -bin histogram of \mathbf{Y} .

Table 1 shows $\tilde{\lambda}$ for instances of Eqn. 24 with $B = 0, 0.1, 0.5$, and 0.8 given $L = 100$, and $n = 5$ in method 3 listed above.

The three different methods yield different values for the leaning, but all the methods lead to the same causal inference, $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$, which agrees with intuition for this example. These methods are meant to be examples of using the data to set δ_y if B is not known. These methods are not expected to be reasonable estimates for δ_x and δ_y in general. For example, method 1 assumes a linear relationship between \mathbf{X} and \mathbf{Y} that may be unreasonable to assume in general. However, Table 1 shows different methods for setting δ_y can lead to the same causal inference. Setting the tolerances requires an understanding of the noise in the time series data. The leaning is meant to be part of an exploratory causal analysis of the time series data and cannot exist independently of other exploratory analysis of the data, including analysis of the noise levels.

This linear impulse response example has only included calculations of leaning using the 1-standard assignment. The 1-standard assignment is expected to be useful for causal inference given Eqn. 24. However, deciding which l -standard assignment to use given empirical, rather than synthetic, data sets may be more difficult. It is expected that a several different l -standard assignments would be

used as part of any exploratory causal analysis using leaning. The next section introduces an example that plots the leanings for a set of different l -assignments and shows the maximum leaning in the set is near the expected value, i.e. near the lag value that appears explicitly in the dynamical system used to create the synthetic data sets.

5.2 Cyclic Linear Example

Consider the linear example dynamical system of

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{Y}\} = \{\{x_t\}, \{y_t\}\} \quad (25)$$

where $t \in [0, L]$,

$$x_t = \sin(t)$$

and

$$y_t = x_{t-1} + B\eta_t$$

with $y_0 = 0$, $B \in \mathbb{R} \geq 0$ and $\eta_t \sim \mathcal{N}(0, 1)$. Again, we will only consider $B \in [0, 1]$. This example is very similar to the previous one, except that the driving system \mathbf{X} is sinusoidal.

Figures 2 and 3 were calculated for an instance of Eqn. 25 with $L = 41$ generated by sampling one period of \mathbf{X} ; i.e. $t \in \{0, f\pi, 2f\pi, 3f\pi, \dots, 2\pi\}$ with $f = 1/20$ implies $L = 41$. Figure 2 shows the weighted mean observed leaning using the 1-standard assignment, λ , is always positive given $\delta_y = B$. So, as was seen in the previous example, the leaning implies $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$, which agrees with intuition for this example.

It has been argued that lagged cross-correlation techniques are the preferred causal inference tool in most situations because of their simplicity [?]. The lagged cross-correlation is defined as

$$\chi_{xy}^l = \frac{\mathbb{E}[(x_t - \mu_x)(y_{t-l} - \mu_y)]}{\sigma_x \sigma_y}, \quad (26)$$

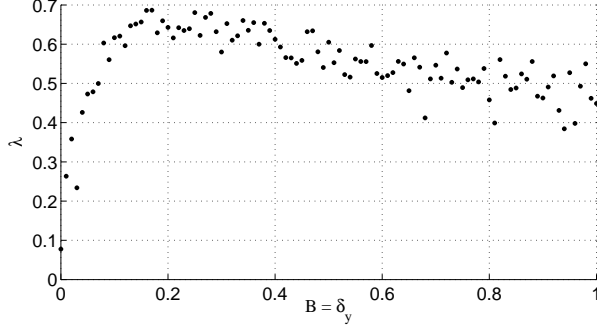


Figure 2: Given $\delta_y = B$, the leaning λ is always positive.

where $E[z_t]$ is the expectation value of $\{z_t\}$, $\mu_{x(y)}$ is the mean of \mathbf{X} (\mathbf{Y}), and $\sigma_{x(y)}$ is the standard deviation of \mathbf{X} (\mathbf{Y}). The cross-correlation is often used for causal inference by introducing a difference quantity [?]

$$\delta\chi_{xy}^l = \chi_{xy}^l - \chi_{yx}^l . \quad (27)$$

The sign of $\delta\chi_{xy}^l$ is used, similar to the leaning approach, to determine the causal inference; i.e. $\delta\chi_{xy}^l > 0$ implies \mathbf{X} “causes” \mathbf{Y} and $\delta\chi_{xy}^l < 0$ implies \mathbf{Y} “causes” \mathbf{X} [?].

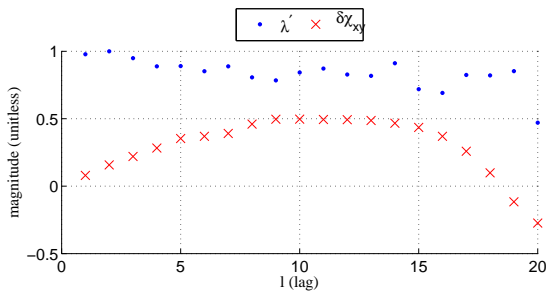


Figure 3: (Color available online.) The unitless, normalized leaning, λ' , can be plotted for different l -standard cause-effect assignments along with the cross correlation, χ for the same lags, l , to show how the two values compare for this simple cyclic example.

Figure 3 shows how $\delta\chi_{xy}^l$ compares to the leaning given $l \in [1, 21]$ for instances of Eqn.

25 with $B = 0.5$. In Figure 3, the leaning has been normalized for presentation clarity as

$$\lambda' = \frac{\lambda_l}{\max_{l \in [1, 15]} \lambda_l} , \quad (28)$$

where λ_l is the weighted mean observed leaning using the l -standard assignment (λ_1 is plotted in Figure 2). Both λ' and $\delta\chi_{xy}^l$ lead to the same causal inference, i.e. \mathbf{X} “drives” \mathbf{Y} , for $l \in [1, 19]$, although only the leaning agrees with intuition for $l = 20$ and $l = 21$ in this example. Thus, both tools agree with intuition for small lags in this simple cyclic example. The leaning, however, has its maximum values near the smallest lags, which is expected given Eqn. 25, while the cross-correlation difference has its maximum values at lags that do not explicitly appear in Eqn. 25.

