For a general metric,

$$D_\mu T^{\mu\nu} = 0,$$

where,

$$\begin{aligned}
D_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda, \\
D_\mu V_{\alpha\beta} &= \partial_\mu V_{\alpha\beta} - \Gamma^\lambda_{\alpha\mu} V_{\lambda\beta} - \Gamma^\lambda_{\beta\mu} V_{\lambda\alpha}, \\
D_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \\
D_\mu V^\mu &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} V^\mu \right), \\
D_\mu V^{\mu\nu} &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} V^{\mu\nu} \right) + \Gamma^\nu_{\lambda\mu} V^{\lambda\mu},
\end{aligned}$$

with

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\nu\sigma} \left(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda} \right).$$

For the metric,

$$\begin{aligned}
g_{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^2 \end{pmatrix}, \\
g^{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau^2} \end{pmatrix},
\end{aligned}$$

we have,

$$\begin{aligned}
\sqrt{-g} &= \tau, \\
\Gamma^0_{33} &= \frac{1}{2} g^{0\sigma} \left(\partial_3 g_{\sigma 3} + \partial_3 g_{\sigma 3} - \partial_\sigma g_{33}\right), \\
&= \frac{1}{2} g^{00} \left(\partial_3 g_{03} + \partial_3 g_{03} - \partial_0 g_{33}\right), \\
&= \frac{-1}{2} g^{00} \left(\partial_0 g_{33}\right) = \tau, \\
\Gamma^3_{30} &= \frac{1}{2} g^{3\sigma} \left(\partial_3 g_{\sigma 0} + \partial_0 g_{\sigma 3} - \partial_\sigma g_{30}\right), \\
&= \frac{1}{2} g^{33} \left(\partial_3 g_{30} + \partial_0 g_{33} - \partial_3 g_{30}\right), \\
&= \frac{1}{2} g^{33} \left(\partial_0 g_{33}\right) = \frac{2\tau}{2\tau^2} = \frac{1}{\tau}, \\
\Gamma^3_{03} &= \Gamma^3_{30} = \frac{1}{\tau}, \\
\text{All other terms} &= 0.
\end{aligned}$$

More usefull relations,

$$\begin{aligned}
\frac{1}{\sigma} \frac{D}{D\tau} \sigma &= -D_\mu u^\mu, \\
D_\mu f u^\mu &= \sigma \frac{D}{D\tau} \frac{f}{\sigma}, \\
D_\mu u^\mu &= \partial_\mu u^\mu + \frac{\gamma}{\tau}, \\
\partial_\mu u^\mu &= \frac{d\gamma}{d\tau} - \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau}, \\
\frac{d\gamma}{d\tau} &= -\frac{u^k}{\gamma} \frac{du_k}{d\tau} - \frac{(u_\eta)^2}{\gamma \tau^3}, \\
\partial_j v^j &= -\frac{1}{\sigma^*} \frac{d\sigma^*}{d\tau},
\end{aligned}$$

where we used the definitions

$$\begin{aligned}
u^\mu \partial_\mu &= \gamma \frac{d}{d\tau}, \\
u^\mu D_\mu &= \frac{D}{D\tau}.
\end{aligned}$$

1 Equation of Motion

The energy-momentum conservation law can be writen as

$$\begin{aligned}
g_{\nu\beta}\frac{1}{\tau}\partial_\mu\left(\tau T^{\mu\nu}\right)+g_{\nu\beta}\Gamma^{\nu}_{\lambda\mu}T^{\lambda\mu} &= 0,\\
g_{\nu\beta}\frac{1}{\tau}\partial_\mu\left(\tau T^{\mu\nu}\right)+g_{\beta0}\tau T^{33}+2g_{\beta3}\frac{T^{03}}{\tau} &= 0,\\
\frac{1}{\tau}\partial^\alpha\left(\tau T_{\alpha\beta}\right)+g_{\beta0}\tau T^{33} &= 0.
\end{aligned}$$

For $\beta=i$, we obtain the momentum conservation law

$$\frac{1}{\tau}\partial^\alpha\left(\tau T_{\alpha i}\right)=0.$$

For $\beta=0$, we obtain the energy conservation law

$$\frac{1}{\tau}\partial^\alpha\left(\tau T_{\alpha 0}\right)+\tau T^{33}=0.$$

2 Energy Conservation

The energy conservation is

$$\begin{aligned}
\frac{1}{\tau}\partial_\mu\left(\tau T^{\mu0}\right)+\tau T^{33} &= 0,\\
\frac{1}{\tau}\partial_\tau\left(\tau T^{00}\right)+\frac{1}{\tau}\partial_i\left(\tau T^{i0}\right)+\tau T^{33} &= 0.
\end{aligned}$$

Integrating,

$$\begin{aligned}
\partial_\tau\left(\int\tau T^{00}\right)+\int\partial_i\left(\tau T^{i0}\right)+\int\tau^2T^{33} &= 0.\\
\frac{d}{d\tau}\left(\int\tau T^{00}\right)+\int\tau^2T^{33} &= 0.
\end{aligned}$$

For the scalling solution,

$$\begin{aligned}
T^{33} &= -g^{33}\left(p+\Pi\right)+\pi^{33}=\frac{1}{\tau^2}\left(p+\Pi\right)+\pi^{33}\\
\frac{d}{d\tau}\left(\int\tau T^{00}\right)+\int p+\Pi+\tau^2\pi^{33} &= 0.
\end{aligned}$$

We can calculate in the SPH parametrization,

$$\begin{aligned}
\int\tau T^{00} &= \sum_\alpha\nu_\alpha\tau\frac{T^{00}_\alpha}{\sigma^*_\alpha},\\
\int\tau^2T^{33} &= \sum_\alpha\nu_\alpha\tau^2\frac{T^{33}_\alpha}{\sigma^*_\alpha}.
\end{aligned}$$

Lets define the variable E_z as

$$E_z(\tau) = \int_{\tau_0}^{\tau} d\tau \int \tau^2 T^{33},$$

where,

$$\begin{aligned} E_z(\tau_0) &= 0, \\ \frac{dE_z(\tau)}{d\tau} &= \int \tau^2 T^{33}. \end{aligned}$$

Then, the quantity below is the one that is actually conserved,

$$\left(\int \tau T^{00}\right) + E_z(\tau) = Const.$$

The term $E_z(\tau)$ must be calculated by Runge-Kutta.

3 Bulk Viscosity

The equation for the bulk viscosity can be expressed, in this metric as,

$$\begin{aligned} \tau_R \frac{D}{D\tau}\left(\frac{\Pi}{\sigma}\right) + \frac{\Pi}{\sigma} &= -\frac{\zeta}{\sigma} D_\mu u^\mu, \\ \tau_R \frac{D}{D\tau}\Pi + \Pi &= -(\zeta + \tau_R \Pi) D_\mu u^\mu, \\ \frac{D}{D\tau}\Pi &= -\frac{\Pi}{\tau_R} - \left(\frac{\zeta}{\tau_R} + \Pi\right) D_\mu u^\mu, \end{aligned}$$

where

$$D_\mu u^\mu = \partial_\mu u^\mu + \frac{\gamma}{\tau}.$$

Finally

$$\frac{d}{d\tau}\left(\frac{\Pi}{\sigma}\right) = -\frac{\tau}{\tau_R \sigma^*}\left[\Pi + \zeta\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right)\right]$$

4 Shear Viscosity

The equation for the shear viscosity is written as (Memory effect model with finite size effect)

$$\tau_R P_{\mu\nu\alpha\beta} \frac{D}{D\tau}\pi^{\alpha\beta} + \pi_{\mu\nu} = \eta P_{\mu\nu\alpha\beta} D^\alpha u^\beta.$$

