DATA130026 Optimization Assignment 4

Due Time: at the beginning of the class Apr. 3, 2022
姓名： 雍崔扬
学号： 21307140051

Requirement for problem $1 \sim 4$

For each of the following optimization problems
(1) show that it is **convex**
(2) write a **CVX code** that solves it
(3) write down the **optimal solution and optimal value** (by running CVX).
You need to write all the above steps and publish your codes (using "publish" in MATLAB) in a PDF file.
You should upload an electronic version on e-learning.

Problem 1

$$\begin{aligned} \min \quad & x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2 \\ \text{s.t.} \quad & \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} + \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} \le 6 \\ & x \succeq 1_3 \end{aligned}$$

Solution:

- **(1) Prove that it is a convex problem:**
  - ① For $f(x) = x_1^2 + 2x_1x_2 + 2x_2^2 + x_3^2$:
    Since $\nabla^2 f(x) \equiv \begin{bmatrix} 1 & 1 & \\ 1 & 2 & \\ & & 1 \end{bmatrix} \succeq 0$ (we know it by **Gershgorin Disk Theorem**)
    The object function f is convex.
  - ② For $g_1(x) = \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} + \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} - 6$:
    Denote $\begin{cases} g_1(x) = g_{11}(x) + g_{12}(x) - 6 \\ g_{11}(x) = \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} \\ g_{12}(x) = \frac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2} \end{cases}$
    - **Consider g_{11}:**
      $$\begin{aligned} g_{11}(x) &= \sqrt{2x_1^2 + x_1x_2 + 4x_2^2 + 4} \\ &= \sqrt{\frac{1}{2}(x_1 + x_2)^2 + \frac{3}{2}x_1^2 + \frac{7}{2}x_2^2 + 4} \\ &= \|y\|_2 \end{aligned}$$
      where $y = \begin{bmatrix} \frac{\sqrt{2}}{2}(x_1 + x_2) \\ \frac{\sqrt{3}}{2}x_1 \\ \frac{\sqrt{7}}{2}x_2 \\ 2 \end{bmatrix}$ is linear combination of $x \in \mathbb{R}^3$
      hence g_{11} is convex on $\mathbb{R}^3$

- **Consider g_{12}:**
  $g_{12} = \dfrac{(x_1 - x_2 + x_3 + 1)^2}{x_1 + x_2}$
  It is the composition of the quadratic-over-line function $\begin{cases} q(x,t) = \frac{x^2}{t} \\ \mathrm{dom}(q) = \mathbb{R} \times \mathbb{R}_{++} \end{cases}$
  and an affine transformation $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
  hence g_{12} is convex on $\{x \in \mathbb{R}^3 : x_1 + x_2 > 0\}$

  Therefore $g_1(x) = g_{11}(x) + g_{12}(x) - 6$ is a convex function.
- ③ $g_2(x) = 1_3 - x$ is an affine function, which is naturally convex.
- By ①②③， we know the problem is convex.

- **(2) CVX code:**

```
cvx_begin
    variable x(3)
    A = [1, 1, 0; 1, 2, 0; 0, 0, 1];
    minimize(quad_form(x, A))
    subject to
        norm([(x(1) + x(2))/sqrt(2); sqrt(3/2)*x(1); sqrt(7/2)*x(2); 2])
            + quad_over_lin(x(1) - x(2) + x(3) + 1, x(1) + x(2)) <= 6;
        x >= 1;
cvx_end
disp(x)
```

- **(3) optimal solution and optimal value:**

$x^\star = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ p^\star = 6$

Problem 2

$$\begin{array}{ll} \min & x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} & (x_1 - x_2)^2 + (x_3 + 2x_4)^4 \leq 25 \\ & x_1 + 2x_2 + 3x_3 + 4x_4 \leq 6 \\ & x \succeq 0_4 \end{array}$$

Solution:

- **(1) Prove that it is a convex problem:**
  - ① The object function $f(x) = x_1 + x_2 + x_3 + x_4$ is an affine function, which is naturally convex.
  - ② For $g_1(x) = (x_1 - x_2)^2 + (x_3 + 2x_4)^4 - 25$:
    Denote $\begin{cases} g_1(x) = g_{11}(x) + g_{12}(x) - 25 \\ g_{11}(x) = (x_1 - x_2)^2 \\ g_{12}(x) = (x_3 + 2x_4)^4 \end{cases}$
    - **Consider g_{11}:**
      $g_{11}(x) = (x_1 - x_2)^2$ is the composition of the **quadratic function** and an affine transformation, which is convex.