The cross-correlation difference technique is also known to be unreliable given nonlinear dynamics [?]. Leanings of data sets generated from nonlinear dynamics will be discussed in Section 5.4. Neither of the previous examples has been physically motivated, so the next section discusses exploratory causal inference of synthetic data sets generated from the well-known dynamics of a physical system.

5.3 RL Circuit Example

Consider a series circuit containing a resistor, inductor, and time varying voltage source related by

$$\frac{dI}{dt} = \frac{V(t)}{L} - \frac{R}{L}I, \quad (29)$$

where I is the current at time t , $V(t) = \sin(t)$ is the voltage at time t , R is the resistance, and L is the inductance. The time series pair for this example is then

$$\bar{\tau}_L = \{\mathbf{V}, \mathbf{I}\} = \{\{V_t\}, \{I_t\}\} \quad (30)$$

where \mathbf{V} is the set of discrete values of $V(t)$ evaluated using $t \in \{0, f\pi, 2f\pi, 3f\pi, \dots, 8\pi\}$

with $f = 1/10$ and \mathbf{I} is the set of discrete values found either by solving Eqn. 29 numerically or by evaluating the analytical solution, i.e.

$$I(t) = \frac{L}{D}e^{-\frac{t}{\tau}} + \frac{R}{D}\sin(t) - \frac{L}{D}\cos(t) \quad (31)$$

with $D = L^2 + R^2$ and $\tau = L/R$, for the same time set used for \mathbf{V} .

Physical intuition is that V drives I , and so we expect to find that $V \xrightarrow{\text{lean}} I$. The weighted mean observed leaning using the 1-standard assignment, λ_1 , can be used to test this expectation. Unlikely, the previous examples, however, there is no noise term in the dynamics (such as B in Eqn. 24 and 25), so setting δ_I will not be as straightforward.

Table 2 shows λ_1 for both the analytical solution and a numerical solution to Eqn. 29 was using the *ode45* integration function in MATLAB. The time series $V(t)$ is created by defining values at fixed points and using linear interpolation to find the time steps required by the ODE solver. Two different physical scenarios are considered in which L and R are constant, $L = 10$ Henries and $R = 5$ Ohms and $L = 5$ Henries and $R = 20$ Ohms.

The previously discussed strategy of increasing δ_I until the leaning becomes undefined and then reporting the leaning calculated using the largest δ_I for which it is defined would lead to the a causal inference that agrees with intuition for this example. Specifically, from Table 2(a) $\delta_I = 10^{-2} \Rightarrow \lambda_1 \approx 0.7 \Rightarrow \mathbf{V} \xrightarrow{\text{lean}} \mathbf{I}$, as expected.

Discussion on setting the tolerance domains has centered on understanding the noise in the system. This example illustrates that the “noise” being considered does not need to be a physical noise source in the system (there are no explicit noise terms in Eqn. 30). For example, the numerical tolerance of the ODE solver was set to 10^{-3} in both Table 2(a) and (b), and the table shows setting $\delta_y = 10^{-3}$ would lead

to causal inferences that agree with intuition in both cases. However, $\delta_y = 0$ leads to causal inferences that do not agree with intuition in both cases even for the analytical solution.

Consider, for example, the peak values of \mathbf{V} . The time steps of these peaks are $\mathbf{T}_{\text{peak}} = \{t|V_t = 1\} = \{6, 26, 46, 66\}$. The values of \mathbf{I} given $\tau = 0.25$ that immediately follow these peaks are $\mathbf{I}_{0.25}^{\text{peak}} = \{I_t|t \in \{7, 27, 47, 67\}\}$. The same values given $\tau = 2$ will be labeled $\mathbf{I}_2^{\text{peak}}$. The standard deviation of the first set is $\sigma_{0.25}^{\text{peak}} \approx 10^{-6}$ and the standard deviation of the second set is $\sigma_2^{\text{peak}} \approx 10^{-2}$. Table 2(a) (for $\sigma_{0.25}^{\text{peak}}$) and (b) (for σ_2^{peak}) shows setting δ_I to the appropriate standard deviation of the peaks would lead to causal inferences that agree with intuition. Rather than physical noise levels, the noise levels used to set the tolerance domains for the leaning calculations is better thought of as the spread in the possible values of an assumed effect that may reasonably be considered due to the same assumed cause.

This example can also illustrate the importance of sample frequency and periods. The leaning calculation requires an assumed cause and effect pair to appear in the data enough times to provide a good estimate of the probability. Thus, data that is sampled for too few periods or too sparsely can lead to counter-intuitive leanings. For example, if there is only a single peak in the assumed driving time series because of poor sampling, then there can only be a single response value, which would be insufficient to reliably provide the conditional probabilities in the leaning calculation for that assumed cause-effect pair. For Eqn. 30 with the analytical solution for \mathbf{I} , if $\delta_I = 10^{-3}$ and $\tau = 0.25$, then $t \in \{0, f\pi, 2f\pi, 3f\pi, \dots, 2\pi\}$ with $f = 1/10$ leads to $\lambda_1 = -0.045$ and $t \in \{0, f\pi, 2f\pi, 3f\pi, \dots, 3\pi\}$ with $f = 2/3$ leads to $\lambda_1 = -0.167$, both of which disagree with intuition.

