SYSTEM STATE:

Encoders:

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \dot{x}(k+1) \\ \dot{y}(k+1) \\ \theta(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \frac{d_l + d_r}{2} * \cos(\theta(k) + \Delta\theta(k)) \\ y(k) + \frac{d_l + d_r}{2} * \sin(\theta(k) + \Delta\theta(k)) \\ \frac{d_l + d_r}{2 * T} * \cos(\theta(k) + \Delta\theta(k)) \\ \frac{d_l + d_r}{2 * T} * \sin(\theta(k) + \Delta\theta(k)) \\ \theta(k) + \frac{d_l - d_r}{R} \end{bmatrix}$$

Now Linearize the Encoder Readings. Let:

$$u(k) = \begin{bmatrix} \Delta D(k) \\ V(k) \\ \Delta\theta(k) \end{bmatrix} = \begin{bmatrix} \frac{d_l + d_r}{2} \\ \frac{d_l + d_r}{2 * T} \\ \frac{d_l - d_r}{R} \end{bmatrix}$$

Now the **State Transition Function** is

$$f(x(k), u(k)) = \begin{bmatrix} x(k) + \Delta D(k) * \cos(\theta(k) + \Delta\theta(k)) \\ y(k) + \Delta D(k) * \sin(\theta(k) + \Delta\theta(k)) \\ V(k) * \cos(\theta(k) + \Delta\theta(k)) \\ V(k) * \sin(\theta(k) + \Delta\theta(k)) \\ \theta(k) + \Delta\theta(k) \end{bmatrix}$$

Take the Jacobian of the State Transition Function w.r.t. ΔΘ(k), V(k) and ΔD(k):

$$\nabla f_u(x(k), u(k)) = \begin{bmatrix} -\Delta D(k) * sin(\theta(k) + \Delta\theta(k)) & 0 & \cos(\theta(k) + \Delta\theta(k)) \\ \Delta D(k) * cos(\theta(k) + \Delta\theta(k)) & 0 & \sin(\theta(k) + \Delta\theta(k)) \\ -V(k) * sin(\theta(k) + \Delta\theta(k)) & \cos(\theta(k) + \Delta\theta(k)) & 0 \\ V(k) * cos(\theta(k) + \Delta\theta(k)) & \sin(\theta(k) + \Delta\theta(k)) & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Use this to find the **Process Noise Covariance Matrix**

$$Q(k) = \nabla f_u(k) * \begin{bmatrix} \sigma_{\Delta\theta}^2 & 0 & 0 \\ 0 & \sigma_V^2 & 0 \\ 0 & 0 & \sigma_{\Delta D}^2 \end{bmatrix} * \nabla f_u(k)^T$$

Then take the Jacobian of the State Transition Function w.r.t. the State Variables:

$$\nabla f_X\big(x(k),u(k)\big) = \begin{bmatrix} 1 & 0 & 0 & 0 & -\Delta D(k) * sin(\theta(k)+\Delta\theta(k)) \\ 0 & 1 & 0 & 0 & \Delta D(k) * cos(\theta(k)+\Delta\theta(k)) \\ 0 & 0 & 0 & 0 & -V(k) * sin(\theta(k)+\Delta\theta(k)) \\ 0 & 0 & 0 & 0 & V(k) * cos(\theta(k)+\Delta\theta(k)) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Use this to find the **Process Error Covariance Matrix** (in the Prediction Phase)

$$P(k+1|k) = \nabla f_X(k) * P(k|k) * \nabla f_X(k)^T + Q(k)$$

To proceed, we first need to find the **Expected Measurement Function**[1] from the Accelerometer to compare to the readings from the Encoder**:**

$$h\big(\hat{x}(k+1|k)\big) = \begin{bmatrix} T^{-2} & 0 & 0 & 0 & 0 \\ 0 & T^{-2} & 0 & 0 & 0 \\ 0 & 0 & T^{-1} & 0 & 0 \\ 0 & 0 & 0 & T^{-1} & 0 \\ 0 & 0 & 0 & 0 & T^{-1} \end{bmatrix} \begin{bmatrix} x(k+1) \\ y(k+1) \\ \dot{x}(k+1) \\ \dot{y}(k+1) \\ \theta(k+1) \end{bmatrix}$$

