数值算法 Homework 11

Due: Dec. 3, 2024
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Problem 1

Find an example of linear system such that the Jacobi method converges while the Gauss–Seidel method diverges.
Justify your claim.

Solution:
考虑以下方阵:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$$

此时我们有:

$$D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad L = \begin{bmatrix} 0 & & \\ -1 & 0 & \\ -2 & -2 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -2 & 2 \\ & 0 & -1 \\ & & 0 \end{bmatrix}$$

Jacobi 迭代矩阵为:

$$B_{\text{Jacobi}} := D^{-1}(L+U) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$$

其特征多项式为 λ^3，特征值为 $0,0,0$，谱半径 $\rho(B_{\text{Jacobi}}) = 0 < 1$
因此 Jacobi 迭代法收敛.

Gauss-Seidel 迭代矩阵为:

$$B_{\text{G-S}} := (D-L)^{-1}U = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -2 & 2 \\ & 0 & -1 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 2 \\ & 0 & -1 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ & 2 & -3 \\ & & 2 \end{bmatrix}$$

其特征多项式为 $\lambda(\lambda-2)^2$，特征值为 $0,2,2$，谱半径 $\rho(B_{\text{G-S}}) = 2 \geq 1$
因此 Gauss-Seidel 迭代发散.

Problem 2

Find an example of positive definite linear system such that the Gauss–Seidel method converges while the Jacobi method diverges.
Justify your claim.

Solution:
考虑以下对称方阵:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

其特征多项式为 $(\lambda-1)^2(\lambda-4)$，特征值为 $1,1,4$，因此是对称正定阵.
此时我们有:

$$D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \quad L = \begin{bmatrix} 0 & & \\ -1 & 0 & \\ -1 & -1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -1 & -1 \\ & 0 & -1 \\ & & 0 \end{bmatrix}$$

Jacobi 迭代矩阵为:

$$B_{\text{Jacobi}} := D^{-1}(L+U) = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

其特征多项式为 $(\lambda-0.5)^2(\lambda+1)$，特征值为 $0.5, 0.5, -1$，谱半径 $\rho(B_{\text{Jacobi}}) = 1$
因此 Jacobi 迭代法发散.

Gauss-Seidel 迭代矩阵为:

$$B_{\text{G-S}} := (D-L)^{-1}U = \begin{bmatrix} 2 & & \\ 1 & 2 & \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 & -1 \\ & 0 & -1 \\ & & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & & \\ -\frac{1}{4} & \frac{1}{2} & \\ -\frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ & 0 & -1 \\ & & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{4} & -\frac{1}{4} \\ & \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

其特征多项式为 $\lambda(\lambda^2 - \frac{5}{8}\lambda + \frac{1}{8})$，特征值为 $0, \frac{5}{16} \pm \frac{\sqrt{7}}{16}i$，谱半径 $\rho(B_{\text{G-S}}) = \frac{\sqrt{2}}{4} < 1$
因此 Gauss-Seidel 迭代收敛.

事实上，对于对称正定线性系统，Gauss-Seidel 迭代法总是收敛的.
(数值线性代数, 定理 $4.2.7$)
若 A 对称正定，则 Gauss-Seidel 迭代法收敛.
或者更一般地，Hermite 正定矩阵 $A \in \mathbb{C}^{n\times n}$ 的 Gauss-Seidel 迭代法一定收敛.
(从这个意义来说，Gauss-Seidel 迭代法要优于 Jacobi 迭代法)

- **邵老师提供的证明:**
  考虑 Hermite 正定线性方程组 $Ax = b$
  回顾之前的记号:

$$A = D - L - U$$

$$D = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \quad L = \begin{bmatrix} 0 & & & \\ -a_{21} & 0 & & \\ \vdots & \ddots & \ddots & \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ & 0 & \ddots & \vdots \\ & & \ddots & -a_{n-1,n} \\ & & & 0 \end{bmatrix}$$

  A 的 Hermite 性保证了 $U = L^H$
  于是 $B = (D-L)^{-1}U = (D-L)^{-1}L^H$
  对于 B 的任意特征对 (λ, x)，我们都有:

$$\begin{gathered} Bx = (D-L)^{-1}L^H x = x\lambda \\ \Leftrightarrow \\ L^H x = (D-L)x\lambda \end{gathered}$$

