# Introduction to Stochastic Process

**Time:** 18:30-21:05

## Problem 1

Let $X$ and $Y$ be independent random variables distributed as exponential with parameters $\lambda$ and $\mu$, respectively.
Let $I$, independent of $X$ and $Y$, be a Bernoulli random variable with success probability $p = \frac{\mu}{\lambda+\mu}$.
Define $W = X - Y$ and $Z = \begin{cases} X, & \text{if } I = 1 \\ -Y, & \text{if } I = 0 \end{cases}$

**(a) Show, by using moment generating functions, that $W$ and $Z$ have the same distribution.**
(5 marks)

**Solution:**

- 计算 $X$ 的矩母函数:

$$
\begin{aligned}
\mathrm{E}[e^{tX}] &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}\mathrm{d}x \\
&= \lambda \int_0^\infty e^{-(\lambda-t)x}\mathrm{d}x \\
&= \frac{\lambda}{\lambda - t}
\end{aligned}
$$

- 计算 $-Y$ 的矩母函数:

$$
\begin{aligned}
\mathrm{E}[e^{-tY}] &= \int_0^\infty e^{-ty}\mu e^{-\mu y}\mathrm{d}y \\
&= \mu \int_0^\infty e^{-(\mu+t)y}\mathrm{d}y \\
&= \frac{\mu}{\mu + t}
\end{aligned}
$$

- 计算 $W = X - Y$ 的矩母函数:

$$
\begin{aligned}
\mathrm{E}[e^{tW}] &= \mathrm{E}[e^{t(X-Y)}] \\
&= \mathrm{E}[e^{tX}] \cdot \mathrm{E}[e^{-tY}] \\
&= \frac{\lambda}{\lambda - t} \cdot \frac{\mu}{\mu + t} \\
&= \frac{\lambda\mu}{(\lambda - t)(\mu + t)}
\end{aligned}
$$

- 计算 $Z = \begin{cases} X, & \text{if } I = 1 \\ -Y, & \text{if } I = 0 \end{cases}$ 的矩母函数:

$$
\begin{aligned}
\mathrm{E}[e^{tZ}] &= p\mathrm{E}[e^{tX}] + (1-p)\mathrm{E}[e^{-tY}] \\
&= \frac{\mu}{\lambda+\mu}\frac{\lambda}{\lambda-t} + \frac{\lambda}{\lambda+\mu}\frac{\mu}{\mu+t} \\
&= \frac{\lambda\mu}{(\lambda - t)(\mu + t)}
\end{aligned}
$$

综上所述，对于任意 $t$ (范围待确定，但这不重要) 都有 $\mathrm{E}[e^{tW}] = \mathrm{E}[e^{tZ}]$
由于矩母函数唯一确定分布，因此 $W \overset{d}{=} Z$.

**(b) Give a simple explanation of the result in part (a)**
**using the memoryless property of exponential distribution.**
(5 marks)

- **Hint:**
  Consider $X$ and $Y$ as the waiting time for the occurrence of type ( I ) and type (II) events and $I$ as the indicator function for the event $X > Y$.

**Solution:**

- **指数分布的无记忆性:**
  对于指数随机变量 $X \sim \exp(\lambda)$,
  有 $\mathrm{P}\{X > t+s | X > s\} = \mathrm{P}\{X > t\} \ \ (\forall\, 0 < s < t)$ 成立.
- **考虑第一种情况:**
  ( I ) 类事件晚于 (II) 类事件发生,
  即事件 $\{X > Y\}$ 发生，也即事件 $\{I = 1\}$ 发生.
  对于任意 $s > 0$ 都有:

$$\begin{aligned}
\mathrm{P}\{Z > s\} &= \mathrm{P}\{I = 1\} \cdot \mathrm{P}\{X > s\} \\
&= \mathrm{P}\{X > Y\} \cdot \mathrm{P}\{X > s\} \quad (\text{应用指数分布的无记忆性}) \\
&= \mathrm{P}\{X > Y\} \cdot \mathrm{P}\{X > Y + s | X > Y\} \\
&= \mathrm{P}\{X > Y + s\} \\
&= \mathrm{P}\{X - Y > s\} \\
&= \mathrm{P}\{W > s\}
\end{aligned}$$

