Equilibrium and Transport Properties of the Hot and Dense Matter created in High-Energy Nuclear Collisions

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I. IDEAL FLUID

In ideal relativistic hydrodynamics (much of the following on ideal fluids is taken from [1]) the conservation of energy-momentum is expressed as

$$\partial_\mu T_0^{\mu\nu} = 0 \tag{1}$$

where the energy momentum tensor is

$$T_0^{\mu\nu} = \epsilon\, u^\mu u^\nu - p\Delta^{\mu\nu} \tag{2}$$

and ϵ is the energy density, p is the pressure, and the projector onto to the space perpendicular to u^μ, $\Delta^{\mu\nu}$, is defined as

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu \tag{3}$$

where $u_\mu \Delta^{\mu\nu} = 0$ and the flow velocity $u^\mu = \gamma(1,\vec{v})$ obeys the constraint $u^\mu u_\mu = 1$. Out flat space metric is $g^{\mu\nu} = diag(+,-,-,-)$. It is convenient to separate the energy-momentum conservation equations for this relativistic ideal fluid in terms of an equation in the direction parallel to the flow velocity

$$\boxed{u_\mu \partial_\nu T_0^{\mu\nu} = D\epsilon + (\epsilon+p)\partial_\mu u^\mu = 0} \tag{4}$$

where $D = u^\beta \partial_\beta$ is the comoving fluid derivative and another one perpendicular to the fluid velocity

$$\boxed{\Delta^\alpha_\nu \partial_\mu T_0^{\mu\nu} = (\epsilon+p)Du^\alpha - \nabla^\alpha p} \tag{5}$$

where $\nabla^\alpha = \Delta^{\mu\alpha}\partial_\mu$.

A. Hyperbolic coordinates

Cartesian coordinates i.e. t, x, y, and z are not the most useful in ultrarelativistic heavy ion collisions due to the approximate boost invariance of the dynamics after the collisions. Thus, it is useful to rewrite Eqs. (4-5) in hyperbolic coordinates where the coordinates are τ, ξ, x, and y (the beam is in the z direction) where the proper time is

$$\tau = \sqrt{t^2 - z^2} \tag{6}$$

and the space-time rapidity is

$$\xi = tanh^{-1}\left(\frac{z}{t}\right) = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right). \tag{7}$$

and the metric is

$$g_{\mu\nu} = diag\left(1,-1,-1,-\tau^2\right), \tag{8}$$

which is no longer uniform. Because the metric is no longer uniform one must take into account the Christoffel symbols when performing derivatives, which in this case are called covariant derivatives. Given the metric in hyperbolic coordinates defined above, the only non-zero components of the Christoffel symbols are

$$\begin{aligned} \Gamma^{\xi}_{\xi\tau} &= \frac{1}{\tau} \\ \Gamma^{\tau}_{\xi\xi} &= \tau \end{aligned} \tag{9}$$

so that the covariant gradient of the flow velocity becomes

$$\nabla_\alpha u^\nu = \partial_\alpha u^\nu + \Gamma^{\nu}_{\alpha\beta} u^\beta \tag{10}$$

and the covariant derivative perpendicular to the flow is

$$\nabla^{\alpha}_{\perp} = \Delta^{\alpha\mu} \nabla_\mu \tag{11}$$

and in this case

$$D = u^\mu \nabla_\mu \tag{12}$$

The variables in terms of the new coordinates are

$$\begin{aligned} \epsilon &= \epsilon\left(\tau, \xi, x, y\right) \\ u^\mu &= u^\mu\left(\tau, \xi, x, y\right) \\ &\dots \end{aligned} \tag{13}$$

Thus, we can rewrite our equations of motion using hyperbolic coordinates

$$\begin{aligned} D\epsilon &= (\epsilon + p)\partial_\mu u^\mu \\ u^\mu \partial_\mu \epsilon &= (\epsilon + p)\left[\frac{\gamma}{\tau} + \frac{\partial\gamma}{\partial\tau} + \partial_i\left(\gamma v^i\right)\right] \\ \frac{\partial\epsilon}{\partial\tau} &= (\epsilon + p)\left[\frac{\gamma}{\tau} + \frac{\partial\gamma}{\partial\tau} + \partial_i\left(\gamma v^i\right)\right] \end{aligned} \tag{14}$$

because ϵ is a scalar so we can take $u^\mu \partial_\mu \epsilon$ in the local rest frame where $u^\mu = (1, 0, 0, 0)$.

$$(\epsilon + p) D u^\alpha = \nabla^\alpha p \tag{15}$$

B. Bjorken expansion in 1+1 dimensions

If we assume "boost-invariance" such that there is no dynamics in the transverse plane

$$u^x = u^y = 0 \tag{16}$$

and

$$u^z = \frac{z}{\tau}. \tag{17}$$

These all translate into the following relations in hyperbolic coordinates

$$\begin{aligned} u^\xi &= -u^t \frac{sinh\xi}{\tau} + u^z \frac{cosh\xi}{\tau} = 0 \\ u^\tau = 1 & \end{aligned} \tag{18}$$

so that the variables ϵ, p, u^μ, only depend on τ and, therefore,

$$\begin{aligned} \epsilon &= \epsilon\left(\tau\right) \\ u^\mu &= \left(1, \vec{0}\right) \\ &\dots \end{aligned} \tag{19}$$

The expansion rate defined using Eq. (10) becomes

$$\nabla_\mu u^\mu = \partial_\mu u^\mu + \Gamma^\xi_{\xi\tau} u^\tau = \frac{1}{\tau} \neq 0 \tag{20}$$

and is non-vanishing even though u^μ is constant. In this simple case, the initial conditions of hydrodynamics at a given initial time $\tau = \tau_0$ are completely specified by the initial energy density $\epsilon(\tau_0)$ so that in ideal hydrodynamics one finds

$$D\epsilon + (\epsilon + p)\,\nabla_\mu u^\mu = \partial_\tau \epsilon + \frac{\epsilon + p}{\tau} = 0. \tag{21}$$

If we have a conformal theory in 4 spacetime dimensions the speed of sound is $c_s^2 = 1/3$

$$c_s^2 = \frac{dp}{d\epsilon} \to \epsilon = 3p \tag{22}$$

then

$$\frac{d\tau}{\tau} = -\frac{d\epsilon}{\epsilon + p} = -\frac{d\epsilon}{4/3\epsilon}. \tag{23}$$

after integration

$$\epsilon(\tau) = \epsilon(\tau_0) \left(\frac{\tau_0}{\tau}\right)^{4/3}. \tag{24}$$

Note that in a conformal plasma, $\epsilon(\tau) = kT^4(\tau)$, where k is a constant that basically counts the number of degrees of freedom. Using the equation above, we see that $T(\tau) = T_0(\tau_0/\tau)^{1/3}$, where $T_0 = (\epsilon(\tau_0)/k)^{1/4}$.

II. VISCOUS CORRECTIONS

If one takes into account dissipative viscous effects then another term must be added to the ideal energy momentum tensor and the total energy momentum tensor becomes

$$T^{\mu\nu} = T_0^{\mu\nu} + \Pi^{\mu\nu}. \tag{25}$$

We also define

$$\Delta^{\mu\nu\alpha\beta} \equiv \frac{1}{2}\left[\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right] \tag{26}$$

which gives the traceless, symmetric projection of a given tensor. In order to use Eq. (26) it's important to show that $\Delta_{\alpha\mu}\Pi^{\mu\nu} = \Pi^\nu_\alpha$:

$$\begin{aligned} \Delta_{\alpha\mu}\Pi^{\mu\nu} &= g_{\alpha\mu}\Pi^{\mu\nu} - u^\alpha \underbrace{u^\mu \Pi^{\mu\nu}}_{=0} \\ &= \Pi^\nu_\alpha \end{aligned} \tag{27}$$

where $u^\mu \Pi^{\mu\nu} = 0$ because we use the Landau-Lifshitz frame where all the momentum density is due to the flow of the energy density such that

$$u_\mu T^{\mu\nu} = \epsilon u^\nu \to u_\mu \Pi^{\mu\nu} = 0. \tag{28}$$

Applying $\Delta^{\alpha\beta}_{\mu\nu}$ to the viscous stress tensor, it returns only the traceless part of Eq. (32)

$$\begin{aligned} \Delta^{\alpha\beta}_{\mu\nu}\Pi^{\mu\nu} &= \frac{1}{2}\left[\Delta^\alpha_\mu \Delta^\beta_\nu + \Delta^\beta_\mu \Delta^\alpha_\nu - \frac{2}{3}\Delta_{\mu\nu}\Delta^{\alpha\beta}\right]\Pi^{\mu\nu} \\ &= \Pi^{\alpha\beta} - \Delta^{\alpha\beta}\frac{\Pi^\mu_\mu}{3} \end{aligned} \tag{29}$$

Let us then define the result of Eq. (29) as

$$\pi^{\alpha\beta} \equiv \Pi^{\alpha\beta} - \Delta^{\alpha\beta}\frac{\Pi^{\mu}_{\mu}}{3} \tag{30}$$

and we can define the term with a trace as

$$\Pi \equiv -\frac{\Pi^{\mu}_{\mu}}{3} \tag{31}$$

so that we can then separate the viscous stress tensor into its traceless part, $\pi^{\alpha\beta}$, and its part with a non-vanishing trace, $\Delta^{\alpha\beta}\Pi$,

$$\boxed{\Pi^{\alpha\beta} = \pi^{\alpha\beta} - \Delta^{\alpha\beta}\Pi}\,. \tag{32}$$

Note that

$$g_{\mu\nu}\pi^{\mu\nu} = 0 \tag{33}$$

because $\pi^{\mu\nu}$ is traceless. Furtheremore, we establish the following relationships

$$\begin{aligned}
\Delta_{\alpha\beta}\Delta^{\alpha\beta} &= 3 && (34)\\
\Delta_{\alpha\beta}\Delta^{\beta\nu} &= \Delta^{\nu}_{\alpha} && (35)\\
\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta}_{\nu\rho} &= \frac{5}{3}\Delta^{\mu} && (36)\\
\partial^{\mu} - \nabla^{\mu} &= u^{\mu}D && (37)
\end{aligned}$$

in Appendix A.

In order to obtain the equations of motion, let's look at what the parallel projection in the flow direction does specifically on the viscous stress tensor

$$\begin{aligned}
u_\nu\partial_\mu\Pi^{\mu\nu} &= \partial_\mu\underbrace{(u_\nu\Pi^{\mu\nu})}_{=0} - \Pi^{\mu\nu}\partial_\mu u_\nu\\
&= -\Pi^{\mu\nu}\partial_\mu u_\nu\\
&= -\Pi^{\mu\nu}\left(u_\mu D + \nabla_\mu\right)u_\nu\\
&= -\underbrace{u_\mu\Pi^{\mu\nu}}_{=0}Du_\nu - \Pi^{\mu\nu}\nabla_\mu u_\nu\\
&= -\Pi^{\mu\nu}\nabla_\mu u_\nu
\end{aligned} \tag{38}$$

Any tensor can be divided into its symmetric $(\dots)$ and anti-symmetric $\{\dots\}$ part

$$\begin{aligned}
A_\mu B_\nu &= \frac{1}{2}\left(A_\mu B_\nu + A_\nu B_\mu\right) + \frac{1}{2}\left(A_\mu B_\nu - A_\nu B_\mu\right)\\
&= A_{(\mu}B_{\nu)} + A_{\{\mu}B_{\nu\}}.
\end{aligned} \tag{39}$$

Because $\Pi^{\mu\nu}$ is symmetric then the anti-symmetric part of $\nabla_{\perp\mu}u_\nu$ can be ignored so we can rewrite Eq. (38) as

$$u_\nu\partial_\mu\Pi^{\mu\nu} = -\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}. \tag{40}$$

If one were to look at $\nabla_\mu u_\nu$ on its own, one could find

$$\begin{aligned}
\partial_\mu u_\nu &= u_\mu Du_\nu + \nabla_\mu u_\nu\\
&= \frac{1}{2}\left(u_\mu Du_\nu + \nabla_\mu u_\nu + u_\nu Du_\mu + \nabla_\nu u_\mu\right) + \frac{1}{2}\left(u_\mu Du_\nu + \nabla_\mu u_\nu - u_\nu Du_\mu - \nabla_\nu u_\mu\right)\\
&= u_\mu Du_\nu + \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu\right) + \frac{1}{2}\left(\nabla_\mu u_\nu - \nabla_\nu u_\mu\right)\\
&= u_\mu Du_\nu + \sigma_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\Theta + \Omega_{\mu\nu}
\end{aligned} \tag{41}$$

where

$$
\begin{aligned}
\sigma_{\mu\nu} &\equiv \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu}\Theta\right) \\
&= \nabla_{(\mu} u_{\nu)} - \frac{1}{3}\Delta_{\mu\nu}\Theta && (42) \\
\Theta &\equiv \partial_\mu u^\mu && (43) \\
\Omega_{\mu\nu} &\equiv \frac{1}{2}\left(\nabla_\mu u_\nu - \nabla_\nu u_\mu\right) && (44)
\end{aligned}
$$

where the shear stress tensor $\sigma_{\mu\nu}$ is symmetric, the expansion rate Θ is symmetric, and the vorticity $\Omega_{\mu\nu}$ is antisymmetric. The term $u_\mu D u_\nu$ is perpendicular to the flow so it does not play a role when $\Pi^{\mu\nu}$ is contracted with $\partial_\mu u_\nu$. Returning to Eq. (26), we can generalize its affect on a given tensor $A_{\alpha\beta}$ as

$$
A^{\langle\mu\nu\rangle} = \Delta^{\mu\nu\alpha\beta} A_{\alpha\beta}. \tag{45}
$$

where $A^{\langle\mu\nu\rangle}$ is the traceless part of $A_{\alpha\beta}$ and the angular brakets $\langle \dots \rangle$ imply a traceless, symmetric tensor. So we can define

$$
\nabla^{\langle\mu} u^{\nu\rangle} \equiv 2\sigma^{\mu\nu} = 2\nabla^{(\mu} u^{\nu)} - \frac{2}{3}\Delta^{\mu\nu}\Theta. \tag{46}
$$

We can also show that

$$
\begin{aligned}
g_{\mu\nu}\nabla^\mu u^\nu &= g_{\mu\nu}\Delta^{\mu\alpha}\partial_\alpha u^\nu \\
&= \Delta^{\mu\alpha}\partial_\alpha u_\mu \\
&= g^{\mu\alpha}\partial_\alpha u_\mu - u^\alpha \underbrace{u^\mu \partial_\alpha u_\mu}_{=0} \\
&= \partial^\mu u_\mu && (47)
\end{aligned}
$$

where $u^\mu \partial_\alpha u_\mu = 0$ because $u^\mu u_\mu = 1$ so $\partial_\alpha \left(u^\mu u_\mu\right) = 0$, then

$$
\partial_\alpha \left(u^\mu u_\mu\right) = u^\mu \partial_\alpha u_\mu + u_\mu \partial_\alpha u^\mu \tag{48}
$$

so $u^\mu \partial_\alpha u_\mu = 0$ must hold. Then,

$$
\Theta = \partial_\mu u^\mu = \nabla_\mu u^\mu = \Delta^{\mu\alpha}\partial_\alpha u_\mu \tag{49}
$$

Returning to Eq. (40) we find

$$
\begin{aligned}
u_\nu \partial_\mu \Pi^{\mu\nu} &= -\Pi^{\mu\nu}\nabla_{(\mu} u_{\nu)} \\
&= -\Pi^{\mu\nu}\left(\sigma_{\mu\nu} + \frac{1}{3}\Delta_{\mu\nu}\Theta\right) && (50)
\end{aligned}
$$

We can also look at the projection in the perpendicular direction

$$
\Delta^\mu_\nu \partial_\alpha \Pi^{\alpha\nu} = \left(\epsilon + p\right) D u^\mu - \nabla^\mu p + \Delta^\mu_\nu \partial_\alpha \Pi^{\alpha\nu} = 0 \tag{51}
$$

Thus, we end up with the follow equations of motion

$$
\boxed{D\epsilon + \left(\epsilon + p\right)\partial_\mu u^\mu - \Pi^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{3}\Pi^{\mu\nu}\Delta_{\mu\nu}\Theta = 0} \tag{52}
$$

$$
\boxed{\left(\epsilon + p\right) D u^\mu - \nabla^\mu p + \Delta^\mu_\nu \partial_\alpha \Pi^{\alpha\nu} = 0}. \tag{53}
$$

We define the entropy density current as

$$
s^\mu = s\, u^\mu \tag{54}
$$

We know from thermodynamics that

$$\begin{aligned} \epsilon + p &= Ts \\ Tds &= d\epsilon. \end{aligned} \tag{55}$$

Moreover, using the entropy four-flux Eq. (54) and taking the derivative, we have

$$\begin{aligned} \partial_\mu s^\mu &= u^\mu \partial_\mu s + s\partial_\mu u^\mu \\ &= Ds + s\partial_\mu u^\mu \\ &= \frac{1}{T}\left[D\epsilon + (\epsilon + p)\partial_\mu u^\mu\right]. \end{aligned} \tag{56}$$

Recalling Eq. (52), we find

$$\boxed{\partial_\mu s^\mu = \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}}. \tag{57}$$

In order for the second law of thermodynamics to hold, $\partial_\mu s^\mu \geq 0$, so $\frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \geq 0$. We can now subsitute in the split version of $\Pi^{\mu\nu}$ as shown in Eq. (32)

$$\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} - \Delta^{\mu\nu}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}. \tag{58}$$

We can then calculate $\pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}$ using the definition found in Eq. (46)

$$\begin{aligned} \pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} &= \pi^{\mu\nu}\sigma_{\langle\mu}u_{\nu\rangle} + \frac{1}{3}\Delta_{\mu\nu}\pi^{\mu\nu}\Theta \\ &= \pi^{\mu\nu}\sigma_{\langle\mu}u_{\nu\rangle} + \frac{1}{3}(\underbrace{g_{\mu\nu}\pi^{\mu\nu}}_{=0} - u_\nu \underbrace{u_\mu\pi^{\mu\nu}}_{=0})\Theta \\ &= \pi^{\mu\nu}\sigma_{\langle\mu}u_{\nu\rangle} \end{aligned} \tag{59}$$

and can calculate $\Delta^{\mu\nu}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)}$ using Eq. (39)

$$\begin{aligned} \Delta^{\mu\nu}\Pi\nabla_{(\mu}u_{\nu)} &= \frac{1}{2}\Delta^{\mu\nu}\Pi\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu\right) \\ &= \frac{1}{2}\Pi\left(\nabla^\nu u_\nu + \nabla^\mu u_\mu\right) \\ &= \Pi\nabla^\mu u_\mu \end{aligned} \tag{60}$$

so Eq. (58)

$$\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = \frac{1}{2}\pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} - \Pi\nabla^\mu u_\mu \tag{61}$$

$$= \pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\Theta \tag{62}$$

Then, Eq. (63) becomes

$$\partial_\mu s^\mu = \frac{1}{2T}\pi^{\mu\nu}\nabla_{\langle\mu}u_{\nu\rangle} - \frac{1}{T}\Pi\nabla^\mu u_\mu \geq 0 \tag{63}$$

and we can also rewrite the equation of motion Eq. (52)

$$\boxed{D\epsilon + (\epsilon + p)\,\partial_\mu = \pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\Theta} \tag{64}$$

A. Acausality of the Relativistic Version of Navier-Stokes Theory

If we assume

$$\begin{aligned} \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} \\ \Pi &= -\zeta\nabla_\alpha u^\alpha \end{aligned} \tag{65}$$

where η is the shear viscosity and ζ is the bulk viscosity, then the second law of thermodynamics implies that $\eta > 0$ and $\zeta > 0$ must hold. In this case, one obtains equations of motion that are similar to the diffusion equation, which is known to have problems with causality. Clearly, in a non-relativistic setting, the lack of causality in the Navier-Stokes equations is not a problem (moreover, the non-relativistic Navier-Stokes equations are stable around hydrostatic equilibrium).

For relativistic fluids, however, causality must be preserved. The assumptions in Eq. (65) lead to acausal behavior because it allows for modes (with large momentum) that propagate faster than the speed of light. One can show that, in a relativistic theory, acausality leads to instabilities. Thus, one cannot solve the relativistic version of Navier-Stokes equations numerically.

In order to show this, let's consider small perturbations of the energy density and fluid velocity in a system that is initially in equilibrium and at rest such that

$$\begin{aligned} \epsilon &= \epsilon_0 + \delta\epsilon(t,x) \\ u^\mu &= (1,\vec{0}) + \delta u^\mu(t,x) \\ p &= p_0 + \delta p(t,x) \\ \eta &= \eta_0 + \delta\eta(t,x) \\ \zeta &= \zeta_0 + \delta\zeta(t,x). \end{aligned} \tag{66}$$

Returning to Eq. (53) and taking just the direction $\mu = y$,

$$\begin{aligned} (\epsilon+p)\,Du^y - \nabla^y p + \Delta^y_\nu \partial_\alpha \Pi^{\alpha\nu} &= \left[\epsilon_0 + \delta\epsilon(t,x) + p_0 + \delta p(t,x)\right]\partial_t \delta u^y(t,x) - \nabla^y\left[p_0 + \delta p(t,x)\right] + \Delta^y_\nu \partial_\alpha \Pi^{\alpha\nu} \\ &= \left[\epsilon_0 + p_0\right]\partial_t \delta u^y(t,x) + \underbrace{\left[\delta\epsilon(t,x) + \delta p(t,x)\right]\partial_t \delta u^y(t,x)}_{\text{non-linear}} - \underbrace{\nabla^y\left[p_0 + \delta p(t,x)\right]}_{=0} + \Delta^y_\nu \partial_\alpha \Pi^{\alpha\nu} \\ &= \left[\epsilon_0 + p_0\right]\partial_t \delta u^y(t,x) + \Delta^y_\nu \partial_\alpha \Pi^{\alpha\nu} + \mathcal{O}(\delta^2). \end{aligned} \tag{67}$$

Then, let's concentrate just on the viscous term

$$\Delta^y_\nu \partial_\alpha \Pi^{\alpha\nu} = \partial_\alpha \Pi^{\alpha y} \tag{68}$$

Let's first look at the case when $\alpha = t$. We know already that $u^\mu \Pi_{\mu\nu} = 0$. In the local rest frame $u^\mu = (1,0)$, so in order for $u^\mu \Pi_{\mu\nu} = 0$ then $\Pi_{0\nu} = 0$ for all ν. Thus, $\partial_t \Pi^{ty} = 0$. Then we can look at the case where $\alpha = x$

$$\begin{aligned} \partial_x \Pi^{xy} &= \partial_x\left(\pi^{xy} - \Delta^{xy}\Pi\right) \\ &= \partial_x\left(2\eta\sigma^{xy} + \underbrace{\zeta\Delta^{xy}\nabla_\alpha u^\alpha}_{\text{non-linear}}\right) \\ &= \partial_x\left[\eta\left(\nabla_x u_y + \underbrace{\nabla_y u_x}_{=0} - \underbrace{\frac{2}{3}\Delta_{xy}\nabla_\alpha u^\alpha}_{\text{non-linear}}\right)\right] \\ &= \partial_x\left[\eta\nabla_x u_y\right] \\ &= \partial_x\left[\left(\eta_0 + \delta\eta(t,x)\right)\nabla_x \delta u_y\right] \\ &= \eta_0 \partial_x^2 \delta u_y \end{aligned} \tag{69}$$

Thus, Eq. (67) becomes

$$\left[\epsilon_0 + p_0\right]\partial_t \delta u^y(t,x) + \eta_0 \partial_x^2 \delta u_y + \mathcal{O}(\delta^2) = 0, \tag{70}$$

which can be rearranged into a diffusion-type evolution equation for the perturbation $\delta u^y(t,x)$

$$\partial_t \delta u^y - \frac{\eta_0}{\epsilon_0 + p_0}\partial_x^2 \delta u_y = \mathcal{O}(\delta^2). \tag{71}$$

Then we can solve Eq. (71) by using a mixed Laplace-Fourier wave ansatz

$$\delta u^y(t,x) = e^{-wt+ikx} f_{w,k}, \tag{72}$$

which we can subsitute into Eq. (71) to find the dispersion relation

$$w = \frac{\eta_0}{\epsilon_0 + p_0} k^2 \tag{73}$$

where we can then estimate the speed of diffusion as

$$v_T(k) = \frac{dw}{dk} = 2\frac{\eta_0}{\epsilon_0 + p_0} k. \tag{74}$$

The problem with Eq. (74) is that the velocity v_T is linearly dipendent on the wavenumber. Thus, as k increases v_T will eventually exceed the speed of light, which violates causality.

A way to resolve this problem is to consider the phenomenological equation proposed by Cattaneo (and later by Israel and Stewart) where the shear tensor, $\pi^{\mu\nu}$, relaxes towards its Navier-Stokes limit. For instance, for π^{xy} we obtain (in the local rest frame where $u^\mu = (1,0)$)

$$\tau_\pi \partial_t \pi^{xy} + \pi^{xy} = 2\,\eta\,\sigma^{xy} \tag{75}$$

where τ_π is the new transport coefficient known as the relaxation time (it is clear from the equation above that this quantity defines the timescale that it takes for flow gradients to be converted into pressure). However, there are some limitations on τ_π and if $\tau_\pi \to 0$ then the problem of acausality arises again. In fact, one can show that the system is causal and stable around hydrostatic equilibrium (where $T^{\mu\nu} = diag(\epsilon, p, p, p)$) if $(\eta/s)/(\tau_\pi T) \leq 3(1 - c_s^2)/4$ [25].

From kinetic theory, one can show that this new transport coefficient is basically the mean free path, i.e, $\tau_\pi \sim \ell_{MFP}$. This shows that τ_π is a microscopic timescale in the dynamics.

B. Equations from Kinetic Theory

In order to avoid issues with acausality, Isreal and Steward derived a set of relativistic fluid-dynamical equations using the 14-moment approximation. The second moment of the Boltzmann equation is used to extract the equations of motion and to determine the transport coefficents. This then gives 14 coefficients that are needed to describe the distribution function. The choice in truncating at the second moment is rather arbitrary and it has actually been shown that the Isreal-Stewart equations are not in good agreement with the numerical soluation of the Boltzmann equation [26].

