DATA130026.01 Optimization Assignment 2

Due Time: at the beginning of the class Mar. 20, 2023

姓名： 雍崔扬

学号： 21307140051

Problem 1: Convexity of Functions

Show that the following functions are convex over the specified domain C:

(a) $f(x_1, x_2, x_3) = -\sqrt{x_1 x_2} + 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$ over $\mathbb{R}^3_{++}$.

Proof:

Function f can be decomposed as $f_1 + f_2$,
where $\begin{cases} f_1(x_1, x_2) = -\sqrt{x_1 x_2} \\ \mathrm{dom}(f_1) = \mathbb{R}^2_{++} \end{cases}$ and $\begin{cases} f_2(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 \\ \mathrm{dom}(f_2) = \mathbb{R}^3_{++} \end{cases}$

- Firstly，we prove that f_1 is a convex function:

  The second derivative of f_1 is $\nabla^2 f_1(x_1, x_2) = \begin{bmatrix} \frac{\sqrt{x_2}}{4\sqrt{x_1^3}} & -\frac{1}{4\sqrt{x_1x_2}} \\ -\frac{1}{4\sqrt{x_1x_2}} & \frac{\sqrt{x_1}}{4\sqrt{x_2^3}} \end{bmatrix}$

  Since $\det[\nabla^2 f_1(x_1, x_2)] = \frac{1}{16}[\frac{\sqrt{x_1x_2}}{\sqrt{x_1^3x_2^3}} - \frac{1}{x_1x_2}] = 0 \quad (\forall\, x \in \mathbb{R}^2_{++})$,

  it holds that $\nabla^2 f_1(x_1, x_2) \succeq 0$，which implies the convexity of f_1

- Secondly，we prove that f_2 is a convex function:

  The second derivative of f_2 is $\nabla^2 f_2(x_1, x_2, x_3) \equiv \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 6 \end{bmatrix}$

  By using **Gershgorin Circle Theorem**,

  we know that the eigenvalues of $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 6 \end{bmatrix}$ are all nonnegative real numbers.

  Therefore，for every $x \in \mathbb{R}^3_{++}$，it holds that $\nabla^2 f_2(x) \succeq 0$，which implies the convexity of f_2

In summary，function f，as the sum of two convex functions，is also convex.
Q.E.D.

(b) $f(x) = ||x||^4$ over $\mathbb{R}^n$.

Proof:

For every $\begin{cases} x_1, x_2 \in \mathbb{R}^n \\ \alpha \in [0, 1] \end{cases}$，it holds that:

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &= ||\alpha x_1 + (1-\alpha)x_2||^4 \\ &\le (||\alpha x_1|| + ||(1-\alpha)x_2||)^4 \\ &= (\alpha||x_1|| + (1-\alpha)||x_2||)^4 \\ &\le \alpha||x_1||^4 + (1-\alpha)||x_2||^4 \\ &= \alpha f(x_1) + (1-\alpha) f(x_2) \end{aligned}$$

Therefore，by definition of convex functions，f is convex over $\mathbb{R}^n$.
Q.E.D.

(c) $f(x) = \sqrt{x^TQx + 1}$ over $\mathbb{R}^n$ where $Q \succeq 0$ is an $n \times n$ matrix.

Proof:
Given that $Q \succeq 0$, there is a lower triangular matrix $L \in \mathbb{R}^{n\times n}$ such that $Q = LL^T$,
which is the **Cholesky decomposition** of Q.
So we have $x^TQx + 1 = x^TLL^Tx + 1 = \begin{bmatrix} L^Tx \\ 1 \end{bmatrix}^T \begin{bmatrix} L^Tx \\ 1 \end{bmatrix}$
Denote $g(x) = \begin{bmatrix} L^Tx \\ 1 \end{bmatrix} = \begin{bmatrix} L^T \\ 0_n^T \end{bmatrix} x + \begin{bmatrix} 0_n \\ 1 \end{bmatrix}$, which is an affine function.
Thus, $f(x) = \sqrt{x^TQx+1} = ||g(x)||_2$,
as the composition of a convex function $||\cdot||_2$ and an affine function g, is also convex.

(Conclusion of function composition, see FDU 最优化方法 3. 凸函数 3.(2))
Q.E.D.

Problem 2: Convex Function Ratio

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and convex and $g : \mathbb{R}^n \to \mathbb{R}$ is positive and concave.
Show that the function $\frac{f^2}{g}$ with domain $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ is convex.
Proof:
Evidently, $\mathrm{dom}(f) \cap \mathrm{dom}(g)$ is a convex set since $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ are both convex sets.
Denote $h = \frac{f^2}{g}$, we have:
$\nabla h = \frac{1}{g^2}(2fg\nabla f - f^2\nabla g)$

$$
\begin{aligned}
\nabla^2 h &= \frac{1}{g^4}\{[2(g\nabla f\nabla f^T + f\nabla g\nabla f^T + fg\nabla^2 f) - (2f\nabla f\nabla g^T + f^2\nabla^2 g)]\cdot g^2 - (2fg\nabla f - f^2\nabla g)\cdot 2g\nabla g^T\} \\
&= \frac{f}{g}\nabla^2 f - \frac{f^2}{g^2}\nabla^2 g + \frac{2}{g^3}(g\nabla f - f\nabla g)(g\nabla f - f\nabla g)^T
\end{aligned}
$$

According to the problem statement, we know that:
for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, it holds that $\begin{cases} f(x) \geq 0 \\ \nabla^2 f(x) \succeq 0 \\ g(x) > 0 \\ \nabla^2 g(x) \preceq 0 \end{cases}$
Therefore, for all $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, it holds that $\nabla^2 h(x) \succeq 0$,
which implies the convexity of $h = \frac{f^2}{g}$
Q.E.D.

