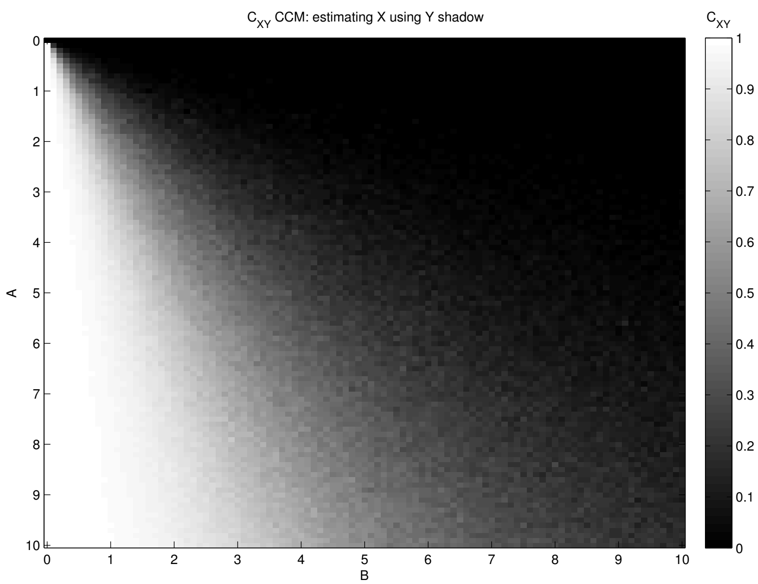
# How is CCM useful?

## 1 Linear Example #1

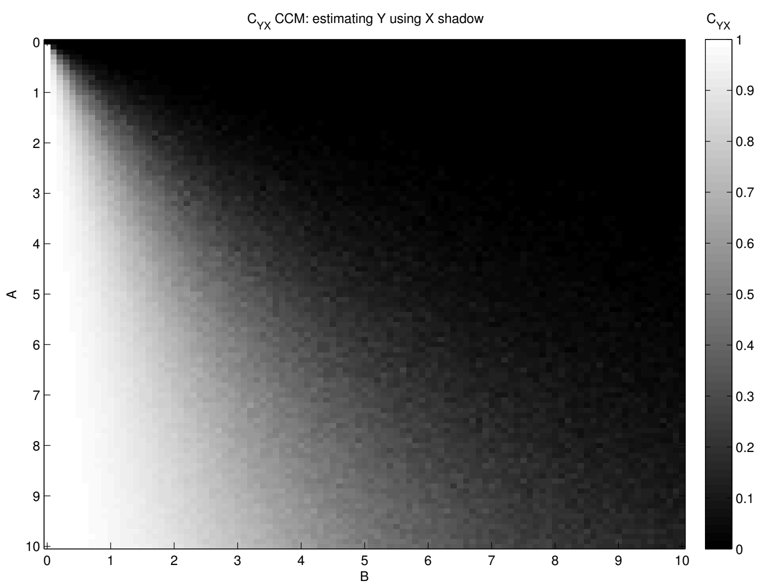
Consider the linear example dynamical system of

$$X_t = \sin(t) \tag{1}$$
$$Y_t = AX_{t-1} + B\eta_t, \tag{2}$$

with $A, B \in \mathbb{R} \geq 0$ and $\eta_t \sim \mathcal{N}(0, 1)$. Specifically, consider $A, B \in [0, 10]$ step through in increments of 0.1. Figure 19 shows $C_{XY}$ and $CYX$.




(a) $C_{XY}$




(b) $C_{YX}$

Figure 1: Changing $A$ and $B$. $C_{XY}$ and $C_{YX}$

Figure 21 shows $\Delta$ for this example.

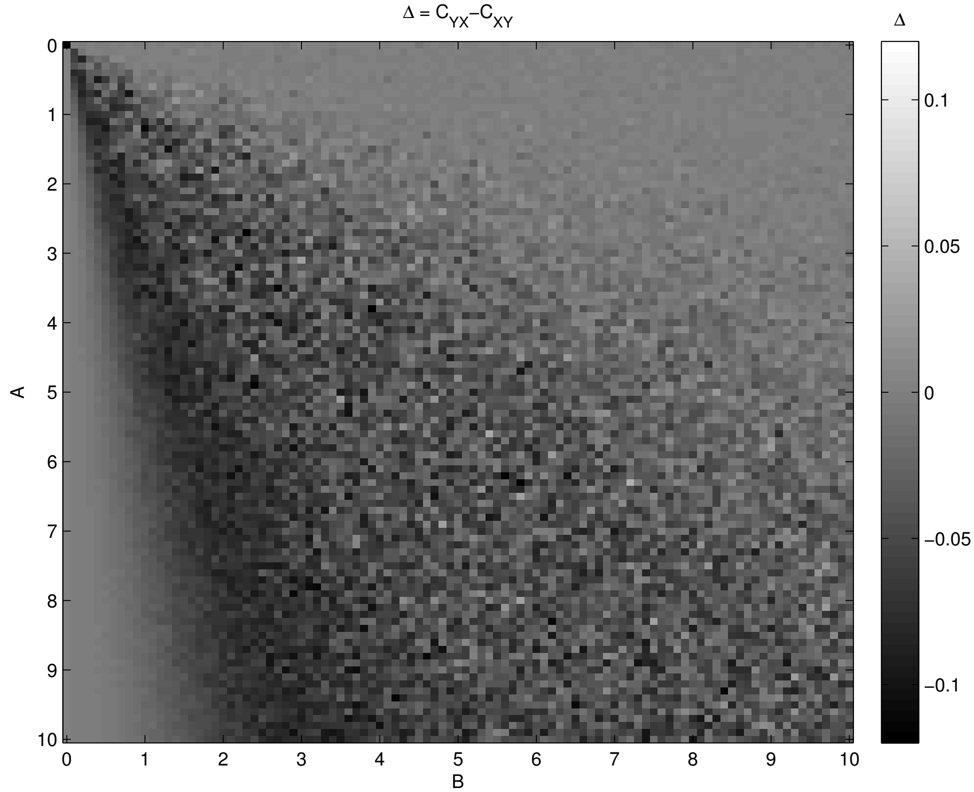



Figure 2: Changing $A$ and $B$. $\Delta$

Consider the convergence of two specific points in the above plots $(A, B) = (2.6, 2.6)$ and $(A, B) = (3.0, 2.6)$.

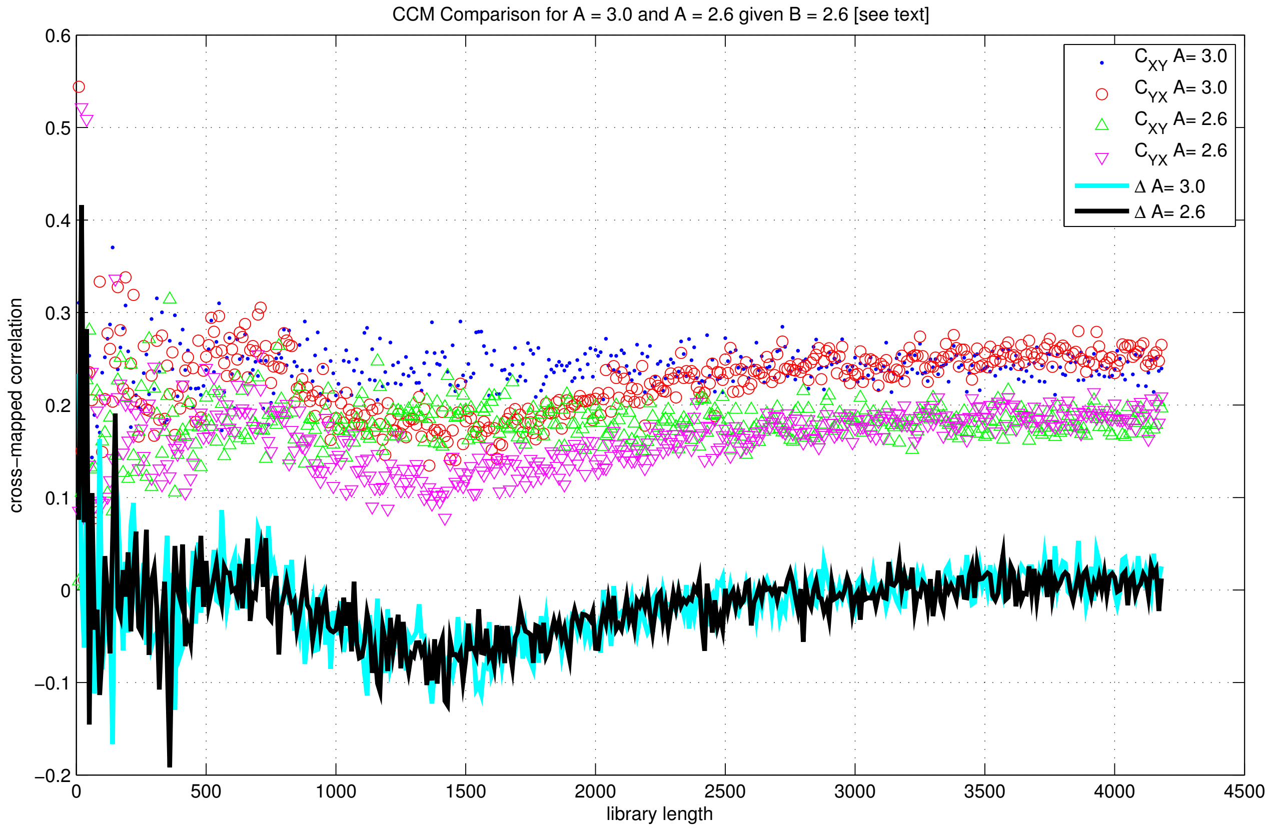



Figure 3:

