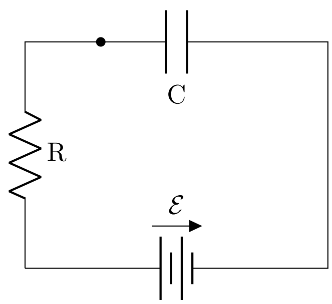
Ampere's Law is actually wrong?



Let $\mathcal{E}$ be the voltage of the battery, R be the resistance of the resistor, and C be the capacitance of the capacitor. Then we can solve for $q(t)$, the charge buildup on the capacitor, and $I(t)$, the current through the circuit.

$$\mathcal{E} = \Delta V_R + \Delta V_c$$
$$\mathcal{E} = \dot{q}R + \frac{q}{C}$$
$$\dot{q} = \frac{1}{R}(\mathcal{E} - \frac{q}{C})$$
$$\frac{1}{\mathcal{E} - \frac{q}{C}}\dot{q} = \frac{1}{R}$$
$$\int \frac{1}{\mathcal{E} - \frac{q}{C}} dq = \int \frac{1}{R} dt$$

$$-Cln|\mathcal{E} - \frac{q}{C}| = \frac{t}{R} + c_1$$
$$ln|\mathcal{E} - \frac{q}{C}| = -\frac{t}{CR} + c_2$$
$$\mathcal{E} - \frac{q}{C} = c_3 e^{-\frac{t}{CR}}$$
$$q = C(\mathcal{E} - c_3 e^{-\frac{t}{CR}})$$

If we let $q = 0$ at $t = 0$, $c_3 = \mathcal{E}$, so

$$q(t) = C\mathcal{E}(1 - e^{-\frac{t}{CR}})$$

Once we have $q(t)$, we can then find the current as $\frac{dq}{dt}$

$$I = \frac{dq}{dt}$$
$$I = \frac{d}{dt} C\mathcal{E}(1 - e^{-\frac{t}{CR}})$$
$$I = \frac{\mathcal{E}}{R} e^{-\frac{t}{CR}}$$

From $I(t)$, we can calculate the path integral of the magnetic field around an Amperian loop about any point in the circuit. To stay consistent, I will look at a circle of radius r about the darkened point between the capacitor and resistor, which I will name $\mathcal{C}$. Ampere's Law tells us

$$\oint_{\mathcal{C}} B \cdot dl = \mu_0 I$$

Which I will leave in this form for now. We know this law should hold regardless of the surface we look at, as long as it is bounded by the curve $\mathcal{C}$.

If we happen to choose a surface that runs between the parallel plates of the capacitors, the current drops to 0 and we get an inconsistent result. Therefore there must be some other quantity that picks up where the current drops off, and the only other quantity in that region is the electric field. We can calculate this electric field E from the charge buildup on the capacitor.

Let d be the distance between the parallel plates, and A be the area of each plate.

$$E = -\frac{dV}{dr}$$

$$E = -\frac{1}{d}\Delta V_C$$

$$E = -\frac{1}{d}\frac{q}{C}$$

For parallel plates, we have $C = \frac{\epsilon_0 A}{d}$

$$E = -\frac{1}{\epsilon_0 A}q$$

We can rearrange to find q in terms of E, and then differentiate to relate it to I.

$$q = -\epsilon_0 AE$$

$$\frac{d}{dt}q = \frac{d}{dt} - \epsilon_0 AE$$

$$I = -\epsilon_0 A\frac{d}{dt}E$$

So, again by Ampere's Law, we have

$$\oint_C B \cdot dl = \mu_0 I$$

$$\oint_C B \cdot dl = -\mu_0\epsilon_0 A\frac{d}{dt}E$$

Now here comes the confusing part. In order for Ampere's Law to hold, these two expressions must be equivalent, which is why I could just substitute our new expression relating I to $\frac{d}{dt}E$ into Ampere's Law. However, this would imply that B is created by a time-varying E in the region where there is no current. The issue comes in the region where there is current, because there is also a time-varying E. This must be the case because there is a time-varying I inside the wire. So Ampere's Law for a section of wire with a time-varying current should include both of these terms, but it only includes I. Therefore Ampere's Law must not hold for a time-varying current.