The Boltzmann equation

$$K^\mu \partial_\mu f_k = C[f] \tag{76}$$

where $C[f]$ is the collision term where only two-to-two collisions ($C[f] = 0$ at equilibrium or when the system is far from equilibrium) are considered

$$C[f] = \frac{1}{2} \int dK' dP dP' W_{KK' \to PP'} \left(f_P f_{P'} \tilde{f}_K \tilde{f}_{K'} - f_K f_{K'} \tilde{f}_P \tilde{f}_{P'} \right) \tag{77}$$

and $K^\mu = (E_k, \mathbf{k})$ is the four-momentum, $E_k = \sqrt{\mathbf{k^2 + m^2}}$, and $f_k\,(x^\mu, K^\mu)$ is the one particle distribution function where $\tilde{f}_K = 1 - af\,(x^\mu, K^\mu)$ where a=1 (or -1) for fermions (or bosons) and a=0 for a Boltzmann gas. Additionally, the Lorentz-invariant measurement is

$$dK = g \frac{d^3 \vec{K}}{(2\pi)^3 E_{\mathbf{k}}}. \tag{78}$$

We can then calculate the expectation values for the conserved particle current

$$N^\mu = \langle K^\mu \rangle = \int dK K^\mu f_K \tag{79}$$

and the energy moment tensor

$$T^{\mu\nu} = \langle K^\mu K^\nu \rangle = \int dK K^\mu K^\nu f_K \tag{80}$$

where one notices that $\langle \ldots \rangle = \int dK\,(\ldots)\,f_K$.

The four-momentum can be decomposed into its perpendicular and parallel to u^μ parts

$$K^\mu = (u \cdot K)u^\mu + K^{\langle\mu\rangle} \tag{81}$$

where $A \cdot B = A_\mu B^\mu$ and $K^{\langle\mu\rangle}$ is similar to Eq. (45), i.e. $K^{\langle\mu\rangle} = \Delta^{\mu\nu} K_\nu$. Using Eq. (81), we can then rewrite Eqs. (79-80)

$$\begin{aligned} N^\mu &= n u^\mu + n^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^{nu} - \Delta^{\mu\nu}\left(P_0 + \Pi\right) + \pi^{\mu\nu} \end{aligned} \tag{82}$$

where the particle density n, the particle diffusion current n^μ, the energy density ϵ, the shear stress tensor $\pi^{\mu\nu}$, the sum of thermodynamic pressure P_0, the bulk viscous pressure Π are defined, respectively

$$\begin{aligned} n &= \langle u \cdot K\rangle \\ n^\mu &= \langle K^{\langle\mu\rangle}\rangle \\ \epsilon &= \langle (u \cdot K)^2\rangle \\ \pi^{\mu\nu} &= \langle K^{\langle\mu} K^{\nu\rangle}\rangle \\ P_0 + \Pi &= -\frac{1}{3}\langle \Delta^{\mu\nu} K_\mu K_\nu\rangle \end{aligned} \tag{83}$$

where P_0 and all other variables indicated by $_0$ are for the ideal case without viscosity. Then the local distribution function is

$$f_{0K} = \left[\exp\left(\beta_0 u \cdot K - \alpha_0\right) + a\right]^{-1} \tag{84}$$

where $\beta_0 = 1/T$ and $\alpha_0 = \mu_B/T$ (the ratio of the chemical potential to the temperature). Eq. (84) is then used to find

$$\begin{aligned} \epsilon = \epsilon_0 &= \langle (u \cdot K)^2\rangle_0 \\ n = n_0 &= \langle u \cdot K\rangle_0 \end{aligned} \tag{85}$$

where $\langle \dots \rangle_0 = \int dK\,(\dots)\, f_{0K}$.

Then one can derive the difference between the thermodynamic pressure and the bulk viscous pressure so that

$$\begin{aligned} P_0 &= -\frac{1}{3}\langle \Delta^{\mu\nu} K_\mu K_\nu\rangle_0 \\ \Pi &= -\frac{1}{3}\langle \Delta^{\mu\nu} K_\mu K_\nu\rangle_\delta \end{aligned} \tag{86}$$

where $\langle \dots \rangle_\delta = \langle \dots \rangle - \langle \dots \rangle_0$.

Up until this point no assumptions beyond those behind the validty of the Boltzmann were employed.

C. Memory Function

1. *Bulk Viscosity: 1+1*

Let us first consider only the bulk channel because it has significantly less transport coefficients. In [21] the equations for SPH for the (1+1) dimension case (where the motion in the transverse direction is ignored and only the longitudinal dynamics are considered) are derived, which we will show here. As always Eq. (1) is needed for the conservation of the energy-momentum tensor. Since we are only consider the bulk viscosity Eq. (25) can be rewritten as

$$T^{\mu\nu} = \left(\epsilon + p + \Pi\right) u^\mu u^\nu - \left(p + \Pi\right) g^{\mu\nu}. \tag{87}$$

In hydrodynamics, fluid cells have a finite volume V^* at point $\vec{r}$. The flow of the fluid inside the cell deforms the volume as a function of time so you have something like

$$\frac{1}{V^*}\frac{dV^*}{dt} = \nabla \cdot \vec{v}. \tag{88}$$

This is known as the continuity equation and can be expressed in covariant form

$$\partial_\mu (\sigma u^\mu) = 0 \tag{89}$$

where σ is the proper reference density

$$\sigma = \frac{1}{V} = \frac{\gamma}{V^*}. \tag{90}$$

In the presence of an irreversible curent the entropy is no longer conserved, rather

$$\partial_\mu s^\mu = -\frac{1}{T}\Pi\partial_\mu u^\mu \tag{91}$$

where the entropy four-flux, s^μ, is shown in Eq. (54). Eq. (91) is derived in Eq. (63). Defining the extensive measure of the entropy (i.e. the total entropy) as

$$\tilde{s} = sV = \frac{s}{\sigma}, \tag{92}$$

we can rewrite Eq. (91) as

$$\begin{aligned}\partial_\mu s^\mu &= \partial_\mu (s u^\mu)\\ &= \partial_\mu (\sigma\tilde{s}u^\mu)\\ &= \sigma u^\mu \partial_\mu \tilde{s} + \tilde{s}\partial_\mu (\sigma u^\mu)\\ &= \sigma u^\mu \partial_\mu \tilde{s}.\end{aligned} \tag{93}$$

Then, defining the extensive measurement inside the fluid cell of the irreversible current, Π, as

$$\tilde{J} = \tilde{\Pi} = \Pi V = \frac{\Pi}{\sigma} \tag{94}$$

and

$$\frac{d}{d\tau} = u^\mu \partial_\mu, \tag{95}$$

we obtain

$$T\frac{d\tilde{s}}{d\tau} = -\tilde{J}F = -\tilde{\Pi}\partial_\mu u^\mu. \tag{96}$$

From Eq. (96) we see that in irreversible thermodynamics the net entropy production in the cell is given by the product of the irreversible displacement in each cell and the corresponding thermodyanmic force field in each cell

$$F = \partial_\alpha u^\alpha. \tag{97}$$

Returning to Eq. (93),

$$\begin{aligned}\partial_\mu s^\mu &= \sigma u^\mu \partial_\mu \tilde{s}\\ &= \sigma u^\mu \partial_\mu \left(\frac{s}{\sigma}\right)\\ &= \sigma u^\mu s \partial_\mu \left(\frac{1}{\sigma}\right) + u^\mu \partial_\mu s\end{aligned} \tag{98}$$

and we can also expand the left hand side of the equation so that

$$\begin{aligned}\partial_\mu s^\mu &= \partial_\mu (s u^\mu)\\ &= u^\mu \partial_\mu (s) + s\partial_\mu (u^\mu)\end{aligned} \tag{99}$$

Thus,

$$\partial_\mu u^\mu = \sigma u^\mu \partial_\mu \left(\frac{1}{\sigma}\right) = \sigma \frac{d}{d\tau}\left(\frac{1}{\sigma}\right), \tag{100}$$

which is our force

$$F = \partial_\alpha u^\alpha = \sigma \frac{d}{d\tau}\left(\frac{1}{\sigma}\right). \tag{101}$$

The significance of Eq. (101) is that the termodynamic force for the fluid cell is the change of the cell volume, which is exactly the physical meaning of the bulk viscosity i.e. the resistance to change of the volume of hte system.

As discussed previously, in relatistic Navier-Stokes it is assumed that the bulk viscosity per volume element is produced by the thermodynamic force without retardation (see Section II A and specifically Eq. (65)). However, as mentioned previously, there are issues with acausality and instability. Thus, we introduce a memory effect to the irreversible current by using the simplest memory function which can be reduced to the differential equation, thus,

$$G(\tau,\tau') \to \frac{1}{\tau_R(\tau')} e^{-\int_{\tau'}^{\tau} \frac{1}{\tau_R(\tau'')} d\tau''} \tag{102}$$

where τ_R is the relaxation time, which gives the time scale for the retardation, and it is, generally, a function of the local proper time $\tau = \tau(\mathbf{r}, t)$ through the termodynamical quantities. Introducing the memory function in the irreversible current,

$$\tilde{\Pi}(\tau) = \int_{-\infty}^{\tau} d\tau' G(\tau,\tau') \frac{\zeta}{\sigma} \partial_\alpha u^\alpha (\tau') \tag{103}$$

When you start with a finite initial time, τ_0, Eq. (103) becomes

$$\tilde{\Pi}(\tau) = \int_{\tau_0}^{\tau} d\tau' G(\tau,\tau') \frac{\zeta}{\sigma} \partial_\alpha u^\alpha (\tau') + e^{-(\tau-\tau_0)/\tau_R} \tilde{\Pi}_0 \tag{104}$$

where Π_0 is an initial condition give at τ_0.

We can use the fundamental theorem of calculate to find the memory function from Eq. (103)

$$\frac{d}{d\tau}\int_{\tau_0}^{\tau} d\tau' G(\tau,\tau') X(\tau') = G(\tau,\tau) X(\tau) + \int_{\tau_0}^{\tau} d\tau' X(\tau') \frac{d}{d\tau} G(\tau,\tau') \tag{105}$$

where $X(\tau) = \frac{\zeta}{\sigma}\partial_\alpha u^\alpha(\tau)$. Then if we assume that the integral in the exponent of the exponential of the Green's function is $F(\tau) - F(\tau') = -\int_{\tau'}^{\tau} \frac{1}{\tau_R(\tau'')} d\tau''$ we find

$$\frac{d}{d\tau}\int_{\tau_0}^{\tau} d\tau' G(\tau,\tau') X(\tau') = \frac{X(\tau)}{\tau_R(\tau)} + F'(\tau) \int_{\tau_0}^{\tau} d\tau' X(\tau') G(\tau,\tau') \tag{106}$$

where $F'(\tau) = -\frac{1}{\tau_R(\tau)}$ so

$$\frac{d}{d\tau}\int_{\tau_0}^{\tau} d\tau' G(\tau,\tau') X(\tau') = \frac{X(\tau)}{\tau_R(\tau)} - \frac{1}{\tau_R(\tau)} \int_{\tau_0}^{\tau} d\tau' X(\tau') G(\tau,\tau'), \tag{107}$$

which gives

$$\tilde{\Pi} + \tau_R \frac{d\tilde{\Pi}}{d\tau} = -\frac{\zeta}{\sigma}\partial_\alpha u^\alpha. \tag{108}$$

Substituting in Eq. (94) so that we can describe Eq. (108) in terms of the density

$$\begin{aligned} -\frac{\zeta}{\sigma}\partial_\alpha u^\alpha &= \tau_R \frac{d}{d\tau}\left(\frac{\Pi}{\sigma}\right) + \frac{\Pi}{\sigma} \\ &= \tau_R \Pi \frac{d}{d\tau}\left(\frac{1}{\sigma}\right) + \tau_R \frac{1}{\sigma}\frac{d\Pi}{d\tau} + \frac{\Pi}{\sigma} \\ &= \tau_R \frac{\Pi}{\sigma}\partial_\mu u^\mu + \tau_R \frac{1}{\sigma}\frac{d\Pi}{d\tau} + \frac{\Pi}{\sigma} \\ \tau_R \frac{d\Pi}{d\tau} + \Pi &= -\left(\zeta + \tau_R \Pi\right) \partial_\mu u^\mu. \end{aligned} \tag{109}$$

Eq. (109) has an extra term in it compared to the linearized kinetic theory (discussed in [22])

$$\tau_R \frac{d\Pi}{d\tau} + \Pi = -\zeta \partial_\mu u^\mu. \tag{110}$$

The difference between Eq. (109) and Eq. (110) is that Eq. (109) takes into account finite size effects whereas Eq. (110) does not. The reason Eq. (109) does not explicitly depend on $1/\sigma$ is because the cell volume is irrelevant because the length is much larger than the mean free path and much smaller than the typical hydrodynamic scale. One interest difference between the two equations is that in Eq. (110) bulk viscosity scalar is always negative whereas in Eq. (109) the bulk viscosity scalar can be both negative or positive and has a minimum at

$$\Pi_{min} = -\frac{\zeta}{\tau_R}. \tag{111}$$

2. Bulk Viscosity: Solving with SPH

At this moment in time it is difficult to calculate a precise quantitative description of the bulk visocisty. The Boltzmann equation only is calcuable for two-body collisions and as of yet a consistent multi-body collision equation for non-Newtonian fluids does not exist. Thus, for now we simply assume that

$$\zeta = as \tag{112}$$

where a is a constant and s is the entropy density in the local rest frame. In order to preseve casaulity the relaxation time cannot go to zero and ζ and τ_R are related through the velocity $v = \sqrt{1/b + \alpha}$ where $\alpha = dp/d\epsilon$ and we take b as a constant such that

$$\tau_R = \frac{\zeta}{\epsilon + p} b. \tag{113}$$

Thus, we can now speak in terms of the paremeters a and b.

Let us remind ourselves of the equations that need to be solved:

$$\begin{aligned} D\epsilon + (\epsilon + p)\,\Theta &= -\Pi\,\Theta & (114)\\ (\epsilon + p)\,Du^\mu &= \nabla^\mu_\perp p - \Delta^\mu_\nu \nabla_\alpha \Delta^{\alpha\nu}\Pi & (115)\\ \tau_R \frac{d\Pi}{d\tau} + \Pi &= -\left(\zeta + \tau_R \Pi\right)\partial_\mu u^\mu. & (116) \end{aligned}$$

Here we have eliminated the shear viscous tensor because we are only considering the bulk viscosity. In order to solve Eqs. (114-116) numerically we will used smoothed particles hydrodynamics or SPH. SPH was initially introduced in astrophysics in [29] and was first extended to heavy-ion collisions in [30].

The idea behind SPH is to obtain an approximte solution in hydrodynamics by discretizing the fluid into a set of effective particles. It can then be interpreted as a physical model of the collective motion in terms of a finite set of dynamical variables. Consider first a distribution $a\left(\mathbf{r}, t\right)$ of an extensive physical quantity A. The behavior of $a\left(\mathbf{r}, t\right)$ contains the effects of whole microscopic degrees of freedom. Because we are interested in the global behavior of $a\left(\mathbf{r}, t\right)$ and how it related to the experimental observables, we introduce a coarse-grain proceedure for $a\left(\mathbf{r}, t\right)$. Thus, we introduce a kernel function, $W(\mathbf{r} - \tilde{\mathbf{r}}, h)$, that maps the original distribution a to a coarse-grained version a_{CG}

$$a_{CG}\left(\mathbf{r}, t\right) = \int a\left(\tilde{\mathbf{r}}, t\right) W(\mathbf{r} - \tilde{\mathbf{r}}, h) d\tilde{\mathbf{r}} \tag{117}$$

where

$$\int W(\mathbf{r} - \tilde{\mathbf{r}}, h) d\tilde{\mathbf{r}} = \mathbf{1}. \tag{118}$$

The parameter h represents the width of W and serves as a cut-off parameter for short wavelength modes

$$W(\mathbf{r}, h) \to 0, \ |\mathbf{r}| \gtrsim h \tag{119}$$

such that

$$\lim_{h\to 0} W(\tilde{\mathbf{r}}, h) = \delta(\tilde{\mathbf{r}}). \tag{120}$$

Then, h is the typical length scale for the coarse-graining, thus, we will take h as the scale for coarse graining in QCD dynamics (i.e. the mean free path of the partons) to obtain the hydrodynamics of QGP (so $h \approx 0.f\ fm$).

Next we have to approximate the coarse-graining distribution, $a_{CG}(\mathbf{r},t)$, by replacing the integral in Eq. (117) with a summation over a finite and discrete set of points $\{\mathbf{r}_\alpha(t), \alpha = 1,\ldots,N_{SPH}\}$ where N_{SPH} is the total number of SPH effective particles. Then we have

$$a_{SPH}(\mathbf{r},t) = \sum_{\alpha=1}^{N_{SPH}} A_\alpha(t) W(|\mathbf{r}-\mathbf{r}_\alpha(t)|) \tag{121}$$

If our choice in $\{A_\alpha(t), \alpha = 1,\ldots,N_{SPH}\}$ and $\{\mathbf{r}_\alpha(t), \alpha = 1,\ldots,N_{SPH}\}$ are appropriate, then Eq. (121) will converge to Eq. (117). Therefore, appropriate choices are vital, thankfully, we can determine $\{A_\alpha(t), \alpha = 1,\ldots,N_{SPH}\}$ and $\{\mathbf{r}_\alpha(t), \alpha = 1,\ldots,N_{SPH}\}$ through the dynamics of the system as we will show in the following.

Using the example of the contuity equation, which can show that it can be solved very simply using SPH. First we choose the reference density σ^*, which must be conserved such that

$$\frac{\partial \sigma^*}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{122}$$

where $\mathbf{j} = \sigma^*\mathbf{v}$ is the current associated with the density σ^*. In SPH formalism we can express the current as

$$\mathbf{j}_{SPH}(\mathbf{r},t) \;=\; \sum_{\alpha=1}^{N_{SPH}} \mathbf{v}_i \nu_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|), \tag{123}$$

so that

$$\nabla\cdot\mathbf{j}_{SPH}(\mathbf{r},t) \;=\; \sum_{\alpha=1}^{N_{SPH}} \mathbf{v}_i \nu_\alpha \nabla W(|\mathbf{r}-\mathbf{r}_\alpha(t)|). \tag{124}$$

The right hand side of Eq. (122) is, however,

$$\sigma^*_{SPH}(\mathbf{r},t) \;=\; \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|) \tag{125}$$

$$\begin{aligned}\frac{\partial \sigma^*_{SPH}(\mathbf{r},t)}{\partial t} \;&=\; \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \frac{d}{dt} W(|\mathbf{r}-\mathbf{r}_\alpha(t)|) \\ &=\; \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \frac{d\mathbf{r}_\alpha(t)}{dt}\cdot\nabla W(|\mathbf{r}-\mathbf{r}_\alpha(t)|),\end{aligned} \tag{126}$$

which comparing with Eq. (124) gives

$$\mathbf{v}_i = \frac{d\mathbf{r}_\alpha(t)}{dt} \tag{127}$$

so we see that Eq. (122) is, indeed, satisfied by SPH.

We can then normalize W (as in Eq. (118)) for Eq. (125) because the v_α's are constant

$$\int_{SPH} \sigma^*(\mathbf{r},t)\, d^3\mathbf{r} = \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha.$$

Then v_α can be interpreted as the conserved quantity at the point $\mathbf{r} = \mathbf{r}_\alpha(t)$. Thus, the distribution $\sigma^*_{SPH}(\mathbf{r},t)$ is a summation of piece-wise distributions carrying the density

$$\sigma^*(\mathbf{r}_\alpha,t) = \nu_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|) \tag{128}$$

for each SPH "particle" and the distribution for each "particle" is

$$a(\mathbf{r}_\alpha,t) = A_\alpha(t) W(|\mathbf{r}-\mathbf{r}_\alpha(t)|). \tag{129}$$

Then we find that A_α is

$$A_\alpha\left(t\right)=\nu_\alpha\frac{a(\mathbf{r}_\alpha,t)}{\sigma^*(\mathbf{r}_\alpha,t)} \tag{130}$$

which is the quantity A carried by the SPH particle at $\mathbf{r}=\mathbf{r}_\alpha(t)$. This then changes Eq. (121) into

$$a_{SPH}\left(\mathbf{r},t\right)=\sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\frac{a(\mathbf{r}_\alpha,t)}{\sigma^*(\mathbf{r}_\alpha,t)}W(|\mathbf{r}-\mathbf{r}_\alpha(t)|). \tag{131}$$

The total amount of A in the system is then found by doing a summation over Eq. (130)

$$A(t)=\sum_{\alpha}^{N_{SPH}}A_\alpha(t). \tag{132}$$

When you consider an ideal fluid you can take the entropy density as the choosen reference density (because it is conserved) and the dynamics of the parameters $\{\mathbf{r}_\alpha(t),\alpha=1,\ldots,N_{SPH}\}$ are determined from the variational principle from the action of ideal hydrodynamics. However, since we are considering a dissipative fluid, the entropy density is no longer conserved and we instead consider the proper reference density σ as show in Eq. (89), which we will use as a reference density for viscous fluids. Here the four-velocity u^μ is defined in terms of the local rest frame of the energy flow, i.e. the Landau frame. The proper reference density can then be written in the SPH form

$$\sigma^*(\mathbf{r},t)=\sum_{\alpha=1}^{N_{SPH}}\nu_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|), \tag{133}$$

where

$$\sigma^*=\sigma u^0 \tag{134}$$

is the specific density in the laboratory frame and

$$\nu_\alpha=\frac{1}{\sigma^*(\mathbf{r}_\alpha,t)} \tag{135}$$

is the inverse of the specific volume of the SPH particle α, and is chosen as an arbitrary constant. The specific volume is interpretated as the volume of the fluid cell. However, as we have shown in Eq. (109) our final results do not depend on the volume so we can simply set $\nu_\alpha=1$ for simplicity's sake. For the kernel $W(\mathbf{r})$ we use the spline function. This only works if the lines of flow in spaced defined by the velocity field u^μ do not cross each other during the time evolution, basically, this excludes turbulence and singularities in the flow lines.

We can now apply SPH to the non-linear memory function in Eq. (109). First we can rewrite Eq. (94) in the SPH form. We can simply substitute in Π for σ^* in Eq. (131)

$$\Pi=\sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\frac{\Pi_\alpha}{\sigma^*_\alpha}W(|\mathbf{r}-\mathbf{r}_\alpha(t)|), \tag{136}$$

where the SPH expression for the viscosity, Π_α, is described using the memory function in Eq. (108)

$$\gamma_\alpha\frac{d\Pi_\alpha}{dt}=-\frac{\zeta}{\tau_R}\left(\partial_\mu u^\mu\right)_\alpha-\frac{1}{\tau_R}\Pi_\alpha \tag{137}$$

where we recall that the proper time can be rewritten as $d\tau=dt/\gamma_\alpha$ and γ_α is the Lorentz factor of the α^{th} particle. Then, we can use the SPH expression for the entropy density, s^*, which is

$$s^*=\sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\left(\frac{s}{\sigma}\right)_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|), \tag{138}$$

where $s = s^*/\gamma$ and Eq. (91) and Eq. (93) to find

$$\begin{aligned}
\sigma D\tilde{s} &= -\frac{1}{T}\Pi\partial_\mu u^\mu \\
\sigma\frac{d\tilde{s}}{d\tau} &= \\
\sigma^*\frac{d\tilde{s}}{dt} &= \\
\frac{d}{dt}\left(\frac{s}{\sigma}\right) &= -\frac{1}{T}\frac{\Pi}{\sigma^*}\partial_\mu u^\mu
\end{aligned} \tag{139}$$

In order to convert Eq. (139) into SPH we can follow these steps for two quantities where $A = B$

$$\begin{aligned}
A &= B \\
\sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \left(\frac{A}{\sigma}\right)_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|) &= \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \left(\frac{B}{\sigma}\right)_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|).
\end{aligned} \tag{140}$$

Using this we can then conver Eq. (139) into SPH

$$\begin{aligned}
\frac{d}{dt}\left(\frac{s}{\sigma}\right) &= -\frac{1}{T}\frac{\Pi}{\sigma^*}\partial_\mu u^\mu \\
\sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \frac{1}{\sigma_\alpha} W(|\mathbf{r}-\mathbf{r}_\alpha(t)|)\frac{d}{dt}\left(\frac{s}{\sigma}\right)_\alpha &= \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \frac{1}{\sigma_\alpha}\frac{1}{T_\alpha}\frac{\Pi_\alpha}{\sigma^*_\alpha}(\partial_\mu u^\mu)_\alpha W(|\mathbf{r}-\mathbf{r}_\alpha(t)|) \\
\frac{d}{dt}\left(\frac{s}{\sigma}\right)_\alpha &= \frac{1}{T_\alpha}\frac{\Pi_\alpha}{\sigma^*_\alpha}(\partial_\mu u^\mu)_\alpha
\end{aligned} \tag{141}$$

We also need to be able to express the momentum conservation equation in terms of SPH variables. Thus, taking Eq. (1) and only the bulk viscosity term from Eq. (25) we can write the space compenent in terms of the reference density (the indentity in Eq. (101) was also used)

$$\begin{aligned}
0 &= \partial_\mu \left[T_0^{\mu\nu} + \Pi^{\mu\nu}\right] \\
&= \partial_\mu \left[\epsilon u^\mu u^\nu - (p+\Pi)\,\Delta^{\mu\nu}\right] \\
&= \partial_\mu \left[u^\mu\,(\epsilon+p+\Pi)\,u^\nu\right] - \partial^\nu\,(p+\Pi) \\
&= \sigma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u^\nu\right) - \partial^\nu\,(p+\Pi)
\end{aligned} \tag{142}$$

We can then switch ν to i because we want to consider only the spatial component so

$$\sigma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u^i\right) = \partial^i\,(p+\Pi) \tag{143}$$

Now in the ideal case the SPH equation of motion can be derived by the variational method. However, that is not possible when we consider viscosity, thus, we create an equation that returns to the ideal case when the visocisty is zero

$$\sigma_\alpha\frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha+\Pi_\alpha}{\sigma_\alpha}u^i_\alpha\right) = \sum_{\beta=1}^{N_{SPH}} \nu_\beta\sigma^*_\alpha\left[\frac{p_\beta+\Pi_\beta}{\left(\sigma^*_\beta\right)^2} + \frac{p_\alpha+\Pi_\alpha}{\left(\sigma^*_\alpha\right)^2}\right] \times \partial^i W(|\mathbf{r}_\alpha-\mathbf{r}_\beta(t)|). \tag{144}$$

If we take $\Pi = 0$ then Eq. (144) returns the equation of motion for an ideal fluid (see [31])

$$\sigma_\alpha\frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha}{\sigma_\alpha}u^i_\alpha\right) = \sum_{\beta=1}^{N_{SPH}} \nu_\beta\sigma^*_\alpha\left[\frac{p_\beta}{\left(\sigma^*_\beta\right)^2} + \frac{p_\alpha}{\left(\sigma^*_\alpha\right)^2}\right] \times \partial^i W(|\mathbf{r}_\alpha-\mathbf{r}_\beta(t)|). \tag{145}$$

