数值算法 Homework 07

Due: Nov. 5, 2024
姓名: 雍崔扬
学号: 21307140051

Problem 1

(Perturbation results for eigenvalue problems)
Let $(\hat{\lambda},\hat{x})$ be an approximate eigenpair of $A\in\mathbb{C}^{n\times n}$, and $r=A\hat{x}-\hat{x}\hat{\lambda}$ be the residual.
Suppose that $\|\hat{x}\|=1$, show that there exists $E\in\mathbb{C}^{n\times n}$ such that:

$$\begin{aligned}(A+E)\hat{x}=\hat{x}\hat{\lambda}\\ \|E\|_2\le\|r\|_2\end{aligned}$$

Solution:
取 $E:=-r\hat{x}^H$
它满足:

$$\begin{aligned}(A+E)\hat{x}&=(A-r\hat{x}^H)\hat{x}\\&=A\hat{x}-r\hat{x}^H\hat{x}\quad(\text{note that }\hat{x}^H\hat{x}=\|\hat{x}\|_2^2=1)\\&=A\hat{x}-r\quad(\text{note that }r=A\hat{x}-\hat{x}\hat{\lambda})\\&=\hat{x}\hat{\lambda}\end{aligned}$$

注意到 $E^HE=(-r\hat{x}^H)^H(-r\hat{x}^H)=\hat{x}r^Hr\hat{x}^H=\|r\|_2^2\hat{x}\hat{x}^H$
其特征值为 $\|r\|_2^2\hat{x}^H\hat{x}=\|r\|_2^2$ (代数重数和几何重数均为 1) 与 0 (代数重数和几何重数均为 $n-1$)
因此我们有:

$$\begin{aligned}\|E\|_2&=\sigma_{\max}(E)\\&=\sqrt{\lambda_{\max}(E^HE)}\\&=\sqrt{\|r\|_2^2}\\&=\|r\|_2\end{aligned}$$

(更深刻地，$E=-r\hat{x}^H$ 的精简 SVD 分解即为 $E=-r\hat{x}^H=\|r\|_2(\frac{-r}{\|r\|_2}\hat{x}^H)$)
这说明一定存在 $E\in\mathbb{C}^{n\times n}$ 使得 $\begin{cases}(A+E)\hat{x}=\hat{x}\hat{\lambda}\\ \|E\|_2\le\|r\|_2\end{cases}$

错误的做法:

- 若 $r=0_n$，则我们有 $A\hat{x}=\hat{x}\hat{\lambda}$
  于是可取 $E=0_{n\times n}$，显然满足命题.
- 若 $r\neq 0_n$，则可取 $E:=\hat{\lambda}I_n-A\hat{x}\hat{x}^H$
  它满足:

$$\begin{aligned}(A+E)\hat{x}&=(A+\hat{\lambda}I_n-A\hat{x}\hat{x}^H)\hat{x}\\&=A\hat{x}+\hat{\lambda}\hat{x}-A\hat{x}\hat{x}^H\hat{x}\quad(\text{note that }\hat{x}^H\hat{x}=\|\hat{x}\|_2^2=1)\\&=A\hat{x}+\hat{x}\hat{\lambda}-A\hat{x}\\&=\hat{x}\hat{\lambda}\end{aligned}$$

  根据 $r=A\hat{x}-\hat{x}\hat{\lambda}$ 可知:

$$
\begin{aligned}
E &= \hat{\lambda} I_n - A\hat{x}\hat{x}^H \\
&= \hat{\lambda} I_n - (\hat{x}\hat{\lambda} + r)\hat{x}^H \\
&= \hat{\lambda}(I_n - \hat{x}\hat{x}^H) - r\hat{x}^H
\end{aligned}
$$

注意到:

$$
\begin{aligned}
E\hat{x} &= [\hat{\lambda}(I_n - \hat{x}\hat{x}^H) - r\hat{x}^H]\hat{x} \\
&= \hat{\lambda}(\hat{x} - \hat{x}\hat{x}^H\hat{x}) - r\hat{x}^H\hat{x} \quad (\text{note that } \hat{x}^H\hat{x} = \|\hat{x}\|_2^2 = 1) \\
&= \hat{\lambda}(\hat{x} - \hat{x}) - r \\
&= -r
\end{aligned}
$$

根据矩阵 2-范数的相容性我们有 $\|r\|_2 = \|-r\|_2 = \|E\hat{x}\|_2 \le \|E\|_2\|\hat{x}\|_2 = \|E\|_2$
但我们证明不了相容性不等式是取等的 (事实上，可使用 Matlab 代码验证 $\|E\|_2 = \|r\|_2$ 并不总是成立)
不过上述错误的做法让我们发现 $E := -r\hat{x}^H$ 的取法更加适合
(因为在 E 的错误的取法中，$I_n - \hat{x}\hat{x}^H$ 对恒等式 $(A+E)\hat{x} = \hat{x}\hat{\lambda}$ 没有贡献)

Problem 2

Let $A_0 \in \mathbb{C}^{n\times n}$ and $\mu_0, \mu_1, \dots, \mu_m \in \mathbb{C}$
Define $A_1, A_2, \dots, A_{m+1}$ by:

$$
\begin{cases} A_k - \mu_k I_n = Q_k R_k \\ A_{k+1} = R_k Q_k + \mu_k I_n \end{cases} (k = 0, 1, \dots, m)
$$

where Q_k are unitary matrices.
Show that:

$$
(A_0 - \mu_0 I_n)(A_0 - \mu_1 I_n)\cdots(A_0 - \mu_m I_n) = (Q_0 Q_1 \cdots Q_m)(R_m \cdots R_1 R_0)
$$

Solution:
由迭代格式容易推出 $A_{k+1} - \mu_k I_n = R_k Q_k = Q_k^H (A_k - \mu_k I_n) Q_k \ (k = 0, 1, \dots, m)$
因此 $A_{k+1} = Q_k^H A_k Q_k \ (k = 0, 1, \dots, m)$
于是我们有:

$$
\begin{aligned}
A_{k+1} &= Q_k^H A_k Q_k \\
&= Q_k^H (Q_{k-1}^H A_{k-1} Q_{k-1}) Q_k \\
&= \cdots \\
&= Q_k^H Q_{k-1}^H \cdots Q_0^H A_0 Q_0 \cdots Q_{k-1} Q_k
\end{aligned}
\qquad (k = 0, 1, \dots, m)
$$