  左乘 x^H 即得:

$$\begin{gathered} x^H L^H x = x^H D x \lambda - x^H L x \lambda \\ \Leftrightarrow \\ \bar{\beta} = \alpha\lambda - \beta\lambda \text{ where } \begin{cases} \alpha = x^H D x \\ \beta = x^H L x \end{cases} \end{gathered}$$

  因此我们有:

$$
\begin{aligned}
\lambda &= \frac{\bar{\beta}}{\alpha - \beta} \\
&= \frac{\mathrm{Re}(\beta) - i \cdot \mathrm{Im}(\beta)}{\alpha - \mathrm{Re}(\beta) - i \cdot \mathrm{Im}(\beta)}
\end{aligned}
$$

注意到 A 是 Hermite 正定阵，因此对角阵 D 的对角元均为正实数.
于是我们有 $\alpha = x^H D x > 0$
此外我们有:

$$
\begin{aligned}
0 &< x^H A x \\
&= x^H (D - L - L^H) x \\
&= x^H D x - x^H L x - x^H L^H x \\
&= \alpha - \beta - \bar{\beta} \\
&= \alpha - 2\mathrm{Re}(\beta)
\end{aligned}
$$

要证明 $|\lambda| < 1$，只需证明 $(\mathrm{Re}(\beta))^2 < (\alpha - \mathrm{Re}(\beta))^2 = \alpha^2 - 2\alpha\mathrm{Re}(\beta) + (\mathrm{Re}(\beta))^2$
即只需证明 $\alpha(\alpha - 2\mathrm{Re}(\beta)) > 0$
而这根据前面 $\alpha > 0$ 和 $\alpha - 2\mathrm{Re}(\beta) > 0$ 的结论可知是成立的.
因此 Gauss-Seidel 迭代格式的系数矩阵 $B = (D - L)^{-1} L^H$ 的任意特征值 λ 都满足 $|\lambda| < 1$
这说明 $\rho(B) < 1$，因此 Gauss-Seidel 迭代法收敛.

Problem 3

Let $B \in \mathbb{C}^{n \times n}, c \in \mathbb{C}^n$
Suppose that $\rho(B) = 0$
Show that the iterative scheme $x^{(k+1)} = Bx^{(k)} + c$ converges to the solution of $x = Bx + c$ for any initial guess $x^{(0)}$ within at most n iterations.

- **(幂零 Jordan 块的性质, Matrix Analysis 引理 $3.1.4$)**
  给定正整数 $n \geq 2$ 和 $x \in \mathbb{C}^n$，记 $\mathbb{C}^n$ 的第 i 个标准单位基向量为 e_i，则我们有:
  - $J_n(0) e_{i+1} = e_i \ (\forall\, i = 1, \dots, n-1)$
  - $J_n(0)^T J_n(0) = \begin{bmatrix} 0 & \\ & I_{n-1} \end{bmatrix}$
  - $[I_n - J_n(0)^T J_n(0)] x = (x^T e_1) e_1$
  - $J_n(0)^k = \begin{cases} I_n & \text{if } k = 0 \\ \begin{bmatrix} & I_{n-k} \\ 0_{k \times k} & \end{bmatrix} & \text{if } 1 \leq k < n \\ 0_{n \times n} & \text{if } k \geq n \end{cases}$

Solution:
根据 $\rho(B) = 0$ 可知 B 的 n 个特征值均为零.
因此可设其 Jordan 标准型为:

$$
S^{-1} B S = J = \begin{bmatrix} J_{n_1}(0) & & & \\ & J_{n_2}(0) & & \\ & & \ddots & \\ & & & J_{n_d}(0) \end{bmatrix}
$$

其中 $S \in \mathbb{C}^{n \times n}$ 为非奇异矩阵，$d \leq 1$ 且 $n_1 + \cdots + n_d = n$
记 $n_{\max} := \max_{1 \leq i \leq d} n_i$ (显然 $n_{\max} \leq n$，当且仅当 $d = 1$ 时取等)
对于任意 $i = 1, \dots, d$ 我们都有:

$$
(J_{n_i}(0))^k = 0_{n_i \times n_i} \quad (\forall\, k \geq n_{\max} \geq n_i)
$$

