For a general metric,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^{\mu\nu}\right)+\Gamma^\nu_{\lambda\mu}T^{\lambda\mu}=0,$$

where,

$$\Gamma^\nu_{\mu\lambda}=\frac{1}{2}g^{\nu\sigma}\left(\partial_\mu g_{\sigma\lambda}+\partial_\lambda g_{\sigma\mu}-\partial_\sigma g_{\mu\lambda}\right)$$

For the metric,

$$\begin{aligned}
g_{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -\tau^2 \end{pmatrix}\\
g^{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -\frac{1}{\tau^2} \end{pmatrix}
\end{aligned}$$

we have,

$$\begin{aligned}
\sqrt{-g} &= \tau,\\
\\
\Gamma^0_{33} &= \frac{1}{2}g^{0\sigma}\left(\partial_3 g_{\sigma 3}+\partial_3 g_{\sigma 3}-\partial_\sigma g_{33}\right),\\
&= \frac{1}{2}g^{00}\left(\partial_3 g_{03}+\partial_3 g_{03}-\partial_0 g_{33}\right),\\
&= \frac{-1}{2}g^{00}\left(\partial_0 g_{33}\right)=\tau,\\
\\
\Gamma^3_{30} &= \frac{1}{2}g^{3\sigma}\left(\partial_3 g_{\sigma 0}+\partial_0 g_{\sigma 3}-\partial_\sigma g_{30}\right),\\
&= \frac{1}{2}g^{33}\left(\partial_3 g_{30}+\partial_0 g_{33}-\partial_3 g_{30}\right),\\
&= \frac{1}{2}g^{33}\left(\partial_0 g_{33}\right)=\frac{2\tau}{2\tau^2}=\frac{1}{\tau},\\
\\
\Gamma^3_{03} &= \Gamma^3_{30}=\frac{1}{\tau},\\
\\
\text{All other terms} &= 0.
\end{aligned}$$

More usefull relations,

$$
\begin{aligned}
u_\mu u^\mu &= 1 \\
u_\mu \frac{du^\mu}{d\tau} + u^\mu \frac{du_\mu}{d\tau} &= 0 \\
2\gamma \frac{d\gamma}{d\tau} + u_i \frac{du^i}{d\tau} + u^i \frac{du_i}{d\tau} &= 0 \\
2\gamma \frac{d\gamma}{d\tau} + u_i \frac{dg^{ij} u_j}{d\tau} + g^{ij} u_j \frac{du_i}{d\tau} &= 0 \\
2\gamma \frac{d\gamma}{d\tau} + 2g^{ij} u_i \frac{du_j}{d\tau} + u_j u_i \frac{dg^{ij}}{d\tau} &= 0 \\
-\frac{1}{\gamma} g^{ij} u_i \frac{du_j}{d\tau} - \frac{u_j u_i}{2\gamma} \frac{dg^{ij}}{d\tau} &= \frac{d\gamma}{d\tau}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\tau\sigma\gamma} u_\mu \partial^\mu \gamma\tau\sigma &= \frac{1}{\tau\sigma} u_\mu \partial^\mu \tau\sigma + \frac{1}{\gamma} u_\mu \partial^\mu \gamma \\
&= \frac{1}{\tau\sigma} u_\mu \partial^\mu \tau\sigma + \frac{d\gamma}{d\tau} \\
\frac{1}{\tau\sigma} u_\mu \partial^\mu \tau\sigma &= \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} - \frac{d\gamma}{d\tau} \\
\frac{1}{\tau\sigma} u_\mu \partial^\mu \tau\sigma &= \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} + \frac{1}{\gamma} g^{ij} u_i \frac{du_j}{d\tau} + \frac{u_j u_i}{2\gamma} \frac{dg^{ij}}{d\tau}
\end{aligned}
$$

1 Equation of Motion

Then, for $\nu = i$,

$$
\begin{aligned}
\frac{1}{\tau} \partial_\mu \left(\tau T^{\mu i} \right) + \Gamma^i_{\lambda\mu} T^{\lambda\mu} &= 0, \\
\frac{1}{\tau} \partial_\mu \left(\tau T^{\mu i} \right) + \frac{2}{\tau} T^{03} \delta_{i3} &= 0, \\
\frac{1}{\tau} \partial^\mu \left(\tau g^{\alpha i} T_{\mu\alpha} \right) + \frac{2}{\tau} T^{03} \delta_{i3} &= 0,
\end{aligned}
$$

