数值算法 Homework 10

Due: Nov. 26, 2024
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Problem 1

(Reference 1) & (Reference 2)
Let A be a Hermitian matrix.
Describe how to transform A to a real tridiagonal matrix by unitary similarity,
before applying the implicit QR algorithm.

Proof:
我们首先使用 Householder 变换将 A 约化为一个 Hermite 三对角阵.
要设计算法将 Hermite 三对角阵约化为对称三对角阵，
首先考虑如何将形如 $\begin{bmatrix} a_1 & b \\ \bar{b} & a_2 \end{bmatrix}$ $(\text{where } a_1, a_2 \in \mathbb{R},\ b \in \mathbb{C})$ 的 2 阶 Hermite 阵化为实对称阵.

设 $b = re^{i\theta}$ (对应地 $b = re^{-i\theta}$)，则我们可以构造如下的酉相似变换:

$$\begin{aligned}
\begin{bmatrix} e^{-i\theta} & \\ & 1 \end{bmatrix} \begin{bmatrix} a_1 & b \\ \bar{b} & a_2 \end{bmatrix} \begin{bmatrix} e^{i\theta} & \\ & 1 \end{bmatrix} &= \begin{bmatrix} e^{-i\theta} \cdot a_1 \cdot e^{i\theta} & e^{-i\theta} \cdot b \cdot 1 \\ 1 \cdot \bar{b} \cdot e^{i\theta} & 1 \cdot a_2 \cdot 1 \end{bmatrix} \quad \left(\text{note that } \begin{cases} b = re^{i\theta} \\ \bar{b} = re^{-i\theta} \end{cases}\right) \\
&= \begin{bmatrix} a_1 & r \\ r & a_2 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & |b| \\ |b| & a_2 \end{bmatrix}
\end{aligned}$$

注意到 $e^{i\theta} = \frac{b}{r} = \frac{b}{|b|}$
因此我们只需取 $D = \text{diag}\{\frac{b}{|b|}, 1\}$ 即有 $D^{-1} \begin{bmatrix} a_1 & b \\ \bar{b} & a_2 \end{bmatrix} D = \begin{bmatrix} a_1 & |b| \\ |b| & a_2 \end{bmatrix}$

Matlab 代码:

```
rng(51);
A = diag(rand(2, 1));
A(1,2) = rand(1, 1) + 1i * rand(1, 1);
A(2,1) = A(1,2)';

disp("A:");
disp(A);

D = diag([A(1,2) / abs(A(1,2)), 1]);

disp("D_inv * A * D:");
disp((D \ A) * D);
```

运行结果:

```
A:
   0.6757 + 0.0000i   0.3433 + 0.6440i
   0.3433 - 0.6440i   0.0447 + 0.0000i

D_inv * A * D:
   0.6757 - 0.0000i   0.7298 + 0.0000i
   0.7298 - 0.0000i   0.0447 + 0.0000i
```

推广到 n 维情况:
我们可以将 Hermite 三对角阵 A 的超对角线提取出来得到 $b \in \mathbb{R}^{n-1}$
逐元素除以模长: `d = b ./ abs(b)`
为一次性构建对角矩阵 D，我们需要不断将 d 中靠后的元素累乘到靠前的元素.
最终 $D = \mathrm{diag}([d^{\mathrm{T}}, 1]) \in \mathbb{C}^{n\times n}$，使得 $D^{-1}AD$ 为对称三对角阵.
Matlab 代码为:

```matlab
% Construct a Hermitian tridiagonal matrix `A`:
rng(51);
n = 5;
A = diag(rand(n, 1)) + diag( ...
    rand(n-1, 1) + 1i * rand(n-1, 1), 1) + diag( ...
    rand(n-1, 1) + 1i * rand(n-1, 1), -1);
A = 0.5 * (A + A');
disp("A:");
disp(A);

% Extract the superdiagonal elements of A
b = diag(A, 1);

% Compute "phase" information
d = b ./ abs(b);

% Loop over the vector `d` to accumulate the "phase" information
for i = n-2:-1:1
    d(i) = d(i) * d(i+1);
end

% Create a diagonal matrix `D` from the modified vector `d`
D = diag([d; 1]);

% Perform the unitary transformation `D_inv * A * D`
% changing the matrix `A` into a real and symmetric matrix `A_tilde`
A_tilde = (D \ A) * D;

% Display the transformed matrix `A_tilde`
disp("D_inv * A * D:");
disp(A_tilde);
```

运行结果:

```
A:
   0.6757 + 0.0000i   0.5996 + 0.0978i   0.0000 + 0.0000i   0.0000 + 0.0000i
0.0000 + 0.0000i
   0.5996 - 0.0978i   0.0447 + 0.0000i   0.2840 - 0.1355i   0.0000 + 0.0000i
0.0000 + 0.0000i
   0.0000 + 0.0000i   0.2840 + 0.1355i   0.3433 + 0.0000i   0.2178 + 0.0790i
0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.2178 - 0.0790i   0.6440 + 0.0000i
0.4568 - 0.0539i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.4568 + 0.0539i
0.2842 + 0.0000i

D_inv * A * D:
   0.6757 + 0.0000i   0.6075 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
0.0000 + 0.0000i
   0.6075 - 0.0000i   0.0447 + 0.0000i   0.3146 - 0.0000i   0.0000 + 0.0000i
0.0000 + 0.0000i
   0.0000 + 0.0000i   0.3146 - 0.0000i   0.3433 + 0.0000i   0.2317 - 0.0000i
0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.2317 + 0.0000i   0.6440 + 0.0000i
0.4599 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.4599 + 0.0000i
0.2842 + 0.0000i
```

Problem 2

Let A and B be $n \times n$ real symmetric matrices.
Suppose that B is positive definite.
Show that AB is diagonalizable, and design an algorithm to compute all eigenvalues and eigenvectors of AB.

Solution:
注意到 B 是对称正定阵，故存在 Cholesky 分解 $B = LL^{\mathrm{T}}$
其中 $L \in \mathbb{R}^{n\times n}$ 为具有正实数对角元的下三角阵 (因为是非奇异矩阵)
因此 $AB = ALL^{\mathrm{T}} = L^{-\mathrm{T}}(L^{\mathrm{T}}AL)L^{\mathrm{T}}$，相似于实对称阵 $L^{\mathrm{T}}AL$
而 $L^{\mathrm{T}}AL$ 作为实对称阵 (自然是正规矩阵) 可正交相似对角化，于是 AB 可相似对角化.

欲计算 AB 的特征值和特征向量，只需先对 B 做 Cholesky 分解得到 $B = LL^{\mathrm{T}}$
并对实对称阵 $L^{\mathrm{T}}AL$ 应用 Wilkinson 位移的对称隐式 QR 算法得到谱分解 $L^{\mathrm{T}}AL = Q\Lambda Q^{\mathrm{T}}$
最后使用回代法求解三角方程组 $L^{\mathrm{T}}X = Q$ 得到 $X = L^{-\mathrm{T}}Q$
根据 $AB = L^{-\mathrm{T}}(L^{\mathrm{T}}AL)L^{\mathrm{T}} = L^{-\mathrm{T}}Q\Lambda Q^{\mathrm{T}}L^{\mathrm{T}} = (L^{-\mathrm{T}}Q)\Lambda(L^{-\mathrm{T}}Q)^{-1}$
可知 Λ 的对角元即为 AB 的特征值，$X = L^{-T}Q$ 的列向量即为对应的特征向量.

Problem 3

Implement the scaling-and-squaring algorithm (with truncated Taylor series) for computing the matrix exponential.
Test the accuracy of your algorithm by a few diagonalizable matrices with known spectral decomposition.

- **(optional)** Implement the Schur–Parlett algorithm and compare the accuracy.

- **(H, optional)** Replace truncated Taylor series by Pade approximants and compare the accuracy.

(1) 秦九韶算法

在超越矩阵函数的逼近中经常涉及计算多项式.
考虑以下矩阵多项式:

$$p(A) = c_d A^d + c_{d-1} A^{d-1} + \cdots + c_1 A + c_0 I$$

其中 $c_0, c_1, \ldots, c_{d-1}, c_d$ $(c_d \neq 0)$ 是给定的实数.