  也就是说，对于任意 $s > 0$ 都有 $\mathrm{P}\{Z \le s\} = \mathrm{P}\{W \le s\}$
- **考虑第二种情况:**
  ( I ) 类事件早于 (II) 类事件发生，或同时发生,
  即事件 $\{X \le Y\}$ 发生，也即事件 $\{I = 0\}$ 发生.
  对于任意 $s \le 0$ 都有:

$$\begin{aligned}
\mathrm{P}\{Z \le s\} &= \mathrm{P}\{I = 0\} \cdot \mathrm{P}\{-Y \le s\} \\
&= \mathrm{P}\{X \le Y\} \cdot \mathrm{P}\{Y \ge -s\} \quad (\text{应用指数分布的无记忆性}) \\
&= \mathrm{P}\{Y \ge X\} \cdot \mathrm{P}\{Y \ge X - s | Y \ge X\} \\
&= \mathrm{P}\{Y \ge X - s\} \\
&= \mathrm{P}\{X - Y \le s\} \\
&= \mathrm{P}\{W \le s\}
\end{aligned}$$

综上所述，我们对于任意 $s \in \mathbb{R}$ 都有 $\mathrm{P}\{Z \le s\} = \mathrm{P}\{W \le s\}$ 成立,
由于累计分布函数 (CDF) 唯一确定分布，因此 $W \overset{d}{=} Z$.

# Problem 2

Let $\{N(t) : t \ge 0\}$ be a Poisson process with rate $\lambda > 0$
that is independent of the non-negative random variable $T$ with mean $\mu$ and variance $\sigma^2$.
Find:

**(a)** $\mathrm{Cov}(T, N(T))$
(5 marks)

- **Lemma: (Poisson 过程的结论)**
  $N(t) \sim \text{Poisson}(\lambda t)$
  因此有 $\begin{cases} \text{E}[N(t)] = \lambda t \\ \text{Var}[N(t)] = \lambda t \\ \text{E}[(N(t))^2] = \lambda t + \lambda^2 t^2 \end{cases}$

**Solution: (全期望公式)**

- 计算 $\text{E}[N(T)]$:
  $$\begin{aligned} \text{E}[N(T)] &= \text{E}[\text{E}[N(T)|T]] \\ &= \text{E}[\lambda T] \\ &= \lambda\mu \end{aligned}$$
- 计算 $\text{E}[TN(T)]$:
  $$\begin{aligned} \text{E}[TN(T)] &= \text{E}[\text{E}[TN(T)|T]] \\ &= \text{E}[T \cdot \lambda T] \\ &= \lambda \text{E}[T^2] \\ &= \lambda[\text{Var}(T) + (\text{E}[T])^2] \\ &= \lambda(\sigma^2 + \mu^2) \end{aligned}$$
- 计算 $\text{Cov}(T, N(T))$:
  $$\begin{aligned} \text{E}[T, N(T)] &= \text{E}[TN(T)] - \text{E}[T] \cdot \text{E}[N(T)] \\ &= \lambda(\sigma^2 + \mu^2) - \mu \cdot \lambda\mu \\ &= \lambda\sigma^2 \end{aligned}$$

---

**(b)** $\text{Var}(N(T))$
(5 marks)

- **Solution 1: (全方差公式)**
  $$\begin{aligned} \text{Var}(N(T)) &= \text{E}[\text{Var}[N(T)|T]] + \text{Var}[\text{E}[N(T)|T]] \\ &= \text{E}[\lambda T] + \text{Var}[\lambda T] \\ &= \lambda\mu + \lambda^2\sigma^2 \end{aligned}$$
- **Solution 2:**
  - 计算 $\text{E}[(N(T))^2]$:
    $$\begin{aligned} \text{E}[(N(T))^2] &= \text{E}[\text{E}[(N(T))^2|T]] \\ &= \text{E}[\lambda T + \lambda^2 T^2] \\ &= \lambda \text{E}[T] + \lambda^2 \text{E}[T^2] \\ &= \lambda\mu + \lambda^2(\mu^2 + \sigma^2) \end{aligned}$$
  - 计算 $\text{Var}(N(T))$:
    $$\begin{aligned} \text{Var}(N(T)) &= \text{E}[(N(T))^2] - (\text{E}[N(T)])^2 \\ &= \lambda\mu + \lambda^2(\mu^2 + \sigma^2) - (\lambda\mu)^2 \\ &= \lambda\mu + \lambda^2\sigma^2 \end{aligned}$$

