DATA130026 Optimization Assignment 5

Due Time: at the beginning of the class Apr. 10, 2022
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Problem 1

Consider the maximization problem

$$\begin{aligned} \max \quad & x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

It is equivalent to a minization problem：

$$\begin{aligned} \min \quad & -(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2) \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- **(1) Is the problem convex?**
  Claim： No, it is not a convex problem.
  Proof：
  Consider the objective function $f(x) = -(x_1^2 + 2x_1x_2 + 2x_2^2 - 3x_1 + x_2)$
  Since $\nabla^2 f(x) \equiv -\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \preceq 0$ (we know it by **Gershgorin Disk Theorem**),
  the objective function f is concave.
  Therefore, the problem is not a convex problem

- **(2) Find all the KKT points of the problem.**
  Solution:
  The Lagrangian function is given by:
  $L(x, \lambda, \nu) = -x_1^2 - 2x_1x_2 - 2x_2^2 + 3x_1 - x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \nu(x_1 + x_2 - 1)$
  The KKT conditions are: $\begin{cases} \nabla_x L(x, \lambda, \nu) = \begin{bmatrix} -2x_1 - 2x_2 + 3 - \lambda_1 + \nu \\ -2x_1 - 4x_2 - 1 - \lambda_2 + \nu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ x_1, x_2 \geq 0 \\ x_1 + x_2 = 1 \\ \lambda_1, \lambda_2 \geq 0 \\ \lambda_1 x_1 = 0 \\ \lambda_2 x_2 = 0 \end{cases}$
  - ① If $\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases}$ then we have $\begin{cases} x_1 = 3 \\ x_2 = -2 \\ \nu = -1 \end{cases}$, but $x = (3, -2)$ is not a feasible point.
  - ② If $\begin{cases} \lambda_1 > 0 \\ \lambda_2 = 0 \end{cases}$ then we have $\begin{cases} x_1 = 0 \\ x_2 = 1 \\ \lambda_1 = 6 \\ \nu = 5 \end{cases}$
  - ③ If $\begin{cases} \lambda_1 = 0 \\ \lambda_2 > 0 \end{cases}$ then we have $\begin{cases} x_1 = 1 \\ x_2 = 0 \\ \lambda_2 = -4 \\ \nu = -1 \end{cases}$, but $\lambda = (0, -4)$ is not dual feasible.
  - ④ If $\begin{cases} \lambda_1 > 0 \\ \lambda_2 > 0 \end{cases}$ then we have $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$, but $x = (0, 0)$ is not a feasible point.

  In conclusion, $x = (0, 1)$ is the only KKT point, whose Lagrange multipliers are $\begin{cases} \lambda = (6, 0) \\ \nu = 5 \end{cases}$

- **(3) Find the optimal solution of the problem.**
  Solution:
  - The objective function is continuous over the feasible region,
    a closed line segment within $\mathbb{R}^2$, thus constituting a nonempty compact set.
    By **Weierstrass theorem**, there exists a optimal solution of the problem.
  - Note that the contraints of the problem are all affine functions.
    By using **Theorem 3.5 in Note 4**,
    it is understood that the KKT conditions are **necessary** for local optimality.
    Therefore, $x = (0, 1)$ is the only local optimum,
    which consequently represents the problem's optimal solution.

Problem 2

Consider the optimization problem (P) $\begin{array}{ll} \min & x_1 - 4x_2 + x_3 \\ \text{s.t.} & x_1 + 2x_2 + 2x_3 = -2 \\ & x_1^2 + x_2^2 + x_3^2 \leq 1 \end{array}$

