VII. NORMAL VECTOR

We calculate the normal 4-vector to the hypersurface as follows:

$$
\begin{aligned}
(n_\tau)_\alpha &= c_\alpha \frac{\partial T_\alpha}{\partial \tau} \\
(n_x)_\alpha &= c_\alpha \frac{\partial T_\alpha}{\partial x} \\
(n_y)_\alpha &= -c_\alpha \frac{\partial T_\alpha}{\partial y}
\end{aligned}
\tag{307}
$$

where the vectors are normalized using

$$
c_\alpha = \frac{1}{\sqrt{\left(\frac{\partial T_\alpha}{\partial \tau}\right)^2 + \left(\frac{\partial T_\alpha}{\partial x}\right)^2 + \left(\frac{\partial T_\alpha}{\partial y}\right)^2}}
\tag{308}
$$

A. Ideal Case

First, calculating the spatial components we use:

$$
\begin{aligned}
\nabla_\alpha T_\alpha &= \left(\frac{\partial T_\alpha}{\partial x}, \frac{\partial T_\alpha}{\partial y}\right) \\
&= \frac{\partial T_\alpha}{\partial s} \nabla_\alpha s_\alpha \\
&= \frac{\partial T_\alpha}{\partial s} \nabla_\alpha \left(\frac{1}{\tau \gamma_\alpha} s^*_\alpha\right).
\end{aligned}
\tag{309}
$$

Up until this point our γ has been defined for a specific SPH particle α. However, to find the normal vector we need a continuous field. Thus, we define:

$$
\tilde{v}_j = \sum_i \nu_i \frac{|\mathbf{v}_i|}{\sigma_i} W(|\mathbf{r}_i - \mathbf{r}_j|)
\tag{310}
$$

where

$$
\gamma = \frac{1}{\sqrt{1 - \tilde{v}^2}}
\tag{311}
$$

and

$$
\nu_i = \left(\frac{s}{\sigma}\right)_i (t = t_0)
\tag{312}
$$

and is a constant over time. Thus,

$$
\begin{aligned}
\nabla_\alpha \left(\frac{1}{\tau \gamma_\alpha} s^*_\alpha\right) &= \frac{1}{\tau \gamma_\alpha} \sum_i \nu_i \nabla_i W(|\mathbf{r} - \mathbf{r}_j|) - \frac{\nabla \gamma}{\tau \gamma_\alpha} \sum_i \nu_i W(|\mathbf{r} - \mathbf{r}_j|) \\
&= \frac{1}{\tau \gamma_\alpha} \sum_i \nu_i \nabla_i W(|\mathbf{r} - \mathbf{r}_j|) - s \nabla \gamma
\end{aligned}
\tag{313}
$$

where

$$
\begin{aligned}
\nabla \gamma &= \left[\frac{\tilde{v}}{(1 - \tilde{v}^2)^{3/2}}\right] \nabla \tilde{v} \\
\nabla \tilde{v} &= \sum_i \nu_i \frac{1}{\sigma_i} \frac{\partial |\mathbf{v}_i|}{\partial \tau} \nabla W(|\mathbf{r}_i - \mathbf{r}_j|) - \left[\sum_i \nu_i \frac{|\vec{v}|}{\sigma_i^2} \left(\sum_i \nu_i \nabla W(|\mathbf{r} - \mathbf{r}_j|)\right) W(|\mathbf{r} - \mathbf{r}_j|)\right]
\end{aligned}
\tag{314}
$$