For, $i = 1$

$$
\begin{aligned}
\frac{1}{\tau} \partial^\mu \left(\tau g^{\alpha 1} T_{\mu\alpha} \right) &= 0, \\
-\frac{1}{\tau} \partial^\mu \left(\tau T_{\mu 1} \right) &= 0, \\
\frac{1}{\tau} \partial^\mu \left(\tau T_{\mu 1} \right) &= 0.
\end{aligned}
$$

For, $i=2$

$$\begin{aligned}
\frac{1}{\tau}\partial^{\mu}\left(\tau g^{\alpha 2}T_{\mu\alpha}\right) &= 0,\\
-\frac{1}{\tau}\partial^{\mu}\left(\tau T_{\mu 2}\right) &= 0,\\
\frac{1}{\tau}\partial^{\mu}\left(\tau T_{\mu 2}\right) &= 0.
\end{aligned}$$

For, $i=3$

$$\begin{aligned}
\frac{1}{\tau}\partial^{\mu}\left(\tau g^{\alpha 3}T_{\mu\alpha}\right)+\frac{2}{\tau}T^{03} &= 0,\\
\frac{1}{\tau}\partial^{\mu}\left(\tau\frac{-1}{\tau^2}T_{\mu 3}\right)+\frac{2}{\tau}T^{03} &= 0,\\
-\frac{1}{\tau}\partial^{\mu}\left(\frac{1}{\tau}T_{\mu 3}\right)+\frac{2}{\tau}T^{03} &= 0,\\
-\frac{1}{\tau^2}\partial^{\mu}\left(T_{\mu 3}\right)-\frac{T_{\mu 3}}{\tau}\partial^{\mu}\left(\frac{1}{\tau}\right)+\frac{2}{\tau}T^{03} &= 0,\\
-\frac{1}{\tau^2}\partial^{\mu}\left(T_{\mu 3}\right)+\frac{g_{\alpha 0}g_{\beta 3}T^{\alpha\beta}}{\tau}\frac{1}{\tau^2}+\frac{2}{\tau}T^{03} &= 0,\\
-\frac{1}{\tau^2}\partial^{\mu}\left(T_{\mu 3}\right)-\frac{T^{03}}{\tau}+\frac{2}{\tau}T^{03} &= 0,\\
-\frac{1}{\tau^2}\partial^{\mu}\left(T_{\mu 3}\right)+\frac{1}{\tau}T^{03} &= 0,\\
-\frac{1}{\tau}\partial^{\mu}\left(\tau T_{\mu 3}\right) &= 0.
\end{aligned}$$

Thus,

$$\frac{1}{\tau}\partial_{\mu}\left(\tau T^{\mu i}\right)+\Gamma^{i}_{\lambda\mu}T^{\lambda\mu}=-\frac{1}{\tau}\partial^{\mu}\left(\tau T_{\mu i}\right)=0.$$

This is the equation of motion used by the SPheRio. We will show this below, using the conservation of entropy in this metric,

$$\begin{aligned}
\frac{1}{\tau}\partial_{\mu}\left(\tau s^{\mu}\right) &= 0,\\
\frac{1}{\tau}\partial^{\mu}\left(\tau s_{\mu}\right) &= 0,\\
\tau s\partial^{\mu}u_{\mu}+u_{\mu}\partial^{\mu}\tau s &= 0,\\
\frac{1}{\tau s}u_{\mu}\partial^{\mu}\tau s &= -\partial^{\mu}u_{\mu},
\end{aligned}$$

and it is usefull to show that,

$$\begin{aligned}
\partial_{\mu}\left(fu^{\mu}\right) &= \tau s u_{\mu}\partial^{\mu}\left(\frac{f}{\tau s}\right),\\
u_{\mu}\partial^{\mu} &= \gamma\frac{d}{d\tau}.
\end{aligned}$$

