

# 统计学基础 I : 数理统计 Assignment 9

姓名: 雍崔扬

学号: 21307140051

习题: E3.2, E3.5, E3.6

## Problem 1 (习题 3.2)

为了验证投币正面出现的概率  $p$  是否为  $\frac{1}{2}$ , 独立地投币 10 次检验如下假设:

$$H_0 : p = \frac{1}{2} \quad \leftrightarrow \quad H_1 : p \neq \frac{1}{2}$$

10 次投币全为正面或全为反面时拒绝原假设.

试问这一检验法的**实际显著性水平** (即第一类错误概率) 是多少?

若正面出现的概率为 0.1 时, 检验法的功效是多少?

**Solution:**

$T(X) = \sum_{i=1}^{10} X_i$  是取自 Bernoulli 分布族的简单随机样本  $X = (X_1, \dots, X_n)$  的充分统计量.

当  $p$  给定时, 我们知道  $T(X) = \sum_{i=1}^{10} X_i \stackrel{d}{=} B(10, p)$

检验法可表述为:

$$\phi(x) = \begin{cases} 1, & T(x) = \sum_{i=1}^{10} x_i = 0 \text{ or } 10 \\ 0, & \text{otherwise} \end{cases}$$

其功效函数为:

$$\begin{aligned} \gamma_\phi(p) &= E_p[\phi(X)] \\ &= P_p\{T(X) = 0\} + P_p\{T(X) = 10\} \\ &= P\{B(10, p) = 0\} + P\{B(10, p) = 10\} \\ &= \binom{10}{0} p^0 (1-p)^{10} + \binom{10}{10} p^{10} (1-p)^0 \\ &= (1-p)^{10} + p^{10} \end{aligned}$$

- 检验法的**实际水平** (即第一类错误概率) 为:

$$\gamma_\phi\left(\frac{1}{2}\right) = \left(1 - \frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10} = \frac{1}{2^9} \approx 0.00195$$

- 当  $p = 0.1$  时, 检验法的功效为:

$$\gamma_\phi(0.1) = (1 - 0.1)^{10} + (0.1)^{10} = (0.9)^{10} + (0.1)^{10} \approx 0.349$$

## Problem 2 (习题 3.5)

设一个观测值  $x$  取自密度函数为  $p(x)$  的总体.

求下列检验问题的水平  $\alpha$  的概率比检验:

$$H_0 : p(x) = p_0(x) = \begin{cases} 4x, & 0 \leq x < \frac{1}{2} \\ 4 - 4x, & \frac{1}{2} \leq x \leq 1 \end{cases} \quad \leftrightarrow \quad H_1 : p(x) = p_1(x) = I_{[0,1]}(x)$$

**定理 3.2.1: (Neyman-Pearson 引理, 数理统计讲义 命题 3.2.2)**

设参数空间为  $\Theta = \{\theta_0, \theta_1\}$

样本  $X$  的分布具有分布密度 (或离散的概率)  $p_0(\cdot) := p_{\theta_0}(\cdot)$  和  $p_1(\cdot) := p_{\theta_1}(\cdot)$

对于假设检验问题  $H_0: \theta = \theta_0 \leftrightarrow H_1: \theta = \theta_1$  和显著水平  $\alpha \in (0, 1)$

• **存在性:**

存在非负常数  $k$  以及**概率比检验**  $\phi_0(x) = \begin{cases} 1, & \text{if } \frac{p_1(x)}{p_0(x)} > k \\ 0, & \text{if } \frac{p_1(x)}{p_0(x)} < k \end{cases}$

满足  $\gamma_{\phi_0}(\theta_0) = \mathbb{E}_{\theta_0}[\phi_0(X)] = \int \phi_0(x)p_{\theta_0}(x)dx = \alpha$

• **充分性:**

存在性中描述的  $\phi_0$  是显著水平  $\alpha$  的**最有效检验法** (MP, most powerful)

即对于显著水平  $\alpha$  的任意检验函数  $\phi \in \Phi_\alpha$  都有:

$$\gamma_{\phi_0}(\theta_1) = \mathbb{E}_{\theta_1}[\phi_0(X)] \geq \mathbb{E}_{\theta_1}[\phi(X)] = \gamma_\phi(\theta_1)$$

• **必要性:**

若  $\phi^*$  为显著水平  $\alpha$  的**最有效检验法**, 则  $\phi^*$  必定是概率比检验.