# Problem 3

Let $Y_1, Y_2, \ldots$ be independent and identically distributed random variables
with cdf $F(y) = y^\alpha$, for all $0 < y < 1$ and $\alpha > 0$.
Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda > 0$, independent of $\{Y_i\}$.
Determine $\text{P}\{Z(t) > z | N(t) > 0\}$ where $Z(t) = \min\{Y_1, Y_2, \ldots, Y_{N(t)}\}$.
(10 marks)

**Solution:**

$$F(y) = \begin{cases} 0, & \text{if } \ y \le 0 \\ y^\alpha, & \text{if } \ 0 < y < 1 \\ 1, & \text{if } \ y = 1 \end{cases}$$

首先对 $N(t)$ 取条件 $N(t) = n$ $(n \ge 1)$，则此时 $Z(t) = Y_{(1)}$
对于任意 $z \in \mathbb{R}$，我们有:

$$\begin{aligned} \mathrm{P}\{Z(t) > z|N(t) = n\} &= \mathrm{P}\{Y_{(1)} > z|N(t) = n\} \\ &= \mathrm{P}\{Y_1 > z, \ldots, Y_{N(t)} > z|N(t) = n\} \\ &= \mathrm{P}\{Y_1 > z, \ldots, Y_n > z\} \\ &= \mathrm{P}\{Y_1 > z\} \cdots \mathrm{P}\{Y_n > z\} \\ &= (1 - F(z))^n \\ &= \begin{cases} 1, & \text{if } \ z \le 0 \\ (1 - z^\alpha)^n, & \text{if } \ 0 < z < 1 \\ 0, & \text{if } \ z \ge 1 \end{cases} \end{aligned}$$

因此对于任意 $z \in \mathbb{R}$，我们有:

$$\begin{aligned} \mathrm{P}\{Z(t) > z|N(t) > 0\} &= \frac{\mathrm{P}\{Z(t) > z, N(t) > 0\}}{\mathrm{P}\{N(t) > 0\}} \\ &= \frac{\sum_{n=1}^\infty \mathrm{P}\{Z(t) > z|N(t) = n\}\mathrm{P}\{N(t) = n\}}{1 - \mathrm{P}\{N(t) = 0\}} \\ &= \frac{\sum_{n=1}^\infty \{(1 - F(z))^n \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!}\}}{1 - e^{-\lambda t} \frac{(\lambda t)^0}{0!}} \\ &= \frac{e^{-\lambda t}}{1 - e^{-\lambda t}} \sum_{n=1}^\infty \frac{((1 - F(z))\lambda t)^n}{n!} \\ &= \frac{e^{(1-F(z))\lambda t} - 1}{e^{\lambda t} - 1} \\ &= \begin{cases} 1, & \text{if } \ z \le 0 \\ \frac{e^{(1-z^\alpha)\lambda t} - 1}{e^{\lambda t} - 1}, & \text{if } \ 0 < z < 1 \\ 0, & \text{if } \ z \ge 1 \end{cases} \end{aligned}$$

# Problem 4

Suppose $\mathbf{N} = \{N(t) : t \ge 0\}$ is a Poisson process with rate $\lambda$
and $\{T_1, T_2, \ldots\}$ is the sequence of arrival time.
Let $f : \mathbb{R}_+ \to \mathbb{R}$, find the value of $\mathrm{E}[\sum_{i=1}^{N(t)} f(T_i)]$.
(15 marks)

- **Lemma:**
  **(到达时刻的联合条件分布, Introduction to Probability Models 定理 $5.2$)**
  假设 $\mathbf{N} = \{N(t) : t \ge 0\}$ 是参数为 $\lambda$ 的 Poisson 过程，记 $T_1, T_2, \ldots$ 为到达时刻.
  给定 $t > 0$，则有 $(T_1, T_2, \ldots, T_n|N(t) = n) \overset{d}{=} (U_{(1)}, U_{(2)}, \ldots, U_{(n)})$
  其中 $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ 为对应于 $U_1, U_2, \ldots, U_n \overset{iid}{\sim} \mathrm{Uniform}(0, t)$ 的次序统计量.
  **上面的结论通常可表述为:**
  在 $(0, t)$ 中已经发生了 $n$ 个事件的条件下，
  事件发生的时间 $T_1, T_2, \ldots, T_n$ (考虑为无次序的随机变量时) 是在 $(0, t)$ 上独立均匀地分布的.