Now we can determine the equation of motion,

$$
\begin{aligned}
-\frac{1}{\tau}\partial^{\mu}\left(\tau T_{\mu i}\right) &= 0,\\
\partial^{\mu}\left(\tau T_{\mu i}\right) &= 0,\\
\partial^{\mu}\left(\tau\left(\varepsilon+p\right)u_{\mu}u_{i}-\tau g_{\mu i}p\right) &= 0,\\
\tau s u_{\mu}\partial^{\mu}\left(\frac{\left(\varepsilon+p\right)}{s}u_{i}\right)-\partial^{\mu}\left(\tau g_{\mu i}p\right) &= 0,\\
\tau s\gamma\frac{d}{d\tau}\left(\frac{\left(\varepsilon+p\right)}{s}u_{i}\right)-\partial^{\mu}\left(\tau g_{\mu i}p\right) &= 0,\\
\tau s\gamma\frac{d}{d\tau}\left(\frac{\left(\varepsilon+p\right)}{s}u_{i}\right)-g_{\mu i}\partial^{\mu}\tau p-\tau p\partial^{\mu}g_{\mu i} &= 0,\\
\tau s\gamma\frac{d}{d\tau}\left(\frac{\left(\varepsilon+p\right)}{s}u_{i}\right)-\partial_{i}\left(\tau p\right) &= 0,\\
\frac{d}{d\tau}\left(\frac{\left(\varepsilon+p\right)}{s}u_{i}\right)-\frac{1}{s\gamma}\partial_{i}p &= 0,\\
\frac{d}{d\tau}\left(\frac{\left(\varepsilon+p\right)}{s}\gamma g_{ij}v^{j}\right)-\frac{1}{s\gamma}\partial_{i}p &= 0,
\end{aligned}
$$

where we used that,

$$\partial^{\mu}g_{\mu i}=0.$$

remembering that the SPH expression for $\partial_i\left(p\right)$ is

$$\left(\partial_i p\right)_{\beta}=\sum_{\alpha}\nu_{\alpha}s^{*}_{\beta}\left(\frac{p_{\alpha}}{\left(s^{*}_{\alpha}\right)^{2}}+\frac{p_{\beta}}{\left(s^{*}_{\beta}\right)^{2}}\right)\nabla_{\beta}W_{\alpha\beta},$$

we finally obtain,

$$
\begin{aligned}
\frac{d}{d\tau}\left(\frac{\left(\varepsilon+p\right)}{s}\gamma g_{ij}v^{j}\right)_{\beta} &= \tau\sum_{\alpha}\nu_{\alpha}\left(\frac{p_{\alpha}}{\left(s^{*}_{\alpha}\right)^{2}}+\frac{p_{\beta}}{\left(s^{*}_{\beta}\right)^{2}}\right)\nabla_{\beta}W_{\alpha\beta},\\
&= \frac{1}{\tau}\sum_{\alpha}\nu_{\alpha}\left(\frac{p_{\alpha}}{\left(\gamma_{\alpha}s_{\alpha}\right)^{2}}+\frac{p_{\beta}}{\left(\gamma_{\beta}s_{\beta}\right)^{2}}\right)\nabla_{\beta}W_{\alpha\beta}.
\end{aligned}
$$

This is exactly the SPH equation of motion calculated in the paper and used by SPheRio. To obtain this expression we used the following SPH definition for the entropy,

$$s^{*}_{\beta}=\tau\gamma_{\beta}s_{\beta}=\sum_{\alpha}\nu_{\alpha}W_{\alpha\beta}.$$

2 Energy Equation

Now we calculate the energy conservation equation, $\nu = 0$

$$
\begin{aligned}
\frac{1}{\tau}\partial_\mu\left(\tau T^{\mu 0}\right) + \Gamma^0_{\lambda\mu}T^{\lambda\mu} &= 0, \\
\frac{1}{\tau}\partial_\mu\left(\tau T^{\mu 0}\right) + \tau T^{33} &= 0, \\
\frac{1}{\tau}\partial_\tau\left(\tau T^{00}\right) + \frac{1}{\tau}\partial_i\left(\tau T^{i0}\right) + \tau T^{33} &= 0.
\end{aligned}
$$

Integrating,

$$
\begin{aligned}
\partial_\tau\left(\int \tau T^{00}\right) + \int \partial_i\left(\tau T^{i0}\right) + \int \tau^2 T^{33} &= 0. \\
\frac{d}{d\tau}\left(\int \tau T^{00}\right) + \int \tau^2 T^{33} &= 0.
\end{aligned}
$$

For the scalling solution,

$$
\begin{aligned}
T^{33} &= -g^{33}p = \frac{1}{\tau^2}p \\
\frac{d}{d\tau}\left(\int \tau T^{00}\right) + \int p &= 0.
\end{aligned}
$$

We can calculate in the SPH parametrization,

$$
\begin{aligned}
\int \tau T^{00} &= \sum_\alpha \nu_\alpha \frac{T^{00}_\alpha}{\gamma_\alpha s_\alpha}, \\
\int \tau^2 T^{33} &= \sum_\alpha \nu_\alpha \tau \frac{T^{33}_\alpha}{\gamma_\alpha s_\alpha}.
\end{aligned}
$$