因此 $A_0 = Q_0 Q_1 \cdots Q_k A_{k+1} Q_k^H \cdots Q_1^H Q_0^H \ (k = 0, 1, \dots, m)$
于是我们有:

$$
\begin{aligned}
A_0 - \mu_k I_n &= Q_0 Q_1 \cdots Q_k A_{k+1} Q_k^H \cdots Q_1^H Q_0^H - \mu_k I_n \\
&= Q_0 Q_1 \cdots Q_k (A_{k+1} - \mu_k I_n) Q_k^H \cdots Q_1^H Q_0^H \\
&= Q_0 Q_1 \cdots Q_k (R_k Q_k) Q_k^H \cdots Q_1^H Q_0^H \\
&= Q_0 Q_1 \cdots Q_k R_k Q_{k-1}^H \cdots Q_1^H Q_0^H
\end{aligned}
\quad (k = 0, 1, \dots, m)
$$

从而得到:

$$
\begin{aligned}
&\text{(note that } A_0 I_n = I_n A_0 \text{, so the multiplication is commutative)}\\
&(A_0-\mu_0 I_n)(A_0-\mu_1 I_n)\cdots(A_0-\mu_{m-1}I_n)(A_0-\mu_m I_n)\\
&=(A_0-\mu_m I_n)(A_0-\mu_{m-1}I_n)\cdots(A_0-\mu_1 I_n)(A_0-\mu_0 I_n)\\
&=(Q_0Q_1\cdots Q_mR_mQ_{m-1}^H\cdots Q_1^HQ_0^H)(Q_0Q_1\cdots Q_{m-1}R_{m-1}Q_{m-2}^H\cdots Q_1^HQ_0^H)\cdots(Q_0Q_1R_1Q_0^H)(Q_0R_0)\\
&=(Q_0Q_1\cdots Q_m)(R_mR_{m-1}\cdots R_1R_0)
\end{aligned}
$$

命题得证.

另一解法 (数学归纳法):

- ① $m=1$ 时，我们有 $A_0-\mu_0 I_n=Q_0R_0$，命题成立.
- ② 假设命题对 $m=m_0$ 的情况成立.
  现考虑 $m=m_0+1$ 的情况，对 A_1 应用归纳假设，即有:

  $$(A_1-\mu_1 I_n)(A_1-\mu_2 I_n)\cdots(A_1-\mu_{m_0+1}I_n)=(Q_1Q_2\cdots Q_{m_0+1})(R_{m_0+1}\cdots R_2R_1)$$

  注意到 $A_1=Q_0^HA_0Q_0$ (参考第一种解法)，故我们有
  $(A_1-\mu I_n)=Q_0^H(A_0-\mu I_n)Q_0$ ($\forall\ \mu\in\mathbb{C}$) 成立.
  因此我们有:

  $$
  \begin{aligned}
  &\text{(note that } A_0 I_n = I_n A_0 \text{, so the multiplication is commutative)}\\
  &(A_0-\mu_0 I_n)\{(A_0-\mu_1 I_n)\cdots(A_0-\mu_{m_0}I_n)(A_0-\mu_{m_0+1}I_n)\}\\
  &=\{(A_0-\mu_1 I_n)\cdots(A_0-\mu_{m_0}I_n)(A_0-\mu_{m_0+1}I_n)\}(A_0-\mu_0 I_n)\\
  &=\{[Q_0(A_1-\mu_1 I_n)Q_0^H]\cdots[Q_0(A_1-\mu_{m_0}I_n)Q_0^H][Q_0(A_1-\mu_{m_0+1}I_n)Q_0^H]\}(Q_0R_0)\\
  &=\{Q_0(A_1-\mu_1 I_n)\cdots(A_1-\mu_{m_0}I_n)(A_1-\mu_{m_0+1}I_n)Q_0^H\}(Q_0R_0)\quad\text{(use inductive assumption)}\\
  &=\{Q_0(Q_1Q_2\cdots Q_{m_0+1})(R_{m_0+1}\cdots R_2R_1)Q_0^H\}(Q_0R_0)\\
  &=(Q_0Q_1Q_2\cdots Q_{m_0+1})(R_{m_0+1}\cdots R_2R_1R_0)
  \end{aligned}
  $$

  因此命题对 $m=m_0+1$ 的情况也成立.

综上所述，命题得证.

Problem 3

Randomly generate a 1000×1000 matrix A with positive entries.
Use the power method to compute $\rho(A)$
Visualize the convergence history.

Solution:
幂法的迭代格式如下:

$$
\begin{array}{c}
\text{Given initial vector } u^{(0)} \text{ such that } \|u^{(0)}\|_\infty=1\\
\hline
y^{(k)}=Au^{(k-1)}\\
\rho_k=\|y^{(k)}\|_\infty\\
u^{(k)}=\dfrac{1}{\rho_k}y^{(k)}\\
\lambda^{(k)}=\dfrac{(u^{(k)})^TAu^{(k)}}{(u^{(k)})^T(u^{(k)})}\ \text{(Rayleigh Quotient)}\\
r^{(k)}=Au^{(k)}-u^{(k)}\lambda^{(k)}\ \text{(Residual)}\\
\text{Converge if } \|r^{(k)}\|_\infty\le\tau\cdot(\|A\|_\infty\|u^{(k)}\|_\infty+\|u^{(k)}\|_\infty|\lambda^{(k)}|) \text{ where } \tau>0 \text{ if predefined tolerance}
\end{array}
$$

其 Matlab 代码为:

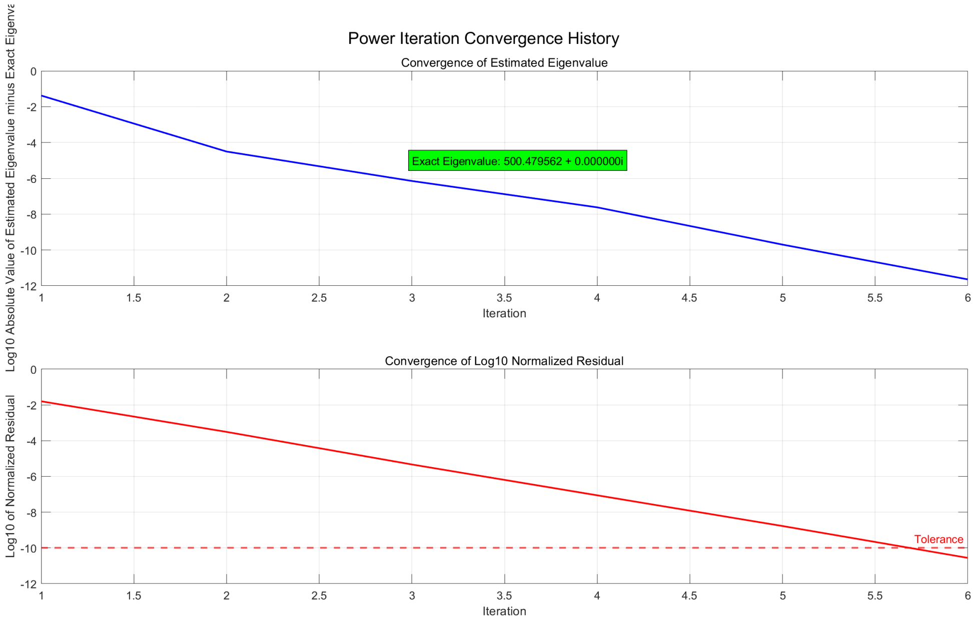
```
function [lambda, u, lambda_history, residual_history] = Power_Iteration(A,
max_iter, tolerance)
    % Computes the dominant eigenvalue and eigenvector of matrix A using the power
iteration method.
    %
    % Inputs:
    %   A - The input matrix (n x n)
    %   max_iter - Maximum number of iterations
    %   tolerance - Convergence tolerance
    %
    % Outputs:
    %   lambda - Estimated dominant eigenvalue
    %   u - Corresponding dominant eigenvector
    %   lambda_history - History of estimated eigenvalues
    %   residual_history - History of normalized residuals

    % Initialize a random vector u
    n = size(A, 1); % Get the dimension of the matrix A
    u = rand(n, 1); % Random initial vector
    u = u / norm(u, inf); % Normalize u such that ||u^(0)||_∞ = 1

    % Initialize histories
    lambda_history = zeros(max_iter, 1);
    residual_history = zeros(max_iter, 1);

    % Power iteration loop
    for k = 1:max_iter
        % Compute y^(k)
        y = A * u;

        % Compute rho_k (the infinity norm of y)
        rho = norm(y, "inf");

        % Update u^(k)
        u = y / rho; % Normalize y to update u

        % Compute the Rayleigh quotient (to estimate the eigenvalue)
        lambda = (u' * A * u) / (u' * u);

        % Compute the residual
        r = A * u - u * lambda;

        % Store histories of current estimate of lambda and normalized residual
        lambda_history(k) = lambda;
        residual_history(k) = norm(r, inf) / (norm(A, inf) * norm(u, inf) + norm(u,
inf) * abs(lambda));

        % Check for convergence
        if residual_history(k) <= tolerance
            fprintf('Power Iteration converged in %d iterations.\n', k);
            break;
        end
    end

    % Trim histories to the actual number of iterations performed
    lambda_history = lambda_history(1:k);
    residual_history = residual_history(1:k);
end
```

函数调用:

```
% Set the random seed for reproducibility
rng(51);

% Define matrix A
n = 1000;
A = rand(n, n);

% Parameters for power iteration
max_iter = 100; % Maximum number of iterations
tolerance = 1e-10; % Convergence tolerance

% Call the power iteration function
[lambda, u, lambda_history, residual_history] = Power_Iteration(A, max_iter,
tolerance);
fprintf('Estimated dominant eigenvalue (Power Iteration): %f + %fi\n', real(lambda),
imag(lambda));

% Use MATLAB's eig() function to compute eigenvalues and eigenvectors
[V, D] = eig(A); % V is the eigenvector matrix, D is the diagonal eigenvalue matrix

% Get the dominant eigenvalue and corresponding eigenvector
[lambda_exact, idx] = max(diag(D)); % Find the maximum eigenvalue
fprintf('Estimated dominant eigenvalue (MATLAB eig): %f + %fi\n',
real(lambda_exact), imag(lambda_exact));

% Plot convergence history
figure;

% Plot estimated eigenvalue history
subplot(2, 1, 1);
plot(1:size(lambda_history), log10(abs(lambda_history - lambda_exact)), 'b-',
'LineWidth', 1.5);
hold on;
title('Convergence of Estimated Eigenvalue');
xlabel('Iteration');
ylabel('Log10 Absolute Value of Estimated Eigenvalue minus Exact Eigenvalue');
grid on;

% Add text box with lambda_exact at a calculated position
text(3, -5, ...
    sprintf('Exact Eigenvalue: %f + %fi', real(lambda_exact), imag(lambda_exact)),
...
    'BackgroundColor', 'green', 'EdgeColor', 'black');

% Plot normalized residual history
subplot(2, 1, 2);
plot(1:size(residual_history), log10(residual_history), 'r-', 'LineWidth', 1.5);
title('Convergence of Log10 Normalized Residual');
xlabel('Iteration');
ylabel('Log10 of Normalized Residual');
grid on;

% Plot tolerance line
yline(log10(tolerance), 'r--', 'Tolerance', 'LineWidth', 1.5); % Tolerance line

% Adjust layout
sgtitle('Power Iteration Convergence History');
```

运行结果:

```
Power Iteration converged in 6 iterations.
Estimated dominant eigenvalue (Power Iteration): 500.479562 + 0.000000i
Estimated dominant eigenvalue (MATLAB eig): 500.479562 + 0.000000i
```



Problem 4

Let A be the 200×200 Hilbert matrix (the (i, j)-entry of A is $a_{ij} := (i+j-1)^{-1}$)
Compute the eigenvalue of A that is closest to 1 using inverse iteration and Rayleigh quotient iteration.
Visualize the convergence history and report the execution time (preferably with detailed profiling)

Solution:

(1) Helper Funtions

构建 Hilbert 矩阵的函数:

```
function A = Generate_Hilbert_Matrix(n)
    % Generate the Hilbert matrix
    A = zeros(n, n);
    for i = 1:n
        for j = 1:n
            A(i, j) = 1 / (i + j - 1);
        end
    end
end
```

部分选主元 Gauss 消去法:

```
function [P, L, U] = Gaussian_Elimination_Partial_Pivoting(A)
    % 获取矩阵的维度
    [n, m] = size(A);
    if n ~= m
        error('矩阵A必须是方阵');
    end
```

```
    % 初始化置换矩阵 P 为单位矩阵
    P = eye(n);

    % 高斯消去过程
    for k = 1:n-1
        % 在第 k 列的 A(k:n, k) 中找到最大值的行索引 p
        [~, p] = max(abs(A(k:n, k)));
        p = p + k - 1; % 调整为在整个矩阵中的行索引

        % 交换第 k 行和第 p 行
        if p ~= k
            A([k, p], :) = A([p, k], :);
            P([k, p], :) = P([p, k], :); % 记录行置换
        end

        % 检查主元是否为零
        if A(k, k) == 0
            error('矩阵是奇异的');
        end

        % Gauss 消去过程：对 A(k+1:n, k) 进行归一化
        A(k+1:n, k) = A(k+1:n, k) / A(k, k);

        % 更新 A(k+1:n, k+1:n)
        A(k+1:n, k+1:n) = A(k+1:n, k+1:n) - A(k+1:n, k) * A(k, k+1:n);
    end

    % 计算 L 和 U 矩阵
    L = tril(A, -1) + eye(n); % L 是单位下三角矩阵
    U = triu(A); % U 是上三角矩阵
end
```

