随机过程导论 Assignment 02

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Problem 1

Suppose $\{X_n : n = 0, 1, \dots\}$ is a two state Markov chain
whose transition probability matrix is given by $P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$
Let $Z_n = (X_{n-1}, X_n)$ be a vector with four states $(0,0), (0,1), (1,0), (1,1)$
Is Z_n a Markov chain?
If yes, determine its transition probability matrix.

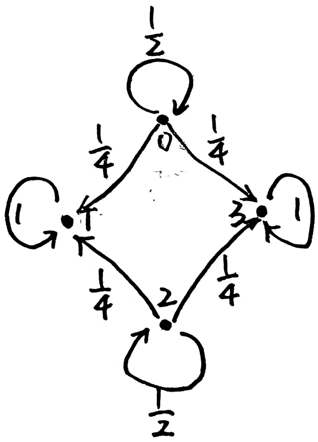
Solution:
我们记 $\{X_n : n = 0, 1, \dots\}$ 的两个状态为 $0, 1$

- **Z_n 是一个 Markov 链.**
  我们记 $z_n = (x_{n-1}, x_n) \ \ (n = 1, 2, \dots)$
  其中 x_{n-1}, x_n 只能在 $\{0, 1\}$ 中取值.
  则对于任意 $n = 1, 2, \dots$ 我们都有:
  $$\begin{aligned} \mathrm{P}\{Z_{n+1} = z_{n+1} | Z_n = z_n, \dots, Z_1 = z_1\} &= \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n, \dots, X_1 = x_1, X_0 = x_0\} \\ &= \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}\} \\ &= \mathrm{P}\{Z_{n+1} = z_{n+1} | Z_n = z_n\} \end{aligned}$$
  说明 $\{Z_n = (X_{n-1}, X_n) : n = 1, 2, \dots\}$ 具有 Markov 性，它是一个 Markov 链.
- **下面确定 $\{Z_n = (X_{n-1}, X_n) : n = 1, 2, \dots\}$ 的状态转移矩阵:**
  $$\begin{aligned} \mathrm{P}\{Z_{n+1} = z_{n+1} | Z_n = z_n\} &= \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n, X_{n-1} = x_{n-1}\} \\ &= \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n\} \\ &= P_{x_n, x_{n+1}} \end{aligned}$$
  $$\text{因此} \begin{cases} \mathrm{P}\{Z_{n+1} = (0,0) | Z_n = (0,0)\} = \mathrm{P}\{Z_{n+1} = (0,0) | Z_n = (1,0)\} = P_{0,0} = \alpha \\ \mathrm{P}\{Z_{n+1} = (0,1) | Z_n = (0,0)\} = \mathrm{P}\{Z_{n+1} = (0,1) | Z_n = (1,0)\} = P_{0,1} = 1-\alpha \\ \mathrm{P}\{Z_{n+1} = (1,0) | Z_n = (0,1)\} = \mathrm{P}\{Z_{n+1} = (1,0) | Z_n = (1,1)\} = P_{1,0} = 1-\beta \\ \mathrm{P}\{Z_{n+1} = (1,1) | Z_n = (0,1)\} = \mathrm{P}\{Z_{n+1} = (1,1) | Z_n = (1,1)\} = P_{1,1} = \beta \end{cases}$$
  于是 $\{Z_n = (X_{n-1}, X_n) : n = 1, 2, \dots\}$ 的状态转移矩阵为:
  $$Q = \begin{bmatrix} \alpha & 1-\alpha & & \\ & & 1-\beta & \beta \\ \alpha & 1-\alpha & & \\ & & 1-\beta & \beta \end{bmatrix}$$

Problem 2

A Markov chain with state space $\{0, 1, 2, 3\}$ has transition probability matrix
$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & & \frac{1}{4} \\ & 1 & & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ & & & 1 \end{bmatrix}$$
(a) Draw the transition diagram.

- **Solution:**



(b) Suppose the initial probability distribution is uniform, i.e. $\mathrm{P}\{X_0=i\}=\frac{1}{4}\ \ (i=0,1,2,3)$ Determine $\mathrm{P}\{X_n=1\}$ and $\mathrm{P}\{X_n=2\}$

- **Solution:**

$$P^{(2)}=P^2=\begin{bmatrix}\frac{1}{4} & \frac{3}{8} & & \frac{3}{8}\\ & 1 & & \\ & \frac{3}{8} & \frac{1}{4} & \frac{3}{8}\\ & & & 1\end{bmatrix}$$

$$P^{(3)}=P^3=\begin{bmatrix}\frac{1}{8} & \frac{7}{16} & & \frac{7}{16}\\ & 1 & & \\ & \frac{7}{16} & \frac{1}{8} & \frac{7}{16}\\ & & & 1\end{bmatrix}$$