于是我们有:

$$
\begin{aligned}
B^k &= (SJS^{-1})^k \\
&= SJ^kS^{-1} \\
&= S \begin{bmatrix} (J_{n_1}(0))^k & & & \\ & (J_{n_2}(0))^k & & \\ & & \ddots & \\ & & & (J_{n_d}(0))^k \end{bmatrix} S^{-1} \quad (\forall\, k \geq n_{\max}) \\
&= S \begin{bmatrix} 0_{n_1\times n_1} & & & \\ & 0_{n_2\times n_2} & & \\ & & \ddots & \\ & & & 0_{n_d\times n_d} \end{bmatrix} S^{-1} \\
&= 0_{n\times n}
\end{aligned}
$$

设 $x = Bx + c$ 的精确解为 $x_\star$
记 $e^{(k)} := x^{(k)} - x_\star \quad (\forall\, k \in \mathbb{N})$
则我们有:

$$
\begin{aligned}
e^{(k+1)} &= x^{(k+1)} - x_\star \\
&= (Bx^{(k)} + c) - (Bx_\star + c) \\
&= B(x^{(k)} - x_\star) \\
&= Be^{(k)}
\end{aligned} \quad (\forall\, k \in \mathbb{N})
$$

于是我们有:

$$
\begin{aligned}
e^{(k)} &= Be^{(k-1)} \\
&= B^2e^{(k-2)} \\
&= \cdots \\
&= B^ke^{(0)}
\end{aligned} \quad (\forall\, k \in \mathbb{Z}_+)
$$

根据 $B^k = 0_{n\times n}\ (\forall\, k \geq n_{\max})$ 可知:

$$
\begin{aligned}
x^{(k)} - x_\star &= e^{(k)} \\
&= B^ke^{(0)} \\
&= 0_{n\times n}e^{(0)} \\
&= 0_n
\end{aligned} \quad (\forall\, k \geq n_{\max})
$$

因此 $x^{(k)} = x_\star\ (\forall\, k \geq n_{\max})$
说明单步线性定常迭代法 $x^{(k+1)} = Bx^{(k)} + c$ 在第 $n_{\max} \leq n$ 步收敛于 $x = Bx + c$ 的精确解.
命题得证.

Problem 4

Let $B, M \in \mathbb{C}^{n\times n}, c \in \mathbb{C}^n$
Suppose that both M and $M - B^HMB$ are positive definite.
Show that the iterative scheme $x^{(k+1)} = Bx^{(k)} + c$ converges to the solution of $x = Bx + c$ for any initial guess $x^{(0)}$

Solution:
设 $x = Bx + c$ 的精确解为 $x_\star$
记 $e^{(k)} := x^{(k)} - x_\star \quad (\forall\, k \in \mathbb{N})$
则我们有:

$$
\begin{aligned}
e^{(k+1)} &= x^{(k+1)} - x_\star \\
&= (Bx^{(k)} + c) - (Bx_\star + c) \\
&= B(x^{(k)} - x_\star) \\
&= Be^{(k)}
\end{aligned} \quad (\forall\, k \in \mathbb{N})
$$

于是我们有:

$$
\begin{aligned}
e^{(k)} &= Be^{(k-1)} \\
&= B^2 e^{(k-2)} \\
&= \cdots \\
&= B^k e^{(0)}
\end{aligned}
\quad (\forall\, k \in \mathbb{Z}_+)
$$

因此单步线性定常迭代法 $x^{(k+1)} = Bx^{(k)} + c$ 收敛 (即 $x^{(k)} \to x_\star\ (n \to \infty)$) 当且仅当 $\lim_{k\to\infty} B^k = 0_{n\times n}$，即当且仅当 $\rho(B) < 1$

注意到 $M, M - B^H MB$ 都是 Hermite 正定阵.
因此对于 B 的任意特征对 (λ, u) 都有:

$$
\begin{aligned}
&u^H Mu > 0 \\
\hline
u^H(M - B^H MB)u &= u^H Mu - (Bu)^H M(Bu) \\
&= u^H Mu - (u\lambda)^H M(u\lambda) \\
&= u^H Mu(1 - |\lambda|^2) \\
&> 0
\end{aligned}
$$