In order to proceed it is useful introduce the following relations

$$\partial_\mu\left(\frac{1}{\sigma}\right) = -\frac{1}{\sigma^2}\partial_\mu\sigma \tag{146}$$

$$\partial_\mu u^\mu = -\frac{1}{\sigma}\partial_\mu\sigma \tag{147}$$

$$\partial^\mu u_\mu = -\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau} \tag{148}$$

as shown in Appendix A and we recall that

$$u^\mu\partial_\mu = \frac{d}{d\tau} = \gamma\frac{d}{dt}. \tag{149}$$

Focusing on the left hand side of Eq. (144) we find

$$\begin{aligned}\sigma_\alpha\frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha+\Pi_\alpha}{\sigma_\alpha}u^i_\alpha\right) &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} + \sigma_\alpha u^i_\alpha(\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{d}{d\tau_\alpha}\left(\frac{1}{\sigma_\alpha}\right) + u^i_\alpha\frac{d}{d\tau_\alpha}(\epsilon_\alpha+p_\alpha) + u^i_\alpha\frac{d\Pi_\alpha}{d\tau_\alpha}\\ &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - \frac{u^i_\alpha}{\sigma_\alpha}(\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{d\sigma_\alpha}{d\tau_\alpha} + u^i_\alpha\frac{d}{d\tau_\alpha}(\epsilon_\alpha+p_\alpha) + u^i_\alpha\frac{d\Pi_\alpha}{d\tau_\alpha}\end{aligned} \tag{150}$$

We can subsitute in Eq. (108)

$$\frac{d\tilde{\Pi}}{d\tau} = -\frac{\zeta}{\sigma\tau_R}\partial_\alpha u^\alpha - \frac{\tilde{\Pi}}{\tau_R} \tag{151}$$

so

$$\begin{aligned}\sigma_\alpha\frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha+\Pi_\alpha}{\sigma_\alpha}u^i_\alpha\right) &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - \frac{u^i_\alpha}{\sigma_\alpha}(\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{d\sigma_\alpha}{d\tau_\alpha} + u^i_\alpha\frac{d}{d\tau_\alpha}(\epsilon_\alpha+p_\alpha)\\ &\quad -u^i_\alpha\left(\frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha}\partial_\mu u^\mu_\alpha + \frac{\Pi_\alpha}{(\tau_R)_\alpha}\right)\\ &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - \frac{u^i_\alpha}{\sigma_\alpha}(\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{d\sigma_\alpha}{d\tau_\alpha} + u^i_\alpha\frac{d}{ds_\alpha}(\epsilon_\alpha+p_\alpha)\frac{ds}{d\tau_\alpha}\\ &\quad -u^i_\alpha\left(\frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha}\partial_\mu u^\mu_\alpha + \frac{\Pi_\alpha}{(\tau_R)_\alpha}\right).\end{aligned} \tag{152}$$

Then, using Eq. (91)

$$\begin{aligned}\partial_\mu s^\mu &= -\frac{1}{T}\Pi\partial_\mu u^\mu\\ \partial_\mu(su^\mu) &=\\ u^\mu\partial_\mu(s) + s\partial_\mu(u^\mu) &=\\ \frac{ds}{d\tau} + s\partial_\mu(u^\mu) &=\\ \frac{ds}{d\tau} &= -\left(\frac{\Pi}{T}+s\right)\partial_\mu u^\mu,\end{aligned} \tag{153}$$

we find

$$\begin{aligned}\sigma_\alpha\frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha+\Pi_\alpha}{\sigma_\alpha}u^i_\alpha\right) &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - \frac{u^i_\alpha}{\sigma_\alpha}(\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{d\sigma_\alpha}{d\tau_\alpha} - u^i_\alpha\frac{d}{ds_\alpha}(\epsilon_\alpha+p_\alpha)\left(\frac{\Pi_\alpha}{T_\alpha}+s_\alpha\right)\partial_\mu u^\mu_\alpha\\ &\quad -u^i_\alpha\left(\frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha}\partial_\mu u^\mu_\alpha + \frac{\Pi_\alpha}{(\tau_R)_\alpha}\right).\end{aligned} \tag{154}$$

Applying Eq. (147),

$$\begin{aligned}
\sigma_\alpha \frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha+\Pi_\alpha}{\sigma_\alpha}u^i_\alpha\right) &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - u^i_\alpha(\epsilon_\alpha+p_\alpha+\Pi_\alpha)\partial_\mu u^\mu_\alpha - u^i_\alpha\frac{d}{ds_\alpha}(\epsilon_\alpha+p_\alpha)\left(\frac{\Pi_\alpha}{T_\alpha}+s_\alpha\right)\partial_\mu u^\mu_\alpha \\
&\quad - u^i_\alpha\left(\frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha}\partial_\mu u^\mu_\alpha + \frac{\Pi_\alpha}{(\tau_R)_\alpha}\right) \\
&= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - u^i_\alpha\frac{\Pi_\alpha}{(\tau_R)_\alpha} \\
&\quad + u^i_\alpha\partial_\mu u^\mu_\alpha\left[\epsilon_\alpha+p_\alpha+\Pi_\alpha - \frac{d}{ds_\alpha}(\epsilon_\alpha+p_\alpha)\left(\frac{\Pi_\alpha}{T_\alpha}+s_\alpha\right) - \frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha}\right]. \qquad (155)
\end{aligned}$$

Then we define

$$A_\alpha \equiv \epsilon_\alpha+p_\alpha+\Pi_\alpha - \frac{dw}{ds_\alpha}\left(\frac{\Pi_\alpha}{T_\alpha}+s_\alpha\right) - \frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha} \qquad (156)$$

Then,

$$\begin{aligned}
\sigma_\alpha \frac{d}{d\tau_\alpha}\left(\frac{\epsilon_\alpha+p_\alpha+\Pi_\alpha}{\sigma_\alpha}u^i_\alpha\right) &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - u^i_\alpha\frac{\Pi_\alpha}{(\tau_R)_\alpha} + A_\alpha u^i_\alpha\partial_\mu u^\mu_\alpha \\
&= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - u^i_\alpha\frac{\Pi_\alpha}{(\tau_R)_\alpha} - A_\alpha u^i_\alpha\frac{u^l_\alpha(g_{lm})_\alpha}{\gamma_\alpha}\frac{d(u^m)_\alpha}{d\tau_\alpha} - A_\alpha u^i_\alpha\frac{\gamma_\alpha}{\sigma^*_\alpha}\frac{d\sigma^*_\alpha}{dt_\alpha} \qquad (157)
\end{aligned}$$

Recalling Eq. (143),

$$\begin{aligned}
\partial^i(p_\alpha+\Pi_\alpha) &= (\epsilon_\alpha+p_\alpha+\Pi_\alpha)\frac{du^i_\alpha}{d\tau_\alpha} - u^i_\alpha\frac{\Pi_\alpha}{(\tau_R)_\alpha} - A_\alpha u^i_\alpha\frac{u^l_\alpha(g_{lm})_\alpha}{\gamma_\alpha}\frac{d(u^m)_\alpha}{d\tau_\alpha} - A_\alpha u^i_\alpha\frac{\gamma_\alpha}{\sigma^*_\alpha}\frac{d\sigma^*_\alpha}{dt_\alpha} \\
\underbrace{\left[(\epsilon_\alpha+p_\alpha+\Pi_\alpha) - \frac{A_\alpha}{\gamma_\alpha}u^l_\alpha u^m_\alpha(g_{mi})_\alpha\right]}_{=M^{lmi}_\alpha}\frac{du^i_\alpha}{d\tau_\alpha} &= \underbrace{u^i_\alpha\frac{\Pi_\alpha}{(\tau_R)_\alpha} + A_\alpha u^i_\alpha\frac{\gamma_\alpha}{\sigma^*_\alpha}\frac{d\sigma^*_\alpha}{dt_\alpha} + \partial^i(p_\alpha+\Pi_\alpha)}_{F^i_\alpha}. \qquad (158)
\end{aligned}$$

In Eq. (158) we were able to separate the equation into force, mass and accelleration terms such that

$$\begin{aligned}
M^{lmi}_\alpha &\equiv (\epsilon_\alpha+p_\alpha+\Pi_\alpha) - \frac{A_\alpha}{\gamma_\alpha}u^l_\alpha u^m_\alpha(g_{mi})_\alpha \qquad (159) \\
F^i_\alpha &\equiv u^i_\alpha\frac{\Pi_\alpha}{(\tau_R)_\alpha} + A_\alpha u^i_\alpha\frac{\gamma_\alpha}{\sigma^*_\alpha}\frac{d\sigma^*_\alpha}{dt_\alpha} + \partial^i(p_\alpha+\Pi_\alpha), \qquad (160)
\end{aligned}$$

which leaves the equation

$$M^{lmi}_\alpha\frac{du^i_\alpha}{d\tau_\alpha} = F^i_\alpha \qquad (161)$$

to solve.

Then we are left with the following coupled differential equations to solve

$$\begin{aligned}
M^{lmi}_\alpha\frac{du^i_\alpha}{d\tau_\alpha} &= F^i_\alpha \qquad (162) \\
\gamma_\alpha\frac{d\Pi_\alpha}{dt} &= -\frac{\zeta}{(\tau_R)_\alpha}(\partial_\mu u^\mu)_\alpha - \frac{1}{(\tau_R)_\alpha}\Pi_\alpha \qquad (163) \\
\frac{d}{dt}\left(\frac{s}{\sigma}\right)_\alpha &= -\frac{1}{T_\alpha}\frac{\Pi_\alpha}{\sigma^*_\alpha}(\partial_\mu u^\mu)_\alpha \qquad (164)
\end{aligned}$$

where we recall that

$$(\partial_\mu u^\mu)_\alpha = -\frac{\gamma_\alpha}{\sigma^*_\alpha}\frac{d\sigma^*_\alpha}{d\tau_\alpha} - \frac{(g_{ij})_\alpha u^i_\alpha}{\gamma_\alpha}\frac{du^j_\alpha}{d\tau_\alpha}. \qquad (165)$$

III. BULK COMPARISON WITH GABRIEL

I'm trying to understand your equations in your code but the problem is, is that some of them are not aligning with your notes and I'm trying to figure out the difference. So from my understanding of the notes and the equations I would think we need to solve the following equations for the bulk viscosity:

$$M_\alpha^{lmi}\frac{du_\alpha^i}{d\tau_\alpha} = F_\alpha^i \tag{166}$$

$$\gamma_\alpha\frac{d\Pi_\alpha}{dt} = -\frac{\zeta}{(\tau_R)_\alpha}\left(\partial_\mu u^\mu\right)_\alpha - \frac{1}{(\tau_R)_\alpha}\Pi_\alpha \tag{167}$$

$$\frac{d}{dt}\left(\frac{s}{\sigma}\right)_\alpha = -\frac{1}{T_\alpha}\frac{\Pi_\alpha}{\sigma_\alpha^*}(\partial_\mu u^\mu)_\alpha \tag{168}$$

where we recall that

$$(\partial_\mu u^\mu)_\alpha = -\frac{\gamma_\alpha}{\sigma_\alpha^*}\frac{d\sigma_\alpha^*}{d\tau_\alpha} - \frac{(g_{ij})_\alpha u_\alpha^i}{\gamma_\alpha}\frac{du_\alpha^j}{d\tau_\alpha}. \tag{169}$$

$$M_\alpha^{lmi} \equiv C - \frac{A_\alpha}{\gamma_\alpha}u_\alpha^l u_\alpha^m (g_{mi})_\alpha \tag{170}$$

$$F_\alpha^i \equiv Bu_\alpha^i + \partial^i\left(p_\alpha + \Pi_\alpha\right), \tag{171}$$

where

$$C \equiv \epsilon_\alpha + p_\alpha + \Pi_\alpha \tag{172}$$

$$A_\alpha \equiv C - \frac{dw}{ds_\alpha}\left(\frac{\Pi_\alpha}{T_\alpha} + s_\alpha\right) - \frac{\zeta_\alpha}{\sigma_\alpha(\tau_R)_\alpha} \tag{173}$$

$$B_\alpha \equiv \frac{\Pi_\alpha}{(\tau_R)_\alpha} + A_\alpha\frac{\gamma_\alpha}{\sigma_\alpha^*}\frac{d\sigma_\alpha^*}{dt_\alpha} \tag{174}$$

Now, comparing with what you have in your code I see the following discrepancies:

- Every time that $(\partial_\mu u^\mu)_\alpha$ comes up in the code it is actually $(\partial_\mu u^\mu)_\alpha^* = t(\partial_\mu u^\mu)_\alpha + \gamma$
- When you definite A_ideal the C term within it is missing the Π_α term
- There appears to be two difference definitions for Π_α but in the equations in your notes you don't distinguish them. One is the "Bulk", which enters only directly into the $\gamma_\alpha\frac{d\Pi_\alpha}{dt}$ equation and the other is "SmoothedBulk", which appears to be defined as $\Pi^* = \frac{\Pi\sigma}{\gamma t}$
- Just double checking here but it looks like also in your $\gamma_\alpha\frac{d\Pi_\alpha}{dt}$ that you absorb the γ_α term into the Π_α because instead of $\gamma_\alpha\frac{d\Pi_\alpha}{dt} = -\frac{\zeta}{(\tau_R)_\alpha}\left(\partial_\mu u^\mu\right)_\alpha - \frac{1}{(\tau_R)_\alpha}\Pi_\alpha$ you have $\frac{d\Pi_\alpha}{dt} = -\frac{\zeta}{(\tau_R)_\alpha}\left(\partial_\mu u^\mu\right)_\alpha - \frac{1}{(\tau_R)_\alpha\gamma_\alpha}\Pi_\alpha$, which would at least explain the γ_α part of the term in the previous point.

My guess is the second item has to be an error (that you've probably already fixed in your current code?).
However, if you know the origin/reason behind the first and third items, I'd really appreciate knowing the background there. :) The fourth item, I just wanted to double check that that was indeed what you are doing.
Thanks!

IV. VISCOSITY WITH HYPERBOLIC COORDINATES

A. Establishing Relationships

Previously, we used Cartesian coordinates but hyperbolic coordinates are more intuitive for heavy-ion collisions. Thus, as in Sec. I A we use hyperbolic coordinates, i.e., τ, ξ, x, and y such that

$$\tau = \sqrt{t^2 - z^2} \tag{175}$$

and the space-time rapidity is

$$\xi = tanh^{-1}\left(\frac{z}{t}\right) = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right). \tag{176}$$

When using hyperbolic coordinates we'll switch from $\partial_\mu \to D_\mu$, thus, the conservation of energy-momentum likes like

$$D_\mu T^{\mu\nu}. \tag{177}$$

The metric is changed to

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^2 \end{pmatrix} \tag{178}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau^2} \end{pmatrix} \tag{179}$$

because the metric is no longer constant, then we must also consider the Christoffel symbols when taking derivatives

$$\begin{aligned} D_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda && (180) \\ D_\mu V^\nu &= \partial_\mu V^\nu - \Gamma^\nu_{\mu\lambda} V^\lambda && (181) \\ D_\mu V_{\alpha\beta} &= \partial_\mu V_{\alpha\beta} - \Gamma^\lambda_{\alpha\mu} V_{\lambda\beta} - \Gamma^\lambda_{\beta\mu} V_{\lambda\alpha} && (182) \\ & && (183) \end{aligned}$$

where

$$\Gamma^\nu_{\mu\lambda} = \frac{1}{2} g^{\nu\sigma}\left(\partial_\mu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\mu} - \partial_\sigma g_{\mu\lambda}\right). \tag{184}$$

Let us assume $\mu = 1$, $\lambda = 1$, and $\nu = 0$, then

$$\begin{aligned} \Gamma^0_{11} &= \frac{1}{2} g^{0\sigma}\left(\partial_1 g_{\sigma 1} + \partial_1 g_{\sigma 1} - \partial_\sigma g_{11}\right) \\ &= 0 \end{aligned} \tag{185}$$

because all terms besides g^{33} or g_{33} are zero when you take the derivative. So let's take $\mu = 3$, $\lambda = 3$, and $\nu = 0$

$$\begin{aligned} \Gamma^0_{33} &= \frac{1}{2} g^{0\sigma}\left(\partial_3 g_{\sigma 3} + \partial_3 g_{\sigma 3} - \partial_\sigma g_{33}\right) \\ &= \frac{1}{2} \underbrace{g^{00}}_{=1} \left(\underbrace{\partial_3 g_{03} + \partial_3 g_{03}}_{=0} - \underbrace{\partial_0 g_{33}}_{=2\tau}\right) \\ &= \tau \end{aligned} \tag{186}$$

where $\sigma = 0$ because of $g^{0\sigma}$ where all other terms for σ go to zero. Note that for other values of ν that the derivative of $\partial_{\nu\neq 0}\tau^2 = 0$ Thus,

$$\begin{aligned} \Gamma^\mu_{\mu 0} = \Gamma^\mu_{0\mu} &= \frac{1}{\tau} \\ \Gamma^0_{\mu\mu} &= \tau. \end{aligned} \tag{187}$$

Then,

$$\begin{aligned} D_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda \\ &= \partial_\mu V^\mu + \frac{1}{\tau} V^\mu \\ &= \frac{1}{\tau}\partial_\mu\left(\tau V^\mu\right). \end{aligned} \tag{188}$$

Also,

$$D_\mu V^{\mu\nu} = \frac{1}{\tau}\partial_\mu\left(\tau V^{\mu\nu}\right) + \Gamma^\nu_{\lambda\mu}V^{\lambda\mu}. \tag{189}$$

We can also use the definitions

$$\begin{aligned} u^\mu\partial_\mu &= \gamma\frac{d}{d\tau} \\ u^\mu D_\mu &= \frac{D}{D\tau} \end{aligned} \tag{190}$$

to rewrite many useful relations that we used in the previous example with Cartesian coordinates. Previously, Eq. (147) becomes

$$\frac{1}{\sigma}\frac{D\sigma}{D\tau} = -D_\mu u^\mu. \tag{191}$$

Subsituting u^μ into Eq. (188) we find

$$\begin{aligned} D_\mu u^\mu &= \frac{1}{\tau}\partial_\mu\left(\tau u^\mu\right) \\ &= \frac{1}{\tau}\left[\tau\partial_\mu\left(u^\mu\right) + u^\mu\partial_\mu\left(\tau\right)\right] \\ &= \partial_\mu u^\mu + \frac{\gamma}{\tau}\frac{d}{d\tau}\left(\tau\right) \\ &= \partial_\mu u^\mu + \frac{\gamma}{\tau} \end{aligned} \tag{192}$$

Previously, we proved Eq. (A9), which becomes

$$\partial^\mu u_\mu = \frac{d\gamma}{d\tau} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}. \tag{193}$$

in hyperbolic coordinates. Furthermore, Eq. (A11) can then be rewritten for hyperbolic coordinates

$$\begin{aligned} \frac{d\gamma}{d\tau} &= -\frac{u_i g^{ij}}{\gamma}\frac{du_j}{d\tau} - \frac{u_i u_j}{2\gamma}\frac{dg^{ij}}{d\tau} \\ &= -\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \frac{u_i u_j}{2\gamma^2}\left(u^\mu\partial_\mu g^{ij}\right) \\ &= -\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \frac{(u_3)^2}{2\gamma^2}\left(u^0\partial_0 g^{33}\right) \\ &= -\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \frac{(u_3)^2}{2\gamma^2}\left(\gamma 2\frac{1}{\tau^3}\right) \\ &= -\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \frac{(u_3)^2}{\gamma\tau^3} \end{aligned} \tag{194}$$

B. Equations of Motion

Returning to Eq. (177) we can rewrite it as

$$
\begin{aligned}
0 &= D_\mu T^{\mu\nu} \\
&= \frac{1}{\tau}\partial_\mu \left(\tau T^{\mu\nu}\right) + \Gamma^\nu_{\lambda\mu} T^{\lambda\mu} \\
&= \frac{1}{\tau}\partial_\mu \left(\tau T^{\mu\nu}\right) + \Gamma^\nu_{33} T^{33} + 2\Gamma^\nu_{30} T^{30} \\
&= \frac{1}{\tau} g_{\alpha\mu}\partial^\alpha \left(\tau T^{\mu\nu}\right) + \Gamma^\nu_{33} T^{33} + 2\Gamma^\nu_{30} T^{30} \\
&= \frac{1}{\tau} g_{\beta\nu} g_{\alpha\mu}\partial^\alpha \left(\tau T^{\mu\nu}\right) + g_{\beta\nu}\Gamma^\nu_{33} T^{33} + 2 g_{\beta\nu}\Gamma^\nu_{30} T^{30} \\
&= \frac{1}{\tau} g_{\beta\nu} g_{\alpha\mu}\partial^\alpha \left(\tau T^{\mu\nu}\right) + g_{\beta 0}\tau T^{33} + \frac{2}{\tau} g_{\beta 3} T^{30} \\
&= \frac{1}{\tau}\partial^\alpha \left(g_{\beta\nu} g_{\alpha\mu}\tau T^{\mu\nu}\right) - T^{\mu\nu}\partial^\alpha \left(g_{\beta\nu} g_{\alpha\mu}\right) + g_{\beta 0}\tau T^{33} + \frac{2}{\tau} g_{\beta 3} T^{30}
\end{aligned} \tag{195}
$$

Taking just the term $T^{\mu\nu}\partial^\alpha \left(g_{\beta\nu} g_{\alpha\mu}\right)$

$$
\begin{aligned}
T^{\mu\nu}\partial^\alpha \left(g_{\beta\nu} g_{\alpha\mu}\right) &= T^{\mu\nu}\partial_0 \left(g_{\beta\nu} g_{0\mu}\right) \\
&= T^{0\nu}\partial_0 \left(g_{\beta\nu}\right) \\
&= -T^{03}\delta_{\beta 3}\partial_0 \left(\frac{1}{\tau^2}\right) \\
&= \frac{2}{\tau^3}\delta_{\beta 3} T^{03} \\
&= \frac{2}{\tau} g_{\beta 3} T^{03}
\end{aligned} \tag{196}
$$

Then, returning to Eq. (195),

$$
\frac{1}{\tau}\partial^\alpha \left(g_{\beta\nu} g_{\alpha\mu}\tau T^{\mu\nu}\right) + g_{\beta 0}\tau T^{33} = 0 \tag{197}
$$

Depending on the value of β, we either obtain the moment conservation law for $\beta = i$

$$
\frac{1}{\tau}\partial^\alpha \left(\tau T_{\alpha i}\right) = 0 \tag{198}
$$

or the energy conservation law for $\beta = 0$

$$
\frac{1}{\tau}\partial^\alpha \left(\tau T_{\alpha 0}\right) + \tau T^{33} = 0 \tag{199}
$$

C. Bulk Viscosity

In cartesian coordinates the bulk viscosity equation is shown in Eq. (108)

$$
\tilde{\Pi} + \tau_R \frac{d\tilde{\Pi}}{d\tau} = -\frac{\zeta}{\sigma}\partial_\alpha u^\alpha . \tag{200}
$$

switch to hyperbolic coordinates we obtain

$$
\tilde{\Pi} + \tau_R \frac{D\tilde{\Pi}}{D\tau} = -\frac{\zeta}{\sigma} D_\alpha u^\alpha \tag{201}
$$

where we recall that $\tilde{\Pi} = \frac{\Pi}{\sigma}$ or we can rewrite it in terms of the density (see Eq. (109) for cartesian coordinates)

$$
\tau_R \frac{D\Pi}{D\tau} + \Pi = -\left(\zeta + \tau_R \Pi\right) D_\mu u^\mu . \tag{202}
$$

where $D_\mu u^\mu$ is shown in Eq. (192).

We can then rewrite Eq. (201)

$$\begin{aligned}\frac{D\tilde{\Pi}}{D\tau} &= -\frac{1}{\tau_R\sigma}\left[\tilde{\Pi}-\zeta D_\alpha u^\alpha\right] \\ &= -\frac{1}{\tau_R\sigma}\left[\tilde{\Pi}-\zeta\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\right]\end{aligned} \tag{203}$$

because Π is a scalar we can rewrite $\frac{D\tilde{\Pi}}{D\tau}=u^\mu D_\mu\tilde{\Pi}=u^\mu\partial_\mu\tilde{\Pi}=\gamma\frac{d}{d\tau}\tilde{\Pi}$ so

$$\frac{d\tilde{\Pi}}{d\tau} = -\frac{1}{\tau_R\sigma^*}\left[\Pi-\zeta\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\right] \tag{204}$$

D. Shear Viscosity

We have already seen the energy momentum tensor that includes shear viscosity in Eq. (25)

$$\begin{aligned}T^{\mu\nu} &= T_0^{\mu\nu}+\Pi^{\mu\nu} \\ &= \epsilon\, u^\mu u^\nu-\Delta^{\mu\nu}(p+\Pi)+\pi^{\mu\nu}\end{aligned} \tag{205}$$

Moreover, we recall Eq. (63)

$$\begin{aligned}\partial_\mu s^\mu &= \frac{1}{T}\Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} \\ &= \frac{1}{T}\left(-\Delta^{\mu\nu}\Pi+\pi^{\mu\nu}\right)\nabla_{(\mu}u_{\nu)}.\end{aligned} \tag{206}$$

The irreversible current is given by

$$\pi_{\mu\nu}=\eta\Delta_{\mu\nu\alpha\beta}\partial^\alpha u^\beta \tag{207}$$

then we can write the memory function for the modified irreversible currents as

$$\pi_{\mu\nu}(\tau)=\Delta_{\mu\nu\alpha\beta}\int_{\tau_0}^{\tau}d\tau' G(\tau,\tau')\,\eta\partial^\alpha u^\beta(\tau')+e^{-(\tau-\tau_0)/\tau_R}\pi^0_{\mu\nu}, \tag{208}$$

which one can consider as the generalized version of the Maxwell-Cattaneo law where a new transport coefficent the relaxation time, τ_R, is introduced, which ensures that causality is not violated. The non-linear term $\tau_R\partial_\mu u^\mu$ is essential to obtain a stable theory [18]. Eq. (208) matches what we saw previously from the bulk viscosity, essentially

$$\tau_R u^\mu D_\mu J+(1+\tau_R D_\mu u^\mu)J=\lambda F \tag{209}$$

where λ is the corresponding transport coefficient, F is the force, and J is the current. In the case of bulk viscosity $J=\tilde{\Pi}$, $\lambda=\zeta$, and $F=\partial_\alpha u^\alpha$, which then returns Eq. (202).