归纳可知 $P^{(n)}=P^n=\begin{bmatrix}\frac{1}{2^n} & \frac{2^n-1}{2^{n+1}} & & \frac{2^n-1}{2^{n+1}}\\ & 1 & & \\ & \frac{2^n-1}{2^{n+1}} & \frac{1}{2^n} & \frac{2^n-1}{2^{n+1}}\\ & & & 1\end{bmatrix}$

(根据 (a) 中的图像可知这个归纳结果是合理的)

$$\begin{aligned}\mathrm{P}\{X_n=1\}&=\sum_{i=1}^{4}\mathrm{P}\{X_n=1|X_0=i\}\mathrm{P}\{X_0=i\}\\&=\sum_{i=1}^{4}P_{i,1}^{(n)}\cdot\frac{1}{4}\\&=\frac{1}{4}(\frac{2^n-1}{2^{n+1}}+1+\frac{2^n-1}{2^{n+1}}+0)\\&=\frac{1}{2}-\frac{1}{2^{n+2}}\end{aligned}$$

$$\begin{aligned}\mathrm{P}\{X_n=2\}&=\sum_{i=1}^{4}\mathrm{P}\{X_n=2|X_0=i\}\mathrm{P}\{X_0=i\}\\&=\sum_{i=1}^{4}P_{i,2}^{(n)}\cdot\frac{1}{4}\\&=\frac{1}{4}(0+0+\frac{1}{2^n}+0)\\&=\frac{1}{2^{n+2}}\end{aligned}$$

Problem 3

Consider independent tosses of a fair die.
Let X_n be the maximum of numbers appearing in the first n throws, $n = 1, 2, \dots$

(a) Verify that $\{X_n : n \geq 1\}$ is a Markov chain.
Describe the state space and the transition probability matrix of this homogeneous Markov Chain.

Solution:
记第 n 次投掷骰子的结果为 ξ_n，其分布为 $\mathrm{P}\{\xi_n = i\} = \frac{1}{6} \quad (i = 1, 2, 3, 4, 5, 6)$
则对于任意 $n \in \mathbb{N}$ 都有 $X_{n+1} = \max\{\xi_1, \dots, \xi_n, \xi_{n+1}\} = \max\{X_n, \xi_{n+1}\}$
显然有 $\mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n, \dots, X_1 = x_1\} = \mathrm{P}\{X_{n+1} = x_{n+1} | X_n = x_n\}$ 成立
说明 $\{X_n : n \geq 1\}$ 具有 Markov 性，因此是一个 Markov 链.

其状态空间为 $\{1, 2, 3, 4, 5, 6\}$，考虑转移概率:

- 当前 n 次投掷处于最大值 m 时，下一步仍停在 m 的概率为:

$$\mathrm{P}_{m,m} = \mathrm{P}\{\xi_{n+1} \leq m\} = \frac{m}{6}$$

- 当前 n 次投掷处于最大值 m 时，下一步变为 $j > m$ 的概率为:

$$\mathrm{P}_{m,m} = \mathrm{P}\{\xi_{n+1} = j\} = \frac{1}{6}$$

因此状态转移矩阵为:

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & & & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ & & & & \frac{5}{6} & \frac{1}{6} \\ & & & & & 1 \end{bmatrix}$$

(b) Find out the classes of all states.
Is there any state closed (a state is closed if, once entered, it cannot be left)?
Is there any state recurrent/transient?
Is there any state positive recurrent?
Write your reasons briefly.

Solution:

- 状态 6 是闭的 (即吸收态)，因为 $\mathrm{P}\{X_{n+1} = 6 | X_n = 6\} = P_{6,6} = 1$
- 状态 6 是常返态 (具体来说，是正常返态)，因为它是一个吸收态.
  状态 $1, 2, 3, 4, 5$ 是瞬时态，
  因为它们都可到达状态 6，可是一旦过程进入状态 6，它就无法返回到其他状态.

Problem 4

Assume a Markov chain X_n has state $0, 1, 2, \dots$

and transition probabilities: $\begin{cases} P_{i,0} = \frac{1}{2i^\alpha} \\ P_{i,i+1} = \begin{cases} 1 & i = 0 \\ 1 - \frac{1}{2i^\alpha} & i = 1, 2, \dots \end{cases} \end{cases}$ where $\alpha > 0$ is a constant.