The term $\eta P_{\mu\nu\alpha\beta} D^{\alpha} u^{\beta}$ is

$$
\begin{aligned}
\frac{\eta}{\sigma} P_{\mu\nu\alpha\beta} D^{\alpha} u^{\beta} &= \frac{\eta}{2\sigma} P_{\mu\alpha} P_{\nu\beta} \left(D^{\alpha} u^{\beta} + D^{\beta} u^{\alpha} \right) - \frac{\eta}{D\sigma} P_{\mu\nu} D_{\beta} u^{\beta}, \\
&= \frac{\eta}{2\sigma} \left(D_{\mu} u_{\nu} + D_{\nu} u_{\mu} \right) - \frac{\eta}{2\sigma} \left(u_{\mu} \frac{D u_{\nu}}{D\tau} + u_{\nu} \frac{D u_{\mu}}{D\tau} \right) - \frac{\eta}{D\sigma} P_{\mu\nu} D_{\beta} u^{\beta}, \\
&= \frac{\eta}{2\sigma} \left(\partial_{\mu} u_{\nu} + \partial_{\nu} u_{\mu} \right) - \frac{\eta\gamma}{2\sigma} \left(u_{\mu} \frac{d u_{\nu}}{d\tau} + u_{\nu} \frac{d u_{\mu}}{d\tau} \right) - \frac{\eta}{D\sigma} P_{\mu\nu} D_{\beta} u^{\beta} \\
&\quad + \frac{\eta}{2\sigma} \left(u_{\mu} u^{\alpha} \Gamma^{\lambda}_{\alpha\nu} u_{\lambda} + u_{\nu} u^{a} \Gamma^{\lambda}_{\alpha\mu} u_{\lambda} \right) - \frac{\eta}{2\sigma} \left(\Gamma^{\lambda}_{\mu\nu} u_{\lambda} + \Gamma^{\lambda}_{\nu\mu} u_{\lambda} \right), \\
&= \frac{\eta}{2\sigma} \left(\partial_{\mu} u_{\nu} + \partial_{\nu} u_{\mu} \right) - \frac{\eta\gamma}{2\sigma} \left(u_{\mu} \frac{d u_{\nu}}{d\tau} + u_{\nu} \frac{d u_{\mu}}{d\tau} \right) - \frac{\eta}{D\sigma} P_{\mu\nu} D_{\beta} u^{\beta} \\
&\quad - \frac{\eta}{2\sigma\tau^{3}} \left(u_{\eta} \right)^{2} \left(u_{\mu} g^{0}_{\nu} + u_{\nu} g^{0}_{\mu} \right) - \frac{\eta\gamma\tau}{\sigma} g^{3}_{\mu} g^{3}_{\nu} - \frac{\eta}{\sigma\tau} u_{\eta} \left(g^{3}_{\mu} g^{0}_{\nu} + g^{0}_{\mu} g^{3}_{\nu} \right),
\end{aligned}
$$

The term $\tau_R P_{\mu\nu\alpha\beta} \frac{D}{D\tau} \left(\frac{\pi^{\alpha\beta}}{\sigma} \right)$ is

$$
\begin{aligned}
\tau_R P_{\mu\nu\alpha\beta} \frac{D}{D\tau} \pi^{\alpha\beta} &= \tau_R P_{\mu\nu\alpha\beta} \frac{D\pi^{\alpha\beta}}{D\tau}, \\
&= \tau_R \frac{D\pi_{\mu\nu}}{D\tau} - \frac{\tau_R}{\sigma} \pi^{\alpha\beta} \frac{D P_{\mu\nu\alpha\beta}}{D\tau},
\end{aligned}
$$

where

$$
\begin{aligned}
-\tau_R \pi^{\alpha\beta} \frac{D P_{\mu\nu\alpha\beta}}{D\tau} &= -\tau_R \pi^{\alpha\beta} \frac{D}{D\tau} \left(P_{\mu\alpha} P_{\nu\beta} \right), \\
&= \tau_R \left(u_{v} \pi^{\alpha}_{\mu} + u_{\mu} \pi^{\alpha}_{\nu} \right) \frac{D u_{\alpha}}{D\tau}, \\
&= \gamma \tau_R \left(u_{v} \pi^{\alpha}_{\mu} + u_{\mu} \pi^{\alpha}_{\nu} \right) \frac{d u_{\alpha}}{d\tau} + \tau_R \left(u_{v} \pi_{\mu 0} + u_{\mu} \pi_{\nu 0} \right) \frac{\left(u_{\eta} \right)^{2}}{\tau^{3}},
\end{aligned}
$$

and the term $\gamma \tau_R \left(u_{v} \pi^{\alpha}_{\mu} + u_{\mu} \pi^{\alpha}_{\nu} \right) \frac{d u_{\alpha}}{d\tau}$ can be simplified as

$$
\begin{aligned}
& \gamma \tau_R \left(u_{v} \pi^{\alpha}_{\mu} + u_{\mu} \pi^{\alpha}_{\nu} \right) \frac{d u_{\alpha}}{d\tau} \\
= \; & \gamma \tau_R \left(u_{v} \pi^{0}_{\mu} + u_{\mu} \pi^{0}_{\nu} \right) \frac{d\gamma}{d\tau} + \gamma \tau_R \left(u_{v} \pi^{j}_{\mu} + u_{\mu} \pi^{j}_{\nu} \right) \frac{d u_{j}}{d\tau}, \\
= \; & -\gamma \tau_R \left(u_{v} \pi^{0}_{\mu} + u_{\mu} \pi^{0}_{\nu} \right) \left(\frac{u^{j}}{\gamma} \frac{d u_{j}}{d\tau} + \frac{\left(u_{\eta} \right)^{2}}{\gamma \tau^{3}} \right) + \tau_R \left(\gamma u_{v} \pi^{j}_{\mu} + \gamma u_{\mu} \pi^{j}_{\nu} \right) \frac{d u_{j}}{d\tau}, \\
= \; & \tau_R \left(\gamma u_{v} \pi^{j}_{\mu} + \gamma u_{\mu} \pi^{j}_{\nu} - u_{v} \pi^{0}_{\mu} u^{j} - u_{\mu} \pi^{0}_{\nu} u^{j} \right) \frac{d u_{j}}{d\tau} - \tau_R \left(u_{v} \pi^{0}_{\mu} + u_{\mu} \pi^{0}_{\nu} \right) \frac{\left(u_{\eta} \right)^{2}}{\tau^{3}}.
\end{aligned}
$$

The term $\tau_R \frac{D\pi_{\mu\nu}}{D\tau}$ is

$$
\begin{aligned}
\tau_R \frac{D\pi_{\mu\nu}}{D\tau} &= \gamma\tau_R \frac{d\pi_{\mu\nu}}{d\tau} - \tau_R u^\lambda \Gamma^\sigma_{\mu\lambda}\pi_{\nu\sigma} - \frac{\tau_R}{\sigma} u^\lambda \Gamma^\sigma_{\nu\lambda}\pi_{\mu\sigma}, \\
&= \gamma\tau_R \frac{d\pi_{\mu\nu}}{d\tau} + \frac{\tau_R}{\tau^3} u_\eta \left(g_{\mu 0}\pi_{\nu 3} + g_{\nu 0}\pi_{\mu 3}\right) - \frac{\tau_R}{\tau^3} u_\eta \left(g_{\mu 3}\pi_{\nu 0} + g_{\nu 3}\pi_{\mu 0}\right) \\
&\quad + \frac{\gamma\tau_R}{\tau^3} \left(g_{\mu 3}\pi_{\nu 3} + g_{\nu 3}\pi_{\mu 3}\right),
\end{aligned}
$$

And we finally obtain

$$
\begin{aligned}
\tau_R P_{\mu\nu\alpha\beta} \frac{D}{D\tau}\pi^{\alpha\beta} &= \gamma\tau_R \frac{d\pi_{\mu\nu}}{d\tau} + \frac{\tau_R}{\tau^3} u_\eta \left(g_{\mu 0}\pi_{\nu 3} + g_{\nu 0}\pi_{\mu 3}\right) - \frac{\tau_R}{\tau^3} u_\eta \left(g_{\mu 3}\pi_{\nu 0} + g_{\nu 3}\pi_{\mu 0}\right) + \frac{\gamma\tau_R}{\tau^3} \left(g_{\mu 3}\pi_{\nu 3} + g_{\nu 3}\pi_{\mu 3}\right) \\
&\quad + \tau_R \left(\gamma u_v \pi^j_\mu + \gamma u_\mu \pi^j_\nu - u_v \pi^0_\mu u^j - u_\mu \pi^0_\nu u^j\right) \frac{du_j}{d\tau}
\end{aligned}
$$

Combining all the terms

$$
\begin{aligned}
& 2 * * * * * * * * * 2 \\
& \gamma\tau_R \frac{d\pi_{\mu\nu}}{d\tau} + \pi_{\mu\nu} \\
& + \tau_R \left(\gamma u_v \pi^j_\mu + \gamma u_\mu \pi^j_\nu - u_v \pi^0_\mu u^j - u_\mu \pi^0_\nu u^j\right) \frac{du_j}{d\tau} \\
& + \frac{\tau_R}{\tau^3} u_\eta \left(g_{\mu 0}\pi_{\nu 3} + g_{\nu 0}\pi_{\mu 3}\right) - \frac{\tau_R}{\tau^3} u_\eta \left(g_{\mu 3}\pi_{\nu 0} + g_{\nu 3}\pi_{\mu 0}\right) + \frac{\gamma\tau_R}{\tau^3} \left(g_{\mu 3}\pi_{\nu 3} + g_{\nu 3}\pi_{\mu 3}\right) \\
= & \frac{\eta}{2} \left(\partial_\mu u_\nu + \partial_\nu u_\mu\right) - \frac{\eta\gamma}{2} \left(u_\mu \frac{du_\nu}{d\tau} + u_\nu \frac{du_\mu}{d\tau}\right) - \frac{\eta}{D} P_{\mu\nu} D_\beta u^\beta \\
& - \frac{\eta}{2\tau^3} \left(u_\eta\right)^2 \left(u_\mu g^0_\nu + u_\nu g^0_\mu\right) - \eta\gamma\tau g^3_\mu g^3_\nu - \frac{\eta}{\tau} u_\eta \left(g^3_\mu g^0_\nu + g^0_\mu g^3_\nu\right)
\end{aligned}
$$