δ_I	λ_1 (<i>ode45</i>)	λ_1 (analytical)	δ_y	λ_1 (<i>ode45</i>)	λ_1 (analytical)
0	-0.132	-0.089	0	-0.132	-0.132
10^{-6}	-0.132	0.493	10^{-6}	-0.132	-0.132
10^{-5}	-0.108	0.548	10^{-5}	-0.120	-0.096
10^{-4}	0.188	0.564	10^{-4}	0.011	0.098
10^{-3}	0.582	0.581	10^{-3}	0.398	0.386
10^{-2}	0.730	0.727	10^{-2}	0.676	0.675
10^{-1}	<i>undefined</i>	<i>undefined</i>	10^{-1}	0.314	0.315
(a) $R = 20 \Omega, L = 5 \text{ H}$			(b) $R = 5 \Omega, L = 10 \text{ H}$		

Table 2: The leaning λ_1 depends on both δ_I and the method for evaluating \mathbf{I} in this example. These two tables show that the values of δ_I for which the leaning starts to agree with intuition can also depend on the physical system parameters (e.g., τ).

The examples so far have all had a linear relationship between the driving signal and the response signal. Of the four broad categories of time series causality tools, transfer entropy [1] and SSR methods [2] are the two categories that can be applied to nonlinear datasets without modification. The conceptual framework of granger causality is not restricted by the linearity of the data set [1], but the original formulation by Granger must be modified to do so [3]. Lagged cross-correlation techniques are known to be unreliable if the data sets are generated by nonlinear dynamics [4, 5]. The next example explores the use of leaning calculations in nonlinear systems.

5.4 Nonlinear Example

Consider the nonlinear dynamical system of

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{Y}\} = \{\{x_t\}, \{y_t\}\} \quad (32)$$

where $t \in [0, L]$,

$$x_t = \sin(t)$$

and

$$y_t = Ax_{t-1}(1 - Bx_{t-1}) + C\eta_t,$$

with $y_0 = 0$, with $A, B, C \in \mathbb{R} \geq 0$ and $\eta_t \sim \mathcal{N}(0, 1)$. Specifically, consider $A, B, C \in$

$[0, 1]$ and $t \in \{0, f\pi, 2f\pi, 3f\pi, \dots, 6\pi\}$ with $f = 1/30$, which implies $L = 181$.

Figure 4 shows the weighted mean observed leaning using the 1-standard assignment, i.e. λ_1 , agrees with intuition over the considered domains of A , B , and C if the tolerance domain for \mathbf{Y} is set to the noise level, i.e. $\delta_y = C$. The result of $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ shows that causal inference using leanings on data sets generated from nonlinear dynamics can be performed similarly, and can lead to similarly intuitive results, as the data sets generated from linear dynamics.

Proponents of SSR time series causality tools have pointed out the limitations of tools like lagged cross-correlation and Granger causality when the dynamics exhibit chaotic behavior [6]. Chaotic dynamics are explored in the next section.

5.5 Coupled Logistic Map Example

Consider the nonlinear dynamical system of

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{Y}\} = \{\{x_t\}, \{y_t\}\} \quad (33)$$

where $t \in [0, L]$,

$$x_t = x_{t-1}(r_x - r_x x_{t-1} - \beta_{xy} y_{t-1})$$

and

$$y_t = y_{t-1}(r_y - r_y y_{t-1} - \beta_{yx} x_{t-1})$$

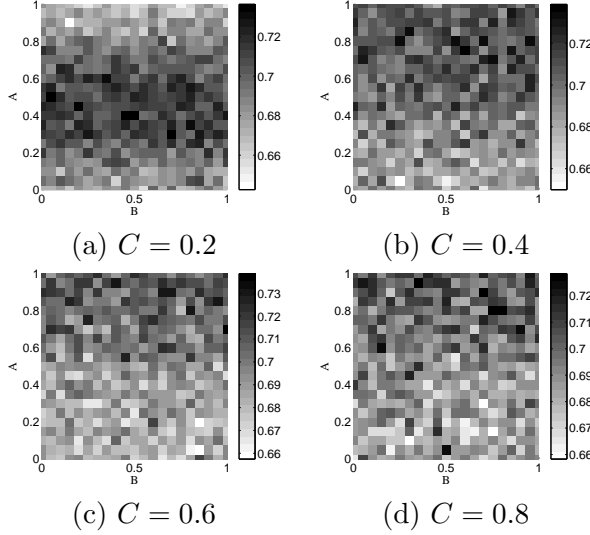


Figure 4: Leaning, λ_1 , is a function of all three unitless parameters in Eqn. 32, A , B , and C . The leaning agrees with intuition in this example for all the tested values of A , B , and C given $\delta_y = C$.

where the parameters $r_x, r_y, \beta_{xy}, \beta_{yx} \in \mathbb{R} \geq 0$. This pair of equations is a specific form of the two-dimensional coupled logistic map system often used to model population dynamics [?] and it was the example system used by Sugihara *et al.* in their introduction of cross convergent mapping (CCM; an SSR time series causality tool) [?].

The idea is that $\beta_{xy} > \beta_{yx}$ intuitively implies \mathbf{Y} “drives” \mathbf{X} more than \mathbf{X} “drives” \mathbf{Y} , and vice versa. Such intuition, however, can be difficult to justify for all instances of Eqn. 33. The x_{t-1} term that appears in y_t can be seen as a function of x_{t-2} with coefficients of $\beta_{yx}r_x$. These product coefficients suggest that if $r_x > r_y$, then \mathbf{X} may be seen as the stronger driver in the system even if $\beta_{yx} < \beta_{xy}$. The same argument can be made, with the appropriate substitutions, to show that \mathbf{Y} may be seen as the stronger driver in the system even if $\beta_{xy} < \beta_{yx}$. As such, there is no clear in-

tuitive causal inference for this system. The conjectures presented in this paragraph, however, are supported by the leaning calculations (using the 1-standard assignment).