前代法:

```
function y = Forward_Sweep(L, b)
    % 前代法求解 Ly = b
    n = length(b);
    for i = 1:n-1
        b(i) = b(i) / L(i, i);  % 对角线归一化
        b(i+1:n) = b(i+1:n) - b(i) * L(i+1:n, i);  % 消去
    end
    b(n) = b(n) / L(n, n);  % 处理最后一行
    y = b;  % 返回结果
end
```

回代法:

```
function x = Backward_Sweep(U, y)
    % 回代法求解 Ux = y
    n = length(y);
    for i = n:-1:2
        y(i) = y(i) / U(i, i);  % 对角线归一化
        y(1:i-1) = y(1:i-1) - y(i) * U(1:i-1, i);  % 消去
    end
    y(1) = y(1) / U(1, 1);  % 处理第一行
    x = y;  % 返回结果
end
```

基于部分选主元 Gauss 消去法求解非奇异线性方程组的函数:

```
% 求解线性方程组 Ax = b
function x = Solve_Linear_System(A, b)

    % 使用部分主元的 Gaussian 消去法计算 PA = LU
    [P, L, U] = Gaussian_Elimination_Partial_Pivoting(A);

    % 使用前代法求解 Ly = Pb
    y = Forward_Sweep(L, P * b);

    % 使用回代法求解 Ux = y
    x = Backward_Sweep(U, y);
end
```

(2) 反幂法

带位移 μ 的反幂法的迭代格式如下:

$$\begin{array}{c}
\text{Given initial vector } z^{(0)} \text{ such that } \|z^{(0)}\|_\infty = 1 \\
\hline
\text{solve } (A-\mu I)y^{(k)} = z^{(k-1)} \\
\rho_k = \|y^{(k)}\|_\infty \\
z^{(k)} = \dfrac{1}{\rho_k} y^{(k)} \\
\lambda^{(k)} = \dfrac{(z^{(k)})^T A z^{(k)}}{(z^{(k)})^T (z^{(k)})} \text{ (Rayleigh Quotient)} \\
r^{(k)} = Az^{(k)} - z^{(k)}\lambda^{(k)} \text{ (Residual)} \\
\text{Converge if } \|r^{(k)}\|_\infty \leq \tau \cdot (\|A\|_\infty \|z^{(k)}\|_\infty + \|z^{(k)}\|_\infty |\lambda^{(k)}|) \text{ where } \tau > 0 \text{ if predefined tolerance}
\end{array}$$

其中线性方程组 $(A-\mu I)y^{(k)} = z^{(k-1)}$ 的求解可通过事先使用部分选主元的 Gauss 消元法计算 $A-\mu I$ 的 LU 分解,
然后通过前代法和回代法求解两个三角方程组来得到 $y^{(k)}$
其 Matlab 代码为:

```
function [lambda, z, lambda_history, residual_history] = Inverse_Iteration(A, mu,
max_iter, tolerance)
    % Computes the dominant eigenvalue and eigenvector of matrix A using inverse
iteration with a shift.
    %
    % Inputs:
    %   A - The input matrix (n x n)
    %   mu - Shift for inverse iteration
    %   max_iter - Maximum number of iterations
    %   tolerance - Convergence tolerance
    %
    % Outputs:
    %   lambda - Estimated dominant eigenvalue
    %   z - Corresponding dominant eigenvector
    %   lambda_history - History of estimated eigenvalues
    %   residual_history - History of normalized residuals

    n = size(A, 1); % Get the dimension of the matrix A
    z = rand(n, 1); % Random initial vector
```

```
    z = z / norm(z, inf); % Normalize to satisfy ||z^(0)||_∞ = 1

    % Initialize histories
    lambda_history = zeros(max_iter, 1);
    residual_history = zeros(max_iter, 1);

    % Obtain the partial pivoted LU decomposition of (A - mu * I)
    [P, L, U] = Gaussian_Elimination_Partial_Pivoting(A - mu * eye(n));

    % Inverse iteration loop
    for k = 1:max_iter
        % Solve (A - mu * I)y^{(k)} = z^{(k-1)}
        y = Forward_Sweep(L, P * z);
        y = Backward_Sweep(U, y);

        % Compute rho_k (the infinity norm of y)
        rho = norm(y, "inf");

        % Update z^(k)
        z = y / rho; % Normalize y to update z

        % Compute the Rayleigh quotient to estimate the eigenvalue
        lambda = (z' * A * z) / (z' * z);

        % Compute the residual
        r = A * z - z * lambda;

        % Store histories of current estimate of lambda and normalized residual
        lambda_history(k) = lambda;
        residual_history(k) = norm(r, inf) / (norm(A, inf) * norm(z, inf) + norm(z,
inf) * abs(lambda));

        % Check for convergence
        if residual_history(k) <= tolerance
            fprintf('Inverse Iteration converged in %d iterations.\n', k);
            break;
        end
    end

    % Trim histories to the actual number of iterations performed
    lambda_history = lambda_history(1:k);
    residual_history = residual_history(1:k);
end
```

(3) Rayleigh 商迭代

$$\text{Given initial vector } z^{(0)} \text{ such that } \|z^{(0)}\|_\infty = 1, \text{ and } \mu_0 \in \mathbb{C}$$

$$\text{solve } (A - \mu_{k-1} I) y^{(k)} = z^{(k-1)}$$

$$\rho_k = \|y^{(k)}\|_\infty$$

$$z^{(k)} = \frac{1}{\rho_k} y^{(k)}$$

$$\lambda^{(k)} = \frac{(z^{(k)})^T A z^{(k)}}{(z^{(k)})^T (z^{(k)})} \text{ (Rayleigh Quotient)}$$

$$r^{(k)} = A z^{(k)} - z^{(k)} \lambda^{(k)} \text{ (Residual)}$$

$$\text{Converge if } \|r^{(k)}\|_\infty \leq \tau \cdot (\|A\|_\infty \|z^{(k)}\|_\infty + \|z^{(k)}\|_\infty |\lambda^{(k)}|) \text{ where } \tau > 0 \text{ if predefined tolerance}$$