- **(1) Given a KKT point of problem (P), must it be an optimal solution?**
  Solution:
  Yes, it must.
  Note that (P) is a convex problem.
  By using **Theorem 3.7 in Note 4**,
  it is understood that KKT conditions are **sufficient** for global optimality.
- **(2) Find the optimal solution of the problem using the KKT conditions.**
  Solution:
  The Lagrangian function is given by:
  $L(x_1, x_2, x_3, \lambda, \nu) = x_1 - 4x_2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 1) + \nu(x_1 + 2x_2 + 2x_3 + 2)$
  The KKT conditions are: $\begin{cases} \nabla_x L(x, \lambda, \nu) = \begin{bmatrix} 1 + 2\lambda x_1 + \nu = 0 \\ -4 + 2\lambda x_2 + 2\nu = 0 \\ 1 + 2\lambda x_3 + 2\nu = 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ x_1^2 + x_2^2 + x_3^3 \leq 1 \\ x_1 + 2x_2 + 2x_3 = -2 \\ \lambda \geq 0 \\ \lambda(x_1^2 + x_2^2 + x_3^2 - 1) = 0 \end{cases}$
  - ① If $\lambda = 0$, then the system has no solution.
  - ② If $\lambda > 0$, then it holds that $x_1^2 + x_2^2 + x_3^2 = 1$
    By solving $\begin{cases} \nabla_x L(x, \lambda, \nu) = \begin{bmatrix} 1 + 2\lambda x_1 + \nu = 0 \\ -4 + 2\lambda x_2 + 2\nu = 0 \\ 1 + 2\lambda x_3 + 2\nu = 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ x_1 + 2x_2 + 2x_3 = -2 \end{cases}$
    we have $\begin{cases} \lambda = \frac{1}{4}(9\nu - 5) \quad (\text{which is positive, implying that } \nu > \frac{5}{9}) \\ x_1 = -2\frac{\nu+1}{9\nu-5} \\ x_2 = -4\frac{\nu-2}{9\nu-5} \\ x_3 = -2\frac{2\nu+1}{9\nu-5} \end{cases}$
    Plugging the solution into $x_1^2 + x_2^2 + x_3^2 = 1$ we have $45\nu^2 - 50\nu - 47 = 0$
    The solutions are $\nu_{1,2} = \frac{5}{9} \pm \frac{2\sqrt{685}}{45}$
    We choose $\nu = \frac{5}{9} + \frac{2\sqrt{685}}{45} \approx 1.7188$ (since $\nu > \frac{5}{9}$)

Thus, the unique solution of KKT conditions is $\begin{cases} x_1 = -0.5194 \\ x_2 = 0.1074 \\ x_3 = -0.8477 \\ \lambda = 2.6173 \\ \nu = 1.7188 \end{cases}$
Therefore, $x = (-0.5194, 0.1074, -0.8477)$ is the only optimal solution.

Problem 3

Consider the optimization problem $(\mathrm{P})\ \min\{a^T x : x^T Q x + 2b^T x + c \leq 0\}$
where $Q \in \mathbb{R}^{n\times n}$ is positive definite, $a(\neq 0_n), b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

- **(1) For which values of Q, b, c is the problem feasible?**
  Solution:
  (P) is feasible if and only if $\min\{g(x)\} \leq 0$
  Since $g(x) = x^T Q x + 2b^T x + c$ is a convex function,
  and the solution of $\nabla g(x) = 2Qx + 2b = 0$ is $x_0 = -Q^{-1}b$,
  we know that $\min\{g(x)\} = g(x_0) = -b^T Q^{-1} b + c$
  Hence the condition is $-b^T Q^{-1} b + c \leq 0$.
- **(2) For which values of Q, b, c are the KKT conditions necessary?**
  Solution:
  - In light of **Theorem 3.4 in Note 4**,
    the KKT conditions are necessary for local optimality within Convex Optimization Problem,
    if the **Slater's condition** holds,
    though Slater's condition is not the only criterion for their necessity.
    This entails that $\min\{g(x)\} < 0$, formally expressed as $-b^T Q^{-1} b + c < 0$.
  - In an effort to proceed with caution,
    we demonstrate that the KKT conditions are not necessary when $-b^T Q^{-1} b + c = 0$:
    Obviously $x_0 = -Q^{-1}b$ is the only feasible point when $-b^T Q^{-1} b + c = 0$,
    thereby making x_0 both the local and global optimum.
    The Lagrangian function is given by:
    $L(x, \lambda) = a^T x + \lambda(x^T Q x + 2b^T x + c)$
    Note that $\nabla_x L(x_0, \lambda) = a + \lambda(2Qx_0 + 2b) = a \neq 0_n$,
    which indicates that $x_0 = -Q^{-1}b$ fails to meet the KKT conditions.
- **(3) For which values of Q, b, c are the KKT conditions sufficient?**
  Solution:
  In light of **Theorem 3.7 in Note 4**,
  the sufficiency of the KKT conditions always holds within convex problems,
  so the answer is the same as (1): $-b^T Q^{-1} b + c \leq 0$
- **(4) Under the condition of part (2), find the optimal solution of (P) using the KKT conditions.**
  Solution:
  It is given that $-b^T Q^{-1} b + c < 0$.
  The Lagrangian function is given by:

$L(x,\lambda) = a^Tx + \lambda(x^TQx + 2b^Tx + c)$

The KKT conditions are: $\begin{cases} \nabla_x L(x,\lambda) = a + \lambda(2Qx + 2b) = 0_n \\ x^TQx + 2b^Tx + c \le 0 \\ \lambda \ge 0 \\ \lambda(x^TQx + 2b^Tx + c) = 0 \end{cases}$

- ① If $\lambda = 0$, then we have $a = 0_n$, which contradicts $a \neq 0_n$
- ② If $\lambda > 0$, then we have $x^TQx + 2b^Tx + c = 0$