(秦九韶算法/Horner 技巧, Matrix Computation 算法 $11.2.1$)
给定 $A \in \mathbb{R}^{n\times n}$ 和 $c_0, c_1, \ldots, c_{d-1}, c_d$ (用向量 $c \in \mathbb{R}^{d+1}$ 传入)
此算法计算多项式 $F = p(A) = c_d A^d + c_{d-1} A^{d-1} + \cdots + c_1 A + c_0 I$:

$$\begin{aligned}
&\text{function: } F = \text{Qin_Jiushao}(A, c)\\
&\qquad F = c_d A + c_{d-1} I\\
&\qquad \text{for } k = d-2 : -1 : 0\\
&\qquad\qquad F = AF + c_k I\\
&\qquad \text{end}\\
&\text{end}
\end{aligned}$$

上述算法需要 $d-1$ 次矩阵乘法.
但和标量的情形不同的是，上述求和过程并不是最优的，我们可以适当增加 "步长".
以 $d = 4$ 为例 (只需要 2 次矩阵乘法):

$$\begin{aligned}
p(A) &= c_4 A^4 + c_3 A^3 + c_2 A^2 + c_1 A + c_0 I_n\\
&= A^2(c_4 A^2 + c_3 A + c_2 I_n) + c_1 A + c_0 I_n
\end{aligned}$$

$$\begin{aligned}
A_1 &= A\\
A_2 &= A^2\\
F_1 &= c_4 A_2 + c_3 A_1 + c_2 I_n\\
F &= A_2 F_1 + c_1 A_1 + c_0 I_n
\end{aligned}$$

一般来说，考虑计算 d 次矩阵多项式:

$$p(A) = c_d A^d + c_{d-1} A^{d-1} + \cdots + c_1 A + c_0 I$$

可取最优步长 $t = \lfloor \sqrt{d} \rfloor$ 并计算:

$$\begin{aligned}
A_1 &= A\\
A_2 &= AA_1\\
A_3 &= AA_2\\
&\cdots\\
A_t &= AA_{t-1}
\end{aligned}$$

这需要 $t-1$ 次矩阵乘法.
记 $r = \lfloor \frac{d}{t} \rfloor$ 并执行以下迭代:

$$\begin{aligned}
&F = c_d A_{d-rt} + c_{d-1} A_{d-rt-1} + \cdots + c_{rt+1} A_1 + c_{rt} I\\
&\text{for } k = r-1 : -1 : 0\\
&\qquad F = A_t F + (c_{kt+t-1} A_{t-1} + \cdots + c_{kt+1} A_1 + c_{kt} I)\\
&\text{end}
\end{aligned}$$

这需要 r 次矩阵乘法.
因此总共需要 $r + t - 1$ 次矩阵乘法.

Matlab 代码为:

```
function F = Qin_Jiushao(A, c)
    % A: Input matrix
    % c: A vector of coefficients for the polynomial, where c(i) corresponds to
the coefficient for A^(d-i)

    d = length(c) - 1;  % Degree of the polynomial
    t = floor(sqrt(d)); % Optimal step length based on the square root of the
degree
    r = floor(d / t);   % Number of iterations required

    % Step 1: Precompute the powers of A, namely A1, A2, ..., At
    A_powers = cell(t, 1);  % Cell array to store powers of A
    A_powers{1} = A;  % A1 = A
    for i = 2:t
        A_powers{i} = A_powers{i-1} * A;  % Ai = A^i, iteratively calculate
powers of A
    end

    % Step 2: Compute the polynomial value using the precomputed powers of A
    % Start by initializing F with the highest degree term
    % F=c_d A_{d-rt} + c_{d-1} A_{d-rt-1} + \dotsm + c_{rt+1} A_1 + c_{rt} I
    n = size(A, 1);
    F = zeros(n, n);
    for j = d-r*t:-1:1
        F = F + c(r*t+j+1) * A_powers{j};
    end
    F = F + c(r*t+1) * eye(n);

    % Iterate backwards to calculate the polynomial
    for k = r-1:-1:0
        % F = A_t F + (c_{kt + t-1} A_{t-1} + \dotsm + c_{kt+1}A_1 + c_{kt}I)
        % Compute the sum term for the current iteration
        sum_term = zeros(n, n);
        for j = t-1:-1:1
            sum_term = sum_term + c(k*t+j+1) * A_powers{j};  % Add each term to
the sum
        end
        sum_term = sum_term + c(k*t+1) * eye(n);
        % Update F using the sum of the current terms and the previous result
        F = A_powers{t} * F + sum_term;
    end
end
```

函数调用:

```
rng(51);
n = 100;
A = randn(n);
A = A / norm(A, 'inf');

% Define random polynomial coefficients
d = 50;
c = rand(1, d+1);
```

```
% Direct computation of the polynomial p(A) = c_d A^d + ... + c_0 I
direct_result = zeros(n, n);
A_powers = cell(d, 1);  % Cell array to store powers of A
A_powers{1} = A;  % A1 = A
for i = 2:d
    A_powers{i} = A_powers{i-1} * A;  % Ai = A^i, iteratively calculate powers
of A
end
for i = d:-1:1
    direct_result = direct_result + c(i+1) * A_powers{i};
end
direct_result = direct_result + c(1) * eye(n);

% Call the Qin_Jiushao function
qin_jiushao_result = Qin_Jiushao(A, c);

% Compare the results
disp('Difference (in Frobenius norm):');
disp(norm(direct_result - qin_jiushao_result, 'fro'));
```

运行结果:

```
Difference (in Frobenius norm):
   5.3512e-16
```

(2) Taylor 逼近

计算 $\exp(A)$ 的**缩放平方算法** (Scaling and Squaring)

$$
\begin{aligned}
&m = \max(0, 1 + \lfloor \log_2(\|A\|_\infty) \rfloor) \\
&E_0 = \text{the Taylor approximation of } \exp(A/2^m) \\
&\text{for } j = 1 : m \\
&\qquad E_j = E_{j-1}^2 \\
&\text{end}
\end{aligned}
$$

Matlab 代码为:

```
function E = scaling_and_squaring_Taylor(A, d)
    % Scaling and Squaring Algorithm to compute exp(A)
    % A: input matrix

    % Step 1: Compute m
    m = max(0, 1 + floor(log2(norm(A, 'inf'))));  % Compute m based on the
infinity norm of A

    % Step 2: Compute the Taylor approximation of exp(A / 2^m)
    % The coefficients for the Taylor expansion of exp(z) are 1, 1, 1/2!, 1/3!,
...
    c = zeros(d+1, 1);  % Coefficients for Taylor expansion
    c(1) = 1;
    for k = 2:d+1
        c(k) = c(k-1) / (k-1);  % Compute the Taylor coefficients
    end
```

```
    % Use Qin_Jiushao to compute the Taylor approximation of exp(A / 2^m)
    E_0 = Qin_Jiushao(A / 2^m, c);  % Apply Qin_Jiushao to compute the
polynomial approximation

    % Step 3: Squaring E_0, m times
    E = E_0;
    for j = 1:m
        E = E * E;  % Square E_0 for m times to get exp(A)
    end
end
```

函数调用:

```
rng(51);
n = 5;

% Generate diagonalizable matrix with known spectral decomposition
[Q, ~] = qr(rand(n, n));
Lambda = randn(n, 1);
A = Q * diag(Lambda) * Q';
disp("Max eigenvalue absolute value:");
disp(max(abs(Lambda)));

% Exact solution
% exact_solution = expm(A);
exact_solution = Q * diag(exp(Lambda)) * Q';
disp("Exact solution:");
disp(exact_solution);

% Taylor approximation
d = 10;
Taylor_approximation = scaling_and_squaring_Taylor(A, d);
disp("Taylor approximation:");
disp(Taylor_approximation);

% Display Difference
disp('Difference (in Frobenius norm):');
disp(norm(exact_solution - Taylor_approximation, 'fro'));
```

运行结果:

```
Max eigenvalue absolute value:
    1.0947

Exact solution:
    2.0455    0.1010   -0.7774   -0.1252    0.4095
    0.1010    1.0922    0.3790   -0.5317    0.6041
   -0.7774    0.3790    2.1706    0.2647    0.6328
   -0.1252   -0.5317    0.2647    2.0280   -0.6589
    0.4095    0.6041    0.6328   -0.6589    1.3701

Taylor approximation:
    2.0455    0.1010   -0.7774   -0.1252    0.4095
    0.1010    1.0922    0.3790   -0.5317    0.6041
   -0.7774    0.3790    2.1706    0.2647    0.6328
   -0.1252   -0.5317    0.2647    2.0280   -0.6589
```

```
    0.4095    0.6041    0.6328   -0.6589    1.3701

Difference (in Frobenius norm):
   1.9342e-13
```

