期中考试 (2024 春)

Problem 1

一共五小问，分别是一段 R 代码，让我们写出输出.

Solution: (从略)

Problem 2

用 R 语言编写一个 `loglik` 函数，计算正态总体的简单随机样本的对数似然函数.

Solution:
正态总体 $\xi \sim N(\mu, \sigma^2)$ 的简单随机样本 $X = (X_1, \ldots, X_n)$ 的对数似然函数为:

$$\begin{aligned} l(\mu, \sigma; x) &= \log\left(\prod_{i=1}^{n} f(x_i|\mu, \sigma)\right) \\ &= \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)\right) \\ &= \sum_{i=1}^{n}\left\{-\log\sigma - \frac{1}{2}\log(2\pi) - \frac{1}{2\sigma^2}(x_i-\mu)^2\right\} \\ &= -n\log\sigma - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 \end{aligned}$$

R 语言实现如下:

```r
loglik <- function(mu, sigma, x) {
  # 检查 sigma 是否为正数
  if (sigma <= 0) {
    stop("sigma 必须大于 0")
  }

  n <- length(x)
  ll <- -n * log(sigma) - (n / 2) * log(2 * pi) - sum((x - mu)^2) / (2 * sigma^2)
  return(ll)
}
```

Problem 3

一个河谷中随机抽取 n 个石头，测量直径得到样本 Y.
假设石头的直径 D 的对数 $\log(D)$ 服从正态分布.
给出总体的可能分布族、样本的可能分布族.

Solution:
已知 $X := \log(D) \sim N(\mu, \sigma^2)$
这等价于 $D \sim \text{Lognormal}(\mu, \sigma^2)$
概率密度函数:

$$\begin{aligned} f_D(y; \mu, \sigma) &= f_X(x; \mu, \sigma) \cdot \left|\frac{\mathrm{d}x}{\mathrm{d}y}\right| \quad (\text{note that } x = \log(y)) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \cdot \frac{1}{y} \\ &= \frac{1}{\sqrt{2\pi}\sigma y} \exp\left(-\frac{(\log y-\mu)^2}{2\sigma^2}\right) \quad (\text{where } y > 0, \sigma > 0) \end{aligned}$$

参数空间 $\Theta=\{(\mu,\sigma):\mu\in\mathbb{R},\sigma>0\}$
总体的可能分布族为对数正态分布族 $\mathcal{F}=\{f_D(y;\mu,\sigma)|(\mu,\sigma)\in\Theta\}$

考虑简单随机样本 $Y=(Y_1,\ldots,Y_n)$，其中:

$$Y_1,\ldots,Y_n\overset{\text{i.i.d.}}{\sim}\text{Lognormal}(\mu,\sigma^2)$$

于是样本的联合概率密度函数为:

$$\begin{aligned}f_Y(y_1,\ldots,y_n)&=\prod_{i=1}^n f_D(y_i|\mu,\sigma)\quad(\text{where } y_i>0\text{ for all } i=1,\ldots,n)\\&=\prod_{i=1}^n\frac{1}{\sqrt{2\pi}\sigma y_i}\exp\left(-\frac{(\log y_i-\mu)^2}{2\sigma^2}\right)\end{aligned}$$

样本分布族为独立同分布的 n 维对数正态分布族:

$$\mathcal{F}^{(n)}=\left\{\prod_{i=1}^n f_D(y_i|\mu,\sigma)\bigg|(\mu,\sigma)\in\Theta\right\}$$

Problem 4

设 $X_1,X_2,X_3\overset{\text{i.i.d.}}{\sim}\exp(\lambda)\ (\lambda>0)$
记 $\begin{cases}Y_1=X_1+X_2+X_3\\Y_2=\frac{X_1}{X_2+X_3}\\Y_3=\frac{X_2+X_3}{X_1+X_2+X_3}\end{cases}$
计算 Y_1,Y_2,Y_3 的联合分布，并说明它们是否独立.