In the Bjorken scalling

$$
\begin{aligned}
& 2 * * * * * * * * * 2 \\
& \gamma\tau_R \frac{d\pi_{\mu\nu}}{d\tau} + \pi_{\mu\nu} \\
& + \tau_R \left(\gamma u_v \pi^j_\mu + \gamma u_\mu \pi^j_\nu - u_v \pi^0_\mu u^j - u_\mu \pi^0_\nu u^j\right) \frac{du_j}{d\tau} \\
= & \frac{\eta}{2} \left(\partial_\mu u_\nu + \partial_\nu u_\mu\right) - \frac{\eta\gamma}{2} \left(u_\mu \frac{du_\nu}{d\tau} + u_\nu \frac{du_\mu}{d\tau}\right) - \frac{\eta}{D} P_{\mu\nu} D_\beta u^\beta
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\gamma\tau_R}{\sigma}\frac{d\pi_{\mu\nu}}{d\tau}+\frac{\pi_{\mu\nu}}{\sigma} \\
& +\frac{\tau_R}{\sigma}\pi_{\mu\nu}\left(\frac{\gamma}{\tau}-\frac{(u_\eta)^2}{\gamma\tau^3}-\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& +\frac{\tau_R}{\sigma}\left(\gamma u_v\pi^j_\mu+\gamma u_\mu\pi^j_\nu-\frac{\pi_{\mu\nu}}{\gamma}u^j-u_v\pi^0_\mu u^j-u_\mu\pi^0_\nu u^j\right)\frac{du_j}{d\tau} \\
& -\frac{\tau_R}{\sigma}\left(g^3_\mu\pi_{\nu 0}+g^3_\nu\pi_{\mu 0}\right)\tau u^\eta-\frac{\tau_R}{\sigma}\left(g^3_\mu\pi_{\nu 3}+g^3_\nu\pi_{\mu 3}\right)\frac{\gamma}{\tau}-\frac{\tau_R}{\sigma}\left(g^0_\mu\pi_{\nu 3}+g^0_\nu\pi_{\mu 3}\right)\frac{u^\eta}{\tau} \\
= \; & \frac{\eta}{2\sigma}\left(\partial_\mu u_\nu+\partial_\nu u_\mu\right)-\frac{\eta\gamma}{2\sigma}\left(u_\mu\frac{du_\nu}{d\tau}+u_\nu\frac{du_\mu}{d\tau}\right) \\
& -\frac{\eta}{D\sigma}P_{\mu\nu}D_\beta u^\beta-\frac{\eta u_\eta u_\eta}{2\sigma\tau^3}\left(u_\mu g^0_\nu+u_\nu g^0_\mu\right) \\
& -\frac{\eta}{\sigma}\left(g^3_\mu g^0_\nu\frac{u_\eta}{\tau}+g^0_\mu g^3_\nu\frac{u_\eta}{\tau}+g^3_\mu g^3_\nu\tau\gamma\right).
\end{aligned}
$$

Considering the components $0i$

$$
\begin{aligned}
& \frac{\gamma\tau_R}{\sigma}\frac{d\pi_{0i}}{d\tau}+\frac{\pi_{0i}}{\sigma} \\
& +\frac{\tau_R}{\sigma}\pi_{0i}\left(\frac{\gamma}{\tau}-\frac{(u_\eta)^2}{\gamma\tau^3}-\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& +\frac{\tau_R}{\sigma}\left(\gamma u_i\pi^j_0+\gamma^2\pi^j_i-\left(\frac{1}{\gamma}+\gamma\right)\pi_{0i}u^j-\pi^{00}u_iu^j\right)\frac{du_j}{d\tau} \\
& -\frac{\tau_R}{\sigma}\tau u^\eta\pi_{00}g^3_i-\frac{\gamma\tau_R}{\sigma\tau}\pi_{03}g^3_i-\frac{\tau_R}{\sigma}\frac{u^\eta}{\tau}\pi_{i3} \\
= \; & \frac{\eta}{2\sigma}\left(\partial_0 u_i+\partial_i\gamma\right)-\frac{\eta\gamma}{2\sigma}\left(\gamma\frac{du_i}{d\tau}+u_i\frac{d\gamma}{d\tau}\right) \\
& +\frac{\eta\gamma}{D\sigma}u_iD_\beta u^\beta-\frac{\eta}{2\sigma\tau^3}\left(u_\eta\right)^2u_i-\frac{\eta}{\sigma}\frac{u_\eta}{\tau}g^3_i,
\end{aligned}
$$

and we have

$$
\begin{aligned}
& \frac{d\pi_{0i}}{d\tau} \\
= \quad & -\frac{\eta}{2\gamma\tau_R} v^j \left(\partial_j u_i + \partial_i u_j\right) \\
& + \frac{\eta}{2\tau_R} \left(\frac{1}{\gamma} - \gamma\right) \frac{du_i}{d\tau} + \left(\frac{\eta}{2\tau_R} - \frac{\eta}{D\tau_R}\right) \frac{u_i u^j}{\gamma} \frac{du_j}{d\tau} \\
& - \left(u_i \pi_0^j + \gamma \pi_i^j - \left(\gamma + \frac{1}{\gamma}\right) \frac{\pi_{0i}}{\gamma} u^j - \frac{\pi^{00}}{\gamma} u_i u^j\right) \frac{du_j}{d\tau} \\
& + \frac{\eta}{D\tau_R} u_i \left(\frac{\gamma}{\tau} - \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& + \frac{\pi_{0i}}{\gamma} \left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\tau} - \frac{1}{\tau_R}\right) \\
& + \left(\frac{\tau u^\eta}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_\eta}{\tau\gamma\tau_R}\right) g_i^3 + \frac{u_\eta}{\gamma\tau}\pi_i^3
\end{aligned}
$$

$2 * * * * * * * * * 2$

$$
\begin{aligned}
& \frac{\gamma\tau_R}{\sigma}\frac{d\pi_{0i}}{d\tau} + \frac{\pi_{0i}}{\sigma} + \tau_R \frac{\pi_{0i}}{\sigma}\left(-\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{\gamma}{\tau}\right) \\
& + \frac{\tau_R}{\sigma}\left(\gamma u_i \pi_0^j + \gamma u_0 \pi_i^j - u_i \pi_0^0 u^j - u_0 \pi_i^0 u^j\right)\frac{du_j}{d\tau} \\
& + \frac{\tau_R}{\sigma\tau^3} u_\eta \pi_{i3} - \frac{\tau_R}{\sigma\tau^3} u_\eta g_{i3} \pi_{00} + \frac{\gamma\tau_R}{\sigma\tau^3} g_{i3}\pi_{03} \\
= \quad & \frac{\eta}{2\sigma}\left(\frac{du_i}{d\tau} - v^j \partial_j u_i - v^j \partial_i u_j\right) - \frac{\eta\gamma^2}{2\sigma}\frac{du_i}{d\tau} + \frac{\eta}{D\sigma}\gamma u_i \left(-\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{\gamma}{\tau}\right) \\
& - \frac{\eta}{\sigma\tau} u_\eta g_i^3 + \frac{\eta}{2\sigma} u_i u^j \frac{du_j}{d\tau}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\sigma}\frac{D}{D\tau}\sigma &= -D_\mu u^\mu, \\
D_\mu f u^\mu &= \sigma \frac{D}{D\tau}\frac{f}{\sigma}, \\
D_\mu u^\mu &= \partial_\mu u^\mu + \frac{\gamma}{\tau}, \\
\partial_\mu u^\mu &= \frac{d\gamma}{d\tau} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}, \\
\frac{d\gamma}{d\tau} &= -\frac{u^k}{\gamma}\frac{du_k}{d\tau} - \frac{(u_\eta)^2}{\gamma\tau^3}, \\
\partial_j v^j &= -\frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau},
\end{aligned}
$$