(3) Padé 逼近

考虑矩阵指数函数 e^A 的计算.
若 $g(z) \approx e^z$，则 $g(A) \approx e^A$
为达到这一目的，有一类非常有用的逼近函数——Padé 函数:

$$R_{p,q}(z) := (D_{p,q}(z))^{-1} N_{p,q}(z)$$

其中:

$$N_{p,q}(z) = \sum_{k=0}^{p} \frac{(p+q-k)!}{(p+q)!} \binom{p}{k} z^k = \sum_{k=0}^{p} \frac{(p+q-k)!p!}{(p+q)!k!(p-k)!} z^k$$

$$D_{p,q}(z) = \sum_{k=0}^{q} \frac{(p+q-k)!}{(p+q)!} \binom{q}{k} (-z)^k = \sum_{k=0}^{q} \frac{(p+q-k)!q!}{(p+q)!k!(q-k)!} (-z)^k$$

特殊地，当 $q = 0$ 时即为 p 阶 Taylor 多项式:

$$R_{p,0}(z) = \sum_{k=0}^{p} \frac{z^k}{k!}$$

不幸的是，Padé 逼近仅在原点附近才非常好，下面的等式说明了这一点:

$$e^A = R_{p,q}(A) + \frac{(-1)^q}{(p+q)!} A^{p+q+1} (D_{p,q}(A))^{-1} \int_0^1 u^p (1-u)^q e^{A(1-u)} du$$

不过我们可以利用缩放平方算法 $e^A = (e^{\frac{A}{2^m}})^{2^m}$ 来克服这一困难.
整个算法的好坏取决于以下逼近的精度:

$$F_{p,q} = \left(R_{p,q} \left(\frac{A}{2^m} \right) \right)^{2^m}$$

可以证明:
若 $\frac{\|A\|_\infty}{2^m} \leq \frac{1}{2}$，则存在矩阵 $\Delta A \in \mathbb{R}^{n \times n}$ 使得:

$$\begin{aligned} F_{p,q} &= e^{A+\Delta A} \\ A\Delta A &= \Delta A A \\ \|\Delta A\|_\infty &\leq \varepsilon(p,q) \|A\|_\infty \\ \varepsilon(p,q) &= 2^{3-(p+q)} \frac{p!q!}{(p+q)!(p+q+1)!} \end{aligned}$$

进而可以证明不等式:

$$\frac{\|e^A - F_{p,q}\|_\infty}{\|e^A\|_\infty} \leq \varepsilon(p,q) \|A\|_\infty \exp\{\varepsilon(p,q)\|A\|_\infty\}$$

参数 p, q 可根据所需的相对精度来确定.
鉴于计算 $F_{p,q}$ 大约需要 $m + \max(p,q)$ 次矩阵乘法，故我们最好令 $p = q$
上述结果构成了有效计算 e^A 并控制误差的方法的基础.

(Matrix Computation 算法 11.3.1)
给定矩阵 $A \in \mathbb{R}^{n\times n}$ 和 $\tau > 0$，此算法计算 $F = e^{A+\Delta A}$ (其中 $\|\Delta A\|_\infty \le \tau \|A\|_\infty$)

$$
\begin{aligned}
&m = \max(0, 1 + \lfloor \log_2(\|A\|_\infty) \rfloor) \\
&A = A/2^m \\
&\text{Let } q \text{ be the smallest nonnegative integer such that } \varepsilon(q,q) \le \tau \\
&N = I \\
&D = I \\
&X = I \\
&c = 1 \\
&\text{for } k = 1 : q \\
&\qquad c = c(q-k+1)/[(2q-k+1)k] \\
&\qquad X = AX \\
&\qquad N = N + cX \\
&\qquad D = D + (-1)^k cX \\
&\text{end} \\
&\text{solve } DF = N \text{ for } F = D^{-1}N \text{ using Gaussian elimination} \\
&\text{for } k = 1 : m \\
&\qquad F = F^2 \\
&\text{end}
\end{aligned}
$$

上述算法大约需要 $2(q + m + \frac{1}{3})n^3$ 的计算量.
此外我们还可利用**秦九韶算法**来加快矩阵多项式 $D = D_{q,q}(A)$ 和 $N = N_{q,q}(A)$ 的计算.

Matlab 代码:

```
function F = scaling_and_squaring_Pade(A, q)
    % Calculate the matrix exponential exp(A) using scaling and squaring method.
    % A: input matrix (n x n)
    % q: number of terms for the Pade series

    % Step 1: Compute m based on the infinity norm of A
    m = max(0, 1 + floor(log2(norm(A, 'inf'))));  % Compute m based on the
infinity norm of A
    A = A / 2^m;  % Scale A by 2^m

    % Step 2: Initialize N, D, X, and c
    n = size(A, 1);  % Size of matrix A
    N = eye(n);      % Identity matrix for N
    D = eye(n);      % Identity matrix for D
    X = eye(n);      % Identity matrix for X
    c = 1;           % Initial coefficient for Pade expansion

    % Step 3: Iterate q times to compute N and D
    for k = 1:q
        % Update coefficient c
        c = c * (q - k + 1) / ((2 * q - k + 1) * k);

        % Update X = A * X
        X = A * X;

        % Update N = N + c * X
        N = N + c * X;
```

```
        % Update D = D + (-1)^k * c * X
        D = D + (-1)^k * c * X;
    end

    % Step 4: Solve D * F = N for F, using Gaussian elimination
    F = D \ N;  % Use MATLAB's backslash operator for solving the system

    % Step 5: Square F m times to apply the scaling and squaring
    for k = 1:m
        F = F * F;  % Square F m times
    end
end
```

函数调用:

```
rng(51);
n = 5;

% Generate diagonalizable matrix with known spectral decomposition
[Q, ~] = qr(rand(n, n));
Lambda = randn(n, 1);
A = Q * diag(Lambda) * Q';
disp("Max eigenvalue absolute value:");
disp(max(abs(Lambda)));

% Exact solution
% exact_solution = expm(A);
exact_solution = Q * diag(exp(Lambda)) * Q';
disp("Exact solution:");
disp(exact_solution);

% Taylor approximation
q = 13;
Pade_approximation = scaling_and_squaring_Pade(A, q);
disp("Pade approximation:");
disp(Pade_approximation);

% Display Difference
disp('Difference (in Frobenius norm):');
disp(norm(exact_solution - Pade_approximation, 'fro'));
```

运行结果:

```
Max eigenvalue absolute value:
    1.0947

Exact solution:
    2.0455    0.1010   -0.7774   -0.1252    0.4095
    0.1010    1.0922    0.3790   -0.5317    0.6041
   -0.7774    0.3790    2.1706    0.2647    0.6328
   -0.1252   -0.5317    0.2647    2.0280   -0.6589
    0.4095    0.6041    0.6328   -0.6589    1.3701

Pade approximation:
    2.0455    0.1010   -0.7774   -0.1252    0.4095
```

```
    0.1010    1.0922    0.3790   -0.5317    0.6041
   -0.7774    0.3790    2.1706    0.2647    0.6328
   -0.1252   -0.5317    0.2647    2.0280   -0.6589
    0.4095    0.6041    0.6328   -0.6589    1.3701

Difference (in Frobenius norm):
   2.8114e-15
```

(4) Schur-Parlett 递推

(Parlett 递推, Matrix Computation 算法 11.1.1)
给定对角元互不相同的上三角阵 $T \in \mathbb{C}^{n\times n}$，计算 $F = f(T)$:

$$
\begin{aligned}
&\text{function: } F = \text{Parlett\_Recursion}(T)\\
&\quad \text{for } i = 1:n \quad \text{(diagonal)}\\
&\quad\quad f_{ii} = f(t_{ii})\\
&\quad \text{end}\\
&\quad \text{for } p = 1:n-1 \quad (p\text{-th superdiagonal})\\
&\quad\quad \text{for } i: n-p \quad \text{(from bottom to top)}\\
&\quad\quad\quad j = i + p\\
&\quad\quad\quad \text{sum} = (f_{ii} - f_{ij})t_{ij}\\
&\quad\quad\quad \text{for } k = i+1 : j-1 \quad \text{(from left to right)}\\
&\quad\quad\quad\quad \text{sum} = \text{sum} + f_{ik}t_{kj} - t_{ik}f_{kj}\\
&\quad\quad\quad \text{end}\\
&\quad\quad\quad f_{ij} = \frac{1}{t_{ii} - t_{jj}}\text{sum}\\
&\quad\quad \text{end}\\
&\quad \text{end}\\
&\text{end}
\end{aligned}
$$