Then the Jacobian of each should be taken w.r.t. the state variables of time step (k+1|k):

$$\nabla h_X\big(\hat{x}(k+1|k)\big) = \begin{bmatrix} T^{-2} & 0 & 0 & 0 & 0 \\ 0 & T^{-2} & 0 & 0 & 0 \\ 0 & 0 & T^{-1} & 0 & 0 \\ 0 & 0 & 0 & T^{-1} & 0 \\ 0 & 0 & 0 & 0 & T^{-1} \end{bmatrix}$$

Then, since there are 5 States being measured,

$$R(k+1) = \begin{bmatrix} {\sigma_{ax}}^2 & 0 & 0 & 0 & 0 \\ 0 & {\sigma_{ax}}^2 & 0 & 0 & 0 \\ 0 & 0 & {\sigma_{av}}^2 & 0 & 0 \\ 0 & 0 & 0 & {\sigma_{av}}^2 & 0 \\ 0 & 0 & 0 & 0 & {\sigma_g}^2 \end{bmatrix}$$

Now we calculate the **Innovation Matrix** (this step combined with the next one by the other MQP team)

$$S(k+1) = \nabla h_X * P(k+1|k) * \nabla h_x^T + R(k+1)$$

Then we can calculate the **Kalman Gain:**

$$K(k+1) = P(k+1|k) * \nabla h_x^T * S^{-1}(k+1)$$

Now we can use this to correct the **Process Error Covariance Matrix** (Correction Phase)

$$P(k+1|k+1) = P(k+1|k) - K(k+1) * S(k+1) * K^T(k+1)$$

Then finally, the **State Estimation**[2] is:

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1)(z(k+1) - h(k+1))$$

[1] See Page 3 for derivation
[2] See Page 3 for z(k+1) reference

Expected Measurement Function Derivation:

The only other instrument we have to compare the values of the state to is the accelerometer. So, we must express the accelerometer data in terms of our state variables.

The accelerometer data consists of:

$$\begin{bmatrix} a_x(k+1) \\ a_y(k+1) \\ \dot{\theta}(k+1) \end{bmatrix}$$

Now instead of expressing acceleration as a double differential of position w.r.t. time, we discretize the equations through "pseudo-integration". This means that for each time step, the process of integrating is equivalent to multiplying by time step T. This results in the following equations:

$$\begin{bmatrix} a_x(k+1) \\ a_y(k+1) \\ a_x(k+1) \\ a_y(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} \frac{x(k+1)}{T^2} \\ \frac{y(k+1)}{T^2} \\ \frac{\dot{x}(k+1)}{T^1} \\ \frac{\dot{y}(k+1)}{T^1} \\ \frac{\theta(k+1)}{T^1} \end{bmatrix}$$

And then by finding this in terms of the state **X**, we arrive at the final equation:

$$h\big(\hat{x}(k+1|k)\big) = \begin{bmatrix} a_x(k+1) \\ a_y(k+1) \\ a_x(k+1) \\ a_y(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} T^{-2} & 0 & 0 & 0 & 0 \\ 0 & T^{-2} & 0 & 0 & 0 \\ 0 & 0 & T^{-1} & 0 & 0 \\ 0 & 0 & 0 & T^{-1} & 0 \\ 0 & 0 & 0 & 0 & T^{-1} \end{bmatrix} \begin{bmatrix} x(k+1) \\ y(k+1) \\ \dot{x}(k+1) \\ \dot{y}(k+1) \\ \theta(k+1) \end{bmatrix}$$

Linearization

Once the system has been linearized as it was in the EKF, it can be written as follows:

$$\boldsymbol{X}(k+1) = \nabla f_X(k) * \boldsymbol{X}(k) + \nabla f_u(k) * u(k) + n(k)$$

$$z(k+1) = \nabla h_X * \boldsymbol{X}(k+1|k) + w(k+1)$$