Thus, for shear viscosity we can take

$$\begin{aligned}J &= \pi_{\mu\nu} \\ \tau_R &= \tau_\pi \\ F &= \sigma_{\mu\nu} \\ \lambda &= \eta\end{aligned} \tag{210}$$

where

$$\begin{aligned}\sigma_{\mu\nu} &= D_{\langle\mu}u_{\nu\rangle} \\ \Theta &= D_\mu u^\mu\end{aligned} \tag{211}$$

which leads to

$$\begin{aligned}\tau_\pi\Delta_{\mu\nu\alpha\beta}u_\lambda D^\lambda\pi^{\alpha\beta}+\pi_{\mu\nu} &= \eta\sigma_{\mu\nu}-\tau_\pi\pi_{\mu\nu}\Theta \\ \tau_\pi\Delta_{\mu\nu\alpha\beta}\frac{D\pi^{\alpha\beta}}{D\tau}+\pi_{\mu\nu} &= \eta\sigma_{\mu\nu}-\tau_\pi\pi_{\mu\nu}\Theta\end{aligned} \tag{212}$$

where

$$
\begin{aligned}
\eta\sigma_{\mu\nu} &= \eta\Delta_{\mu\nu\alpha\beta}D^\alpha u^\beta \\
&= \eta\frac{1}{2}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta}+\Delta_{\mu\beta}\Delta_{\nu\alpha}-\frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right]D^\alpha u^\beta \\
&= \eta\frac{1}{2}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta}+\Delta_{\mu\beta}\Delta_{\nu\alpha}\right]D^\alpha u^\beta-\frac{\eta}{3}\Delta_{\mu\nu}D_\beta u^\beta \\
&= \frac{\eta}{2}\Delta_{\mu\alpha}\Delta_{\nu\beta}\left[D^\alpha u^\beta+D^\beta u^\alpha\right]-\frac{\eta}{3}\Delta_{\mu\nu}D_\beta u^\beta
\end{aligned} \tag{213}
$$

where the last step can achieved because you can simply rearrange the dummy indices. Then we can use the chain rule:

$$
\begin{aligned}
\Delta_{\nu\beta}D^\alpha u^\beta &= D^\alpha\underbrace{\left(\Delta_{\nu\beta}u^\beta\right)}_{=0}-u^\beta D^\alpha\left(\Delta_{\nu\beta}\right) \\
&= u^\beta D^\alpha\left(u_\nu u_\beta\right) \\
&= u^\beta u_\beta D^\alpha\left(u_\nu\right)+u^\beta u_\nu D^\alpha\left(u_\beta\right) \\
&= D^\alpha u_\nu
\end{aligned} \tag{214}
$$

Recall that the derivative can be decomposed into its perpendicular and parallel part

$$
D^\mu = D^\mu_\perp + u^\mu\frac{D}{D\tau} \tag{215}
$$

so that

$$
\begin{aligned}
\Delta_{\mu\alpha}\Delta_{\nu\beta}D^\alpha u^\beta &= \Delta_{\mu\alpha}D^\alpha u_\nu \\
&= D^\perp_\mu u_\nu \\
&= D_\mu u_\nu - u_\mu\frac{Du_\nu}{D\tau}
\end{aligned} \tag{216}
$$

Then, returning to Eq. (213) we obtain

$$
\begin{aligned}
\eta\sigma_{\mu\nu} &= \frac{\eta}{2}\left[D_\mu u_\nu+D_\nu u_\mu\right]-\frac{\eta}{2}\left[u_\mu\frac{Du_\nu}{D\tau}+u_\nu\frac{Du_\mu}{D\tau}\right]-\frac{\eta}{3}\Delta_{\mu\nu}D_\beta u^\beta \\
&= \frac{\eta}{2}\left[\partial_\mu u_\nu+\partial_\nu u_\mu\right]-\frac{\eta}{2}\left[\Gamma^\lambda_{\mu\nu}u_\lambda+\Gamma^\lambda_{\nu\mu}u_\lambda\right]-\frac{\eta\gamma}{2}\left[u_\mu\frac{du_\nu}{d\tau}+u_\nu\frac{du_\mu}{d\tau}\right]+\frac{\eta}{2}\left[u_\mu u^\alpha\Gamma^\lambda_{\alpha\nu}u_\lambda+u_\nu u^\alpha\Gamma^\lambda_{\alpha\mu}u_\lambda\right] \\
&\quad -\frac{\eta}{3}\Delta_{\mu\nu}D_\beta u^\beta
\end{aligned} \tag{217}
$$

Taking the term $\Gamma^\lambda_{\mu\nu}u_\lambda$

$$
\begin{aligned}
\Gamma^\lambda_{\mu\nu}u_\lambda &= \gamma\Gamma^0_{\mu\nu}+\Gamma^3_{\mu\nu}u_3 \\
&= \gamma\tau g^3_\nu g^3_\mu+\frac{u_3}{\tau}\left[g^3_\nu g^0_\mu+g^0_\nu g^3_\mu\right],
\end{aligned} \tag{218}
$$

and evaluating the term $u_\mu u^\alpha\Gamma^\lambda_{\alpha\nu}u_\lambda$

$$
\begin{aligned}
u_\mu u^\alpha\Gamma^\lambda_{\alpha\nu}u_\lambda &= u_\mu u^\alpha\Gamma^3_{\alpha\nu}u_3+\gamma u_\mu u^\alpha\Gamma^0_{\alpha\nu} \\
&= u_\mu\frac{\gamma}{\tau}u_3 g^3_\nu+u_\mu\frac{1}{\tau}u^3u_3g^0_\nu+u_\mu\gamma u^3\tau g^3_\nu
\end{aligned} \tag{219}
$$

where $u^3=g^{33}u_3=-\frac{1}{\tau^2}u_3$ and $u_3=g_{33}u^3=-\tau^2u^3$, so

$$
\begin{aligned}
u_\mu u^\alpha\Gamma^\lambda_{\alpha\nu}u_\lambda &= u_\mu\frac{\gamma}{\tau}u_3g^3_\nu-u_\mu\frac{1}{\tau^3}(u_3)^2g^0_\nu-u_\mu\frac{\gamma}{\tau}u_3g^3_\nu \\
&= -u_\mu\frac{1}{\tau^3}(u_3)^2g^0_\nu
\end{aligned} \tag{220}
$$

Then,

$$\begin{aligned}\eta\sigma_{\mu\nu} &= \frac{\eta}{2}\left[\partial_\mu u_\nu + \partial_\nu u_\mu\right] - \frac{\eta\gamma}{2}\left[u_\mu \frac{du_\nu}{d\tau} + u_\nu \frac{du_\mu}{d\tau}\right] - \frac{\eta}{3}\Delta_{\mu\nu} D_\beta u^\beta \\ &\quad -\eta\gamma\tau g^3_\nu g^3_\mu - \eta\frac{u_3}{\tau}\left[g^3_\nu g^0_\mu + g^0_\nu g^3_\mu\right] - \frac{\eta (u_3)^2}{2\tau^3}\left(u_\mu g^0_\nu + u_\nu g^0_\mu\right).\end{aligned} \tag{221}$$

Returning to Eq. (212) we evaluate the term

$$\tau_\pi \Delta_{\mu\nu\alpha\beta}\frac{D\pi^{\alpha\beta}}{D\tau} = \tau_\pi \frac{D\pi_{\mu\nu}}{D\tau} - \tau_\pi \pi^{\alpha\beta}\frac{D\Delta_{\mu\nu\alpha\beta}}{D\tau} \tag{222}$$

In Appendix A we show the following relationships, which will be needed to evaluate Eq. (222)

$$D_\mu g_{\alpha\beta} = 0 \tag{223}$$

$$\pi^{\alpha\beta}\frac{D}{D\tau}\Delta_{\alpha\beta} = 0 \tag{224}$$

$$\Delta_{\mu\alpha}\Delta_{\nu\beta} = \Delta_{\mu\beta}\Delta_{\nu\alpha} \tag{225}$$

where Eq. (225) is valid when $\Delta_{\mu\alpha}\Delta_{\nu\beta}$ are not contracted with another tensor/vector. Thus, returning to Eq. (222),

$$\begin{aligned}\pi^{\alpha\beta}\frac{D\Delta_{\mu\nu\alpha\beta}}{D\tau} &= \frac{1}{2}\pi^{\alpha\beta}\frac{D}{D\tau}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\mu\beta}\Delta_{\nu\alpha} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right] \\ &= \pi^{\alpha\beta}\frac{D}{D\tau}\left[\Delta_{\mu\alpha}\Delta_{\nu\beta}\right] \\ &= \pi^{\alpha\beta}\left[\underbrace{\frac{D}{D\tau}\left(g_{\mu\alpha}g_{\nu\beta}\right)}_{=0} - \frac{D}{D\tau}\left(u_\mu u_\alpha g_{\nu\beta}\right) - \frac{D}{D\tau}\left(g_{\mu\alpha}u_\nu u_\beta\right) + \frac{D}{D\tau}\left(u_\mu u_\alpha u_\nu u_\beta\right)\right] \\ &= -\pi^{\alpha\beta}\frac{D}{D\tau}\left(u_\mu u_\alpha g_{\nu\beta}\right) - \pi^{\alpha\beta}\frac{D}{D\tau}\left(g_{\mu\alpha}u_\nu u_\beta\right) + u_\mu \underbrace{u_\alpha \pi^{\alpha\beta}}_{=0}\frac{D}{D\tau}\left(u_\nu u_\beta \pi^{\alpha\beta}\right) + u_\nu \underbrace{u_\beta \pi^{\alpha\beta}}_{=0}\frac{D}{D\tau}\left(u_\mu u_\alpha\right) \\ &= -u_\mu g_{\nu\beta}\pi^{\alpha\beta}\frac{D}{D\tau}\left(u_\alpha\right) - \underbrace{u_\alpha \pi^{\alpha\beta}}_{=0}\frac{D}{D\tau}\left(u_\mu g_{\nu\beta}\right) - \pi^{\alpha\beta}\frac{D}{D\tau}g_{\mu\alpha}u_\nu u_\beta \\ &= -\left(u_\mu \pi^\alpha_\nu + u_\nu \pi^\alpha_\mu\right)\frac{Du_\alpha}{D\tau}\end{aligned} \tag{226}$$

where

$$\begin{aligned}\frac{Du_\alpha}{D\tau} &= u^\mu D_\mu u_\alpha \\ &= u^\mu\left[\partial_\mu u_\alpha - \Gamma^\lambda_{\mu\alpha}u_\lambda\right] \\ &= \gamma\frac{du_\alpha}{d\tau} - u^\mu \Gamma^\lambda_{\mu\alpha}u_\lambda \\ &= \gamma\frac{du_\alpha}{d\tau} - \left[u^3 u_3 \frac{1}{\tau}g_{\alpha 0} + \frac{\gamma}{\tau}u_3 g_{\alpha 3} + u^3\gamma\tau g_{\alpha 3}\right] \\ &= \gamma\frac{du_\alpha}{d\tau} + \frac{(u_3)^2}{\tau^3}g_{\alpha 0}\end{aligned} \tag{227}$$

so

$$-\pi^{\alpha\beta}\frac{D\Delta_{\mu\nu\alpha\beta}}{D\tau} = \left(u_\mu \pi^\alpha_\nu + u_\nu \pi^\alpha_\mu\right)\gamma\frac{du_\alpha}{d\tau} + \left(u_\mu \pi_{\nu 0} + u_\nu \pi_{\mu 0}\right)\frac{(u_3)^2}{\tau^3}. \tag{228}$$

We can then further simplify the term

$$\begin{aligned}
\left(u_\mu \pi^\alpha_\nu + u_\nu \pi^\alpha_\mu\right)\gamma\frac{du_\alpha}{d\tau} &= \left(u_\mu \pi^0_\nu + u_\nu \pi^0_\mu\right)\gamma\frac{d\gamma}{d\tau} + \left(u_\mu \pi^j_\nu + u_\nu \pi^j_\mu\right)\gamma\frac{du_j}{d\tau} \\
&= -\left(u_\mu \pi^0_\nu + u_\nu \pi^0_\mu\right)\left[u^j\frac{du_j}{d\tau} + \frac{(u_3)^2}{\tau^3}\right] + \left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu\right)\frac{du_j}{d\tau} \\
&= \left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu - u_\mu \pi^0_\nu u^j - u_\nu \pi^0_\mu u^j\right)\frac{du_j}{d\tau} - \left(u_\mu \pi^0_\nu + u_\nu \pi^0_\mu\right)\frac{(u_3)^2}{\tau^3}
\end{aligned} \tag{229}$$

This leads to

$$-\pi^{\alpha\beta}\frac{D\Delta_{\mu\nu\alpha\beta}}{D\tau} = \tau_\pi\left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu - u_\mu \pi^0_\nu u^j - u_\nu \pi^0_\mu u^j\right)\frac{du_j}{d\tau}. \tag{230}$$

We can then discuss the term $\tau_\pi \frac{D\pi_{\mu\nu}}{D\tau}$

$$\begin{aligned}
\tau_\pi\frac{D\pi_{\mu\nu}}{D\tau} &= \gamma\tau_\pi\frac{d\pi_{\mu\nu}}{d\tau} - \tau_\pi u^\lambda \Gamma^\sigma_{\mu\lambda}\pi_{\nu\sigma} - \tau_\pi u^\lambda \Gamma^\sigma_{\nu\lambda}\pi_{\mu\sigma} \\
&= \gamma\tau_\pi\frac{d\pi_{\mu\nu}}{d\tau} + \frac{\tau_\pi}{\tau^3}u_3\left(g_{\mu0}\pi_{\nu3} + g_{\nu0}\pi_{\mu3}\right) + \frac{\tau_\pi}{\tau}u_3\left(g_{\mu3}\pi_{\nu0} + g_{\nu3}\pi_{\mu0}\right) + \frac{\gamma\tau_\pi}{\tau^3}\left(g_{\mu3}\pi_{\nu3} + g_{\nu3}\pi_{\mu3}\right).
\end{aligned} \tag{231}$$

Thus,

$$\begin{aligned}
\tau_\pi\Delta_{\mu\nu\alpha\beta}\frac{D\pi^{\alpha\beta}}{D\tau} &= \gamma\tau_\pi\frac{d\pi_{\mu\nu}}{d\tau} + \frac{\tau_\pi}{\tau^3}u_3\left(g_{\mu0}\pi_{\nu3} + g_{\nu0}\pi_{\mu3}\right) + \frac{\tau_\pi}{\tau}u_3\left(g_{\mu3}\pi_{\nu0} + g_{\nu3}\pi_{\mu0}\right) + \frac{\gamma\tau_\pi}{\tau^3}\left(g_{\mu3}\pi_{\nu3} + g_{\nu3}\pi_{\mu3}\right) \\
&\quad + \tau_\pi\left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu - u_\mu \pi^0_\nu u^j - u_\nu \pi^0_\mu u^j\right)\frac{du_j}{d\tau}.
\end{aligned} \tag{232}$$

The final component $\tau_\pi\pi_{\mu\nu}\Theta$ can be rewritten as

$$\begin{aligned}
\tau_\pi\pi_{\mu\nu}\Theta &= \tau_\pi\pi_{\mu\nu}D_\alpha u^\alpha \\
&= \tau_\pi\pi_{\mu\nu}\left(\partial_\alpha u^\alpha + \frac{\gamma}{\tau}\right)
\end{aligned} \tag{233}$$

Returning to our original memory function for the shear viscosity Eq. (212)

$$\tau_\pi\Delta_{\mu\nu\alpha\beta}\frac{D\pi^{\alpha\beta}}{D\tau} + \pi_{\mu\nu} = \eta\sigma_{\mu\nu} - \tau_\pi\pi_{\mu\nu}\Theta \tag{234}$$

$$\begin{aligned}
\gamma\tau_\pi\frac{d\pi_{\mu\nu}}{d\tau} + \pi_{\mu\nu} + \frac{\tau_\pi}{\tau^3}u_3\left(g_{\mu0}\pi_{\nu3} + g_{\nu0}\pi_{\mu3}\right) &+ \frac{\tau_\pi}{\tau}u_3\left(g_{\mu3}\pi_{\nu0} + g_{\nu3}\pi_{\mu0}\right) + \frac{\gamma\tau_\pi}{\tau^3}\left(g_{\mu3}\pi_{\nu3} + g_{\nu3}\pi_{\mu3}\right) \\
+\tau_\pi\left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu - u_\mu \pi^0_\nu u^j - u_\nu \pi^0_\mu u^j\right)\frac{du_j}{d\tau} &= \frac{\eta}{2}\left[\partial_\mu u_\nu + \partial_\nu u_\mu\right] - \frac{\eta\gamma}{2}\left[u_\mu\frac{du_\nu}{d\tau} + u_\nu\frac{du_\mu}{d\tau}\right] - \frac{\eta}{3}\Delta_{\mu\nu}\left(\partial_\beta u^\beta + \frac{\gamma}{\tau}\right) \\
-\eta\gamma\tau g^3_\nu g^3_\mu - \eta\frac{u_3}{\tau}\left[g^3_\nu g^0_\mu + g^0_\nu g^3_\mu\right] &- \frac{\eta(u_3)^2}{2\tau^3}\left(u_\mu g^0_\nu + u_\nu g^0_\mu\right) - \tau_\pi\pi_{\mu\nu}\left(\partial_\alpha u^\alpha + \frac{\gamma}{\tau}\right)
\end{aligned} \tag{235}$$

$$\begin{aligned}
&\gamma\tau_\pi\frac{d\pi_{\mu\nu}}{d\tau} + \pi_{\mu\nu} + \tau_\pi\left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu - u_\mu \pi^0_\nu u^j - u_\nu \pi^0_\mu u^j\right)\frac{du_j}{d\tau} \\
&+\frac{\tau_\pi}{\tau^3}u_3\left(g_{\mu0}\pi_{\nu3} + g_{\nu0}\pi_{\mu3}\right) + \frac{\tau_\pi}{\tau}u_3\left(g_{\mu3}\pi_{\nu0} + g_{\nu3}\pi_{\mu0}\right) + \frac{\gamma\tau_\pi}{\tau^3}\left(g_{\mu3}\pi_{\nu3} + g_{\nu3}\pi_{\mu3}\right) \\
= &\ \frac{\eta}{2}\left[\partial_\mu u_\nu + \partial_\nu u_\mu\right] - \frac{\eta\gamma}{2}\left[u_\mu\frac{du_\nu}{d\tau} + u_\nu\frac{du_\mu}{d\tau}\right] - \frac{\eta}{3}\Delta_{\mu\nu}\left(\partial_\beta u^\beta + \frac{\gamma}{\tau}\right) \\
&-\eta\gamma\tau g^3_\nu g^3_\mu - \eta\frac{u_3}{\tau}\left[g^3_\nu g^0_\mu + g^0_\nu g^3_\mu\right] - \frac{\eta(u_3)^2}{2\tau^3}\left(u_\mu g^0_\nu + u_\nu g^0_\mu\right) - \tau_\pi\pi_{\mu\nu}\left(\partial_\alpha u^\alpha + \frac{\gamma}{\tau}\right)
\end{aligned} \tag{236}$$

so

$$
\begin{aligned}
& \gamma\tau_\pi \frac{d\pi_{\mu\nu}}{d\tau} + \pi_{\mu\nu} + \tau_\pi \left(\gamma u_\mu \pi^j_\nu + \gamma u_\nu \pi^j_\mu - u_\mu \pi^0_\nu u^j - u_\nu \pi^0_\mu u^j\right) \frac{du_j}{d\tau} \\
& + \tau_\pi \pi_{\mu\nu} \left(\frac{\gamma}{\tau} - \frac{(u_3)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& + \frac{\tau_\pi}{\tau^3} u_3 \left(g_{\mu 0}\pi_{\nu 3} + g_{\nu 0}\pi_{\mu 3}\right) + \frac{\tau_\pi}{\tau} u_3 \left(g_{\mu 3}\pi_{\nu 0} + g_{\nu 3}\pi_{\mu 0}\right) + \frac{\gamma\tau_\pi}{\tau^3}\left(g_{\mu 3}\pi_{\nu 3} + g_{\nu 3}\pi_{\mu 3}\right) \\
= & \ \frac{\eta}{2}\left[\partial_\mu u_\nu + \partial_\nu u_\mu\right] - \frac{\eta\gamma}{2}\left[u_\mu \frac{du_\nu}{d\tau} + u_\nu \frac{du_\mu}{d\tau}\right] - \frac{\eta}{3}\Delta_{\mu\nu}\left(\partial_\beta u^\beta + \frac{\gamma}{\tau}\right) \\
& - \eta\left[\frac{u_3}{\tau} g^3_\nu g^0_\mu + \frac{u_3}{\tau} g^0_\nu g^3_\mu + \gamma\tau g^3_\nu g^3_\mu\right] - \frac{\eta (u_3)^2}{2\tau^3}\left(u_\mu g^0_\nu + u_\nu g^0_\mu\right)
\end{aligned} \tag{237}
$$

Then taking $\mu = 0$ and $\nu = i$ we have

$$
\begin{aligned}
& \gamma\tau_\pi \frac{d\pi_{0i}}{d\tau} + \pi_{0i} + \tau_\pi \left(\gamma u_0 \pi^j_i + \gamma u_i \pi^j_0 - u_0 \pi^0_i u^j - u_i \pi^0_0 u^j\right) \frac{du_j}{d\tau} \\
& + \tau_\pi \pi_{0i} \left(\frac{\gamma}{\tau} - \frac{(u_3)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& + \frac{\tau_\pi}{\tau^3} u_3 \left(g_{00}\pi_{i3} + g_{i0}\pi_{03}\right) + \frac{\tau_\pi}{\tau} u_3 \left(g_{03}\pi_{i0} + g_{i3}\pi_{00}\right) + \frac{\gamma\tau_\pi}{\tau^3}\left(g_{03}\pi_{i3} + g_{i3}\pi_{03}\right) \\
= & \ \frac{\eta}{2}\left[\partial_0 u_i + \partial_i u_0\right] - \frac{\eta\gamma}{2}\left[u_0 \frac{du_i}{d\tau} + u_i \frac{du_0}{d\tau}\right] - \frac{\eta}{3}\Delta_{0i}\left(\partial_\beta u^\beta + \frac{\gamma}{\tau}\right) \\
& - \eta\left[\frac{u_3}{\tau} g^3_i g^0_0 + \frac{u_3}{\tau} g^0_i g^3_0 + \gamma\tau g^3_i g^3_0\right] - \frac{\eta (u_3)^2}{2\tau^3}\left(u_0 g^0_i + u_i g^0_0\right)
\end{aligned} \tag{238}
$$

so $u_0 = u^0 = \gamma$ and $g^i_0 = g_{0i} = g^{0i} = g^0_i = 0$

$$
\begin{aligned}
& \gamma\tau_\pi \frac{d\pi_{0i}}{d\tau} + \pi_{0i} + \tau_\pi \left(\gamma^2 \pi^j_i + \gamma u_i \pi^j_0 - \left(\frac{1}{\gamma} + \gamma\right) \pi^0_i u^j - u_i \pi^0_0 u^j\right) \frac{du_j}{d\tau} \\
& + \tau_\pi \pi_{0i} \left(\frac{\gamma}{\tau} - \frac{(u_3)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
& + \frac{\tau_\pi}{\tau^3} u_3 \pi_{i3} + \frac{\tau_\pi}{\tau} u_3 g_{i3}\pi_{00} + \frac{\gamma\tau_\pi}{\tau^3} g_{i3}\pi_{03} \\
= & \ \frac{\eta}{2}\left[\partial_0 u_i + \partial_i \gamma\right] - \frac{\eta\gamma}{2}\left[\gamma \frac{du_i}{d\tau} + u_i \frac{d\gamma}{d\tau}\right] + \frac{\eta\gamma}{3} u_i \left(\partial_\beta u^\beta + \frac{\gamma}{\tau}\right) \\
& - \frac{\eta (u_3)^2}{2\tau^3} u_i - \eta \frac{u_3}{\tau} g^3_i
\end{aligned} \tag{239}
$$

$$
\begin{aligned}
&\gamma\tau_\pi \frac{d\pi_{0i}}{d\tau} + \tau_\pi \left(\gamma^2 \pi_i^j + \gamma u_i \pi_0^j - \left(\frac{1}{\gamma} + \gamma \right) \pi_i^0 u^j - u_i \pi_0^0 u^j \right) \frac{du_j}{d\tau} \\
=\; & \frac{\eta}{2} \left[\partial_0 u_i + \partial_i \gamma \right] - \frac{\eta\gamma}{2} \left[\gamma \frac{du_i}{d\tau} + u_i \frac{d\gamma}{d\tau} \right] - \frac{\eta\gamma}{3} u_i \left(\frac{u^j}{\gamma} \frac{du_j}{d\tau} + \frac{(u_3)^2}{\gamma\tau^3} \right) \\
&+ \frac{\eta\gamma}{3} u_i \left(\frac{\gamma}{\tau} - \frac{3(u_3)^2}{2\gamma\tau^3} - \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} \right) \\
&+ \tau_\pi \pi_{0i} \left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi} - \frac{\gamma}{\tau} \right) \\
&+ \frac{\tau_\pi \pi_i^3}{\tau} u_3 + \left[\tau_\pi \tau u^3 \pi_{00} + \frac{\gamma\tau_\pi}{\tau} \pi_{03} - \frac{\eta u_3}{\tau} \right] g_i^3 \\
=\; & \frac{\eta}{2} \left[\partial_0 u_i + \partial_i \gamma \right] - \frac{\eta\gamma}{2} u_i \frac{d\gamma}{d\tau} - \frac{\eta}{3} \frac{u_i (u_3)^2}{\tau^3} \\
&- \frac{\eta\gamma^2}{2} \frac{du_i}{d\tau} - \frac{\eta}{3} u_i u^j \frac{du_j}{d\tau} \\
&+ \frac{\eta\gamma}{3} u_i \left(\frac{\gamma}{\tau} - \frac{3(u_3)^2}{2\gamma\tau^3} - \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} \right) \\
&+ \tau_\pi \pi_{0i} \left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi} - \frac{\gamma}{\tau} \right) \\
&+ \frac{\tau_\pi \pi_i^3}{\tau} u_3 + \left[\tau_\pi \tau u^3 \pi_{00} + \frac{\gamma\tau_\pi}{\tau} \pi_{03} - \frac{\eta u_3}{\tau} \right] g_i^3 \\
=\; & \frac{\eta}{2} \left[\partial_0 u_i + \partial_i \gamma \right] - \frac{\eta}{3} \frac{u_i (u_3)^2}{\tau^3} + \frac{\eta}{2} \frac{u_i (u_3)^2}{\tau^3} \\
&- \frac{\eta\gamma^2}{2} \frac{du_i}{d\tau} + \frac{\eta}{6} u_i u^j \frac{du_j}{d\tau} \\
&+ \frac{\eta\gamma}{3} u_i \left(\frac{\gamma}{\tau} - \frac{3(u_3)^2}{2\gamma\tau^3} - \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} \right) \\
&+ \tau_\pi \pi_{0i} \left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi} - \frac{\gamma}{\tau} \right) \\
&+ \frac{\tau_\pi \pi_i^3}{\tau} u_3 + \left[\tau_\pi \tau u^3 \pi_{00} + \frac{\gamma\tau_\pi}{\tau} \pi_{03} - \frac{\eta u_3}{\tau} \right] g_i^3
\end{aligned}
\tag{240}
$$