- **定理** $1.2.2$**: (数理统计讲义 命题** $1.3.15$**)**
  给定 $\begin{cases}X_1\sim\text{Gamma}(\alpha_1,\lambda)\\X_2\sim\text{Gamma}(\alpha_2,\lambda)\\X_1\perp X_2\end{cases}$ 则有 $\begin{cases}Y_1=X_1+X_2\sim\text{Gamma}(\alpha_1+\alpha_2,\lambda)\\Y_2=\frac{X_1}{X_2}\sim\text{Beta Prime}(\alpha_1,\alpha_2)\\Y_1\perp Y_2\end{cases}$
  - **推论:**
    给定 $\begin{cases}X_1\sim\text{Gamma}(\alpha_1,\lambda)\\X_2\sim\text{Gamma}(\alpha_2,\lambda)\\X_1\perp X_2\end{cases}$ 则有 $\begin{cases}Y_1=X_1+X_2\sim\text{Gamma}(\alpha_1+\alpha_2,\lambda)\\Y_2=\frac{X_1}{X_1+X_2}\sim\text{Beta}(\alpha_1,\alpha_2)\\Y_1\perp Y_2\end{cases}$

  根据上述定理，我们知道:

  $$\begin{cases}Y_1\sim\text{Gamma}(3,\lambda)\\Y_2\sim\text{Beta Prime}(1,2)\\Y_1\perp Y_2\\Y_3\sim\text{Beta}(2,1)\\Y_1\perp Y_3\end{cases}$$

  但注意到 $Y_2=(1-Y_3)/Y_3$，故 Y_2,Y_3 之间是完全确定的关系，并不独立.

Solution:
注意到 $Y_2=(1-Y_3)/Y_3$，故 Y_2,Y_3 之间是完全确定的关系.
对 Y_1,Y_2,Y_3 直接求 Jacobi 矩阵是奇异的，因此我们应当先对 Y_1,Y_2 求.

设 $U=X_2+X_3$，易知 $U\sim\text{Gamma}(2,\lambda)$.
现在 $\begin{cases}Y_1=g_1(X_1,U)=X_1+U\\Y_2=g_2(X_1,U)=\frac{X_1}{U}\end{cases}$
我们有 $\begin{cases}X_1=h_1(Y_1,Y_2)=\frac{Y_1Y_2}{1+Y_2}\\U=h_2(Y_1,Y_2)=\frac{Y_1}{1+Y_2}\end{cases}$
且具有联合概率密度:

$$\begin{aligned}f_{X_1,U}(x_1,u)&=f_{X_1}(x_1)\cdot f_U(u)\\&=\frac{\lambda}{\Gamma(1)}x_1^{1-1}e^{-\lambda x_1}\cdot\frac{\lambda^2}{\Gamma(2)}u^{2-1}e^{-\lambda u}\\&=\lambda^3ue^{-\lambda(x_1+u)}\end{aligned}$$

而 Jacobi 矩阵为:

$$J(x_1,u)=\begin{vmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial u}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial u}\end{vmatrix}=\begin{vmatrix}1 & 1\\ \frac{1}{u} & -\frac{x_1}{u^2}\end{vmatrix}=-\frac{x_1+u}{u^2}\neq 0\ \ (\forall\, x_1,u>0)$$

因此 Y_1,Y_2 联合地连续，且联合概率密度函数为:

$$\begin{aligned}f_{Y_1,Y_2}(y_1,y_2)&=f_{X_1,U}(h_1(y_1),h_2(y_2))\cdot|J(h_1(y_1),h_2(y_2))|^{-1}\\&=\lambda^3ue^{-\lambda(x_1+u)}\cdot\left|-\frac{\frac{y_1y_2}{1+y_2}+\frac{y_1}{1+y_2}}{(\frac{y_1}{1+y_2})^2}\right|^{-1}\\&=\lambda^3y_1^2e^{-\lambda y_1}\cdot\frac{1}{(1+y_2)^3}\\&=\frac{\lambda^3}{\Gamma(3)}y_1^2e^{-\lambda y_1}\cdot\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)}\frac{1}{(1+y_2)^3}\\&=\mathrm{P}\{\mathrm{Gamma}(3,\lambda)=y_1\}\cdot\mathrm{P}\{\mathrm{Beta\ Prime}(1,2)=y_2\}\end{aligned}$$

表明 $\begin{cases}Y_1\sim\mathrm{Gamma}(3,\lambda)\\Y_2\sim\mathrm{Beta\ Prime}(1,2)\\Y_1\perp Y_2\end{cases}$

由于 Y_3 完全由 Y_2 决定: $Y_3=(Y_2+1)^{-1}$.
这意味着 Y_1,Y_2,Y_3 的联合分布是奇异的，其联合 PDF 包含 Dirac Delta 函数 $\delta(\cdot)$:

$$\begin{aligned}f_{Y_1,Y_2,Y_3}&=f_{Y_1,Y_2}(y_1,y_2)\cdot\delta\left(y_3-\frac{1}{y_2+1}\right)\\&=\lambda^3y_1^2e^{-\lambda y_1}\cdot\frac{1}{(1+y_2)^3}\cdot\delta\left(y_3-\frac{1}{y_2+1}\right)\end{aligned}$$

支持集为 $y_1,y_2>0,y_3=(y_2+1)^{-1}$.