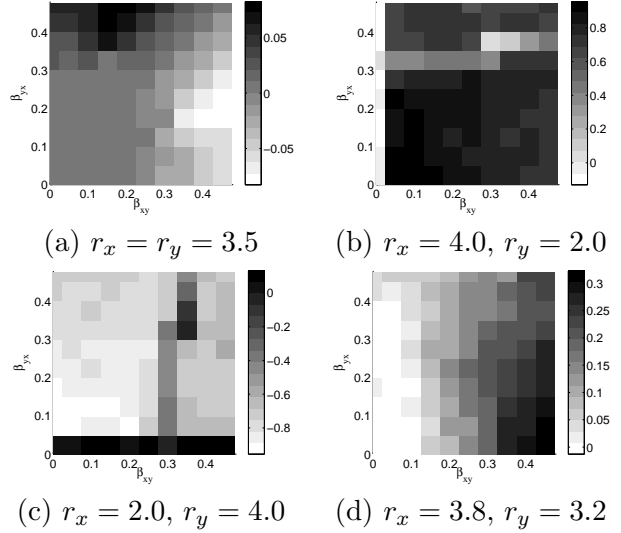


Figure 5: Leaning, λ_1 , is a function of four unitless parameters in Eqn. 33, r_x , r_y , β_{xy} , and β_{yx} (along with the initial conditions x_0 and y_0 , which are fixed in this example). The tolerance domains are set as $\delta_x = \sigma_{x_t - \langle x_t \rangle}$ and $\delta_y = \sigma_{y_t - \langle y_t \rangle}$. The leaning is defined using the 1-standard assignment, so $\lambda_1 > 0 \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$.

Figure 5 shows four instances of Eqn. 33 with different values for r_x and r_y . Each instance has a library length of $L = 500$ and initial conditions of $x_0 = 0.4$ and $y_0 = 0.4$. There is no clear, intuitive driver in this example, so both tolerance domains must be set in the leaning calculation. The leaning is calculated using the 1-standard cause-effect assignment and estimated tolerance domains of $\delta_x = \sigma_{x_t - \langle x_t \rangle}$ and $\delta_y = \sigma_{y_t - \langle y_t \rangle}$.

Figure 5(a) shows the intuition of $\beta_{xy} < \beta_{yx} \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ can be true if $r_x = r_y$. However, Figure 5(b) and (c) shows $r_x > r_y \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ and $r_x < r_y \Rightarrow \mathbf{Y} \xrightarrow{\text{lean}} \mathbf{X}$ can be

strong enough implications to make the values of β_{xy} and β_{yx} irrelevant over the considered domains. Figure 5(d) shows this effect can be pronounced even in instances of Eqn. 33 where r_x and r_y are close.

The complexity of determining causal relationships in this system may make the system less of a convincing example of the leaning calculation than the previous examples. However, Figure 5 shows the weighted mean observed leaning using the 1-standard cause-effect assignment can provide causal inferences that may be considered intuitively justifiable, even if the system does not have an unequivocal driver.

All of the previous examples have ignored the concept of causal confounders. The presence of confounders in the system is a serious problem for causal inference in general [?, ?]. Time series causality usually seeks to answer the less general causal inference question of “Given these two times series, which may be considered the stronger driver?” Nevertheless, some bivariate time series causality tools consider causal inference in systems with potential confounders by trying to relate the estimated bivariate driving relationships within a collection of more than two time series data sets (e.g., see CCM [?]). The next example explores in the use of leaning calculations in such a scenario.

5.6 Impulse with Multiple Noisy Responses Example

Consider the multivariate system of

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\} = \{\{x_t\}, \{y_t\}, \{z_t\}\} \quad (34)$$

where $t \in [0, L]$,

$$x_t = \begin{cases} 2 & t = 1 \\ 0 & \forall t \in \{t \mid t \neq 1 \text{ and } t \bmod 5 \neq 0\} \\ 2 & \forall t \in \{t \mid t \bmod 5 = 0\} \end{cases}$$

and

$$y_t = x_{t-1} + B\eta_t ,$$

and either (case 0)

$$z_t = y_{t-1} \quad (35)$$

or (case 1)

$$z'_t = y_{t-1} + y_t = y_{t-1} + x_{t-1} + B\eta_t \quad (36)$$

or (case 2)

$$z''_t = y_{t-1} + x_{t-1} + z_{t-1} \quad (37)$$

with $y_0 = 0$, $B \in \mathbb{R} \geq 0$, $\eta_t \sim \mathcal{N}(0, 1)$, and $L = 500$.

In case 0, \mathbf{Z} depends directly on \mathbf{Y} and indirectly on \mathbf{X} (through \mathbf{Y} , which depends directly on \mathbf{X}). The intuitive causal inference is then $\mathbf{Y} \xrightarrow{\text{lean}} \mathbf{Z}$ and $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Z}$. Case 1, despite the additional \mathbf{Y} dependence in \mathbf{Z} , has the same intuitive causal inference as case 0. In case 2, however, \mathbf{Z} depends directly on itself and both \mathbf{Y} and \mathbf{X} . Case 2 also has the same intuitive causal inference.

Figure 6 shows the weighted mean observed leaning using the 1-standard cause-effect assignment (with $\delta_x = 0$ and $\delta_y = \delta_z = B$), λ_1 , may lead to causal inferences that do not agree with intuition for case 0 and case 2, even though case 1 agrees with intuition for all points within the considered noise levels domain. For case 0, the leaning calculation implies $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$ and $\mathbf{Y} \xrightarrow{\text{lean}} \mathbf{Z}$ as expected, but it also seem to imply that no causal inference can be made about the relationship between \mathbf{X} and \mathbf{Z} . For case 2, the leaning calculation also implies no causal inference can be made about the relationship between \mathbf{X} and \mathbf{Z} , but, unlike case 0, it also implies $\mathbf{Z} \xrightarrow{\text{lean}} \mathbf{Y}$, which is counter-intuitive.

