Astr 511: Galaxies as galaxies

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Lecture 7:

Dynamics II: Galaxies as Collisonless Systems

Modeling Galaxies

In the previous lecture we learned how to qualitatively (and quantitatively) understand observed properties of galaxies by considering some (static) potentials and orbits that those potentials admit.

But we haven't tackled the more difficult problem: how does one find self-consistent, equilibrium, solutions of the Poisson equation equation $\nabla^2\Phi = 4\pi G\rho$, that we can compare to the data? How do we "build" a model of a galaxy?

This is the problem we'll turn to in this lecture. But first...

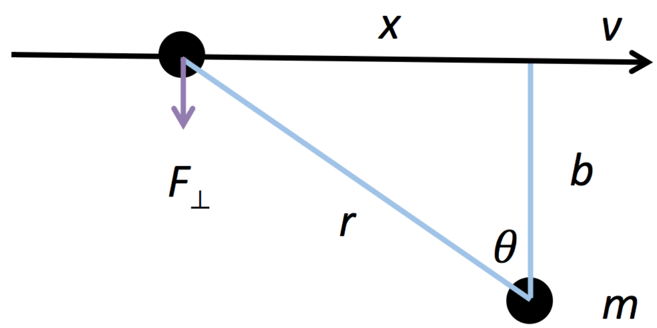
Galaxies in Continuum Approximation

How do we understand the internal dynamics of galaxies? We could go back to basic, and treat them as N-body systems consisting of (at least) $N \sim 10^{10}$ objects. That will, however, be quickly revealed as impractical (to put it mildly...).

Taking a hint from fluid dynamics, we try a different approach: could we consider them as made up from a smooth, continous, fluid instead? That is, how accurately can we approximate a galaxy of N identical stars of mass m as **a smooth density distribution plus a gravitational field?**

To answer this question, let's observe the motion of an individual star as its orbit carries it once across the galaxy. Let's find an order-of-magnitude estimate of the difference between the actual velocity of this star, and the velocity that it would have had if the masses of other stars were smoothly distributed.

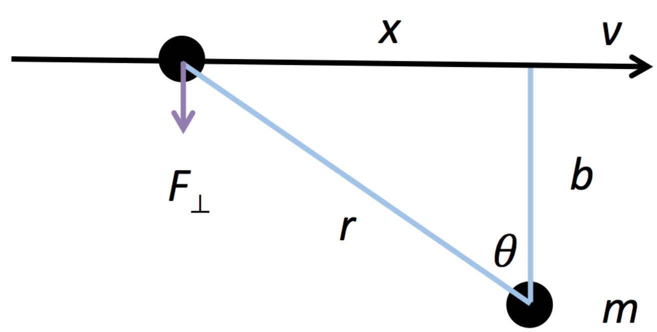
Two-body Scattering



Consider the situation above, where a **subject star** (top) scatters off of a **field star** (bottom) with an **impact parameter** b (the distance of closes approach). Assume the deflection is *small* (i.e., the trajectory roughly remains to be a straight line). To an order of magnitude, the deflection δv will be given by:

$$\delta v = F_{\perp} \times \delta t_{encounter} = \left(\frac{Gm}{b^2}\right) \times \left(\frac{2b}{v}\right) = \frac{2Gm}{bv} \tag{1}$$

Two-body Scattering



When does this line of reasoning break down? Definitely when $\delta v \approx v$. What does that equate to in terms of the impact parameter b?

$$b \lesssim b_{\mathrm{min}} \equiv \frac{2Gm}{v^2} \approx \frac{R}{N} \qquad (2)$$

where the last equality comes from $v^2 = \frac{GM}{R} = \frac{GNm}{R}$, the circular velocity at radius R.

Galaxies as Collisionless Systems

What is $b_{\rm min}$ for the Milky Way? Plugging in $R = 10$kpc and $N = 10^{10}$ we get:

$$b_{\rm min,MW} \simeq = 3 \times 10^7 \text{m} \simeq 50 R_\odot \tag{3}$$

Your intuition already tells you this kind of encounter will be infrequent. How infrequent?