Using Eqs. (246,248), we find

$$
\begin{aligned}
&\gamma\tau_\pi \frac{d\pi_{0i}}{d\tau} \\
=\; & -\tau_\pi \left(\gamma^2 \pi_i^j + \gamma u_i \pi_0^j - \left(\frac{1}{\gamma} + \gamma \right) \pi_i^0 u^j - u_i \pi_0^0 u^j \right) \frac{du_j}{d\tau} \\
&- \frac{\eta}{2} v^j \left[\partial_j u_i + \partial_i u_j \right] \\
&+ \frac{\eta}{2} \left(1 - \gamma^2 \right) \frac{du_i}{d\tau} + \frac{\eta}{6} u_i u^j \frac{du_j}{d\tau} \\
&+ \frac{\eta\gamma}{3} u_i \left(\frac{\gamma}{\tau} - \frac{3(u_3)^2}{2\gamma\tau^3} - \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} \right) \\
&+ \tau_\pi \pi_{0i} \left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*} \frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau} - \frac{1}{\tau_\pi} \right) \\
&+ \frac{\tau_\pi \pi_i^3}{\tau} u_3 + \left[\tau_\pi \tau u^3 \pi_{00} + \frac{\gamma\tau_\pi}{\tau} \pi_{03} - \frac{\eta u_3}{\tau} \right] g_i^3
\end{aligned}
\tag{241}
$$

E. Entropy Production

Using Eqs. (56,64), we can write the entropy as

$$TD_\mu(su^\mu) = \pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\Theta \tag{242}$$

in hyperbolic coordinates. Then,

$$\begin{aligned}
TD_\mu(su^\mu) &= \pi^{\mu\nu}\sigma_{\mu\nu} - \Pi D_\mu u^\mu \\
&= \pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right) \\
T\sigma\frac{D}{D\tau}\left(\frac{s}{\sigma}\right) &= \pi^{\mu\nu}\sigma_{\mu\nu} - \Pi\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right) \\
\frac{d}{d\tau}\left(\frac{s}{\sigma}\right) &= \frac{\pi^{\mu\nu}}{T\gamma\sigma}\sigma_{\mu\nu} - \frac{\Pi}{T\gamma\sigma}\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right) \\
\frac{1}{\sigma}\frac{ds}{d\tau} + \frac{s}{\sigma}\frac{d\sigma}{d\tau} &= \\
\frac{ds}{d\tau} + \frac{s}{\gamma}D_\mu u^\mu &= \frac{\pi^{\mu\nu}}{T\gamma}\sigma_{\mu\nu} - \frac{\Pi}{T\gamma}\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right) \\
\frac{ds}{d\tau} &= \frac{\pi^{\mu\nu}}{T\gamma}\sigma_{\mu\nu} - \frac{1}{\gamma}\left(\frac{\Pi}{T} + s\right)\left(\partial_\mu u^\mu + \frac{\gamma}{\tau}\right)
\end{aligned} \tag{243}$$

However, $\sigma_{\mu\nu} = \Delta_{\mu\nu\alpha\beta}D^\alpha u^\beta$ such that

$$\begin{aligned}
\frac{\pi^{\mu\nu}}{T\gamma}\sigma_{\mu\nu} &= \frac{\Delta^{\mu\nu}_{\alpha\beta}\pi^{\alpha\beta}}{T\gamma}D_\mu u_\nu \\
&= \frac{\pi^{\mu\nu}}{T\gamma}D_\mu u_\nu \\
&= \frac{\pi^{\mu\nu}}{T\gamma}\left(\partial_\mu u_\nu - \Gamma^\lambda_{\mu\nu}u_\lambda\right) \\
&= \frac{\pi^{\mu\nu}}{T\gamma}\partial_\mu u_\nu - \frac{\tau\pi^{33}\gamma}{T} - \frac{2\pi^{03}u_3}{T\gamma}
\end{aligned} \tag{244}$$

where $\frac{\pi^{\mu\nu}}{T\gamma}\partial_\mu u_\nu$

$$\frac{\pi^{\mu\nu}}{T\gamma}\partial_\mu u_\nu = \frac{\pi^{00}}{T\gamma}\partial_0\gamma + \frac{\pi^{i0}}{T\gamma}\partial_i\gamma + \frac{\pi^{0i}}{T\gamma}\partial_0 u_i + \frac{\pi^{ij}}{T\gamma}\partial_i u_j \tag{245}$$

Recall that $u^\mu = (\gamma, \gamma\vec{v})$ so that $\gamma = \frac{1}{\sqrt{1-\vec{v}^2}}$ and

$$u^\mu\partial_\mu = \gamma\frac{d}{d\tau} = \gamma\partial_0 + \gamma v^i\partial_i. \tag{246}$$

Thus,

$$\begin{aligned}
\pi^{\mu\nu}\partial_\mu u_\nu &= \pi^{00}\left(\frac{d\gamma}{d\tau} - v^j\partial_j\gamma\right) + \pi^{i0}\partial_i\gamma + \pi^{0i}\left(\frac{du_i}{d\tau} - v^j\partial_j u_i\right) + \pi^{ij}\partial_i u_j \\
&= \left(\pi^{0k} - \pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau} - \pi^{00}\frac{(u_3)^2}{\gamma\tau^3} + \left[\pi^{i0} - \pi^{00}v^j\right]\partial_i\gamma + \left(\pi^{ij} - \pi^{0j}v^i\right)\partial_i u_j.
\end{aligned} \tag{247}$$

Before we continue it is useful to show that $\partial_j u_i = -v^n\partial_j u_n$

$$\begin{aligned}
\partial_j\left(u^m u_m\right) &= 0 \\
u^m\partial_j u_m &= 0 \\
\gamma\partial_j\gamma + u^n\partial_j u_n &= 0 \\
\gamma\partial_j\gamma &= -u^n\partial_j u_n \\
&= -\gamma v^n\partial_j u_n \\
\partial_j\gamma &= -v^n\partial_j u_n
\end{aligned} \tag{248}$$

so that

$$\pi^{\mu\nu}\partial_\mu u_\nu \;=\; \left(\pi^{0k}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}-\pi^{00}\frac{(u_3)^2}{\gamma\tau^3}+\left(\pi^{ij}+\pi^{00}v^iv^j-\pi^{i0}v^j-\pi^{0j}v^i\right)\partial_i u_j . \tag{249}$$

Thus,

$$\pi^{\mu\nu}D_\mu u_\nu = \left(\pi^{0k}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}-\pi^{00}\frac{(u_3)^2}{\gamma\tau^3}+\left(\pi^{ij}+\pi^{00}v^iv^j-\pi^{i0}v^j-\pi^{0j}v^i\right)\partial_i u_j-\pi^{33}\tau\gamma-\pi^{03}\frac{2u_3}{\tau} \tag{250}$$

and

$$\begin{aligned}\frac{ds}{d\tau} \;&=\; -\frac{1}{\gamma}\left(\frac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\\ &+\frac{1}{T\gamma}\left[\left(\pi^{0k}-\pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau}-\pi^{00}\frac{(u_3)^2}{\gamma\tau^3}+\left(\pi^{ij}+\pi^{00}v^iv^j-\pi^{i0}v^j-\pi^{0j}v^i\right)\partial_i u_j-\pi^{33}\tau\gamma-\pi^{03}\frac{2u_3}{\tau}\right]\end{aligned} \tag{251}$$

F. Momentum Conservation

The equation of motion in hyperbolic coordinates including shear viscosity is

$$\sigma\gamma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u_i\right)+\frac{1}{\tau}\partial^\mu\left(\tau\pi_{\mu i}\right)=\partial_i\left(p+\Pi\right) \tag{252}$$

where

$$\sigma\gamma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u_i\right) \;=\; \gamma\frac{\epsilon+p+\Pi}{\sigma}\frac{du_i}{d\tau}-u_i\frac{\epsilon+p+\Pi}{\sigma}\frac{\gamma}{\sigma}\frac{d\sigma}{d\tau}+u_i\gamma\frac{d}{d\tau}\left(\epsilon+p\right)+u_i\gamma\frac{d\Pi}{d\tau} \tag{253}$$

We can then substitute in Eqs. (191,192), Eq. (202), Eq. (243)

$$\begin{aligned}\frac{\gamma}{\sigma}\frac{d\sigma}{d\tau}=D_\mu u^\mu \;&=\; \partial_\mu u^\mu+\frac{\gamma}{\tau}\\ \gamma\frac{d\Pi}{d\tau}=\frac{D\Pi}{D\tau} \;&=\; -\frac{\Pi}{\tau_\pi}-\frac{1}{\tau_\pi}\left(\zeta+\tau_\pi\Pi\right)\left[\partial_\mu u^\mu+\frac{\gamma}{\tau}\right]\\ \frac{ds}{d\tau} \;&=\; \frac{\pi^{\mu\nu}}{T\gamma}\sigma_{\mu\nu}-\frac{1}{\gamma}\left(\frac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\end{aligned} \tag{254}$$

so that

$$\begin{aligned}\sigma\gamma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u_i\right) \;&=\; \gamma\frac{\epsilon+p+\Pi}{\sigma}\frac{du_i}{d\tau}-u_i\frac{\epsilon+p+\Pi}{\sigma}\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\\ &+u_i\gamma\frac{d}{d\tau}\left(\epsilon+p\right)\\ &-u_i\frac{\Pi}{\tau_\pi}-\frac{u_i}{\tau_\pi}\left(\zeta+\tau_\pi\Pi\right)\left[\partial_\mu u^\mu+\frac{\gamma}{\tau}\right]\end{aligned} \tag{255}$$

In the Appendix A we show that

$$\gamma\frac{d}{d\tau}\left(\epsilon+p\right)=\frac{dw}{ds}\left[-\left(\frac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)-\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu\right], \tag{256}$$

which means we then have

$$
\begin{aligned}
\sigma\gamma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u_i\right) &= \gamma\left(\epsilon+p+\Pi\right)\frac{du_i}{d\tau}-u_i\left(\epsilon+p+\Pi\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\\
&\quad +u_i\frac{dw}{ds}\left[-\left(\frac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)-\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu\right]\\
&\quad -u_i\frac{\Pi}{\tau_\pi}-\frac{u_i}{\tau_\pi}\left(\zeta+\tau_\pi\Pi\right)\left[\partial_\mu u^\mu+\frac{\gamma}{\tau}\right]\\
&= \gamma\left(\epsilon+p+\Pi\right)\frac{du_i}{d\tau}-u_i\frac{\Pi}{\tau_\pi}+u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu\\
&\quad +u_i\left[\epsilon+p-\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)-\frac{\zeta}{\tau_\pi}\right]\left[\partial_\mu u^\mu+\frac{\gamma}{\tau}\right]
\end{aligned}
\tag{257}
$$

Let

$$
A=\epsilon+p-\frac{dw}{ds}\left(\frac{\Pi}{T}+s\right)-\frac{\zeta}{\tau_\pi}
\tag{258}
$$

then,

$$
\begin{aligned}
\sigma\gamma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u_i\right) &= \gamma\left(\epsilon+p+\Pi\right)\frac{du_i}{d\tau}-u_i\frac{\Pi}{\tau_\pi}+u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu\\
&\quad +u_iA\left[\partial_\mu u^\mu+\frac{\gamma}{\tau}\right]\\
&= \gamma\left(\epsilon+p+\Pi\right)\frac{du_i}{d\tau}-u_i\frac{\Pi}{\tau_\pi}+u_i\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu\\
&\quad -u_iA\left[\frac{u^j}{\gamma}\frac{du_j}{d\tau}+\frac{\left(u_3\right)^2}{\gamma\tau^3}+\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}-\frac{\gamma}{\tau}\right]..
\end{aligned}
\tag{259}
$$

Returning to Eq. (264) and looking at the term $\frac{1}{\tau}\partial^\mu\left(\tau\pi_{\mu i}\right)$

$$
\frac{1}{\tau}\partial^\mu\left(\tau\pi_{\mu i}\right)=\partial^\mu\pi_{\mu i}+\frac{\pi_{\mu i}}{\tau}\partial^\mu\tau
\tag{260}
$$

Here we can separate ∂^μ into its components

$$
\partial^\mu=\partial^0+\partial^j.
\tag{261}
$$

Then,

$$
\begin{aligned}
\frac{1}{\tau}\partial^\mu\left(\tau\pi_{\mu i}\right) &= \partial^0\pi_{0i}+\partial^j\pi_{ji}+\frac{\pi_{0i}}{\tau}\underbrace{\partial^0\tau}_{=1}+\frac{\pi_{ji}}{\tau}\underbrace{\partial^j\tau}_{=0}\\
&= \partial^0\pi_{0i}+\partial^j\pi_{ji}+\frac{\pi_{0i}}{\tau}\\
&= \frac{d\pi_{0i}}{d\tau}-v^j\partial_j\pi_{0i}+\partial^j\pi_{ji}+\frac{\pi_{0i}}{\tau}
\end{aligned}
\tag{262}
$$

Substituting in Eq. (241)

$$
\begin{aligned}
\frac{1}{\tau}\partial^{\mu}\left(\tau\pi_{\mu i}\right) \;=\;& -v^{j}\partial_{j}\pi_{0i}+\partial^{j}\pi_{ji}+\left(\gamma\pi_{i}^{j}+u_{i}\pi_{0}^{j}-\left(\frac{1}{\gamma}+\gamma\right)\frac{\pi_{i}^{0}}{\gamma}u^{j}-u_{i}\frac{\pi_{0}^{0}}{\gamma}u^{j}\right)\frac{du_{j}}{d\tau}\\
&-\frac{\eta}{2\tau_{\pi}\gamma}v^{j}\left[\partial_{j}u_{i}+\partial_{i}u_{j}\right]\\
&+\frac{\eta}{2\tau_{\pi}}\left(\frac{1}{\gamma}-\gamma\right)\frac{du_{i}}{d\tau}+\frac{\eta}{6\tau_{\pi}\gamma}u_{i}u^{j}\frac{du_{j}}{d\tau}\\
&+\frac{\eta}{3\tau_{\pi}}u_{i}\left(\frac{\gamma}{\tau}-\frac{3(u_{3})^{2}}{2\tau^{3}}-\frac{1}{\sigma^{*}}\frac{d\sigma^{*}}{d\tau}\right)\\
&+\frac{\pi_{0i}}{\gamma}\left(\frac{(u_{3})^{2}}{\gamma\tau^{3}}+\frac{\gamma}{\sigma^{*}}\frac{d\sigma^{*}}{d\tau}-\frac{1}{\tau_{\pi}}\right)\\
&+\frac{\pi_{i}^{3}}{\gamma\tau}u_{3}+\left[\frac{\tau u^{3}}{\gamma}\pi_{00}+\frac{1}{\tau}\pi_{03}-\frac{\eta u_{3}}{\tau\tau_{\pi}\gamma}\right]g_{i}^{3}
\end{aligned}
\tag{263}
$$

Returning to momentum conservation, we can substitute in Eq. (259)

$$
\sigma\gamma\frac{d}{d\tau}\left(\frac{\epsilon+p+\Pi}{\sigma}u_{i}\right)+\frac{1}{\tau}\partial^{\mu}\left(\tau\pi_{\mu i}\right)=\partial_{i}\left(p+\Pi\right)
\tag{264}
$$

and substituting in Eqs. (259,263)

$$
\begin{aligned}
\partial_{i}\left(p+\Pi\right) \;=\;& \frac{d\pi_{0i}}{d\tau}-v^{j}\partial_{j}\pi_{0i}+\partial^{j}\pi_{ji}+\frac{\pi_{0i}}{\tau}+\gamma\left(\epsilon+p+\Pi\right)\frac{du_{i}}{d\tau}-u_{i}\frac{\Pi}{\tau_{\pi}}+u_{i}\frac{dw}{ds}\frac{\pi^{\mu\nu}}{T}D_{\mu}u_{\nu}\\
&-u_{i}A\left[\frac{u^{j}}{\gamma}\frac{du_{j}}{d\tau}+\frac{\left(u_{3}\right)^{2}}{\gamma\tau^{3}}+\frac{\gamma}{\sigma^{*}}\frac{d\sigma^{*}}{d\tau}-\frac{\gamma}{\tau}\right]\\
=\;& \frac{d\pi_{0i}}{d\tau}-v^{j}\partial_{j}\pi_{0i}+\partial^{j}\pi_{ji}+\frac{\pi_{0i}}{\tau}+\gamma\left(\epsilon+p+\Pi\right)\frac{du_{i}}{d\tau}-u_{i}\frac{\Pi}{\tau_{\pi}}+\frac{u_{i}}{T}\frac{dw}{ds}\left(\pi^{0k}-\pi^{00}\frac{u^{j}}{\gamma}\right)\frac{du_{j}}{d\tau}\\
&+\frac{u_{i}}{T}\frac{dw}{ds}\left[-\pi^{00}\frac{\left(u_{3}\right)^{2}}{\gamma\tau^{3}}+\left(\pi^{ij}+\pi^{00}v^{i}v^{j}-\pi^{i0}v^{j}-\pi^{0j}v^{i}\right)\partial_{i}u_{j}-\pi^{33}\tau\gamma-\pi^{03}\frac{2u_{3}}{\tau}\right]\\
&-u_{i}A\left[\frac{u^{j}}{\gamma}\frac{du_{j}}{d\tau}+\frac{\left(u_{3}\right)^{2}}{\gamma\tau^{3}}+\frac{\gamma}{\sigma^{*}}\frac{d\sigma^{*}}{d\tau}-\frac{\gamma}{\tau}\right]
\end{aligned}
\tag{265}
$$

where $\frac{d\pi_{0i}}{d\tau}$ can be found in Eq. (241). Then

$$\begin{aligned}
\partial_i (p+\Pi) &= \gamma(\epsilon+p+\Pi)\frac{du_i}{d\tau} - u_i\frac{\Pi}{\tau_\pi} + \frac{u_i}{T}\frac{dw}{ds}\left(\pi^{0k} - \pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau} \\
&+\frac{u_i}{T}\frac{dw}{ds}\left[-\pi^{00}\frac{(u_3)^2}{\gamma\tau^3} + \left(\pi^{ij} + \pi^{00}v^i v^j - \pi^{i0}v^j - \pi^{0j}v^i\right)\partial_i u_j - \pi^{33}\tau\gamma - \pi^{03}\frac{2u_3}{\tau}\right] \\
&-u_i A\left[\frac{u^j}{\gamma}\frac{du_j}{d\tau} + \frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right] \\
&-v^j\partial_j\pi_{0i} + \partial^j\pi_{ji} - \left(\gamma\pi_i^j + u_i\pi_0^j - \left(\frac{1}{\gamma}+\gamma\right)\frac{\pi_i^0}{\gamma}u^j - u_i\frac{\pi_0^0}{\gamma}u^j\right)\frac{du_j}{d\tau} \\
&-\frac{\eta}{2\tau_\pi\gamma}v^j\left[\partial_j u_i + \partial_i u_j\right] \\
&+\frac{\eta}{2\tau_\pi}\left(\frac{1}{\gamma}-\gamma\right)\frac{du_i}{d\tau} + \frac{\eta}{6\tau_\pi\gamma}u_i u^j\frac{du_j}{d\tau} \\
&+\frac{\eta}{3\tau_\pi}u_i\left(\frac{\gamma}{\tau} - \frac{3(u_3)^2}{2\tau^3} - \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
&+\frac{\pi_{0i}}{\gamma}\left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi}\right) \\
&+\frac{\pi_i^3}{\gamma\tau}u_3 + \left[\frac{\tau u^3}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_3}{\tau\tau_\pi\gamma}\right]g_i^3
\end{aligned} \tag{266}$$

G. Force Equation

Using Eq. (266), we first separate out the acceleration terms:

$$\begin{aligned}
\gamma(\epsilon+p+\Pi)\frac{du_j}{d\tau} - u_i A\frac{u^j}{\gamma}\frac{du_j}{d\tau} - \left(\gamma\pi_i^j + u_i\pi_0^j - \left(\frac{1}{\gamma^2}+1\right)\pi_i^0 u^j - \pi_0^0 u_i u^j\right)\frac{du_j}{d\tau} + \frac{u_i}{T}\frac{dw}{ds}\left(\pi^{0k} - \pi^{00}\frac{u^j}{\gamma}\right)\frac{du_j}{d\tau} \\
+\frac{\eta}{2\tau_\pi}\left(\frac{1}{\gamma}-\gamma\right)\frac{du_j}{d\tau} + \frac{\eta}{6\gamma\tau_\pi}u_i u^j\frac{du_j}{d\tau}
\end{aligned} \tag{267}$$

so that we write

$$M_i^j\frac{du_j}{d\tau} \tag{268}$$

where

$$M_i^j = \gamma C_{tot} g_j^i + A_{tot} u^j u_i + m_i^j \tag{269}$$

and

$$\begin{aligned}
C_{tot} &= \epsilon + p + \Pi - \frac{\eta}{2\tau_\pi}\left(\frac{1}{\gamma^2} - 1\right) \\
A_{tot} &= \frac{1}{\gamma}\left[\frac{\eta}{6\tau_\pi} - \frac{1}{T}\frac{dw}{ds}\pi^{00} + \pi_0^0 - A\right] \\
m_i^j &= \frac{1}{T}\frac{dw}{ds}\pi^{0j}u_i - \gamma\pi_i^j - u_i\pi_0^j + \left(\frac{1}{\gamma^2}+1\right)\pi_i^0 u^j
\end{aligned} \tag{270}$$

Returning to Eq. (266), we can take the remaining force terms

$$
\begin{aligned}
&\partial_i (p+\Pi) + u_i \frac{\Pi}{\tau_\pi} - \frac{u_i}{T}\frac{dw}{ds}\left(\pi^{ij} + \pi^{00}v^i v^j - \pi^{i0}v^j - \pi^{0j}v^i\right)\partial_i u_j + \frac{u_i}{T}\frac{dw}{ds}\left[\pi^{00}\frac{(u_3)^2}{\gamma\tau^3} + \pi^{33}\tau\gamma + \pi^{03}\frac{2u_3}{\tau}\right] \\
&+u_i A\left[\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right] \\
&+v^j\partial_j\pi_{0i} - \partial^j\pi_{ji} + \frac{\eta}{2\tau_\pi\gamma}v^j\left[\partial_j u_i + \partial_i u_j\right] \\
&-\frac{\eta}{3\tau_\pi}u_i\left(\frac{\gamma}{\tau} - \frac{3(u_3)^2}{2\tau^3} - \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
&-\frac{\pi_{0i}}{\gamma}\left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi}\right) \\
&-\frac{\pi_i^3}{\gamma\tau}u_3 - \left[\frac{\tau u^3}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_3}{\tau\tau_\pi\gamma}\right]g_i^3
\end{aligned}
\tag{271}
$$

so that

$$
\begin{aligned}
&\partial_i (p+\Pi) + u_i \frac{\Pi}{\tau_\pi} + v^j\partial_j\pi_{0i} - \partial^j\pi_{ji} + \frac{\eta}{2\tau_\pi\gamma}v^j\left[\partial_j u_i + \partial_i u_j\right] \\
&+\frac{u_i}{T}\frac{dw}{ds}\left[\pi^{00}\frac{(u_3)^2}{\gamma\tau^3} + \pi^{33}\tau\gamma + \pi^{03}\frac{2u_3}{\tau}\right] \\
&-\frac{u_i}{T}\frac{dw}{ds}\left(\pi^{ij} + \pi^{00}v^i v^j - \pi^{i0}v^j - \pi^{0j}v^i\right)\partial_i u_j \\
&+u_i A\left[\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right] + \frac{\eta}{3\tau_\pi}u_i\left[\frac{3(u_3)^2}{2\tau^3} + \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right] \\
&-\frac{\pi_{0i}}{\gamma}\left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi}\right) \\
&-\frac{\pi_i^3}{\gamma\tau}u_3 - \left[\frac{\tau u^3}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_3}{\tau\tau_\pi\gamma}\right]g_i^3 .
\end{aligned}
\tag{272}
$$

We can then write our parameterized equation of motion

$$
M_i^j\frac{du_j}{d\tau} = B_{tot}u_i + F_i + \partial_i(p+\Pi) + v^j\partial_j\pi_{0i} - \partial^j\pi_{ji}
\tag{273}
$$

where

$$
\begin{aligned}
B_{tot} &= \frac{\Pi}{\tau_\pi} + \frac{1}{T}\frac{dw}{ds}\left[\pi^{00}\frac{(u_3)^2}{\gamma\tau^3} + \pi^{33}\tau\gamma + \pi^{03}\frac{2u_3}{\tau}\right] - \frac{1}{T}\frac{dw}{ds}\left(\pi^{ij} + \pi^{00}v^i v^j - \pi^{i0}v^j - \pi^{0j}v^i\right)\partial_i u_j \\
&\quad +A\left[\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right] + \frac{\eta}{3\tau_\pi}u_i\left[\frac{3(u_3)^2}{2\tau^3} + \frac{1}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right] \\
F_i &= \frac{\eta}{2\tau_\pi\gamma}v^j\left[\partial_j u_i + \partial_i u_j\right] - \frac{\pi_{0i}}{\gamma}\left(\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{1}{\tau_\pi}\right) - \frac{\pi_i^3}{\gamma\tau}u_3 - \left[\frac{\tau u^3}{\gamma}\pi_{00} + \frac{1}{\tau}\pi_{03} - \frac{\eta u_3}{\tau\tau_\pi\gamma}\right]g_i^3
\end{aligned}
\tag{274}
$$

H. Shear Viscosity Components

As mentioned before the shear stress tensor is tracelss,

$$
\pi_\mu^\mu = 0
\tag{275}
$$

symmetric,

$$\pi^{\mu\nu} = \pi^{\nu\mu} \tag{276}$$

and we have the orthogonality relation

$$u_\mu \pi^{\mu\nu} = 0. \tag{277}$$

Between those three we can find relations between the components of the shear stress tensor that help to decrease the number of equations that we will eventually need to solve. First, using the symmetry we can immediately eliminate 6 components that we do not need to solve

$$\begin{aligned}
\pi^{01} &= \pi^{10} \\
\pi^{02} &= \pi^{20} \\
\pi^{03} &= \pi^{30} \\
\pi^{12} &= \pi^{21} \\
\pi^{13} &= \pi^{31} \\
\pi^{23} &= \pi^{32}.
\end{aligned} \tag{278}$$

Next, because the shear stress tensor is traceless we can further elimminate one more component

$$\begin{aligned}
g_{\mu\nu} &= 0 \\
\pi^{00} - \pi^{11} - \pi^{22} - \tau^2 \pi^{33} &= 0 \\
\pi^{33} = \frac{1}{\tau^2}\left(\pi^{00} - \pi^{11} - \pi^{22}\right).
\end{aligned} \tag{279}$$

Finally, we can eliminate four more components using the orthogonality relation

$$\begin{aligned}
u_\mu \pi^{\mu\nu} &= 0 \\
u_0 \pi^{0\nu} + u_i \pi^{i\nu} &= 0 \\
\pi^{0\nu} &= -\frac{u_\nu}{\gamma}\pi^{i\nu} \\
\pi^{0j} &= -\frac{u_j}{\gamma}\pi^{ij}
\end{aligned} \tag{280}$$

and

$$\begin{aligned}
u_\nu u_\mu \pi^{\mu\nu} &= 0 \\
\gamma^2 \pi^{00} + u_i u_j \pi^{ij} + u_i u_0 \pi^{i0} + u_0 u_j \pi^{0j} &= 0 \\
\gamma^2 \pi^{00} + u_i u_j \pi^{ij} - u_i u_0 \frac{u_j}{\gamma}\pi^{ij} - u_0 u_j \frac{u_i}{\gamma}\pi^{ij} &= 0 \\
\gamma^2 \pi^{00} + u_i u_j \pi^{ij} &= 0 \\
\pi^{00} &= -\frac{u_i u_j}{\gamma^2}\pi^{ij}.
\end{aligned} \tag{281}$$

Then we can combine all 5 components into one equation

$$
\begin{aligned}
\pi^{00} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{u^i u^j}{\gamma^2}\pi_{ij} \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{v^i v^j}{\gamma^2}\pi_{ij} + 2\frac{v^3}{\gamma^2}\sum_{j=1}^{2}v^j\pi_{3j} + \left(\frac{v^3}{\gamma^2}\right)^2\pi_{33} \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{v^i v^j}{\gamma^2}\pi_{ij} + 2\frac{v^3}{\gamma^2}\sum_{j=1}^{2}v^j\pi_{3j} + \left(\frac{v^3\tau}{\gamma^2}\right)^2\left(\pi^{00}-\pi^{11}-\pi^{22}\right) \\
\pi^{00}\left[1-\left(\frac{v^3\tau}{\gamma}\right)^2\right] &= \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{v^i v^j}{\gamma^2}\pi_{ij} + 2\frac{v^3}{\gamma^2}\sum_{j=1}^{2}v^j\pi_{3j} - \left(\frac{v^3\tau}{\gamma^2}\right)^2\left(\pi^{11}+\pi^{22}\right) \\
\pi^{00} &= \frac{1}{1-\left(\frac{v^3\tau}{\gamma}\right)^2}\left[\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{v^i v^j}{\gamma^2}\pi_{ij} + 2\frac{v^3}{\gamma^2}\sum_{j=1}^{2}v^j\pi_{3j} - \left(\frac{v^3\tau}{\gamma^2}\right)^2\left(\pi^{11}+\pi^{22}\right)\right]
\end{aligned}
\tag{282}
$$

I. Solving Shear Viscosity

Returning to Eq. (237),

$$
\begin{aligned}
\frac{d\pi_{\mu\nu}}{d\tau} &= \frac{\eta}{2\gamma\tau_\pi}\left[\partial_\mu u_\nu + \partial_\nu u_\mu\right] - \frac{\eta}{2\tau_\pi}\left[u_\mu\frac{du_\nu}{d\tau} + u_\nu\frac{du_\mu}{d\tau}\right] - \frac{\eta}{3\gamma\tau_\pi}\Delta_{\mu\nu}\left(\partial_\beta u^\beta + \frac{\gamma}{\tau}\right) \\
&\quad -\frac{\eta}{\gamma\tau_\pi}\left[\frac{u_3}{\tau}g^3_\nu g^0_\mu + \frac{u_3}{\tau}g^0_\nu g^3_\mu + \gamma\tau g^3_\nu g^3_\mu\right] - \frac{\eta(u_3)^2}{2\tau^3\gamma\tau_\pi}\left(u_\mu g^0_\nu + u_\nu g^0_\mu\right) \\
&\quad -\frac{1}{\gamma\tau_\pi}\pi_{\mu\nu} - \frac{1}{\gamma}\left(\gamma u_\mu\pi^j_\nu + \gamma u_\nu\pi^j_\mu - u_\mu\pi^0_\nu u^j - u_\nu\pi^0_\mu u^j\right)\frac{du_j}{d\tau} \\
&\quad -\frac{\pi_{\mu\nu}}{\gamma}\left(\frac{\gamma}{\tau} - \frac{(u_3)^2}{\gamma\tau^3} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau}\right) \\
&\quad -\frac{1}{\tau^3\gamma}u_3\left(g_{\mu 0}\pi_{\nu 3} + g_{\nu 0}\pi_{\mu 3}\right) - \frac{1}{\tau\gamma}u_3\left(g_{\mu 3}\pi_{\nu 0} + g_{\nu 3}\pi_{\mu 0}\right) - \frac{1}{\tau^3}\left(g_{\mu 3}\pi_{\nu 3} + g_{\nu 3}\pi_{\mu 3}\right)
\end{aligned}
\tag{283}
$$

we can then separate out the transverse and longitudinal components.