**Solution:**

$$
\begin{aligned}
\mathrm{E}[\sum_{i=1}^{N(t)} f(T_i)] &= \mathrm{E}[\mathrm{E}[\sum_{i=1}^{N(t)} f(T_i)|\mathbf{N}]] \quad (\text{应用引理}) \\
&= \mathrm{E}[\mathrm{E}[\sum_{i=1}^{N(t)} f(U_{(i)})|\mathbf{N}]] \\
&= \mathrm{E}[\sum_{i=1}^{N(t)} \mathrm{E}[f(U_{(i)})|\mathbf{N}]] \\
&= \mathrm{E}[\sum_{i=1}^{N(t)} \mathrm{E}[f(U_{(i)})]] \\
&= \mathrm{E}[\sum_{i=1}^{N(t)} \mathrm{E}[f(U_i)]] \qquad (\text{去掉次序})
\end{aligned}
$$

其中 $U_{(1)},\ldots,U_{(N(t))}$ 为 $U_1,\ldots,U_{N(t)} \overset{iid}{\sim} \text{Uniform}(0,t)$ 对应的次序统计量.
记 $C = \mathrm{E}[f(U_i)] = \int_0^t f(u)\cdot\frac{1}{t}du = \frac{1}{t}\int_0^t f(u)du$
则有:

$$
\begin{aligned}
\mathrm{E}[\sum_{i=1}^{N(t)} f(T_i)] &= \mathrm{E}[\sum_{i=1}^{N(t)} \mathrm{E}[f(U_i)]] \\
&= \mathrm{E}[\sum_{i=1}^{N(t)} C] \\
&= C\mathrm{E}[N(t)] \\
&= C\cdot\lambda t \\
&= \frac{1}{t}\int_0^t f(u)du\cdot\lambda t \\
&= \lambda\int_0^t f(u)du
\end{aligned}
$$

# Problem 5

Cars pass a junction on the highway at a Poisson rate of one per minute.
Suppose that each car has $0.05$ probability of speeding.

**(a) Find the probability that more than one car are speeding through this junction in an hour.**
(5 marks)

- **Lemma: (Poisson 过程的拆分)**
  在之前的假设下，记 $N_i(t)\ (i=1,\ldots,k)$ 为到时刻 $t$ 为止类型 $i$ 事件发生的个数,
  则 $N_i(t)\ (i=1,\ldots,k)$ 是具有均值 $\mathrm{E}[N_i(t)] = \lambda\int_0^t p_i(t)ds$ 的**独立 Poisson 随机变量**,
  即 $N_i(t) \sim \text{Poisson}(\lambda\int_0^t p_i(t)ds)$ 且它们相互独立.
  这样我们就拆分得到了一系列非齐次 Poisson 过程 $\{N_i(t): t\geq 0\}$..
  - **推论:**
    特殊地，若 $p_1(\cdot),p_2(\cdot),\ldots,p_k(\cdot)$ 都是常数函数,
    则对于任意 $i=1,2,\ldots,k$ 我们有 $N_i(t) = \text{Poisson}(\lambda p_i)$ 且它们相互独立,
    即 $\{N_i(t): t\geq 0\}$ 是参数为 $\lambda p_i$ 的独立 Poisson 过程.

**Solution:**
记 $\begin{cases}\lambda = 1 \\ p = 0.05\end{cases}$
分别记加速和不加速的车辆为 ( I ) 型和 (II) 型,
记其计数过程为 $\mathbf{N}_1 = \{N_1(t): t\geq 0\}$ 和 $\mathbf{N}_2 = \{N_2(t): t\geq 0\}$.

根据引理可知,
$\mathbf{N}_1$ 和 $\mathbf{N}_2$ 分别是速率为 $\lambda p = 0.05$ 和 $\lambda(1-p) = 0.95$ 的 Poisson 过程，且相互独立.