## 2 Linear Example #2

Consider the linear example dynamical system of

$$\begin{aligned} X_t &= \sin(t) & (3)\\ Y_t &= AX_{t-1} + B\eta_t & (4)\\ Z_t &= Y_{t-1}, & (5) \end{aligned}$$

with $A, B \in \mathbb{R} \geq 0$ and $\eta_t \sim \mathcal{N}(0,1)$. Specifically, consider $A, B \in [0,5]$ step through in increments of 0.1.

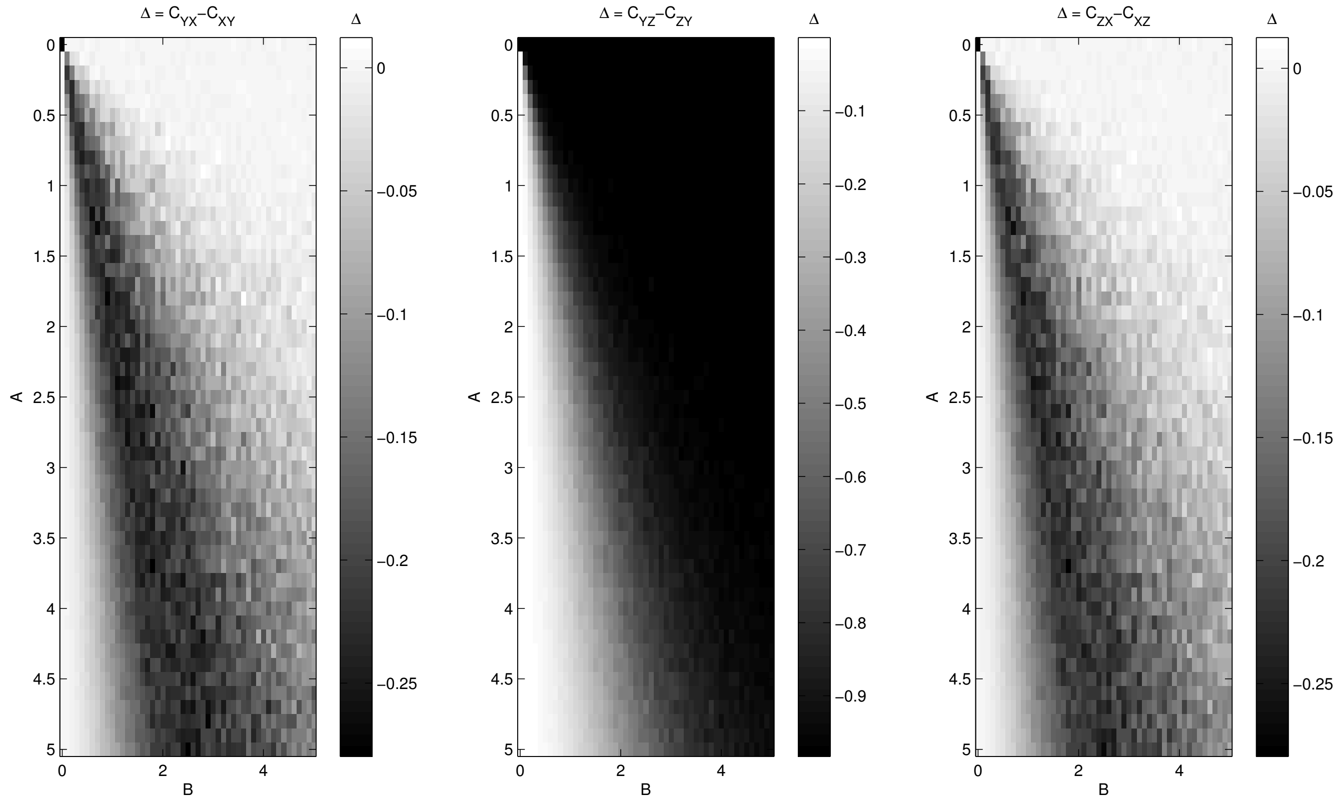



Figure 4:

# 3 Non-Linear Example

Consider the nonlinear example dynamical system of

$$X_t = \sin(t) \tag{6}$$

$$Y_t = AX_{t-1}(1 - BX_{t-1}) + C\eta_t, \tag{7}$$

with $A, B, C \in \mathbb{R} \geq 0$ and $\eta_t \sim \mathcal{N}(0, 1)$. Specifically, consider $A, B, C \in [0, 5]$ step through in increments of 0.5.

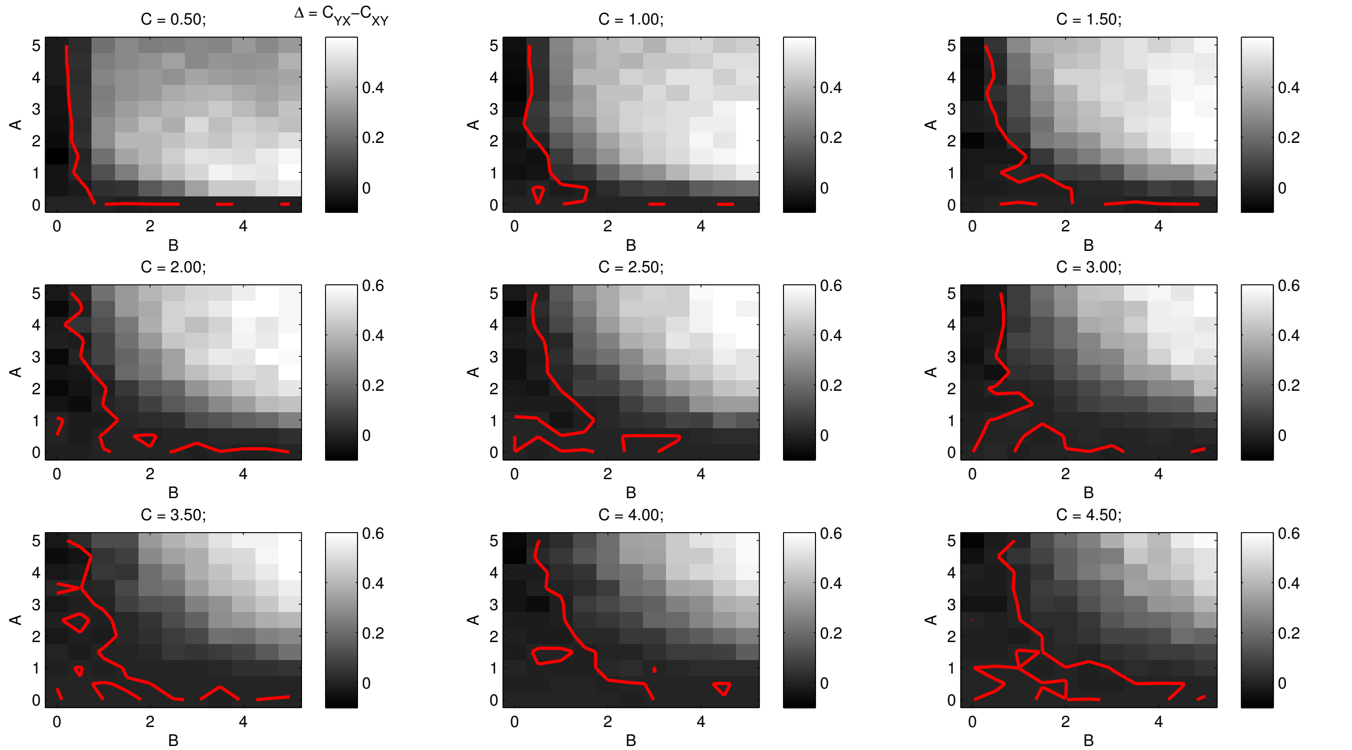



Figure 5:

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# 4 RL Circuit Example

The continuous system is

$$\frac{dI}{dt} = \frac{V(t)}{L} - \frac{R(t)}{L} I, \tag{8}$$

where $I$ is the current at time $t$, $V(t)$ is the voltage at time $t$, $R(t)$ is the resistance at time $t$, and $L$ is the inductance (which is also constant in these examples), and it can be approximated as

$$\dot{I} = \frac{V(t)}{L} - \frac{R(t)}{L} I \Rightarrow I_{t+1} - I_t = \frac{V_t}{L} - \frac{R_t}{L} I_t. \tag{9}$$

Rearranging leads to

$$I_{t+1} = \frac{V_t}{L} + I_t \left(1 - \frac{R_t}{L}\right), \tag{10}$$

$$V_t = L\left(I_{t+1} - I_t\left(1 - \frac{R_t}{L}\right)\right), \tag{11}$$

and

$$R_t = L\left(I_t - I_{t+1} + \frac{V_t}{L}\right). \tag{12}$$

All of the plots of $I$ seen below are produced by using MATLAB's *ode45* to solve Eqn. 8 (i.e. not using the discrete approximation shown). The time series $V(t)$ and $R(t)$ are created by defining values at fixed points and using linear interpolation (i.e. MATLAB's *interp1*) to find the time steps required by the ODE solver (i.e. MATLAB's *ode45*).