上述算法所需的计算量为 $\frac{2}{3}n^3$

Matlab 代码:

```matlab
function F = Parlett_Recursion(T)
    % Given upper triangular matrix T, compute F = exp(T) using Parlett
Recursion

    n = size(T, 1); % Size of the matrix T
    F = zeros(n, n); % Initialize the result matrix F

    % Step 1: Compute the diagonal elements of F
    for i = 1:n
        F(i, i) = exp(T(i, i));  % Apply f to each diagonal element
    end

    % Step 2: Compute the off-diagonal elements of F using the recursion
    for p = 1:n-1 % Iterate over superdiagonals
        for i = 1:n-p % Iterate over rows for the p-th superdiagonal
            j = i + p;  % j is the column index in the p-th superdiagonal

            % Start computing the sum for f_{ij}
```

```
            sum = (F(i, i) - F(i, j)) * T(i, j);

            % Sum over the elements in between (i+1 to j-1)
            for k = i+1:j-1
                sum = sum + F(i, k) * T(k, j) - T(i, k) * F(k, j);
            end

            % Compute the final off-diagonal element f_{ij}
            F(i, j) = sum / (T(i, i) - T(j, j));
        end
    end
end
```

函数调用:

```
rng(51);
n = 5;

% Generate diagonalizable matrix with known spectral decomposition
[Q, ~] = qr(rand(n, n));
Lambda = randn(n, 1);
A = Q * diag(Lambda) * Q';

% Minimum difference between eigenvalues
sorted_Lambda = sort(Lambda);
differences = diff(sorted_Lambda);
min_diff = min(differences);
disp("Minimum difference between eigenvalues:");
disp(min_diff);

% Exact solution
% exact_solution = expm(A);
exact_solution = Q * diag(exp(Lambda)) * Q';
disp("Exact solution:");
disp(exact_solution);

% Calculate Schur decomposition
[Q_schur, T] = schur(A);
exp_T = Parlett_Recursion(T);
parlett_solution = Q_schur * exp_T * Q_schur';
disp("Schur-Parlett solution:");
disp(parlett_solution);

% Display Difference
disp('Difference (in Frobenius norm):');
disp(norm(exact_solution - parlett_solution, 'fro'));
```

运行结果:

```
Minimum difference between eigenvalues:
    0.0187

Exact solution:
    2.0455    0.1010   -0.7774   -0.1252    0.4095
    0.1010    1.0922    0.3790   -0.5317    0.6041
   -0.7774    0.3790    2.1706    0.2647    0.6328
```

```
   -0.1252   -0.5317    0.2647    2.0280   -0.6589
    0.4095    0.6041    0.6328   -0.6589    1.3701

Schur-Parlett solution:
    2.0455    0.1010   -0.7774   -0.1252    0.4095
    0.1010    1.0922    0.3790   -0.5317    0.6041
   -0.7774    0.3790    2.1706    0.2647    0.6328
   -0.1252   -0.5317    0.2647    2.0280   -0.6589
    0.4095    0.6041    0.6328   -0.6589    1.3701

Difference (in Frobenius norm):
   1.0243e-14
```

Problem 4

Let A and E be Hermitian matrices with $AE = EA$.
Try to give an upper bound on $\|\exp(A+E) - \exp(A)\|_2$
Make sure your upper bound tends to zero when $\|E\|_2 \to 0$

Solution:
考虑初值问题:

$$\begin{aligned} \dot{X}(t) &= AX(t) \\ X(0) &= I \end{aligned}$$

其中 $A, X(t) \in \mathbb{R}^{n\times n}$
它具有唯一解 $X(t) = e^{At}$
于是我们有:

$$\begin{aligned} e^{(A+E)t} - e^{At} &= e^{At}(e^{Et} - 1) \\ &= e^{At}\int_0^t Ee^{Es}ds \\ &= \int_0^t e^{A(t-s)}Ee^{(A+E)s}ds \end{aligned}$$

根据上式可知:

$$\|e^{(A+E)t} - e^{At}\|_2 \le \|E\|_2 \int_0^t \|e^{A(t-s)}\|_2 \|e^{(A+E)s}\|_2 ds$$

注意到 $A \in \mathbb{C}^{n\times n}$ 为 Hermite 阵 (自然是正规矩阵)，因而存在谱分解 $A = U\Lambda U^{\mathrm{H}}$
其中 $U \in \mathbb{C}^{n\times n}$ 为酉矩阵，$\Lambda := \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ 为对角阵.
于是我们有:

$$\begin{aligned} e^{At} &= e^{U\Lambda U^{\mathrm{H}}t} \\ &= Ue^{\Lambda t}U^{\mathrm{H}} \\ &= U\mathrm{diag}\{e^{\lambda_1 t}, \dots, e^{\lambda_n t}\}U^{\mathrm{H}} \end{aligned}$$

进而有:

$$
\begin{aligned}
\|e^{At}\|_2 &= \|U\mathrm{diag}\{e^{\lambda_1 t},\dots,e^{\lambda_n t}\}U^{\mathrm{H}}\|_2 \\
&= \|\mathrm{diag}\{e^{\lambda_1 t},\dots,e^{\lambda_n t}\}\|_2 \\
&\le \exp\{\max\{\mathrm{Re}(\lambda_i):1\le i\le n\}\cdot t\} \\
&\le e^{\rho(A)t} \quad (\text{note that } A \text{ is Hermitian, so that } \|A\|_2=\sigma_{\max}(A)=\rho(A)) \\
&= e^{\|A\|_2 t}
\end{aligned}
$$

其中 $\rho(A)=\max_{1\le i\le n}|\lambda_i|$ 代表 A 的谱半径.
注意到 E 也为 Hermite 阵，且 $AE=EA$，因此它们可以同时酉对角化.
类似地，可推得 $\|e^{(A+E)t}\|_2 \le e^{\|A+E\|_2 t} \le e^{(\|A\|_2+\|E\|_2)t}$

因此我们有:

$$
\begin{aligned}
\|e^{(A+E)t}-e^{At}\|_2 &\le \|E\|_2\int_0^t \|e^{A(t-s)}\|_2\|e^{(A+E)s}\|_2 ds \\
&\le \|E\|_2\int_0^t e^{\|A\|_2(t-s)}e^{(\|A\|_2+\|E\|_2)s}ds \\
&= \|E\|_2 e^{\|A\|_2 t}\int_0^t e^{\|E\|_2 s}ds \\
&= \|E\|_2 e^{\|A\|_2 t}\frac{1}{\|E\|_2}(e^{\|E\|_2 t}-1) \\
&= e^{\|A\|_2 t}(e^{\|E\|_2 t}-1)
\end{aligned}
$$

取 $t=1$ 即有:

$$
\|e^{(A+E)}-e^{A}\|_2 \le e^{\|A\|_2}(e^{\|E\|_2}-1)
$$

当 $\|E\|_2\to 0$ 时，上界 $e^{\|A\|_2}(e^{\|E\|_2}-1)$ 趋于 0

Problem 5

Have a quick glance at the paper "From Random Polygon to Ellipse: An Eigenanalysis"
by A. N. Elmachtoub and C. F. Van Loan (available on eLearning).
Reproduce the experiments in this paper.

(0) Helper Functions

生成元素和分别为 0 的单位向量 $x,y\in\mathbb{R}^n$ 的函数:

```
function [x, y] = generate_unit_vectors(n)
    % Step 1: Generate two random vectors of length n
    x_raw = randn(n, 1); % Random vector for x
    y_raw = randn(n, 1); % Random vector for y

    % Step 2: Adjust to make the sum of components zero
    x_raw = x_raw - mean(x_raw); % Ensure sum(x) = 0
    y_raw = y_raw - mean(y_raw); % Ensure sum(y) = 0

    % Step 3: Normalize the vectors to have unit 2-norm
    x = x_raw / norm(x_raw); % Normalize x
    y = y_raw / norm(y_raw); % Normalize y
end
```

生成平均矩阵的函数:

```
% Function to create the Averaging Matrix M_n
function M = averaging_matrix(n)
    % Create an n x n matrix of zeros
    M = 0.5 * (diag(ones(n,1), 0) + diag(ones(n-1,1), 1));
    M(n, 1) = 0.5;
end
```

绘制多边形的函数:

```
% Function to display the polygon given two vectors
function display_polygon(x, y, k, ax)
    % Ensure the polygon is closed by connecting the first and last points
    x = [x; x(1)];
    y = [y; y(1)];

    % Plot the polygon in the specified subplot axis
    plot(ax, x, y, 'b-o', 'LineWidth', 2, 'MarkerFaceColor', 'b');
    title(ax, ['Iteration: ', num2str(k)]);
    axis(ax, 'equal');
    grid(ax, 'on');
end
```

(1) A First Try

Averaging matrix M_n is defined as:

$$M_n := \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n\times n}$$

Denote $\mathcal{P}(x, y)$ as the initial polygon, where $x, y \in \mathbb{R}^n$ are unit l_2-norm vectors.
After k averaging steps we have the polygon $\mathcal{P}(M_n^k x, M_n^k y)$
Algorithm 1:

Input: unit vectors $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ whose components sum to zero
Display $\mathcal{P}(x^{(0)}, y^{(0)})$
for $k = 1$: max_iter
    $x^{(k)} = M_n x^{(k-1)}$
    $y^{(k)} = M_n y^{(k-1)}$
    Display $\mathcal{P}(x^{(k)}, y^{(k)})$ for some k that we are interested in
end

Matlab 代码:

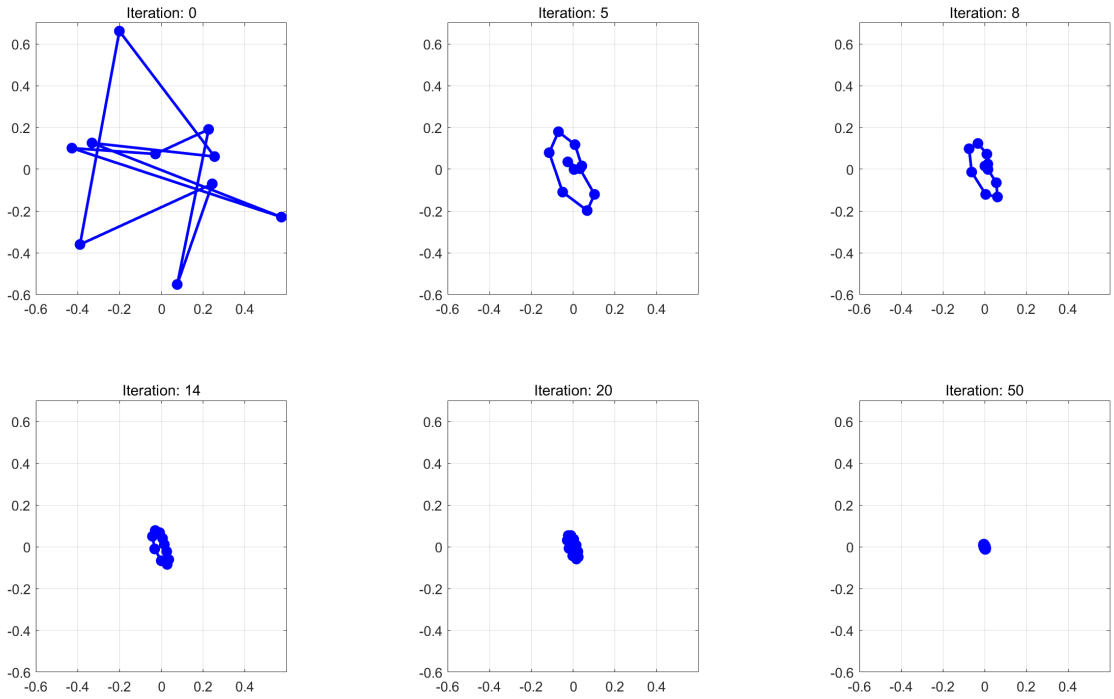
```
% Implementing Algorithm 1
function averaging_algorithm(x, y, max_iter)
    % Step 1: Create the averaging matrix M_n
```

```
    n = size(x, 1);
    M = averaging_matrix(n);

    % Create a figure for the subplots
    figure;
    subplot(2, 3, 1);
    display_polygon(x, y, 0, gca);

    % Define the set of iterations where we are interested in displaying the
polygon
    interested_set = [5, 8, 14, 20, 50];
    plot_index = 2;

    % Step 2: Iterate through the algorithm
    for k = 1:max_iter
        % Apply the averaging matrix to x and y
        x = M * x;
        y = M * y;

        % Check if the current iteration is in the interested set
        if ismember(k, interested_set)
            subplot(2, 3, plot_index); % Create a subplot (2x3 grid)
            display_polygon(x, y, k, gca); % Plot in the current axis
            plot_index = plot_index + 1;
        end
    end
end
```

函数调用:

```
rng(51);
n = 10; % Size of the averaging matrix
max_iter = 200; % Number of iterations

% Generate initial unit vectors
[x, y] = generate_unit_vectors(n);

% Run the averaging algorithm
averaging_algorithm(x, y, max_iter);
```

运行结果:
可以观察到多边形的各个顶点趋近于中心点 (即原点，因为 $x, y \in \mathbb{R}^n$ 做了中心化)



(2) A Second Try

We introduce a normalization step to keep the polygon $\mathcal{P}(x^{(k)}, y^{(k)})$ from collapsing down to a point,
Specifically, at each iteration k, we rescale the vectors $x^{(k)}$ and $y^{(k)}$ to ensure that they remain unit vectors.

Algorithm 2:

Input: unit vectors $x^{(0)}, y^{(0)} \in \mathbb{R}^n$ whose components sum to zero
Display $\mathcal{P}(x^{(0)}, y^{(0)})$
for $k = 1 : \text{max\_iter}$
$u = M_n x^{(k-1)}$
$x^{(k)} = u/\|u\|_2$
$v = M_n y^{(k-1)}$
$y^{(k)} = v/\|v\|_2$
Display $\mathcal{P}(x^{(k)}, y^{(k)})$ for some k that we are interested in
end

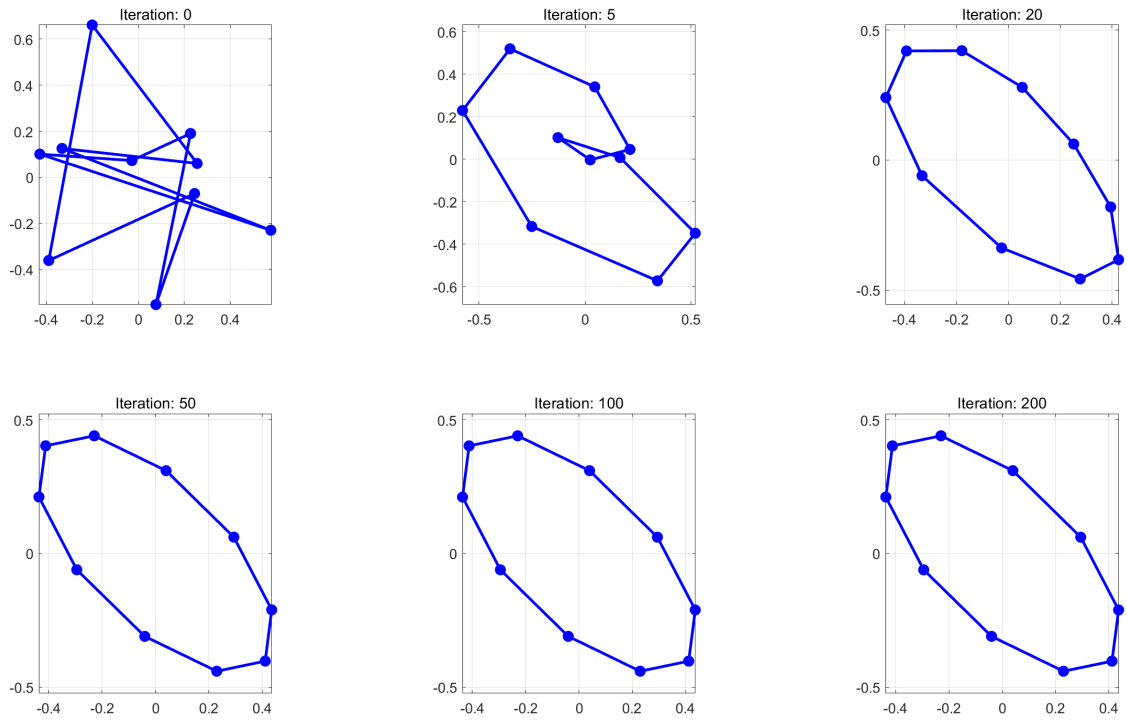
```
% Implementing Algorithm 2 (i.e. Algorithm 1 with normalization)
function averaging_algorithm(x, y, max_iter)
    % Step 1: Create the averaging matrix M_n
    n = size(x, 1);
    M = averaging_matrix(n);

    % Create a figure for the subplots
    figure;
    subplot(2, 3, 1);
    display_polygon(x, y, 0, gca);

    % Define the set of iterations where we are interested in displaying the
polygon
    interested_set = [5, 20, 50, 100, 200];
```

```
    plot_index = 2;

    % Step 2: Iterate through the algorithm
    for k = 1:max_iter
        % Apply the averaging matrix to x and y, and normalize them
        x = M * x;
        x = x / norm(x);
        y = M * y;
        y = y / norm(y);

        % Check if the current iteration is in the interested set
        if ismember(k, interested_set)
            subplot(2, 3, plot_index); % Create a subplot (2x3 grid)
            display_polygon(x, y, k, gca); % Plot in the current axis
            plot_index = plot_index + 1;
        end
    end
end
```

函数调用:

```
rng(51);
n = 10; % Size of the averaging matrix
max_iter = 200; % Number of iterations

% Generate initial unit vectors
[x, y] = generate_unit_vectors(n);

% Run the averaging algorithm
averaging_algorithm(x, y, max_iter);
```

运行结果:



猜想 (Conjecture):
No matter how random the initial $\mathcal{P}(x^{(0)}, y^{(0)})$, the edges of the $\mathcal{P}(x^{(k)}, y^{(k)})$ eventually "uncross",
and in the limit the vertices appear to arrange themselves around an ellipse
that is tilted 45 degrees from the coordinate axes.