(a) Is this Markov chain irreducible?

- **Solution:**
  - 一方面，任意状态 $i \geq 1$ 都能到达状态 0；
  - 另一方面，从状态 0 可以到达状态 1，从状态 1 可以到达状态 2，以此类推...
    所以从状态 0 可以到达任意状态 $i \geq 1$

  因此 $\{X_n : n \in \mathbb{N}\}$ 的所有状态之间都是互通的，因而是不可约的.

(b) Assume $X_0 = 0$ and let T be the first return time to 0
(i.e. the first time after the initial time the chain is back at the origin)
Determine for which $\alpha > 0$ is $1 - f_0 = \mathrm{P}\{\text{no return}\} = \lim_{n\to\infty} \mathrm{P}\{T > n\} = 0$

- **Solution:**

$$\begin{aligned} 1 - f_0 &= \mathrm{P}\{\text{no return}\} \\ &= \lim_{n\to\infty} \mathrm{P}\{T > n\} \\ &= \lim_{n\to\infty} \mathrm{P}\{X_n = n | X_0 = 0\} \\ &= \lim_{n\to\infty} (\prod_{i=0}^{n-1} P_{i,i+1}) \\ &= \lim_{n\to\infty} \{1 \cdot \prod_{i=1}^{n-1} (1 - \frac{1}{2i^\alpha})\} \end{aligned}$$

  要寻找 $\alpha > 0$ 使得 $1 - f_0 = \lim_{n\to\infty} \prod_{i=1}^{n-1} (1 - \frac{1}{2i^\alpha}) = 0$,

  等价于寻找 $\alpha > 0$ 使得 $\lim_{n\to\infty} \log\{\prod_{i=1}^{n-1} (1 - \frac{1}{2i^\alpha})\} = \lim_{n\to\infty} \sum_{i=1}^{n-1} \log(1 - \frac{1}{2i^\alpha}) = -\infty$

  根据 $\log(1 - x) \approx -x \ \ (x \to 0)$ 可知,

  等价于寻找 $\alpha > 0$ 使得 $\lim_{n\to\infty} \sum_{i=1}^{n-1} (-\frac{1}{2i^\alpha}) = -\infty$

  即等价于寻找 $\alpha > 0$ 使得 $\lim_{n\to\infty} \sum_{i=1}^{n-1} \frac{1}{i^\alpha} = \sum_{i=1}^{\infty} \frac{1}{i^\alpha} = \infty$

  我们知道级数 $\sum_{i=1}^{\infty} \frac{1}{i^\alpha}$ 在 $0 < \alpha \leq 1$ 时发散，在 $\alpha > 1$ 时收敛,

  因此当且仅当 $0 < \alpha \leq 1$ 时我们有 $1 - f_0 = \mathrm{P}\{\text{no return}\} = \lim_{n\to\infty} \mathrm{P}\{T > n\} = 0$ 成立.

(c) Depending on α, determine which classes are recurrent.

- **Solution:**
  根据 (b) 的结论，我们知道:
  状态 0 在 $0 < \alpha \leq 1$ 时是常返态，在 $\alpha > 1$ 时是瞬时态.

  由于所有状态都是互通的，因此它们要么都是常返态，要么都是瞬时态.
  因此所有状态在 $0 < \alpha \leq 1$ 时是常返态，在 $\alpha > 1$ 时是瞬时态.

Problem 5

Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a simple random walk.
Each step it has equal probability to walk to the left and to the right.
The first passage time $\tau(m) = \min\{n \geq 0 : X_n = m\}$ is the first time the process ever walks to position m

Suppose that $X_0 = 0$
Denote $G_m(z) = \mathrm{E}[z^{\tau(m)}]$ as the probability generating function of $\tau(m)$
Without loss of generality, assume that $m > 0$