# 5 Changing V(t)

Consider the situation where $R(t)$ is constant.

**Physical intuition is that $V$ drives $I$, so we expect to find $V$ CCM causes $I$ ($C_{VI} > C_{IV}$).**

For this example, the voltage is described by

$$V(t) = A_v \sin(f_v t + \phi_v) + O_v, \tag{13}$$

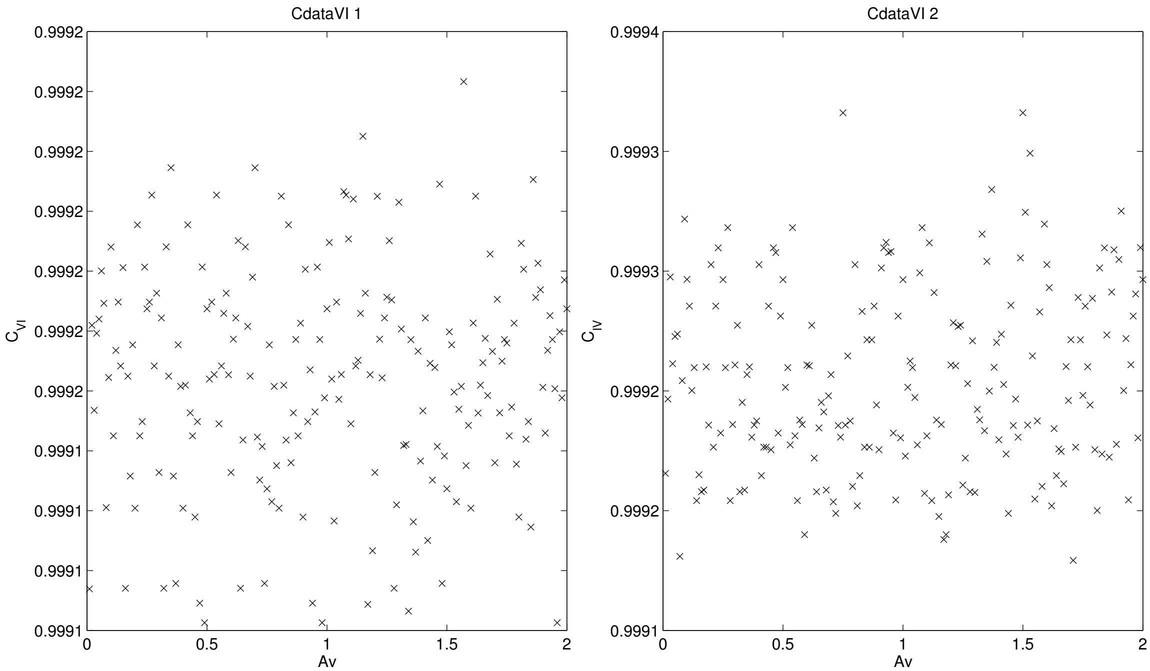
where $A_v$ is the amplitude, $f_v$ is the frequency, $\phi_v$ is the phase, and $O_v$ is the offset voltage.

## 5.1 Changing $A_v$

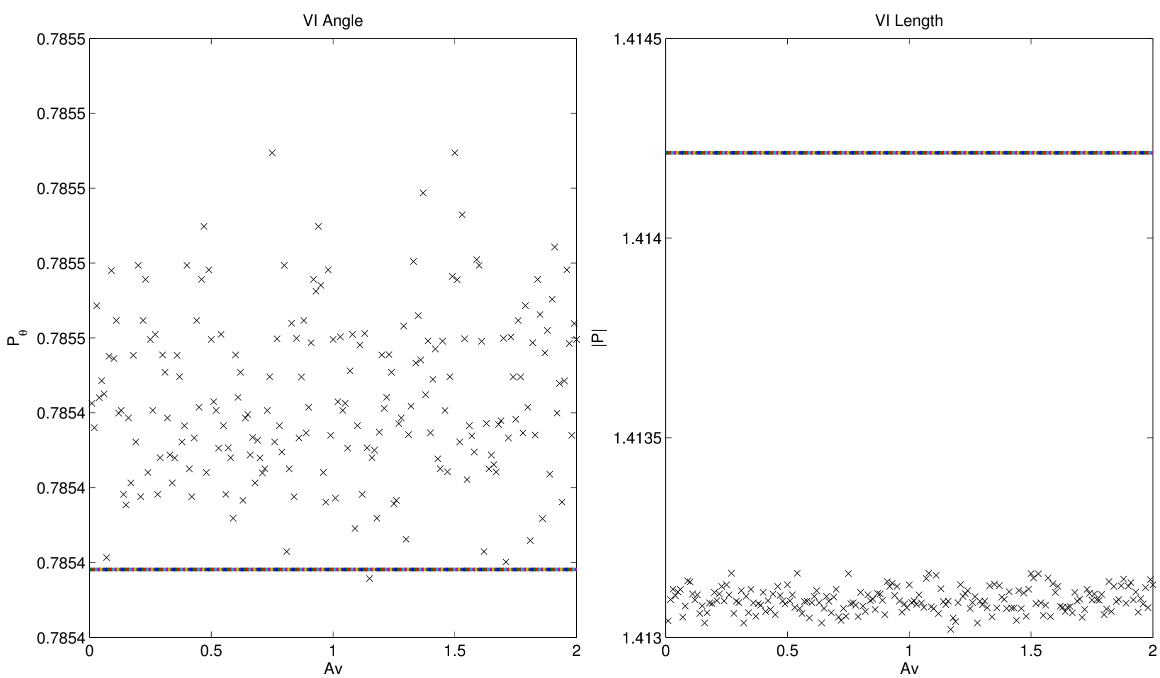
Consider evaluating the CCM correlations $C_{VI}$ and $C_{IV}$ for each $A_v \in [0.01, 2.0]$ in steps of 0.01. For reference, both V(t) and I(t) are plotted for different $A_v$ in Figure 6.

Figure 6: Reference plots for changing $A_v$.

The CCM correlations are each plotted in Figure 7 along with the corresponding PAI elements $P_\theta$ and $|P|$.




(a) $C_{VI}$ and $C_{IV}$




(b) $P_\theta$ and $|P|$

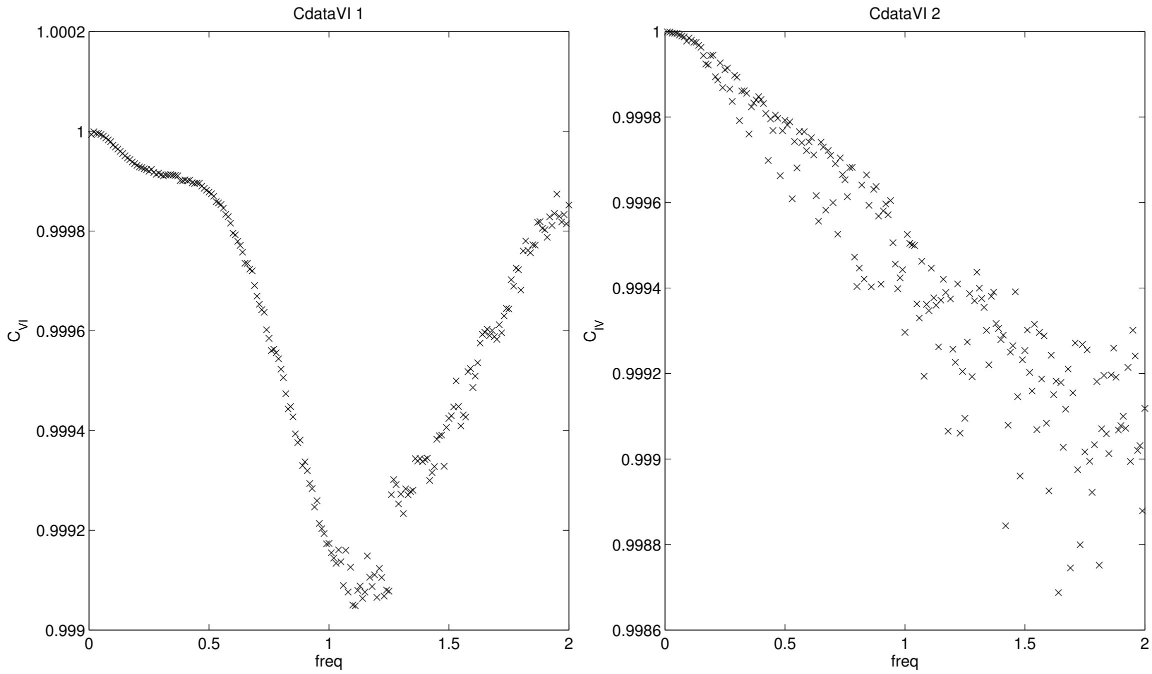
Figure 7: Changing $A_v$.

