Introduction to Stochastic Process

Time: 18:30-21:05

Problem 1

[**Sheldon M. Ross: Introduction to Probability Models** Exercise 24 (p169) & Exercise 51(p92)]
A coin, having probability p of landing on hea\mathrm{d}s,
is continually flipped until at least one head and one tail have been obtained.
The flips of the coin are made independently.

(1) Find the expected number of flips needed. [10 marks]

- **Lemma:**
  Let X be a random variable that follows a geometric distribution with success probability p, denoted as $X \sim \mathrm{Geo}(p)$.
  The probability mass function (pmf) of X is given by:

$$\mathrm{pmf}(i) = (1-p)^{i-1}p \quad (i = 1, 2, \dots)$$

  Therefore the expectation of X is:

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{i=1}^{\infty} i \cdot \mathrm{pmf}(i) \\
&= \sum_{i=1}^{\infty} i \cdot (1-p)^{i-1}p \\
&= p\sum_{i=1}^{\infty} i \cdot q^{i-1} \qquad (q := 1-p) \\
&= p\sum_{i=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}q}(q^i) \\
&= p\frac{\mathrm{d}}{\mathrm{d}q}\left(\sum_{i=1}^{\infty} q^i\right) \\
&= p\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{q}{1-q}\right) \\
&= \frac{p}{(1-q)^2} \\
&= \frac{1}{p}
\end{aligned}$$

Solution: [(Reference)]
Consider the two scenarios based on the first coin toss result:

- **First toss is a head (probability p):** We then wait for the first tail.
- **First toss is a tail (probability $1-p$):** We then wait for the first head.

Denote N as the total number of flips required.

- The conditional distribution $(N-1|\text{first toss is a head})$
  follows a Geometric distribution with success probability $1-p$,
  thus $\mathbb{E}[N-1|\text{first toss is a head}] = \frac{1}{1-p}$.

- The conditional distribution $(N-1|\text{first toss is a tail})$ follows a Geometric distribution with success probability p, thus $\mathbb{E}[N-1|\text{first toss is a tail}] = \frac{1}{p}$.

Therefore, the expected total number of flips can be calculated as follows:

$$\begin{aligned}\mathbb{E}[N] &= 1+\mathbb{E}[N-1] \\ &= 1+p\cdot\mathbb{E}[N-1|\text{first toss is a head}]+(1-p)\cdot\mathbb{E}[N-1|\text{first toss is a tail}] \\ &= 1+\frac{p}{1-p}+\frac{1-p}{p} \\ &= \frac{1}{p(1-p)}-1\end{aligned}$$

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(2) Find the expected number of flips that land on hea\mathrm{d}s. [10 marks]

Solution: (Reference)
Consider the two scenarios based on the first coin toss result:

- **First toss is a head (probability p):** the outcome must be $HH\cdots HT$
- **First toss is a tail (probability $1-p$):** the outcome must be $TT\cdots TH$

Denote N' as the number of flips that land on hea\mathrm{d}s after the first toss.

- The conditional distribution $(N'+1|\text{first toss is a head})$ follows a Geometric distribution with success probability $1-p$, thus $\mathbb{E}[N'|\text{first toss is a head}] = \frac{1}{1-p}-1$.
- The conditional distribution $(N'|\text{first toss is a tail}) \equiv 1$ follows a one-point distribution at 1 thus $\mathbb{E}[N'|\text{first toss is a tail}] = 1$

Therefore, the expected number of flips that land on hea\mathrm{d}s can be calculated as follows:

$$\begin{aligned}\mathbb{E}[N] &= p\cdot[1+\mathbb{E}[N'|\text{first toss is a head}]]+(1-p)\cdot[0+\mathbb{E}[N'|\text{first toss is a tail}]] \\ &= p\cdot(1+\frac{1}{1-p}-1)+(1-p)\cdot 1 \\ &= \frac{p}{1-p}+(1-p)\end{aligned}$$

Problem 2

Suppose $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)$, find the distribution of $X_1|_{X_2=x_2}$.

Solution:
(Deriving the conditional distributions of a multivariate normal distribution)
Denote $Z := X_1 + AX_2$ (assume that Σ_{22} is nonsingular).