- **Consider g_{12}:**
  $g_{12}(x)=(x_3+2x_4)^4$ is the composition of the **quartic function** and an affine transformation, which is convex.

  Therefore $g_1(x)=g_{11}(x)+g_{12}(x)-25$ is a convex function.
- ③ For $g_2(x)=x_1+2x_2+3x_3+4x_4-6$:
  It is an affine function, which is naturally convex.
- ④ For $g_3(x)=0_4-x$:
  It is an affine function, which is naturally convex.
- By ①②③④, we know the problem is convex.

- **(2) CVX code:**

```
cvx_begin
cvx_precision high
    variable x(4)
    minimize(sum(x))
    subject to
        (x(1) - x(2))^2 + (x(3) + 2*x(4))^4 <= 25
        [1, 2, 3, 4] * x <= 6
        x >= 0
cvx_end
disp(x)
```

- **(3) optimal solution and optimal value:**

  The output of the CVX code is $x^\star=\begin{bmatrix}0.9089\\0.9213\\0.9213\\0.9177\end{bmatrix}\times 10^{-14},\ p^\star=3.66923\times 10^{-14}$

  which we could take to mean $x^\star=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\ p^\star=0$

Problem 3

$$\begin{aligned}\min\quad & |2x_1+3x_2+x_3|+\|x\|_2^2+\sqrt{2x_1^2+4x_1x_2+7x_2^2+10x_2+6}\\ \text{s.t.}\quad & \frac{x_1^2+1}{x_2}+2x_1^2+5x_2^2+10x_3^2+4x_1x_2+2x_1x_3+2x_2x_3\le 17\\ & \max\{x_1+x_2,x_3,x_1-x_3\}\le 19\\ & x_1\ge 0\\ & x_2\ge 1\end{aligned}$$

Solution:

- **(1) Prove that it is a convex problem:**
  - ① For objective function f:

    Denote $\begin{cases}f(x)=f_1(x)+f_2(x)+f_3(x)\\ f_1(x)=|2x_1+3x_2+x_3|\\ f_2(x)=\|x\|_2^2\\ f_3(x)=\sqrt{2x_1^2+4x_1x_2+7x_2^2+10x_2+6}\end{cases}$

- **Consider f_1:**
  f_1 is the composition of the absolute value function and an affine transformation, which is convex.
- **Consider f_2:**
  f_2 is the squared Euclidean norm, which is convex.
- **Consider f_3:**
  $$\begin{aligned} f_3 &= \sqrt{2x_1^2+4x_1x_2+7x_2^2+10x_2+6} \\ &= \sqrt{2(x_1+x_2)^2+5(x_2+1)^2+1} \\ &= \|y\|_2 \end{aligned} \quad \text{where } y = \begin{bmatrix} \sqrt{2}(x_1+x_2) \\ \sqrt{5}(x_2+1) \\ 1 \end{bmatrix}$$
  hence f_3 is convex.

  Therefore, $f(x) = f_1(x) + f_2(x) + f_3(x)$ is a convex function.

- ② For inequality constraint function g_1:

  Denote $\begin{cases} g_1(x) = g_{11}(x) + g_{12}(x) - 17 \\ g_{11}(x) = \frac{x_1^2+1}{x_2} \\ g_{12}(x) = 2x_1^2+5x_2^2+10x_3^2+4x_1x_2+2x_1x_3+2x_2x_3 \end{cases}$

  - **Consider g_{11}:**
    $$g_{11} = \frac{x_1^2+1}{x_2} = \begin{bmatrix} x_1 \\ 1 \end{bmatrix}^T (x_2 I_2)^{-1} \begin{bmatrix} x_1 \\ 1 \end{bmatrix}$$
    It is the composition of **matrix fractional function** $\begin{cases} f(x, A) = x^T A^{-1} x \\ \mathrm{dom}(f) = \mathbb{R}^n \times \mathbb{S}_{++}^n \end{cases}$ (with $n = 2$)
    and an affine transformation $q(x) = (\begin{bmatrix} x_1 \\ 1 \end{bmatrix}, x_2 I_2)$
    Hence g_{11} is convex.
  - **Consider g_{12}:**
    $\nabla^2 g_{12}(x) \equiv \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 1 & 10 \end{bmatrix} \succeq 0$ (we know it by 计算主子式，**Gershgorin Disk Theorem** 用不了)
    Hence g_{12} is convex.

  Therefore, $g_1(x) = g_{11}(x) + g_{12}(x) - 17$ is a convex function.