## 5.2 Changing $f_v$

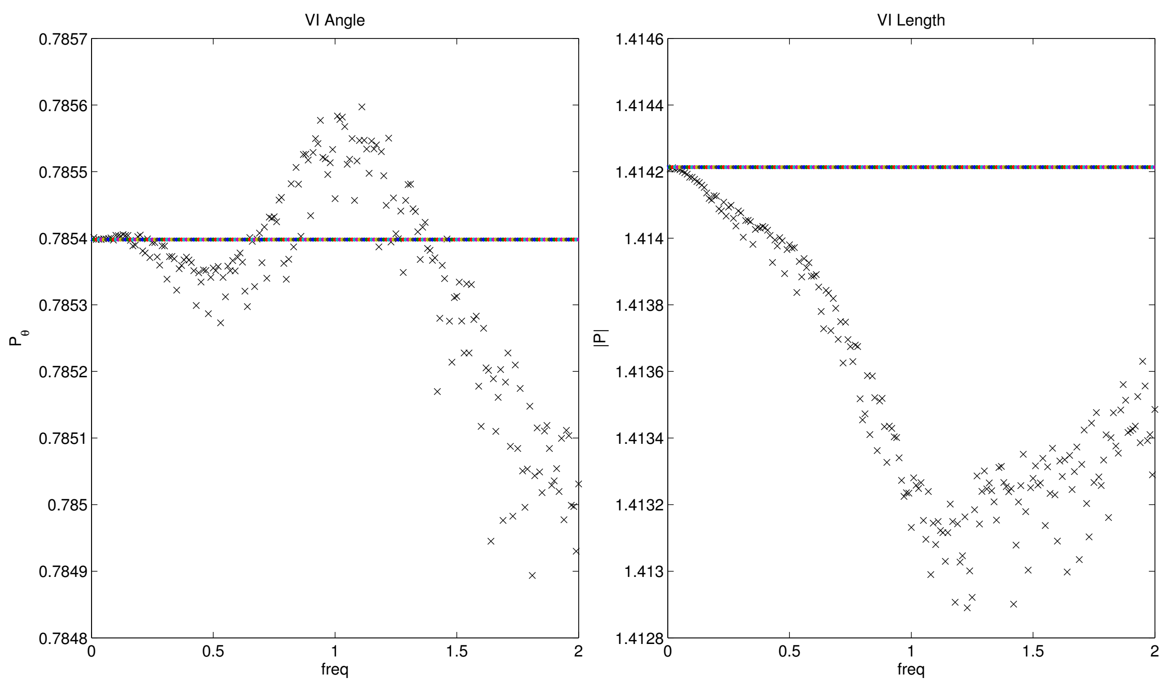
Consider evaluating the CCM correlations $C_{VI}$ and $C_{IV}$ for each $f_v \in [0.01, 2.0]$ in steps of 0.01. For reference, both V(t) and I(t) are plotted for different $f_v$ in Figure 8.

Figure 8: Reference plots for changing $f_v$.

The CCM correlations are each plotted in Figure 9 along with the corresponding PAI elements $P_\theta$ and $|P|$.




(a) $C_{VI}$ and $C_{IV}$




(b) $P_\theta$ and $|P|$

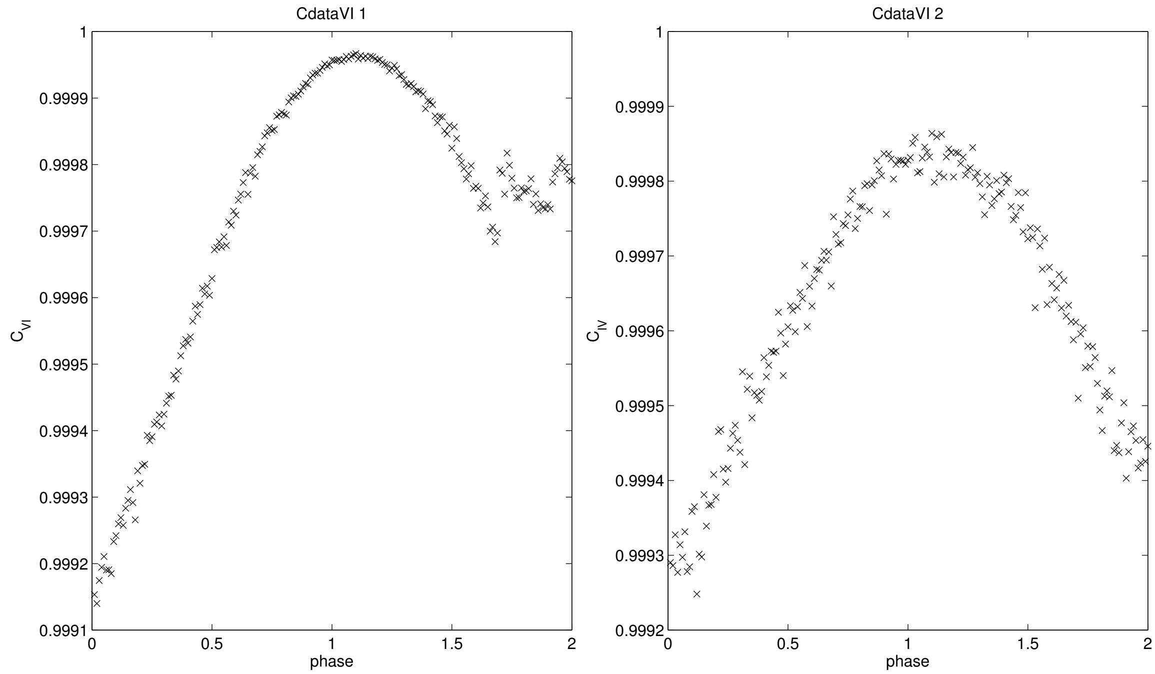
Figure 9: Changing $f_v$.

## 5.3 Changing $\phi_v$

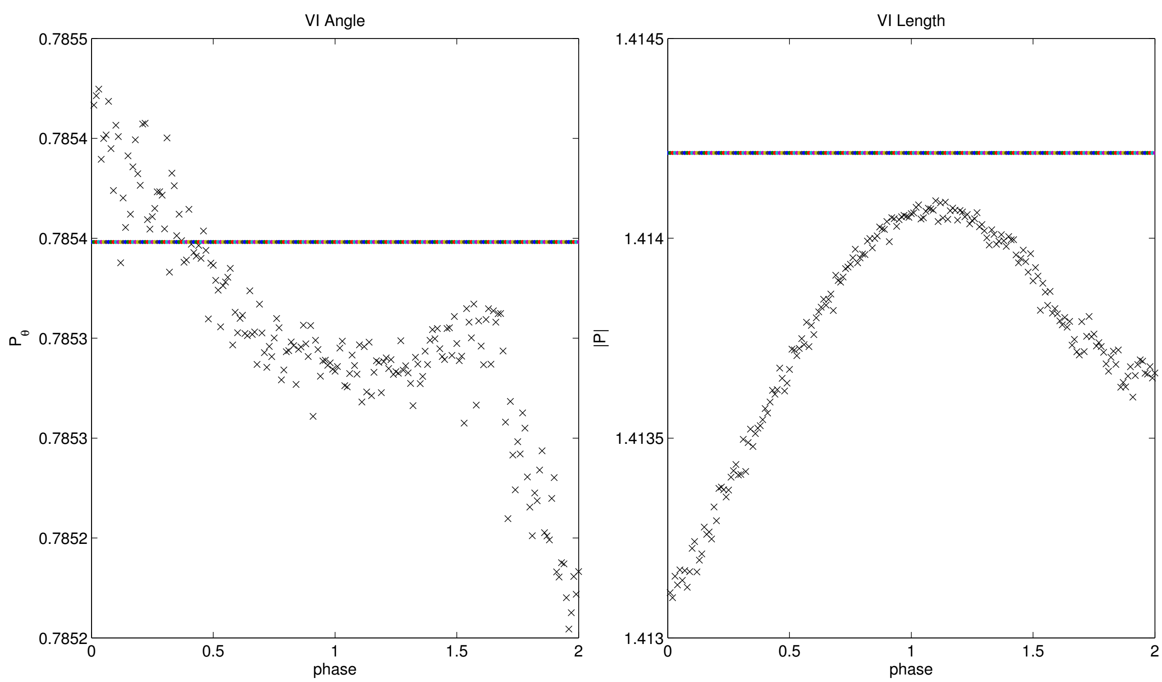
Consider evaluating the CCM correlations $C_{VI}$ and $C_{IV}$ for each $\phi_v \in [0.01, 2.0]$ in steps of 0.01. For reference, both V(t) and I(t) are plotted for different $\phi_v$ in Figure 10.