Transeverse components ($i = 1, 2$ and $j = 1, 2$):

$$
\begin{aligned}
\frac{d\pi_{ij}}{d\tau} &= -\frac{\pi_{ij}}{\gamma\tau_\pi} + \frac{\eta}{2\gamma\tau_\pi}\left[\partial_i u_j + \partial_j u_i\right] - \frac{\eta}{2\tau_\pi}\left[u_i\frac{du_j}{d\tau} + u_j\frac{du_i}{d\tau}\right] \\
&\quad - \left(u_i\pi^k_j + u_j\pi^k_\mu\right)\frac{du_k}{d\tau} + \left(u_i\pi^0_j + u_j\pi^0_i + \frac{\eta}{3\gamma\tau_\pi}\Delta_{ij}\right)v^k\frac{du_k}{d\tau} \\
&\quad + \left[\frac{\pi_{ij}}{\gamma} + \frac{\eta}{3\gamma\tau_\pi}\Delta_{ij}\right]\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{(u_3)^2}{\gamma\tau^3} - \frac{\gamma}{\tau}\right)
\end{aligned}
\tag{284}
$$

Longitudinale components ($i = 1, 2$ and $j = 3$):

$$
\begin{aligned}
\frac{d\pi_{i3}}{d\tau} &= \frac{\eta}{2\gamma\tau_\pi}\left[\partial_i u_3 + \partial_3 u_i\right] - \frac{\eta}{2\tau_\pi}\left[u_i\frac{du_3}{d\tau} + u_3\frac{du_i}{d\tau}\right] \\
&\quad - \left(u_i\pi^k_3 + u_3\pi^k_i\right)\frac{du_k}{d\tau} - \left(u_i\pi^0_3 - u_3\pi^0_i - \frac{\eta}{3\gamma\tau_\pi}u_i u_3\right)v^k\frac{du_k}{d\tau} \\
&\quad + \left[\frac{\pi_{i3}}{\gamma} - \frac{\eta}{3\gamma\tau_\pi}u_i u_3\right]\left(\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} + \frac{(u_3)^2}{\gamma\tau^3} - \frac{\gamma}{\tau}\right) \\
&\quad + \frac{\tau}{\gamma}u_3\pi_{i0} + \left[\frac{1}{\tau} - \frac{1}{\gamma\tau_\pi}\right]\pi_{i3}
\end{aligned}
\tag{285}
$$

J. Ideal Limit

We can now take Eq. (269) and Eq. (273) for the ideal case where

$$\left[\gamma(\epsilon+p)g^i_j + \frac{-A}{\gamma}u^j u_i\right]\frac{du_j}{d\tau} = A\left[\frac{(u_3)^2}{\gamma\tau^3} + \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{\gamma}{\tau}\right]u_i + \partial_i p \tag{286}$$

Setting the bulk viscosity to zero we can check the ideal equations:

$$M^{lmi}_\alpha \frac{du^i_\alpha}{d\tau} = F^i_\alpha \tag{287}$$

$$\frac{d}{dt}\left(\frac{s}{\sigma}\right)_\alpha = 0 \tag{288}$$

$$M^{lmi}_\alpha \equiv (\epsilon_\alpha + p_\alpha) - \frac{A_\alpha}{\gamma_\alpha}u^l_\alpha u^m_\alpha g_{mi} \tag{289}$$

$$F^i_\alpha \equiv A_\alpha u^i_\alpha \frac{\gamma_\alpha}{\sigma^*_\alpha}\frac{d\sigma^*_\alpha}{dt} + \partial^i p_\alpha \tag{290}$$

where

$$A_\alpha \equiv \epsilon_\alpha + p_\alpha - s_\alpha \frac{d}{ds_\alpha}(\epsilon_\alpha + p_\alpha) \tag{291}$$

V. INITIAL CONDITIONS

In the code the initial conditions -given in terms of an energy density profile- need to be converted into the SPH method and also must be converted into the term $\left(\frac{s}{\sigma}\right)_\alpha$ as seen in the sigma calculation from Eq. (138)

$$s^* = \gamma t s = \sum_{\alpha=1}^{N_{SPH}} \nu_\alpha \left(\frac{s}{\sigma}\right)_\alpha W(|\mathbf{r} - \mathbf{r}_\alpha(t)|) \tag{292}$$

where s^* is the lab frame and s is in the rest frame. In our calculations $\nu_\alpha = 1$ so we need to obtain $\left(\frac{s}{\sigma}\right)_\alpha$, which is constant over time and is vital for calculating the thermodynamic quantities through out the hydrodynamical evolution. We can already use $s^{SPH}_{alpha} = s^{an}(t,x,y)$ where $s^{an}(t,x,y)$ is the analytical entropy calculated from a given energy density field using the equation of state.

For instance, in the ideal case where $\epsilon = 3p$ and $p = cT^4$ where c is a constant, then

$$\begin{aligned} s &= \frac{\epsilon+p}{T} \\ &= \frac{\epsilon+p}{(p/c)^{0.25}} \\ &= 4/3(3c)^{0.25}\epsilon^{0.75}. \end{aligned} \tag{293}$$

However, we still need to obtain σ_α. Because σ_α is the density we can assume that $\sigma_\alpha = \frac{1}{\Delta x \Delta y \gamma_\alpha t}$ where $\Delta x \Delta y$ is the grid size in the x and y direction. Thus, we initially take

$$\left(\frac{s}{\sigma}\right)_\alpha = s_\alpha(\epsilon^{an}_\alpha)\Delta x \Delta y \gamma_\alpha t \tag{294}$$

However, being that we're turning a field into grid there is some error if we were to add up all of the entropy densities such that

$$S^{tot}_{SPH} = \sum_i^N s^{SPH}_i \Delta x \Delta y \tag{295}$$

and compared it to the analytical total entropy

$$S_{an}^{tot} = \int d^3\vec{r} s_{an}^{*}(\vec{r}, t_0). \tag{296}$$

Therefore, we also take the fraction $S_{an}^{tot}/S_{SPH}^{tot}$ and multiply each $\left(\frac{s}{\sigma}\right)_\alpha$ by it to take that into account. Thus, our final $\left(\frac{s}{\sigma}\right)_\alpha$ is

$$\left(\frac{s}{\sigma}\right)_\alpha^{final} = \left(\frac{s}{\sigma}\right)_\alpha \frac{S_{an}^{tot}}{S_{SPH}^{tot}} \tag{297}$$

Typically, the initial flow is set to zero such that $u_x = u_y = u_\eta = 0$. But if not that can also be calculated on the grid.

It can shown that as $h \to 0$ and Δx, $\Delta y \to 0$

A. Gubser 2+1 Initial Conditions

Recenter Gubser and Yarom derived an analyitcal solution for 2+1 ideal hydrodynamics [32]. They found the following analitical solution for the energy density profile

$$\epsilon = \frac{\epsilon_0}{\tau^{4/3}} \frac{(2q)^{8/3}}{\left[1 + 2q^2\left(\tau^2 + x_\perp^2\right) + q^4\left(\tau^2 - x_\perp^2\right)\right]^{4/3}} \tag{298}$$

where

$$x_\perp^2 = x^2 + y^2 \tag{299}$$

and q and ϵ_0 are both constants that we set equal to 1. The flow can be described by

$$\begin{aligned} u_x &= \frac{\sinh\left(\kappa(t,x,y)\right) x}{x_\perp} \\ u_y &= \frac{\sinh\left(\kappa(t,x,y)\right) y}{x_\perp} \end{aligned} \tag{300}$$

where

$$\kappa(t,x,y) = \operatorname{arctanh}\left(\frac{2qtx_\perp}{1 + q^2t^2 + q^2x_\perp}\right) \tag{301}$$

Taking the initial time step at $t = 1fm$ we were able to match his analytical solution within our code as see in Figs. 1-2

The initial condition at $t = 1fm$ requires a relatively small h, which in terms requires a great deal of particles. This is because of how steep the initial energy density profile is. However, most average computer have a limit to how many SPH particles they can handle (because of the amount of available dynamic memory at one period of time). Thus, we found that we were able to use a grid that range from $x = -8$ to 8 fm and $y = -8$ to 8 fm with a separation space of $\Delta x = \Delta y = 0.05$ fm, which gave a total of 103041 SPH particles. For this we found that $h = 0.12$ was appropriate and provided the best results. Smaller $h's$ would have required a greater number of particles, whereas larger h's take a longer computational time and also cause probably because they can miss the behavior at the center of the energy density.

The analytical solution has a behavior of a peak that dies out very quickly. Because of that the behavior can be seen within about $t = 2fm$. Additionally, following about $t = 2.4$ the fluid becomes very dilute so it's difficult for such a small h to calculate the behavior well.

In principle, if one were to have no computational limits and could include as small as a grid spacing as possible and correspondingly small h then we could obtain the analyitical solution for the entropy.

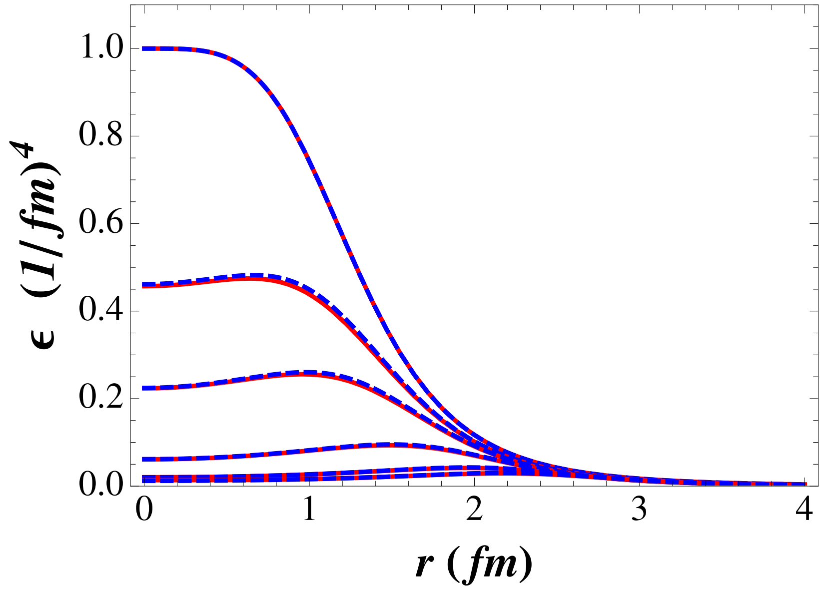

FIG. 1: Comparison of our results for the energy density and the Eq. (298) for times t=1, 1.2, 1.4, 1.8, 2.2, and 2.4 fm.

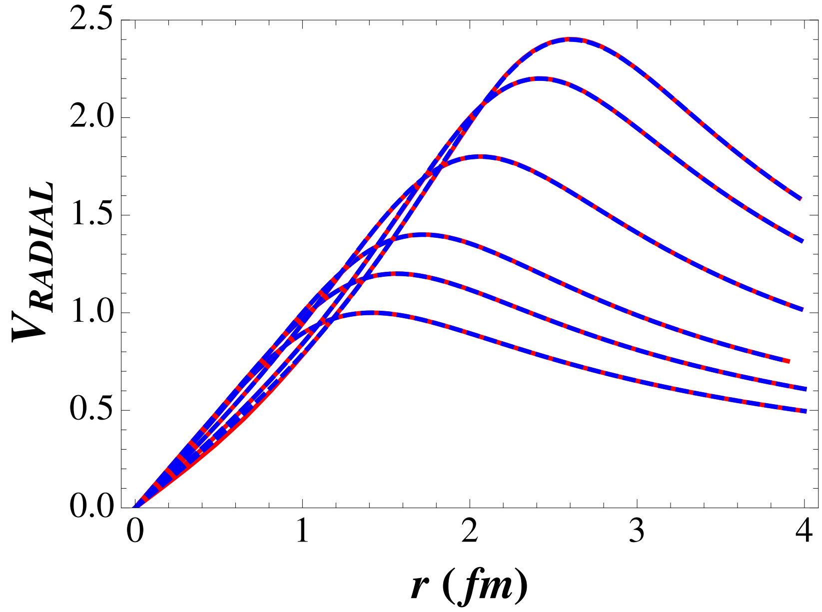

FIG. 2: Comparison of our results for the radial velocity and the Eq. (300) for times t=1, 1.2, 1.4, 1.8, 2.2, and 2.4 fm.

B. CGC Initial Conditions

For CGC event-by-event initial conditions we use the code described in [33]. A quick installation can be done as follow

- Download mckln-jnh.tar.bz2 (originally, this is `mckln-3.52.tar.bz2` from `http://physics.baruch.cuny.edu/files/CGC/CGC_IC.html` but I haved changed a couple of subroutines) and unzip it into a director on you computer.
- Enter the director `mckln-3.52` in your terminal and type:

```
./configure
make
```

hitting enter after each line

- Then type:

  ```
  cd src/
  ./kln.exe
  ```

  to execute the program.

  The results of that first run will be within `mckln-3.52/src` in the files `dens1.dat` and `dens2.dat` for the averaged number density of the collision. The columns in `dens1.dat` are : Rapidity$\big|$x$\big|$y$\big|\rho$ and the folder `mckln-3.52/src/eventbyevent` contains the event-by-event results with the columns x$\big|$y$\big|\rho^{4/3}$ where the energy density is $\epsilon = c\rho^{4/3}$ and c is a constant. c by knowing that the central energy density for zero impact parameter $\epsilon\,(\tau = \tau_0, 0, 0, 0)$ correspond to a temperature T_i via the equation of state. Then the temperature can be used to match the experimental data for the multiplicities at the end of the run.

1. *Editing Output*

In order to change the configuration of the ouptu you will need to open the file `mckln-3.52/src/MakeDensity` and edit the subroutine `void MakeDensity::dumpDensityperEvent(int ev, const int iy)`. The output is set up to automatically print out in terms x$\big|$y$\big|\rho^{4/3}$ into the corresponding file for each event in the `mckln-3.52/src/eventbyevent` folder.

Note that every time you run the code the files in the `mckln-3.52/src/eventbyevent` are automatically deleted.

2. *Changing Parameters*

- Impact Parameter: Within `mckln-3.52/src/main.cxx` in `int main(int argc, char *argv[])` change the value of `double b` on line 54. You'll also want to change `double bmin` and `double bmax` in

  ```
  void mc_make_density(double ecm, double lambda, double kglue, int mode, int Et,
  int nevent, int rank, int size, int massnumber)
  ```

  on lines 139-193. Note that if you want to plot other values such as the eccentricity you will need to change the corresponding values in the other functions within main() e.g. line 253-254 in

  ```
  void mc_eccentricity(double ecm, double lambda, double kglue, int mode,int Et,
  int nevent, int rank, int size, int massnumber).
  ```

  Obviously, you do not have to worry about the functions outside of main.

- CGC vs. Glauber: Within `mckln-3.52/src/main.cxx` in

  ```
  void mc_make_density(double ecm, double lambda, double kglue, int mode, int Et,
  int nevent, int rank, int size, int massnumber)
  ```

  change the value of `int isKLN` on line 123. Note that if you want to plot other values such as the eccentricity you will need to change the corresponding values in the other functions within main() e.g. line 235 in

  ```
  void mc_eccentricity(double ecm, double lambda, double kglue, int mode,int Et,
  int nevent, int rank, int size, int massnumber).
  ```

  Obviously, you do not have to worry about the functions outside of main.

- Number of Events: Within `mckln-3.52/src/main.cxx` in `int main(int argc, char *argv[])` change the value of `int nev` on line 55

- Mass number: Within `mckln-3.52/src/main.cxx` in `int main(int argc, char *argv[])` change the value of `int massnumber` on line 49

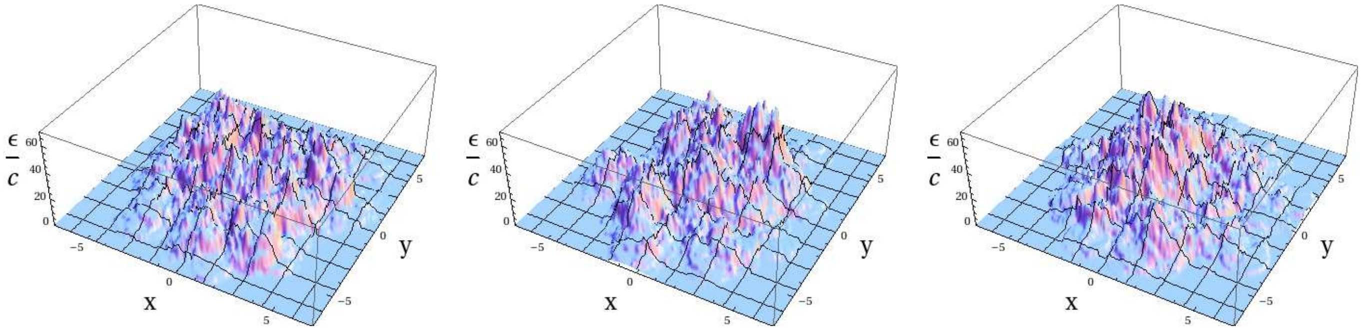

FIG. 3: Three events of CGC initial conditions for b=0 fm.

```
ListPlot3D[{}, AxesLabel → {x, y, ε/c}, PlotRange → {{-7, 7}, {-7, 7}, {0, 70}}]    ListPlot3D[{}, AxesLabel → {x, y, ε/c}, PlotRange → {{-7, 7}, {-7, 7}, {0, 70}}]    ListPlot3D[{}, AxesLabel → {x, y, ε/c}, PlotRange → {{-7, 7}, {-7, 7}, {0, 70}}]
```

FIG. 4: Three events of CGC initial conditions for b=3 fm.

- Grid size/step: Within `mckln-3.52/src/main.cxx` in

  ```
  void mc_make_density(double ecm, double lambda, double kglue, int mode, int Et,
  int nevent, int rank, int size, int massnumber)
  ```

  change the value of `const double maxx`, `const double maxx`, `const double maxx`, and `const double maxx`, on lines 125-128. Note that if you want to plot other values such as the eccentricity you will need to change the corresponding values in the other functions within main() e.g. line 237-240 in

  ```
  void mc_eccentricity(double ecm, double lambda, double kglue, int mode,int Et,
  int nevent, int rank, int size, int massnumber).
  ```

  Obviously, you do not have to worry about the functions outside of main.

- $\sqrt{s_{NN}}$ Incident center of mass energy: Within `mckln-3.52/src/main.cxx` in `int main(int argc, char *argv[])` change the value of `double ecm` on line 53

- Rapidity bin: Within `mckln-3.52/src/main.cxx` in

  ```
  void mc_make_density(double ecm, double lambda, double kglue, int mode, int Et,
  int nevent, int rank, int size, int massnumber)
  ```

  change the values of `int ny`, `double ymin`, and `ymax` on lines 197-198. Note that if you want to plot other values such as the eccentricity you will need to change the corresponding values in the other functions within main() e.g. line 310-311 in

  ```
  void mc_eccentricity(double ecm, double lambda, double kglue, int mode,int Et,
  int nevent, int rank, int size, int massnumber).
  ```

  Obviously, you do not have to worry about the functions outside of main.

3. Results

As discussed above in *Editing Output*, $\rho^{4/3}$ is proportional to the energy density. Thus, it is useful to plot our results to see if they give reasonable initial conditions. We plot three random events for the following situations: CGC with b=0 fm Fig. 3, CGC with b=3 fm Fig. 4, Glauber with b=0 fm Fig. 5, and Glauber with b=3 fm Fig. 6.

As one can see in the graphs, the glauber model as smoother initial conditions than the CGC model.

ListPlot3D[{}, AxesLabel → {x, y, $\frac{\epsilon}{c}$}, PlotRange → {{-7, 7}, {-7, 7}, {0, 20}}] ListPlot3D[{}, AxesLabel → {x, y, $\frac{\epsilon}{c}$}, PlotRange → {{-7, 7}, {-7, 7}, {0, 20}}] ListPlot3D[{}, AxesLabel → {x, y, $\frac{\epsilon}{c}$}, PlotRange → {{-7, 7}, {-7, 7}, {0, 20}}]

FIG. 5: Three events of Glauber initial conditions for b=0 fm.

ListPlot3D[{}, AxesLabel → {x, y, $\frac{\epsilon}{c}$}, PlotRange → {{-7, 7}, {-7, 7}, {0, 20}}] ListPlot3D[{}, AxesLabel → {x, y, $\frac{\epsilon}{c}$}, PlotRange → {{-7, 7}, {-7, 7}, {0, 20}}] ListPlot3D[{}, AxesLabel → {x, y, $\frac{\epsilon}{c}$}, PlotRange → {{-7, 7}, {-7, 7}, {0, 20}}]

FIG. 6: Three events of Glauber initial conditions for b=3 fm.

C. h and Grid Size

When doing SPH we have a smoothing factor known as h. It's important to pick an h that is not too large (because it slows down the program significantly) but also not too small because the statistics can break down then if there are not enough SPH particles. One might assume then that it makes the most sense to increase the number of SPH particles and have a small h, however, this can also cause problems in terms of the run time of the program. So in order to optimize the program we want to have a small enough h and few enough SPH particles so that we get accurate statistics but also still have a quick run time.

Because of this we have tried various h's with a constant grid size (i.e. the number of SPH particles was held constant). In the case of Glauber Initial Conditions we used $dx = dy = 0.1$ which translated into about 25,000 SPH particles. Using $h = 0.2$ we obtained very reasonable results first trying the averaged initial conditions for 200 events. A good check is to start with the average value of the energy density for all events and run it through the code. In running these test we found that the averaged initial conditions break down roughly 1fm/c after the event-by-event initial conditions so it's a very good test for the stability of our chosen paramters.

Using the assumption $h = 0.2$ with about 25,000 SPH particles from glauber initial conditions we were able to run our code until $t = 10fm/c$ with complete stability as seen in Fig. 7.

Using the same assumption $h = 0.2$ with about 22,000 SPH particles from glauber initial conditions we were able to run our code until $t = 10fm/c$ as seen in Fig. 8. However, you can see that there are still some bumps in Fig. 8 at both $t = 1fm/c$ and $t = 10fm/c$ the question is: are those just effects of the initial condition propogating those bumps or is it due to a too small h and not enough SPH particles. First we tested what happens when we include a larger $h = 0.5$. The results are shown in Fig. 9 8. What we see, though, is that both the initial condition and at $t = 10fm/c$ that they are smooth. Because the initial condition is aleady smoothed out due to a larger h, it's hard to know for sure if the smoothness in the final states is because of a more stable hydro code or simply because of the smoother initial condition. Thus, we also select and event and watch it's evolution over time.