For the temporal component

$$\begin{aligned}
\frac{\partial T_\alpha}{\partial \tau} &= \frac{\partial T_\alpha}{\partial s_\alpha}\frac{\partial s_\alpha}{\partial \tau} \\
&= \frac{\partial T_\alpha}{\partial s_\alpha}\frac{\partial}{\partial \tau}\left[\frac{1}{\tau\gamma_\alpha}s^*_\alpha\right] \\
&= \frac{\partial T_\alpha}{\partial s_\alpha}\left[-\frac{s^*_\alpha}{\gamma_\alpha\tau^2}+\frac{s^*_\alpha}{\tau\gamma^2_\alpha}\frac{\partial\gamma}{\partial\tau}+\frac{1}{\tau\gamma_\alpha}\frac{\partial s^*_\alpha}{\partial\tau}\right] \\
&= \frac{\partial T_\alpha}{\partial s_\alpha}\left[s_\alpha\left(-\frac{1}{\tau}+\frac{1}{\gamma_\alpha}\frac{\partial\gamma}{\partial\tau}\right)+\frac{1}{\tau\gamma_\alpha}\frac{\partial s^*_\alpha}{\partial\tau}\right]
\end{aligned} \tag{315}$$

where we recall that for the ideal case:

$$\begin{aligned}
s^*_\alpha &= \sigma^*_\alpha = \sum_i \nu_i W(|\mathbf{r}-\mathbf{r}_j|) \\
\frac{\partial s^*_\alpha(\vec{r})}{\partial\tau} &= \frac{\partial\sigma^*_\alpha(\vec{r})}{\partial\tau} = \sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\frac{d\vec{r}_\alpha}{d\tau}\cdot\nabla W(|\mathbf{r}-\mathbf{r}_\alpha|) \\
&= \sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\vec{v}_\alpha\cdot\nabla W(|\mathbf{r}-\mathbf{r}_\alpha|).
\end{aligned} \tag{316}$$

a more rigorous way to write this, though, is

$$s = \frac{s^*_\alpha}{\gamma\tau} = \frac{\sigma^*_\alpha}{\gamma\tau}\left(\frac{s}{\sigma}\right)_i \tag{317}$$

where $\left(\frac{s}{\sigma}\right)_i = 1$, which is constant in time. Taking the time derivative of γ

$$\frac{\partial\gamma}{\partial\tau} = \left[\frac{\tilde{v}}{(1-\tilde{v}^2)^{3/2}}\right]\frac{\partial\tilde{v}}{\partial\tau} \tag{318}$$

$$\frac{\partial\tilde{v}}{\partial\tau} = \sum_i\nu_i\frac{1}{\sigma_i}\frac{\partial|\vec{v}|}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)+\sum_i\nu_i\frac{|\vec{v}|}{\sigma_i}\vec{v}\cdot\nabla W(|\mathbf{r}-\mathbf{r}_j|)-\sum_i\nu_i\frac{|\vec{v}|}{\sigma_i^2}\frac{\partial\sigma_i}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|) \tag{319}$$

VIII. BULK VISCOSITY

In the case of Bulk Viscsity we can no longer use the relationship that $s^* = \sigma^*$ instead $s^*_i = \sigma^*_i\left(\frac{s}{\sigma}\right)_i(t)$, thus, the equation needs to be rewritten. Starting with the spatial compenents we use

$$\begin{aligned}
\nabla_\alpha T_\alpha &= \left(\frac{\partial T_\alpha}{\partial x},\frac{\partial T_\alpha}{\partial y}\right) \\
&= \frac{\partial T_\alpha}{\partial s}\nabla_\alpha s_\alpha \\
&= \frac{\partial T_\alpha}{\partial s}\nabla_\alpha\left(\frac{1}{\tau\gamma_\alpha}s^*_\alpha\right).
\end{aligned} \tag{320}$$

where

$$\begin{aligned}
\nabla_\alpha\left(\frac{1}{\tau\gamma_\alpha}s^*_\alpha\right) &= \frac{1}{\tau\gamma_\alpha}\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\nabla_i W(|\mathbf{r}-\mathbf{r}_j|)-\frac{\nabla\gamma}{\tau\gamma_\alpha}\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)W(|\mathbf{r}-\mathbf{r}_j|) \\
&= \frac{1}{\tau\gamma_\alpha}\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\nabla_i W(|\mathbf{r}-\mathbf{r}_j|)-\frac{s^*\nabla\gamma}{\tau\gamma_\alpha} \\
&= \frac{1}{\tau\gamma_\alpha}\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\nabla_i W(|\mathbf{r}-\mathbf{r}_j|)-s\nabla\gamma
\end{aligned} \tag{321}$$

notice that in this case the $\left(\frac{s}{\sigma}\right)_i(t)$ has a time dependence but not a spatial dependence, so it is not affected by the gradient. Thus, we can use the same form for $\nabla\gamma$ as seen in Eq. (314).