Figure 10: Reference plots for changing $\phi_v$.

The CCM correlations are each plotted in Figure 11 along with the corresponding PAI elements $P_\theta$ and $|P|$.




(a) $C_{VI}$ and $C_{IV}$




(b) $P_\theta$ and $|P|$

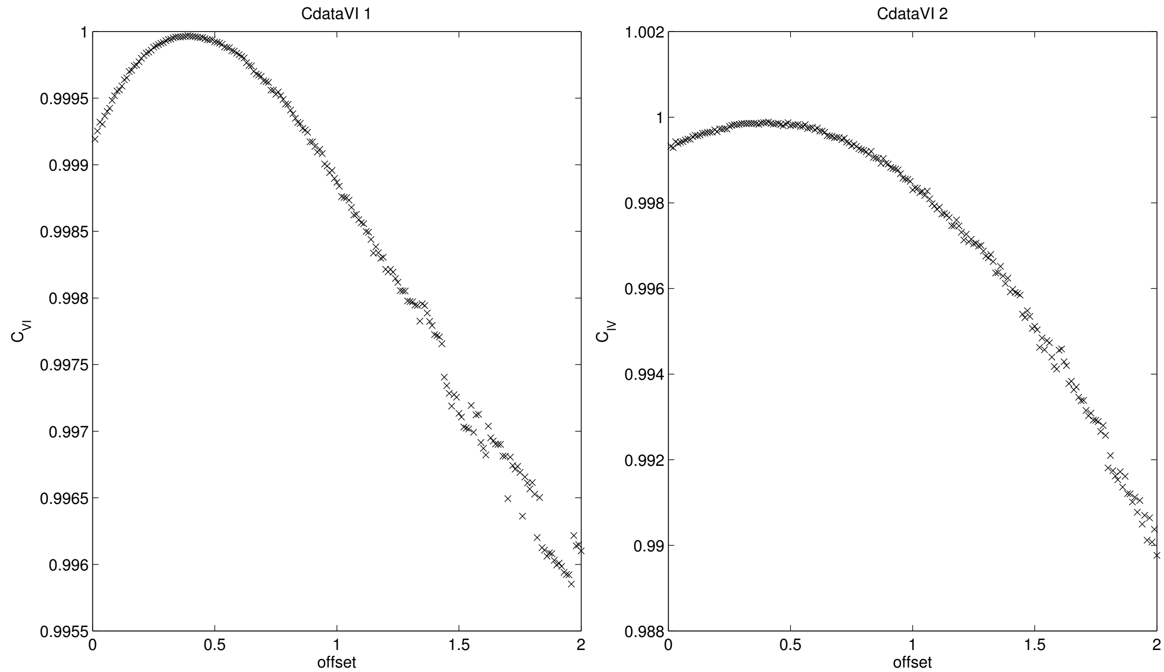
Figure 11: Changing $\phi_v$.

## 5.4 Changing $O_v$

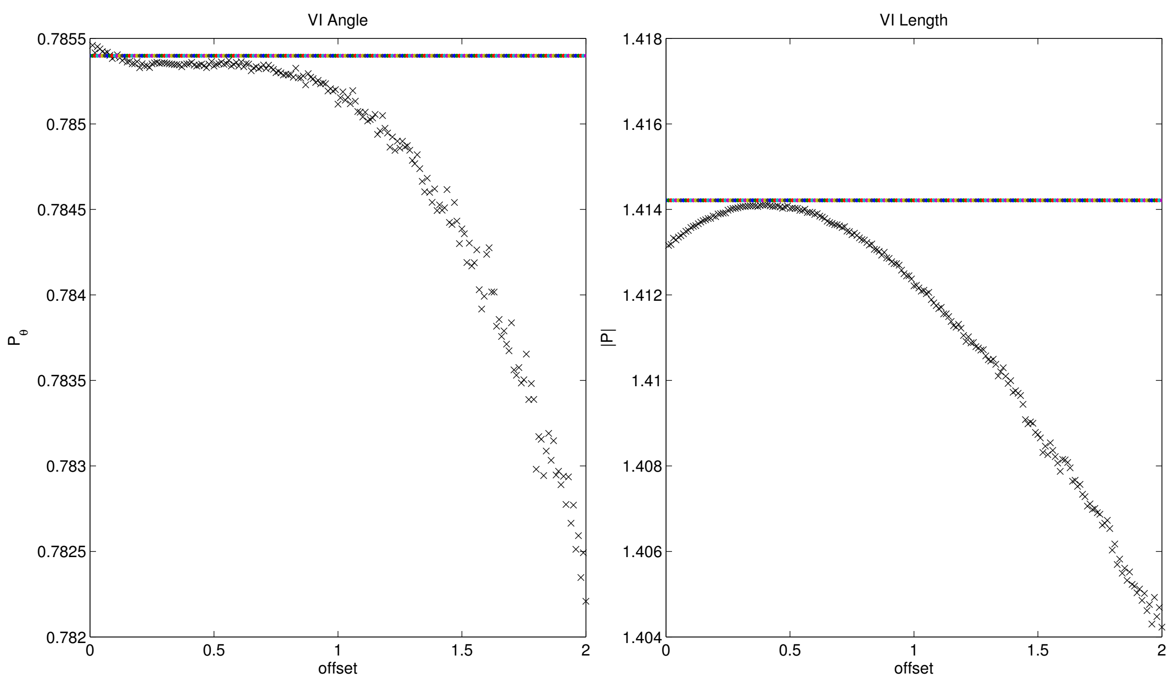
Consider evaluating the CCM correlations $C_{VI}$ and $C_{IV}$ for each $O_v \in [0.01, 2.0]$ in steps of 0.01. For reference, both V(t) and I(t) are plotted for different $O_v$ in Figure 12.

Figure 12: Reference plots for changing $O_v$.

The CCM correlations are each plotted in Figure 13 along with the corresponding PAI elements $P_\theta$ and $|P|$.