If we consider the probability of a disk of cross section $4\pi b^2_{\rm min,MW}$ to experience a collision while crossing a spherical galaxy with density $n = \frac{3N}{4\pi R^3}$, we find:

$$t_{scatter} \simeq \left(\frac{R}{b_{\rm min}}\right) \frac{t_{cross}}{N} = N t_{cross} \tag{4}$$

where $t_{cross} \simeq 50$Myr and $N \approx 10^{10}$. Therefore, **the Milky Way is a collisionless system**. I.e., stars almost never "collide" (strongly scatter) in the MW!

Relaxation time

How long does it take for a star's velocity to change appreciably due to the two-body scattering it experiences?

As the star crosses, these interactions accumulate. They do not change the mean velocity (i.e. $\Delta v_\perp = 0$), but only the variance $(\Delta v_\perp)^2$. In one crossing, a star experiences:

$$\delta n = \left(\frac{N}{\pi R^2}\right)(2\pi b)\, db = \frac{2N}{R^2} b db \tag{5}$$

encounters with impact parameters between b and $b + db$. The total change in $(\Delta v_\perp)^2$ is therefore:

$$\int d(\Delta v_\perp)^2 = \int_{b_{\rm min}}^{R} \left(\frac{2Gm}{bv}\right)^2 \frac{2Nb}{R^2} db = 8N \left(\frac{Gm}{Rv}\right)^2 \int_{b_{\rm min}}^{R} \frac{db}{b} \tag{6}$$

$$(\Delta v_\perp)^2 = 8N \left(\frac{Gm}{Rv}\right)^2 \ln \Lambda \tag{7}$$

where $\ln \Lambda = \ln \frac{R}{b_{\rm min}} = \ln N$ is know as the **Coulomb logarithm**.

Relaxation time

So, putting it all together, we have:

$$(\Delta v_\perp)^2 = \frac{8 \ln N}{N} \left(\frac{GNm}{R}\right)^2 \frac{1}{v^2} = \frac{8 \ln N}{N} v^2 \tag{8}$$

$$\frac{(\Delta v_\perp)^2}{v^2} \approx 10 \frac{\ln N}{N} \tag{9}$$

meaning that it takes about $\sim .1 \times N/lnN$ crossings for $(\Delta v_\perp)^2$ to become comparable to v^2. Expressed in terms of timescales:

$$t_{\text{relax}} = \frac{N}{10 \ln N} t_{\text{cross}} \tag{10}$$

where t_{relax} is the **relaxation time** – the time it takes the system to "forget" its initial conditions.

Relaxation time

Relaxation time:

$$t_{\text{relax}} = \frac{N}{10\ln N} t_{\text{cross}} = \frac{1}{10\ln N} t_{\text{scatter}} \tag{11}$$

Plugging in the numbers for the Milky Way, we find that $t_{\text{relax}} \approx 2 \times 10^6$ Gyr, **i.e., the Milky Way (and galaxies in general) are not relaxed systems**. In other words, **global galaxy properties that we observe are largely a consequence of their formation**.

N.b.: typically, in collisionless systems we have:

$$t_{\text{cross}} \ll t_H \precsim t_{\text{form}} \ll t_{relax} \ll t_{scatter} \ll t_{coll} \tag{12}$$

Equilibria of Colisionless Systems

- Binney & Tremaine, Chapter 4.

The Distribution Function (DF)

The positions and motions of stars can be described by a phase-space distribution function $f(\mathbf{x}, \mathbf{v}, t)$ (aka the phase-space number density).

The **distribution function (DF)** fully encodes the state of a dynamical system (i.e., we know where all parcels of matter are, and how they're moving).

For example, the density distribution is an integral over the velocities:

$$\rho = \int f(x, v, t) d\mathbf{v} \tag{13}$$

etc.