- ③ For $g_2(x) = \max\{x_1+x_2, x_3, x_1-x_3\} - 19$:
  g_2 is the composition of the max function and an affine transformation, and plus a constant, which is convex.
- ④ For $g_3(x) = -x_1$ and $g_4(x) = 1 - x_2$:
  They are both affine functions, which are naturally convex.
- By ①②③④，we know the problem is convex.

- **(2) CVX code:**

```
cvx_begin
    A = [2, 2, 1; 2, 5, 1; 1, 1, 10];
    variable x(3)
    minimize(abs([2, 3, 1] * x) ...
                + sum_square(x) ...
                + norm([sqrt(2)*(x(1)+x(2)); sqrt(5)*(x(2)+1); 1]))
    subject to
        quad_over_lin([x(1); 1], x(2)) + quad_form(x, A) <= 17
        max([x(1) + x(2); x(3); x(1) - x(3)]) <= 19
        x(1) >= 0
        x(2) >= 1
cvx_end
disp(x)
```

- **(3) optimal solution and optimal value:**

$$x^\star = \begin{bmatrix} 0 \\ 1 \\ -0.5 \end{bmatrix},\ p^\star = 8.54583$$

Problem 4

$$\begin{aligned}
\min \quad & \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 + |x_1+5| + |x_2+5| + |x_3+5| \\
\text{s.t.} \quad & [(x_1^2+x_2^2+x_3^2+1)^2+1]^2 + x_1^4 + x_2^4 + x_3^4 \le 100 \\
& \max\{x_1^2+4x_1x_2+9x_2^2, x_1, x_2\} \le 40 \\
& x_1 \ge 1 \\
& x_2 \ge 1
\end{aligned}$$

Solution:

- **(1) Prove that it is a convex problem:**
  - ① For object function f:

    Denote $\begin{cases} f(x) = f_1(x) + f_2(x) \\ f_1(x) = \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 \\ f_2(x) = |x_1+5| + |x_2+5| + |x_3+5| \end{cases}$

    - **Consider** f_1**:**

      $f_1(x) = \frac{x_1^4}{x_2^2} + \frac{x_2^4}{x_1^2} + 2x_1x_2 = (\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1})^2 \overset{\Delta}{=} (q(x))^2$ where $q(x) = \frac{x_1^2}{x_2} + \frac{x_2^2}{x_1}$

      We note that $q(x)$ is the sum of two **quadratic-over-linear function**

      and $q(x) > 0$ for all x that satisfies the constraints $\begin{cases} x_1 \ge 1 \\ x_2 \ge 1 \end{cases}$

      Thus, f_1 is the composition of the **nondecreasing quadratic function** $t^2 \quad (t > 0)$ and a convex function $q(x)$, which is convex.

    - **Consider** f_2**:**

      $f_2(x) = |x_1+5| + |x_2+5| + |x_3+5|$ is the composition of l_1 **norm** $\|\cdot\|_1$ and an affine transformation, which is convex.

    Therefore, $f(x) = f_1(x) + f_2(x)$ is convex.

  - ② For inequality constraint function g_1:

    Denote $\begin{cases} g_1(x) = g_{11}(x) + g_{12}(x) - 100 \\ g_{11}(x) = [(x_1^2+x_2^2+x_3^2+1)^2+1]^2 \\ g_{12}(x) = x_1^4 + x_2^4 + x_3^4 \end{cases}$

- **Consider g_{11}:**

  Denote $\begin{cases} g_{11}(x) = [(x_1^2 + x_2^2 + x_3^2 + 1)^2 + 1]^2 = q_2(q_2(q_1(x))) \\ q_1(x) = x_1^2 + x_2^2 + x_3^2 \\ q_2(t) = (t+1)^2 \quad (t > 0) \end{cases}$

  $g_{11} = q_2 \circ q_2 \circ q_1$

  We note that $q_2 \circ q_1$ is the composition of a nondecreasing convex function q_2 and a convex function q_1, which is convex.

  Hence g_{11} is the composition of the nondecreasing convex function q_2 and a convex function $q_2 \circ q_1$, which is convex.

- **Consider g_{12} :**

  $g_{12} = x_1^4 + x_2^4 + x_3^4$ a sum of quartic monomials,

  where each term is a **quartic function** in a single variable,

  hence it is a convex function. (还可以通过这是 l_4-范数的四次方来说明凸性)

Therefore, $g_1(x) = g_{11}(x) + g_{12}(x) - 100$ is a convex function.