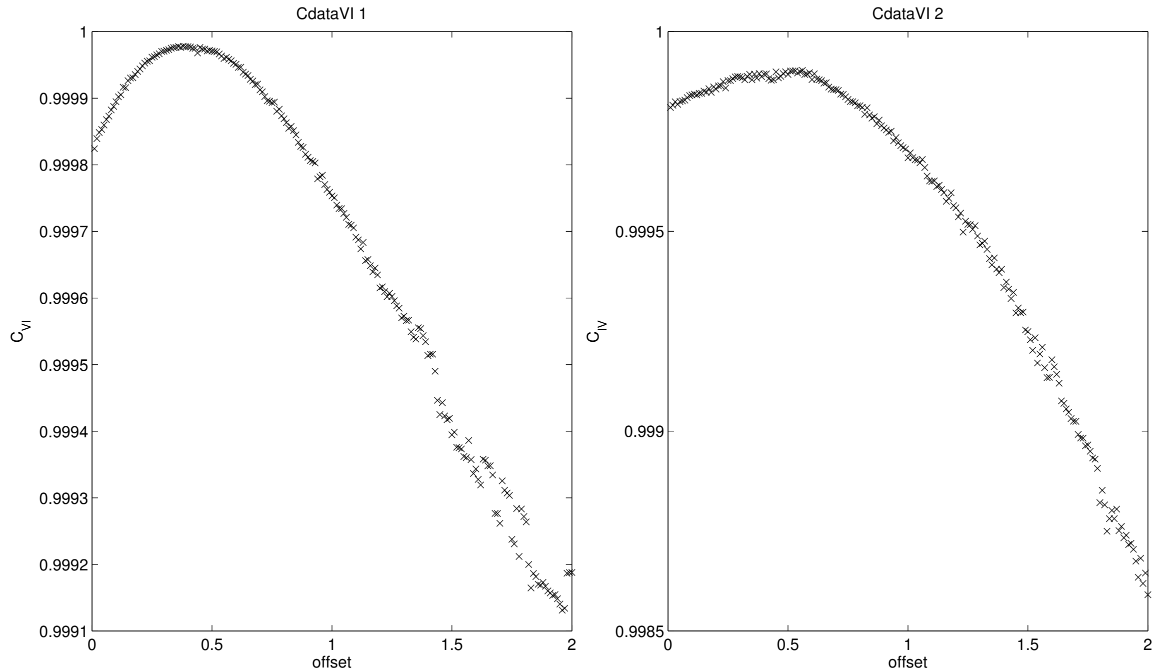

(a) $C_{VI}$ and $C_{IV}$



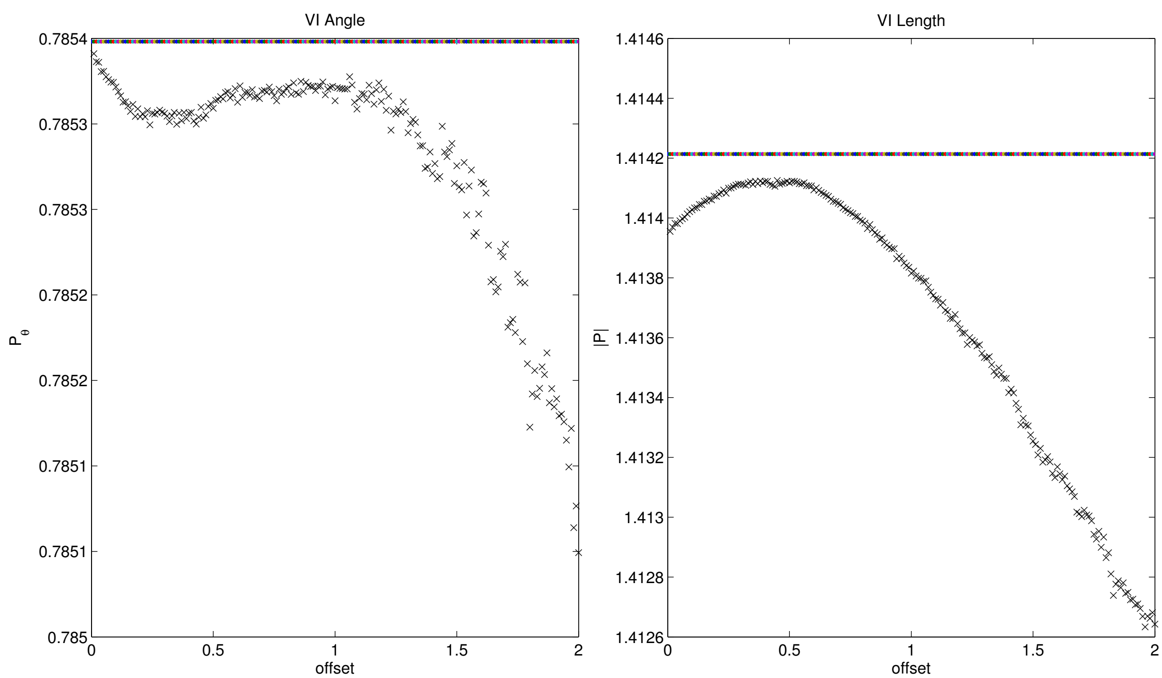

(b) $P_\theta$ and $|P|$

Figure 13: Changing $O_v$.

Figure 14 shows the effect of increasing the library length from $2 \times 10^3$ (i.e. `tspan = [0:0.5:1000];`) to $10^4$ (i.e. `tspan = [0:0.5:5000];`), and Figure 15 extends the above plots to $O_v \in [0.01, 10.0]$ in steps of 0.05.

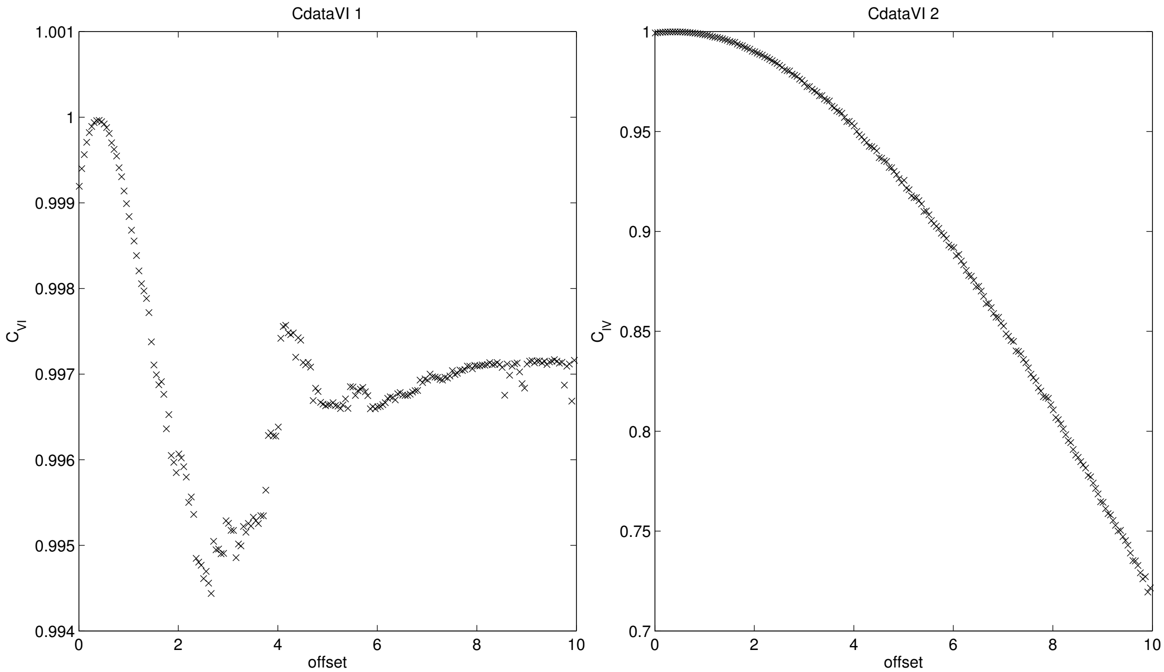



(a) $C_{VI}$ and $C_{IV}$

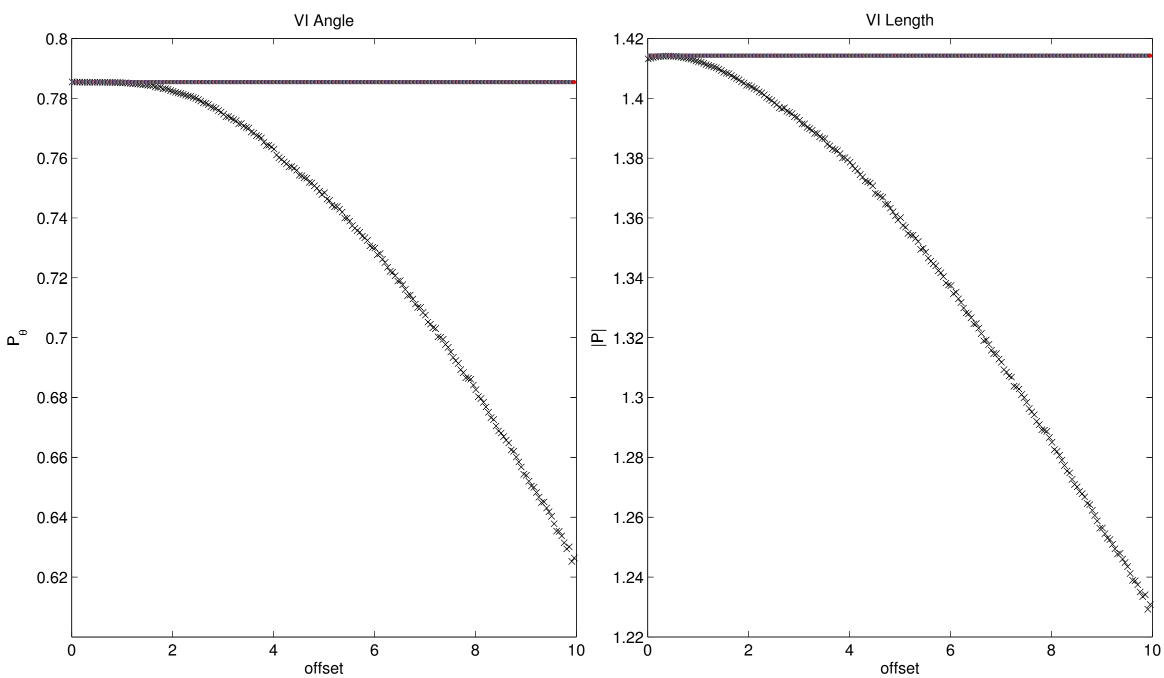



(b) $P_\theta$ and $|P|$

Figure 14: Changing $O_v$ (longer library length).




