DATA130026 Optimization Assignment 3

Due Time: at the beginning of the class Mar. 27 2023
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Problem 1

Let $C \in \mathbb{R}^n$ be a convex set.
Let f be a convex function over C and let g be a strictly convex function over C.
Show that the sum function $f+g$ is strictly convex over C.

Proof:

Denote $h \overset{\Delta}{=} f+g$ as the sum function.
For every $\begin{cases} x_1, x_2 \in C \\ \alpha \in (0,1) \end{cases}$, it holds that:
$$\begin{aligned} h(\alpha x_1 + (1-\alpha)x_2) &= f(\alpha x_1 + (1-\alpha)x_2) + g(\alpha x_1 + (1-\alpha)x_2) \\ &< \alpha f(x_1) + (1-\alpha) f(x_2) + \alpha g(x_1) + (1-\alpha) g(x_2) \\ &= \alpha h(x_1) + (1-\alpha) h(x_2) \end{aligned}$$
which implies the strict convexity of $h = f+g$ over the convex set C.

Problem 2

Show that the function $f(x) = \frac{||Ax-b||_2^2}{1-x^Tx}$ is convex on $\{x \in \mathbb{R}^n \mid x^Tx \leq 1\}$.
Proof 1:
$$\begin{aligned} \mathrm{epi}(f) &= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \begin{cases} x^Tx \leq 1 \\ t \geq \frac{||Ax-b||_2^2}{1-x^Tx} \end{cases} \} \\ &= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \begin{cases} ||x||_2^2 \leq 1 \\ ||Ax-b||_2^2 + t \cdot (||x||_2^2 - 1) \leq 0 \end{cases} \} \\ &= [\{x \in \mathbb{R}^n : ||x||_2^2 \leq 1\} \times \mathbb{R}] \bigcap \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : ||Ax-b||_2^2 + t \cdot (||x||_2^2 - 1) \leq 0\} \end{aligned}$$

- **Consider the first set:**
  $C_1 \overset{\Delta}{=} \{x \in \mathbb{R}^n : ||x||_2^2 \leq 1\} \times \mathbb{R} = B(0_n, 1) \times \mathbb{R}$
  We observe that C_1 is the direct product of the unit Euclidean ball $B(0_n, 1)$ and $\mathbb{R}$,
  which implies that C_1 is a **convex set**.
- **Consider the second set:**
  $C_2 \overset{\Delta}{=} \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : ||Ax-b||_2^2 + t \cdot (||x||_2^2 - 1) \leq 0\}$
  Denote $\begin{cases} g(x,t) = ||Ax-b||_2^2 + t \cdot (||x||_2^2 - 1) \\ \mathrm{dom}(g) = \mathbb{R}^n \times \mathbb{R} \end{cases}$
  We observe that $g(x,t)$ is convex with respect to $x \in \mathbb{R}^n$,
  and is affine (hence convex) with respect to $t \in \mathbb{R}$
  So the function $g(x,t)$ is a **convex function**.
  Thus, $C_2 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : g(x,t) \leq 0\}$ is the 0-sublevel set of the convex function $g(x,t)$,
  which implies that C_2 is a **convex set**.

Therefore, $\text{epi}(f) = C_1 \cap C_2$, as the intersection of two convex sets, is also a convex set, which implies the convexity of function f.
Q.E.D.

Proof 2:

$$\begin{aligned}\text{epi}(f) &= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \begin{cases} x^T x \le 1 \\ t \ge \frac{||Ax-b||_2^2}{1-x^T x} \end{cases}\} \\ &= \{(x,t,s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \begin{cases} ||x||_2^2 \le s \\ ||Ax-b||_2^2 \le t(1-s) \\ t \ge 0 \\ s \le 1 \end{cases}\}\end{aligned}$$

We observe that $||x||_2^2 \le s$ and $||Ax-b||_2^2 \le t(1-s)$ are both **second-order cone constraints**,
while $t \ge 0$ and $s \le 1$ are affine constraints.
Therefore, $\text{epi}(f)$ is a convex set, which implies the convexity of function f.
Q.E.D.