于是我们有 $|\lambda| < 1$ 成立.
根据 λ 的任意性可知 $\rho(B) = \max_{\lambda\in\text{eig}(B)} |\lambda| < 1$
因此单步线性定常迭代法 $x^{(k+1)} = Bx^{(k)} + c$ 收敛

Problem 5

Numerically solve the 2D Laplace equation:

$$
\frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial x^2} u(x,y) = 0
$$

over the unit square $[0,1]^2$ with boundary conditions:

$$
\begin{aligned}
u(0,y) &\equiv 0 \\
u(1,y) &\equiv 0 \\
u(x,0) &= \sin(\pi x) \quad (\forall\, x \in [0,1]) \\
u(x,1) &\equiv 0
\end{aligned}
$$

Use the Jacobi method or the Gauss-Seidel method to solve the discretized system.
Visualize the solution and the convergence history.

Solution:

(1) 理论推导

考虑二维 Poisson 方程:

$$
-\frac{\partial^2}{\partial x^2} u(x,y) - \frac{\partial^2}{\partial x^2} u(x,y) = f(x,y) \quad (0 < x, y < 1)
$$

(本题中函数 $f(x,y) \equiv 0$)
我们可以使用有限差分去逼近上述方程:

$$
\begin{aligned}
h &:= \frac{1}{n+1} \\
-\frac{\partial^2}{\partial x^2} u(x,y)\Big|_{x=x_i, y=y_j} &\approx \frac{2u_{ij} - u_{i-1,j} - u_{i+1,j}}{h^2} \quad (i,j = 1,\dots,n) \\
-\frac{\partial^2}{\partial y^2} u(x,y)\Big|_{x=x_i, y=y_j} &\approx \frac{2u_{ij} - u_{i,j-1} - u_{i,j+1}}{h^2} \quad (i,j = 1,\dots,n)
\end{aligned}
$$

将上述近似相加得到:

$$-\left\{\frac{\partial^2}{\partial x^2}u(x,y)+\frac{\partial^2}{\partial y^2}u(x,y)\right\}\Big|_{x=x_i,y=y_j}\approx\frac{4u_{ij}-u_{i-1,j}-u_{i+1,j}-u_{i,j-1}-u_{i,j+1}}{h^2}$$

其中截断误差是 $O(h^2)$ 级别的.
于是我们得到一组具有 n 个未知量 u_{ij} $(1\le i,j\le n)$ 的 n^2 维线性方程组:

$$4u_{ij}-u_{i-1,j}-u_{i+1,j}-u_{i,j-1}-u_{i,j+1}=h^2f_{ij}\quad(1\le i,j\le n)$$

(2) Jacobi 迭代

(二维 Poisson 方程 Jacobi 法单步, 应用数值线性代数, 算法 6.2)

$$\begin{aligned}&\text{for } i=1:n\\&\quad\text{for } j=1:n\\&\qquad u_{i,j}^{(k+1)}=\frac{1}{4}(u_{i-1,j}^{(k)}+u_{i+1,j}^{(k)}+u_{i,j-1}^{(k)}+u_{i,j+1}^{(k)}+h^2f_{ij})\\&\quad\text{end}\\&\text{end}\end{aligned}$$

注意到所有的新值 $u_{i,j}^{(k+1)}$ 都可以彼此独立的计算.
因此当 $u_{i,j}^{(k+1)}$ 被存放在一个 $n+2$ 阶矩阵 U 中 (这个矩阵补全了边界值) 时,
我们可以只用一条 Matlab 语句来实现上述算法.

```
U(2:n+1, 2:n+1) = 0.25 * (U(1:n,2:n+1) + U(3:n+2,2:n+1) + U(2:n+1,1:n) + U(2:n+1,3:n+2) +
h^2 * F);
```

其中 $h=\frac{1}{n+1}$，而 $f_{i,j}$ 的值存储在 n 阶矩阵 F 中 (本题中函数 $f(x,y)\equiv 0$)

