# 随机过程导论 Assignment 01

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## Problem 1

Prove that the distribution of $S := \sum_{i=1}^n \xi_i$ is Gamma$(n, \lambda)$
where $\xi_i$ are i.i.d. exponential random variables with parameter $\lambda$.
**Proof:**

- **Firstly, we derive the MGF of $\text{Gamma}(n, \lambda)$:**
  $$\begin{aligned} M_X(t) &= \text{E}[e^{tX}] \\ &= \int_0^\infty e^{tx} \cdot \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{\Gamma(n)} dx \\ &= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty e^{(t-\lambda)x}(\lambda x)^{n-1} dx \\ &= \frac{\lambda^n}{\Gamma(n)(\lambda-t)^n} \int_0^\infty e^{-(\lambda-t)x}((\lambda-t)x)^{n-1} d(\lambda-t)x \\ &= \frac{\lambda^n}{\Gamma(n)(\lambda-t)^n} \cdot \Gamma(n) \\ &= (\frac{\lambda}{\lambda-t})^n \quad (t<\lambda) \end{aligned}$$
- **Secondly, we prove that $S \sim \text{Gamma}(n, \lambda)$:**
  Since $\{\xi_i\} \stackrel{\text{iid}}{\sim} \exp(\lambda) = \text{Gamma}(1, \lambda)$, it holds that:
  $$\begin{aligned} M_S(t) &= \prod_{i=1}^n M_{\xi_i}(t) \\ &= \prod_{i=1}^n \frac{\lambda}{\lambda-t} \\ &= (\frac{\lambda}{\lambda-t})^n \end{aligned}$$
  Therefore， $S \sim \text{Gamma}(n, \lambda)$
  **Q.E.D.**

## Problem 2

Prove the law of total variance: $\text{Var}(X) = \text{E}(\text{Var}(X|Y)) + \text{Var}(\text{E}(X|Y))$
**Proof:**

- Calculate $\text{E}(\text{Var}(X|Y))$:
  $$\begin{aligned} \text{E}(\text{Var}(X|Y)) &= \text{E}[\text{E}(X^2|Y) - (\text{E}(X|Y))^2] \\ &= \text{E}[\text{E}(X^2|Y)] - \text{E}[(\text{E}(X|Y))^2] \\ &= \text{E}(X^2) - \text{E}[(\text{E}(X|Y))^2] \end{aligned}$$
- Calculate $\text{Var}(\text{E}(X|Y))$:
  $$\begin{aligned} \text{Var}(\text{E}(X|Y)) &= \text{E}[(\text{E}(X|Y))^2] - (\text{E}[\text{E}(X|Y)])^2 \\ &= \text{E}[(\text{E}(X|Y))^2] - (\text{E}(X))^2 \end{aligned}$$

Therefore, it holds that:

$$\begin{aligned}\mathrm{E}(\mathrm{Var}(X|Y)) + \mathrm{Var}(\mathrm{E}(X|Y)) &= \mathrm{E}(X^2) - \mathrm{E}[(\mathrm{E}(X|Y))^2] + \mathrm{E}[(\mathrm{E}(X|Y))^2] - (\mathrm{E}(X))^2 \\ &= \mathrm{E}(X^2) - (\mathrm{E}(X))^2 \\ &= \mathrm{Var}(X)\end{aligned}$$

**Q.E.D.**

# Problem 3

The length of the password for a random account in a particular system is modelled as

$L = N + 8$

where $N$ is assumed to following a binomial distribution with mean $2$ and variance $1.6$.

A hacking program can crack a password with length $L$ in $T$ minutes where

$$\begin{cases}\mathrm{E}(T|L) = 2L - 4 \\ \mathrm{Var}(T|L) = \frac{L(L-1)}{4}\end{cases}.$$

Determine the mean and variance of $T$.

**Solution:**

By $\begin{cases}\mathrm{E}(N) = 2 \\ \mathrm{Var}(N) = 1.6\end{cases}$ we know that $\begin{cases}\mathrm{E}(L) = 10 \\ \mathrm{Var}(L) = 1.6\end{cases}$

hence $\mathrm{E}(L^2) = 101.6$

- Determine $\mathrm{E}(T)$:

$$\begin{aligned}\mathrm{E}(T) &= \mathrm{E}(\mathrm{E}(T|L)) \\ &= \mathrm{E}(2L - 4) \\ &= 2\mathrm{E}(L) - 4 \\ &= 16\end{aligned}$$

- Determine $\mathrm{Var}(T)$:

$$\begin{aligned}\mathrm{Var}(T) &= \mathrm{E}(\mathrm{Var}(T|L)) + \mathrm{Var}(\mathrm{E}(T|L)) \\ &= \mathrm{E}(\frac{L(L-1)}{4}) + \mathrm{Var}(2L - 4) \\ &= \frac{1}{4}\mathrm{E}(L^2) - \frac{1}{4}\mathrm{E}(L) + 4\mathrm{Var}(L) \\ &= \frac{1}{4}101.6 - \frac{1}{4}10 + 4 \times 1.6 \\ &= 29.3\end{aligned}$$

# Problem 4

Let $X$ denote the size of a surgical claim and $Y$ denote the size of the associated hospital claim.

