The first step we took when we evaluated a path integral was to define the path using parametric equations, like

$$x(t) = cos(t) \quad y(t) = sin(t)$$

for movement around a circle. So the initial step I took was to figure out how to define a surface using parametrics, and I came to the conclusion that to define a surface we need two variables. I thought of it like adding two parametric equations that define motion along lines. The first could be $x(a) = (a, 0, 0)$ and the second could be $x(b) = (0, b, 0)$. Adding the two would give $x(a, b) = (a, b, 0)$, so at any point we could move in the $\hat{i}$ direction by changing a, and we could move in the $\hat{j}$ direction by changing b. Another way I tried to justify this to myself was by using ideas from linear algebra. If you were standing on any surface, if you looked at a small enough section of it up close, the surface would seem locally flat. A basis for $\mathbb{R}^2$ has two linearly independent vectors, so as long as a and b describe motion in different directions it should make a surface.

So, a surface can be defined by three parametric equations each with two variables. For the x-y plane, these equation could look like.

$$x(a, b) = a \quad y(a, b) = b \quad z(a, b) = 0$$

If we wanted to only look at a section of the plane, say for $0 \leq x \leq 2$ and $0 \leq y \leq 1$, we can bound a and b, and in this case we bound a to the same values as x and b to the values of y.

Next we need to find $\vec{n}$ and A. For this plane, the normal vector will always be in the z-direction, and the total surface area is 2, but that's only easy to find in this case. While sitting in the airport waiting for my flight I thought of a way to generalize finding $\vec{n}$ and A in the same step. Earlier I mentioned how varying a and b would change the position in distinct directions. If we pick a random point P on the surface, then, going back to the locally flat argument, any point close to P can be approximated by adding P to a linear combination of the a direction vector and b direction vector at P, so the normal vector at that point should be perpendicular to both, and can be found with the cross product. Now I just had to figure out what the a and b directions were.

Only focusing on the a direction, one way to find it would be to start at a point P, say given by $(x(a, b), y(a, b), z(a, b))$, and take a small step in the a direction Δa. The new point P^* would be given by $(x(a + \Delta a, b), y(a + \Delta a, b), z(a + \Delta a, b))$. The direction from P to P^* would be given by $P^* - P$. Already this is looking like a derivative, and dividing by Δa and taking the limit as $a \to 0$, we should get the a direction vector (I'll call this $\vec{a}$)

$$\vec{a} = \lim_{a \to 0} \frac{(x(a + \Delta a, b), y(a + \Delta a, b), z(a + \Delta a, b)) - (x(a, b), y(a, b), z(a, b))}{\Delta a}$$

By moving the subtraction inside each component, distributing the division across components, and moving the limit inside each component, we get derivatives with respect to a.

$$\vec{a}(a,b) = (\frac{d}{da}x(a,b), \frac{d}{da}y(a,b), \frac{d}{da}z(a,b))$$

Really $\vec{a}$ is a function of a and b, because for almost any surface it will change depending on where you are. Because b is given as an argument when evaluating $\vec{a}$ at some point, and taking the derivative with respect to a only changes the first input in each position function, b should be able to be treated like a constant when taking the derivative. So, for this plane example,

$$\vec{a}(a,b) = (\frac{d}{da}a, \frac{d}{da}b, \frac{d}{da}0)$$

$$\vec{a}(a,b) = (1, 0, 0)$$

And the same should hold true for $\vec{b}$

$$\vec{b}(a,b) = (\frac{d}{db}x(a,b), \frac{d}{db}y(a,b), \frac{d}{db}z(a,b))$$

$$\vec{b}(a,b) = (\frac{d}{db}a, \frac{d}{db}b, \frac{d}{db}0)$$

$$\vec{b}(a,b) = (0, 1, 0)$$

I think these results make sense for our plane example, where a causes a change in the $\hat{i}$ direction and b causes a change in the $\hat{j}$ direction. I mentioned earlier that we can find $\vec{n}$ with a cross product, so

$$\vec{n}(a,b) = \vec{a}(a,b) \times \vec{b}(a,b)$$

$$\vec{n}(a,b) = (0, 0, 1)$$

This also makes sense, because $\vec{n}$ is what we dot with E, and if the surface is in the x-y plane, the normal vector should be only in the $\hat{k}$ direction so that only the $\hat{k}$ component of E matters. Around this point I realized that we've actually aready found A, because we are using the cross product, which returns a vector perpendicular to the first two and with magnitude equal to the swept out area! So actually this $\vec{n}$ is already $A\hat{n}$.