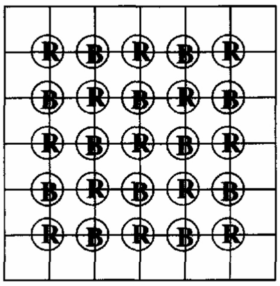
Matlab 代码为:

```
% Jacobi Iteration to solve 2D Poisson Problem
function [U, errorHistory] = Poisson_Jacobi(U, F, maxIter, tolerance)

    n = size(U, 1) - 2; % Get the grid size excluding the boundary points (n x n grid)
    h = 1 / (n + 1); % Grid spacing
    h_square_times_F = h^2 * F; % Precompute h^2 * F for efficiency in the Jacobi update

    % Initialize error history array
    errorHistory = zeros(maxIter, 1);

    for iter = 1:maxIter
        % Update interior points using the Jacobi method
        U_new = 0.25 * (U(1:n,2:n+1) + U(3:n+2,2:n+1) + U(2:n+1,1:n) + U(2:n+1,3:n+2) +
h_square_times_F);

        % Calculate the error (difference between new and old solution)
        error = max(max(abs(U_new - U(2:n+1, 2:n+1))));

        % Store the error in the error history array
        errorHistory(iter) = error;

        % Update U for the next iteration
        U(2:n+1, 2:n+1) = U_new;

        % Check for convergence
        if error < tolerance
            fprintf('Jacobi: Convergence reached after %d iterations.\n', iter);
            errorHistory = errorHistory(1:iter); % Trim the error history to the actual
number of iterations
            break;
        end
```

```
    end
end
```

(3) Gauss-Seidel 迭代

红黑序:
(红节点的坐标之和为偶数，黑节点的坐标之和为奇数)
我们首先使用来自黑节点的旧数据更新所有的红节点，
再使用来自红节点的新数据更新所有的黑节点.



于是得到如下算法:
(二维 Poisson 方程红黑序 Gauss-Seidel 法单步, 应用数值线性代数, 算法 6.4**)**

$$
\begin{aligned}
&\text{for all nodes } (i,j) \text{ that are red (i.e. } i+j \text{ is even)}\\
&\qquad u_{i,j}^{(k+1)} = \frac{1}{4}(u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} + h^2 f_{ij})\\
&\text{end}\\
&\text{for all nodes } (i,j) \text{ that are black (i.e. } i+j \text{ is odd)}\\
&\qquad u^{(k+1)_{i,j}} = \frac{1}{4}(u_{i-1,j}^{(k+1)} + u_{i+1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k+1)} + h^2 f_{ij})\\
&\text{end}
\end{aligned}
$$

Matlab 代码为:

```
% Red-Black Gauss-Seidel Iteration to solve 2D Poisson Problem
function [U, errorHistory] = Poisson_RedBlack_GaussSeidel(U, F, maxIter, tolerance)

    n = size(U, 1) - 2; % Get the grid size excluding the boundary points (n x n grid)
    h = 1 / (n + 1); % Grid spacing
    h_square_times_F = h^2 * F; % Precompute h^2 * F for efficiency in the Gauss-Seidel
update

    % Initialize error history array
    errorHistory = zeros(maxIter, 1); % Store the error at each iteration

    for iter = 1:maxIter
        % Save the old version for convergence check
        U_old = U;

        % Update red nodes (i + j is even)
        for i = 2:n+1
            for j = 2:n+1
                if mod(i + j, 2) == 0 % Red node (i+j is even)
                    U(i,j) = 0.25 * (U(i-1,j) + U(i+1,j) + U(i,j-1) + U(i,j+1) +
h_square_times_F(i-1,j-1));
                end
            end
        end

        % Update black nodes (i + j is odd)
```

```
        for i = 2:n+1
            for j = 2:n+1
                if mod(i + j, 2) == 1 % Black node (i+j is odd)
                    U(i,j) = 0.25 * (U(i-1,j) + U(i+1,j) + U(i,j-1) + U(i,j+1) +
h_square_times_F(i-1,j-1));
                end
            end
        end

        % Calculate the error (difference between new and old solution)
        error = max(max(abs(U(2:n+1, 2:n+1) - U_old(2:n+1, 2:n+1))));

        % Store the error in the error history array for convergence tracking
        errorHistory(iter) = error;

        % Check for convergence: If error is below tolerance, break the loop
        if error < tolerance
            fprintf('Gauss-Seidel: Convergence reached after %d iterations.\n', iter);
            errorHistory = errorHistory(1:iter); % Trim the error history to the actual
number of iterations
            break; % Exit the loop if convergence is reached
        end
    end
end
```

