DATA130026.01 Optimization Assignment 1

Due Time: at the beginning of the class Mar. 13, 2024
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Problem 1: Convexity of Sets

For each of the following sets, determine whether they are convex or not, explaining your choice.

- **(a)** $C_1 = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$
  - **Claim:**
    C_1 (which is the serface of an unit norm ball) is not convex.
  - **Counterexample:**
    Consider any $x_0 \in C_1$, we note that $\begin{cases} -x_0 \in C_1 \\ \frac{1}{2}x_0 + \frac{1}{2}(-x_0) = 0_n \notin C_1 \end{cases}$'
    which implies that C_1 is not convex.
- **(b)** $C_2 = \{x \in \mathbb{R}^n : \max\limits_{i=1,...,n} x_i \leq 1\}$
  - **Claim:**
    C_2 is a convex set.
  - **Proof:**
    Consider any $\begin{cases} x, y \in C_2 \\ \alpha \in (0,1) \\ i = 1, \ldots, n \end{cases}$,
    we have $\alpha x_i + (1-\alpha)y_i \leq \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1$
    Therefore, $\alpha x + (1-\alpha)y \in C_2$, demonstrating the convexity of C_2.
- **(c)** $C_3 = \{x \in \mathbb{R}^n : \min\limits_{i=1,...,n} x_i \leq 1\}$
  - **Claim:**
    C_3 is not a convex set.
  - **Counterexample:**
    Denote e_i as the i-th standard unit basis vector of $\mathbb{R}^n$ space.
    Let $\begin{cases} x = 2 \cdot 1_n - e_1 \\ y = 2 \cdot 1_n - e_2 \end{cases}$ (where 1_n represents the n-dimensional all-ones vector)
    We observe that $\begin{cases} x, y \in C_3 \\ \frac{1}{2}x + \frac{1}{2}y = [\frac{3}{2}, \frac{3}{2}, \underbrace{2, \ldots, 2}_{n-2}]^T \notin C_3, \end{cases}$
    which implies that C_3 is not convex.
- **(d)** $C_4 = \{x \in \mathbb{R}^n_{++} : \prod_{i=1}^n x_i \leq 1\}$
  - **Claim:**
    C_4 is not a convex set.

- **Counterexample:**
  Let $\begin{cases} x = 1_n \\ y = 1_n - \frac{1}{2}e_1 + e_2 \end{cases}$
  We observe that $\begin{cases} x, y \in C_4 \\ \frac{1}{2}x + \frac{1}{2}y = [\frac{3}{4}, \frac{3}{2}, \underbrace{1, \dots, 1}_{n-2}]^T \notin C_4, \end{cases}$
  which implies that C_4 is not convex.

Problem 2: Convex Closed Cones

Let K_1 and K_2 be two closed convex cones in $\mathbb{R}^n$.

- **(a)** Provide an example that $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$ is not closed.
  Example:
  Let K_1 be the second-order cone in $\mathbb{R}^3$, i.e. $K_1 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \leq x_3^3\}$
  Let $K_2 = \{t(1, 0, -1) : t \geq 0\}$ be a ray in $\mathbb{R}^3$
  Both of them are closed convex cones.
  Obviously, the sequence $\{n \cdot (1, 0, -1) + (-n, 1 + \frac{1}{n}, \sqrt{n^2 + (1 + \frac{1}{n})^2})\} \subseteq K_1 + K_2$,
  and it holds that:
  $\lim_{n \to +\infty} \{n \cdot (1, 0, -1) + (-n, 1 + \frac{1}{n}, \sqrt{n^2 + (1 + \frac{1}{n})^2})\} = (0, 1, 0)$
  It can be proven that the limit point $(0, 1, 0)$ is not an element of $K_1 + K_2$,
  because the system
  $\begin{cases} x_1^2 + x_2^2 \leq x_3^2 \\ x_1 + t \cdot 1 = 0 \\ x_2 + t \cdot 0 = 1 \\ x_3 + t \cdot (-1) = 0 \end{cases}$ has no solution.