- First, we choose A so that Z and X_2 are independent:

$$\begin{aligned}\mathrm{Cov}(Z,X_2) &= \mathrm{Cov}(X_1+AX_2,X_2) \\ &= \mathrm{Cov}(X_1,X_2)+A\mathrm{Cov}(X_2,X_2) \\ &= \Sigma_{12}+A\Sigma_{22} \\ &= 0\end{aligned}$$

which yields $A=-\Sigma_{12}\Sigma_{22}^{-1}$ (this is actually block Gaussian elimination).
Therefore Z and X_2 are uncorrelated.
Since they are jointly Gaussian, uncorrelatedness implies independence.

- Secondly, determine the expectation of $X_1|_{X_2=x_2}$:

$$\begin{aligned}\mathbb{E}[X_1|X_2=x_2]&=\mathbb{E}[Z-AX_2|X_2=x_2]\\&=\mathbb{E}[Z|X_2=x_2]-Ax_2\\&=\mathbb{E}[Z]-Ax_2\\&=(\mu_1+A\mu_2)-Ax_2\\&=\mu_1+A(\mu_2-x_2)\\&=\mu_1-\Sigma_{12}\Sigma_{22}^{-1}(\mu_2-x_2)\end{aligned}$$

- Thirdly, determine the variance of $X_1|_{X_2=x_2}$:

$$\begin{aligned}\mathrm{Var}(X_1|X_2=x_2)&=\mathrm{Var}(Z-AX_2|X_2=x_2)\\&=\mathrm{Var}(Z|X_2=x_2)\\&=\mathrm{Var}(Z)\\&=\mathrm{Var}(X_1+AX_2)\\&=\mathrm{Var}(X_1)+\mathrm{Var}(AX_2)+\mathrm{Cov}(X_1,AX_2)+\mathrm{Cov}(AX_2,X_1)\\&=\Sigma_{11}+A\Sigma_{22}A^{\mathrm{T}}+\Sigma_{12}A^{\mathrm{T}}+A\Sigma_{21}\\&=\Sigma_{11}+\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\\&=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\end{aligned}$$

  which is the **Schur complement** of $(1,1)$ block of $\Sigma=\begin{bmatrix}\Sigma_{11}&\Sigma_{12}\\\Sigma_{21}&\Sigma_{22}\end{bmatrix}$

Therefore, $X_1|_{X_2=x_2}\sim N(\mu_1-\Sigma_{12}\Sigma_{22}^{-1}(\mu_2-x_2),\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

Problem 3

[**Sheldon M. Ross: Introduction to Probability Models** Example 5.18 (P319)]
Assume that the number of cars that enters into a parking lot in the interval $[0,t]$
follows a Poisson process with rate λ.
Moreover, assume the parking time of each car follows the distribution with $\mathrm{CDF}\ G(x)$
and is independent of its arrival time.
Find the distribution of the number of cars that drive out of the parking lot in the interval $[0,t]$.

- **Lemma: (Poisson 过程的拆分, 原书命题 5.4)**
  考虑一个参数为 $\lambda>0$ 的 Poisson 过程 $\mathbf{N}=\{N(t):t\geq 0\}$:
  假设有 k 种可能类型的事件,
  而一个事件被分类为类型 i $(i=1,\dots,k)$ 事件的概率依赖于事件发生的时间.
  特别地，我们假设若一个事件在时刻 t 发生,
  则它将以概率 $p_i(t)$ 被分类到类型 i $(i=1,\dots,k)$ (独立于以前发生的任何事件),
  其中 $\{p_i(t)\}_{i=1}^k$ 满足 $\sum_{i=1}^k p_i(t)=1$
  记 $N_i(t)$ $(i=1,\dots,k)$ 为到时刻 t 为止类型 i 事件发生的个数,
  则 $N_i(t)$ $(i=1,\dots,k)$ 是具有均值 $\mathbb{E}[N_i(t)]=\lambda\int_0^t p_i(t)\mathrm{d}s$ 的**独立 Poisson 随机变量**,
  即 $N_i(t)\sim\mathrm{Poisson}(\lambda\int_0^t p_i(t)\mathrm{d}s)$ 且它们相互独立.
  这样我们就拆分得到了一系列非齐次 Poisson 过程 $\{N_i(t):t\geq 0\}$.

Solution:
对于任意给定的 t，我们将进入停车场的车辆分为两类:

- 第一类: 在时刻 t 之前 (包含时刻 t) 离开了停车场;

- 第二类: 在时刻 t 之前 (包含时刻 t) 没有离开停车场;

现在考虑一个在时刻 $s \leq t$ 进入停车场的车辆.
如果它的停车时间小于等于 $t-s$，则它属于第一类，其概率为 $G(t-s)$;
否则，它属于第二类，其概率为 $1-G(t-s)$.