(4) 运行结果

函数调用:

```
% Parameters
n = 50; % Number of grid points in each direction
maxIter = 5000; % Maximum number of iterations
tolerance = 1e-6; % Convergence tolerance

% Grid spacing
h = 1 / (n + 1);

% The source term (right-hand side of Poisson equation)
F = zeros(n, n);

% Create the grid (including boundary)
U = zeros(n+2, n+2); % Solution grid (with boundaries included)

% Set initial condition at y = 0
x = linspace(0, 1, n+2); % x grid points, including boundary points
U(2:n+1, 1) = sin(pi * x(2:n+1)); % Set u(x, 0) = sin(pi * x)

% Use Jacobi iteration to solve 2D Poisson problem
[U_Jacobi, errorHistory_Jacobi] = Poisson_Jacobi(U, F, maxIter, tolerance);

% Visualization of the solution
[X, Y] = meshgrid(0:h:1, 0:h:1); % Create meshgrid for plotting
surf(X, Y, U_Jacobi); % Plot the solution
title('Solution of the 2D Laplace Equation (Jacobi)');
xlabel('x');
ylabel('y');
zlabel('u(x, y)');
colorbar;

% Use Gauss-Seidel iteration to solve 2D Poisson problem
[U_GS, errorHistory_GS] = Poisson_RedBlack_GaussSeidel(U, F, maxIter, tolerance);
```

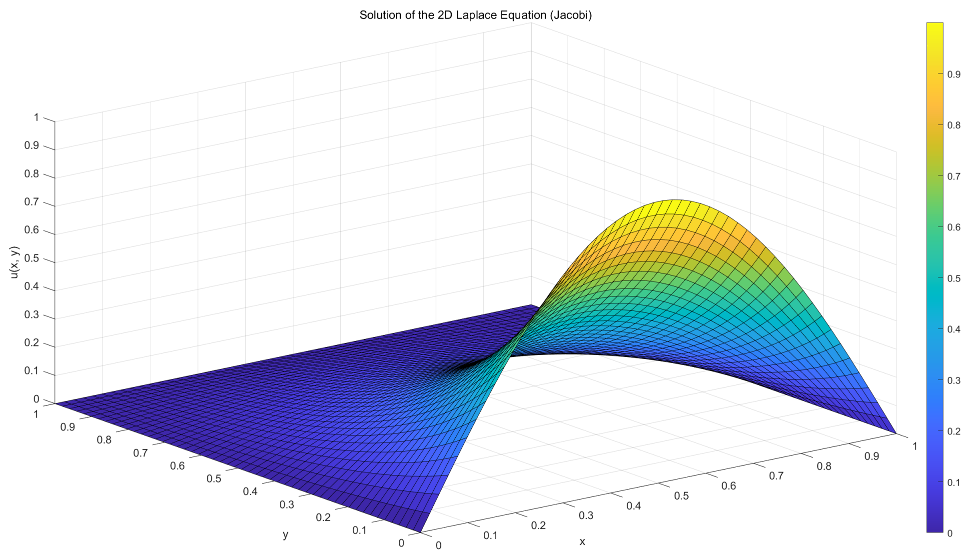
```
% Visualization of the solution
[X, Y] = meshgrid(0:h:1, 0:h:1); % Create meshgrid for plotting
surf(X, Y, U_GS); % Plot the solution
title('Solution of the 2D Laplace Equation (Gauss-Seidel)');
xlabel('x');
ylabel('y');
zlabel('u(x, y)');
colorbar;

% Plot the error history
figure;
plot(1:length(errorHistory_Jacobi), log10(errorHistory_Jacobi), 'r-', 'LineWidth', 1.5);
hold on;
plot(1:length(errorHistory_GS), log10(errorHistory_GS), 'b-', 'LineWidth', 1.5);

% Plot the tolerance line
tolerance_line = log10(tolerance);
plot([1, max(length(errorHistory_Jacobi), length(errorHistory_GS)) + 100],
[tolerance_line, tolerance_line], ...
            'k--', 'LineWidth', 1);
xlabel('Iteration');
ylabel('log10(Error)');
title('Convergence History for Jacobi and Gauss-Seidel Methods');
legend({'Jacobi', 'Gauss-Seidel', 'Tolerance'}, 'Location', 'northeast');
grid on;
```