记 $t = 60\ (\text{min})$
则 $N_1(t) \sim \text{Poisson}(\lambda pt) = \text{Poisson}(3)$
于是 $(0,t)$ 时间段内加速车辆数超过 $1$ 的概率为:

$$\begin{aligned}
\text{P}\{N_1(t) \geq 2\} &= 1 - \text{P}\{N_1(t) = 0\} - \text{P}\{N_1(t) = 1\} \\
&= 1 - e^{-3}\frac{3^0}{0!} - e^{-3}\frac{3^1}{1!} \\
&= 1 - 4e^{-3} \\
&\approx 0.801
\end{aligned}$$

---

**(b) Given that ten cars have been speeding through the junction in an hour,**
**what is the expected number of cars that had passed the junction?**
(5 marks)

**Solution:**
记 $t = 60\ (\text{min})$
我们知道 $\begin{cases} N(t) \sim \text{Poisson}(\lambda t) = \text{Poisson}(60) \\ N_1(t) \sim \text{Poisson}(\lambda pt) = \text{Poisson}(3) \\ N_2(t) \sim \text{Poisson}(\lambda(1-p)t) = \text{Poisson}(57) \end{cases}$
于是我们有:

$$\begin{aligned}
\text{E}[N(t)|N_1(t) = 10] &= \text{E}[N_1(t) + N_2(t)|N_1(t) = 10] \quad (\mathbf{N}_1 \perp \mathbf{N}_2) \\
&= 10 + \text{E}[N_2(t)] \\
&= 10 + 57 \\
&= 67
\end{aligned}$$

---

**(c) If 50 cars have passed the junction in an hour,**
**what is the probability that 5 of them were speeding?**
(5 marks)

**Solution:**
记 $t = 60\ (\text{min})$
我们首先计算给定 $N(t) = n$ 条件下 $N_1(t)$ 的分布:
对于任意 $k = 0, 1, \ldots, n$ 都有:

$$\begin{aligned}
\text{P}\{N_1(t) = k|N(t) = n\} &= \frac{\text{P}\{N_1(t) = k, N_2(t) = n-k\}}{\text{P}\{N(t) = n\}} \\
&= \frac{\text{P}\{N_1(t) = k\} \cdot \text{P}\{N_2(t) = n-k\}}{\text{P}\{N(t) = n\}} \\
&= \frac{e^{-\lambda pt}\frac{(\lambda pt)^k}{k!} \cdot e^{-\lambda(1-p)t}\frac{(\lambda(1-p)t)^{n-k}}{(n-k)!}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} \\
&= \text{P}\{B(n,p) = k\}
\end{aligned}$$

说明 $(N_1(t)|N(t) = n) \overset{d}{=} B(n,p)$
于是我们有:

$$\begin{aligned}\mathrm{P}\{N_1(t)=5|N(t)=50\} &= \mathrm{P}\{B(50,0.05)=5\}\\ &= \binom{50}{5}(0.05)^5(1-0.05)^{45}\\ &\approx 0.0658\end{aligned}$$

# Problem 6

Let $\{X(t):t\geq 0\}$ be a compound Poisson process:
$X(t)=\sum_{i=1}^{N(t)}Y_i$ with Poisson rate $\lambda=1$
and $Y_1,Y_2,...$ are $\text{i.i.d.}$ random variables distributed according to $\begin{cases}\mathrm{P}\{Y=1\}=\frac{1}{3}\\ \mathrm{P}\{Y=2\}=\frac{2}{3}\end{cases}$
Calculate $\mathrm{P}\{X(2)=3\}$.
(10 marks)

**Solution:**

$$\begin{aligned}\mathrm{P}\{X(2)=3\} &= \mathrm{P}\{\sum_{i=1}^{N(2)}Y_i=3\}\\ &= \sum_{n=0}^{\infty}\mathrm{P}\{\sum_{i=1}^{N(2)}Y_i=3|N(2)=n\}\mathrm{P}\{N(2)=n\} \quad (\{Y_i\}\perp N(t))\\ &= \sum_{n=0}^{\infty}\mathrm{P}\{\sum_{i=1}^{n}Y_i=3\}\mathrm{P}\{N(2)=n\}\end{aligned}$$

注意到当 $n=0,1$ 或 $n\geq 4$ 时都有 $\mathrm{P}\{\sum\limits_{i=1}^{n}Y_i=3\}=0$ 成立
(基于题干信息 $\begin{cases}\mathrm{P}\{Y=1\}=\frac{1}{3}\\ \mathrm{P}\{Y=2\}=\frac{2}{3}\end{cases}$)
因此我们有:

$$\begin{aligned}\mathrm{P}\{X(2)=3\} &= \sum_{n=0}^{\infty}\mathrm{P}\{\sum_{i=1}^{n}Y_i=3\}\mathrm{P}\{N(2)=n\}\\ &= 0+\mathrm{P}\{\sum_{i=1}^{2}Y_i=3\}\mathrm{P}\{N(2)=2\}+\mathrm{P}\{\sum_{i=1}^{3}Y_i=3\}\mathrm{P}\{N(2)=3\}+0\\ &= \binom{2}{1}\mathrm{P}\{Y_1=1\}\mathrm{P}\{Y_2=2\}\mathrm{P}\{\mathrm{Poisson}(2)=2\}+\binom{3}{3}(\mathrm{P}\{Y_1=1\})^3\mathrm{P}\{\mathrm{Poisson}(2)=3\}\\ &= 2\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot e^{-2}\frac{2^2}{2!}+1\cdot(\frac{1}{3})^3\cdot e^{-2}\frac{2^3}{3!}\\ &= \frac{8}{9}e^{-2}+\frac{4}{81}e^{-2}\\ &= \frac{76}{81}e^{-1}\\ &\approx 0.127\end{aligned}$$

# Problem 7

A pedestrian wants to cross a road at a spot where cars go by
in accordance with a Poisson process with rate $\lambda$.
He/She will begin to cross the first time he/she sees that
there will be no any cars passing for the next $c$ time units.
(Assume that he/she is able to forsee that.)

Let $N$ denote the number of cars that pass before he finally crosses the road.
Let $R$ denote the time at which he finally starts to cross the road.
Let $\{X_i\}$ denote the inter-arrival times of cars (i.e. waiting times of Poisson Process)

**(a) Determine the distribution of $N$ and calculate $\mathrm{E}[N]$.**
(5 marks)

- **Lemma 1:**
  计算指数随机变量 $X \sim \exp(\lambda)$ 的 $\mathrm{CDF}$:

$$\begin{aligned} F(x) &= \int_0^x \lambda e^{-\lambda t} dt \\ &= \int_0^{\lambda x} e^{-u} du \\ &= -e^{-u}|_0^{\lambda x} \\ &= -e^{-\lambda x} + e^0 \\ &= 1 - e^{-\lambda x} \end{aligned}$$

- **Lemma 2:**
  计算几何随机变量 $X \sim \mathrm{Geo}(p)$ 的均值:

$$\begin{aligned} \mathrm{E}(X) &= \sum_{i=1}^{\infty} i \cdot \mathrm{pmf}(i) \\ &= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1} p \\ &= p \sum_{i=1}^{\infty} i \cdot q^{i-1} \qquad (q := 1-p) \\ &= p \sum_{i=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}q}(q^i) \\ &= p \frac{\mathrm{d}}{\mathrm{d}q}\left(\sum_{i=1}^{\infty} q^i\right) \\ &= p \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{q}{1-q}\right) \\ &= \frac{p}{(1-q)^2} \\ &= \frac{1}{p} \end{aligned}$$

**Solution:**
$N$ 相当于首次等待时间超过 $c$ 的车辆出现之前，已经经过的车辆数.
我们知道 $\{X_i\}$ 作为 Poisson 过程的等待时间，有 $\{T_i\} \overset{iid}{\sim} \exp(\lambda)$ 成立.
事件 $\{N = n\}$ 发生等价于事件 $\{T_1 \leq c, \ldots, T_n \leq c, T_{n+1} > c\}$ 发生.
因此对于任意 $n = 0, 1, \ldots$ 我们有:

$$\begin{aligned} \mathrm{P}\{N = n\} &= \mathrm{P}\{T_1 \leq c, \ldots, T_n \leq c, T_{n+1} > c\} \\ &= \mathrm{P}\{T_1 \leq c\} \cdots \mathrm{P}\{T_n \leq c\} \mathrm{P}\{T_{n+1} > c\} \\ &= (1 - e^{-\lambda c})^n \cdot e^{-\lambda c} \end{aligned}$$

说明 $N+1$ 服从成功概率为 $e^{-\lambda c}$ 的几何分布 $\mathrm{Geo}(e^{-\lambda c})$.
根据 Lemma 2 可知 $\mathrm{E}[N+1] = \frac{1}{e^{-\lambda c}} = e^{\lambda c}$,
故 $\mathrm{E}[N] = e^{\lambda c} - 1$.

