StaRPulS
(Stellar Radial Pulsation Simulator)

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About this document ...

Hereby the basic principles and methods underlying the stellar radial oscillation code StaRPulS are described. StaRPulS is a langrangian code, which include a time dependent description for radiative as well as convective energy transfer and is capable of doing linear stability analysis and full hydrodynamic simulations of the radial oscillations seen in stars.

The Partial Differential Equations

The basic equations describing the stellar matter, radiation and convective processes solved by the STaRPulS code is somewhat different from those presented in its predecessor (Olivier & Wood 2005).

Firstly the radiation contribution to the total internal energy and pressure has been formally seperated from the gas contributions. Radiative energy transfer is treated in a similar manner as that of Castor (1972), but in the Eddington approximation, i.e. $K/J = \frac{1}{3}$. Here J is the zeroth order moment of the radiation field (equivalent to the mean intensity) and K the second order moment of the radiation field.

Also formally nuclear energy generation has been included in the total energy equation. The code, in its current form, assumes a uniform composition throughout the stellar model. As such the code cannot properly model the effect of nuclear energy generation variation in stellar cores in general. Modelling this effect in fully convective low mass objects, such as brown dwarfs, is in principle feasible though.

Following Kollath et. al. (2002) a correction factor accounting for radiative losses at small Peclet numbers have been included in the expression for the turbulent source and convective flux term as well.

The main equations, in terms of the time variable t, the lagrangian variable M_r, and the Stokes derivative $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r}$, governing the motion and thermodynamics of stellar envelope are:

$$\frac{\partial\, r^3}{\partial\, M_r} - \frac{3}{4\pi\rho} = 0 \tag{1}$$

$$\frac{d\, r}{d\, t} - v_r = 0 \tag{2}$$

$$\frac{d\, v_r}{d\, t} + 4\pi r^2 \frac{\partial\, P_{tot}}{\partial\, M_r} + \frac{G\, M_r}{r^2} = 0 \tag{3}$$

$$\frac{d}{d\, t}\left(e_g + e_r + \varpi^2\right) - \frac{P_{tot}}{\rho^2}\frac{d\,\rho}{d\, t} + \frac{\partial\, L_{tot}}{\partial\, M_r} - \epsilon_{nuc} = 0 \tag{4}$$

$$\frac{d\, e_r}{d\, t} - \frac{P_r}{\rho^2}\frac{d\,\rho}{d\, t} + \frac{\partial\, L_r}{\partial\, M_r} - C_r = 0 \tag{5}$$

$$\frac{d\, \varpi^2}{d\, t} - \frac{P_t + P_{tv}}{\rho^2}\frac{d\,\rho}{d\, t} + \frac{\partial\, L_t}{\partial\, M_r} - C_\varpi = 0 \tag{6}$$

where $P_{tot} = P_g + P_r + P_t + P_{tv}$ and $L_{tot} = L_c + L_r + L_t$. Here ρ is the density, P_g is the local gas pressure, e_g the internal gas energy per unit mass, ϵ_{nuc} the nuclear energy generation rate, r the radius and v_r the radial velocity (G the gravitational constant). L_r is the radiative luminosity and $e_r = \frac{J}{\rho}$ is the radiation energy density (energy per unit mass of matter). κ the mean Rosseland opacity and c speed of light. The radiation pressure is $P_r = \frac{1}{3}\rho e_r = \frac{1}{3}J$, and the radiative luminosity is given by $L_r = 4\pi r^2 H$, where is H is the second order moment of the radiation field. Here we modelled it in the diffusion approximation (e.g. see Castor 1972) leading to the following equation relating L_r and the radiation pressure gradient:

$$\frac{1}{c}\kappa L_r - 16\pi^2 r^4 \frac{\partial\, P_r}{\partial\, M_r} = 0 \tag{7}$$

The term C_r in equation 5 is given by

$$C_r = c\,\kappa\left(a\, T^4 - \rho e_r\right) \tag{8}$$

where T is the temperature and a is the radiation energy density constant. The remaining dependent variables in equations 3 to 6 characterize the turbulent convective flow. The turbulent pressure P_t is given by:

$$P_t = \alpha_p \rho \varpi^2 \tag{9}$$

where ϖ^2 is the mean specific turbulent energy and it is a measure of the strength of the turbulence. The parameter $\alpha_p = \frac{2}{3}$ by definition. The turbulent viscous pressure term P_{tv}, responsible for dissipation of the kinetic energy of pulsation is given by:

$$P_{tv} = -\frac{16}{3}\pi\,\mu\,\rho^2\, r^3 \frac{\partial}{\partial\, M_r}\left(\frac{v_r}{r}\right) \tag{10}$$

where $\mu = \alpha_\mu \varpi \Lambda$ is the turbulent viscosity with Λ being some scale length and α_μ a free parameter. Other relevant terms in equations 4 and 6 are:

$$C_\varpi = S_\varpi - D_\varpi \tag{11}$$

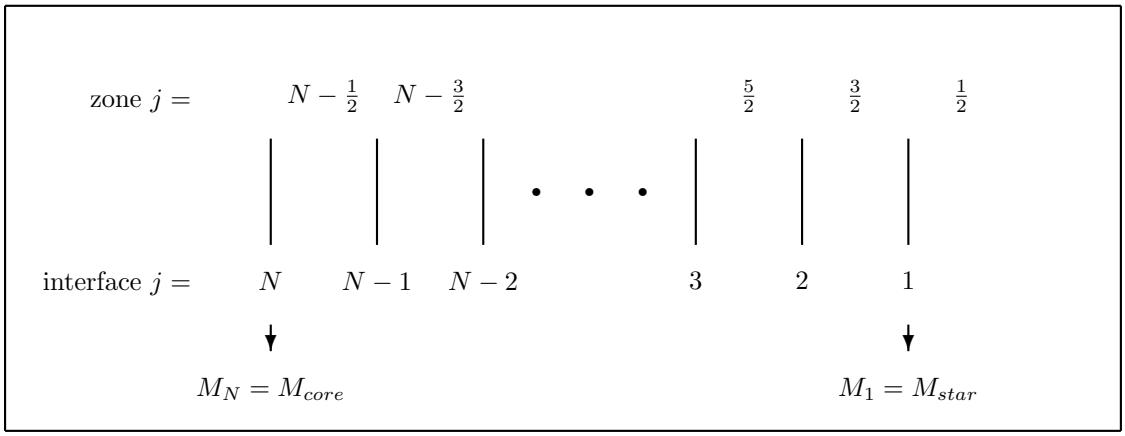

Figure 1: Stellar envelope mesh.

$$D_\varpi = \frac{\alpha_d}{\Lambda}\varpi^3 \tag{12}$$

$$S_\varpi = \frac{\nabla_{ad}}{H_P}\,\Pi \tag{13}$$

$$L_c = 4\,\pi\,r^2\,\rho\,\Pi \tag{14}$$

$$L_t = -4\,\pi\,\alpha_t\Lambda\,\varpi r^2\rho^2\frac{\partial\,\varpi^2}{\partial\,M_r} \tag{15}$$

$$\Pi = 4\pi\alpha_s\,\varpi\Lambda r^2\rho c_P\left(\left(\frac{\partial\,T}{\partial\,M_r}\right)_{ad} - \frac{\partial\,T}{\partial\,M_r}\right) f_{pec} \tag{16}$$

$$f_{pec} = \frac{1}{1+\alpha_r Pe^{-1}} \tag{17}$$

$$Pe = \frac{3}{4}\frac{\varpi\Lambda\kappa\rho^2 c_P}{acT^3} \tag{18}$$

where H_P is the local pressure scale-height (gas+radiation) and $\left(\frac{\partial\,T}{\partial\,M_r}\right)_{ad} = -\frac{T\nabla_{ad}}{4\pi r^2\rho H_P}$. The free parameters α_d, α_s, α_t and α_r in the convection model parametrize/control the effects of molecular disspation of turbulent energy at small scales, efficiency of the convection, convective overshoot and radiative losses respectively.

Like most stellar convection models in use, the model presented above contains an unknown scale-length, in this case represented by Λ. There is no guidance as to which length scale to use. For consistency with previous work in the field one can take the scale length Λ proportional to the local hydrostatic pressure scale-height, with radiation and turbulent pressure ignored:

$$\Lambda = \alpha_\Lambda\frac{r^2 P_g}{G\,M_r\,\rho} \tag{19}$$

where α_Λ is an additional free parameter. Without loss of generality one can take $\alpha_\Lambda = 1$ because given any set of free parameters (α_Λ,α_s,α_d,α_t,α_r,α_μ), the set (1,$\alpha_s\alpha_\Lambda$,α_d/α_Λ,$\alpha_t\alpha_\Lambda$,α_r/α_Λ,$\alpha_\mu\alpha_\Lambda$) will yield exactly the same convection model. StaRPuLS automatically sets $\alpha_\Lambda = 1$ and the user is required to supply the free parameters α_s, α_d, α_t, α_r and α_μ (more about this in the section on installation and operation of the code).

The Difference Equations

Imagine one can divide the stellar envelope into $N-1$ mass zones as depicted in Figure 1, bounded by N interfaces. Zones are denoted by half-integer values and the interfaces between zones by whole integer values.

In general a zone $j-\frac{1}{2}$ is bounded by the interfaces j and $j-1$. The whole stellar envelope is bounded by the interfaces $j=N$ at the interior and $j=1$ at the surface. One can also define "buffer" zone $j=\frac{1}{2}$ just outside the stellar model surface, which is useful when inplementing the surface boundary conditions (see later). Thus in total there are N interfaces and N zones in a the stellar model.