  By diminishing x in $\begin{cases} a + \lambda(2Qx + 2b) = 0_n \ \Rightarrow \ x = -\frac{1}{2\lambda}Q^{-1}(a + 2\lambda b) \\ x^TQx + 2b^Tx + c = 0 \end{cases}$

  we have $\frac{1}{4\lambda^2}a^TQ^{-1}a + c - b^TQ^{-1}b = 0$

  Thus $\lambda = \frac{1}{2}\sqrt{\frac{a^TQ^{-1}a}{b^TQ^{-1}b - c}}$ (discarding the negative root)

  Therefore $x = -\frac{1}{2\lambda}Q^{-1}(a + 2\lambda b) = -Q^{-1}(\sqrt{\frac{b^TQ^{-1}b - c}{a^TQ^{-1}a}}a + b)$ is the only **KKT point**.

  By (2), we know that $x = -Q^{-1}(\sqrt{\frac{b^TQ^{-1}b - c}{a^TQ^{-1}a}}a + b)$ is the only **local optimum**,

  which consequently represents the problem's **optimal solution**.

Problem 4

Consider the optimization problem $\begin{array}{ll} \min & x_1^2 - x_2^2 - x_3^2 \\ \text{s.t.} & x_1^4 + x_2^4 + x_3^4 \le 2 \end{array}$

- **(1) Is the problem convex?**

  Solution:

  Since $\nabla^2 f(x) \equiv \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$ is not positive semi-definite,

  the objective function f is not convex.

  Hence the problem is not convex.

- **(2) Find all the KKT points of the problem.**

  Solution:

  The Lagrangian is given by:

  $L(x,\lambda) = x_1^2 - x_2^2 - x_3^2 + \lambda(x_1^4 + x_2^4 + x_3^4 - 2)$

  The KKT conditions are: $\begin{cases} \nabla_x L(x,\lambda) = \begin{bmatrix} x_1(2\lambda x_1^2 + 1) \\ x_2(2\lambda x_2^2 - 1) \\ x_3(2\lambda x_3^2 - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ x_1^4 + x_2^4 + x_3^4 \le 2 \\ \lambda \ge 0 \\ \lambda(x_1^4 + x_2^4 + x_3^4 - 2) = 0 \end{cases}$

  - ① If $\lambda = 0$, then we have $x_1 = x_2 = x_3 = 0$, which is primal feasible.

    Therefore $x = (0,0,0)$ is a KKT point.

  - ② If $\lambda > 0$, then we have $x_1^4 + x_2^4 + x_3^4 = 2$

    Given $x_1(1 + 2\lambda x_1^2) = 0$, we deduce $x_1 = 0$,

    which implies that x_2, x_3 cannot be zero at the same time.

    - If $x_2, x_3 \neq 0$, then $x_2 = \pm\frac{1}{\sqrt{2\lambda}}, x_3 = \pm\frac{1}{\sqrt{2\lambda}}$

      By $x_1^4 + x_2^4 + x_3^4 = 2$, it follows that $\lambda = \frac{1}{2}$, which is dual feasible.

      Therefore $x = (0, \pm1, \pm1)$ are KKT points.

        - If $x_2 = 0, x_3 \neq 0$
          By $x_1^4 + x_2^4 + x_3^4 = 2$, it follows that $x_3 = \pm\sqrt[4]{2}$
          Given $x_3(2\lambda x_3^2 - 1) = 0$, we deduce $\lambda = \frac{\sqrt{2}}{4}$, which is dual feasible.
          Therefore $x = (0, 0, \pm\sqrt[4]{2})$ are KKT points.
        - If $x_2 \neq 0, x_3 = 0$
          By $x_1^4 + x_2^4 + x_3^4 = 2$, it follows that $x_2 = \pm\sqrt[4]{2}$
          Given $x_2(2\lambda x_2^2 - 1) = 0$, we deduce $\lambda = \frac{\sqrt{2}}{4}$, which is dual feasible.
          Therefore $x = (0, \pm\sqrt[4]{2}, 0)$ are KKT points.

  In conclusion, there are 9 KKT points：
  $(0,0,0), (0, \pm\sqrt[4]{2}, 0), (0, 0, \pm\sqrt[4]{2}), (0, \pm 1, \pm 1)$
- **(3) Find the optimal solution of the problem.**
  Within the scope of the problem, it is clear that all feasible points meet the criteria of regularity.
  Under this assumption of regularity, the KKT conditions are necessary for local optimality, hence the optimal solutions only exist in KKT points.
    - $f(0,0,0) = 0$
    - $f(0, \pm\sqrt[4]{2}, 0) = f(0, 0, \pm\sqrt[4]{2}) = -\sqrt{2}$
    - $f(0, \pm 1, \pm 1) = -2$

  So the optimal solution of the problem is $x = (0, \pm 1, \pm 1)$

THE END