**(b) Compute $\mathrm{E}[X_i|X_i < c]$.**

(10 marks)

[probability - Crossing the road - Mathematics Stack Exchange]

**Solution:**

首先计算条件概率:

对于任意 $0 < x < c$ 都有:

$$\begin{aligned} \mathrm{P}\{X_i = x|X_i < c\} &= \frac{\mathrm{P}\{X_i = x,\ X_i < c\}}{\mathrm{P}\{X_i < c\}} \\ &= \frac{\mathrm{P}\{X = x\}}{\mathrm{P}\{X_i < c\}} \\ &= \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}} \end{aligned}$$

现在计算条件期望:

$$\begin{aligned} \mathrm{E}[X_i|X_i < c] &= \int_0^c x \cdot \mathrm{P}\{X_i = x|X_i < c\} \mathrm{d}x \\ &= \int_0^c x \cdot \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}} \mathrm{d}x \\ &= \frac{1}{\lambda(1 - e^{-\lambda c})} \int_0^{\lambda c} u e^{-u} du \\ &= \frac{1}{\lambda(1 - e^{-\lambda c})} (-1 - u) e^{-u}|_0^{\lambda c} \\ &= \frac{1}{\lambda(1 - e^{-\lambda c})} [1 - (\lambda c + 1) e^{-\lambda c}] \\ &= \frac{e^{\lambda c} - (\lambda c + 1)}{\lambda(e^{\lambda c} - 1)} \end{aligned}$$

---

**(c) Express $R$ in terms of $N$ and $\{X_i\}$ and determine $\mathrm{E}[R]$**

(10 marks)

- **Hint:** use the results in (b)

**Solution:**

我们有 $R = \sum_{i=1}^{N} X_i$ 成立.

$$\begin{aligned} \mathrm{E}[R] &= \mathrm{E}[\sum_{i=1}^{N} X_i | X_1 < c, \ldots, X_N < c, X_{N+1} \geq c] \\ &= \mathrm{E}[\mathrm{E}[\sum_{i=1}^{N} X_i | N, X_1 < c, \ldots, X_N < c, X_{N+1} \geq c]] \\ &= \mathrm{E}[\sum_{i=1}^{N} \mathrm{E}[X_i|X_i < c]] \qquad (\{X_i\} \text{ 相互独立}) \\ &= \mathrm{E}[\sum_{i=1}^{N} \frac{e^{\lambda c} - (\lambda c + 1)}{\lambda(e^{\lambda c} - 1)}] \quad (\text{代入(b)结论}\mathrm{E}[X_i|X_i < c] = \frac{e^{\lambda c} - (\lambda c + 1)}{\lambda(e^{\lambda c} - 1)}) \\ &= \mathrm{E}[N] \cdot \frac{e^{\lambda c} - (\lambda c + 1)}{\lambda(e^{\lambda c} - 1)} \quad (\text{代入(a)结论}\mathrm{E}[N] = e^{\lambda c} - 1) \\ &= (e^{\lambda c} - 1) \cdot \frac{e^{\lambda c} - (\lambda c + 1)}{\lambda(e^{\lambda c} - 1)} \\ &= \frac{e^{\lambda c} - 1}{\lambda} - c \end{aligned}$$

**(d) Determine** $\mathrm{E}[X_{N+1}]$**.**
(5 marks)

**Solution:**
首先计算条件概率:
对于任意 $x \geq c$ 都有:

$$\begin{aligned}
\mathrm{P}\{X_{N+1} = x | X_{N+1} > c\} &= \frac{\mathrm{P}\{X_{N+1} = x, X_{N+1} > c\}}{\mathrm{P}\{X_{N+1} > c\}} \\
&= \frac{\mathrm{P}\{X_{N+1} = x\}}{\mathrm{P}\{X_{N+1} > c\}} \\
&= \frac{\lambda e^{-\lambda x}}{e^{-\lambda c}}
\end{aligned}$$

现在计算条件期望:

$$\begin{aligned}
\mathrm{E}[X_{N+1}] &= \mathrm{E}[X_{N+1} | X_{N+1} > c] \\
&= \int_c^\infty x \frac{\lambda e^{-\lambda x}}{e^{-\lambda c}} \mathrm{d}x \\
&= \frac{1}{\lambda e^{-\lambda c}} \int_{\lambda c}^\infty u e^{-u} du \\
&= \frac{1}{\lambda e^{-\lambda c}} [-(1+u)e^{-u}]|_{\lambda c}^\infty \\
&= \frac{1}{\lambda e^{-\lambda c}} [0 + (1+\lambda c)e^{-\lambda c}] \\
&= \frac{1+\lambda c}{\lambda}
\end{aligned}$$

**The End**