An actuary is using a model in which $\begin{cases}\mathrm{E}(X) = 5 \\ \mathrm{E}(X^2) = 27.4 \\ \mathrm{E}(Y) = 7 \\ \mathrm{E}(Y^2) = 51.4 \\ \mathrm{Var}(X + Y) = 8\end{cases}$

The actuary had assumed that $\mathrm{E}(X|Y) = aY + b$.

Determine the values of $a$ and $b$.

**Proof:**

- Equation one:
$$\begin{aligned}\mathrm{E}(X) &= \mathrm{E}(\mathrm{E}(X|Y)) \\ &= \mathrm{E}(aY+b) \\ &= a\mathrm{E}(Y)+b \\ &= 7a+b \\ &= 5\end{aligned}$$

- Equation two:
$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{E}[(X+Y-\mathrm{E}(X+Y))^2] \\ &= \mathrm{E}[X^2+2XY+Y^2]-2\mathrm{E}[X+Y]\cdot\mathrm{E}[X+Y]+(\mathrm{E}(X+Y))^2 \\ &= \mathrm{E}(X^2)+2\mathrm{E}(XY)+\mathrm{E}(Y^2)-(\mathrm{E}(X)+\mathrm{E}(Y))^2 \\ &= 27.4+2\mathrm{E}(Y\cdot\mathrm{E}(X|Y))+51.4-(5+7)^2 \\ &= 2\mathrm{E}(aY^2+bY)-65.2 \\ &= 2[a\mathrm{E}(Y^2)+b\mathrm{E}(Y)]-65.2 \\ &= 102.8a+14b-65.2 \\ &= 8\end{aligned}$$

Hence, we have $\begin{cases}7a+b=5 \\ 102.8a+14b-65.2=8\end{cases}$

Therefore, $\begin{cases}a=\frac{2}{3} \\ b=\frac{1}{3}\end{cases}$

## Problem 5

In a game of odd-man-out each of three players tosses a coin independently.
If the outcomes of their tosses are all the same then they start over and re-toss their coins.
Until the outcomes differ then the game ends,
and the player whose outcome differs from those of any other players wins.
Suppose that one of the players, Bob, uses a biased coin with probability $p$ of landing head;
and the other players use fair coins.

- **(a) Find the probability that Bob will win.**

  **Solution:**
  Denote Bob as $B$ and the other players as $A_1, A_2$
  $$\begin{aligned}\mathrm{P}(\text{Bob wins}) &= \mathrm{P}(\text{Bob's result is unique} \mid \text{Outcomes differ}) \\ &= \frac{\mathrm{P}(B=H, A_1=A_2=T)+\mathrm{P}(B=T, A_1=A_2=H)}{1-\mathrm{P}(\text{Outcomes same})} \\ &= \frac{p\times(1-\frac{1}{2})^2+(1-p)\times(\frac{1}{2})^2}{1-[p\times(\frac{1}{2})^2+(1-p)\times(1-\frac{1}{2})^2]} \\ &= \frac{1}{3}\end{aligned}$$

- **(b) Determine the expected number of rounds that they will toss.**
  **Solution:**
  Denote the number of rounds as $N$
  The probability of a effective round is:
  $$p_0 \overset{\Delta}{=} 1-\mathrm{P}(\text{Outcomes same}) = 1-[p\times(\tfrac{1}{2})^2+(1-p)\times(1-\tfrac{1}{2})^2] = \tfrac{3}{4}$$

Evidently $N \sim \mathrm{Geo}(p_0)$ (Geometric distribution)

$$
\begin{aligned}
\mathrm{E}(N) &= \sum_{i=1}^{\infty} i \cdot (1-p_0)^{i-1} p_0 \\
&= p_0 \sum_{i=1}^{\infty} i \cdot q^{i-1} \qquad (q \overset{\Delta}{=} 1 - p_0) \\
&= p_0 \sum_{i=1}^{\infty} \frac{d}{dq}(q^i) \\
&= p_0 \frac{d}{dq}(\frac{q}{1-q}) \\
&= \frac{p_0}{(1-q)^2} \\
&= \frac{1}{p_0} \\
&= \frac{4}{3}
\end{aligned}
$$

## Problem 6

John von Neumann gave the following procedure to make a fair toss from a biased coin with unknown probability of tossing a Head:

1. Toss the coin twice.
2. If the outcomes are the same, ignore the outcomes and start over from step (1).
3. If the outcomes differ, use the first outcome and ignore the second outcome.

Show that this procedure will result in a fair toss.

**Proof:**

Denote the probability of tossing a Head as $p$, and the results of two tosses as $R_1, R_2$

The probability of a final Head outcome from the above procedure is :

$$
\begin{aligned}
\mathrm{P}(\text{final Head outcome}) &= \mathrm{P}(\text{First outcome is Head} \mid \text{Outcomes differ}) \\
&= \frac{\mathrm{P}(R_1 = H, R_2 = T)}{1 - \mathrm{P}(\text{Outcomes same})} \\
&= \frac{p(1-p)}{1 - [p^2 + (1-p)^2]} \\
&= \frac{1}{2}
\end{aligned}
$$

Similarly, by replacing $p$ as $1-p$, we know that $\mathrm{P}(\text{final Tail outcome}) = \frac{1}{2}$

Therefore the procedure will result in a fair toss.

**Q.E.D.**