Considering the components ij

$$
\begin{aligned}
& 2*********2 \\
\frac{d}{d\tau}\frac{\pi_{ij}}{\sigma} \quad = \quad & -\frac{\tau\pi_{ij}}{\sigma^*\tau_R} \\
& -\frac{\tau}{\sigma^*}\left(\gamma u_j\pi_i^k+\gamma u_i\pi_j^k-u_j\pi_{0i}u^k-u_i\pi_{0j}u^k\right)\frac{du_k}{d\tau} \\
& +\frac{\eta\tau}{2\sigma^*\tau_R}\left(\partial_i u_j+\partial_j u_i\right)-\frac{\eta\gamma\tau}{2\sigma^*\tau_R}\left(u_i\frac{du_j}{d\tau}+u_j\frac{du_i}{d\tau}\right)-\frac{\eta\tau}{D\sigma^*\tau_R}P_{ij}D_\beta u^\beta \\
& +\frac{1}{\sigma^*\tau^2}u_\eta\left(g_{i3}\pi_{j0}+g_{j3}\pi_{i0}\right)-\frac{\gamma}{\sigma^*\tau^2}\left(g_{i3}\pi_{j3}+g_{j3}\pi_{i3}\right) \\
& -\frac{\eta\gamma}{\sigma^*\tau^2\tau_R}g_{i3}g_{j3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\pi_{ij}}{\sigma}\right) \quad = \quad & -\frac{\pi_{ij}}{\sigma\gamma\tau_R} \\
& +\frac{\eta}{2\sigma\gamma\tau_R}\left(\partial_i u_j+\partial_j u_i\right)-\frac{\eta}{2\sigma\tau_R}\left(u_i\frac{du_j}{d\tau}+u_j\frac{du_i}{d\tau}\right) \\
& -\frac{1}{\sigma}\left(u_i\pi_j^k+u_j\pi_i^k\right)\frac{du_k}{d\tau}+\left(u_i\frac{\pi_{0j}}{\gamma\sigma}+u_j\frac{\pi_{0i}}{\gamma\sigma}\right)u^k\frac{du_k}{d\tau} \\
& -\frac{\eta}{D\sigma\gamma\tau_R}P_{ij}D_\beta u^\beta-\frac{\eta}{\sigma\tau^3\tau_R}g_{3i}g_{3j}-\frac{1}{\sigma\tau^3}\left(g_{3i}\pi_{j3}+g_{3j}\pi_{i3}\right).
\end{aligned}
$$

5 IS full

new term

$$
-\frac{1}{2}\pi^{\mu\nu}\frac{\eta T}{\tau_R}D_\mu\left(\frac{\tau_R}{\eta T}u^\mu\right)=-\frac{1}{2}\pi^{\mu\nu}D_\alpha u^\alpha-\frac{1}{2}\pi^{\mu\nu}u^\alpha\frac{\eta T}{\tau_R}D_\alpha\left(\frac{\tau_R}{\eta T}\right)
$$

where,

$$
-\frac{1}{2}\pi^{\mu\nu}\frac{\eta T}{\tau_R}u^\alpha D_\alpha\left(\frac{\tau_R}{\eta T}\right)=\frac{1}{2}\pi^{\mu\nu}\frac{1}{T}u^\alpha D_\alpha T+\frac{1}{2}\pi^{\mu\nu}\frac{\tau_R}{\eta}u^\alpha D_\alpha\left(\frac{\eta}{\tau_R}\right)
$$

We assume that we have a massless and that

$$
\tau_R=b\frac{\eta}{P}
$$

Then

$$
-\frac{1}{2}\pi^{\mu\nu}\frac{\eta T}{\tau_R}u^\alpha D_\alpha\left(\frac{\tau_R}{\eta T}\right)=\left(\frac{1}{\varepsilon+P}+\frac{1}{P}\right)\frac{\pi^{\mu\nu}}{2}u^\alpha D_\alpha P
$$

In the massless limit

$$
\begin{aligned}
-\frac{1}{2}\pi^{\mu\nu}\frac{\eta T}{\tau_R}u^{\alpha}D_{\alpha}\left(\frac{\tau_R}{\eta T}\right) &= \frac{5}{4P}\frac{\pi^{\mu\nu}}{2}u^{\alpha}D_{\alpha}\frac{\varepsilon}{3} \\
&= \frac{5}{4P}\frac{\pi^{\mu\nu}}{6}u^{\alpha}D_{\alpha}\varepsilon \\
&= \frac{5}{4P}\frac{\pi^{\mu\nu}}{6}\left(-\varepsilon-P-\Pi\right)\theta+\frac{5}{4P}\frac{\pi^{\mu\nu}}{6}\pi^{\alpha\beta}D_{\alpha}u_{\beta} \\
&= \frac{5}{4P}\frac{\pi^{\mu\nu}}{6}\left(-\varepsilon-P-\Pi\right)\theta
\end{aligned}
$$

we have that

$$
\begin{aligned}
\frac{dP}{dT}\frac{dT}{d\tau} &= \frac{dP}{d\tau} \\
&= \frac{dP}{ds}\frac{ds}{d\tau} \\
\frac{dw}{ds} &= \frac{d\varepsilon}{ds}+\frac{dP}{ds} \\
&= T+\frac{dP}{ds} \\
\frac{dP}{ds} &= \frac{dw}{ds}-T \\
\frac{dT}{d\tau} &= \frac{1}{s}\left(\frac{dw}{ds}-T\right)\frac{ds}{d\tau}
\end{aligned}
$$

6 Entropy production

The equation for the entropy is

$$
\begin{aligned}
TD_{\mu}\left(su^{\mu}\right) &= -\Pi D_{\mu}u^{\mu}+\pi^{\mu\nu}D_{\mu}u_{\nu}, \\
\sigma\frac{D}{D\tau}\left(\frac{s}{\sigma}\right) &= -\frac{\Pi}{T}D_{\mu}u^{\mu}+\frac{\pi^{\mu\nu}}{T}D_{\mu}u_{\nu}, \\
\frac{d}{d\tau}\left(\frac{s}{\sigma}\right) &= -\frac{\Pi}{T\gamma\sigma}\left(\partial_{\mu}u^{\mu}+\frac{\gamma}{\tau}\right)+\frac{\pi^{\mu\nu}}{T\gamma\sigma}D_{\mu}u_{\nu}.
\end{aligned}
$$

Separating the first term,

$$
\begin{aligned}
\gamma\frac{ds}{d\tau}-\frac{\gamma s}{\sigma}\frac{d\sigma}{d\tau} &= -\frac{\Pi}{T}\left(\partial_{\mu}u^{\mu}+\frac{\gamma}{\tau}\right)+\frac{\pi^{\mu\nu}}{T}D_{\mu}u_{\nu}, \\
\gamma\frac{ds}{d\tau} &= -\left(\frac{\Pi}{T}+s\right)\left(\partial_{\mu}u^{\mu}+\frac{\gamma}{\tau}\right)+\frac{\pi^{\mu\nu}}{T}D_{\mu}u_{\nu}.
\end{aligned}
$$

We have to calculate the last term

$$\begin{aligned}
\frac{1}{T}\pi^{\mu\nu}D_{\mu}u_{\nu} &= \frac{1}{T}\pi^{\mu\nu}\partial_{\mu}u_{\nu}-\frac{1}{T}\pi^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}u_{\lambda}, \\
&= \frac{1}{T}\pi^{\mu\nu}\partial_{\mu}u_{\nu}-\frac{\tau\gamma}{T}\pi^{33}-\frac{2u_{\eta}}{T\tau}\pi^{03}.
\end{aligned}$$

The term $\pi^{\mu\nu}\partial_{\mu}u_{\nu}$ is

$$\begin{aligned}
\pi^{\mu\nu}\partial_{\mu}u_{\nu} &= \pi^{00}\partial_0\gamma+\pi^{0i}\partial_0 u_i+\pi^{i0}\partial_i\gamma+\pi^{ij}\partial_i u_j, \\
&= \pi^{00}\left(\frac{d\gamma}{d\tau}-v^j\partial_j\gamma\right)+\pi^{0i}\left(\frac{du_i}{d\tau}-v^j\partial_j u_i\right)+\pi^{i0}\partial_i\gamma+\pi^{ij}\partial_i u_j, \\
&= \left(\pi^{0k}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}-\pi^{00}\frac{(u_{\eta})^2}{\gamma\tau^3}+\left(\pi^{mn}+\pi^{00}v^m v^n-\pi^{m0}v^n-\pi^{0n}v^m\right)\partial_m u_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
\pi^{\mu\nu}D_{\mu}u_{\nu} &= -\tau\gamma\pi^{33}-\frac{2u_{\eta}}{\tau}\pi^{03}-\frac{(u_{\eta})^2}{\gamma\tau^3}\pi^{00} \\
&\quad +\left(\pi^{0j}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}+\left(\pi^{mn}+\pi^{00}v^m v^n-\pi^{m0}v^n-\pi^{0n}v^m\right)\partial_m u_n.
\end{aligned}$$

or

$$\begin{aligned}
\pi^{\mu\nu}D_{\mu}u_{\nu} &= -\frac{\gamma}{\tau^3}\pi_{33}+\frac{2u_{\eta}}{\tau^3}\pi_{03}-\frac{(u_{\eta})^2}{\gamma\tau^3}\pi^{00} \\
&\quad +\left(\pi_0^j-\pi_{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}+\left(\pi^{mn}+\pi^{00}v^m v^n-\pi^{m0}v^n-\pi^{0n}v^m\right)\partial_m u_n.
\end{aligned}$$

7 Momentum Conservation

The equation of motion is

$$\sigma\gamma\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}u_i\right)+\frac{1}{\tau}\partial^{\mu}\left(\tau\pi_{\mu i}\right)=\partial_i\left(p+\Pi\right).$$

