Changing Reference Frames

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1 Lorentz Transformations

1.1 Finding the Matrix

If we assume the speed of light is invariant under changes of reference frame, then the vector $\begin{bmatrix} ct \\ t \end{bmatrix}$ should be an eigenvector of whatever transformation we are doing, likewise for $\begin{bmatrix} -ct \\ t \end{bmatrix}$. From our reference frame, our spacetime path is along $\begin{bmatrix} 0 \\ t \end{bmatrix}$, but an object moving past us at velocity v is $\begin{bmatrix} vt \\ t \end{bmatrix}$. From the object's perspective, it is at $\begin{bmatrix} 0 \\ t \end{bmatrix}$, and we are at $\begin{bmatrix} -vt \\ t \end{bmatrix}$. So, we have a system of equations with some $A \in M_{2\times 2}(\mathbb{R})$.

$$\lambda_1 \begin{bmatrix} ct \\ t \end{bmatrix} = A \begin{bmatrix} ct \\ t \end{bmatrix} \tag{1}$$

$$\lambda_2 \begin{bmatrix} -ct \\ t \end{bmatrix} = A \begin{bmatrix} -ct \\ t \end{bmatrix} \tag{2}$$

$$\gamma \begin{bmatrix} -vt \\ t \end{bmatrix} = A \begin{bmatrix} 0 \\ t \end{bmatrix} \tag{3}$$

$$\alpha \begin{bmatrix} 0 \\ t \end{bmatrix} = A \begin{bmatrix} vt \\ t \end{bmatrix} \tag{4}$$

If we let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then we can write all four of these equations, and dividing by t where it's common, we have

$$\begin{aligned} \lambda_1 c &= a_{11}c + a_{12} \\ \lambda_1 &= a_{21}c + a_{22} \\ -\lambda_2 c &= -a_{11}c + a_{12} \\ -\lambda_2 &= -a_{21}c + a_{22} \end{aligned}$$

$$\begin{aligned} -\gamma v &= a_{12} \\ \gamma &= a_{22} \\ 0 &= a_{11}v + a_{12} \\ \alpha &= a_{21}v + a_{22} \end{aligned}$$

First we have that $a_{12} = -\gamma v$, and $a_{22} = \gamma$. We can then substitute in the first result to the seventh equation to find $a_{11} = \gamma$ Now using these three results, we have $\lambda_1 = \gamma\left(1 - \frac{v}{c}\right)$ from the first equation, and then we can solve for a_{21} in the second, giving $a_{21} = -\gamma\frac{v}{c^2}$.

$$A = \begin{bmatrix} \gamma & -\gamma v \\ -\gamma\frac{v}{c^2} & \gamma \end{bmatrix}$$

We then have the added property that spacetime is conserved, so $det(A) = 1$ is a requirement.

$$\gamma^2 - \gamma^2\frac{v^2}{c^2} = 1$$

$$\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This quantity γ is known as the Lorentz Factor, and it adjusts time, velocity, and distance under changes of reference frames. Now let's look at the application of this matrix A to these vectors and see what's actually happening with this Lorentz Factor.

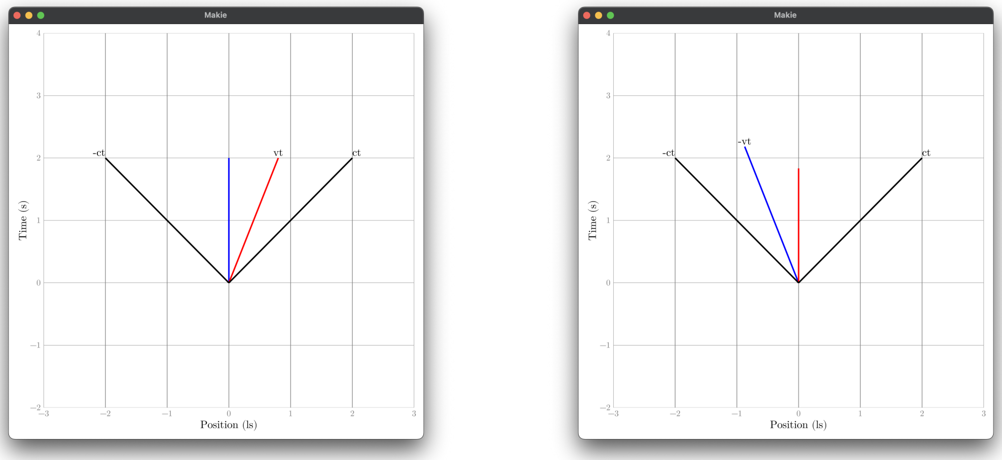
1.2 What is Actually Happening

First, let's restate the main two transformations here. What we're really looking at is a change of basis, so we have forward and backward transformations. The matrix we just found I'll call the forward transformation that goes from what I'll call the blue basis ($\mathcal{B}$) to the red basis ($\mathcal{R}$). We can also find the backward transformation as the inverse of the forward, i.e. from $\mathcal{R}$ to $\mathcal{B}$. Because the determinant is one, and we only have a 2×2 matrix, we can swap the diagonal entries and negate the non-diagonal entries.