Time evolution of a collisionless system

The time evolution of $f(\mathbf{x}, \mathbf{v}, t)$ is governed by Newtonian dynamics:

$$\nabla^2 \Phi = 4\pi G \rho \tag{14}$$

Assuming that stars can be neither created nor destroyed, the **continuity equation**:

$$\frac{\partial f}{\partial t} + \nabla \cdot (f \mathbf{v}) = 0 \tag{15}$$

can be applied to $f(\mathbf{x}, \mathbf{v}, t)$. In six-dimensional space described by $w_i = (\mathbf{x}, \mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$,

$$\frac{\partial f(\mathbf{w}, t)}{\partial t} + \sum_{i=1}^{6} \frac{\partial \left(f(\mathbf{w}, t) \dot{w}_i \right)}{\partial w_i} = 0. \tag{16}$$

The collisonless Boltzmann Equation

$$\frac{\partial (f\dot{w}_i)}{\partial w_i} = \dot{w}_i \frac{\partial f}{\partial w_i} + f \frac{\partial \dot{w}_i}{\partial w_i} \tag{17}$$

Note that the last term is either $(\partial v_i / \partial x_i)$, or $(\partial \dot{v}_i / \partial v_i)$.

This term is always zero: in the first case because v_i and x_i are independent coordinates, and in the second case because $\dot{v}_i = -(\partial \Phi / \partial x_i)$, and Φ does not depend on velocity (because it's gravitational potential). Hence,

$$\frac{\partial f(\mathbf{w}, t)}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f(\mathbf{w}, t)}{\partial w_i} = 0. \tag{18}$$

The collisonless Boltzmann Equation (CBE)

We therefore obtain the **collisionless Boltzmann Equation**:

$$\frac{\partial f}{\partial t} + \mathbf{x}\frac{\partial f}{\partial \mathbf{x}} + \mathbf{v}\frac{\partial f}{\partial \mathbf{v}} = 0 \quad \text{or compactly} \quad \frac{df}{dt} = 0 \tag{19}$$

This is the equation of motion of self-gravitating collisionless fluid. In other forms:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left[v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\right] = 0 \tag{20}$$

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = \nabla \Phi \frac{\partial f}{\partial \mathbf{v}} \tag{21}$$

The collisonless Boltzmann Equation (CBE)

The last (vector) notation is the most useful one for expressing the collisonless Boltzmann equation in arbitrary coordinate systems

The CBE is very difficult to solve directly (and hence not terribly useful from that standpoint), but forms a) the basis for deriving the Jeans equations, and b) the starting point for N-body methods (N-body codes are essentially Monte-Carlo solvers of the CBE).

A side note: the radiative transfer equation is also a special case of the general Boltzmann Equation (in the limit that all particles move at the same speed).

The Moment Equations

Some insights can be obtained by integrating the CBE multiplied by powers of the coordinates and/or velocities. By doing so we will end up with differential equations for the evolution of various **moments** of the distribution function.

For example, let us integrate the CBE over all velocities:

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{22}$$

The Moment Equations

For example, let us integrate the CBE over all velocities:

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{23}$$

How do we evaluate these integrals? Two rules:

1. Derivative wrt $\mathbf{x}$, or a function of $\mathbf{x}$, can be taken out of the integral as **v** and **x** are independent, and

2. Let us introduce the notation

$$\int g(\mathbf{v}) f \mathrm{d}^3\mathbf{v} = < g > \int f \mathrm{d}^3\mathbf{v} \tag{24}$$

   where

$$\nu(\mathbf{x}) = \int f \mathrm{d}^3\mathbf{v} \tag{25}$$

   is the number density as a function of position.

The Moment Equations

Then

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{26}$$

with

$$\overline{v}_i \equiv \frac{1}{\nu} \int f v_i \mathrm{d}^3\mathbf{v}, \tag{27}$$

becomes

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu \overline{v}_i)}{\partial x_i} = 0. \tag{28}$$

This is just the continuity equation for the stellar number density in real space!

More interesting results are obtained by multiplying the CBE with higher powers of $\mathbf{v}$.

The Moment Equations

E.g. take the first velocity moment of the CBE. Then

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{29}$$

becomes

$$\frac{\partial}{\partial t} \int f v_j \mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{30}$$

We can use the divergence theorem to manipulate the last term

$$\int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = -\int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\mathbf{v} = -\int \delta_{ij} f \mathrm{d}^3\mathbf{v} = -\delta_{ij} \nu, \tag{31}$$

Note that

$$v_j \frac{\partial f}{\partial v_i} = -f \frac{\partial v_j}{\partial v_i} + \frac{\partial (v_j f)}{\partial v_i} \tag{32}$$

and the last term must be 0 when the integration surface is expendend to infinity (where f must vanish).