运行结果:

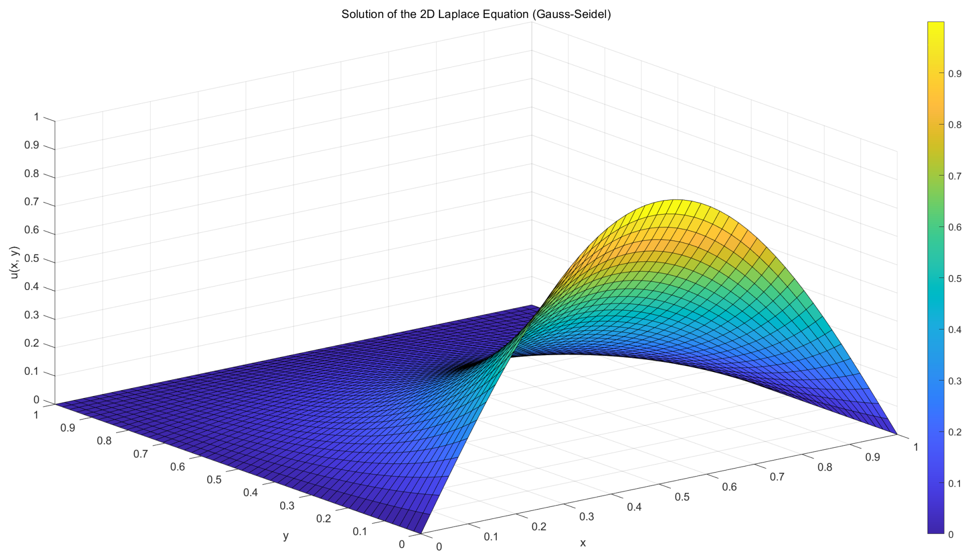
```
Jacobi: Convergence reached after 3374 iterations.
Gauss-Seidel: Convergence reached after 1870 iterations.
```

这验证了一个重要的结论:
求解二维 Poisson 问题时 Gauss-Seidel 迭代每步的效果相当于 Jacobi 迭代的两步.