注意到:

$$\begin{aligned}Y_2&=\frac{1-Y_3}{Y_3}=\frac{1}{Y_3}-1\\\frac{\mathrm{d}y_2}{\mathrm{d}y_3}&=-\frac{1}{Y_3^2}\end{aligned}$$

于是 Y_3 的边际分布可通过 Y_2 的分布得到:

$$\begin{aligned}f_{Y_3}(y_3)&=f_{Y_2}(y_2)\cdot\left|\frac{\mathrm{d}y_2}{\mathrm{d}y_3}\right|\\&=\frac{2}{(1+y_2)^3}\cdot\left|-\frac{1}{y_3^2}\right|\\&=\frac{2}{(1+1/y_3-1)^3}\cdot\frac{1}{y_3^2}\\&=2y_3\\&=\frac{\Gamma(3)}{\Gamma(2)\Gamma(1)}y_3^{2-1}(1-y_3)^{1-1}\\&=\mathrm{P}\{\mathrm{Beta}(2,1)=y_3\}\end{aligned}$$

支持集为 $0<y_3<1$.
表明 $Y_3\sim\mathrm{Beta}(2,1)$.

Problem 5

考虑简单随机样本 $X=(X_1,\dots,X_n)$，计算 $\mathrm{Cov}(\hat{F}_n(x),\hat{F}_n(y))$
其中 $\hat{F}_n(\cdot)$ 是经验分布函数:

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^n I_{[X_i,\infty)}(x) = \begin{cases} 0, & x < X_{(1)} \\ \frac{k}{n}, & X_{(k)} \le n < X_{(k+1)} \quad (1 \le k \le n-1) \\ 1, & x \ge X_{(n)} \end{cases}$$

Solution:
记 $I_i(x) = \mathbb{1}\{X_i \le x\}$，根据定义我们有:

$$\begin{aligned} \hat{F}_n(x) &= \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{X_i \le x\} = \frac{1}{n}\sum_{i=1}^n I_i(x) \\ \hat{F}_n(y) &= \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{X_i \le y\} = \frac{1}{n}\sum_{i=1}^n I_i(y) \end{aligned}$$

于是我们有:

$$\begin{aligned} \operatorname{Cov}(\hat{F}_n(x), \hat{F}_n(y)) &= \operatorname{Cov}\left(\frac{1}{n}\sum_{i=1}^n I_i(x), \frac{1}{n}\sum_{j=1}^n I_j(y)\right) \\ &= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \operatorname{Cov}(I_i(x), I_j(y)) \quad (\text{note that } \operatorname{Cov}(I_i(x), I_j(y)) = 0 \text{ when } i \ne j) \\ &= \frac{1}{n^2}\sum_{i=1}^n \operatorname{Cov}(I_i(x), I_i(y)) \\ &= \frac{n}{n^2}\operatorname{Cov}(I_1(x), I_1(y)) \\ &= \frac{1}{n}(\mathbb{E}[I_1(x)I_1(y)] - \mathbb{E}[I_1(x)]\mathbb{E}[I_1(y)]) \\ &= \frac{1}{n}(\mathrm{P}\{X_1 \le x, X_1 \le y\} - \mathrm{P}\{X_1 \le x\}\mathrm{P}\{X_1 \le y\}) \\ &= \frac{1}{n}(F(\min\{x,y\}) - F(x)F(y)) \end{aligned}$$

特别地，当 $x = y$ 时我们有 $\operatorname{Var}(\hat{F}_n(x)) = F(x)(1-F(x))/n$.

Problem 6

若离散型或连续型的含参数 θ 的分布族 $\mathscr{F}_X = \{F_X(\theta) : \theta \in \Theta\}$
其概率质量函数 (或概率密度函数) 可表示为:

$$p(x;\theta) = C(\theta)\exp\left\{\sum_{i=1}^k Q_i(\theta)T_i(x)\right\}h(x) \quad (\theta \in \Theta)$$

其中 $\{T_i(x)\}_{i=1}^k$ 和 $h(x)$ 仅为 x 的函数,
而 $C(\theta)$ 和 $\{Q_i(\theta)\}_{i=1}^k$ 仅为 θ 的函数,
则我们称分布族 $\mathscr{F}_X$ 为**指数型分布族** (简称指数族)

(1) 判断负二项分布族是否是指数型分布族?
Solution:
负二项分布随机变量 X 表示在独立重复 Bernoulli 试验中，在出现 r 次成功之前观测到的失败次数.
(当 $r = 1$ 时，退化为几何分布 $\mathrm{Geo}(p)$)
设单次试验的成功概率为 p，则有:

$$\mathrm{P}\{X = x\} = \binom{x+r-1}{x}p^r(1-p)^x \quad (x = 0, 1, \ldots, \ r \ge 1)$$