Recall that for the temporal component

$$\frac{\partial T_\alpha}{\partial \tau} = \frac{\partial T_\alpha}{\partial s_\alpha}\left[s_\alpha\left(-\frac{1}{\tau}+\frac{1}{\gamma_\alpha}\frac{\partial\gamma}{\partial\tau}\right)+\frac{1}{\tau\gamma_\alpha}\frac{\partial s^*_\alpha}{\partial\tau}\right] \tag{322}$$

where

$$\begin{aligned}\frac{\partial s^*_\alpha}{\partial\tau} &= \frac{\partial}{\partial\tau}\left[\sum_i \nu_i\left(\frac{s}{\sigma}\right)_i(t)W(|\mathbf{r}-\mathbf{r}_j|)\right]\\ &= \sum_i \nu_i\left[\frac{\partial}{\partial\tau}\left(\frac{s}{\sigma}\right)_i(t)\right]W(|\mathbf{r}-\mathbf{r}_j|)+\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\ |\vec{v}|\cdot\nabla W(|\mathbf{r}-\mathbf{r}_j|).\end{aligned} \tag{323}$$

We recall that for the bulk viscosity $\frac{\partial}{\partial\tau}\left(\frac{s}{\sigma}\right)_i(t)$ has been defined already in Eq. (139)

$$\frac{d}{dt}\left(\frac{s}{\sigma}\right) = -\frac{1}{T}\frac{\Pi}{\sigma^*}\partial_\mu u^\mu. \tag{324}$$

and that

$$\frac{\partial\gamma}{\partial\tau} = \left[\frac{\tilde{v}}{(1-\tilde{v}^2)^{3/2}}\right]\left\{\sum_i\nu_i\frac{1}{\sigma_i}\frac{\partial|\vec{v}|}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)+\sum_i\nu_i\frac{|\vec{v}|}{\sigma_i}\vec{v}\cdot\nabla W(|\mathbf{r}-\mathbf{r}_j|)-\sum_i\nu_i\frac{|\vec{v}|}{\sigma_i^2}\frac{\partial\sigma_i}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)\right\} \tag{325}$$

Appendix A: Important relations

In this Appendix we prove Eqs. (34-37,146).

1. $\Delta_{\alpha\beta}\Delta^{\alpha\beta}=3$

First, we prove Eq. (34), i.e. $\Delta_{\alpha\beta}\Delta^{\alpha\beta}=3$

$$\begin{aligned}\Delta_{\alpha\beta}\Delta^{\alpha\beta} &= (g_{\alpha\beta}-u_\alpha u_\beta)\left(g^{\alpha\beta}-u^\alpha u^\beta\right)\\ &= \underbrace{g_{\alpha\beta}g^{\alpha\beta}}_{=4}-u_\alpha u_\beta g^{\alpha\beta}-u^\alpha u^\beta g_{\alpha\beta}+\underbrace{u_\alpha u_\beta u^\alpha u^\beta}_{=1}\\ &= 5-2u^\alpha u_\alpha\\ &= 3\end{aligned} \tag{A1}$$

Eq. (35), i.e. $\Delta_{\alpha\beta}\Delta^{\beta\nu}=\Delta_\alpha^\nu$, is shown by

$$\begin{aligned}\Delta_{\alpha\beta}\Delta^{\beta\nu} &= (g_{\alpha\beta}-u_\alpha u_\beta)\left(g^{\beta\nu}-u^\beta u^\nu\right)\\ &= g_{\alpha\beta}g^{\beta\nu}-u_\alpha u_\beta g^{\beta\nu}-u^\beta u^\nu g_{\alpha\beta}+u_\alpha u_\beta u^\beta u^\nu\\ &= g_\alpha^\nu-u_\alpha u^\nu\\ &= \Delta_\alpha^\nu\end{aligned} \tag{A2}$$