根据引理，我们知道:

- $[0,t]$ 时间内驶离停车场的车辆数 (即第一类车辆数) $X(t)$ 的分布是
均值为 $\mathbb{E}[X(t)] = \lambda \int_0^t G(t-s)\mathrm{d}s = -\lambda \int_t^0 G(x)\mathrm{d}x = \lambda \int_0^t G(x)\mathrm{d}x$ 的 Poisson 分布.
- 类似地，在时刻 t 仍在停车的车辆数 (即第二类车辆数) $Y(t)$ 的分布是
均值为 $\mathbb{E}[Y(t)] = \lambda \int_0^t [1-G(t-s)]\mathrm{d}s = \lambda t - \lambda \int_0^t G(y)\mathrm{d}y$ 的 Poisson 分布.
- 而且 $X(t) \perp Y(t)$

Problem 4

Denote by $\mathbf{X} = \{X(t) : t \geq 0\}$ a compound Poisson process.
Compute $\operatorname{Cov}(X(s), X(t))$.

- **Lemma:**
设 $X_1, X_2, \ldots$ 是一列独立同分布的随机变量，具有均值 $\mathrm{E}[X] = \mu$ 和方差 $\operatorname{Var}(X) = \sigma^2$
假设它们与取非负整数值的随机变量 N 独立.
我们称 $S = \sum_{i=1}^N X_i$ 为**复合随机变量**，有 $\begin{cases} \mathrm{E}[S] = \mu \mathrm{E}[N] \\ \operatorname{Var}[S] = \sigma^2 \mathrm{E}[N] + \mu^2 \operatorname{Var}(N) \end{cases}$ 成立.

Solution:
设 $X(t)$ 具有形式 $X(t) = \sum_{i=1}^{N(t)} \xi_i$
其中 $\mathbf{N} = \{N(t) : t \geq 0\}$ 是均值为 $\lambda > 0$ 的 Poisson 过程,
而 $\{\xi_i\}$ 是一列均值为 μ，方差为 σ^2 的独立同分布的随机变量，且与 $\mathbf{N}$ 独立.
不失一般性地，假设 $0 < s < t$.

$$
\begin{aligned}
\operatorname{Cov}(X(s), X(t)) &= \operatorname{Cov}(X(s), X(t) - X(s) + X(s)) \\
&= \operatorname{Cov}(X(s), X(t) - X(s)) + \operatorname{Cov}(X(s), X(s)) \quad (\text{note that } X(s) \perp X(t) - X(s)) \\
&= 0 + \operatorname{Var}(X(s)) \\
&= \mathbb{E}\left[\operatorname{Var}\left(\sum_{i=1}^{N(s)} \xi_i \,\Big|\, \mathbf{N}\right)\right] + \operatorname{Var}\left[\mathbb{E}\left[\sum_{i=1}^{N(s)} \xi_i \,\Big|\, \mathbf{N}\right]\right] \\
&= \mathbb{E}[N(s)\sigma^2] + \operatorname{Var}[N(s)\mu] \\
&= \lambda s \cdot \sigma^2 + \lambda s \cdot \mu^2 \\
&= (\mu^2 + \sigma^2)\lambda s
\end{aligned}
$$

Problem 5

Consider independent tosses of a fair die.
Let X_n be the maximum of numbers appearing in the first n throws, $n = 1, 2, \ldots$

(a) Verify that $\{X_n : n \geq 1\}$ is a Markov chain.
Describe the state space and the transition probability matrix of this homogeneous Markov Chain.

Solution:
记第 n 次投掷骰子的结果为 ξ_n，其分布为 $\mathrm{P}\{\xi_n = i\} = \frac{1}{6} \ \ (i = 1,2,3,4,5,6)$
则对于任意 $n \in \mathbb{N}$ 都有 $X_{n+1} = \max\{\xi_1,\ldots,\xi_n,\xi_{n+1}\} = \max\{X_n,\xi_{n+1}\}$
显然有 $\mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n,\ldots,X_1 = x_1\} = \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n\}$ 成立
说明 $\{X_n : n \geq 1\}$ 具有 Markov 性，因此是一个 Markov 链.