These results may imply that λ_1 is unable identify confounded driving (i.e., situations in which the effect of the driving variable is mediated by another variable). For example, in case 0, the driving of \mathbf{Z} by \mathbf{X} is occurs through the interaction of \mathbf{Y} and \mathbf{Z} . For case 0, λ_1

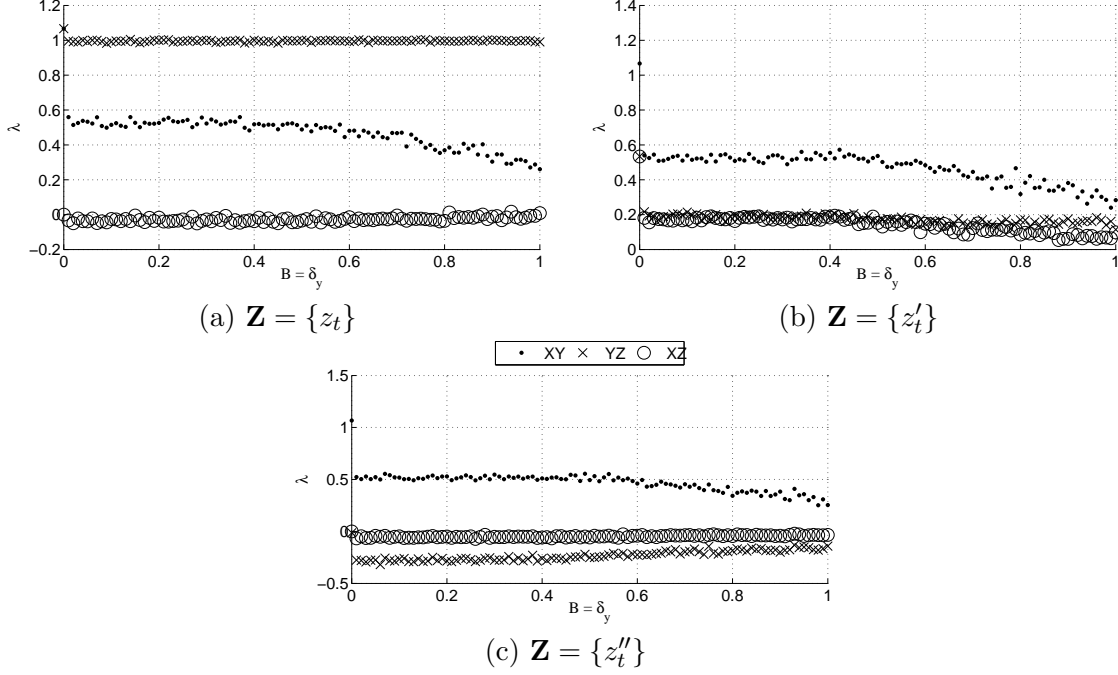


Figure 6: The (unitless) weighted mean observed leaning using the 1-standard cause-effect assignment, λ_1 only leads to causal inferences that agree with intuition for all points within the considered noise level domain, B , in this example for case 1. B is the unitless noise parameter found in Eqn. 34, and the symmetric tolerance domains for \mathbf{Y} and \mathbf{Z} are set to this value.

implies $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y} \xrightarrow{\text{lean}} \mathbf{Z}$ but does not imply $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Z}$. For case 2, λ_1 implies $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y} \xleftarrow{\text{lean}} \mathbf{Z}$, which may imply that λ_1 is not a reliable causal inference tool in autoregressive systems.

The results of Figure 6 may also be considered an indication that the cause-effect assignment is insufficient. It was previously mentioned that exploratory causal analysis using the leaning would involve comparing several different cause-effect assignments. The set of tested cause-effect assignment need not only include l -standard assignment. Consider the weighted mean observed leaning, λ_{AR}^{xy} using the 1-AR cause-effect assignment, i.e., $\{C, E\} = \{x_{t-1} \text{ and } y_{t-1}, y_t\}$. Table 3 shows this leaning calculation, using $\delta_y = \delta_z = B = 0.6$, for the same bivariate relationships shown in Figure 6.

	case 0	case 1	case 2
λ_{AR}^{xy}	0.150	0.159	0.169
λ_{AR}^{yz}	-0.002	0.133	0.447
λ_{AR}^{xz}	0.691	0.030	0.735

Table 3: The leaning calculation depends strongly on the cause-effect assignment. The table shows the weighted mean observed leaning using the 1-AR assignment and may be compared with Figure 6, which showed this leaning calculation using the 1-standard assignment.

Table 3 implies $\mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y} \xrightarrow{\text{lean}} \mathbf{Z}$ for case 1 and 2, as expected, but not for case 0 (which λ_1 did imply). Thus, the leaning calculations are part of an exploratory causal anal-

ysis and must be considered using several different cause-effect assignments when trying to understand the potential causal structure of a set of times series data. The cause-effect assignments can also be expanded beyond the bivariate and autoregressive definitions, e.g., $\{C, E\} = \{x_{t-1} \text{ and } y_{t-1} \text{ and } z_{t-1}, y_t\}$, but such extensions will not be considered in this article.

6 Empirical Data

Empirical data sets with known (or assumed) causal relationships may be used to understand how exploratory causal inference using leanings might be done if the system dynamics are unknown (or sufficiently complicated to make first principle numerical comparisons cumbersome).