- **(b)** Show that if $(-K_1) \cap K_2 = \{0_n\}$ then $K_1 + K_2$ is a closed convex cone.
  Proof:
  - Firstly, we prove that $K_1 + K_2$ is a **convex cone**:
    Consider any $z_1, z_2 \in K_1 + K_2$, we can decompose z_1, z_2 as $x_1 + y_1$ and $x_2 + y_2$, where
    $\begin{cases} x_1, x_2 \in K_1 \\ y_1, y_2 \in K_2 \end{cases}$
    For every $\alpha_1, \alpha_2 \geq 0$, since
    $\begin{cases} \alpha_1 x_1 + \alpha_2 x_2 \in K_1 \\ \alpha_1 y_1 + \alpha_2 y_2 \in K_2 \end{cases}$
    we have $\alpha_1 z_1 + \alpha_2 z_2 = (\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 y_1 + \alpha_2 y_2) \in K_1 + K_2$.
    Therefore, $K_1 + K_2$ is a **convex cone**.
  - Secondly, we prove that $K_1 + K_2$ is **closed**:
    Lemma:
    Suppose K is a cone.
    - ① $K^* := \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0,\ x \in K\}$ is a closed convex cone;
    - ② $K^{**} = \text{cl}(\text{conv}(K))$
      (Therefore $K^{**} = K$ holds if and only if K is a closed convex cone);
    - ③ If $K_1 \subseteq K_2$, then $K_1^* \supseteq K_2^*$;
    - ④ For any arbitrary cone K_1, K_2, it holds that $(K_1 + K_2)^* = K_1^* \cap K_2^*$;

- ⑤ If closed convex cones K_1, K_2 satisfy $K_1 \cap K_2 \supset \{0_n\}$, then $(K_1 \cap K_2)^* = K_1^* + K_2^*$;

Given that K_1, K_2 are closed convex sets,
the condition $(-K_1) \cap K_2 = \{0_n\}$ implies that $(-K_1^{**}) \cap K_2^{**} = \{0_n\}$,
which means there is no homogeneous hyperplane that can seperate K_1^* and K_2^*.
Therefore, the intersection between K_1^* and K_2^* is not merely $\{0_n\}$,
i.e., $K_1^* \cap K_2^* \supset \{0_n\}$.

By using Lemma ①②④⑤, we have:

$$\begin{aligned}\mathrm{cl}(K_1 + K_2) &= (K_1 + K_2)^{**} \\ &= (K_1^* \cap K_2^*)^* \\ &= K_1^{**} + K_2^{**} \\ &= K_1 + K_2\end{aligned}$$

Therefore, $K_1 + K_2$ is **closed**.

In summary, $K_1 + K_2$ is a closed convex set.

Proof of Lemma:

- **Lemma ①②③** could be found in **Convex Optimization (S.Boyd) 2.6.1**;
- **Lemma ④:**
  For any arbitrary cone K_1, K_2, it holds that $(K_1 + K_2)^* = K_1^* \cap K_2^*$
  Proof:
  - **Proving $K_1^* \cap K_2^* \subseteq (K_1 + K_2)^*$:**
    If $y \in K_1^* \cap K_2^*$, then for every
    $\begin{cases} x_1 \in K_1 \\ x_2 \in K_2 \end{cases}$ it holds that:
    $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle \geq 0$
    so that $y \in (K_1 + K_2)^*$, which implies $K_1^* \cap K_2^* \subseteq (K_1 + K_2)^*$.
  - **Proving $(K_1 + K_2)^* \subseteq K_1^* \cap K_2^*$:**
    If $y \in (K_1 + K_2)^*$, then for every $x_1 \in K_1$, it holds that:
    $\langle x_1, y\rangle = \langle x_1 + 0_n, y\rangle \geq 0$
    so that $y \in K_1^*$, which implies $(K_1 + K_2)^* \subseteq K_1^*$
    Similarly, we have $(K_1 + K_2)^* \subseteq K_2^*$.
    Therefore, it holds that $(K_1 + K_2)^* \subseteq K_1^* \cap K_2^*$.

Q.E.D.

- **Lemma ⑤:**
  If closed convex cones K_1, K_2 satisfy $K_1 \cap K_2 \supset \{0_n\}$, then $(K_1 \cap K_2)^* = K_1^* + K_2^*$
  Proof:
  - **Proving $K_1^* + K_2^* \subseteq (K_1 \cap K_2)^*$**
    If $y = y_1 + y_2 \in K_1^* + K_2^*$ where
    $\begin{cases} y_1 \in K_1^* \\ y_2 \in K_2^*\text{'} \end{cases}$
    then for every $x \in K_1 \cap K_2$, it holds that:
    $\langle x, y\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle \geq 0$
    so that $y \in (K_1 \cap K_2)^*$, which implies $K_1^* + K_2^* \subseteq (K_1 \cap K_2)^*$