The Moment Equations

Eq.(31) can be substituted into (30) giving

$$\frac{\partial(\nu\overline{v_j})}{\partial t} + \frac{\partial(\nu\overline{v_i v_j})}{\partial x_i} + \nu\frac{\partial\Phi}{\partial x_j} = 0, \tag{33}$$

where

$$\overline{v_i v_j} \equiv \frac{1}{\nu}\int v_i v_j f \mathrm{d}^3\mathbf{v}. \tag{34}$$

This is an equation of momentum conservation.

The Moment Equations

Each velocity can be expressed as a sum of the mean value (aka streaming motion) and the so-called peculiar velocity

$$v_i = \overline{v_i} + w_i \tag{35}$$

where $\overline{w_i} = 0$ by definition. Then

$$\sigma_{ij}^2 \equiv \overline{w_i w_j} = \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v}_i \overline{v}_j. \tag{36}$$

At each point $\mathbf{x}$ the symmetric tensor σ^2 defines an ellipsoid whose principal axes run parallel to σ^2's eigenvectors and whose semi-axes are proportional to the square roots of σ^2's eigenvalues. This is called the **velocity ellipsoid** at $\mathbf{x}$. We have encountered it when discussing the LOSVD in Lecture 5, and we'll encounter it again when we examine the kinematics of the Milky Way.

The Jeans Equations

Taken together, the continuity equation:

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} = 0. \tag{37}$$

and the momentum equation

$$\nu \frac{\partial \overline{v_j}}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i} \tag{38}$$

are commonly know as the **Jeans Equations**. They are analogous to Euler equations of fluid dynamics. The term $-\nu\sigma_{ij}^2$ is a **stress tensor** – it describes anisotropic pressure.

Note that **the system is not closed** (like it is in gases): there is no "equation of state"! The multiplication by higher powers of $\mathbf{v}$ doesn't help – need an *ansatz*. In practice one assumes a particular form for σ_{ij}^2, e.g. for isotropic velocity dispersion $\sigma_{ij}^2 = \sigma^2 \delta_{ij}$

The Jeans Equations

Specialization for an axially symmetric system:

First express the CBE in cylindrical coordinates

$$\frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0 \tag{39}$$

With $\dot{R} \equiv v_R$, $\dot{\phi} \equiv v_\phi/R$, and $\dot{z} \equiv v_z$, and

$$\dot{v}_R = -\frac{\partial \Phi}{\partial R} + \frac{v_\phi^2}{R} \tag{40}$$

$$\dot{v}_\phi = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{v_R v_\phi}{R} \tag{41}$$

$$\dot{v}_z = -\frac{\partial \Phi}{\partial z} \tag{42}$$

we get

The Jeans Equations

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R} \right] \frac{\partial f}{\partial v_R} - \frac{v_R v_\phi}{R} \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 \quad (43)$$

where it was assumed that $\partial / \partial \phi \equiv 0$.

Now we multiply by v_R, v_z and v_ϕ, and integrate over all velocities to get (assuming steady state)

$$\begin{aligned} \frac{\partial(\nu \overline{v_R^2})}{\partial R} + \frac{\partial \nu \overline{v_R v_z}}{\partial z} + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) &= 0, \\ \frac{\partial(\nu \overline{v_R v_\phi}}{\partial R} + \frac{\partial(\nu \overline{v_\phi v_z})}{\partial z} + \frac{2\nu}{R} \overline{v_\phi v_R} &= 0, \qquad (44) \\ \frac{\partial(\nu \overline{v_R v_z})}{\partial R} + \frac{\partial(\nu \overline{v_z^2})}{\partial z} + \frac{\nu \overline{v_R v_z}}{R} + \nu \frac{\partial \Phi}{\partial z} &= 0. \end{aligned}$$

Lovely! And powerful.