Figure 7 shows a time series pair with causal “truth” from the UCI Machine Learning Repository (MLR) [1]. This data repository is a collection of data sets (not necessarily time series data) with known, intuitive, or assumed causal relationships meant for use in the testing of causal discovery algorithms in machine learning [1].

Figure 7(a) and (b) are times series of the daily snowfall (the expected response) and mean temperature (the expected driver) from July 1 1972 to December 31 2009 at Whistler, BC, Canada (Latitude: 50°04′04.000″ N, Longitude: 122°56′50.000″ W, Elevation: 1835.00 meters). From [1], “Common sense tells us that X [mean temperature] causes Y [snow fall] (with maybe very small feedback of Y on X). Confounders are present (e.g., day of the year).” This data is pair 87 of the MLR [1], and the data collection procedure is described there.

As stated previously, the primary difficulty in using the leaning for exploratory causal analysis is the determination of the cause-effect

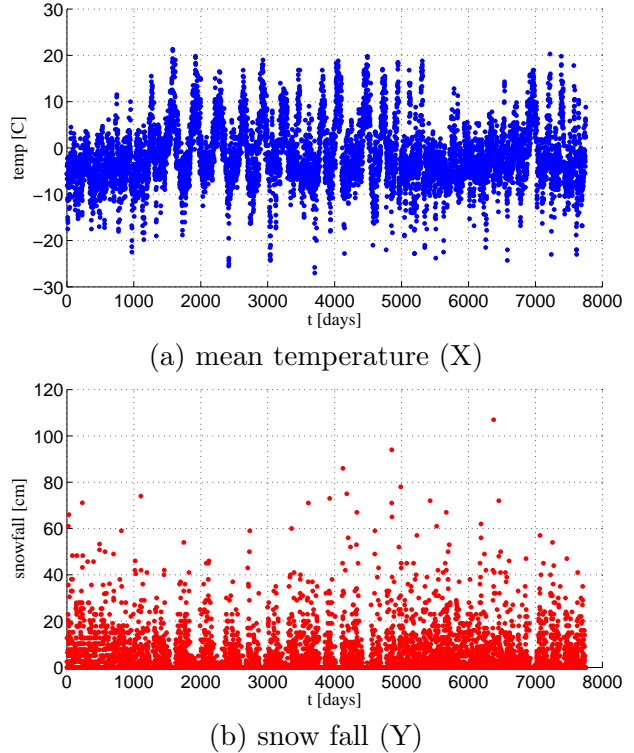


Figure 7: (Color available online.) This times series data pair is expected to have the causal relationship of $X \rightarrow Y$ for the pair of subplots [(a),(b)], where X or Y is marked in parenthesis for each time series.

assignment and tolerance domains. The above data is meant only to illustrate the use of leanings, so while a thorough analysis of the noise in the system should precede the leaning calculations, such a step is avoided here for brevity.

The symmetric tolerance domains are estimated using the maximum standard deviations of the n sets of binned points of an n -bin histogram of the normalized time series data \mathbf{X}' and \mathbf{Y}' , where $\mathbf{X}' = \frac{\mathbf{X} - \langle \mathbf{X} \rangle}{\sigma_{\mathbf{X}}}$, $\mathbf{Y}' = \frac{\mathbf{Y} - \langle \mathbf{Y} \rangle}{\sigma_{\mathbf{Y}}}$, and $n = \lfloor 0.1L \rfloor$ (i.e., n is the closest integer that is not larger than 10% of the library length L). This estimation is similar to the *n-bin mean standard deviation* technique discussed in Sec. 5.1).

The cause-effect assignment will be set naively because, again, the purpose of this article is not to study these particular time series in detail. To reiterate the previous comment regarding tolerance domains, detailed study would be required to have confidence in using leanings for exploratory causal analysis. However, the convenience of having causal “truths” is that we can take the naive approach of simply testing many different cause-effect assignments and compare the results to the expected causal inference.

Table 4 shows the weighted mean observed leaning, λ_l^{xy} , using the 1-standard cause-effect assignment with $l \in [0, 15]$ and using σ_x and σ_y estimated in the manner described above. The times series pair in the table, τ_1 , is shown in Figure 7.

Table 4 shows the leanings, using the given tolerance domains, imply causal inferences that agree with the causal truths for the tested pair with 1-standard assignments. Estimating the tolerance domains with $\sigma_{\mathbf{X}-\langle\mathbf{X}\rangle}$ and $\sigma_{\mathbf{Y}-\langle\mathbf{Y}\rangle}$ causes more negative values in the set of leanings shown for τ_1 .

Table 4 also present the problem of determining which l -standard assignment to use for the causal inference. If it is decided that the causal inference depend on

$$\lambda_{max}^{xy} = \lambda_{l'}^{xy} \quad (38)$$

where

$$|\lambda_{l'}^{xy}| = \max_l |\lambda_l^{xy}|, \quad (39)$$

then for τ_1 , $\lambda_{max}^{xy} = 0.040 \Rightarrow \mathbf{X} \xrightarrow{lean} \mathbf{Y}$, which agrees with the causal truth.

The different causal inferences implied by the leanings shown in Table 4 illustrate the need to consider leaning calculations as part of an exploratory causal analysis. The naive approach shown here has made no attempt to determine the most appropriate cause-effect assignment or tolerance domains for each individual time series pair. Yet, the causal truth

l	τ_1
1	0.040
2	0.036
3	0.032
4	0.030
5	0.028
6	0.025
7	0.024
8	0.024
9	0.023
10	0.022
11	0.020
12	0.018
13	0.018
14	0.019
15	0.017

Table 4: **–MAKE THIS A FIGURE–** The weighted mean observed leaning, λ_l^{xy} , using the 1-standard cause-effect assignment for $l \in [0, 15]$. The time series pair $\tau_1 = (\mathbf{X}, \mathbf{Y})$ with \mathbf{X} and \mathbf{Y} shown in Figure 7(a) and (b), respectively. The expected causal inference is $\mathbf{X} \xrightarrow{lean} \mathbf{Y}$; i.e., the expectation is $\lambda_l^{xy} > 0$ for every entry in this table.

is still recovered for some of the tested leaning calculations.