(3) A Third Try

Consider the eigenvalues and eigenvectors of M_n:

$$M_n := \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n\times n}$$

(注意到 M_n 是一个 Frobenius 友阵加上单位阵最后乘以 $\frac{1}{2}$，而这个 Frobenius 友阵的特征值和特征向量我们是知道的)
记 $\omega = \exp\left(\frac{2\pi}{n}\right)$，则 M_n 的特征值和特征向量为:

$$\lambda_k = \frac{1}{2}\left(1+\omega^{k-1}\right)$$

$$x_k = \begin{bmatrix} 1 \\ \omega^{k-1} \\ \omega^{2(k-1)} \\ \vdots \\ \omega^{(n-1)(k-1)} \end{bmatrix} \quad (k = 1, \ldots, n)$$

Algorithm 3:

Input: $\theta_x, \theta_y \in \mathbb{R}$

$$\tau = \begin{bmatrix} 0 \\ 2\pi/n \\ 4\pi/n \\ \vdots \\ 2(n-1)\pi/n \end{bmatrix}, \quad c = \sqrt{2/n}\cos(\tau), \quad s = \sqrt{2/n}\sin(\tau)$$

$x^{(0)} = \cos(\theta_x)c + \sin(\theta_x)s$
$y^{(0)} = \cos(\theta_y)c + \sin(\theta_y)s$
Display $\mathcal{P}(x^{(0)}, y^{(0)})$
for $k = 1 : \text{max\_iter}$
    $u = M_n x^{(k-1)}$
    $x^{(k)} = u/\|u\|_2$
    $v = M_n y^{(k-1)}$
    $y^{(k)} = v/\|v\|_2$
    Display $\mathcal{P}(x^{(k)}, y^{(k)})$ for some k that we are interested in
end

```
% Implementing Algorithm 3
function averaging_algorithm(x, y, max_iter)
    % Step 1: Create the averaging matrix M_n
    n = size(x, 1);
    M = averaging_matrix(n);

    % Create a figure for the subplots
    figure;
    subplot(2, 3, 1);
    display_polygon(x, y, 0, gca);

    % Define the set of iterations where we are interested in displaying the
polygon
    interested_set = [2, 4, 5, 7, 9];
    plot_index = 2;

    % Step 2: Iterate through the algorithm
    for k = 1:max_iter
        % Apply the averaging matrix to x and y, and normalize them
        x = M * x;
        x = x / norm(x);
        y = M * y;
        y = y / norm(y);

        % Check if the current iteration is in the interested set
        if ismember(k, interested_set)
            subplot(2, 3, plot_index); % Create a subplot (2x3 grid)
            display_polygon(x, y, k, gca); % Plot in the current axis
            plot_index = plot_index + 1;
        end
    end
end
```

绘制多边形的函数修改为:

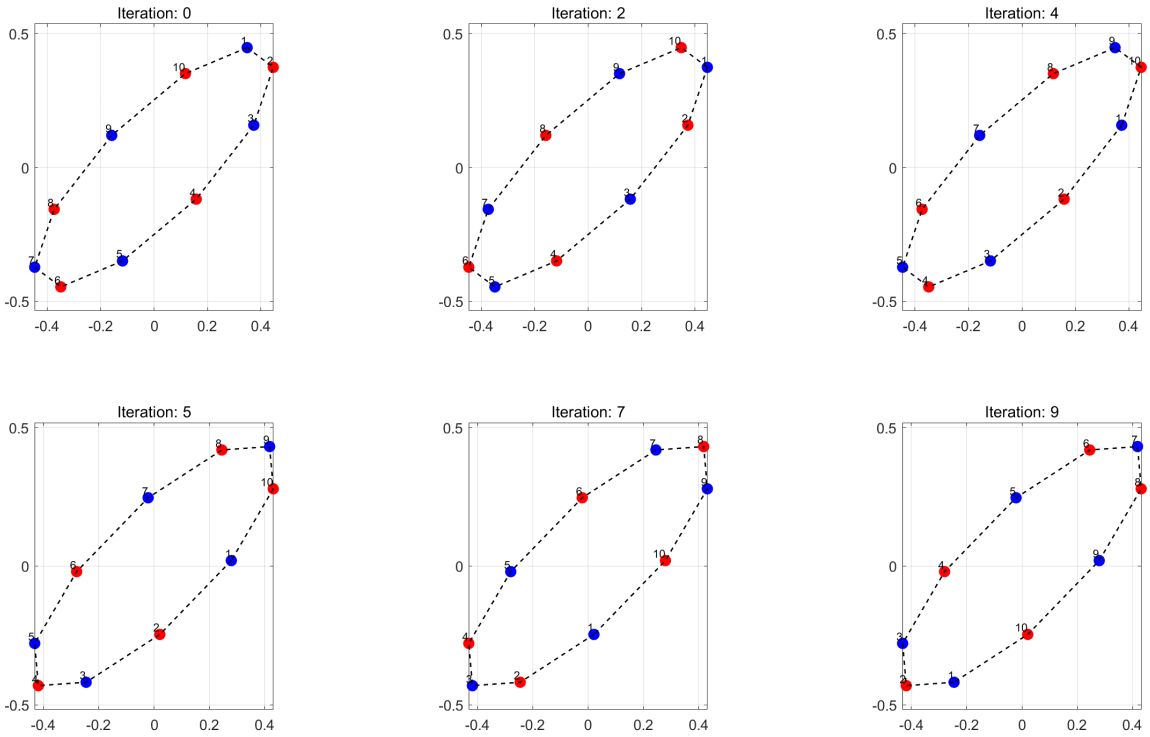
```
function display_polygon(x, y, k, ax)
    % Ensure the polygon is closed by connecting the first and last points
    x = [x; x(1)];
    y = [y; y(1)];

    % Identify even and odd indices
    even_indices = 2:2:length(x);  % Even indices (2, 4, 6, ...)
    odd_indices = 1:2:length(x);   % Odd indices (1, 3, 5, ...)

    % Plot even-indexed points in red
    plot(ax, x(even_indices), y(even_indices), 'ro', 'LineWidth', 2,
'MarkerFaceColor', 'r');
    hold(ax, 'on');  % Hold on to the current plot

    % Plot odd-indexed points in blue
    plot(ax, x(odd_indices), y(odd_indices), 'bo', 'LineWidth', 2,
'MarkerFaceColor', 'b');

    % Connect the points with a dashed line
    plot(ax, x, y, 'k--', 'LineWidth', 1);  % Dashed line (k-- for black dashed)

    % Add the title and adjust the grid and axis
```

```
    title(ax, ['Iteration: ', num2str(k)]);
    axis(ax, 'equal');
    grid(ax, 'on');

    % Add index labels next to each point
    for i = 1:length(x)-1  % Exclude the last repeated point for closing the
polygon
        text(ax, x(i), y(i), num2str(i), 'VerticalAlignment', 'bottom',
'HorizontalAlignment', 'right', 'FontSize', 8);
    end

    hold(ax, 'off');  % Release the hold
end
```

函数调用:

```
rng(51);
n = 10; % Size of the averaging matrix
max_iter = 200; % Number of iterations

% Generate initial unit vectors
tau = 0:n-1;
tau = (2 * pi / n) * tau';
c = sqrt(2/n) * cos(tau);
s = sqrt(2/n) * sin(tau);

theta_x = rand(1);
theta_y = rand(1);
x = cos(theta_x) * c + sin(theta_x) * s;
y = cos(theta_y) * c + sin(theta_y) * s;

% Run the averaging algorithm
averaging_algorithm(x, y, max_iter);
```

运行结果:



猜想 (Conjecture):
For any input values $\theta_x, \theta_y \in \mathbb{R}$ in Algorithm 3,
the even indexed polygons $\mathcal{P}(x^{(2k)}, y^{(2k)})$ are all the same
and the odd-indexed polygons $\mathcal{P}(x^{(2k+1)}, y^{(2k+1)})$ are all the same.