We remind that τ is the time-like coordinate and not the proper time. Now, we can open the equation of motion. The term

$$\begin{aligned}\sigma\gamma\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}u_i\right) &= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau}\\ &\quad -u_i\left(\varepsilon+p+\Pi\right)\frac{\gamma}{\sigma}\frac{d\sigma}{d\tau}\\ &\quad +u_i\gamma\frac{d}{d\tau}\left(\varepsilon+p\right)\\ &\quad +u_i\gamma\frac{d\Pi}{d\tau},\\ &= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau}\\ &\quad +u_i\left(\varepsilon+p+\Pi\right)\left(\partial^\mu u_\mu+\frac{\gamma}{\tau}\right)\\ &\quad -u_i\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)\left(\partial^\mu u_\mu+\frac{\gamma}{\tau}\right)\\ &\quad -u_i\frac{\Pi}{\tau_R}-u_i\left(\frac{\zeta}{\tau_R}+\Pi\right)\left(\partial^\mu u_\mu+\frac{\gamma}{\tau}\right)\\ &\quad +u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu.\end{aligned}$$

We conclude that

$$\begin{aligned}\sigma\gamma\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}u_i\right) &= \gamma\left(\varepsilon+p+\Pi\right)g_i^j\frac{du_j}{d\tau}-A\frac{u_iu^j}{\gamma}\frac{du_j}{d\tau}\\ &\quad -u_iA\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{\left(u_\eta\right)^2}{\gamma\tau^3}-\frac{\gamma}{\tau}\right)\\ &\quad -u_i\frac{\Pi}{\tau_R}+u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu,\end{aligned}$$

where we defined

$$A=\varepsilon+p-\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)-\frac{\zeta}{\tau_R}.$$

The term

$$\begin{aligned}\frac{1}{\tau}\partial^\mu\left(\tau\pi_{\mu i}\right) &= \partial^\mu\pi_{\mu i}+\frac{1}{\tau}\pi_{\mu i}\partial^\mu\tau,\\ &= \frac{d\pi_{0i}}{d\tau}-v^j\partial_j\pi_{0i}+\partial^j\pi_{ji}+\frac{\pi_{0i}}{\tau}.\end{aligned}$$

The term $\frac{d\pi_{0i}}{d\tau}$ was calculated and is

$$
\begin{aligned}
& \frac{d\pi_{0i}}{d\tau} \\
= \quad & -\frac{\eta}{2\gamma\tau_R} v^j \left(\partial_j u_i + \partial_i u_j\right) \\
& + \frac{\eta}{2\tau_R}\left(\frac{1}{\gamma} - \gamma\right)\frac{du_i}{d\tau} + \left(\frac{\eta}{2\tau_R} - \frac{\eta}{D\tau_R}\right)\frac{u_i u^j}{\gamma}\frac{du_j}{d\tau} \\
& - \left(u_i \pi_0^j + \gamma \pi_i^j - \left(\gamma + \frac{1}{\gamma}\right)\frac{\pi_{0i}}{\gamma} u^j - \frac{\pi^{00}}{\gamma} u_i u^j\right)\frac{du_j}{d\tau} \\
& + \frac{\eta}{D\tau_R} u_i \left(\frac{\gamma}{\tau} - \frac{\left(u_\eta\right)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& + \frac{\pi_{0i}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{\left(u_\eta\right)^2}{\gamma\tau^3} - \frac{\gamma}{\tau} - \frac{1}{\tau_R}\right) \\
& + \left(\frac{\tau u^\eta}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_\eta}{\tau\gamma\tau_R}\right) g_i^3 + \frac{u_\eta}{\gamma\tau}\pi_i^3
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& *\,*\,*\,*\,*\,*\,* \\
& \frac{1}{\tau}\partial^\mu\left(\tau\pi_{\mu i}\right) + u_i \frac{dw}{ds}\frac{\pi^{\mu\nu}}{T} D_\mu u_\nu \\
= \quad & -v^j \partial_j \pi_{0i} + \partial_j \pi_i^j - \frac{\eta}{2\gamma\tau_R} v^j \left(\partial_j u_i + \partial_i u_j\right) \\
& + \frac{\eta}{2\tau_R}\left(\frac{1}{\gamma} - \gamma\right)\frac{du_i}{d\tau} + \left(\frac{\eta}{2\tau_R} - \frac{\eta}{D\tau_R}\right)\frac{u_i u^j}{\gamma}\frac{du_j}{d\tau} \\
& - \left(u_i \pi_0^j + \gamma \pi_i^j - \left(\gamma + \frac{1}{\gamma}\right)\frac{\pi_{0i}}{\gamma} u^j - \frac{\pi^{00}}{\gamma} u_i u^j\right)\frac{du_j}{d\tau} \\
& + \frac{\eta}{D\tau_R} u_i \left(\frac{\gamma}{\tau} - \frac{\left(u_\eta\right)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& + \frac{\pi_{0i}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{\left(u_\eta\right)^2}{\gamma\tau^3} - \frac{1}{\tau_R}\right) \\
& + \left(\frac{\tau u^\eta}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_\eta}{\tau\gamma\tau_R}\right) g_i^3 + \frac{u_\eta}{\gamma\tau}\pi_i^3 \\
& \frac{u_i}{T}\frac{dw}{ds}\left(-\frac{\gamma}{\tau^3}\pi_{33} + \frac{2u_\eta}{\tau^3}\pi_{03} - \frac{\left(u_\eta\right)^2}{\gamma\tau^3}\pi^{00}\right) \\
& \frac{u_i}{T}\frac{dw}{ds}\left(+\left(\pi_0^j - \pi_{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau} + \left(\pi^{mn} + \pi^{00} v^m v^n - \pi^{m0} v^n - \pi^{0n} v^m\right)\partial_m u_n\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{\tau}\partial^{\mu}\left(\tau\pi_{\mu i}\right)+u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_{\mu}u_{\nu} \\
= \quad & -v^j\partial_j\pi_{0i}+\partial^j\pi_{ji}-\frac{\eta}{2\gamma^2\tau_R}u^j\left(\partial_j u_i+\partial_i u_j\right) \\
& +\frac{u_i}{T}\frac{dw}{ds}\left(\pi^{0j}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}-u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}\Gamma^{\lambda}_{\mu\nu}u_{\lambda}-\frac{u_i}{T}\frac{dw}{ds}\frac{\left(u_{\eta}\right)^2\pi^{00}}{\gamma\tau^3} \\
& +\frac{u_i}{T}\frac{dw}{ds}\left(\pi^{mn}+\pi^{00}v^m v^n-\pi^{m0}v^n-\pi^{0n}v^m\right)\partial_m u_n \\
& +\frac{\eta}{2\tau_R}\left(\frac{1}{\gamma}-\gamma\right)\frac{du_i}{d\tau}+\left(\frac{\eta}{2\tau_R}-\frac{\eta}{D\tau_R}\right)\frac{u_i u^j}{\gamma}\frac{du_j}{d\tau} \\
& -\left(u_i\pi^j_0+\gamma\pi^j_i-\left(\gamma+\frac{1}{\gamma}\right)\frac{\pi_{0i}}{\gamma}u^j-\frac{\pi^{00}}{\gamma}u_i u^j\right)\frac{du_j}{d\tau} \\
& u_i\left(\frac{\eta\gamma}{D\tau\tau_R}-\left(\frac{\eta}{D\tau_R}+\frac{\eta}{2\tau_R}\right)\frac{\left(u_{\eta}\right)^2}{\gamma\tau^3}-\frac{\eta}{D\tau_R}\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& +\frac{\pi_{0i}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{\left(u_{\eta}\right)^2}{\gamma\tau^3}-\frac{1}{\tau_R}\right) \\
& +\left(\frac{\tau u^{\eta}}{\gamma}\pi_{00}+\frac{1}{\tau}\pi_{03}-\frac{\eta u_{\eta}}{\tau\gamma\tau_R}\right)g^3_i+\frac{u_{\eta}}{\gamma\tau}\pi^3_i.
\end{aligned}
$$

8 Separation of the Terms

8.1 Acceleration Terms

We will collect only the acceleration terms

$$
\begin{aligned}
& \gamma\left(\varepsilon+p+\Pi\right)g^j_i\frac{du_j}{d\tau}-A\frac{u_i u^j}{\gamma}\frac{du_j}{d\tau} \\
& +\frac{u_i}{T}\frac{dw}{ds}\left(\pi^{0j}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau} \\
& +\frac{\eta}{2\tau_R}\left(\frac{1}{\gamma}-\gamma\right)\frac{du_i}{d\tau}+\left(\frac{\eta}{2\tau_R}-\frac{\eta}{D\tau_R}\right)\frac{u_i u^j}{\gamma}\frac{du_j}{d\tau} \\
& -\left(u_i\pi^j_0+\gamma\pi^j_i-\left(\frac{1}{\gamma}+\gamma\right)\frac{\pi_{0i}}{\gamma}u^j-\frac{\pi^{00}}{\gamma}u_i u^j\right)\frac{du_j}{d\tau}
\end{aligned}
$$