Some Applications of the Jeans Equations

- Asymmetric drift
- The local mass density
- The shape of local velocity ellipsoid
- Spheroidal components with isotropic velocity dispersion
- Halo mass density profile

Application: Asymmetric drift

Observations indicate that stars with large $\overline{v_R^2}$ rotate more slowly:

$$\overline{v_\phi} = v_c - \overline{v_R^2}/D \tag{45}$$

with $D \approx 120$ km/s. This can be explained using the v_R Jeans equation.

From the v_R Jeans equation at $z = 0$, with an assumed symmetry around the equatorial plane, $\partial\nu/\partial z = 0$, and definitions $\sigma_\phi^2 = \overline{v_\phi^2} - \overline{v_\phi}^2$ and $v_c^2 = R(\partial\Phi/\partial R)$:

$$\overline{v_\phi} = v_c - \frac{\overline{v_R^2}}{2v_c}\zeta, \tag{46}$$

where

$$\zeta = \frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu\overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}}\frac{\partial(\overline{v_R v_z})}{\partial z} \tag{47}$$

How large is each of these terms?

Asymmetric drift

$$\zeta = \frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu \overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}} \frac{\partial(\overline{v_R v_z})}{\partial z} \tag{48}$$

1. We know that locally $\overline{v_z^2}/\overline{v_R^2} \approx \sigma_\phi^2/\overline{v_R^2} \approx 0.45$

2. $R(\partial(\overline{v_R v_z})/\partial z)/\overline{v_R^2}$ is somewhere between 0 and 0.55

3. The largest term is $\partial \ln(\nu \overline{v_R^2})/\partial \ln R \approx 2(\partial \ln \nu/\partial \ln R) \approx R_\odot/R_d \approx$ 2.4, where it was assumed that $v_R^2 \propto \nu$ and that $\nu(R) \propto \exp(-R/R_d)$.

Asymmetric drift

Hence,

$$\zeta = 0.45 - 1 - 4.8 - x = -5.35 - x \tag{49}$$

where $0 < x < 0.55$. That is, ζ is uncertain to within only 10%.

These arguments can be inverted, and the measured value of ζ (from asymmetric drift slope) can be used to infer $R_\odot/R_d$ (or, more generally, $\partial \ln \nu / \partial \ln R$).

If there were no density gradient, there would be no asymmetric drift!

Using the CBE to Build Galaxy Models

- Binney & Tremaine, Chapter 4.

- Barnes, Galaxies, https://www.ifa.hawaii.edu/~barnes/ast626_09/scbe.pdf

Integrals of Motion

An **integral of motion** is any function $I(\mathbf{r}, \mathbf{v})$ of the phase space coordinates $(\mathbf{r}, \mathbf{v})$ that satisfies:

$$\frac{d}{dt} I(\mathbf{r(t)}, \mathbf{v(t)}) = 0 \tag{50}$$

along *all* orbits $(\mathbf{r(t)}, \mathbf{v(t)})$. We're already familiar with a few integrals of motion – e.g., the total energy, $E = 1/2|\mathbf{v}|^2 + \Phi(r)$ is an integral in time-independent potentials, the angular momentum, $\mathbf{L}$, is an integral in spherical systems, and the $\hat{z}$ component of the angular momentum, L_z is an integral in axisymmetric systems.

Jeans Theorem

Any integral of motion is the solution of the time-independent CBE. Proof:

$$\frac{dI}{dt} = 0 = \frac{\partial I}{\partial \mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial I}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = \ \mathbf{v} \cdot \frac{\partial I}{\partial \mathbf{r}} - \nabla\Phi \cdot \frac{\partial I}{\partial \mathbf{v}} = 0 \qquad (51)$$

and this is the same condition satisfied by the steady state ($\frac{\partial f}{\partial t} = 0$) solution of the CBE:

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f - \nabla\Phi\frac{\partial f}{\partial \mathbf{v}} = 0 \qquad (52)$$

when we substitute $f \rightarrow I$. Q.E.D.

Integrals of Motion and the CBE

Any steady-state solution of the CBE depends on the phase space coordinates only through integrals of motion in the given potential, and any function of the integrals yields a steady state solution of the CBE. Proof:

- $\rightarrow$ Suppose f is a steady state solution of the CBE. Then, as we've seen on the previous slide, it will satisfy the condition to be an integral of motion.