**Solution:**

为方便起见, 我们考虑概率比统计量  $\frac{p_0(X)}{p_1(X)}$  (其分母恒不为 0) 而不是  $\frac{p_1(X)}{p_0(X)}$ :

$$\frac{p_0(x)}{p_1(x)} = \begin{cases} 4x, & 0 \leq x < \frac{1}{2} \\ 4 - 4x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

它的取值范围为  $[0, 2]$

**Neyman-Pearson 引理**所描述的检验法形如:

$$\phi(x) = \begin{cases} 1, & \text{if } \frac{p_0(x)}{p_1(x)} < k \\ 0, & \text{if } \frac{p_0(x)}{p_1(x)} > k \end{cases}$$

其中  $k \in (0, 2)$

为确定阈值  $k$ , 考虑计算  $P_{H_0}\{\frac{p_0(X)}{p_1(X)} < k\}$ :

$$\begin{aligned} P_{H_0}\left\{\frac{p_0(X)}{p_1(X)} < k\right\} &= \int_{-\infty}^{\infty} P_{H_0}\left\{\frac{p_0(X)}{p_1(X)} < k \mid X = x\right\} \cdot P_{H_0}\{X = x\}dx \\ &= \int_0^1 \mathbb{1}\left\{\frac{p_0(x)}{p_1(x)} < k\right\} \cdot p_0(x)dx \\ &= \int_0^1 \mathbb{1}\left\{0 \leq x < \frac{k}{4} \text{ or } 1 - \frac{k}{4} < x \leq 1\right\} \cdot p_0(x)dx \\ &= \int_0^{\frac{k}{4}} 1 \cdot 4x dx + \int_{1-\frac{k}{4}}^1 1 \cdot (4 - 4x) dx \\ &= 2x^2 \Big|_0^{\frac{k}{4}} + (4x - 2x^2) \Big|_{1-\frac{k}{4}}^1 \\ &= \frac{k^2}{8} + \frac{k^2}{8} \\ &= \frac{k^2}{4} \end{aligned}$$

令  $P_{H_0}\{\frac{p_0(X)}{p_1(X)} < k\} = \frac{k^2}{4} = \alpha$  解得  $k = 2\sqrt{\alpha}$  (舍弃负值)

因此概率比检验为:

$$\phi(x) = \begin{cases} 1, & \text{if } \frac{p_0(x)}{p_1(x)} < 2\sqrt{\alpha} \\ 0, & \text{if } \frac{p_0(x)}{p_1(x)} > 2\sqrt{\alpha} \end{cases}$$

### Problem 3 (习题 3.6)

设样本  $X = (X_1, \dots, X_n)$  取自指数分布  $\exp\left(\frac{1}{\theta}\right)$

试求下列检验问题的显著水平  $\alpha = 0.05$  的 MP (Most Powerful) 检验:

- 考虑指数分布族的充分统计量  $T(X) = \sum_{i=1}^n X_i$   
样本分布为  $p(x) = \frac{1}{\theta^n} \exp\left\{-\frac{1}{\theta} \sum_{i=1}^n x_i\right\} = \frac{1}{\theta^n} \exp\left\{-\frac{1}{\theta} T(x)\right\}$

$$\frac{p_1(x)}{p_0(x)} = \frac{\frac{1}{\theta_1^n} \exp\left\{-\frac{1}{\theta_1} T(x)\right\}}{\frac{1}{\theta_0^n} \exp\left\{-\frac{1}{\theta_0} T(x)\right\}} = \frac{\theta_0^n}{\theta_1^n} \exp\left\{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) T(x)\right\}$$