$$\mu_k = \lambda^{(k)}$$

其中线性方程组 $(A - \mu_{k-1} I) y^{(k)} = z^{(k-1)}$ 的求解可通过部分选主元的 Gauss 消元法计算 $A - \mu_{k-1} I$ 的 LU 分解，
然后通过前代法和回代法求解两个三角方程组来得到 $y^{(k)}$

```
function [lambda, z, lambda_history, residual_history] =
Rayleigh_Quotient_Iteration(A, mu0, max_iter, tolerance)
    % Computes the dominant eigenvalue and eigenvector of matrix A using Rayleigh
Quotient Iteration.
    %
    % Inputs:
    %   A - The input matrix (n x n)
    %   mu0 - Initial shift for the first iteration
    %   max_iter - Maximum number of iterations
    %   tolerance - Convergence tolerance
    %
    % Outputs:
    %   lambda - Estimated dominant eigenvalue
    %   z - Corresponding dominant eigenvector
    %   lambda_history - History of estimated eigenvalues
    %   residual_history - History of normalized residuals

    n = size(A, 1); % Get the dimension of the matrix A
    z = rand(n, 1); % Random initial vector
    z = z / norm(z, inf); % Normalize to satisfy ||z^(0)||_∞ = 1

    % Initialize histories
    lambda_history = zeros(max_iter, 1);
    residual_history = zeros(max_iter, 1);

    % Set initial shift
    mu = mu0;

    % Rayleigh Quotient Iteration loop
    for k = 1:max_iter
        % Solve (A - mu * I)y^{(k)} = z^{(k-1)}
        [P, L, U] = Gaussian_Elimination_Partial_Pivoting(A - mu * eye(n));
        y = Forward_Sweep(L, P * z);
        y = Backward_Sweep(U, y);

        % Compute rho_k (the infinity norm of y)
        rho = norm(y, "inf");

        % Update z^(k)
        z = y / rho; % Normalize y to update z
```

```
        % Compute the Rayleigh quotient to estimate the eigenvalue
        lambda = (z' * A * z) / (z' * z);

        % Compute the residual
        r = A * z - z * lambda;

        % Store histories of current estimate of lambda and normalized residual
        lambda_history(k) = lambda;
        residual_history(k) = norm(r, inf) / (norm(A, inf) * norm(z, inf) + norm(z,
inf) * abs(lambda));

        % Check for convergence
        if residual_history(k) <= tolerance
            fprintf('Rayleigh Quotient Iteration converged in %d iterations.\n', k);
            break;
        end

        % Update mu for the next iteration
        mu = lambda; % Set mu to the last estimated eigenvalue
    end

    % Trim histories to the actual number of iterations performed
    lambda_history = lambda_history(1:k);
    residual_history = residual_history(1:k);
end
```

(4) 函数调用

我们对 200 阶的 Hilbert 矩阵应用位移为 $\mu = 1$ 的反幂法和初始位移为 $\mu_0 = 1$ 的 Rayleigh 商迭代法，期望能够找到与 1 最接近的特征值.
作为对比，我们使用 Matlab 的 `eig()` 函数的计算结果作为精确解.
函数调用:

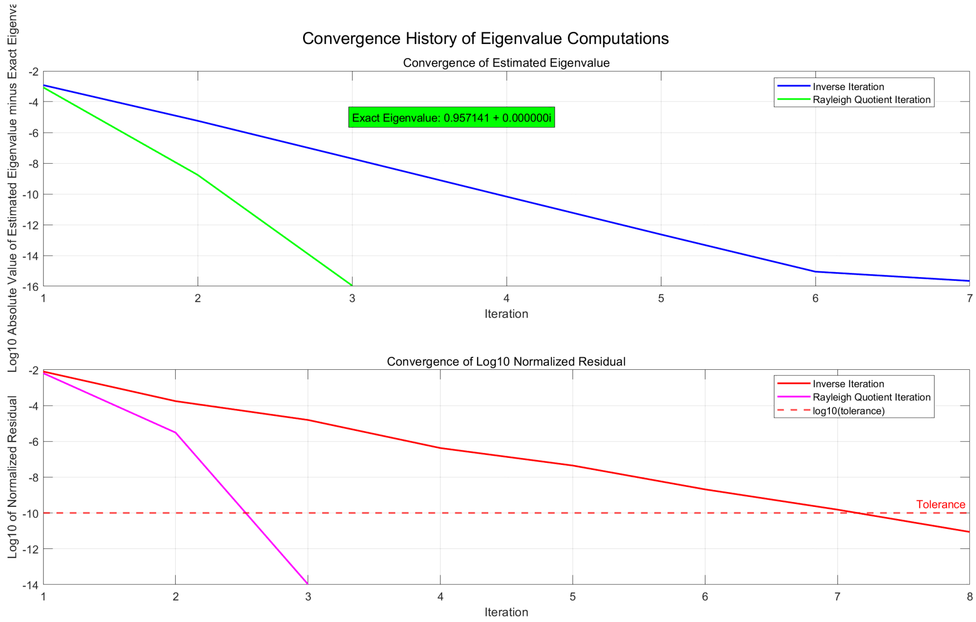
```
% Define parameters
n = 200; % Size of the Hilbert matrix
mu = 1;  % Shift for inverse iteration
max_iter = 100; % Maximum number of iterations
tolerance = 1e-10; % Convergence tolerance

% Generate the Hilbert matrix
A = Generate_Hilbert_Matrix(n);

% Use MATLAB's eig() function to find the closest eigenvalue to mu
[V, D] = eig(A);
[~, idx] = min(abs(diag(D - mu * eye(size(D, 1)))));
lambda_exact = D(idx, idx);
fprintf('Estimated dominant eigenvalue (MATLAB eig): %f + %fi\n',
real(lambda_exact), imag(lambda_exact));

% Run inverse iteration
[lambda_inverse, z_inverse, lambda_history_inverse, residual_history_inverse] =
Inverse_Iteration(A, mu, max_iter, tolerance);
fprintf('Estimated dominant eigenvalue (Inverse Iteration): %f + %fi\n', ...
         real(lambda_inverse), imag(lambda_inverse));
```

```
% Run Rayleigh Quotient iteration
[lambda_Rayleigh, z_Rayleigh, lambda_history_Rayleigh, residual_history_Rayleigh] =
Rayleigh_Quotient_Iteration(A, mu, max_iter, tolerance);
fprintf('Estimated dominant eigenvalue (Inverse Iteration): %f + %fi\n', ...
         real(lambda_Rayleigh), imag(lambda_Rayleigh));

% Plot convergence history
figure;

% Plot estimated eigenvalue history for Inverse Iteration
subplot(2, 1, 1);
plot(1:length(lambda_history_inverse), log10(abs(lambda_history_inverse -
lambda_exact)), 'b-', 'LineWidth', 1.5);
hold on;
plot(1:length(lambda_history_Rayleigh), log10(abs(lambda_history_Rayleigh -
lambda_exact)), 'g-', 'LineWidth', 1.5);
title('Convergence of Estimated Eigenvalue');
xlabel('Iteration');
ylabel('Log10 Absolute Value of Estimated Eigenvalue minus Exact Eigenvalue');
grid on;
legend('Inverse Iteration', 'Rayleigh Quotient Iteration', 'Location', 'best');

% Add text box with lambda_exact at a calculated position
text(3, -5, ...
    sprintf('Exact Eigenvalue: %f + %fi', real(lambda_exact), imag(lambda_exact)),
...
    'BackgroundColor', 'green', 'EdgeColor', 'black');

% Plot normalized residual history for Inverse Iteration
subplot(2, 1, 2);
plot(1:length(residual_history_inverse), log10(residual_history_inverse), 'r-',
'LineWidth', 1.5);
hold on;
plot(1:length(residual_history_Rayleigh), log10(residual_history_Rayleigh), 'm-',
'LineWidth', 1.5);
title('Convergence of Log10 Normalized Residual');
xlabel('Iteration');
ylabel('Log10 of Normalized Residual');
grid on;

% Plot tolerance line
yline(log10(tolerance), 'r--', 'Tolerance', 'LineWidth', 1.5); % Tolerance line
legend('Inverse Iteration', 'Rayleigh Quotient Iteration', 'log10(tolerance)' ,
'Location', 'best');

% Adjust layout
sgtitle('Convergence History of Eigenvalue Computations');
```