(a) $C_{VI}$ and $C_{IV}$




(c) $P_\theta$ and $|P|$

Figure 15: Changing $O_v$ (larger domain for $O_v$).

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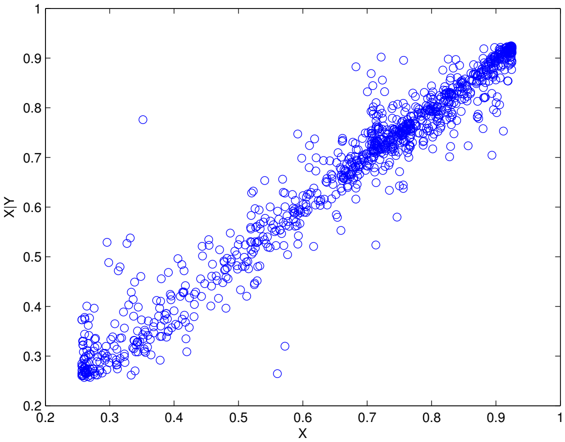
# 6 PAI

Consider the system (Sugihara Figure 3 C and D)

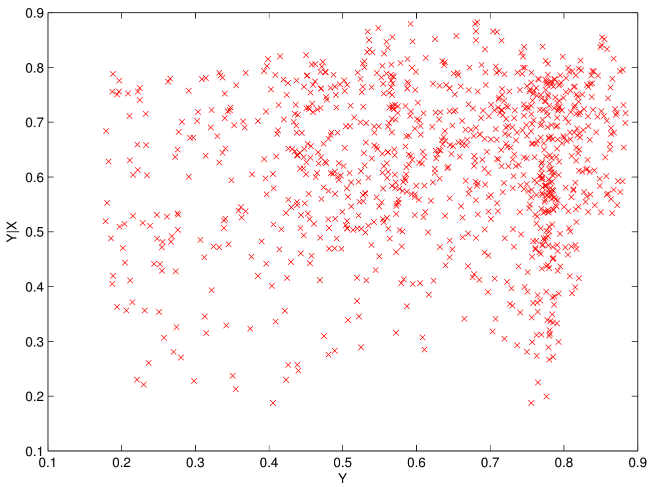
$$X_{t+1} = X_t \left( r_x - r_x X_t - \beta_{xy} Y_t \right) \quad (14)$$

$$Y_{t+1} = Y_t \left( r_y - r_y Y_t - \beta_{yx} X_t \right), \quad (15)$$

with $r_y = r_y = 3.7$, $X_0 = 0.2$, $Y_0 = 0.4$, $\beta_{xy} = 0$, and $\beta_{yx} = 0.32$. Plots of the correlation between $X$ and $X|Y$, as well as, $Y$ and $Y|X$ are shown below.




(a) $C_{XY}$



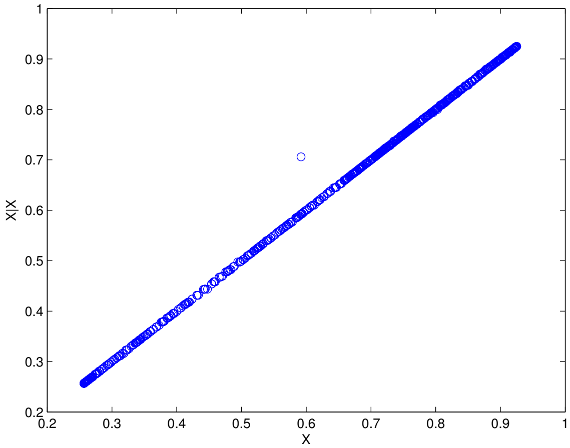

(b) $C_{YX}$

Figure 16: Changing $A$ and $B$. $C_{XY}$ and $C_{YX}$

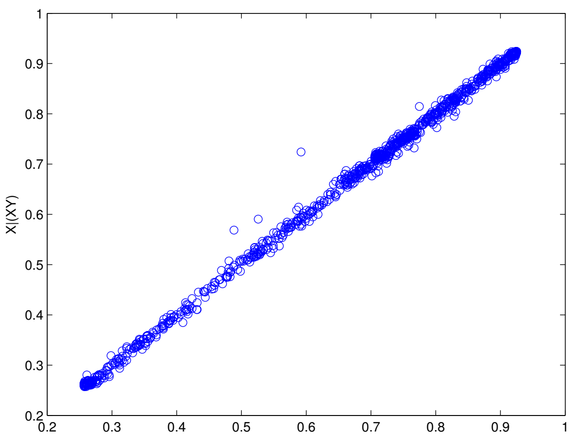
This leads to

| $C_{XY}$ | $C_{YX}$ | $\Delta = C_{YX} - C_{XY}$ |
|---|---|---|
| 0.973921 | 0.110698 | -0.8632 |

But, these are not the only correlations that can be tested:

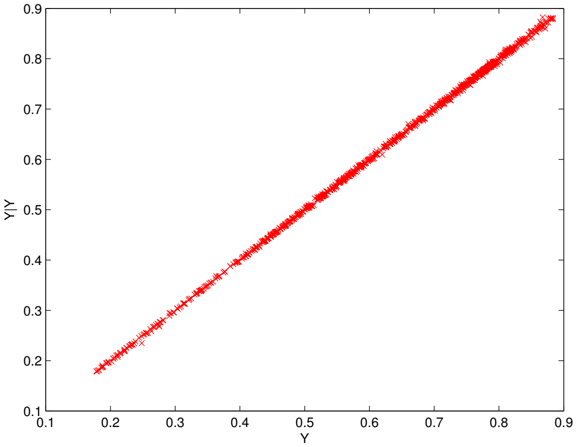



(a) $C_{XX}$

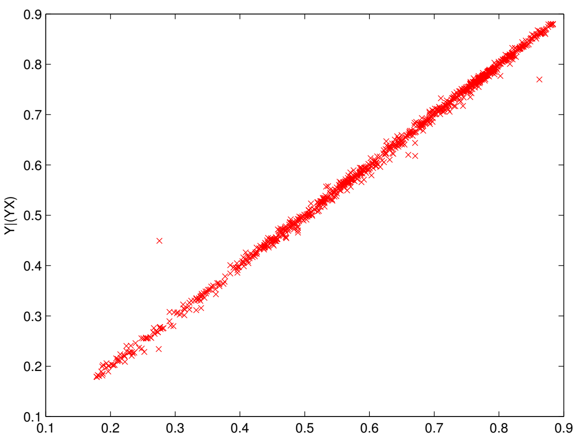



(b) $C_{X(XY)}$

Figure 17: Changing $A$ and $B$. $C_{XY}$ and $C_{YX}$




(c) $C_{XX}$




(d) $C_{X(XY)}$

Figure 18: Changing $A$ and $B$. $C_{XY}$ and $C_{YX}$

This leads to

| $C_{XX}$ | $C_{X(XY)}$ | $C_{YY}$ | $C_{Y(YX)}$ | $\Delta = C_{Y(YX)} - C_{X(XY)}$ |
|---|---|---|---|---|
| 0.999841 | 0.998989 | 0.999908 | 0.998693 | -2.9548e-04 |

Consider a comparison of PAI and CCM given the linear example system from above.