Just to make sure it works, we can define $\vec{E}(x,y,z) = (0,0,1)$, and plug into the integral.

$$\Phi_E = \iint \vec{E} \cdot d\vec{A}$$

$$= \iint (0,0,1) \cdot (0,0,1)$$

$$= \iint 1$$

Whoops. What am I integrating with respect to? I think it should be da for one integral and db for the other, but I needed to find where those came from.

The only place it makes sense for them to be is somewhere in $d\vec{A}$, and really what this is is a local linear approximation.

When we look for $\vec{a}$, what we really want is to get $P^* - P$ (i.e. ΔP) for a small change Δa. This is an LLA!

$$\Delta P = P'(a,b)\Delta a$$

$$dP = \frac{d}{da}P(a,b)da$$

Here what I've been calling $\vec{a}$ is really dP, and the same is true with respect to b

$$\vec{a}(a,b) = (\frac{d}{da}x(a,b), \frac{d}{da}y(a,b), \frac{d}{da}z(a,b))da$$

$$\vec{b}(a,b) = (\frac{d}{db}x(a,b), \frac{d}{db}y(a,b), \frac{d}{db}z(a,b))db$$

So $\vec{n}$ for our plane example is really

$$\vec{a}(a,b) \times \vec{b}(a,b)$$

$$(0,1,0)da \times (0,0,1)db$$

Scalars distribute across the product but only to one term ($c(v \times w) = cv \times w$) because the determinant of a matrix is linear in each row separately.

$$(0,0,1)dadb$$

Now we can try that integral again

$$\Phi_E = \iint \vec{E} \cdot d\vec{A}$$

$$= \iint (0,0,1) \cdot (0,0,1)dadb$$

$$= \iint (0,0,1) \cdot (0,0,1)dadb$$

Here we can take $dadb$ out of the dot product.

$$= \iint [(0,0,1) \cdot (0,0,1)]dadb$$

$$= \iint 1dadb$$

Earlier we said $0 \leq a \leq 2$ and $0 \leq b \leq 1$, so those are our bounds for integration

$$= \int_0^1 \int_0^2 1dadb$$

And evaluating one integral after another we get our final answer of 2!

Doing this integral in a different coordinate system, say polar, is slightly more difficult because we have to convert polar area to Cartesian area. We can do this by finding a transformation that maps polar coordinates to their Cartesian counterparts, and then taking the determinant of that matrix. First, let us list out our parametric equations again.

$$r = R \quad \phi = t \quad \theta = s$$

Next we need to find the conversion from $dtds$ to $dxdy$. To do this, let us write out x, y, and z in terms of r, θ, and ϕ, and then sub in to get them in terms of t and s.

$$x = rsin(\phi)cos(\theta) \quad y = rsin(\phi)sin(\theta) \quad z = rcos(\phi)$$

Now we can find the area conversion factor from $dtds$ to $dxdy$. We do this by taking the partial derivatives of the old basis with respect to the new basis.

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} sin(\phi)cos(\theta) & rcos(\phi)cos(\theta) & -rsin(\phi)sin(\theta) \\ sin(\phi)sin(\theta) & rcos(\phi)sin(\theta) & rsin(\phi)cos(\theta) \\ cos(\phi) & -rsin(\phi) & 0 \end{vmatrix}$$

$$\begin{aligned} =& cos(\phi)[r^2 sin(\phi)cos(\phi)cos^2(\theta) + r^2 sin(\phi)cos(\phi)sin^2(\theta)] \\ &- (-rsin(\phi))[rsin^2(\phi)cos^2(\theta) + rsin^2(\phi)sin^2(\theta)] \\ &= r^2 sin(\phi)cos^2(\phi) + r^2 sin^3(\phi) \\ &= r^2 sin(\phi) \end{aligned}$$

So we get that $dxdy = r^2 sin(\theta)d\theta d\phi$

$$\begin{aligned} \Phi_E &= \oiint \vec{E} \cdot d\vec{A} \\ &= \iint \left(\frac{q}{4\pi\epsilon_0 r^2}, 0, 0\right) \cdot (1,0,0)\, dxdy \\ &= \iint \frac{q}{4\pi\epsilon_0 r^2} r^2 sin(\theta)\, dtds \\ &= \int_{-\pi}^{\pi} \int_0^{\pi} \frac{q}{4\pi\epsilon_0} sin(t)\, dtds \\ &= \frac{q}{4\pi\epsilon_0} \int_{-\pi}^{\pi} [-cos(t)]_0^{\pi}\ ds \\ &= \frac{q}{4\pi\epsilon_0} \int_{-\pi}^{\pi} 2\, ds \\ &= \frac{q}{4\pi\epsilon_0} [2s]_{-\pi}^{\pi} \\ &= \frac{q}{4\pi\epsilon_0} 4\pi \\ &= \frac{q}{\epsilon_0} \end{aligned}$$

Nice! We have shown Gauss's law holds for a sphere!