Lets define the variable E_z as

$$
E_z(\tau) = \int_{\tau_0}^{\tau} d\tau \int \tau^2 T^{33},
$$

where,

$$
\begin{aligned}
E_z(\tau_0) &= 0, \\
\frac{dE_z(\tau)}{d\tau} &= \int \tau^2 T^{33}.
\end{aligned}
$$

Then, the quantity below is the one that is actually conserved,

$$
\left(\int \tau T^{00}\right) + E_z(\tau) = Const.
$$

The term $E_z(\tau)$ must be calculated by Runge-Kutta.

3 With Bulk Viscosity

The extention of the result of the first section to include bulk viscosity is trivial,

$$\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}g_{ij}u^j\right)=\frac{1}{\sigma\gamma}\partial_i\left(p+\Pi\right).$$

In the expression above we also changed the reference density to the especific volume. This modification is also trivial. It is usefull to remind the following relations,

$$\begin{aligned}
\frac{1}{\tau\sigma}u_\mu\partial^\mu\tau\sigma &= -\partial^\mu u_\mu \\
\frac{1}{\sigma}u_\mu\partial^\mu\sigma+\frac{\gamma}{x_0} &= -\partial^\mu u_\mu \\
\sigma^* &= \gamma\tau\sigma \\
\partial_\mu\left(fu^\mu\right) &= \tau\sigma u_\mu\partial^\mu\left(\frac{f}{\tau\sigma}\right) \\
u_\mu\partial^\mu &= \gamma\frac{d}{d\tau}.
\end{aligned}$$

and,

$$\begin{aligned}
\frac{\gamma}{\sigma^*}u_\mu\partial^\mu\frac{\sigma^*}{\gamma} &= \frac{\gamma}{\sigma^*}\gamma\frac{d}{d\tau}\frac{\sigma^*}{\gamma} \\
&= \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\gamma^2\frac{d}{d\tau}\frac{1}{\gamma} \\
&= \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{d\gamma}{d\tau} \\
\gamma &= \left(1-u^iu_i\right)^{1/2} \\
&= \left(1-g^{ij}u_iu_j\right)^{1/2} \\
\frac{d\gamma}{d\tau} &= \frac{d}{d\tau}\left(1-g^{ij}u_iu_j\right)^{1/2} \\
&= \frac{-1/2}{\left(1-g^{ij}u_iu_j\right)^{1/2}}\frac{d}{d\tau}\left(g^{ij}u_iu_j\right) \\
&= -\frac{1}{\gamma}\left[\frac{g^{ij}u_i}{2}\frac{du_j}{d\tau}+\frac{g^{ij}u_j}{2}\frac{du_i}{d\tau}+\frac{u_iu_j}{2}\frac{dg^{ij}}{d\tau}\right] \\
&= -\frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau}-\frac{u_iu_j}{2\gamma}\frac{dg^{ij}}{d\tau} \\
\frac{\gamma}{\sigma^*}u_\mu\partial^\mu\frac{\sigma^*}{\gamma} &= \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau}+\frac{u_iu_j}{2\gamma}\frac{dg^{ij}}{d\tau} \\
&= -\partial^\mu u_\mu
\end{aligned}$$

We remind that τ is the time-like coordinate and not the proper time, as can be seen in the last relation above. The equation for the bulk viscosity can be expressed, in this metric as,

$$\begin{aligned}
\tau_R \gamma \frac{d\Pi}{d\tau} + \Pi &= -\frac{\zeta}{\tau} \partial_\mu \left(\tau u^\mu\right) \\
&= -\zeta \partial_\mu u^\mu - \frac{\zeta \gamma}{\tau}
\end{aligned}$$

The equation for the entropy is,

$$\begin{aligned}
\frac{1}{\tau} \partial_\mu \left(\tau s^\mu\right) &= -\frac{\Pi}{T} \frac{1}{\tau} \partial_\mu \left(\tau u^\mu\right) \\
\partial_\mu \left(\tau s^\mu\right) &= -\frac{\Pi}{T} \partial_\mu \left(\tau u^\mu\right) \\
\gamma \frac{d}{d\tau} \left(\frac{s}{\sigma}\right) &= -\frac{\Pi}{T} u_\mu \partial^\mu \left(\frac{1}{\sigma}\right) \\
&= \frac{\Pi}{T\sigma^2} u_\mu \partial^\mu \sigma \\
&= -\frac{\Pi}{T\sigma} \partial^\mu u_\mu - \frac{\Pi}{T\sigma} \frac{\gamma}{\tau}
\end{aligned}$$