- ③ For g_2:

  $g_2 = \max\{x_1^2 + 4x_1x_2 + 9x_2^2, x_1, x_2\} - 40$

  $\quad = \max\{(x_1 + 2x_2)^2 + 5x_2^2, x_1, x_2\} - 40$

  It is the composition of the max function (which is nondecreasing and convex)

  and a convex function $q(x) = \begin{bmatrix} (x_1 + 2x_2)^2 + 5x_2^2 \\ x_1 \\ x_2 \end{bmatrix}$, plus a constant,

  hence it is a convex function.

- ④ For $f_3(x) = -x_1 + 1$ and $f_4(x) = -x_2 + 1$

  They are both affine functions, which are naturally convex.

- By ①②③④， we know the problem is convex.

- **(2) CVX code:**

```
cvx_begin
    variable x(3)
    minimize(square_pos(quad_over_lin(x(1), x(2)) + quad_over_lin(x(2),
x(1)))
                + norm(x + 5, 1))
    subject to
        square_pos(square_pos(sum_square(x) + 1) + 1)
            + x(1)^4 + x(2)^4 + x(3)^4 <= 100
        max([(x(1) + 2*x(2))^2 + 5*x(2)^2; x(1); x(2)]) <= 40
        x(1) >= 1
        x(2) >= 1
cvx_end
disp(x)
```

- **(3) optimal solution and optimal value:**

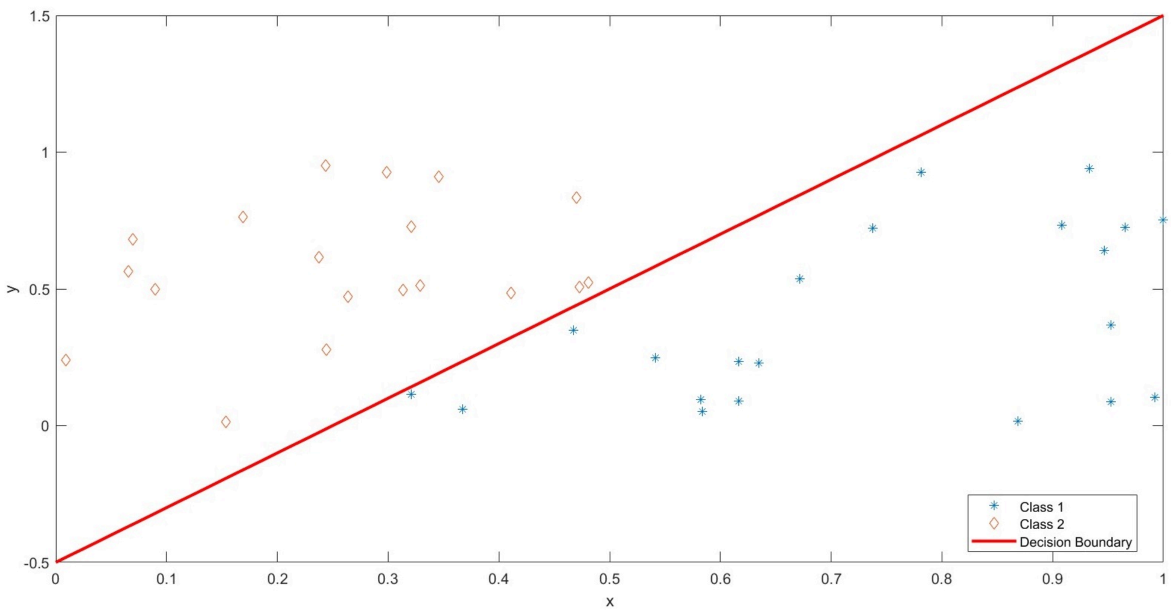
$x^\star = \begin{bmatrix} 1 \\ 1 \\ -0.3.1463 \end{bmatrix}$, $p^\star = 17.8537$

Problem 5

Suppose that we are given 40 points in the plane.
Each of these points belongs to one of two classes.
The points are generated and plotted by the following MATLAB commands:

```
id = 21307140051;
rand('seed', id);
x = rand(40, 1);
y = rand(40, 1);
class = (2*x < y + 0.5) + 1;
A1 = [x(class == 1), y(class == 1)];
A2 = [x(class == 2), y(class == 2)];

% Plot points of class 1
plot(A1(:,1), A1(:,2), '*', 'MarkerSize', 6)
hold on
% Plot points of class 2
plot(A2(:,1), A2(:,2), 'd', 'MarkerSize', 6)

% Plot the decision boundary line y + 0.5 = 2*x
x_line = 0:0.01:1; % Generate x values from 0 to 1 for the line
y_line = 2*x_line - 0.5; % Calculate corresponding y values using the line equation
plot(x_line, y_line, 'r', 'LineWidth', 2) % Plot the line in red color

% Adjust plot
xlabel('x')
ylabel('y')
legend('Class 1', 'Class 2', 'Decision Boundary', 'Location', 'Best')
hold off
```