运行结果:

```
Estimated dominant eigenvalue (MATLAB eig): 0.957141 + 0.000000i
Inverse Iteration converged in 8 iterations.
Estimated dominant eigenvalue (Inverse Iteration): 0.957141 + 0.000000i
Rayleigh Quotient Iteration converged in 3 iterations.
Estimated dominant eigenvalue (Inverse Iteration): 0.957141 + 0.000000i
```

从收敛历史图像中可以看出:
反幂法全局线性收敛，而 Rayleigh 商迭代局部超线性收敛 (具体来说是三次收敛，因为 Hilbert 阵是对称阵)



Problem 5

Write a program to compute all eigenvectors of an upper triangular matrix with distinct diagonal entries.
Compare your result with the one produced by a math library (e.g., `eig()` in MATLAB/Octave)

Solution:
考虑 $n=4$ 的情况，假设我们想计算 $(2,2)$ 位置的特征值 $\lambda_2 = T(2,2)$ 的特征向量，则我们可以这样做:

$$\begin{aligned}
&\underline{\text{Solve } (T-\lambda_2 I)x = 0_n} \\
&x = (T-\lambda_2 I)^{-1} 0_n \\
&= \begin{bmatrix} * & * & * & * \\ & 0 & * & * \\ & & * & * \\ & & & * \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \infty \\ \infty \\ 0 \\ 0 \end{bmatrix} (\text{normalize}) \propto \begin{bmatrix} * \\ 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

根据这个思路我们可以将右端的零向量的第二个元素置为 1，将 $T(2,2)$ 置为 1，即求解:

$$x = \begin{bmatrix} * & * & * & * \\ & 1 & * & * \\ & & * & * \\ & & & * \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

注意到这实际上相当于求解一个更小阶数的上三角方程组，最后在解的后面补 0 得到原三角方程组的解:

$$\tilde{x} = \begin{bmatrix} * & * \\ & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} * \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} \tilde{x} \\ 0_2 \end{bmatrix} = \begin{bmatrix} * \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

将上述思路一般化，我们便得到如下算法:

```
function [eigenvectors, eigenvalues] = Upper_Triangular_Eigenvectors(T)
    % Upper_Triangular_Eigenvectors computes the eigenvectors of an upper triangular
matrix T.
    %
    % Inputs:
    %   T - an n x n upper triangular matrix
    %
    % Outputs:
    %   eigenvector - a matrix containing the normalized eigenvectors as columns
    %   eigenvalue - a vector containing the eigenvalues on the diagonal of T

    n = size(T, 1); % Get the size of the matrix (number of rows)
    eigenvalues = diag(T); % Extract the eigenvalues from the diagonal of T
    eigenvectors = zeros(n, n); % Initialize the matrix to store eigenvectors
    eigenvectors(1, 1) = 1; % Set the first entry of the first eigenvector

    for i = 2:n
        % Create the matrix U by subtracting the eigenvalue from T
        U = T(1:i, 1:i) - eigenvalues(i) * eye(i);
        U(i, i) = 1; % Set the pivot element to 1 to facilitate back substitution

        % Create the right-hand side vector b with random values and set the last
entry to 1
        b = [zeros(i-1, 1); 1];

        % Solve the linear system U * x = b using backward substitution
        x = Backward_Sweep(U, b);

        % Normalize the eigenvector and store it
        eigenvectors(1:i, i) = x / norm(x);
    end
end
```

函数调用:

```
rng(51);
n = 100;
T = triu(rand(n, n) + 1i * rand(n, n));

% Call the function to compute eigenvalues and eigenvectors
[eigenvectors, eigenvalues] = Upper_Triangular_Eigenvectors(T);

% Display the results
disp('Frobenius norm of T * eigenvectors - eigenvectors * eigenvalues')
residual = T * eigenvectors - eigenvectors * diag(eigenvalues);
disp(norm(residual, "fro"));
```

运行结果:

```
Frobenius norm of T * eigenvectors - eigenvectors * eigenvalues
   2.6505e-15
```

Problem 6 (optional)

Investigate the behavior of the power method applied to the following matrices:
(where $\lambda \in \mathbb{C}$ is a given constant)

$$A = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} \quad B = \begin{bmatrix} \lambda & 1 \\ & -\lambda \end{bmatrix}$$

Solution:
首先我们观察到:
(特征向量的求解可以利用 Problem 5 的算法)

- $A = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}$ 作为一个 2 阶 Jordan 块，其特征值为 λ，代数重数为 2，几何重数为 1，特征向量为 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $B = \begin{bmatrix} \lambda & 1 \\ & -\lambda \end{bmatrix}$ 作为一个上三角阵，其特征值为 λ 和 $-\lambda$
  当 $\lambda = 0$ 时，就归结为 A 的情况，我们不赘述了;
  当 $\lambda \neq 0$ 时，其特征值 λ 和 $-\lambda$ 都是单重的，特征向量分别为 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 和 $\begin{bmatrix} -\frac{1}{\lambda} \\ 1 \end{bmatrix}$(或者说 $\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$)

容易归纳证明 $A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ & \lambda^k \end{bmatrix}$ $(\forall\, k \in \mathbb{Z}_+)$:

$$\frac{A = \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix}}{A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ & \lambda^k \end{bmatrix} \Rightarrow A^{k+1} = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ & \lambda^k \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ & \lambda^{k+1} \end{bmatrix}}$$

任意给定的非零初始向量 $u^{(0)} = [a, b]^T$ $(a, b \in \mathbb{C})$ 我们都有:

$$A^k u^{(0)} = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ & \lambda^k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda^k a + k\lambda^{k-1} b \\ \lambda^k b \end{bmatrix} = \lambda^k \begin{bmatrix} a + \frac{k}{\lambda} b \\ b \end{bmatrix} \ (\forall\, k \in \mathbb{Z}_+)$$

若 $b = 0$ (根据初始向量非零的假设我们有 $a \neq 0$)，则初始向量天然就是 A 关于特征值 λ 的特征向量.
若 $b \neq 0$，则当 $k \to \infty$ 时，$\frac{1}{\lambda^k} A^k u^{(0)} \to \begin{bmatrix} \infty \\ b \end{bmatrix}$，即方向趋近于与 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 平行.
因此对 A 应用幂法，无论使用什么非零初始向量，在方向上都是收敛的.