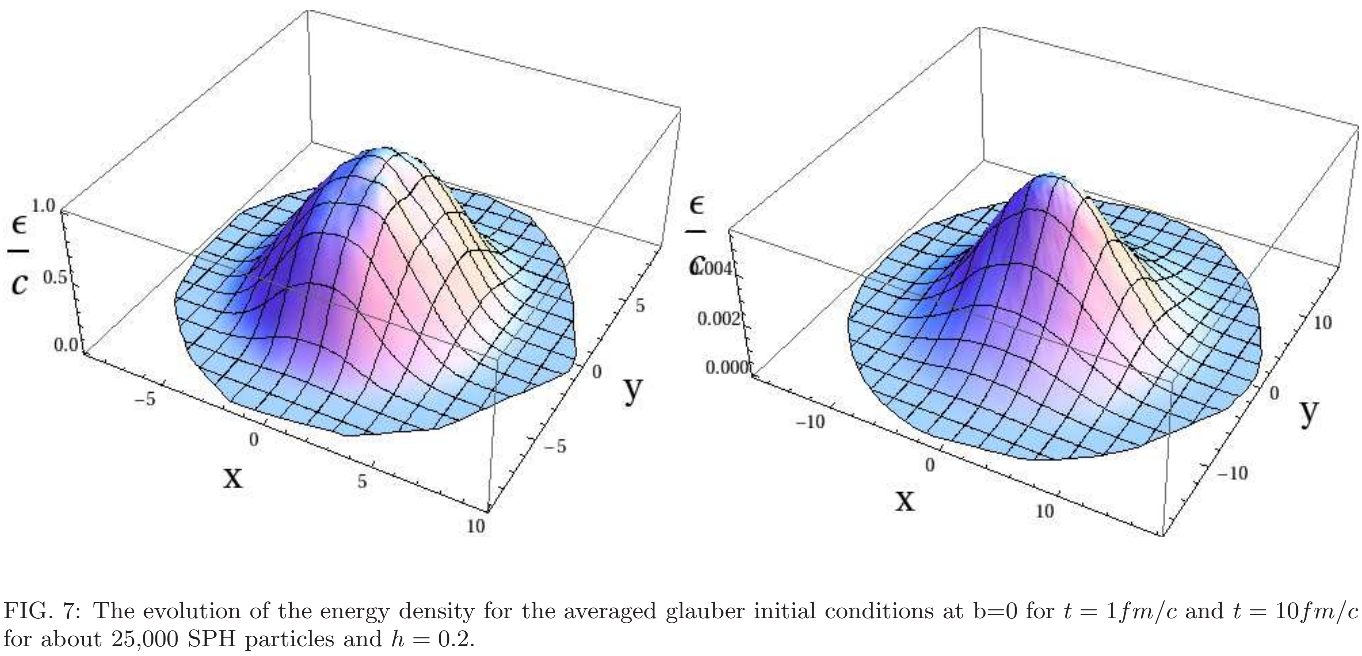

FIG. 7: The evolution of the energy density for the averaged glauber initial conditions at b=0 for $t = 1fm/c$ and $t = 10fm/c$ for about 25,000 SPH particles and $h = 0.2$.

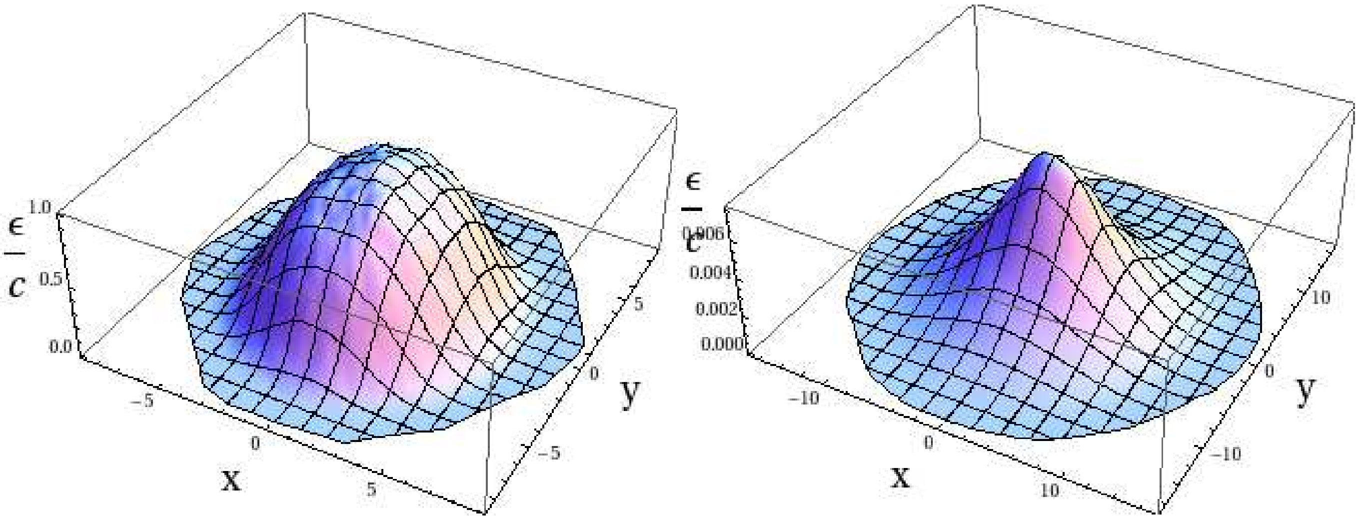

FIG. 8: The evolution of the energy density for the averaged CGC initial conditions at b=0 for $t = 1fm/c$ and $t = 10fm/c$ for about 22,000 SPH particles and $h = 0.2$.

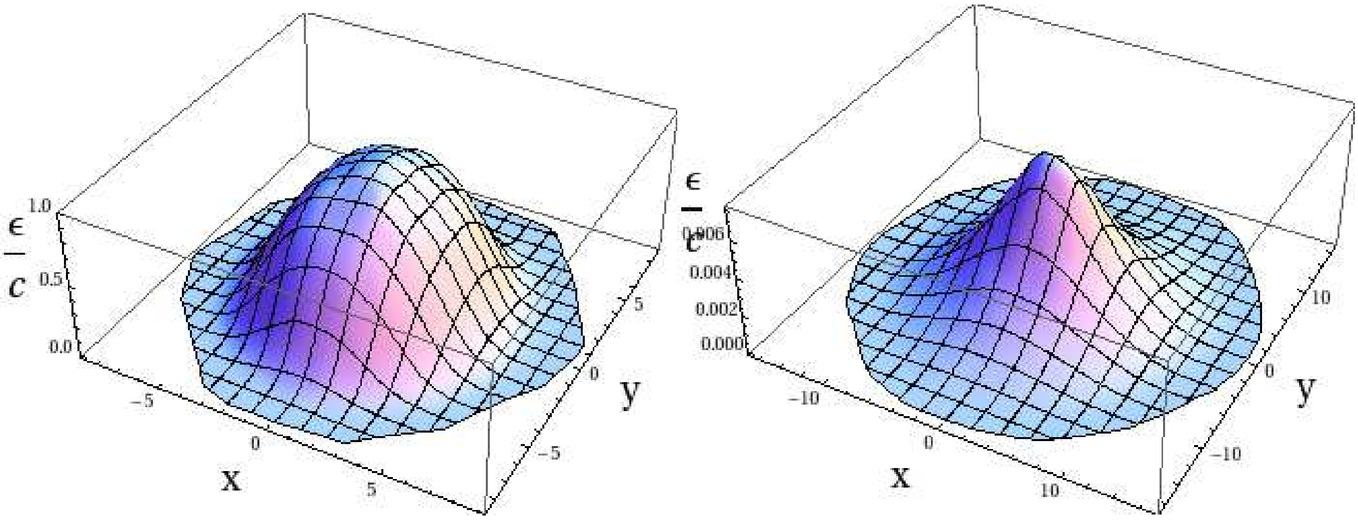

FIG. 9: The evolution of the energy density for the averaged CGC initial conditions at b=0 for $t = 1fm/c$ and $t = 10fm/c$ for about 22,000 SPH particles and $h = 0.5$.

We select the follow CGC event at $b = 0$ as shown in Fig./ we then compare $h = 0.3$ and $h = 0.5$ to see if there is a difference or any clear errors that occur. The results for the initial time are shown in Fig. 11 and for t=10 fm/c in Fig. 12.

VI. NUMERICAL METHODS

Hydrodynamic codes can take a significant amount of computation time, especially once you consider the 3-dimension case, viscous corrections, and event-by-event initial conditions. Because of that it is important to find numerical methods that are precise but still offer enough speed that the code will run in a resonable amount of time.

In vSPHRIO we include three different subroutines that allow you to integrate the equations of motion over time. There is a trade between precision and speed that the user can decide on.

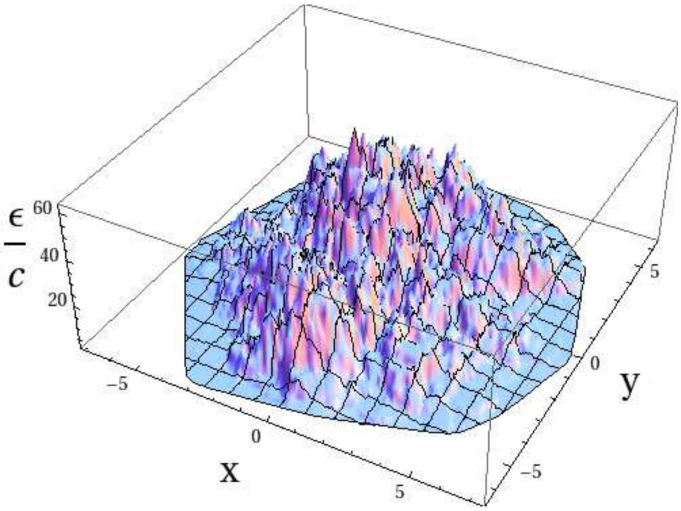

FIG. 10: The original event at $b = 0$ for about 22,000 SPH particles.

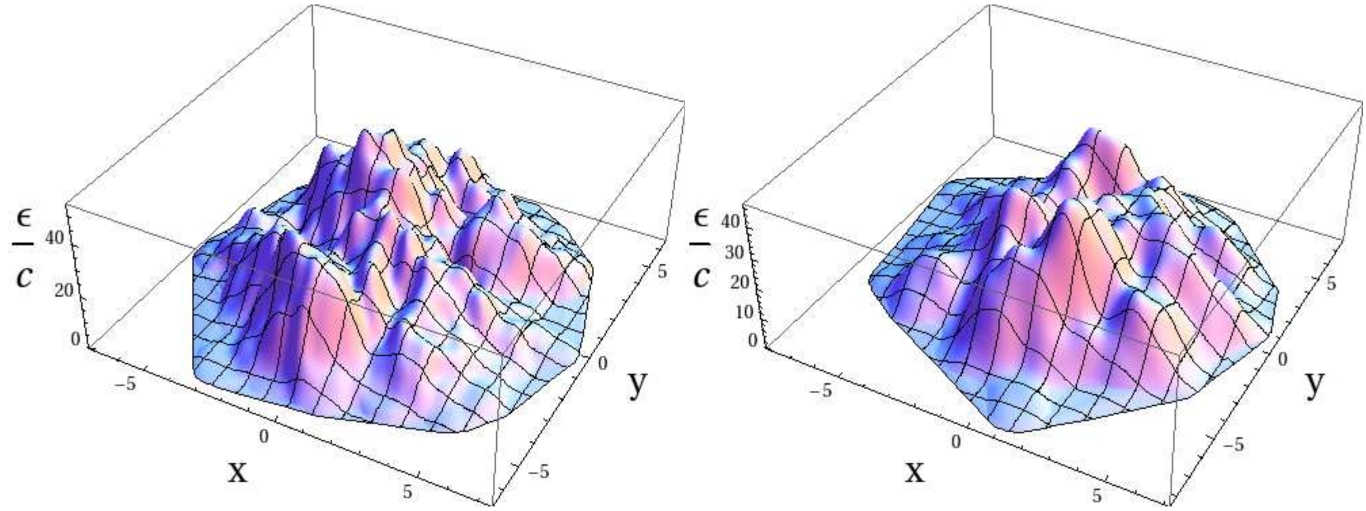

FIG. 11: The initial time after discretizing the event into SPH particles for $h = 0.3$ (left) and $h = 0.5$ (right).

A. AdamsBashforth methods

The Adams-Bashforth method is a linear multi-step method that relies on the past derivatives to calculate the next time step. The general equation is

$$y_{n+2} = y_{n+1} + h\left(1.5 f\left(t_{n+1}, y_{n+1}\right) - 0.5 f\left(t_n, y_n\right)\right). \tag{302}$$

Because Eq. (302) is dependent on the derivative in the previous time step we need a method to calculate the initial time step. For that we simply use Euler

$$y_1 = y_0 + h f\left(t_0, y_0\right) \tag{303}$$

and from that point on we can use Eq. (302) for all the subsequent time steps. Of our three subroutines this is by far the quickest. However, it also has the worst energy conservation.

Within the code it is called using:

```
adams<d>(dt,cc,&equationsofmotion<d>,linklist)
```

where d is the number of dimensions and $equationsofmotion$ is the function used to calculate the equations of motion e.g. $idealhydro3$ for ideal hydrodynamics.

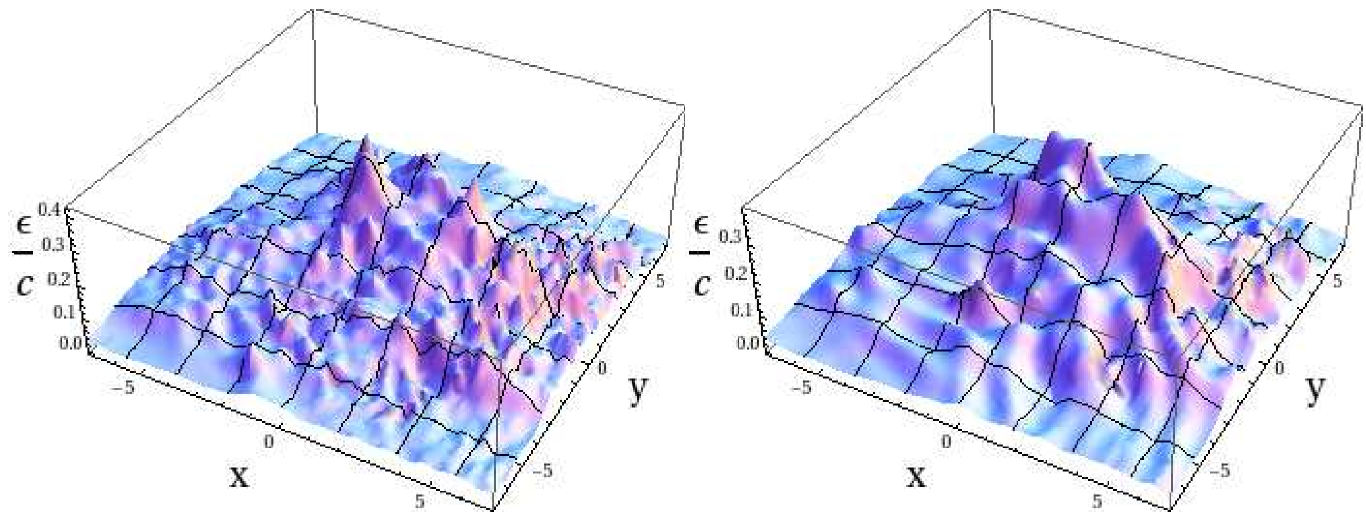

FIG. 12: The $t = 10fm/c$ after discretizing the event into SPH particles for $h = 0.3$ (left) and $h = 0.5$ (right).

B. 2^{nd} order Runge Kutta method

Runge Kutta methods are iterative numerical methods to solve ordinanry differential equations. The quicker one that we've employed here is the 2nd order Runge Kutta method and it is described by

$$y_{n+1} = y_n + hf\left(t_n + 0.5h, y_n + 0.5hf(t_n, y_n)\right). \tag{304}$$

As you can see at each time step the derivative needs to be calculated twice (as compared to the Adams-Bashforth method where it is only calculated once). Hence, the run time being double that of the Adams Bashforth method. However, the precision is also better as detailed before.

Within the code it is called using:

```
rungeKutta2<d>(dt,&equationsofmotion<d>,linklist)
```

where d is the number of dimensions and $equationsofmotion$ is the function used to calculate the equations of motion e.g. $idealhydro3$ for ideal hydrodynamics.

C. 4^{th} order Runge Kutta

The 4^{th} order Runge Kutta is the same as above, however, in this case four derivatives are calculated at each time step:

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4)\right) \\ t_{n+1} &= t_n + h \end{aligned} \tag{305}$$

where

$$\begin{aligned} k_1 &= hf\left(t_n, y_n\right) \\ k_2 &= hf\left(t_n + 0.5h, y_n + 0.5k_1\right) \\ k_3 &= hf\left(t_n + 0.5h, y_n + 0.5k_2\right) \\ k_4 &= hf\left(t_n + h, y_n + k_3\right). \end{aligned} \tag{306}$$

Because of the extra derivatives at each time step it is not recommended to call this subroutine unless a very precise solution is needed.

Within the code it is called using:

```
rungeKutta4<d>(dt,&equationsofmotion<d>,linklist)
```

where d is the number of dimensions and $equationsofmotion$ is the function used to calculate the equations of motion e.g. $idealhydro3$ for ideal hydrodynamics.

	Adams-Bashforth	Runge Kutta 2	Runge Kutta 4
real time	13m2.506s	25m49.686s	51m52.719s
usr time	12m57.830s	25m42.770s	51m40.940s
energy loss	-0.00120366%	0.000466988%	-9.18957e-09%

TABLE I: Energy loss and run time of various numerical methods

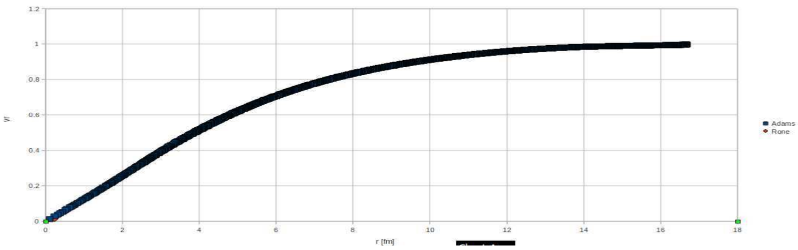

FIG. 13: Comparison of our results for the radial velocity for various numerical integration methods to other well-known codes after $t = 8.6$ fm.

D. Test run

In order to compare our three subroutines we did a test run for a 2+1 ideal gas (with an ideal Equation of State) where $h = 0.2$ and the time step is $\Delta t = 0.01$ fm. Our initial time was $t_0 = 0.6$ fm and we ran until $t = 8.6$ fm. The initial condition was that taken from TECHQM and we did not consider freezeout, just pure hydro.

The results of our test run are shown in Table I. As you can see, the Adams-Bashforth method is certainly the quickest because it only requires that you calculate one derivative at each time step. However, it also has the worst energy conservation. In this case it gains on the order of 0.001%. Although, that is still a very reasonable percision and, considering that the run time is roughly half that of the 2^{nd} order Runge Kutta method, it is considered our default method. However, when better precision is needed, he 2^{nd} Runge Kutta method is still quite quick and the precision improves by almost an entire order of magnitude. Finally, we also include the 4^{th} order Runge Kutta, which runs significantly slower than the other two methods (4 times slower than Adams-Bashforth and just over double the time of the 2^{nd} order Runge Kutta method) but the precision is significantly better and energy is only gained to the $9 * 10^{-9}$%, so it is included if a very precise calculation is needed.

Using this initial set-up we also compared the validity of our code using various numerical methods and compared them to well-known codes as shown in Figs. (13-14). As shown in the Figs. (13-14) the results are indentical for both the radial velocity and the energy density at $t = 8.6$ fm

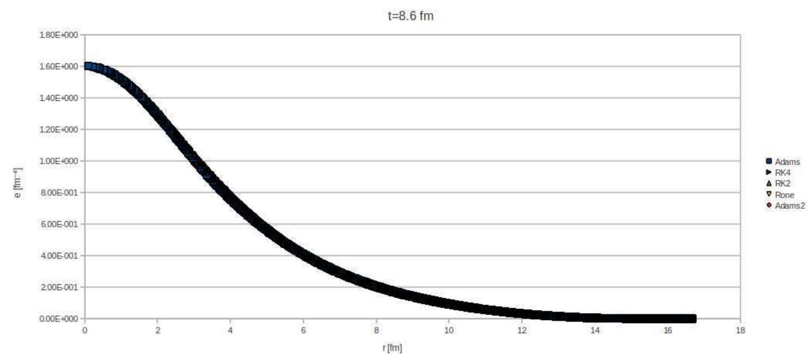

FIG. 14: Comparison of our results for the energy density for various numerical integration methods to other well-known codes after $t = 8.6$ fm.

VII. NORMAL VECTOR

We calculate the normal 4-vector to the hypersurface as follows:

$$\begin{aligned}
(n_\tau)_\alpha &= c_\alpha \frac{\partial T_\alpha}{\partial \tau} \\
(n_x)_\alpha &= c_\alpha \frac{\partial T_\alpha}{\partial x} \\
(n_y)_\alpha &= c_\alpha \frac{\partial T_\alpha}{\partial y}
\end{aligned} \tag{307}$$

where

$$c_\alpha = \frac{1}{\sqrt{(\partial_\mu T)_\alpha (\partial^\mu T)_\alpha}} \tag{308}$$

A. Ideal

First, calculating the spatial components we use:

$$\begin{aligned}
\nabla_\alpha T_\alpha &= \left(\frac{\partial T_\alpha}{\partial x}, \frac{\partial T_\alpha}{\partial y}\right) \\
&= \frac{\partial T_\alpha}{\partial s} \nabla_\alpha s_\alpha \\
&= \frac{\partial T_\alpha}{\partial s} \nabla_\alpha \left(\frac{1}{\tau\gamma_\alpha} s^*_\alpha\right).
\end{aligned} \tag{309}$$

Up until this point our γ has been defined for a specific SPH particle α. However, to find the normal vector we need a continuous field. Thus, we define:

$$\tilde{v}_j = \frac{\sum_i \nu_i |\vec{v}| W(|\mathbf{r} - \mathbf{r}_j|)}{\sum_i \nu_i W(|\mathbf{r} - \mathbf{r}_j|)} \tag{310}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \tilde{v}^2}} \tag{311}$$

and

$$\nu_i = \left(\frac{s}{\sigma}\right)_i (t = t_0) \tag{312}$$

and is a constant over time. Thus,

$$\nabla_\alpha \left(\frac{1}{\tau\gamma_\alpha} s^*_\alpha\right) = \frac{1}{\tau\gamma_\alpha} \sum_i \nu_i \nabla_i W(|\mathbf{r} - \mathbf{r}_j|) - \frac{\nabla\gamma}{\tau\gamma_\alpha} \sum_i \nu_i W(|\mathbf{r} - \mathbf{r}_j|) \tag{313}$$

where

$$\begin{aligned}
\nabla\gamma &= \left[\frac{\tilde{v}}{(1 - \tilde{v}^2)^{3/2}}\right] \nabla\tilde{v} \\
\nabla\tilde{v} &= \frac{\sum_i \nu_i |\vec{v}| \nabla W(|\mathbf{r} - \mathbf{r}_j|)}{\sum_i \nu_i W(|\mathbf{r} - \mathbf{r}_j|)} - \frac{\left[\sum_i \nu_i \nabla W(|\mathbf{r} - \mathbf{r}_j|)\right]\left[\sum_i \nu_i |\vec{v}| W(|\mathbf{r} - \mathbf{r}_j|)\right]}{\left[\sum_i \nu_i W(|\mathbf{r} - \mathbf{r}_j|)\right]^2}
\end{aligned} \tag{314}$$

For the temporal component

$$\begin{aligned}
\frac{\partial T_\alpha}{\partial \tau} &= \frac{\partial T_\alpha}{\partial s_\alpha}\frac{\partial s_\alpha}{\partial \tau} \\
&= \frac{\partial T_\alpha}{\partial s_\alpha}\frac{\partial}{\partial \tau}\left[\frac{1}{\tau\gamma_\alpha}s_\alpha^*\right] \\
&= \frac{\partial T_\alpha}{\partial s_\alpha}\left[\frac{s_\alpha^*}{\gamma_\alpha}+\frac{s_\alpha^*}{\tau\gamma_\alpha^2}\frac{\partial\gamma}{\partial\tau}+\frac{1}{\tau\gamma_\alpha}\frac{\partial s_\alpha^*}{\partial\tau}\right] \\
&= \frac{1}{\gamma_\alpha}\frac{\partial T_\alpha}{\partial s_\alpha}\left[s_\alpha^*\left(1+\frac{1}{\tau\gamma_\alpha}\frac{\partial\gamma}{\partial\tau}\right)+\frac{1}{\tau}\frac{\partial s_\alpha^*}{\partial\tau}\right]
\end{aligned} \tag{315}$$

where we recall that for the ideal case:

$$\begin{aligned}
s_\alpha^* &= \sigma_\alpha^* = \sum_i \nu_i W(|\mathbf{r}-\mathbf{r}_j|) \\
\frac{\partial s_\alpha^*(\vec{r})}{\partial\tau} &= \frac{\partial\sigma_\alpha^*(\vec{r})}{\partial\tau} = \sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\frac{d\vec{r}_\alpha}{d\tau}\cdot\nabla W(|\mathbf{r}-\mathbf{r}_\alpha|) \\
&= \sum_{\alpha=1}^{N_{SPH}}\nu_\alpha\vec{v}_\alpha\cdot\nabla W(|\mathbf{r}-\mathbf{r}_\alpha|).
\end{aligned} \tag{316}$$

a more rigorous way to write this, though, is

$$\frac{s_\alpha^*}{\gamma\tau} = \frac{\sigma_\alpha^*}{\gamma\tau}\left(\frac{s}{\sigma}\right)_i \tag{317}$$

where $\left(\frac{s}{\sigma}\right)_i = 1$, which is constant in time. Taking the time derivative of γ

$$\frac{\partial\gamma}{\partial\tau} = \left[\frac{\tilde{v}}{(1-\tilde{v}^2)^{3/2}}\right]\frac{\partial\tilde{v}}{\partial\tau} \tag{318}$$

$$\frac{\partial\tilde{v}}{\partial\tau} = \frac{\sum_i\nu_i\frac{\partial|\vec{v}|}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)}{\sum_i\nu_iW(|\mathbf{r}-\mathbf{r}_j|)}+\frac{\sum_i\nu_i|\vec{v}|\frac{\partial}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)}{\sum_i\nu_iW(|\mathbf{r}-\mathbf{r}_j|)}-\frac{\left[\sum_i\nu_i\frac{\partial}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)\right]\left[\sum_i\nu_i|\vec{v}|W(|\mathbf{r}-\mathbf{r}_j|)\right]}{\left[\sum_i\nu_iW(|\mathbf{r}-\mathbf{r}_j|)\right]^2} \tag{319}$$

VIII. BULK VISCOSITY

In the case of Bulk Viscsity we can no longer use the relationship that $s^* = \sigma^*$ instead $s_i^* = \sigma_i^*\left(\frac{s}{\sigma}\right)_i(t)$, thus, the equation needs to be rewritten. Starting with the spatial compenents we use

$$\begin{aligned}
\nabla_\alpha T_\alpha &= \left(\frac{\partial T_\alpha}{\partial x},\frac{\partial T_\alpha}{\partial y}\right) \\
&= \frac{\partial T_\alpha}{\partial s}\nabla_\alpha s_\alpha \\
&= \frac{\partial T_\alpha}{\partial s}\nabla_\alpha\left(\frac{1}{\tau\gamma_\alpha}s_\alpha^*\right).
\end{aligned} \tag{320}$$

where

$$\nabla_\alpha\left(\frac{1}{\tau\gamma_\alpha}s_\alpha^*\right) = \frac{1}{\tau\gamma_\alpha}\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\nabla_iW(|\mathbf{r}-\mathbf{r}_j|)-\frac{\nabla\gamma}{\tau\gamma_\alpha}\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)W(|\mathbf{r}-\mathbf{r}_j|) \tag{321}$$

notice that in this case the $\left(\frac{s}{\sigma}\right)_i(t)$ has a time dependence but not a spatial dependence, so it is not affected by the gradient. Thus, we can use the same form for $\nabla\gamma$ as seen in Eq. (314).