Separating the first term,

$$\begin{aligned}
\frac{1}{\sigma} \gamma \frac{ds}{d\tau} - \frac{s}{\sigma^2} \gamma \frac{d\sigma}{d\tau} &= -\frac{\Pi}{T\sigma} \partial^\mu u_\mu - \frac{\Pi}{T\sigma} \frac{\gamma}{\tau} \\
\frac{1}{\sigma} \gamma \frac{ds}{d\tau} + \frac{s}{\sigma} \partial^\mu u_\mu + \frac{s}{\sigma} \frac{\gamma}{\tau} &= -\frac{\Pi}{T\sigma} \partial^\mu u_\mu - \frac{\Pi}{T\sigma} \frac{\gamma}{\tau} \\
\gamma \frac{ds}{d\tau} &= -\left(\frac{\Pi}{T} + s\right) \left(\partial^\mu u_\mu + \frac{\gamma}{\tau}\right)
\end{aligned}$$

Now, we can open the equation of motion,

$$\sigma \gamma \frac{d}{d\tau} \left(\frac{(\varepsilon + p + \Pi)}{\sigma} u_i\right) = \partial_i \left(p + \Pi\right)$$

then,

$$
\begin{aligned}
\sigma\gamma\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}u_i\right) &= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau} \\
&\quad -u_i\left(\varepsilon+p+\Pi\right)\frac{\gamma}{\sigma}\frac{d\sigma}{d\tau} \\
&\quad +u_i\gamma\frac{d}{d\tau}\left(\varepsilon+p\right) \\
&\quad +u_i\gamma\frac{d\Pi}{d\tau} \\
&= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau} \\
&\quad +u_i\left(\varepsilon+p+\Pi\right)\partial^\mu u_\mu \\
&\quad +u_i\left(\varepsilon+p+\Pi\right)\frac{\gamma}{\tau} \\
&\quad +u_i\frac{dw}{ds}\gamma\frac{ds}{d\tau} \\
&\quad -u_i\left[\frac{\Pi}{\tau_R}+\frac{\zeta}{\tau_R}\partial_\mu u^\mu+\frac{\zeta\gamma}{\tau_R\tau}\right] \\
&= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau} \\
&\quad +u_i\left(\varepsilon+p+\Pi\right)\partial^\mu u_\mu \\
&\quad +u_i\left(\varepsilon+p+\Pi\right)\frac{\gamma}{\tau} \\
&\quad -u_i\frac{dw}{ds}\left[\left(\frac{\Pi}{T}+s\right)\left(\partial^\mu u_\mu+\frac{\gamma}{\tau}\right)\right] \\
&\quad -u_i\left[\frac{\Pi}{\tau_R}+\frac{\zeta}{\tau_R}\partial_\mu u^\mu+\frac{\zeta\gamma}{\tau_R\tau}\right]
\end{aligned}
$$

We conclude that,

$$
\begin{aligned}
\sigma\gamma\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}u_i\right) &= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau} \\
&\quad +u_i\left(\varepsilon+p+\Pi\right)\partial^\mu u_\mu \\
&\quad +u_i\left(\varepsilon+p+\Pi\right)\frac{\gamma}{\tau} \\
&\quad -u_i\frac{dw}{ds}\left[\left(\frac{\Pi}{T}+s\right)\left(\partial^\mu u_\mu+\frac{\gamma}{\tau}\right)\right] \\
&\quad -u_i\left[\frac{\Pi}{\tau_R}+\frac{\zeta}{\tau_R}\partial_\mu u^\mu+\frac{\zeta\gamma}{\tau_R\tau}\right] \\
&= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau} \\
&\quad +\partial^\mu u_\mu\left(\varepsilon+p+\Pi-\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)-\frac{\zeta}{\tau_R}\right)u_i \\
&\quad +u_i\left(\varepsilon+p+\Pi-\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)-\frac{\zeta}{\tau_R}\right)\frac{\gamma}{\tau} \\
&\quad -u_i\frac{\Pi}{\tau_R}
\end{aligned}
$$

We define the quantitie

$$
A=(\varepsilon+p+\Pi)-\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)-\frac{\zeta}{\tau_R}
$$

and remind that

$$
\partial^\mu u_\mu=-\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau}-\frac{u_iu_j}{2\gamma}\frac{dg^{ij}}{d\tau}
$$