(1)  $H_0 : \theta = 2 \leftrightarrow H_1 : \theta = 4$

**Solution:**

$$\frac{p_1(x)}{p_0(x)} = \frac{\theta_0^n}{\theta_1^n} \exp\left\{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) T(x)\right\} = \frac{1}{2^n} \exp\left\{\frac{1}{4} T(x)\right\}$$

注意到  $\frac{p_1(x)}{p_0(x)}$  关于  $T(x)$  是严格单调递增的.

**Neyman-Pearson 引理**所描述的检验法形如:

$$\phi(x) = \begin{cases} 1, & \text{if } \frac{p_1(x)}{p_0(x)} > k' \\ 0, & \text{if } \frac{p_1(x)}{p_0(x)} < k' \end{cases} = \begin{cases} 1, & \text{if } T(x) > k \\ 0, & \text{if } T(x) < k \end{cases}$$

为确定阈值  $k$ , 我们求解  $P_{H_0}\{T(x) > k\} = \alpha$ :

$$\begin{aligned} P_{H_0}\{T(x) > k\} &= P\left\{\text{Gamma}\left(n, \frac{1}{2}\right) > k\right\} \\ &= 1 - F(k) \end{aligned}$$

其中  $F(t)$  为  $T(X)$  的累计分布函数, 即:

$$F(t) = \int_0^t \frac{\left(\frac{1}{2}\right)^n}{\Gamma(n)} s^{n-1} e^{-\frac{1}{2}s} ds$$

令  $P_{H_0}\{T(x) > k\} = 1 - F(k) = \alpha$  解得  $k = F^{-1}(1 - \alpha)$

因此概率比检验为:

$$\phi(x) = \begin{cases} 1, & T(x) = \sum_{i=1}^n x_i > k = F^{-1}(1 - \alpha) \\ 0, & T(x) = \sum_{i=1}^n x_i < k = F^{-1}(1 - \alpha) \end{cases}$$

(2)  $H_0 : \theta = 2 \leftrightarrow H_1 : \theta = 1$

**Solution:**

$$\frac{p_1(x)}{p_0(x)} = \frac{\theta_0^n}{\theta_1^n} \exp\left\{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) T(x)\right\} = 2^n \exp\left\{-\frac{1}{2} T(x)\right\}$$

注意到  $\frac{p_1(x)}{p_0(x)}$  关于  $T(x)$  是严格单调递减的.

**Neyman-Pearson 引理**所描述的检验法形如:

$$\phi(x) = \begin{cases} 1, & \text{if } \frac{p_1(x)}{p_0(x)} > k' \\ 0, & \text{if } \frac{p_1(x)}{p_0(x)} < k' \end{cases} = \begin{cases} 1, & \text{if } T(x) < k \\ 0, & \text{if } T(x) > k \end{cases}$$

为确定阈值  $k$ , 我们求解  $P_{H_0}\{T(x) < k\} = \alpha$ :

$$\begin{aligned} P_{H_0}\{T(x) < k\} &= P\left\{\text{Gamma}\left(n, \frac{1}{2}\right) < k\right\} \\ &= F(k) \end{aligned}$$

其中  $F(t)$  为  $T(X)$  的累计分布函数, 即:

$$F(t) = \int_0^t \frac{\left(\frac{1}{2}\right)^n}{\Gamma(n)} s^{n-1} e^{-\frac{1}{2}s} ds$$

令  $P_{H_0}\{T(x) < k\} = F(k) = \alpha$  解得  $k = F^{-1}(\alpha)$

因此概率比检验为:

$$\phi(x) = \begin{cases} 1, & T(x) = \sum_{i=1}^n x_i < k = F^{-1}(\alpha) \\ 0, & T(x) = \sum_{i=1}^n x_i > k = F^{-1}(\alpha) \end{cases}$$

**The End**