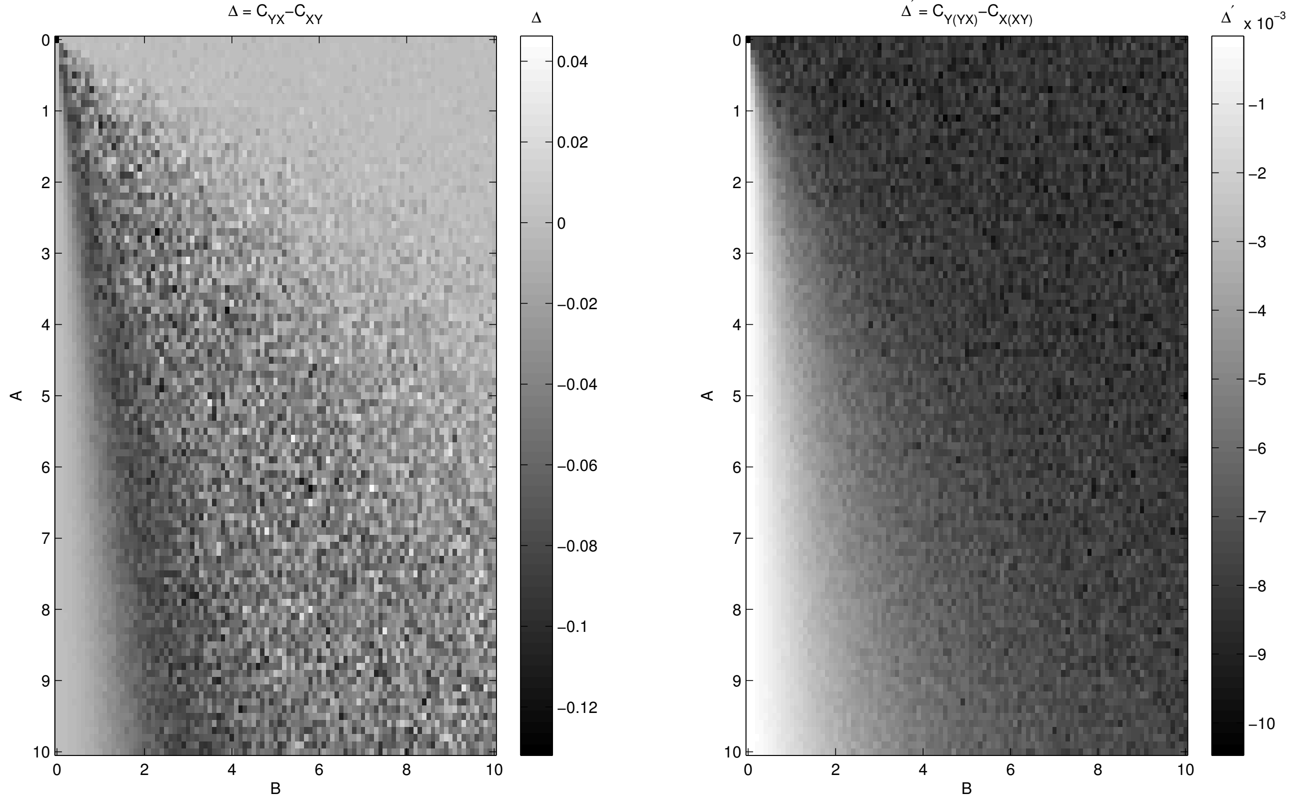



Figure 19: Linear Example 1 PAI

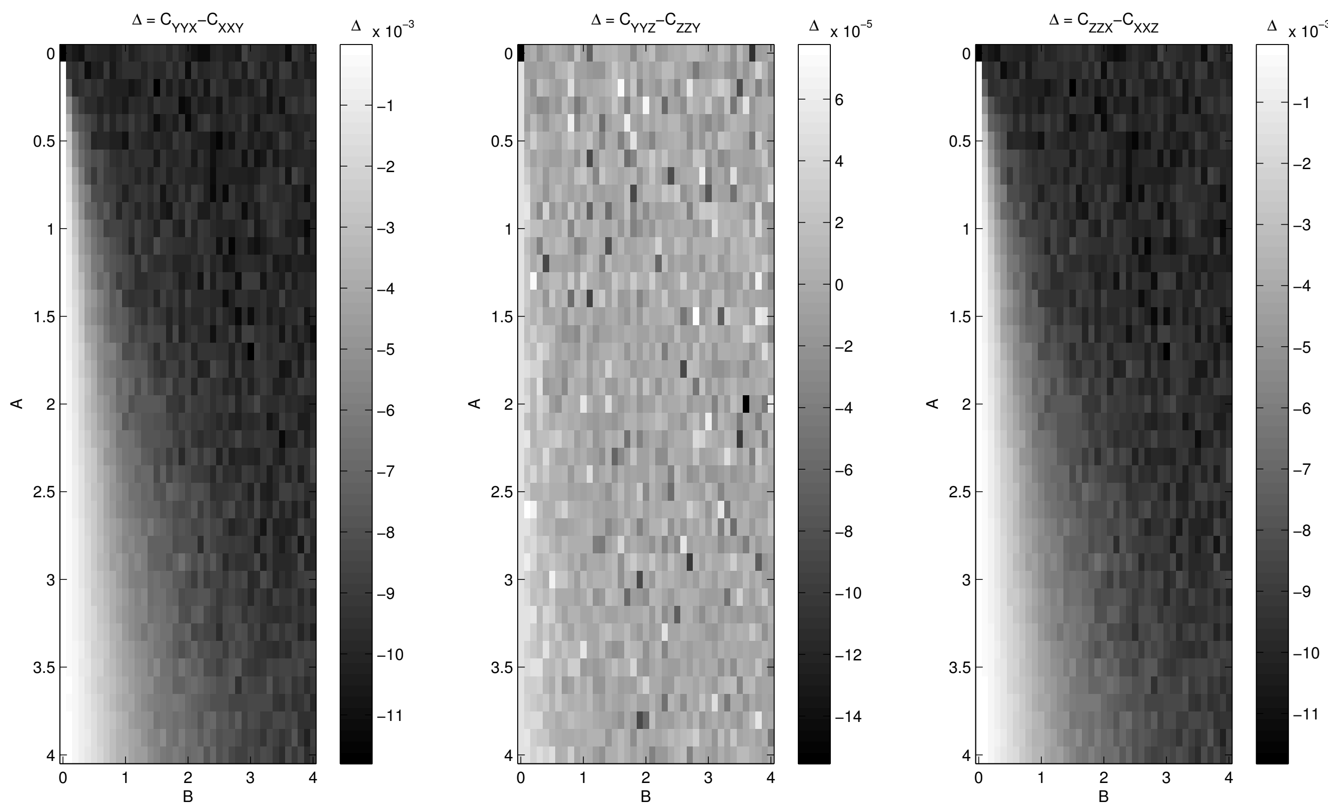



Figure 20: Linear Example 2 PAI

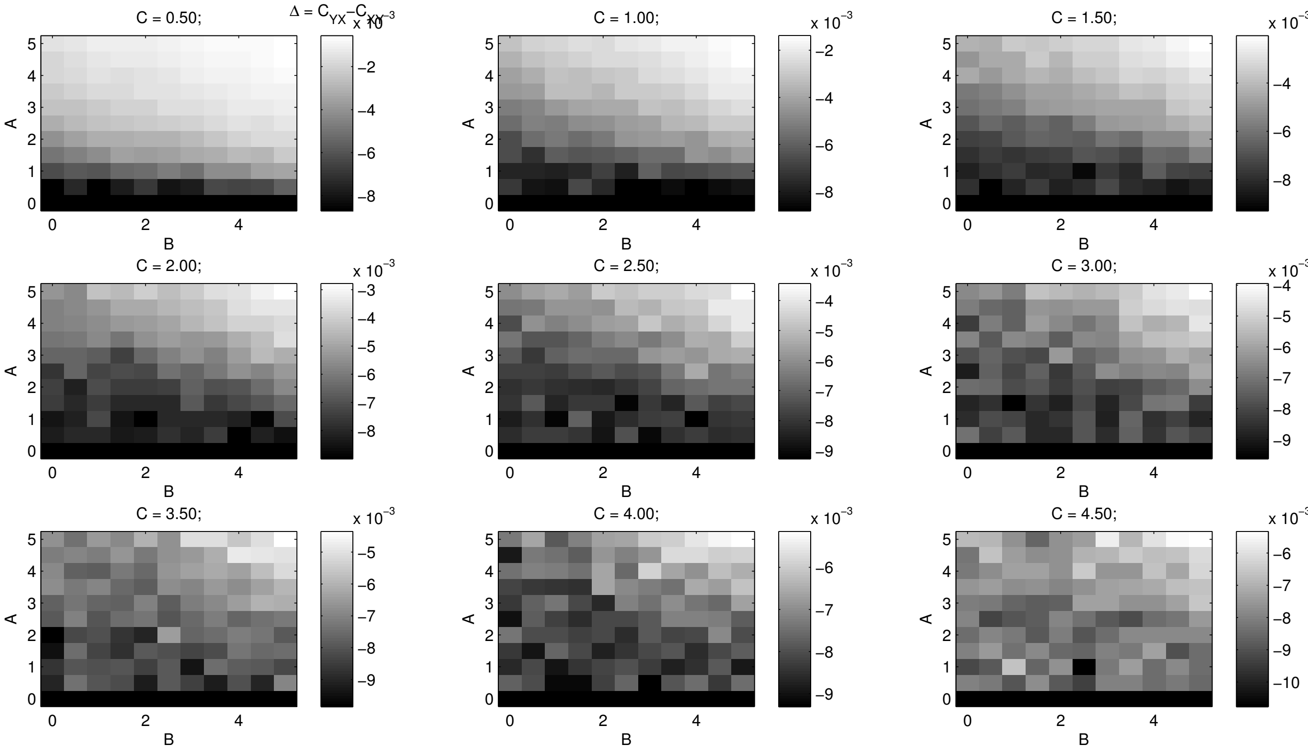



Figure 21: Non Linear Example PAI