- **Proving** $(K_1 \cap K_2)^* \subseteq K_1^* + K_2^*$
  By using Lemma ① and ③, we have:
  $$\begin{cases} K_1^* \subseteq (K_1^* \cup K_2^*) & \Rightarrow \quad K_1 = K_1^{**} \supseteq (K_1^* \cup K_2^*)^* \\ K_2^* \subseteq (K_1^* \cup K_2^*) & \Rightarrow \quad K_2 = K_2^{**} \supseteq (K_1^* \cup K_2^*)^* \end{cases} \quad \Rightarrow \quad (K_1 \cap K_2) \supseteq (K_1^* \cup K_2^*)^*$$
  Therefore, it holds that $(K_1 \cap K_2)^* \supseteq (K_1^* \cup K_2^*)^{**}$
  By using Lemma ① and ②, we have:
  $$\begin{aligned} (K_1^* \cup K_2^*)^{**} &= \mathrm{cl}(\mathrm{conv}(K_1^* \cup K_2^*)) \\ &= \mathrm{conv}(\mathrm{cl}(K_1^* \cup K_2^*)) \\ &= \mathrm{conv}(K_1^* \cup K_2^*) \qquad (\text{since } K_1^* \cup K_2^* \text{ is closed}) \\ &= K_1^* + K_2^* \end{aligned}$$
  Therefore, $(K_1 \cap K_2)^* \subseteq K_1^* + K_2^*$

Warning: The proof of Lemma ⑤ may be flawed, since I haven't used the condition $K_1 \cap K_2 \supset \{0_n\}$

Problem 3: Convexity of Various Sets

Determine the convexity of the following sets, with explanations.

- **(a)** The set of points closer to a given point than a given set,
  i.e., $\{x : \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$ where $S \subset \mathbb{R}^n$.
  - **Claim:**
    $C \overset{\Delta}{=} \{x : \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$ is a convex set.
  - **Explanation:**
    Suppose that $S \neq \emptyset$.
    Consider any $y \in S$,
    x is closer to x_0 than y if and only if:
    $$\begin{aligned} &||x - x_0||_2 \leq ||x - y||_2 \\ \Leftrightarrow\ & (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y) \\ \Leftrightarrow\ & x^T x - 2x_0^T x + x_0^T x_0 \leq x^T x - 2y^T x + y^T y \\ \Leftrightarrow\ & 2(y - x_0)^T x \leq y^T y - x_0^T x_0 \end{aligned}$$
    So we have:
    $$\begin{aligned} C &= \{x : \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\} \\ &= \bigcap_{y \in S} \{x : 2(y - x_0)^T x \leq y^T y - x_0^T x_0\} \end{aligned}$$
    - If $y \neq x_0$,
      then $\{x : 2(y - x_0)^T x \leq y^T y - x_0^T x_0\}$ is a halfspace, which is a convex set;
    - If $y = x_0$ (this can happen if $x_0 \in S$),
      then $\{x : 2(y - x_0)^T x \leq y^T y - x_0^T x_0\} = \mathbb{R}^n$, which is a convex set;

    Therefore, C, as the intersection of convex sets, is also a convex set.

- **(b)** The set of points closer to one set than another,
  i.e., $\{x : \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\}$ where $S, T \subset \mathbb{R}^n$.
  - **Claim:**
    $C \overset{\Delta}{=} \{x : \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\}$ is a convex set.

- **Explanation:**
  Assume that $\mathrm{dist}()$ is defined by the Euclidean norm,
  and S, T are both nonempty sets.
  Consider any $t \in T$,
  we denote $C(t, S)$ as the set of points closer to the set S than to the point t.
  Similarly to the derivation in (a),
  we know that $C(t, S) = \bigcap\limits_{s \in S} \{x : 2(s-t)^T x \geq s^T s - t^T t\}$, as the intersection of convex sets, is a convex set.
  Therefore, $C = \bigcap\limits_{t \in T} C(t, S)$, as the intersection of convex sets, is also a convex set.

- **(c)** The set $\{x : x + S_2 \subset S_1\}$ where $S_1, S_2 \subset \mathbb{R}^n$ with S_1 convex.
  - **Claim:**
    $C \stackrel{\Delta}{=} \{x : x + S_2 \subset S_1\}$ is a convex set.
  - **Explanation:**
    we observe that $C = \{x : x + S_2 \subset S_1\} = \bigcap\limits_{z \in S_2} \{x : x + z \in S_1\}$
    - Denote $C(z) \stackrel{\Delta}{=} \{x : x + z \in S_1\}$,
      we first prove that for any $z \in S_2$, $C(z)$ is a convex set:
      For every
      $\begin{cases} x_1, x_2 \in C(z) \\ \alpha \in (0,1) \end{cases}$
      , we have
      $\alpha x_1 + (1-\alpha) x_2 + z = \alpha(x_1 + z) + (1-\alpha)(x_2 + z) \in S_1$,
      so $\alpha x_1 + (1-\alpha) x_2 \in C(z)$, which implies $C(z)$ is a convex set (for all $z \in S_2$).
    - Then we know that $C = \bigcap\limits_{z \in S_2} C(z) = \bigcap\limits_{z \in S_2} \{x : x + z \in S_1\}$,
      as the intersection of convex sets, is also a convex set.