Recall that for the temporal component

$$\frac{\partial T_\alpha}{\partial \tau} = \frac{1}{\gamma_\alpha}\frac{\partial T_\alpha}{\partial s_\alpha}\left[s^*_\alpha\left(1+\frac{1}{\tau\gamma_\alpha}\frac{\partial\gamma}{\partial\tau}\right)+\frac{1}{\tau}\frac{\partial s^*_\alpha}{\partial\tau}\right] \tag{322}$$

where

$$\begin{aligned}\frac{\partial s^*_\alpha}{\partial \tau} &= \frac{\partial}{\partial\tau}\left[\sum_i \nu_i\left(\frac{s}{\sigma}\right)_i(t)W(|\mathbf{r}-\mathbf{r}_j|)\right]\\ &= \sum_i \nu_i\left[\frac{\partial}{\partial\tau}\left(\frac{s}{\sigma}\right)_i(t)\right]W(|\mathbf{r}-\mathbf{r}_j|)+\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\left[\frac{\partial}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)\right].\end{aligned} \tag{323}$$

We recall that for the bulk viscosity $\frac{\partial}{\partial\tau}\left(\frac{s}{\sigma}\right)_i(t)$ has been defined already in Eq. (139)

$$\frac{d}{dt}\left(\frac{s}{\sigma}\right) = -\frac{1}{T}\frac{\Pi}{\sigma^*}\partial_\mu u^\mu. \tag{324}$$

Thus, the entire equation for the temporal dependence is:

$$\frac{\partial T_\alpha}{\partial \tau} = \frac{1}{\gamma_\alpha}\frac{\partial T_\alpha}{\partial s_\alpha}\left[s^*_\alpha\left(1+\frac{1}{\tau\gamma_\alpha}\frac{\partial\gamma}{\partial\tau}\right)+\frac{1}{\tau}\left(\sum_i \nu_i\left[\frac{\partial}{\partial\tau}\left(\frac{s}{\sigma}\right)_i(t)\right]W(|\mathbf{r}-\mathbf{r}_j|)+\sum_i\nu_i\left(\frac{s}{\sigma}\right)_i(t)\left[\frac{\partial}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)\right]\right)\right]$$

where

$$\frac{\partial\gamma}{\partial\tau} = \left[\frac{\tilde v}{(1-\tilde v^2)^{3/2}}\right]\left\{\frac{\sum_i\nu_i\frac{\partial|\vec v|}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)}{\sum_i\nu_i W(|\mathbf{r}-\mathbf{r}_j|)}+\frac{\sum_i\nu_i|\vec v|\frac{\partial}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)}{\sum_i\nu_i W(|\mathbf{r}-\mathbf{r}_j|)}-\frac{\left[\sum_i\nu_i\frac{\partial}{\partial\tau}W(|\mathbf{r}-\mathbf{r}_j|)\right]\left[\sum_i\nu_i|\vec v|W(|\mathbf{r}-\mathbf{r}_j|)\right]}{\left[\sum_i\nu_i W(|\mathbf{r}-\mathbf{r}_j|)\right]^2}\right\} \tag{325}$$

Appendix A: Important relations

In this Appendix we prove Eqs. (34-37,146).

1. $\Delta_{\alpha\beta}\Delta^{\alpha\beta}=3$

First, we prove Eq. (34), i.e. $\Delta_{\alpha\beta}\Delta^{\alpha\beta}=3$

$$\begin{aligned}\Delta_{\alpha\beta}\Delta^{\alpha\beta} &= \left(g_{\alpha\beta}-u_\alpha u_\beta\right)\left(g^{\alpha\beta}-u^\alpha u^\beta\right)\\ &= \underbrace{g_{\alpha\beta}g^{\alpha\beta}}_{=4}-u_\alpha u_\beta g^{\alpha\beta}-u^\alpha u^\beta g_{\alpha\beta}+\underbrace{u_\alpha u_\beta u^\alpha u^\beta}_{=1}\\ &= 5-2u^\alpha u_\alpha\\ &= 3\end{aligned} \tag{A1}$$

Eq. (35), i.e. $\Delta_{\alpha\beta}\Delta^{\beta\nu}=\Delta^\nu_\alpha$, is shown by

$$\begin{aligned}\Delta_{\alpha\beta}\Delta^{\beta\nu} &= \left(g_{\alpha\beta}-u_\alpha u_\beta\right)\left(g^{\beta\nu}-u^\beta u^\nu\right)\\ &= g_{\alpha\beta}g^{\beta\nu}-u_\alpha u_\beta g^{\beta\nu}-u^\beta u^\nu g_{\alpha\beta}+u_\alpha u_\beta u^\beta u^\nu\\ &= g^\nu_\alpha-u_\alpha u^\nu\\ &= \Delta^\nu_\alpha\end{aligned} \tag{A2}$$

2. $\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta}_{\nu\rho} = \frac{5}{3}\Delta^{\mu}_{\rho}$

Eq. (35), i.e. $\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta}_{\nu\rho} = \frac{5}{3}\Delta^{\mu}_{\rho}$, is shown by

$$\begin{aligned}
\Delta^{\mu\nu}_{\alpha\beta}\Delta^{\alpha\beta}_{\nu\rho} &= \frac{1}{4}\left[\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta} - \frac{2}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right]\left[\Delta^{\alpha}_{\nu}\Delta^{\beta}_{\rho} + \Delta^{\beta}_{\nu}\Delta^{\alpha}_{\rho} - \frac{2}{3}\Delta^{\alpha\beta}\Delta_{\nu\rho}\right] \\
&= \frac{1}{4}\left[\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}\Delta^{\alpha}_{\nu}\Delta^{\beta}_{\rho} + \Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}\Delta^{\beta}_{\nu}\Delta^{\alpha}_{\rho} - \frac{2}{3}\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}\Delta^{\alpha\beta}\Delta_{\nu\rho} + \Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta}\Delta^{\alpha}_{\nu}\Delta^{\beta}_{\rho} + \Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta}\Delta^{\beta}_{\nu}\Delta^{\alpha}_{\rho} - \frac{2}{3}\Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta}\Delta^{\alpha\beta}\Delta_{\nu\rho}\right. \\
&- \left.\frac{2}{3}\Delta^{\alpha}_{\nu}\Delta^{\beta}_{\rho}\Delta^{\mu\nu}\Delta_{\alpha\beta} - \frac{2}{3}\Delta^{\beta}_{\nu}\Delta^{\alpha}_{\rho}\Delta^{\mu\nu}\Delta_{\alpha\beta} + \frac{4}{6}\Delta^{\alpha\beta}\Delta_{\nu\rho}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] \\
&= \frac{1}{4}\left[\Delta^{\mu}_{\rho} + 3\Delta^{\mu}_{\rho} - \frac{2}{3}\Delta^{\mu}_{\rho} + 3\Delta^{\mu}_{\rho} + \Delta^{\mu}_{\rho} - \frac{2}{3}\Delta^{\mu}_{\rho} - \frac{2}{3}\Delta^{\mu}_{\rho} - \frac{2}{3}\Delta^{\mu}_{\rho} + \frac{4}{9}3\Delta^{\mu}_{\rho}\right] \\
&= \frac{5}{3}\Delta^{\mu}_{\rho}
\end{aligned} \tag{A3}$$

3. $\partial^{\mu} - \nabla^{\mu} = u^{\mu} D$

To find a relationship between $\nabla^{\mu}_{\perp}$, ∂^{μ}, and $u^{\mu}D$, which will be useful when we solve the equations of motion, we prove Eq. (37)

$$\begin{aligned}
\partial^{\mu} - \nabla^{\mu} &= \partial_{\alpha} g^{\mu\alpha} - \Delta^{\mu\alpha}\partial_{\alpha} \\
&= \left[g^{\mu\alpha} - \Delta^{\mu\alpha}\right]\partial_{\alpha} \\
&= u^{\mu}u^{\alpha}\partial_{\alpha} \\
&= u^{\mu} D
\end{aligned} \tag{A4}$$

4. $\partial_{\mu}\left(\frac{1}{\sigma}\right) = -\frac{1}{\sigma^2}\partial_{\mu}\sigma$

Proving Eq. (146), i.e. $\partial_{\mu}\left(\frac{1}{\sigma}\right) = -\frac{1}{\sigma^2}\partial_{\mu}\sigma$,

$$\begin{aligned}
\partial_{\mu}\sigma &= \partial_{\mu}\left(\sigma^2\frac{1}{\sigma}\right) \\
&= \frac{1}{\sigma}\partial_{\mu}\left(\sigma^2\right) + \sigma^2\partial_{\mu}\left(\frac{1}{\sigma}\right) \\
&= 2\partial_{\mu}\sigma + \sigma^2\partial_{\mu}\left(\frac{1}{\sigma}\right) \\
\partial_{\mu}\left(\frac{1}{\sigma}\right) &= -\frac{1}{\sigma^2}\partial_{\mu}\sigma
\end{aligned} \tag{A5}$$

Then, multiplying by σ we find

$$\begin{aligned}
\partial_{\mu}u^{\mu} &= \sigma\partial_{\mu}\left(\frac{1}{\sigma}\right) \\
&= -\frac{1}{\sigma}\partial_{\mu}\sigma
\end{aligned} \tag{A6}$$

5. $\partial^{\mu}u_{\mu} = -\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau} - \frac{u_i u_j}{2\gamma}\frac{dg^{ij}}{d\tau}$

In order to prove Eq. (148), i.e. $\partial^{\mu}u_{\mu} = -\frac{\gamma}{\sigma^*}\frac{d\sigma^*}{d\tau} - \frac{g^{ij}u_i}{\gamma}\frac{du_j}{d\tau} - \frac{u_i u_j}{2\gamma}\frac{dg^{ij}}{d\tau}$, we start with

$$\begin{aligned}
\partial^\mu u_\mu &= -\frac{1}{\sigma}\frac{d\sigma}{d\tau} \\
&= -\frac{\gamma}{\sigma^*}\frac{d}{d\tau}\left(\frac{\sigma^*}{\gamma}\right) \\
&= -\frac{\gamma}{\sigma^*}\gamma\frac{d}{dt}\left(\frac{\sigma^*}{\gamma}\right) \\
&= -\gamma^2\frac{d}{dt}\left(\frac{1}{\gamma}\right) - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{dt} \qquad \text{(A7)}
\end{aligned}$$

However,

$$\begin{aligned}
\frac{d\gamma}{d\tau} &= \frac{d}{d\tau}\left(\frac{\gamma^2}{\gamma}\right) \\
&= 2\frac{d\gamma}{d\tau} + \gamma^2\frac{d}{d\tau}\left(\frac{1}{\gamma}\right) \\
\gamma\frac{d}{d\tau}\left(\frac{1}{\gamma}\right) &= -\frac{1}{\gamma}\frac{d\gamma}{d\tau} \qquad \text{(A8)}
\end{aligned}$$

so that Eq. (A7) becomes

$$\partial^\mu u_\mu = \frac{d\gamma}{dt} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{dt}. \qquad \text{(A9)}$$

We then need to understand $\frac{d\gamma}{d\tau}$ and to do so we remind ourselves of the properties of u^μ

$$\begin{aligned}
u^\mu &= \gamma\left(1, \frac{dr^i}{d\tau}\right) \\
\frac{du^\mu}{d\tau} &= \left(\frac{d\gamma}{d\tau}, \frac{du^i}{d\tau}\right) \qquad \text{(A10)}
\end{aligned}$$

Then,

$$\begin{aligned}
u_\mu u^\mu &= 1 \\
u_\mu \frac{u^\mu}{d\tau} + u^\mu \frac{u_\mu}{d\tau} &= 0 \\
2\gamma\frac{d\gamma}{d\tau} + u_i\frac{du^i}{d\tau} + u^i\frac{du_i}{d\tau} &= \\
2\gamma\frac{d\gamma}{d\tau} + u_i\frac{d(g^{ij}u_j)}{d\tau} + g^{ij}u_j\frac{du_i}{d\tau} &= \\
2\gamma\frac{d\gamma}{d\tau} + u_i g^{ij}\frac{du_j}{d\tau} + u_i u_j\frac{dg^{ij}}{d\tau} + g^{ij}u_j\frac{du_i}{d\tau} &= \\
2\gamma\frac{d\gamma}{d\tau} + 2u_i g^{ij}\frac{du_j}{d\tau} + u_i u_j\frac{dg^{ij}}{d\tau} &= \\
\frac{d\gamma}{d\tau} &= -\frac{u_i g^{ij}}{\gamma}\frac{du_j}{d\tau} - \frac{u_i u_j}{2\gamma}\frac{dg^{ij}}{d\tau} \qquad \text{(A11)}
\end{aligned}$$

so

$$\partial^\mu u_\mu = -\frac{u_i g^{ij}}{\gamma}\frac{du_j}{d\tau} - \frac{u_i u_j}{2\gamma}\frac{dg^{ij}}{d\tau} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{dt}. \qquad \text{(A12)}$$

Note that in our current coordinates $g^{\mu\nu} = diag(+,-,-,-)$ that $\frac{dg^{ij}}{d\tau} = 0$. We have left it up until this point for reference when we switch coordinate systems. Thus,

$$\partial^\mu u_\mu = -\frac{u_i g^{ij}}{\gamma}\frac{du_j}{d\tau} - \frac{\gamma}{\sigma^*}\frac{d\sigma^*}{dt}. \qquad \text{(A13)}$$

6. $D_\mu g_{\alpha\beta} = 0$

Here we want to prove Eq. (223)

$$\begin{aligned} D_\mu g_{\alpha\beta} &= \partial_\mu - \Gamma^\lambda_{\alpha\mu} g_{\lambda\beta} - \Gamma^\lambda_{\beta\mu} g_{\lambda\alpha} \\ &= 2\tau - (-\tau - \tau + \tau) - (-\tau - \tau + \tau) \\ &= 0 \end{aligned} \tag{A14}$$

7. $\pi^{\alpha\beta} \frac{D}{D\tau} \Delta_{\alpha\beta} = 0$

Here we want to prove Eq. (224)

$$\begin{aligned} \pi^{\alpha\beta} \frac{D}{D\tau} \Delta_{\alpha\beta} &= \pi^{\alpha\beta} \frac{D}{D\tau} \left(g_{\alpha\beta} - u_\alpha u_\beta \right) \\ &= -\pi^{\alpha\beta} \frac{D}{D\tau} \left(u_\alpha u_\beta \right) \\ &= -u_\alpha \pi^{\alpha\beta} \frac{D}{D\tau} \left(u_\beta \right) - u_\beta \pi^{\alpha\beta} \frac{D}{D\tau} \left(u_\alpha \right) \\ &= 0 \end{aligned} \tag{A15}$$

8. $\Delta_{\mu\alpha} \Delta_{\nu\beta} = \Delta_{\mu\beta} \Delta_{\nu\alpha}$

Here we want to prove Eq. (225), so we start with $\Delta_{\mu\alpha} \Delta_{\nu\beta}$

$$\begin{aligned} \Delta_{\mu\alpha} \Delta_{\nu\beta} &= \left(g_{\mu\alpha} - u_\mu u_\alpha \right) \left(g_{\nu\beta} - u_\nu u_\beta \right) \\ &= g_{\mu\alpha} g_{\nu\beta} - u_\mu u_\alpha g_{\nu\beta} - g_{\mu\alpha} u_\nu u_\beta + u_\mu u_\alpha u_\nu u_\beta . \end{aligned} \tag{A16}$$

Now, comparing that with

$$\begin{aligned} \Delta_{\mu\beta} \Delta_{\nu\alpha} &= \left(g_{\mu\beta} - u_\mu u_\beta \right) \left(g_{\nu\alpha} - u_\nu u_\alpha \right) \\ &= g_{\mu\beta} g_{\nu\alpha} - u_\mu u_\beta g_{\nu\alpha} - g_{\mu\beta} u_\nu u_\alpha + u_\mu u_\beta u_\nu u_\alpha \\ &= g_{\mu\alpha} g^\alpha_\beta g_{\nu\beta} g^\beta_\alpha - u_\mu u_\beta g_{\nu\beta} g^\beta_\alpha - g_{\mu\alpha} g^\alpha_\beta u_\nu u_\alpha + u_\mu u_\beta u_\nu u_\alpha \\ &= g_{\mu\alpha} g_{\nu\beta} - u_\mu u_\alpha g_{\nu\beta} - g_{\mu\alpha} u_\nu u_\beta + u_\mu u_\alpha u_\nu u_\beta . \end{aligned} \tag{A17}$$

Thus,

$$\Delta_{\mu\alpha} \Delta_{\nu\beta} = \Delta_{\mu\beta} \Delta_{\nu\alpha} .$$

$$\textbf{9.}\quad \gamma\tfrac{d}{d\tau}\left(\epsilon+p\right)=\tfrac{dw}{ds}\left[-\left(\tfrac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\tfrac{\gamma}{\tau}\right)D_\mu u^\mu-\tfrac{\pi^{\mu\nu}}{T}D_\mu u_\nu\right]$$

Here we prove Eq. (256). First, let $w=\epsilon+p$, then

$$\begin{aligned}
\gamma\frac{d}{d\tau}\left(\epsilon+p\right) &= \frac{d}{d\tau}\left(Ts\right)\\
&= T\frac{ds}{d\tau}+s\frac{dT}{d\tau}\\
&= T\frac{ds}{d\tau}+s\left[\frac{dT}{ds}\frac{ds}{d\tau}\right]\\
&= T\frac{ds}{d\tau}\left[1+\frac{s}{T}\frac{dT}{ds}\right]\\
&= T\frac{ds}{d\tau}\left[1+\frac{s}{T}\frac{d}{ds}\left(\frac{w}{s}\right)\right]\\
&= T\frac{ds}{d\tau}\left[1+\frac{s}{T}\left(\frac{1}{s}\frac{dw}{ds}+w\frac{d}{ds}\left(\frac{1}{s}\right)\right)\right]\\
&= T\frac{ds}{d\tau}\left[1+\frac{s}{T}\left(\frac{1}{s}\frac{dw}{ds}-\frac{w}{s^2}\right)\right]\\
&= T\frac{ds}{d\tau}\left[1+\frac{s}{T}\left(\frac{1}{s}\frac{dw}{ds}-\frac{T}{s}\right)\right]\\
&= \frac{ds}{d\tau}\frac{dw}{ds}
\end{aligned}\tag{A18}$$

where we recall that $\frac{ds}{d\tau}$ is from Eq. (243)

$$\frac{ds}{d\tau}=\frac{\pi^{\mu\nu}}{T\gamma}D_\mu u_\nu-\frac{1}{\gamma}\left(\frac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)\tag{A19}$$

Thus,

$$\gamma\frac{d}{d\tau}\left(\epsilon+p\right)=\frac{dw}{ds}\left[-\left(\frac{\Pi}{T}+s\right)\left(\partial_\mu u^\mu+\frac{\gamma}{\tau}\right)-\frac{\pi^{\mu\nu}}{T}D_\mu u_\nu\right].\tag{A20}$$

Appendix B: Full Equations

In [28], a general derivation of relativistic fluid dynamics from the Botzmann equation is performed where the method of moments is used in a novel way that is consistent with Chapman-Enskog theory. From kinetic theory, the equation for the shear viscosity assuming zero chemical potential (we do not take into account the baryonic current) is

$$\begin{aligned}
\Delta^{\mu\nu}_{\alpha\beta}D\pi^{\alpha\beta}+\frac{\pi^{\mu\nu}}{\tau_\pi}=2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu}+2\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha}-\delta_{\pi\pi}\pi^{\mu\nu}\Theta\ -\ \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha}+\lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}+\phi_6\Pi\pi^{\mu\nu}+\phi_7\pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle}\\
+\eta_1\omega_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda}+\eta_2\Theta\sigma^{\mu\nu}+\eta_3\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle}\ +\ \eta_4\sigma_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda}+\eta_6F^{\langle\mu}F^{\nu\rangle}+\eta_9\nabla_\perp^{\langle\mu}F^{\nu\rangle}
\end{aligned}\tag{B1}$$

where $F^\mu=\nabla^\mu_\perp p$ and our notation is such that, for instance, $\sigma^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle}=\Delta^{\mu\nu}_{\alpha\beta}\sigma^{\lambda\alpha}\sigma_\lambda^\beta$ and for the bulk viscosity

$$\begin{aligned}
D\Pi+\frac{\Pi}{\tau_\Pi}=-\frac{\zeta}{\tau_\Pi}\Theta\ +\ \delta_{\Pi\Pi}\Pi\Theta+\lambda_{\Pi\pi}\Pi^{\mu\nu}\sigma_{\mu\nu}+\phi_1\Pi^2+\\
\phi_3\pi_{\mu\nu}\pi^{\mu\nu}+\omega_{\mu\nu}\omega^{\mu\nu}\ +\ \zeta_1\sigma_{\mu\nu}\sigma^{\mu\nu}+\zeta_2\Theta^2+\zeta_4F\cdot F+\zeta_7\nabla_\mu F^\mu
\end{aligned}\tag{B2}$$

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[1] P. Romatschke, Int. J. Mod. Phys. **E19**, 1-53 (2010). [arXiv:0902.3663 [hep-ph]].

[2] D. Molnar and M. Gyulassy, Nucl. Phys. A **697**, 495 (2002) [Erratum-ibid. A **703**, 893 (2002)].
[3] L. D. McLerran and R. Venugopalan, Phys. Rev. D **49** (1994) 2233.
[4] L. D. McLerran and R. Venugopalan, Phys. Rev. D **49** (1994) 3352.
[5] D. Kharzeev, E. Levin and M. Nardi, Nucl. Phys. A **730** (2004) 448 [Erratum-ibid. A **743** (2004) 329].
[6] H. J. Drescher, A. Dumitru, A. Hayashigaki and Y. Nara, Phys. Rev. C **74** (2006) 044905.
[7] P. F. Kolb, U. W. Heinz, P. Huovinen, K. J. Eskola and K. Tuominen, Nucl. Phys. A **696** (2001) 197.
[8] B. Alver *et al.* [PHOBOS Collaboration], arXiv:nucl-ex/0702036.
[9] J. Adams *et al.* [STAR Collaboration], Phys. Rev. Lett. **92**, 062301 (2004).
[10] M. Luzum and P. Romatschke, Phys. Rev. C **78**, 034915 (2008) [Erratum-ibid. C **79**, 039903 (2009)].
[11] J. Noronha-Hostler, J. Noronha and C. Greiner, Phys. Rev. Lett. **103**, 172302 (2009).
[12] P. Huovinen and P. Petreczky, Nucl. Phys. A **837**, 26 (2010).
[13] O. . J. Socolowski, F. Grassi, Y. Hama and T. Kodama, Phys. Rev. Lett. **93**, 182301 (2004).
[14] R. Andrade, F. Grassi, Y. Hama, T. Kodama, O. Socolowski, Jr., Phys. Rev. Lett. **97**, 202302 (2006).
[15] Y. Hama, R. P. G. Andrade, F. Grassi, O. Socolowski, Jr., T. Kodama, B. Tavares, S. S. Padula, Nucl. Phys. **A774**, 169-178 (2006).
[16] C. Anderlik, L. P. Csernai, F. Grassi, W. Greiner, Y. Hama, T. Kodama, Z. I. Lazar, V. K. Magas *et al.*, Phys. Rev. **C59**, 3309-3316 (1999).
[17] F. Cooper and G. Frye, Phys. Rev. D **10**, 186 (1974).
[18] G. S. Denicol, T. Kodama, T. Koide, P. .Mota, J. Phys. G **G35**, 115102 (2008). [arXiv:0807.3120 [hep-ph]].
[19] G. S. Denicol, T. Koide and D. H. Rischke, Phys. Rev. Lett. **105**, 162501 (2010).
[20] L. B. Lucy, Astrophys. J. **82**, 1013 (1977); J. J. Monaghan, Annu. Rev. Astron. Astrophys. **30**, 543 (1992). C. E. Aguiar, T. Kodama, T. Osadae, and Y. Hama, J. Phys. G 27, 75 (2001).
[21] G. S. Denicol, T. Kodama, T. Koide, P. .Mota, Phys. Rev. **C78**, 034901 (2008);
[22] T. Koide, G. S. Denicol, P. .Mota,T. Kodama, Phys. Rev. **C75**, 034909 (2007);
[23] G. S. Denicol, T. Kodama, T. Koide, J. Phys. G **G37**, 094040 (2010). [arXiv:1002.2394 [nucl-th]].
[24] H. B. Meyer, Phys. Rev. D **76**, 101701 (2007); Phys. Rev. Lett. **100**, 162001 (2008).
[25] S. Pu, T. Koide, and D. H. Rischke, Phys. Rev. D **81**, 114039 (2010).
[26] P. Huovinen, D. Molnar, Phys. Rev. **C79**, 014906 (2009). [arXiv:0808.0953 [nucl-th]]; D. Molnar, P. Huovinen, Nucl. Phys. **A830**, 475C-478C (2009). [arXiv:0907.5014 [nucl-th]]; A. El, Z. Xu, C. Greiner, Phys. Rev. **C81**, 041901 (2010). [arXiv:0907.4500 [hep-ph]].
[27] G. S. Denicol, X. -G. Huang, T. Koide, D. H. Rischke,
[arXiv:1003.0780 [hep-th]].
[28] G. S. Denicol, H. Niemi, E. Molnar, D. H. Rischke, To appear soon.
[29] L.B. Lucy, A.J. **82**, 1013 (1977); J.J. Monaghan, Annu. Rev. Astron. Astrophys. **30**, 543 (1992).
[30] C.E. Aguiar, T. Kodama, T. Osada, and Y. Hama, J. Phys. **27**, 75 (2001).
[31] Y. Hama, T. Kodama, O. Socolowski, Jr., Braz. J. Phys. **35**, 24-51 (2005). [hep-ph/0407264].
[32] S. S. Gubser and A. Yarom, Nucl. Phys. B **846**, 469 (2011) [arXiv:1012.1314 [hep-th]].
[33] J. L. ALbacete and A. Dumitru, arXiv:1011.5161 [hep-ph].