下面考虑 $B = \begin{bmatrix} \lambda & 1 \\ & -\lambda \end{bmatrix}$ $(\lambda \neq 0)$ 的情况:

$$\frac{B = \begin{bmatrix} \lambda & 1 \\ & -\lambda \end{bmatrix} \quad B^2 = \begin{bmatrix} \lambda^2 & 0 \\ & \lambda^2 \end{bmatrix}}{B^{2k-1} = \begin{bmatrix} \lambda^{2k-1} & \lambda^{2k-2} \\ & -\lambda^{2k-1} \end{bmatrix}}$$

$$\Rightarrow$$

$$B^{2k} = \begin{bmatrix} \lambda^{2k-1} & \lambda^{2k-2} \\ & -\lambda^{2k-1} \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ & -\lambda \end{bmatrix} = \begin{bmatrix} \lambda^{2k} & 0 \\ & \lambda^{2k} \end{bmatrix}$$

$$\Rightarrow$$

$$B^{2k+1} = \begin{bmatrix} \lambda^{2k} & 0 \\ & \lambda^{2k} \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ & -\lambda \end{bmatrix} = \begin{bmatrix} \lambda^{2k+1} & \lambda^{2k} \\ & -\lambda^{2k+1} \end{bmatrix}$$

容易归纳得到 $B^m = \begin{bmatrix} \lambda^m & \frac{1}{2}[1-(-1)^m]\lambda^{m-1} \\ & (-1)^m\lambda^m \end{bmatrix} \ (\forall\, m \in \mathbb{Z}_+)$
我们将其分为两个子列讨论，分别是 $\{B^{2k-1} := \begin{bmatrix} \lambda^{2k-1} & \lambda^{2k-2} \\ & -\lambda^{2k-1} \end{bmatrix}\}$ 和 $\{B^{2k} := \begin{bmatrix} \lambda^{2k} & 0 \\ & \lambda^{2k} \end{bmatrix}\}$
任意给定的初始向量 $u^{(0)} = [a,b]^T \ (a,b \in \mathbb{C})$，则我们有:

- 对于第一个子列:

$$B^{2k-1}u^{(0)} = \begin{bmatrix} \lambda^{2k-1} & \lambda^{2k-2} \\ & -\lambda^{2k-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda^{2k-1}a + \lambda^{2k-2}b \\ -\lambda^{2k-1}b \end{bmatrix} = \lambda^{2k-2} \begin{bmatrix} \lambda a + b \\ -\lambda b \end{bmatrix}$$

  我们发现对于任意 $k \in \mathbb{Z}_+$，向量 $B^{2k-1}u^{(0)}$ 都平行于 $\begin{bmatrix} \lambda a + b \\ -\lambda b \end{bmatrix}$ (特征向量 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 和 $\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$ 的线性组合)

- 对于第二个子列:

$$B^{2k}u^{(0)} = \begin{bmatrix} \lambda^{2k} & 0 \\ & \lambda^{2k} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda^{2k}a \\ \lambda^{2k}b \end{bmatrix} = \lambda^{2k} \begin{bmatrix} a \\ b \end{bmatrix}$$

  我们发现对于任意 $k \in \mathbb{Z}_+$，向量 $B^{2k}u^{(0)}$ 都平行于 $\begin{bmatrix} a \\ b \end{bmatrix}$ (特征向量 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 和 $\begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$ 的线性组合)

容易验证 $\begin{bmatrix} \lambda a + b \\ -\lambda b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix}$ 在 $a \neq 0$ 的情况下是有解的，其解是 $\begin{cases} c = \frac{b}{2a} \\ \lambda = -\frac{b}{2a} \end{cases}$
因此如果我们选取的初始向量 $u^{(0)} = [a,b]^T \ (a,b \in \mathbb{C})$ 满足 $b = -2a\lambda$,
那么两个子列的极限向量是平行的，此时幂法的序列在方向上是收敛的.

但在其他情况下，这两个子列的极限向量不平行，导致幂法的序列在方向上发散.
此时我们可以使用**子空间迭代法**，使用 $X^{(0)} = [u^{(0)}, Bu^{(0)}]$ 作为初始向量组
最终收敛得到的 $X_{\text{converge}} \in \mathbb{C}^{n\times 2}$ 满足 $\text{span}\{X_{\text{converge}}\} \approx \text{span}\{p_1, p_2\}$
其中特征向量 $p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 和 $p_2 = \begin{bmatrix} 1 \\ -\lambda \end{bmatrix}$
我们最终可以从 X_{converge} 的列空间中分离出特征向量 p_1, p_2，这里不赘述了.

Problem 7 (optional)

Implement the naive QR algorithm and visualize the componentwise convergence of a small example.

Solution:
显式 QR 迭代格式为:

$$\frac{\text{Given } A_0 = A \in \mathbb{C}^{n\times n}}{\begin{array}{c} A_{k-1} = Q_k R_k \\ A_k = R_k Q_k \end{array}}$$

我们将收敛准则设为 A 的对角元的变化幅度小于某一阈值时停止迭代，就得到如下算法:

```
function [A_history, Q_history, R_history] = naive_QR(A, max_iter, tolerance)
    % naive_QR performs the naive QR iteration on matrix A
    %
    % Inputs:
    %   A - an n x n matrix
    %   max_iter - maximum number of iterations
    %
    % Outputs:
```

```
    %   A - the final matrix after max_iter iterations
    %   Q_history - history of Q matrices
    %   R_history - history of R matrices

    % Initialize
    n = size(A, 1);
    A_history = zeros(n, n, max_iter); % To store R matrices
    Q_history = zeros(n, n, max_iter); % To store Q matrices
    R_history = zeros(n, n, max_iter); % To store R matrices

    A_history(:, :, 1) = A;

    for k = 1:max_iter
        % Perform QR factorization
        [Q, R] = qr(A_history(:, :, k));

        % Store Q and R
        Q_history(:, :, k) = Q;
        R_history(:, :, k) = R;

        % Update A_k for the next iteration
        A_history(:, :, k+1) = R * Q;

        if norm(diag(A_history(:, :, k+1)) - diag(A_history(:, :, k)), "fro") /
norm(diag(A_history(:, :, k+1)), "fro") < tolerance
            fprintf("naive QR converge at %d-th iteration\n", k);
            break;
        end
    end

    A_history = A_history(:, :, 1:k+1);
    Q_history = Q_history(:, :, 1:k);
    R_history = R_history(:, :, 1:k);
end
```