If we parametrize this as

$$
M^j_i\frac{du_j}{d\tau}.
$$

We have

$$
\begin{aligned}
M_i^j &= \gamma \left[\varepsilon + p + \Pi + \frac{\eta}{2\gamma\tau_R} \left(\frac{1}{\gamma} - \gamma \right) \right] g_i^j \\
+ \quad & \left(-A + \frac{\eta}{2\tau_R} - \frac{\eta}{D\tau_R} + \pi^{00} - \frac{1}{T} \frac{dw}{ds} \pi^{00} \right) \frac{u_i u^j}{\gamma} \\
& + \left(-u_i \pi_0^j + -\gamma \pi_i^j + \left(\frac{1}{\gamma} + \gamma \right) \frac{\pi_{0i}}{\gamma} u^j + \frac{u_i}{T} \frac{dw}{ds} \pi^{0j} \right)
\end{aligned}
$$

We can write it in the form,

$$
M_i^j = \gamma C_{total} g_i^j + A_{total} u^j u_i + m_i^j,
$$

with

$$
\begin{aligned}
C_{total} &= \varepsilon + p + \Pi - \frac{\eta}{2\tau_R} \left(\frac{\gamma^2 - 1}{\gamma^2} \right), \\
A_{total} &= \frac{1}{\gamma} \left[\frac{\eta}{2\tau_R} - \left(A + \frac{\eta}{D\tau_R} \right) + \pi_{00} \left(1 - \frac{1}{T} \frac{dw}{ds} \right) \right], \\
m_i^j &= -\gamma \pi_i^j - u_i \pi_0^j + \left(1 + \frac{1}{\gamma^2} \right) \pi_{i0} u^j + \frac{1}{T} \frac{dw}{ds} \pi_0^j u_i.
\end{aligned}
$$

8.2 Force Terms

Now we will collect the force terms (carefull with the sign)

$$
\begin{aligned}
& *\,*\,*\,*\,*\,*\,* \\
& \partial_i (p + \Pi) + u_i A \left(\frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} + \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\tau} \right) + u_i \frac{\Pi}{\tau_R} \\
& + v^j \partial_j \pi_{0i} - \partial_j \pi_i^j + \frac{\eta}{2\gamma\tau_R} v^j (\partial_j u_i + \partial_i u_j) \\
& - \frac{\eta}{D\tau_R} u_i \left(\frac{\gamma}{\tau} - \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} \right) \\
& - \frac{\pi_{0i}}{\gamma} \left(\frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} + \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{1}{\tau_R} \right) \\
& + \left(-\frac{u_\eta}{\gamma} \pi_{00} + \pi_{03} - \frac{\eta u_\eta}{\gamma\tau_R} \right) \frac{g_{i3}}{\tau^3} + \frac{u_\eta}{\gamma\tau} \pi_i^3 \\
& - \frac{u_i}{T} \frac{dw}{ds} \left(-\frac{\gamma}{\tau^3} \pi_{33} + \frac{2u_\eta}{\tau^3} \pi_{03} - \frac{(u_\eta)^2}{\gamma\tau^3} \pi^{00} \right) \\
& - \frac{u_i}{T} \frac{dw}{ds} \left(\left(\pi_0^j - \pi_{00} v^j \right) \frac{du_j}{d\tau} + \left(\pi^{mn} + \pi^{00} v^m v^n - \pi^{m0} v^n - \pi^{0n} v^m \right) \partial_m u_n \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\partial_i\left(p+\Pi\right)+v^j\partial_j\pi_{0i}-\partial^j\pi_{ji}+\frac{\eta}{2\gamma^2\tau_R}u^j\left(\partial_j u_i+\partial_i u_j\right)\\
&+u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}\Gamma^{\lambda}_{\mu\nu}u_\lambda+\frac{u_i}{T}\frac{dw}{ds}\frac{\left(u_\eta\right)^2\pi^{00}}{\gamma\tau^3}\\
&-\frac{u_i}{T}\frac{dw}{ds}\left(\pi^{mn}+\pi^{00}v^m v^n-\pi^{m0}v^n-\pi^{0n}v^m\right)\partial_m u_n\\
&+u_i\left(A+\frac{\eta}{D\tau_R}\right)\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{\left(u_\eta\right)^2}{\gamma\tau^3}-\frac{\gamma}{\tau}\right)+u_i\frac{\Pi}{\tau_R}\\
&-\frac{\pi_{0i}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{\left(u_\eta\right)^2}{\gamma\tau^3}-\frac{1}{\tau_R}\right)\\
&-\left(\frac{\tau u^\eta}{\gamma}\pi_{00}+\frac{1}{\tau}\pi_{03}-\frac{\eta u_\eta}{\tau\gamma\tau_R}\right)g_i^3-\frac{u_\eta}{\gamma\tau}\pi_i^3.
\end{aligned}
$$

If we parametrize the equation of motion as before

$$
M_i^j\frac{du_j}{d\tau}=B_{total}u_i+F_i+\partial_i\left(p+\Pi\right)+v^j\partial_j\pi_{0i}-\partial_j\pi_i^j,
$$

where

$$
\begin{aligned}
B_{total} &= \left(A+\frac{\eta}{D\tau_R}\right)\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{\left(u_\eta\right)^2}{\gamma\tau^3}-\frac{\gamma}{\tau}\right)+\frac{\Pi}{\tau_R}\\
&+\frac{1}{T\tau^3}\frac{dw}{ds}\left(\gamma\pi_{33}-2u_\eta\pi_{30}+\frac{\left(u_\eta\right)^2}{\gamma}\pi_{00}\right)\\
&-\frac{1}{T}\frac{dw}{ds}\left(\pi^{ij}+\frac{\pi_{00}}{\gamma^2}u^i u^j-\frac{\pi_0^i}{\gamma}u^j-\frac{\pi_0^j}{\gamma}u^i\right)\partial_i u_j,\\
F_i &= \frac{\eta}{2\gamma\tau_R}v^j\left(\partial_i u_j+\partial_j u_i\right)-\frac{\pi_{0i}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{\left(u_\eta\right)^2}{\gamma\tau^3}-\frac{1}{\tau_R}\right)\\
&+\left(\pi_{03}-\frac{u_\eta}{\gamma}\pi_{00}-\frac{\eta u_\eta}{\gamma\tau_R}\right)\frac{g_{3i}}{\tau^3}+\frac{u_\eta}{\gamma\tau^3}\pi_{i3}.
\end{aligned}
$$

9 derivatives

$$
\begin{aligned}
\partial_i u_j &= \gamma\partial_i v_j+v_j\partial_i\left(1+v_k v^k\right)^{-1/2}\\
&= \gamma g_{jj}\partial_i v^j-\gamma u_j u_k\partial_i v^k
\end{aligned}
$$

$$
\partial_i v^j=\frac{1}{\gamma}g^{jj}\partial_i u_j+\frac{v^j v^k}{\gamma}\partial_i u_k
$$

9.1 General Comment about solving shear

When solving the equation for the shear viscosity we dont need to solve all the components. Whe can use the orthogonality relation

$$u_\mu \pi^{\mu\nu} = 0,$$

and the traceless condition

$$\pi^\mu_\mu = 0$$

Thus,

$$\begin{aligned}
u_\mu \pi^{\mu\nu} &= 0 \\
u_0 \pi^{0\nu} + u_i \pi^{i\nu} &= 0 \\
\pi^{0\nu} &= -\frac{u_i}{u_0}\pi^{i\nu} \\
\pi^{0j} &= -\frac{u_i}{u_0}\pi^{ij} \\
&\quad * * * \\
u_\nu u_\mu \pi^{\mu\nu} &= 0 \\
u_0 u_0 \pi^{00} + u_i u_j \pi^{ij} - u_i u_0 \frac{u_j}{u_0}\pi^{ij} - u_0 u_j \frac{u_i}{u_0}\pi^{ij} &= 0 \\
u_0 u_0 \pi^{00} - u_i u_j \pi^{ij} &= 0 \\
\pi^{00} &= \frac{u_i u_j}{u_0^2}\pi^{ij} \\
&\quad * * * \\
g_{\mu\nu}\pi^{\mu\nu} &= 0 \\
\pi^{00} - \pi^{11} - \pi^{22} - \tau^2 \pi^{33} &= 0 \\
\frac{1}{\tau^2}\left(\pi^{00} - \pi^{11} - \pi^{22}\right) &= \pi^{33}
\end{aligned}$$

This implies that once we know the space part of the shear stress π^{ij} we can detrmine all the other components.