$$[A]_{\mathcal{B}}^{\mathcal{R}} = F = \gamma \begin{bmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{bmatrix} \qquad [A]_{\mathcal{R}}^{\mathcal{B}} = B = \gamma \begin{bmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{bmatrix}$$

And as a note, when I say blue or red basis, I mean that the color specified is considered the observer, with $x = 0$ for all t.

Let's work through an example. On the y-axis is time in seconds, starting at $t = 0$ and progressing upwards. The x-axis has position in light-seconds, or the distance light travels in one second.



(a) Initial State

(b) After Transformation

Imagine you are floating in space, and at time $t = 0$ you send out two photons, one to your right and one to your left. You also release a baseball that travels away from you with velocity $v = 0.4c$, or 40% the speed of light. Your path is plotted in diagram (a) as a blue line, the two photons are the black lines with positions ct and $-ct$, and the baseball's path is shown as the red line with position vt. At $t = 2$ seconds, each photon is now 2 light-seconds away, and the baseball has traveled 0.8 light-seconds.

Now, let's see the world from the baseball's perspective. Applying the transformation on these paths results in diagram (b), and the light paths have been scaled to be correct in the baseball's reference frame. The key thing to notice is that the blue path is now longer in both the time and position axes. Now, while in your reference frame you consider yourself to be at the spacetime coordinates $t = 2$ and $x = 0$, from the baseball's reference frame, your spacetime coordinates are actually $t = 2\gamma$ and $x = -2v\gamma$.

1.3 Ramblings of a Deranged Lunatic

A fitting title for the entire contents of my overleaf account.

1. If spacetime is conserved, does that mean a kind of spacetime-velocity is also conserved? Like a combined speed through space and time?
2. I don't even want to look at acceleration. Does $F = \frac{dp}{dt}$ still hold?
3. Do any conservation laws hold? Are energy and momentum conserved within reference frames?

Maybe there is some light after all.

It is pretty apparent that if, from your perspective, an object's speed is lower than c, then, from its perspective, your speed will also be lower than c. A more interesting concern is if we have two objects where, from your perspective, it would appear that relative to each other they are moving faster than c. Consider a brachiosaurus moving at $0.6c$ to the right, and a snail moving at $0.6c$ to the left. Without relativity one might conclude that, from the brachiosaurus's perspective, the snail moves at $1.2c$ to the left, but we know this is not true. Instead, it will appear that you are moving at $0.6c$ to the left, and the snail is moving somewhere between c and $0.6c$ to the left. The odd thing is that then, from the brachiosaurus's perspective, you and the snail no longer have a relative velocity of $0.6c$, but instead something far less. Perhaps the solution to this lies in the way the coordinates transform–that although you and the snail now appear to have a relative velocity far under $0.6c$ over distance, the snail now moves faster (or slower) through time. Let's do some math!

$$T = \gamma \begin{bmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{bmatrix}$$

You	Brachiosaurus	Snail
$x_0 = \begin{bmatrix} 0 \\ t \end{bmatrix}$	$x_b = \begin{bmatrix} vt \\ t \end{bmatrix}$	$x_s = \begin{bmatrix} -vt \\ t \end{bmatrix}$
$Tx_0 = \gamma \begin{bmatrix} -vt \\ t \end{bmatrix}$	$Tx_b = \gamma(1 - \frac{v^2}{c^2}) \begin{bmatrix} 0 \\ t \end{bmatrix}$	$Tx_s = \gamma(1 + \frac{v^2}{c^2}) \begin{bmatrix} -c^2 \frac{v^2-v}{c^2+v^2} t \\ t \end{bmatrix}$

So we find the new velocity of the snail as $v_s = -c^2 \frac{v^2-v}{c^2+v^2}$, and when we plug in $v = -0.6c$ we find that it's speed relative to the brachiosaurus is roughly $-0.705c$