- $\leftarrow$ Conversely, if I_1 through I_n are n integrals, and if f is any function of n variables, then:

$$\frac{d}{dt} f[I_1(\mathbf{x}, \mathbf{v}), ..., I_n(\mathbf{x}, \mathbf{v})] = \sum_{m=1}^{n} \frac{\partial f}{\partial I_m} \frac{dI_m}{dt} = 0 \tag{53}$$

  so f satisfies the CBE (hint: write out df/dt to see that another form of the CBE is $df/dt = 0$).

A recipe to build galaxies!

Why is this interesting? Because it gives us a **recipe to build (idealized) models galaxies**, and collisionless stellar systems in general, without resorting to N-body techniques!

These models can frequently capture many of the qualitative properties we observe in real galaxies (density profiles, kinematical properties).

They may also be starting points for N-body simulations.

Example: Isotropic models of spherical galaxies

In the case of isotropic (i.e., no preferred direction) models of spherical galaxies, the distribution function $f(\mathbf{x}, \mathbf{v}) = f(E)$ can only be a function of specific energy, $E = v^2/2 + \Phi(\mathbf{r})$.

Recipe ingredients:

#1 The Poisson equation in spherical coordinates:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho(r) \tag{54}$$

(note: we often take the boundary condition that $\Phi \rightarrow 0$ as $r \rightarrow$ inf)

Example: Isotropic models of spherical galaxies

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Recipe ingredients:

#2 The expression for density (an integral of the DF over the (isotropic) velocity field):

$$\rho = 4\pi \int_0^{v_e} dv v^2 f(v^2/2 + \Phi(r)) \quad (55)$$

where $v_e = \sqrt{-2\Phi(r)}$ is the escape velocity at radius r. Alternatively, we can switch to energy as the independent variable and we have:

$$\rho = 4\pi \int_{\Phi}^{0} dE \sqrt{2E - 2\Phi} f(E) \quad (56)$$

From f to ρ: Plummer Model

Given any functional form for $f(E)$ which is non-negative for all $E < 0$, use either (55) or (56) to calculate the function $\rho(\Phi)$, and insert the result into (54).

Example: **Plummer model**:

$$f(E) = \begin{cases} F \cdot (-E)^{7/2}, & E < 0, \\ 0, & E \geq 0, \end{cases}$$

where F is a constant. We use (56) to obtain $\rho(\Phi)$ and plug the result into (54):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = K(-\Phi)^5 \tag{57}$$

where K is another constant. The solution gives us a model with the density profile:

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-5/2} \tag{58}$$

where M is the total mass and a is the characteristic scale (related to the constant F).

From f to ρ: King Model

The Plummer model was originally devised to describe observations of star cluster.