It is convenient, for example, to define the variables $V(\equiv 1/\rho)$, T, ϖ in the zones and L_t, L, and r at the interfaces. We shall denote a variable in a zone or at an interface by a subscript such as in $T^n_{j-\frac{1}{2}}$ and r^n_j. The superscript n shall refer to the value of the variable at some time $t=t_n$. We also need to calculate the value of some variables in a zone (at an interface) which is not usually defined there. We will use some average based on the values at (in) the two bounding interfaces (adjacent zones). A geometric or arithmetic average is used, which ever seems the more appropriate. It is also necessary to define time-averages during the time-step $\Delta t_n = t_n - t_{n-1}$ denoted by $n-\frac{1}{2}$. These time-averages are usually calculated based on energy conservation considerations and numerical stability of the difference equations. As an example, the time-average velocity is written as:

$$v_j^{n-\frac{1}{2}} = \theta_v\, v_j^n + (1-\theta_v)\; v_j^{n-1} \tag{20}$$

where $0 \le \theta_v \le 1$. Any other time-averages that appear in the difference equations are written in a similar form as the time-average velocity in equation 20 with $\theta = \frac{1}{2}$ exactly. The time-averages for r^2 and its inverse however (see equation 25) are chosen according to Fraley scheme (see Fraley 1968) to ensure conservation of total energy if $\theta_v = \frac{1}{2}$:

$$\left(r^2\right)_j^{n-\frac{1}{2}} = \frac{1}{3}\left(r_j^n r_j^n + r_j^n r_j^{n-1} + r_j^{n-1} r_j^{n-1}\right) \tag{21}$$

$$\left(\frac{1}{r^2}\right)_j^{n-\frac{1}{2}} = \frac{1}{r_j^n r_j^{n-1}} \tag{22}$$

The mass-steps across an interface $j-1$ and zone $j-\frac{1}{2}$ are also defined by $\Delta M_{j-1} = \frac{1}{2}[M_j - M_{j-2}]$ and $\Delta M_{j-\frac{1}{2}} = [M_j - M_{j-1}]$ respectively. Note that in this notation ΔM_{j-1} and $\Delta M_{j-\frac{1}{2}}$ are negative. Also we explicitly set the surface buffer zone $\Delta M_{\frac{1}{2}} = \Delta M_{\frac{3}{2}}$.

We then approximate the pulsation differential equations and the modified velocity-entropy correlation function, Π, on this mesh by the following system of difference equations:

$$\frac{\left(r_j^n\right)^3 - \left(r_{j-1}^n\right)^3}{\Delta M_{j-\frac{1}{2}}} - \frac{3}{4\pi} V^n_{j-\frac{1}{2}} = 0 \tag{23}$$

$$\frac{r_{j-1}^n - r_{j-1}^{n-1}}{\Delta t_n} - v_{j-1}^{n-\frac{1}{2}} = 0 \tag{24}$$

$$\frac{v_{j-1}^n - v_{j-1}^{n-1}}{\Delta t_n} + 4\pi \left(r^2\right)^{n-\frac{1}{2}}_{j-1} \frac{\left(P_{tot}\right)^{n-\frac{1}{2}}_{j-\frac{1}{2}} - \left(P_{tot}\right)^{n-\frac{1}{2}}_{j-\frac{3}{2}}}{\Delta M_{j-1}} + G\, M_{j-1} \left(\frac{1}{r^2}\right)^{n-\frac{1}{2}}_{j-1} = 0 \tag{25}$$

$$\frac{\left(e_{tot}\right)^n_{j-\frac{1}{2}} - \left(e_{tot}\right)^{n-1}_{j-\frac{1}{2}}}{\Delta t_n} + \left(P_{tot}\right)^{n-\frac{1}{2}}_{j-\frac{1}{2}} \frac{V^n_{j-\frac{1}{2}} - V^{n-1}_{j-\frac{1}{2}}}{\Delta t_n} + \frac{L_j^{n-\frac{1}{2}} - L_{j-1}^{n-\frac{1}{2}}}{\Delta M_{j-\frac{1}{2}}} - \left(\epsilon_{nuc}\right)^{n-\frac{1}{2}}_{j-\frac{1}{2}} = 0 \tag{26}$$

$$\frac{\left(e_r\right)^n_{j-\frac{1}{2}} - \left(e_r\right)^{n-1}_{j-\frac{1}{2}}}{\Delta t_n} + 4\pi \left(P_r\right)^n_{j-\frac{1}{2}} \frac{\left(r_j^n\right)^2 v_j^n - \left(r_{j-1}^n\right)^2 v_{j-1}^n}{\Delta M_{j-\frac{1}{2}}} + \frac{\left(L_r\right)^n_j - \left(L_r\right)^n_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \left(C_r\right)^n_{j-\frac{1}{2}} = 0 \tag{27}$$