函数调用:

```
rng(51);
n = 5;
A = randn(n, n) + 1i * randn(n, n);
max_iter = 50; % Maximum number of iterations
tolerance = 1e-2;

% Perform naive QR iteration
[A_history, Q_history, R_history] = naive_QR(A, max_iter, tolerance);

% Prepare for visualization of convergence
n = size(A, 1);
convergence_history = zeros(size(A_history, 3), n);

% Store the diagonal elements of A_k at each iteration
for k = 1:size(A_history, 3)
    convergence_history(k, :) = diag(A_history(:, :, k)); % Diagonal of A_k
end

% Compute exact eigenvalues
exact_eigenvalues = eig(A);
sorted_exact_eigenvalues = zeros(size(exact_eigenvalues, 1), 1);
for i = 1:size(exact_eigenvalues)
```

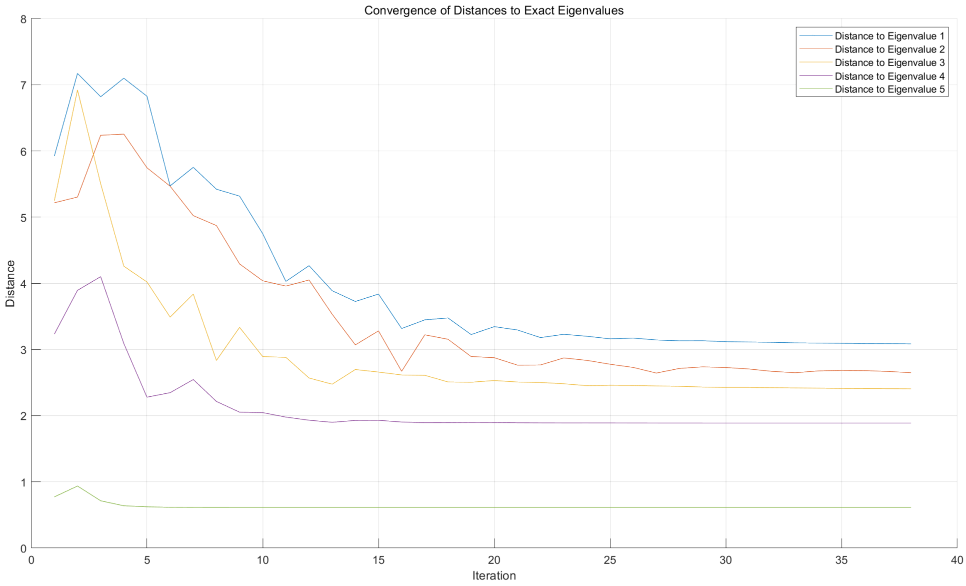
```
    [~, idx] = min(abs(convergence_history(end, :) - exact_eigenvalues(i)));
    sorted_exact_eigenvalues(idx) = exact_eigenvalues(i);
end

% Plotting the convergence of eigenvalues in the complex plane
figure;
hold on;
colors = lines(n); % Generate distinct colors for each eigenvalue

for i = 1:n
    % Plot the convergence path for each eigenvalue
    plot(real(convergence_history(:, i)), imag(convergence_history(:, i)), 'x-', ...
        'DisplayName', sprintf('Eigenvalue %d', i), 'Color', colors(i, :), ...
        'LineWidth', 1, 'MarkerSize', 5);

    % Plot the exact eigenvalue
    plot(real(sorted_exact_eigenvalues(i)), imag(sorted_exact_eigenvalues(i)), 'ro',
'MarkerSize', 10, ...
        'DisplayName', sprintf('Exact Eigenvalue %d', i));
end

hold off;
xlabel('Real Part');
ylabel('Imaginary Part');
title('Componentwise Convergence of QR Algorithm in Complex Plane');
legend show;
grid on;
axis equal; % Equal scaling for both axes

% Calculate the distance to exact eigenvalues
distance_history = zeros(size(convergence_history));

for k = 1:size(convergence_history, 1)
    distance_history(k, :) = abs(convergence_history(k, :) -
sorted_exact_eigenvalues.') + abs(sorted_exact_eigenvalues.');
end

% Plotting the distance convergence
figure;
hold on;
for i = 1:n
    plot(1:size(A_history, 3), distance_history(:, i), 'DisplayName',
sprintf('Distance to Eigenvalue %d', i), 'Color', colors(i, :));
end

hold off;
xlabel('Iteration');
ylabel('Distance');
title('Convergence of Distances to Exact Eigenvalues');
legend show;
grid on;
```

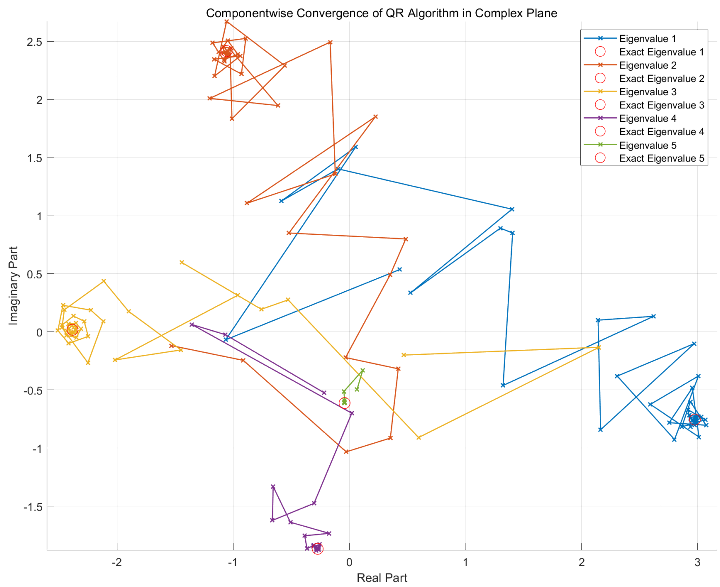
运行结果:

(存疑: 似乎与 "QR 迭代更倾向于优先收敛于模最大特征值" 的事实不符)

(可能是跟特征值分离度有关，模最小的特征值和模第二小的特征值分离度最大)

```
naive QR converge at 37-th iteration
```


Convergence of Distances to Exact Eigenvalues
Distance to Eigenvalue 1
Distance to Eigenvalue 2
Distance to Eigenvalue 3
Distance to Eigenvalue 4
Distance to Eigenvalue 5
Distance
Iteration


Componentwise Convergence of QR Algorithm in Complex Plane
Eigenvalue 1
Exact Eigenvalue 1
Eigenvalue 2
Exact Eigenvalue 2
Eigenvalue 3
Exact Eigenvalue 3
Eigenvalue 4
Exact Eigenvalue 4
Eigenvalue 5
Exact Eigenvalue 5
Imaginary Part
Real Part