(值得注意的是，由于最后一次一定是成功的，故组合数是 $\binom{x+r-1}{x}$，而不是 $\binom{x+r}{x}$)

- 当 r 已知，p 未知时:

$$
\begin{aligned}
\mathrm{P}\{X=x\} &= \binom{x+r-1}{x} p^r (1-p)^x \quad (x=0,1,\dots,\ r\geq 1) \\
&= p^r \cdot \exp\left(\log(1-p)\cdot x\right)\cdot \binom{x+r-1}{x} \\
&= C(p)\cdot \exp\left(Q(p)T(x)\right)\cdot h(x)
\end{aligned}
$$

其中:

$$
\begin{cases}
C(p) = p^r \\
Q(p) = \log(1-p) \\
T(x) = x \\
h(x) = \binom{x+r-1}{x}
\end{cases}
$$

此时负二项分布族是指数型分布族.

- 当 r 未知时，无论 p 是否未知，由于 $\binom{x+r-1}{x}$ 无法分离 x 和 r，故此时负二项分布族不是指数型分布族.

---

(2) 证明指数型分布族满足:

$$
\mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2} Q_j(\theta)T_j(X)\right] + \mathrm{Var}_\theta\left[\sum_{j=1}^k \frac{\partial}{\partial\theta_i} Q_j(\theta)T_j(X)\right] = -\frac{\partial^2}{\partial\theta_i^2}\log\left(C(\theta)\right)
$$

Solution:
我们记:

$$
\begin{aligned}
H(\theta) &:= \int \exp\left\{\sum_{j=1}^k Q_j(\theta)T_j(x)\right\} h(x)\mathrm{d}x \\
&= \frac{1}{C(\theta)}\int C(\theta)\exp\left\{\sum_{i=1}^k Q_i(\theta)T_i(x)\right\}h(x)\mathrm{d}x \\
&= \frac{1}{C(\theta)}\int p(x;\theta)\mathrm{d}x \\
&= \frac{1}{C(\theta)}
\end{aligned}
$$

我们有:

$$
\begin{aligned}
\frac{\partial}{\partial\theta_i}H(\theta) &= \frac{\partial}{\partial\theta_i}\int \exp\left\{\sum_{j=1}^k Q_j(\theta)T_j(x)\right\}h(x)\mathrm{d}x \\
&= \frac{1}{C(\theta)}\cdot C(\theta)\int\left(\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(x)\right)\exp\left\{\sum_{j=1}^k Q_j(\theta)T_j(x)\right\}h(x)\mathrm{d}x \\
&= H(\theta)\cdot\int\left(\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(x)\right)p(x;\theta)\mathrm{d}x \\
&= H(\theta)\cdot\mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(X)\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial\theta_i^2}H(\theta) &= \frac{\partial^2}{\partial\theta_i^2}\int \exp\left\{\sum_{j=1}^k Q_j(\theta)T_j(x)\right\}h(x)\mathrm{d}x \\
&= \frac{1}{C(\theta)}\cdot C(\theta)\int\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(x) + \left(\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(x)\right)^2\right]\exp\left\{\sum_{j=1}^k Q_j(\theta)T_j(x)\right\}h(x)\mathrm{d}x \\
&= H(\theta)\int\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(x) + \left(\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(x)\right)^2\right]p(x;\theta)\mathrm{d}x \\
&= H(\theta)\left(\mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(x)\right] + \mathbb{E}_\theta\left[\left(\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(x)\right)^2\right]\right)
\end{aligned}
$$

现对 $\log H(\theta) = -\log C(\theta)$ 求二阶导可得:

$$
\begin{aligned}
\frac{\partial^2}{\partial\theta_i^2}\log H(\theta) &= \frac{1}{H(\theta)}\frac{\partial^2}{\partial\theta_i^2}H(\theta) - \frac{1}{(H(\theta))^2}\left(\frac{\partial}{\partial\theta_i}H(\theta)\right)^2 \\
&= \left(\mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(x)\right] + \mathbb{E}_\theta\left[\left(\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(x)\right)^2\right]\right) - \left(\mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(X)\right]\right)^2 \\
&= \mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(x)\right] + \mathrm{Var}_\theta\left[\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(X)\right]
\end{aligned}
$$