The NASA OMNI data set consists of hourly-averaged time series measurements of several different space weather parameters from 1963 to present, collected from more than twenty different satellites, along with sunspot number and several different geomagnetic indices, including D_{ST} , collected from the NOAA National Geophysical Data Center [?]. The disturbance storm time, D_{st} , is a measure of geomagnetic activity [?]. The magnetic field measurements in OMNI data sets, i.e., B_x , B_y , and B_z in GSE coordinates [?, ?, ?, ?, ?, ?], are believed to be drivers of D_{st} [?].

Let $\mathbf{P}_x^L = \{\{B_x(t')\}, \{D_{st}(t')\} \mid t' \in [t'_0, L]\}$ be an ordered subset of the available time series data $\{\{B_x(t)\}, \{D_{st}(t)\} \mid t \in [0, N]\}$ where N is the number of hourly data points in the OMNI data set. Let \mathbf{P}_y^L and \mathbf{P}_z^L be defined analogously. If $\lambda_1^{x(y)(z)}$ is the weighted mean observed leaning for $\mathbf{P}_{x(y)(z)}^L$ using the 1-standard cause-effect assignment, then n samples of $\mathbf{P}_{x(y)(z)}^L$, each with a different t'_0 , would produce a set of leanings, $\{\lambda_1^{x(y)(z)}\}$ from which the causal inference could be drawn.

Let $L = 500$ and $n = 10^4$. The symmetric tolerance domains are naively set with $f\sigma_{|B_{x(y)(z)} - \langle B_{x(y)(z)} \rangle|}$ and $f\sigma_{|D_{st} - \langle D_{st} \rangle|}$ for each sampled times series of length L with $f=0.05$. The starting points for each sampled time series are sampled from a uniform distribution over $[0, N - L]$. Figure 8 shows the causal inference drawn from each set of leanings agrees with intuition, i.e., $B_{x(y)(z)} \xrightarrow{\text{leans}} D_{st}$, if the causal inference is based on, e.g., the mean value from the set of leanings.

7 Spurious Leanings

Consider the linear system of

$$\bar{\tau}_L = \{\mathbf{X}, \mathbf{Y}\} = \{\{x_t\}, \{y_t\}\} \quad (40)$$

where $t \in [0, L]$,

$$x_t = \begin{cases} 2 & t = 1 \\ 0 & \forall t \in \{t \mid t \neq 1 \text{ and } t \bmod 5 \neq 0\} \\ 2 & \forall t \in \{t \mid t \bmod 5 = 0\} \end{cases}$$

and

$$y_t = \eta_t$$

with $\eta_t \sim \mathcal{N}(0, 1)$. The first time series, \mathbf{X} , is the periodic impulse that drove the example system in Eqn. 24. The second time series, \mathbf{Y} , is just standard Gaussian noise applied at each time step.

There is no intuitive causal relationship in Eqn. 40. However, Figure 9 shows the weighted mean observed leaning using the 1-standard assignment will lead to causal inferences for different σ_y , given $\sigma_x = 0$. The causal inference becomes inconclusive as the library length L is increased; i.e. the leaning moves towards zero for the tested tolerance domains as the library length of Eqn. 40 is increased. The use of leanings for causal inference with Eqn. 40 may imply a spurious relationship.

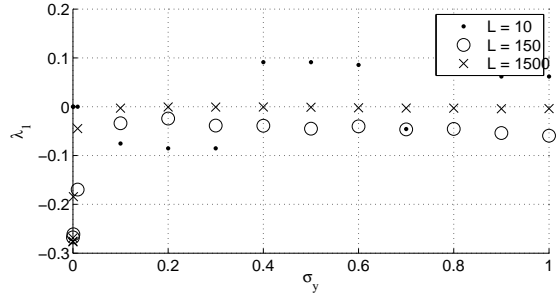


Figure 9: Eqn. 40 leads to spurious leanings, i.e. weighted mean observed leanings using the 1-standard assignment, λ_1 , that depend on both the tolerance domain σ_y and the library length L .

Consider the slightly more humorous example of $\tau = \{\mathbf{X}, \mathbf{Y}\}$ where \mathbf{X} is the total yearly revenue generated by arcades in the United States between the years of 2000 and 2009 and