Problem 3

Suppose $f(x)$ is a continuous differentiable function.
(a) Show that if in addition f is convex, then $\nabla f(x)$ is monotone, i.e.
$(\nabla f(x) - \nabla f(y))^T (x-y) \ge 0 \;\; (\forall\, x, y)$.
(而且是单调递增)
Proof:
According to the **first-order convexity condition**,
for every $x, y \in \text{dom}(f)$, it holds that $\begin{cases} f(y) - f(x) \ge \nabla f(x)^T (y-x) \\ f(x) - f(y) \ge \nabla f(y)^T (x-y) \end{cases}$
By adding the two inequalities, we have $(\nabla f(x) - \nabla f(y))^T (x-y) \ge 0$
Therefore, $\nabla f(x)$ is monotone on $\text{dom}(f)$.

(b) Show that if in addition f is m strongly convex (i.e. $f(x) - \frac{m}{2}||x||_2^2$ is convex)
then $\nabla f(x)$ is m strongly monotone, i.e. $(\nabla f(x) - \nabla f(y))^T (x-y) \ge m||x-y||_2^2 \;\; (\forall\, x, y)$.
Proof:
By using the conclusion of (a), we know for every $x, y \in \text{dom}(f)$,
it holds that $[(\nabla f(x) - mx) - (\nabla f(y) - my)]^T (x-y) \ge 0$
Therefore, $(\nabla f(x) - \nabla f(y))^T (x-y) \ge m||x-y||_2^2$, which means ∇f is m strongly monotone.

Problem 4

Define $\phi(c) = \inf\limits_x \{f(x) : g(x) \le c\}$
Suppose $f(x)$ and $g(x)$ are convex functions.
Show that $\phi(c)$ is a convex and non-increasing function, and calculate its domain.
Proof:
Obviously
$\text{dom}(\phi) = \{c \in \mathbb{R} : \phi(c) < +\infty\} = \{c \in \mathbb{R} : g(x) \le c \text{ for some } x \in \text{dom}(f) \cap \text{dom}(g)\}$

- ① Firstly, we show that $\phi(c)$ is a **non-increasing** function:
  For every $c_1, c_2 \in \mathrm{dom}(\phi)$ such that $c_2 > c_1$,
  it holds that
  $\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le c_1\} \subseteq \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le c_2\}$
  which implies that $\phi(c_1) = \inf\limits_x\{f(x) : g(x) \le c_1\} \ge \inf\limits_x\{f(x) : g(x) \le c_2\} = \phi(c_2)$
  Thus, $\phi(c)$ is a **non-increasing** function.

- ② Secondly, we show that $\phi(c)$ is a **convex** function:
  For every given $c_1, c_2 \in \mathrm{dom}(\phi)$, consider any
  $\begin{cases} \alpha \in [0,1] \\ x_1 \in \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le c_1\} \\ x_2 \in \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le c_2\} \end{cases}$
  Since g is convex, we have:
  $\begin{aligned} g(\alpha x_1 + (1-\alpha)x_2) &\le \alpha g(x_1) + (1-\alpha)g(x_2) \\ &\le \alpha c_1 + (1-\alpha)c_2 \end{aligned}$
  hence $\alpha x_1 + (1-\alpha)x_2 \in \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le \alpha c_1 + (1-\alpha)c_2\}$
  From defination of ϕ and the convexity of f we get:
  $\begin{aligned} \phi(\alpha c_1 + (1-\alpha)c_2) &\le f(\alpha x_1 + (1-\alpha)x_2) \\ &\le \alpha f(x_1) + (1-\alpha)f(x_2) \end{aligned}$
  Now take the first $\inf$ for $x_1 \in \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le c_1\}$,
  and take the second $\inf$ for $x_2 \in \{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : g(x) \le c_2\}$,
  we know that $\phi(\alpha c_1 + (1-\alpha)c_2) \le \alpha\phi(c_1) + (1-\alpha)\phi(c_2) \ \ (\forall\, \alpha \in [0,1])$
  Thus, $\phi(c)$ is a **convex** function.

THE END