(4) A Fourth Try

It is natural to ask what happens to the normalized polygon averaging process,
if we use analternative to the l_2-norm for normalization.
A change in norm will only affect the size of the vertex vectors, not their direction.
Algorithm 4:

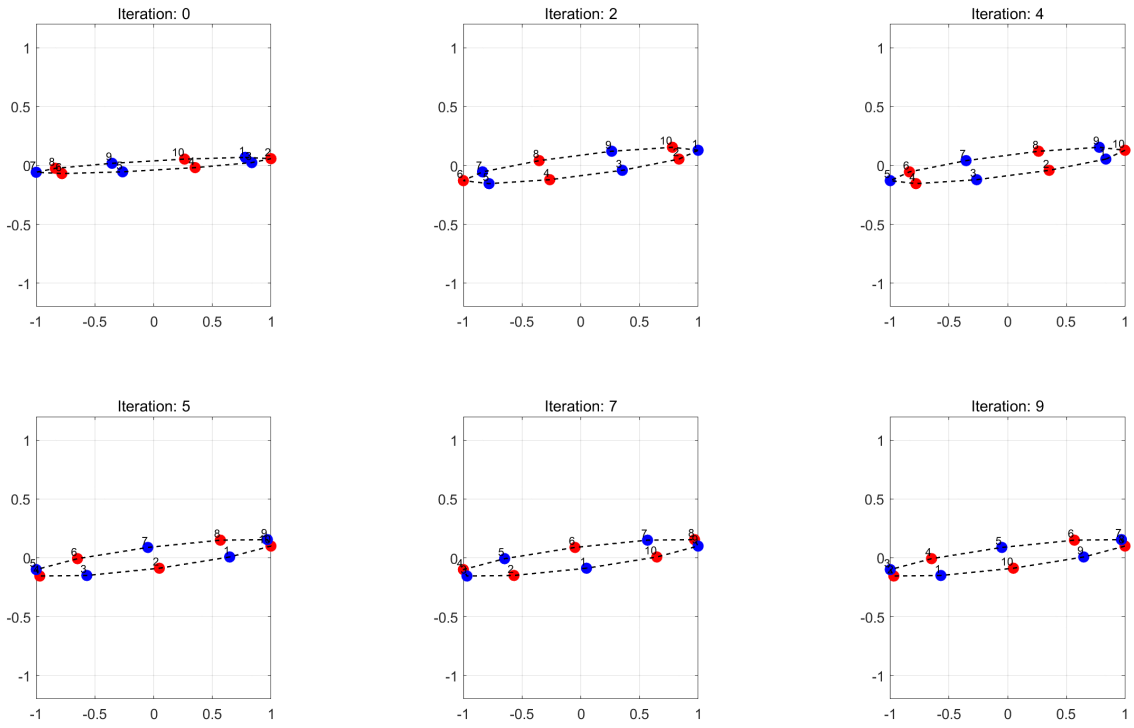
```
% Implementing Algorithm 4
function averaging_algorithm(x, y, max_iter, p, q)
    % Step 1: Create the averaging matrix M_n
    n = size(x, 1);
    M = averaging_matrix(n);

    % Create a figure for the subplots
    figure;
    subplot(2, 3, 1);
    display_polygon(x, y, 0, gca);

    % Define the set of iterations where we are interested in displaying the
polygon
    interested_set = [2, 4, 5, 7, 9];
    plot_index = 2;

    % Step 2: Iterate through the algorithm
    for k = 1:max_iter
        % Apply the averaging matrix to x and y, and normalize them
        x = M * x;
        x = x / norm(x, p);
        y = M * y;
        y = y / norm(y, q);

        % Check if the current iteration is in the interested set
        if ismember(k, interested_set)
            subplot(2, 3, plot_index); % Create a subplot (2x3 grid)
            display_polygon(x, y, k, gca); % Plot in the current axis
            plot_index = plot_index + 1;
        end
    end
end
```

函数调用:

```
rng(51);
n = 10; % Size of the averaging matrix
max_iter = 200; % Number of iterations
p = 'inf';
q = 1;

% Generate initial unit vectors
```

```
tau = 0:n-1;
tau = (2 * pi / n) * tau';
c = sqrt(2/n) * cos(tau);
s = sqrt(2/n) * sin(tau);

theta_x = rand(1);
theta_y = rand(1);
x = cos(theta_x) * c + sin(theta_x) * s;
x = x / norm(x, p);
y = cos(theta_y) * c + sin(theta_y) * s;
y = y / norm(x, q);

% Run the averaging algorithm
averaging_algorithm(x, y, max_iter, p, q);
```

运行结果: A more "rapid" look
(对 x 使用 $\|\cdot\|_\infty$ 范数标准化，对 y 使用 $\|\cdot\|_1$ 范数标准化)



Problem 6 (optional)

In the lecture we have discussed how to solve Sylvester matrix equation $AX - XB = C$ using the Bartels–Stewart algorithm (through Schur decompositions).
What happens the matrices are real, and if only real Schur decompositions are permitted?

Solution:
给定 $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{m\times n}$
Sylvester 方程组 $AX - XB = C$ 可以等价表示为 $\mathrm{vec}(X) = (I_n \otimes A - B^{\mathrm{T}} \otimes I_m)^{-1}\mathrm{vec}(C)$

$$AX - XB = C$$
$$\Leftrightarrow$$
$$(I_n \otimes A - B^{\mathrm{T}} \otimes I_m)\mathrm{vec}(X) = \mathrm{vec}(C)$$

(Sylvester 定理, Matrix Analysis 定理 $2.4.4.1$)
Sylvester 方程组有唯一解，当且仅当系数矩阵 $(I_n \otimes A - B^{\mathrm{T}} \otimes I_m)$ 非奇异，即当且仅当 A, B 没有公共特征值.

假设 $A \in \mathbb{C}^{m,m}$ 和 $B \in \mathbb{C}^{n \times n}$ 没有公共特征值.
直接求解 $(I_n \otimes A - B^{\mathrm{T}} \otimes I_m)\mathrm{vec}(X) = \mathrm{vec}(C)$ 的代价是 $O(m^3 n^3)$，难以接受.
我们可以对 A 做 Schur 分解 $A = U_1 T_1 U_1^{\mathrm{H}}$ (代价为 $O(m^3)$)，得到:

$$AX - XB = U_1 T_1 U_1^{\mathrm{H}} X - XB = C$$
$$\Leftrightarrow$$
$$T_1(U_1^{\mathrm{H}} X) - (U_1^{\mathrm{H}} X)B = U_1^{\mathrm{H}} C$$
$$\Leftrightarrow$$
$$T_1 Y - YB = \tilde{C} \text{ where } \begin{cases} Y = U_1^{\mathrm{H}} X \in \mathbb{C}^{m \times n} \\ \tilde{C} = U_1^{\mathrm{H}} C \in \mathbb{C}^{m \times n} \end{cases}$$

我们也可以对 B 做 Schur 分解 $B = U_2 T_2 U_2^{\mathrm{H}}$ (代价为 $O(n^3)$)，得到:

$$AX - XB = AX - XU_2 T_2 U_2^{\mathrm{H}} = C$$
$$\Leftrightarrow$$
$$A(XU_2) - (XU_2)T_2 = CU_2$$
$$\Leftrightarrow$$
$$AY - YT_2 = \tilde{C} \text{ where } \begin{cases} Y = XU_2 \in \mathbb{C}^{m \times n} \\ \tilde{C} = CU_2 \in \mathbb{C}^{m \times n} \end{cases}$$

因此 Schur 分解预处理代价是 $O(\min(m^3, n^3))$
我们可以比较 A, B 的阶数，哪个阶数小就对哪个做 Schur 分解.