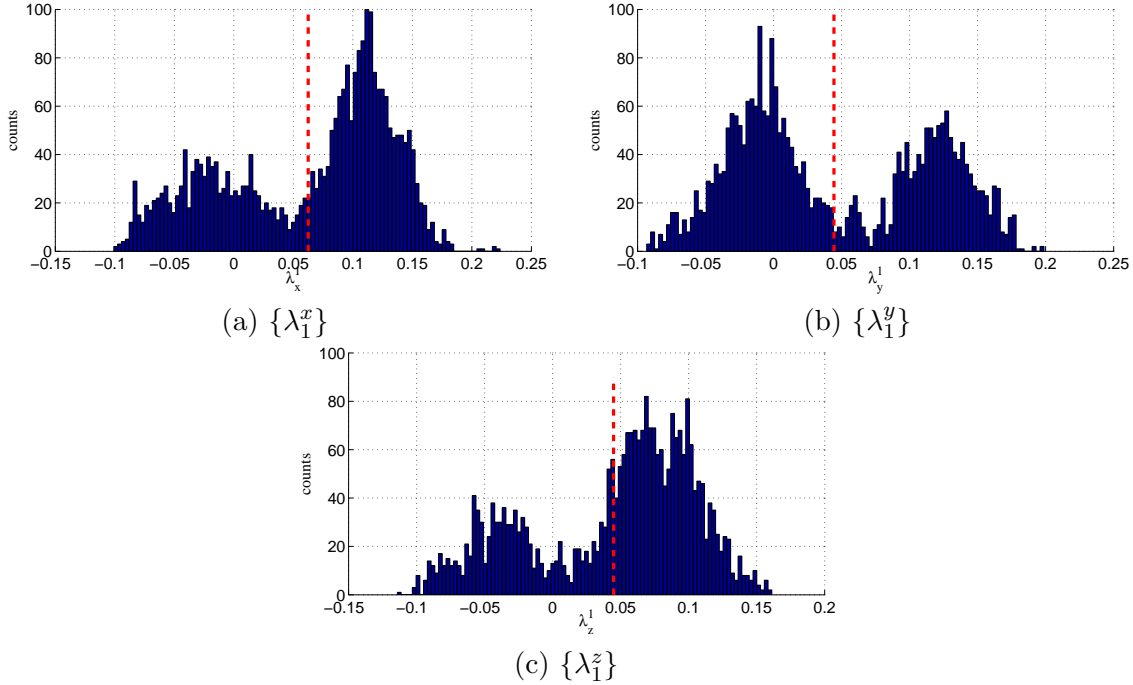


Figure 8: (Color available online) Histograms of the three sets of $n = 10^4$ (unitless) weighted mean observed leanings, using the 1-standard cause-effect assignment, i.e., $\lambda_{x(y)(z)}^1$, show causal inferences that agree with intuition for the OMNI data set. The red dashed lines show the algebraic means of the sets.

\mathbf{Y} is the number of computer science doctorates awarded yearly in the United States over the same time period [?]. There is no intuitive causal relationship between \mathbf{X} and \mathbf{Y} . The weighted mean observed leaning, λ_1 , using the 1-standard assignment with tolerance domains of $f\sigma_{|\mathbf{X}-\langle\mathbf{X}\rangle|}$, $f\sigma_{|\mathbf{Y}-\langle\mathbf{Y}\rangle|}$ and $f = 0.05$ is $\lambda_1 \approx 0.11 \Rightarrow \mathbf{X} \xrightarrow{\text{lean}} \mathbf{Y}$. Thus, causal inference using leanings may imply spurious relationships between time series.

The spurious leanings shown in these examples may imply causal relationships that do not exist in the system. Leaning calculations may be part of an exploratory causal inference analysis, but care must to be taken to ensure a causal relationship is actually present in the data, even if the directionality (or other features) of that relationship are unknown. The

relationship between leanings and causality as it is typically understood in physics (i.e., involving interventions into the systems under investigation, e.g., through experiments [?]) is not currently known. This article is exploring the use of leanings as part of an exploratory causal analysis in times series data, not as a definition or proof of causality in a dynamical system.

8 Conclusion

Causal inference using observational data alone is a difficult task []. This problem is important in many fields, but in physics in particular, there are often subfields for which direct experimentation is not technologically feasible. For example, studying space weather relies en-

tirely on observation data for both forecasting events and understanding the dynamics of the system. The exploratory causal inference proposed in this article is meant to be one of the first steps in understanding such data.

Exploratory causal analysis, as it has been described here, involves many different techniques, including Granger causality (GC), transfer entropy (TE), cross correlation (CC), and state space reconstruction (SSR) techniques. Each of these techniques as well known shortcomings. GC is model dependent [], TE can be computationally difficult [], CC can be unreliable [], and SSR relies on setting lag times and embedding dimensions, which is not straightforward []. Causal leaning has been introduced to overcome many of these shortcomings, however, it relies on setting tolerance domains and cause-effect assignments, which, as discussed above, may require in-depth data analysis to determine. Thus, there is no single *correct* technique for causal inference in time series data, and it is posited that exploratory causal analysis of time series data sets should require the use and comparison of several different techniques.

No attempt has been made to understand causal leanings in term of current philosophical causality studies. For example, there is no exploration of how causal leanings are associated with token or *prima facie* causality []. Leanings have also been grouped under the broad term of time series causality studies, which implies that technique is distinct from other data causality studies, including direct acyclic graph (DAG) [?] and temporal logic [?] techniques. Causal leanings have been introduced only as a practical tool and connections with the broader fields of data causality and causality foundations are left for future work. For example, leanings may be a subset of the more general temporal logic presented by Kleinberg [].

There are many open questions regarding

the use of leanings for causal inference that have not been explored in this article. There has been no discussion of how to interpret the magnitude of the leaning calculations. For example, if there are two weighted mean observed leanings λ_1 and λ_2 with different cause-effect assignments such that $\lambda_1 > \lambda_2$, then does the first cause-effect assignment represent a “stronger” driver than the second? How should leanings of 0, 2, or -2 be interpreted with respect to the cause-effect assignments? Can leanings calculated using different cause-effect assignment even be compared? Such questions are still open.

There has also been no exploration of using leanings as part of statistical tests, as is often done with GC []. The use of histograms in Figure 8 may be considered first steps toward statistical interpretations of leanings, but there are still many open questions to be addressed in future work.

Finally, this article has discussed the use of leanings as part of an exploratory causal analysis of time series data. Exactly how such an analysis should be conducted is, however, still an open question. For example, given a report of GC, TE, SSR, and leanings (all calculated in various ways), how should the results be interpreted holistically? There are many potentially confusing scenarios in which, e.g., two techniques lead to opposite causal inferences. The most reasonable time series causality techniques to use for a given exploratory causal analysis may depend strongly on the data itself, but general guidelines for such analysis is, as far as we know, still an open question.