Problem 3: Convexity of Optimal Solution Sets

Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subset \mathbb{R}^n$.
Then the set of optimal solutions of the problem $\min\limits_{x\in C} f(x)$ which we denote by $X^\star$ is convex.
In addition, if f is strictly convex over C ,
then there exists at most one optimal solution of the problem.
Proof:

- **(1) Proof of the first statement:**
  Denote $p^\star = \inf\limits_{x\in C} f(x)$ as the optimal value of the problem $\min\limits_{x\in C} f(x)$
  Then we can describe $X^\star$ as the $p^\star$**-sublevel set** of the convex function f,
  i.e., $X^\star = \{x \in C : f(x) \leq p^\star\}$
  which implies the convexity of $X^\star$

- **(2) Proof of the second statement:**
  We prove it by contradiction:
  Suppose there are $x_1, x_2 \in X^\star$ such that $x_1 \neq x_2$
  We construct $\begin{cases} g(t) = f(x_1 + t(x_2 - x_1)) \\ \mathrm{dom}(g) = [0,1] \end{cases}$, it holds that:
  $\nabla g(t) = (x_2 - x_1)^T \nabla f(x_1 + t(x_2 - x_1))$
  $\nabla^2 g(t) = (x_2 - x_1)^T \nabla^2 f(x_1 + t(x_2 - x_1))(x_2 - x_1)$
  Due to the **strict convexity** of function f and the **convexity** of its domain C,
  for all $t \in [0,1]$, it holds that $\begin{cases} x_1 + t(x_2 - x_1) \in C \\ \nabla^2 f(x_1 + t(x_2 - x_1)) \succ 0 \end{cases}$
  Thus, for all $t \in [0,1]$, it holds that $\nabla^2 g(t) \succ 0$, which implies the **strict convexity** of g
  So for all $t \in (0,1)$, we have:
  $$\begin{aligned} f(x_1 + t(x_2 - x_1)) &= g(t) \\ &< (1-t)g(0) + tg(1) \\ &= (1-t)f(x_1) + tf(x_2) \\ &= (1-t)p^\star + tp^\star \\ &= p^\star \end{aligned}$$
  which contradicts the definition of $p^\star$ as the optimal value, i.e., $p^\star = \inf\limits_{x \in C} f(x)$
  Therefore, we conclude that the second statement is true.

Q.E.D.

Problem 4: Convexity of Solution Sets under Quadratic Constraints

Let $C \subset \mathbb{R}^n$ be the solution set of a quadratic inequality $C = \{x \in \mathbb{R}^n : x^T Ax + b^T x + c \leq 0\}$ with $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) Show that C is convex if $A \succeq 0$.

Proof:

Denote $\begin{cases} f(x) = x^T Ax + b^T x + c \\ \mathrm{dom}(f) = \mathbb{R}^n \end{cases}$, which is obviously a convex function if $A \succeq 0$
The set $C = \{x \in \mathbb{R}^n : x^T Ax + b^T x + c \leq 0\}$ can be decribed as the 0-sublevel set of f, which implies the convexity of C.

(b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda gg^T \succeq 0$ for some $\lambda \in \mathbb{R}$.

Proof:

Suppose there is a $\lambda_0 \in \mathbb{R}$ such that $A + \lambda_0 gg^T \succeq 0$
Denote $C_0 = \{x \in \mathbb{R}^n : x^T(A + \lambda_0 gg^T)x + b^T x + c - \lambda_0 h^T h \leq 0\}$
By using the conclusion of (a), we know that C_0 is a convex set.
Thus, $C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$ is also a convex set.
Below, we prove that $C \cap \{x \in \mathbb{R}^n : g^T x + h = 0\} = C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$:

- ① For every $x \in C \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$, it holds that:
  $$\begin{aligned} x^T(A + \lambda_0 gg^T)x + b^T x + c - \lambda_0 h^T h &= (x^T Ax + b^T x + c) + \lambda_0(x^T gg^T x - h^T h) \\ &\leq 0 + 0 \\ &= 0 \end{aligned}$$
  meaning that $x \in C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$
  Thus, $C \cap \{x \in \mathbb{R}^n : g^T x + h = 0\} \subseteq C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$

- ② For every $x \in C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$, it holds that:
  $$\begin{aligned} x^T A x + b^T x + c &= (x^T A x + b^T x + c) + \lambda_0 (x^T g g^T x - h^T h) \\ &= x^T (A + \lambda_0 g g^T) x + b^T x + c - \lambda_0 h^T h \\ &\leq 0 \end{aligned}$$
  meaning that $x \in C \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$
  Thus, $C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\} \subseteq C \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$

Therefore, $C \cap \{x \in \mathbb{R}^n : g^T x + h = 0\} = C_0 \cap \{x \in \mathbb{R}^n : g^T x + h = 0\}$ is also a convex set.
Q.E.D.