不失一般性，假设 $m > n$，对 $B \in \mathbb{C}^{n \times n}$ 做 Schur 分解 $B = UTU^{\mathrm{H}}$:

$$AX - XB = AX - XUTU^{\mathrm{H}} = C$$
$$\Leftrightarrow$$
$$A(XU) - (XU)T = CU$$
$$\Leftrightarrow$$
$$AY - YT = \tilde{C} \text{ where } \begin{cases} Y = XU \in \mathbb{C}^{m \times n} \\ \tilde{C} = CU \in \mathbb{C}^{m \times n} \end{cases}$$

我们可以首先解出 Y 的第 1 列 Ye_1:
(记 $T = [t_{ij}]_{i,j=1}^{n}$，则 $Te_1 = t_{11}e_1$)

$$AYe_1 - YTe_1 = \tilde{C}e_1$$
$$\Leftrightarrow$$
$$A(Ye_1) - t_{11}Ye_1 = \tilde{C}e_1$$

这相当于求解一个线性方程组，得到 Y 的第 1 列 (Ye_1)
然后利用回代法的思想求解 Y 的剩余各列即可.
上述算法称为 **Bartels-Stewart 算法**

(Bartels-Stewart 算法, Matrix Computation 算法 $7.6.2$)
给定矩阵 $A \in \mathbb{C}^{m \times m}, C \in \mathbb{C}^{m \times n}$ 和上三角阵 $T \in \mathbb{C}^{n \times n}$ (其中 $m < n$，且 A, T 没有公共特征值)
本算法用方程 $AY - YT = C$ 的解覆盖 C:

$$
\begin{aligned}
&\text{function: } Y = \text{Bartels-Stewart}(T, B, C)\\
&\quad \text{for } k = 1 : n\\
&\quad\quad C(1:m,k) = C(1:m,k) + C(1:p,1:k-1)T(1:k-1,k)\\
&\quad\quad \text{solve } (A - T(k,k)I_m)y = C(1:m,k) \text{ to get } y\\
&\quad\quad C(1:m,k) = y\\
&\quad \text{end}\\
&\quad Y = C\\
&\text{end}
\end{aligned}
$$

对于实 Schur 型，要解决 2×2 对角块的问题只需使用 Gauss 消去法将两列放在一起解.
(或者使用复运算，它可以拆成实部和虚部的实运算)

Problem 7 (optional)

Use truncated SVD to compress some grayscale images.
If you only have colored images, you can convert them to grayscale using:

$$\text{gray} = \alpha \cdot \text{red} + \beta \cdot \text{green} + \gamma \cdot \text{blue}$$

Common choices of the constants are $(\alpha, \beta, \gamma) = (0.299, 0.587, 0.114)$ and $(\alpha, \beta, \gamma) = (0.2126, 0.7152, 0.0722)$.
You are also encouraged to think about how to compress colored images.

(1) Grayscale Image

SVD 压缩灰度图像的函数:

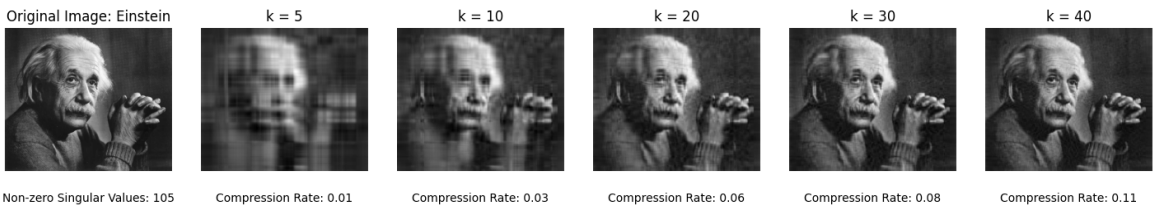
```
def svd_compress(image, k):
    """
    Compress an image using Singular Value Decomposition (SVD).
    Only the first `k` singular values are kept.

    Parameters:
    - image: 2D numpy array representing the grayscale image.
    - k: The number of singular values to retain for compression.

    Returns:
    - compressed_image: The compressed image as a 2D numpy array.
    - num_singular_values: The number of singular values used.
    - compression_rate: The compression rate for the compressed image.
    """
    # Perform SVD decomposition
    U, S, Vt = np.linalg.svd(image, full_matrices=False)

    # Keep only the first k singular values
    U_k = U[:, :k]
    S_k = np.diag(S[:k])  # Create a diagonal matrix with the first k singular
values
    Vt_k = Vt[:k, :]

    # Reconstruct the compressed image using the reduced SVD matrices
    compressed_image = np.dot(U_k, np.dot(S_k, Vt_k))

    # Calculate compression rate
    m, n = image.shape
```

```
    compression_rate = (k * (m + n + k)) / (m * n)

    # Return the compressed image, number of singular values used, and the
compression rate
    return compressed_image, k, compression_rate
```

函数调用:

```
if __name__ == "__main__":
    # Open the image, convert it to grayscale, and then convert it to a numpy
array
    image_path = 'DIP_Fig02.41(c)(einstein high contrast).tif'
    image_name = 'Einstein'
    image = Image.open(image_path).convert('L')  # Convert the image to
grayscale ('L' mode)
    image = np.array(image)  # Convert the grayscale image to a numpy array

    # Choose the number of singular values to use for compression
    k_values = [5, 10, 20, 30, 40]  # Example k-values to test
    compressed_images = []
    singular_values_list = []
    compression_rates_list = []

    # Generate compressed images for each k value
    for k in k_values:
        compressed_image, num_singular_values, compression_rate =
svd_compress(image, k)
        compressed_images.append(compressed_image)
        singular_values_list.append(num_singular_values)
        compression_rates_list.append(compression_rate)
        print(f"Number of singular values used for k={k}:
{num_singular_values}")
        print(f"Compression rate for k={k}: {compression_rate:.2f}")

    # Plot the original image and all compressed images
    plot_compressed_images(image, compressed_images, k_values, image_name,
sum(singular_values_list), compression_rates_list)
```

运行结果:
(图片来源: Digital Image Process (3rd Edition, R. Gonzalez) Figure 2.41)



(2) RGB Image

SVD 压缩彩色图像的函数:

```
def svd_compress_color(image, k):
    """
    Compress a color image using Singular Value Decomposition (SVD).
```

```
    Only the first `k` singular values are kept for each RGB channel.

    Parameters:
    - image: 3D numpy array representing the color image (height, width, 3).
    - k: The number of singular values to retain for compression.

    Returns:
    - compressed_image: The compressed color image as a 3D numpy array.
    - num_singular_values: The number of non-zero singular values used for each
channel.
    - compression_rate: The compression rate for each channel.
    """
    # Get image dimensions
    m, n, _ = image.shape

    # Initialize the compressed image
    compressed_image = np.zeros_like(image)

    # List to store compression rates for each channel
    compression_rates = []
    total_non_zero_singular_values = 0

    # Apply SVD compression on each RGB channel
    for i in range(3):  # Loop through each channel (0 = Red, 1 = Green, 2 =
Blue)
        channel = image[:, :, i]
        U, S, Vt = np.linalg.svd(channel, full_matrices=False)

        # Number of non-zero singular values
        num_singular_values = np.sum(S > 1e-10)  # Consider values larger than a
small threshold as non-zero

        # Keep only the first k singular values
        U_k = U[:, :k]
        S_k = np.diag(S[:k])  # Create a diagonal matrix with the first k
singular values
        Vt_k = Vt[:k, :]

        # Reconstruct the compressed channel using the reduced SVD matrices
        compressed_channel = np.dot(U_k, np.dot(S_k, Vt_k))

        # Store the compressed channel back into the image
        compressed_image[:, :, i] = compressed_channel

        # Update total number of non-zero singular values
        total_non_zero_singular_values += num_singular_values

        # Calculate the compression rate for this channel
        compression_rate = (k * (m + n + k)) / (m * n)
        compression_rates.append(compression_rate)

    # Return the compressed image, number of singular values, and compression
rates
    return compressed_image, total_non_zero_singular_values, compression_rates
```

函数调用:

```
if __name__ == "__main__":
    # Open the image, convert it to RGB, and then convert it to a numpy array
    image_path = 'DIP_Fig06.46(a)(lenna_original_RGB).tif'
    image_name = 'Lenna'
    image = Image.open(image_path).convert('RGB')  # Ensure the image is in RGB
mode
    image = np.array(image)  # Convert the image to a numpy array

    # Choose the number of singular values to use for compression
    k_values = [10, 100, 200, 300, 500]  # Example k-values to test
    compressed_images = []
    singular_values_list = []
    compression_rates_list = []

    # Generate compressed images for each k value
    for k in k_values:
        compressed_image, num_singular_values, compression_rates =
svd_compress_color(image, k)
        compressed_images.append(compressed_image)
        singular_values_list.append(num_singular_values)
        compression_rates_list.append(np.mean(compression_rates))  # Mean
compression rate for all channels
        print(f"Number of singular values used for k={k}:
{num_singular_values}")
        print(f"Average compression rate for k={k}:
{np.mean(compression_rates):.2f}")

    # Plot the original image and all compressed images
    plot_compressed_images_color(image, compressed_images, k_values, image_name,
sum(singular_values_list), compression_rates_list)
```

运行结果:
(图片来源: Digital Image Process (3rd Edition, R. Gonzalez) Figure 6.46)

Poor girl.