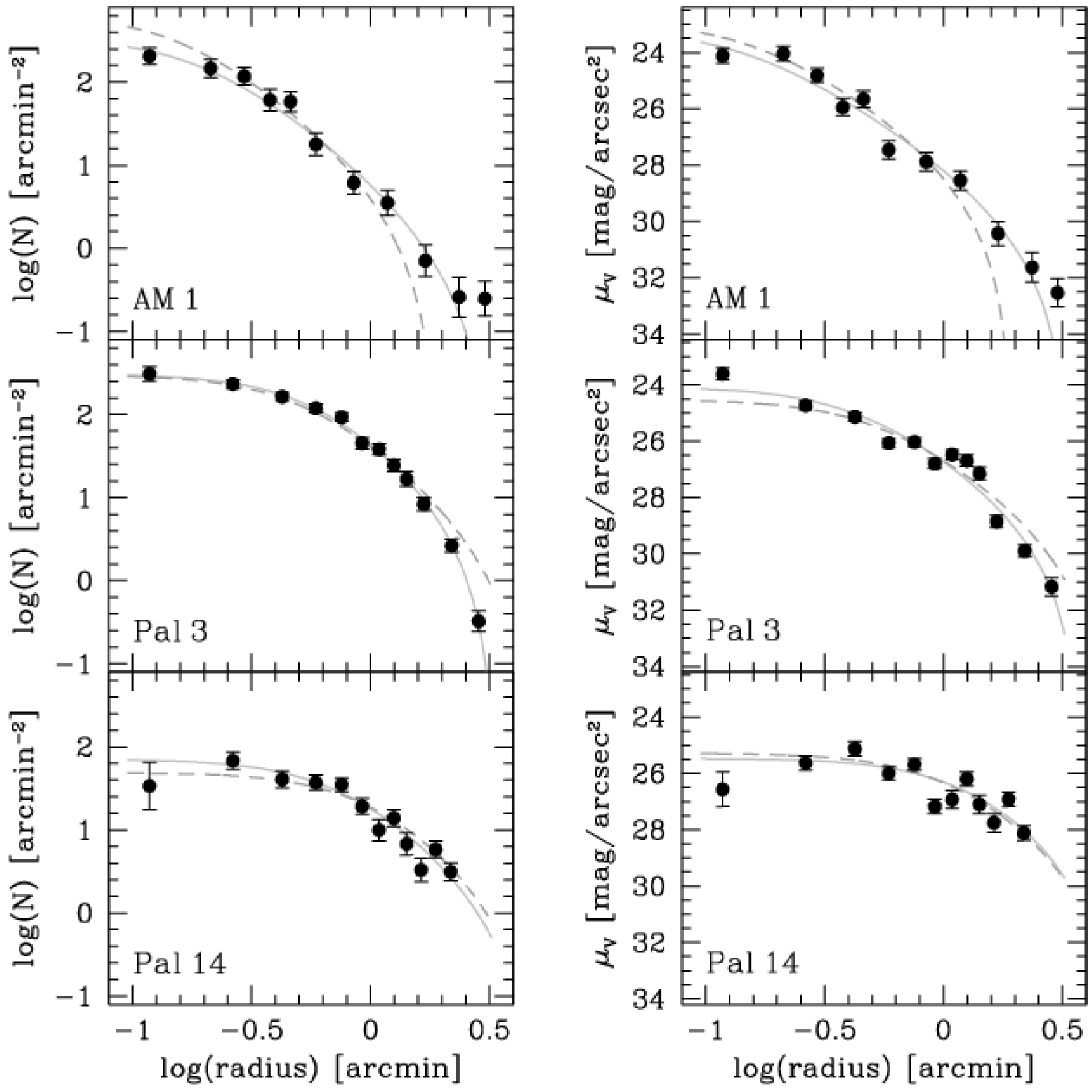
Another popular model, originally introduced by Ivan King (King 1966) to explain the observations of globular clusters is the **King model**:

$$f(E) = \begin{cases} \rho_1 (2\pi\sigma)^{-3/2} \exp(-(E - E_0)/\sigma^2 - 1), & E < E_0, \\ 0, & E \geq E_0, \end{cases}$$

where ρ_1, σ, and E_0 are parameters of the model. Just like with the Plummer model, the Poisson equation can be solved to derive $\rho(r)$ (see Eq. 4.111 in BT).

The three parameters above can be related to the central brightness, Σ_0, the **core radius** r_c, at which the brightnes drops to 50% of central, and **tidal radius**, r_t, at which the brightness vanishes.

Example: King Profiles



Hilker (2005)

From ρ to f

Given any functional form for $\rho(r)$ which is non-negative everywhere, it can be shown that:

$$f(E) = \frac{1}{\sqrt{8}\pi^2} \frac{d}{dE} \int_E^0 d\Phi \frac{\rho'(\Phi)}{\sqrt{\Phi - E}} \tag{59}$$

where $\rho'(Phi) = d\rho(\Phi)/d\Phi$ and $\rho(\Phi)$ can be obtained by solving the Poisson equation and inverting (see BT, Chapter 4, for details).

The above equation is useful in constructing isotropic models of spherical systems with known density profiles (e.g., inferred from observations). In some cases, these can be derived analytically (e.g., Jaffe 1983, or Henrquist 1990).

Generally, it can be solved numerically – it's frequently used to set up initial conditions for spherical systems in N-body calculations.