where `id` is my student id `21307140051`.
The rows of $A_1 \in \mathbb{R}^{n_1 \times 2}$ are the points of class 1
and the rows of $A_2 \in \mathbb{R}^{n_2 \times 2}$ are the points of class 2.
Here is the plot of two classes of points and the line $2x = y + 0.5$:



Write a CVX-based code for finding the maximum-margin line separating the two classes of points.

Solution:

We have the following convex fomulation of the problem:

$$\min_{w,\beta} \ \frac{1}{2}||w||_2^2$$

$$\text{s.t.} \quad b_i(w^T x_i + \beta) \geq 1, \ i = 1, 2, \ldots, 40$$

where $b_i = \begin{cases} 1, & x_i \in A_1 \\ -1, & x_i \in A_2 \end{cases}$

- **Here is the CVX code that solves it:**

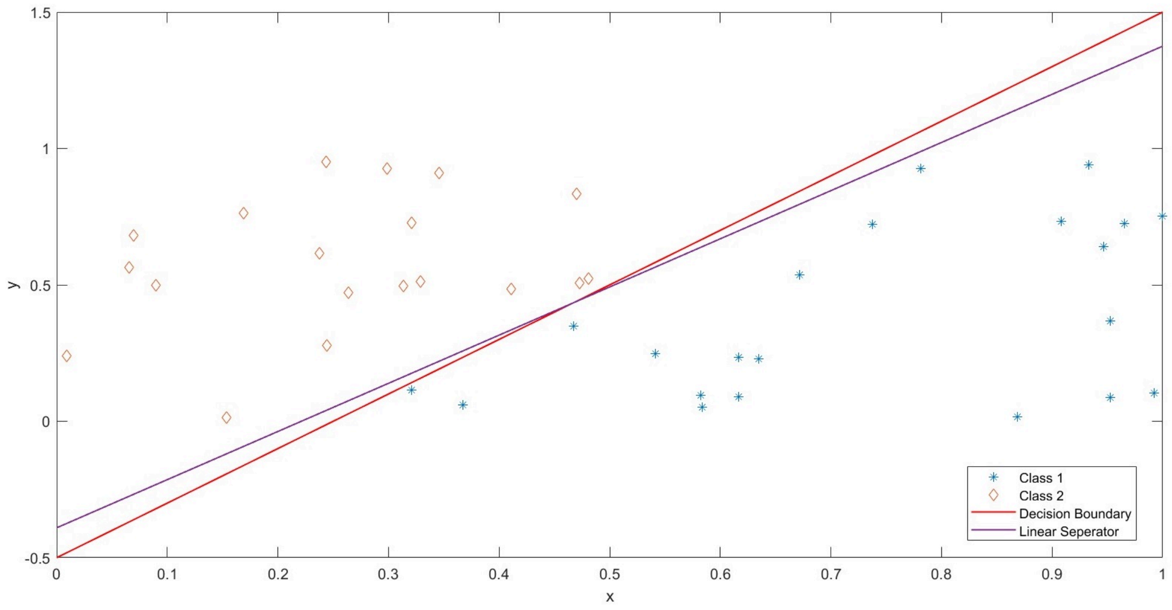
```
% Store all points in matrix A
A=[x y];

% label class 1 as 1, label class 2 as -1
b(class==1) = 1;
b(class==2) = -1;

% CVX code
cvx_begin
    variables w(2) beta0
    minimize(norm(w))
    subject to
        b'.*(A*w + ones(40,1)*beta0) >= 1;
cvx_end

% display the parameters of the linear seperator
disp(w)
disp(beta0)
```

- **The result is:**

$$w = \begin{bmatrix} 28.8340 \\ -16.3307 \end{bmatrix}, \ \beta = -6.3768$$

- **Plot the linear seperator:**

```
% plot the linear seperator
x_line = linspace(0,1);
y_line = -(w(1)*x_line + beta0)/w(2);
plot(x_line, y_line, 'LineWidth', 1)
```


1.5
1
0.5
0
-0.5
y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
Class 1
Class 2
Decision Boundary
Linear Seperator

The End