Actually, we didn't use the traceless condition up to now. This means that we only need to solve 5 of the space components of π^{ij} (since it is also a symmetric tensor). That is

$$
\begin{aligned}
\pi^{00} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{u^i u^j}{\gamma^2}\pi_{ij} \\
\pi^{00} &= \sum_{i=1}^{2}\sum_{j=1}^{2} v^i v^j \pi_{ij} + 2\frac{u^3}{\gamma}\sum_{j=1}^{2}\frac{u^j}{\gamma}\pi_{3j} + v^3 v^3\left(\pi_{00}-\pi_{11}-\pi_{22}\right)\tau^2 \\
\pi^{00}\left(1-(v^\eta\tau)^2\right) &= \sum_{i=1}^{2}\sum_{j=1}^{2} v^i v^j \pi_{ij} + 2\frac{u^3}{\gamma}\sum_{j=1}^{2}\frac{u^j}{\gamma}\pi_{3j} - (v^\eta\tau)^2\left(\pi_{11}+\pi_{22}\right) \\
\pi^{00} &= \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{u_i u_j}{u_0^2}\pi^{ij}\right) + \frac{u_\eta}{u_0^2}2\left(\sum_{i=1}^{2} u_i\pi^{i3}\right) + \left(\frac{u_\eta}{u_0}\right)^2\pi^{33} \\
\pi^{00} &= \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{u_i u_j}{u_0^2}\pi^{ij}\right) + \frac{u_\eta}{u_0^2}2\left(\sum_{i=1}^{2} u_i\pi^{i3}\right) + \left(\frac{u_\eta}{u_0\tau}\right)^2\left(\pi^{00}-\pi^{11}-\pi^{22}\right) \\
\pi^{00}\left(1-\left(\frac{u_\eta}{u_0\tau}\right)^2\right) &= \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{u_i u_j}{u_0^2}\pi^{ij}\right) + \frac{u_\eta}{u_0^2}2\left(\sum_{i=1}^{2} u_i\pi^{i3}\right) - \left(\frac{u_\eta}{u_0\tau}\right)^2\left(\pi^{11}+\pi^{22}\right) \\
\pi^{00} &= \frac{1}{1-\left(\frac{u_\eta}{u_0\tau}\right)^2}\left[\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{u_i u_j}{u_0^2}\pi^{ij}\right) + \frac{u_\eta}{u_0^2}2\left(\sum_{i=1}^{2} u_i\pi^{i3}\right) - \left(\frac{u_\eta}{u_0\tau}\right)^2\left(\pi^{11}+\pi^{22}\right)\right]
\end{aligned}
$$

10 Solving Shear Viscosity

10.1 Transverse components

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\pi_{ij}}{\sigma}\right) = & -\frac{\pi_{ij}}{\sigma\gamma\tau_R} \\
& +\frac{\eta}{2\sigma\gamma\tau_R}\left(\partial_i u_j + \partial_j u_i\right) - \frac{\eta}{2\sigma\tau_R}\left(u_i\frac{du_j}{d\tau} + u_j\frac{du_i}{d\tau}\right) \\
& -\frac{1}{\sigma}\left(u_i\pi_j^k + u_j\pi_i^k\right)\frac{du_k}{d\tau} + \left(u_i\frac{\pi_{j0}}{\sigma} + u_j\frac{\pi_{i0}}{\sigma} + \frac{\eta}{D\gamma\tau_R}\frac{P_{ij}}{\sigma}\right)v^k\frac{du_k}{d\tau} \\
& +\frac{\eta}{D\sigma\gamma\tau_R}P_{ij}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{(u_\eta)^2}{\gamma\tau^3} - \frac{\gamma}{\tau}\right).
\end{aligned}
$$

10.2 Longitudinal components

$i = 1, 2$ and $j = 3$

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\pi_{i3}}{\sigma}\right) &= -\frac{\pi_{i3}}{\sigma\gamma\tau_R} \\
&+\frac{\eta}{2\sigma\gamma\tau_R}\left(\partial_i u_\eta+\partial_3 u_i\right)-\frac{\eta}{2\sigma\tau_R}\left(u_i\frac{du_\eta}{d\tau}+u_\eta\frac{du_i}{d\tau}\right) \\
&-\frac{1}{\sigma}\left(u_i\pi_3^k+u_\eta\pi_i^k\right)\frac{du_k}{d\tau}+\left(u_i\frac{\pi_{03}}{\sigma}+u_\eta\frac{\pi_{0i}}{\sigma}-\frac{\eta}{D\gamma\tau_R}\frac{u_\eta u_i}{\sigma}\right)v^k\frac{du_k}{d\tau} \\
&-\frac{\eta}{D\sigma\gamma\tau_R}u_\eta u_i\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{(u_\eta)^2}{\gamma\tau^3}-\frac{\gamma}{\tau}\right) \\
&+\frac{1}{\sigma\tau}\pi_{i3}.
\end{aligned}
$$

11 SPH representation

$$
\begin{aligned}
\left(v^j\partial_j\pi_{0i}\right)_\alpha &= \sigma_\alpha^*\sum_\beta\nu_\beta\left[\frac{(\pi_{0i})_\beta}{\sigma_\beta^{*2}}+\frac{(\pi_{0i})_\alpha}{\sigma_\alpha^{*2}}\right]v_\alpha^j\partial_j W_{\alpha\beta} \\
\left(\partial_j\pi_i^j\right)_\alpha &= \sigma_\alpha^*\sum_\beta\nu_\beta\left[\frac{\left(\pi_i^j\right)_\beta}{\sigma_\beta^{*2}}+\frac{\left(\pi_i^j\right)_\alpha}{\sigma_\alpha^{*2}}\right]\partial_j W_{\alpha\beta}
\end{aligned}
$$

12 Bjorken Scalling

The scalling ansatz is

$$u_\eta = 0$$

with all the derivatives in the longitudinal direction equal to zero. In the Bjorken scalling approximation, the equation for velocity should look like

$$M_{ij}\frac{du_j}{d\tau} = Bu_i + F_i + \partial_i\left(p+\Pi\right) + v^j\partial_j\pi_{0i} + \partial_j\pi_{ij},$$

where

$$M_{ij} = \gamma C g_i^j + \frac{A}{\gamma}u_j u_i + m_{ij},$$

with

$$
\begin{aligned}
C_{ideal} &= \varepsilon + p, \\
C_{bulk} &= \Pi, \\
C_{shear} &= \frac{\eta}{2\tau_R}\frac{1-\gamma^2}{\gamma^2}, \\
C &= C_{ideal} + C_{bulk} + C_{shear}. \\
A_{ideal} &= \varepsilon + p - \frac{dw}{ds}s, \\
A_{bulk} &= -\frac{\zeta}{\tau_R} - \frac{dw}{ds}\frac{\Pi}{T}, \\
A_{shear} &= -\frac{\eta}{\tau_R}\left(\frac{1}{2}-\frac{1}{D}\right) - \pi^{00}\left(1-\frac{1}{T}\frac{dw}{ds}\right), \\
A &= A_{ideal} + A_{bulk} + A_{shear}, \\
m_{ij} &= \gamma\pi_{ij} + u_i\pi_{j0} - \left(1+\frac{1}{\gamma^2}\right)\pi_{i0}u_j - \frac{1}{T}\frac{dw}{ds}\pi_{j0}u_i.
\end{aligned}
$$

and

$$
\begin{aligned}
B_{ideal} &= A_{ideal}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right) \\
B_{bulk} &= A_{bulk}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right) + \frac{\Pi}{\tau_R} \\
B_{shear} &= \frac{\eta}{D\tau_R}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right) \\
&\quad + \frac{1}{T}\frac{dw}{ds}\frac{\gamma\pi_{33}}{\tau^3} - \frac{1}{T}\frac{dw}{ds}\left(\pi_{ij} + \frac{\pi_{00}}{\gamma}u_iu_j - \frac{\pi_{i0}}{\gamma}u_j - \frac{\pi_{j0}}{\gamma}u_i\right)\partial_i u_j, \\
B &= B_{ideal} + B_{bulk} + B_{shear} \\
F_i &= \frac{\eta}{2\gamma^2\tau_R}u^j\left(\partial_i u_j + \partial_j u_i\right) - \frac{\pi_{0i}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau_R}\right).
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{s}{\sigma}\right) &= -\frac{\Pi\tau}{T\sigma^*}\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right) \\
&\quad - \frac{\tau}{T\sigma^*}\left(\frac{\gamma}{\tau^3}\pi_{33} + \left(\pi_{j0} + \pi_{00}v^j\right)\frac{du_j}{d\tau}\right) \\
&\quad + \frac{\tau}{T\sigma^*}\left(\pi_{ij} + \frac{\pi_{00}}{\gamma^2}u_iu_j - \frac{\pi_{i0}}{\gamma}u_j - \frac{\pi_{j0}}{\gamma}u_i\right)\partial_i u_j
\end{aligned}
$$