(a) Show that $G_m(z) = (G_1(z))^m$

- **Lemma：(Catalan 数)**
  将 n 个 -1 和 $n+1$ 个 $+1$ 排成一列，
  使得任意前 i 项和 $(i = 1, 2, \ldots, 2n)$ 都小于 1 的排列总数是 **Catalan 数** $C_n = \frac{1}{n+1}\binom{2n}{n}$
  其递推公式为 $C_n = \sum\limits_{i=0}^{n-1} C_i \cdot C_{n-1-i}$
- **Solution:**
  记 $\tau(i,j) = \min\{n \geq 0 : X_n = j, \text{ given that } X_0 = i\}$
  记 $s(i,j) = \sum\limits_{k=i}^{j-1} \tau(k, k+1)$
  - **首先证明** $\tau(i, i+1) \overset{d}{=} \tau(j, j+1) \ \ (\forall\, i, j \in \mathbb{Z})$:
    任意给定 $i \in \mathbb{Z}$，我们注意到 $\tau(i, i+1)$ 只可能取奇数值 $2n+1$ $(n \geq 0)$
    对于任意 $n \geq 0$ 都有:
    $$\begin{aligned} \mathrm{P}\{\tau(i,i+1) = 2n+1\} &= \mathrm{P}\{X_{2n+1} = 1, X_k \leq 0 \text{ for all } k = 1, 2, \ldots, 2n | X_0 = 0\} \\ &= C_n(\frac{1}{2})^n(\frac{1}{2})^{n+1} \\ &= C_n(\frac{1}{2})^{2n+1} \end{aligned}$$
    根据 i 的任意性可知 $\tau(i, i+1) \overset{d}{=} \tau(j, j+1) \ \ (\forall\, i, j \in \mathbb{Z})$

  我们注意到，从状态 0 出发的过程要到达状态 $m > 0$，
  必然要依次到达状态 $1, \ldots, m-1$
  因此我们有:
  $$\begin{aligned} \tau(0,m) &= \min\{n \geq 0 : X_n = m, \text{ given that } X_0 = 0\} \\ &= \sum_{i=0}^{m-1} \min\{n \geq 0 : X_{s(0,i)+n} = j, \text{ given that } X_{s(0,i)} = i\} \\ &= \sum_{i=0}^{m-1} \min\{n \geq 0 : X_n = j, \text{ given that } X_0 = i\} \quad (\text{利用 Markov 性}) \\ &= \tau(0,1) + \tau(1,2) + \cdots + \tau(m-1, m) \end{aligned}$$
  根据 Markov 性我们还知道 $\tau(0,1), \ldots, \tau(m-1, m)$ 是相互独立的.
  因此我们有 $\tau(m) = \tau(0,m) \overset{d}{=} \sum\limits_{i=1}^{m} \xi_i$，
  其中 $\{\xi_i\}$ 独立同分布，$\xi_i \overset{d}{=} \tau(0,1) = \tau(1)$

因此对于任意 z 我们都有：

$$
\begin{aligned}
G_m(z) &= \mathrm{E}[z^{\tau(m)}] \\
&= \mathrm{E}[z^{\sum_{i=1}^{m}\xi_i}] \\
&= \prod_{i=1}^{m}\mathrm{E}[z^{\xi_i}] \\
&= \prod_{i=1}^{m}\mathrm{E}[z^{\tau(1)}] \\
&= \prod_{i=1}^{m}G_1(z) \\
&= (G_1(z))^m
\end{aligned}
$$

(b) Use the result of part (a), or otherwise, to find $G_m(z)$

- **Solution:**

  对 X_1 取条件，从分布意义上可得 $\tau(1) = \begin{cases} 1 & \text{if } X_1 = 1 \\ 1+\tau(2) & \text{if } X_1 = -1 \end{cases}$

  其中 $\tau(2)$ 是从 -1 出发首次到 1 所需步数的计数，
  相当于一个新过程从 0 出发首次到 2 所需步数的计数.

  故对于任意给定 z 我们都有：$z^{\tau(1)} = \begin{cases} z & \text{if } X_1 = 1 \\ z^{1+\tau(2)} & \text{if } X_1 = -1 \end{cases}$

  因此我们有：

$$
\begin{aligned}
G_1(z) &= \mathrm{E}[z^{\tau(1)}] \\
&= z \cdot \frac{1}{2} + z \cdot \mathrm{E}[z^{\tau(2)}] \cdot \frac{1}{2} \\
&= \frac{z}{2} + \frac{z}{2}G_2(z) \\
&= \frac{z}{2} + \frac{z}{2}(G_1(z))^2
\end{aligned}
$$

  求解 $G_1(z) = \frac{z}{2} + \frac{z}{2}(G_1(z))^2$ 可得 $G_1(z) = \frac{1}{z} \pm \sqrt{\frac{1}{z^2} - 1} = \frac{1}{z}(1 \pm \sqrt{1-z^2})$

  (注意写成**奇函数**的形式)

  由于 $G_1(z)$ 是一个生成函数，故对于任意 $z \in (0,1)$ 应有 $G_1(z) < 1$ 成立.
  因此我们取 $G_1(z) = \frac{1}{z}(1 - \sqrt{1-z^2})$

  故 $G_m(z) = (G_1(z))^m = (\frac{1}{z}(1 - \sqrt{1-z^2}))^m$