注意到 $\frac{\partial^2}{\partial\theta_i^2}\log H(\theta) = -\frac{\partial^2}{\partial\theta_i^2}\log C(\theta)$
因此我们有:

$$
\mathbb{E}_\theta\left[\sum_{j=1}^k \frac{\partial^2}{\partial\theta_i^2}Q_j(\theta)T_j(X)\right] + \mathrm{Var}_\theta\left[\sum_{j=1}^k \frac{\partial}{\partial\theta_i}Q_j(\theta)T_j(X)\right] = -\frac{\partial^2}{\partial\theta_i^2}\log\left(C(\theta)\right)
$$

Problem 7

设 $X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathrm{Uniform}(0,\theta)$
记极差 $R = X_{(n)} - X_{(1)}$ 和两极中心 $V = \frac{1}{2}(X_{(1)} + X_{(n)})$
计算 R, V 的联合概率密度函数.

- > **定理 $1.3.4$: (次序统计量的联合概率密度函数, 数理统计讲义 命题 $1.4.7$)**
  > 设 $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ 为对应于简单随机样本 $X = (X_1, X_2, \ldots, X_n)$ 的次序统计量,
  > (我们可以看作存在映射关系 $T(X_1, X_2, \ldots, X_n) = (X_{(1)}, X_{(2)}, \ldots, X_{(n)})$)
  > 总体分布具有分布函数 F 和概率密度函数 f.
  > 则对于任意 $\begin{cases} 1 \le r \le n \\ 1 \le j_1 < j_2 < \cdots < j_r \le n \end{cases}$
  > $(X_{(j_1)}, X_{(j_2)}, \ldots, X_{(j_r)})$ 具有联合概率密度函数:
  >
  > $$
  > \begin{aligned}
  > &f_{X_{(j_1)},X_{(j_2)},\ldots,X_{(j_r)}}(y_{j_1}, y_{j_2}, \ldots, y_{j_r}) \\
  > &= \frac{n!}{(j_1-1)!(j_2-j_1-1)!\ldots(j_r-j_{r-1}-1)!(n-j_r)!} \\
  > &\quad \times [F(y_{j_1})]^{j_1-1}[F(y_{j_2})-F(y_{j_1})]^{j_2-j_1-1}\cdots[F(y_{j_r})-F(y_{j_{r-1}})]^{j_r-j_{r-1}-1}[1-F(y_{j_r})]^{n-j_r} \\
  > &\quad \times f(y_{j_1})f(y_{j_2})\cdots f(y_{j_r}) \\
  > &\quad \times I(y_{j_1} < y_{j_2} < \cdots < y_{j_r})
  > \end{aligned}
  > $$

Solution:
根据引理可知 $X_{(1)}, X_{(n)}$ 的联合概率密度函数为:

$$
\begin{aligned}
&f_{X_{(1)},X_{(n)}}(x_1, x_n) \\
&= \frac{n!}{(1-1)!(n-1-1)!(n-n)!}[F(x_1)]^{1-1}[F(x_n)-F(x_1)]^{n-1-1}[1-F(x_n)]^{n-n}f(x_1)f(x_n)I(0<x_1<x_n<\theta) \\
&= n(n-1)(F(x_n)-F(x_1))^{n-2}f(x_1)f(x_2)I(0<x_1<x_n<\theta) \\
&= n(n-1)\left(\frac{x_n}{\theta}-\frac{x_1}{\theta}\right)^{n-2}\cdot\frac{1}{\theta}\cdot\frac{1}{\theta}\cdot I(0<x_1<x_n<\theta) \\
&= \frac{n(n-1)}{\theta^n}(x_n-x_1)^{n-2}I(0<x_1<x_n<\theta)
\end{aligned}
$$

记极差 $R = X_{(n)} - X_{(1)}$ 和两极中心 $V = \frac{1}{2}(X_{(1)} + X_{(n)})$.
反变换为:

$$
X_{(1)} = V - \frac{R}{2}, \quad X_{(n)} = V + \frac{R}{2}
$$

Jacobi 行列式为:

$$
J(x_1, x_n) = \begin{vmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_n} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_n} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -1
$$

于是 R, V 的联合概率密度函数为:

$$
\begin{aligned}
f_{R,V}(r,v) &= f_{X_{(1)},X_{(n)}}(x_1,x_2)\cdot|J(x_1,x_n)|^{-1} \\
&= \frac{n(n-1)}{\theta^n}(x_n-x_1)^{n-2}I(0<x_1<x_n<\theta)\cdot|-1|^{-1} \\
&= \frac{n(n-1)}{\theta^n}r^{n-2}I(0<v-\frac{r}{2}<v+\frac{r}{2}<\theta)
\end{aligned}
$$

The End