12.1 Transverse components

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\pi_{ij}}{\sigma}\right) &= -\frac{\pi_{ij}}{\sigma\gamma\tau_R} \\
&+\frac{\eta}{2\sigma\gamma\tau_R}\left(\partial_i u_j+\partial_j u_i\right)-\frac{\eta}{2\sigma\tau_R}\left(u_i\frac{du_j}{d\tau}+u_j\frac{du_i}{d\tau}\right) \\
&+\frac{\gamma\tau}{\sigma^*}\left(u_i\pi_{jk}+u_j\pi_{ik}\right)\frac{du_k}{d\tau}+\frac{\gamma\tau}{\sigma^*}\left(u_i\pi_{j0}+u_j\pi_{ik}+\frac{\eta}{D\gamma\tau_R}P_{ij}\right)v^k\frac{du_k}{d\tau} \\
&+\frac{\eta\tau}{D\tau_R\sigma^*}P_{ij}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{\gamma}{\tau}\right).
\end{aligned}
$$

12.2 General Comment about solving shear with Scalling

When solving the equation for the shear viscosity we dont need to solve all the components. Whe can use the orthogonality relation

$$u_\mu\pi^{\mu\nu}=0,$$

and the traceless condition

$$\pi^\mu_\mu=0$$

Thus,

$$
\begin{aligned}
u_\mu\pi^{\mu\nu} &= 0 \\
u_0\pi^{0\nu}+u_i\pi^{i\nu} &= 0 \\
\pi^{0\nu} &= -\frac{u_i}{u_0}\pi^{i\nu} \\
\pi^{0j} &= -\frac{u_i}{u_0}\pi^{ij} \\
& *** \\
u_\nu u_\mu\pi^{\mu\nu} &= 0 \\
u_0u_0\pi^{00}+u_iu_j\pi^{ij}-u_iu_0\frac{u_j}{u_0}\pi^{ij}-u_0u_j\frac{u_i}{u_0}\pi^{ij} &= 0 \\
u_0u_0\pi^{00}-u_iu_j\pi^{ij} &= 0 \\
\pi^{00} &= \frac{u_iu_j}{u_0^2}\pi^{ij} \\
& *** \\
g_{\mu\nu}\pi^{\mu\nu} &= 0 \\
\pi^{00}-\pi^{11}-\pi^{22}-\tau^2\pi^{33} &= 0 \\
\frac{1}{\tau^2}\left(\pi^{00}-\pi^{11}-\pi^{22}\right) &= \pi^{33}
\end{aligned}
$$

This implies that once we know the space part of the shear stress π^{ij} we can detrmine all the other components.

Actually, we didn't use the traceless condition up to now. This means that we only need to solve 5 of the space components of π^{ij} (since it is also a symmetric tensor). That is

$$\begin{aligned}\pi^{00} &= \frac{u_i u_j}{\gamma^2}\pi^{ij} \\ \pi^{00} &= \left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{u_i u_j}{u_0^2}\pi^{ij}\right)\end{aligned}$$

and

$$\begin{aligned}\pi^{03} &= -\frac{u_i}{u_0}\pi^{i3} \\ &= -\left(\sum_{i=1}^{2}\frac{u_i}{u_0}\pi^{i3}\right)\end{aligned}$$

13 Comparison - index up

$$M^{ij}\frac{du^j}{d\tau} = B_{total}u^i + F^i - \partial_i\left(p+\Pi\right) + v^j\partial_j\pi^{i0} - \partial_j\pi^{ij},$$

where

$$M^{ij} = \gamma C_{total}\delta^{ij} + A_{total}u^j u^i + m^{ij},$$

with

$$\begin{aligned}C_{total} &= \varepsilon + p + \Pi + \frac{\eta}{2\tau_R}\frac{1-\gamma^2}{\gamma^2}, \\ C_{shear} &= \frac{\eta}{2\tau_R}\frac{1-\gamma^2}{\gamma^2}, \\ A_{total} &= \frac{1}{\gamma}\left[\left(A + \frac{\eta}{D\tau_R}\right) - \frac{\eta}{2\tau_R} - \pi^{00}\left(1 - \frac{1}{T}\frac{dw}{ds}\right)\right], \\ A_{shear} &= \frac{\eta}{D\gamma\tau_R} - \frac{\eta}{2\gamma\tau_R} - \pi^{00}\left(1 - \frac{1}{T}\frac{dw}{ds}\right), \\ m^{ij} &= \gamma\pi^{ij} + u^i\pi^{j0} - \left(1 + \frac{1}{\gamma^2}\right)\pi^{i0}u^j - \frac{1}{T}\frac{dw}{ds}\pi^{j0}u^i.\end{aligned}$$

and

$$
\begin{aligned}
B_{total} &= \left(A+\frac{\eta}{D\tau_R}\right)\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{\gamma}{\tau}\right)+\frac{\Pi}{\tau_R} \\
&\quad +\frac{1}{T}\frac{dw}{ds}\gamma\tau\pi^{33}-\frac{1}{T}\frac{dw}{ds}\left(-\pi^{ij}-\frac{\pi_{00}}{\gamma^2}u^iu^j+\frac{\pi^{i0}}{\gamma}u^j+\frac{\pi^{j0}}{\gamma}u^i\right)\partial_i u^j, \\
B_{shear} &= \frac{\eta}{D\tau_R}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{\gamma}{\tau}\right)+\frac{1}{T}\frac{dw}{ds}\gamma\tau\pi^{33}-\frac{1}{T}\frac{dw}{ds}\left(-\pi^{ij}-\frac{\pi_{00}}{\gamma^2}u^iu^j+\frac{\pi^{i0}}{\gamma}u^j+\frac{\pi^{j0}}{\gamma}u^i\right)\partial_i u^j \\
F^i &= \frac{\eta}{2\gamma^2\tau_R}u^j\left(\partial_i u^j+\partial_j u^i\right)-\frac{\pi^{i0}}{\gamma}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{1}{\tau_R}\right).
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{s}{\sigma}\right) &= -\frac{\Pi\tau}{T\sigma^*}\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right) \\
&\quad -\frac{\tau}{T\sigma^*}\gamma\tau\pi^{33} \\
&\quad +\frac{\tau}{T\sigma^*}\left(-\pi^{j0}+\pi^{00}v^j\right)\frac{du^j}{d\tau} \\
&\quad -\frac{\tau}{T\sigma^*}\left(\pi^{ij}+\frac{\pi^{00}}{\gamma^2}u^iu^j-\frac{\pi^{i0}}{\gamma}u^j-\frac{\pi^{j0}}{\gamma}u^i\right)\partial_i u^j
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\pi^{ij}}{\sigma}\right) &= -\frac{\pi^{ij}}{\sigma\gamma\tau_R} \\
&\quad -\frac{\eta\tau}{2\sigma^*\tau_R}\left(\partial_i u^j+\partial_j u^i\right)-\frac{\eta\gamma\tau}{2\tau_R\sigma^*}\left(u^i\frac{du^j}{d\tau}+u^j\frac{du^i}{d\tau}\right) \\
&\quad +\frac{\gamma\tau}{\sigma^*}\left(u^i\pi^{jk}+u^j\pi^{ik}\right)\frac{du^k}{d\tau}-\frac{\gamma\tau}{\sigma^*}\left(u^i\pi^{j0}+u^j\pi^{i0}+\frac{\eta}{D\gamma\tau_R}P^{ij}\right)v^k\frac{du^k}{d\tau} \\
&\quad +\frac{\eta\tau}{D\tau_R\sigma^*}P^{ij}\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{\gamma}{\tau}\right).
\end{aligned}
$$

$$
\begin{aligned}
\left(v^j\partial_j\pi^{0i}\right)_\alpha &= \sigma^*_\alpha\sum_\beta \nu_\beta\left[\frac{\left(\pi^{0i}\right)_\beta}{\sigma^{*2}_\beta}+\frac{\left(\pi^{0i}\right)_\alpha}{\sigma^{*2}_\alpha}\right]v^j_\alpha\partial_j W_{\alpha\beta} \\
\left(\partial_j\pi^{ij}\right)_\alpha &= \sigma^*_\alpha\sum_\beta \nu_\beta\left[\frac{\left(\pi^{ij}\right)_\beta}{\sigma^{*2}_\beta}+\frac{\left(\pi^{ij}\right)_\alpha}{\sigma^{*2}_\alpha}\right]\partial_j W_{\alpha\beta}
\end{aligned}
$$

Also,

$$
\partial_i u^j=\gamma\partial_i v^j+\gamma^3 v^j v^k\partial_i v^k.
$$

13.1 General Comment about solving shear

When solving the equation for the shear viscosity we dont need to solve all the components. Whe can use the orthogonality relation

$$u_\mu \pi^{\mu\nu} = 0,$$

and the traceless condition

$$\pi^\mu_\mu = 0$$

Thus,

$$\begin{aligned} \pi^{0j} &= \frac{u^i}{\gamma}\pi^{ij} \\ \pi^{00} &= \frac{u^i u^j}{\gamma^2}\pi^{ij} \\ \pi^{33} &= \frac{1}{\tau^2}\left(\pi^{00} - \pi^{11} - \pi^{22}\right) \end{aligned}$$