We also remind that the index in the velocity field is very important. For example,

$$
\begin{aligned}
u^i &= \gamma\left(1,\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{d\eta}{d\tau}\right) \\
u_i &= \gamma\left(1,-\frac{dx}{d\tau},-\frac{dy}{d\tau},-\tau^2\frac{d\eta}{d\tau}\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sigma\gamma\frac{d}{d\tau}\left(\frac{(\varepsilon+p+\Pi)}{\sigma}u_i\right) &= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau}+Au_i\partial^\mu u_\mu+u_iA\frac{\gamma}{\tau}-u_i\frac{\Pi}{\tau_R}, \\
&= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau}-Au_i\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau}+\frac{u_iu_j}{2\gamma}\frac{dg^{ij}}{d\tau}\right) \\
&\quad +u_iA\frac{\gamma}{\tau}-u_i\frac{\Pi}{\tau_R}, \\
&= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau}-Au_i\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{1}{\gamma}g^{ij}u_i\frac{du_j}{d\tau}+\frac{1}{2\gamma}u_ju_i\frac{dg^{ij}}{d\tau}\right) \\
&\quad +u_iA\frac{\gamma}{\tau}-u_i\frac{\Pi}{\tau_R}, \\
&= \gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau}-\frac{A}{\gamma}g^{lm}u_iu_l\frac{du_m}{d\tau}-Au_i\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{1}{2\gamma}u_ju_i\frac{dg^{ij}}{d\tau}-\frac{\gamma}{\tau}\right) \\
&\quad -u_i\frac{\Pi}{\tau_R},
\end{aligned}
$$

The acceleration terms,

$$
\begin{gathered}
\gamma\left(\varepsilon+p+\Pi\right)\frac{du_i}{d\tau} \\
-\frac{A}{\gamma}g^{lm}u_iu_l\frac{du_m}{d\tau}
\end{gathered}
$$

and the force terms

$$
\begin{gathered}
Au_i\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}+\frac{1}{2\gamma}u_lu_m\frac{dg^{lm}}{d\tau}-\frac{\gamma}{\tau}\right) \\
+u_i\frac{\Pi}{\tau_R} \\
+\partial_i\left(p+\Pi\right)
\end{gathered}
$$

Which we can write in the following form,

$$
M_i^m\frac{du_m}{d\tau}=F_i
$$

where,

$$
\begin{aligned}
M_i^j &= \gamma C\delta_i^m-\frac{A}{\gamma}g^{lm}u_iu_l \\
F_i &= Bu_i+\partial_i\left(p+\Pi\right)
\end{aligned}
$$

and

$$
\begin{aligned}
C &= \varepsilon+p+\Pi \\
B &= \left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{\gamma}{\tau}+\frac{1}{2\gamma}u_lu_m\frac{dg^{lm}}{d\tau}\right)A+\frac{\Pi}{\tau_R}
\end{aligned}
$$

4 Bjorken Scalling

In the Bjorken scalling

$$u_\eta = 0.$$

then,

$$\begin{aligned}
A &= (\varepsilon + p + \Pi) - \frac{dw}{ds}\left(\frac{\Pi}{T} + s\right) - \frac{\zeta}{\tau_R} \qquad \text{(unchanged)} \\
B &= \left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right) A + \frac{\Pi}{\tau_R} \\
C &= \varepsilon + p + \Pi \qquad \text{(unchanged)}
\end{aligned}$$

And the dynamics in the x-y plane is given by

$$\left(\gamma C \delta^{ij} + \frac{A}{\gamma} u^i u^j\right) \frac{du^j}{d\tau} = B u^i - \partial_i (p + \Pi)$$

and we remind that

$$\begin{aligned}
\gamma \frac{d}{d\tau}\left(\frac{s}{\sigma}\right) &= -\frac{\Pi}{T\sigma}\partial^\mu u_\mu - \frac{\Pi}{T\sigma}\frac{\gamma}{\tau} \\
\tau_R \gamma \frac{d}{d\tau}\frac{\Pi}{\sigma} + \frac{\Pi}{\sigma} &= -\frac{\zeta}{\sigma}\partial_\mu u^\mu - \frac{\zeta\gamma}{\sigma\tau}
\end{aligned}$$

the four divergence of the velocity is unchanged

$$\partial^\mu u_\mu = -\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{u^j}{\gamma}\frac{du^j}{d\tau}$$

All the rest is unchanged.