其状态空间为 $\{1,2,3,4,5,6\}$，考虑转移概率:

- 当前 n 次投掷处于最大值 m 时，下一步仍停在 m 的概率为:

$$\mathrm{P}_{m,m} = \mathrm{P}\{\xi_{n+1} \leq m\} = \frac{m}{6}$$

- 当前 n 次投掷处于最大值 m 时，下一步变为 $j > m$ 的概率为:

$$\mathrm{P}_{m,m} = \mathrm{P}\{\xi_{n+1} = j\} = \frac{1}{6}$$

因此状态转移矩阵为:

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & & & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ & & & & \frac{5}{6} & \frac{1}{6} \\ & & & & & 1 \end{bmatrix}$$

(b) Find out the classes of all states.
Is there any state closed (a state is closed if, once entered, it cannot be left)?
Is there any state recurrent/transient?
Is there any state positive recurrent?
Write your reasons briefly.

Solution:

- 状态 6 是闭的 (即吸收态)，因为 $\mathrm{P}\{X_{n+1} = 6 | X_n = 6\} = P_{6,6} = 1$
- 状态 6 是常返态 (具体来说，是正常返态)，因为它是一个吸收态.
  状态 $1,2,3,4,5$ 是瞬时态，
  因为它们都可到达状态 6，可是一旦过程进入状态 6，它就无法返回到其他状态.

Problem 6

For all integer $n \geq 3$, is the origin point for the n-dimensional symmetric random walk recurrent or transient?
Prove your answer using Stirling's formula: $k! \sim \sqrt{2\pi k} \cdot k^k e^{-k}$

Solution:
对于维度 $n \geq 3$ 的情况，即使是对称的随机游动也是瞬时的.
考察状态 0，尝试确定 $\sum_{m=0}^{\infty} P_{00}^{(m)}$ 是有限还是无穷大.

- 奇数次行走后肯定不能回到原点 0，因此 $P_{00}^{(2k-1)} = 0 \ \ (\forall\ k = 1,2,\ldots)$

- 偶数次行走最终回到原点 0 当且仅当沿着每个维度，向正方向的步数等于向负方向的步数.
  设对于第 j 个维度，正向步数和负向步数均为 r_j，满足 $2\sum_{j=1}^n r_j = 2k$.
  对应的概率为:

$$
\begin{aligned}
P_{00}^{(2k)} &= \frac{1}{(2n)^{2k}} \sum_{r_1+\cdots+r_n=k} \frac{(2k)!}{\prod_{j=1}^n (r_j! r_j!)} \\
&= \frac{(2k)!}{(2n)^{2k}} \sum_{r_1+\cdots+r_n=k} \frac{1}{(\prod_{j=1}^n r_j!)^2}
\end{aligned}
$$

注意到存在某个正常数 $C_1 > 0$ 使得:

$$
\max_{r_1+\cdots+r_n=k} \frac{1}{\prod_{j=1}^n r_j!} \leq \frac{C_1}{(\lfloor k/n \rfloor !)^n}
$$

于是我们有:

$$
\begin{aligned}
P_{00}^{(2k)} &= \frac{(2k)!}{(2n)^{2k}} \sum_{r_1+\cdots+r_n=k} \frac{1}{(\prod_{j=1}^n r_j!)^2} \\
&= \frac{(2k)!}{(2n)^{2k}} \max_{r_1+\cdots+r_n=k} \frac{1}{\prod_{j=1}^n r_j!} \cdot \frac{1}{k!} \sum_{r_1+\cdots+r_n=k} \frac{k!}{\prod_{j=1}^n r_j!} \\
&\leq \frac{(2k)!}{(2n)^{2k}} \cdot \frac{C_1}{(\lfloor k/n \rfloor !)^n} \cdot \frac{1}{k!} \quad (\text{note that } k! \sim \sqrt{2\pi k} \cdot k^k e^{-k}) \\
&\lesssim C_1 \frac{(2k)^{2k}}{(2n)^{2k}} \cdot \frac{n^k}{k^k} \cdot \frac{1}{(k/n)^{k+n/2}} \cdot \frac{\sqrt{2}}{(2\pi)^{-n/2}} \\
&\leq \frac{C_2}{k^{n/2}}
\end{aligned}
$$

其中 $C_2 > 0$ 为某个正常数.
因此我们有:

$$
\sum_{m=0}^{\infty} P_{00}^{(m)} = \sum_{k=1}^{\infty} P_{00}^{(2k)} \leq C_2 \sum_{k=1}^{\infty} k^{-n/2} < \infty
$$

故状态 0 是瞬时态.

The End