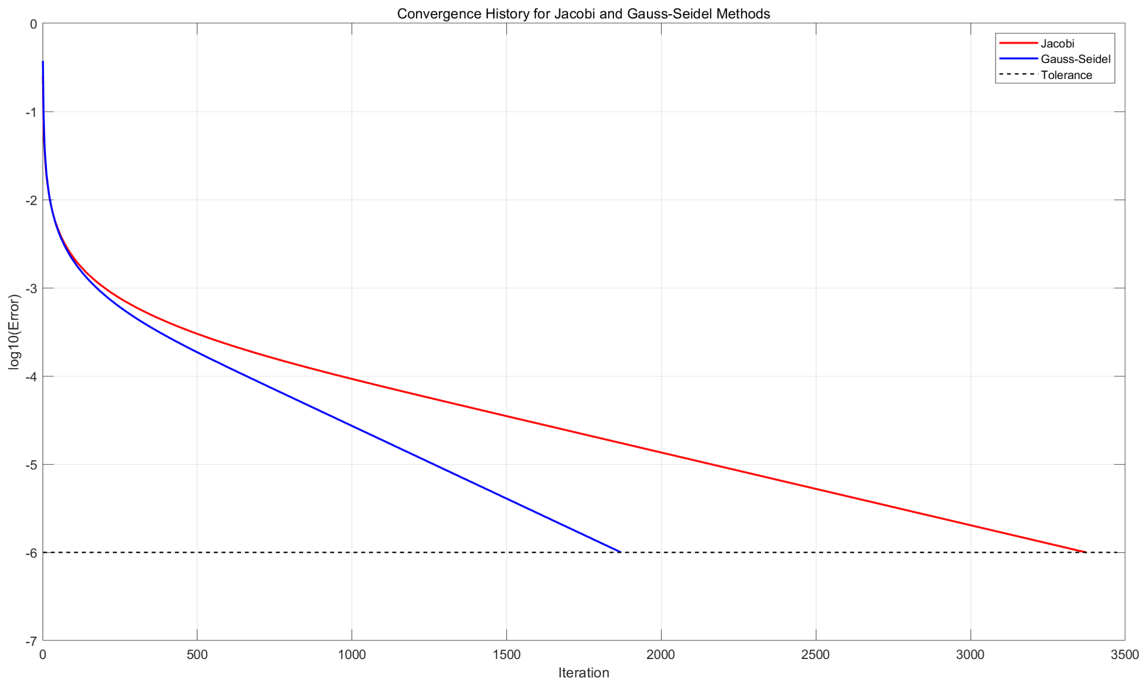
- **① Jacobi 迭代:**



- **② Gauss-Seidel 迭代:**



- **③ 对比:**
  这验证了一个重要的结论:
  求解二维 Poisson 问题时 Gauss-Seidel 迭代每步的效果相当于 Jacobi 迭代的两步.



Problem 6 (optional)

Let A be a real symmetric nonsingular matrix with positive diagonal entries.
Suppose that the Gauss-Seidel method for solving $Ax = b$ converges for any initial guess.
Show that A is positive definite.
(Hint: Show that $C = A - B^T AB$ is positive definite, where $B = I - (D - L)^{-1}A$)

Proof:
将 A 拆分为 $A = D - L - U$ (分别是对角部分、严格下三角部分和严格上三角部分)
根据 A 的对称性可知 $U = L^T$
因此 Gauss-Seidel 迭代矩阵为:

$$\begin{aligned} B &:= (D-L)^{-1}U \\ &= (D-L)^{-1}(D-L-A) \\ &= I-(D-L)^{-1}A \end{aligned}$$

根据题设可知无论起始点如何选取，Gauss-Seidel 迭代法总是收敛的，这意味着 $\rho(B) < 1$
即对于 B 的任意特征值 λ 都有 $|\lambda| < 1$ 成立.

我们定义:

$$
\begin{aligned}
&C := A - B^TAB \\
&\quad = A - (I - (D-L)^{-1}A)^TA(I - (D-L)^{-1}A) \\
&\quad = A - A + A^T(D-L)^{-T}A + A(D-L)^{-1}A - A^T(D-L)^{-T}A(D-L)^{-1}A \\
&\quad = A^T[(D-L)^{-T} + (D-L)^{-1} - (D-L)^{-T}A(D-L)^{-1}]A \\
&\quad = A^T(D-L)^{-T}[(D-L) + (D-L)^T - A](D-L)^{-1}A \\
&\quad = [(D-L)^{-1}A]^T[D + (D-L-L^T) - A](D-L)^{-1}A \quad (\text{note that } A = D-L-U = D-L-L^T) \\
&\quad = [(D-L)^{-1}A]^TD(D-L)^{-1}A
\end{aligned}
$$

根据题设可知 A 的对角元均为正实数，因此 D 为正定的对角阵
进而得到 $C = [(D-L)^{-1}A]^TD(D-L)^{-1}A$ 对称正定.
即对于任意 $x \neq 0_n \in \mathbb{R}^n$ 都有 $x^TCx > 0$ 成立.

下面我们证明 A 的正定性:

$$
\begin{aligned}
A &= C + B^TAB \\
&= C + B^T(C + B^TAB)B \\
&= C + B^TCB + (B^2)^TA(B^2) \\
&= C + B^TCB + (B^2)^T(C + B^TAB)(B^2) \\
&= \cdots \\
&= C + \sum_{k=1}^{\infty}(B^k)^TC(B^k)
\end{aligned}
$$

由于 C 是对称正定的，且对于任意 $k \in \mathbb{Z}_+$，合同变换后的矩阵 $(B^k)^TC(B^k)$ 都是对称正定的，
故 A 也是对称正定的.
命题得证.

> **失败的尝试:**
> 对于 B 的任意特征对 (λ, x) 我们都有:
>
> $$
> \begin{aligned}
> x^TCx &= x^T(A - B^TAB)x \\
> &= x^TAx - (Bx)^TA(Bx) \\
> &= x^TAx - (x\lambda)^TA(x\lambda) \\
> &= x^TAx(1-\lambda^2) \\
> &> 0
> \end{aligned}
> $$
>
> 根据 $|\lambda| < 1$ 可知 $x^TAx > 0$
> **(存疑: 但是如何推广到一般的 x?)**