$$\frac{1}{c}\kappa^n_{j-1}\left(L_r\right)^n_{j-1} + 16\pi^2 \left(r^n_{j-1}\right)^4 \frac{\left(P_r\right)^n_{j-\frac{1}{2}} - \left(P_r\right)^n_{j-\frac{3}{2}}}{\Delta M_{j-1}} = 0 \tag{28}$$

$$\frac{\left(\varpi^n_{j-\frac{1}{2}}\right)^2 - \left(\varpi^{n-1}_{j-\frac{1}{2}}\right)^2}{\Delta t_n} + 4\pi \left(P_t + P_{tv}\right)^n_{j-\frac{1}{2}} \frac{\left(r_j^n\right)^2 v_j^n - \left(r_{j-1}^n\right)^2 v_{j-1}^n}{\Delta M_{j-\frac{1}{2}}} + \frac{\left(L_t\right)^n_j - \left(L_t\right)^n_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \left(C_\varpi\right)^n_{j-\frac{1}{2}} = 0 \tag{29}$$

$$\Pi^n_{j-1} + 4\pi\alpha_s \varpi^n_{j-1} \left(r^n_{j-1}\right)^2 \frac{\left(c_P\right)^n_{j-1}}{V^n_{j-1}} \Lambda^n_{j-1} \left[\frac{T^n_{j-\frac{1}{2}} - T^n_{j-\frac{3}{2}}}{\Delta M_{j-1}} - \left(\left(\frac{\partial T}{\partial M_r}\right)_{ad}\right)^n_{j-1}\right] \left(f_{pec}\right)^n_{j-1} = 0 \tag{30}$$

with

$$\left(\left(\frac{\partial T}{\partial M_r}\right)_{ad}\right)^n_{j-1} = -\frac{G M_{j-1} T^n_{j-1} \left(\nabla_{ad}\right)^n_{j-1}}{4\pi \left(r^n_{j-1}\right)^4 \left(\left(P_g\right)^n_{j-1} + \left(P_r\right)^n_{j-1}\right)} \tag{31}$$

$$(f_{pec})^n_{j-1} = \left[1 + \frac{4}{3}\alpha_r \frac{ac\left(T^n_{j-1}\right)^3\left(V^n_{j-1}\right)^2}{\varpi^n_{j-1}\Lambda^n_{j-1}\kappa^n_{j-1}\left(c_P\right)^n_{j-1}}\right]^{-1} \tag{32}$$

$$\Lambda^n_{j-1} = \alpha_\Lambda \frac{\left(r^n_{j-1}\right)^2 (P_g)^n_{j-1} V^n_{j-1}}{G\, M_{j-1}} \tag{33}$$

The total internal energy is expressed as:

$$(e_{tot})^n_{j-\frac{1}{2}} = (e_g)^n_{j-\frac{1}{2}} + (e_r)^n_{j-\frac{1}{2}} + \left(\varpi^n_{j-\frac{1}{2}}\right)^2 \tag{34}$$

while the total pressure is expressed as:

$$(P_{tot})^n_{j-\frac{1}{2}} = (P_g)^n_{j-\frac{1}{2}} + (P_r)^n_{j-\frac{1}{2}} + (P_t)^n_{j-\frac{1}{2}} + (P_{tv})^n_{j-\frac{1}{2}} + (P_{av})^n_{j-\frac{1}{2}} \tag{35}$$

where:

$$(P_g)^n_{j-\frac{1}{2}} = P_g\left(V^n_{j-\frac{1}{2}}, T^n_{j-\frac{1}{2}}\right) \tag{36}$$

$$(P_r)^n_{j-\frac{1}{2}} = \frac{1}{3} {e_r}^n_{j-\frac{1}{2}} / V^n_{j-\frac{1}{2}} \tag{37}$$

$$(P_t)^n_{j-\frac{1}{2}} = \alpha_p\left(\varpi^n_{j-\frac{1}{2}}\right)^2 / V^n_{j-\frac{1}{2}} \tag{38}$$

$$(P_{tv})^n_{j-\frac{1}{2}} = -16\pi\alpha_\mu \varpi^n_{j-\frac{1}{2}} \Lambda^n_{j-\frac{1}{2}} \frac{\left(r^n_{j-\frac{1}{2}}\right)^3}{\left(V^n_{j-\frac{1}{2}}\right)^2} \frac{\left[\frac{v^n_j}{r^n_j} - \frac{v^n_{j-1}}{r^n_{j-1}}\right]}{\Delta M_{j-\frac{1}{2}}} \tag{39}$$

where

$$\Lambda^n_{j-\frac{1}{2}} = \alpha_\Lambda \frac{\left(r^n_{j-\frac{1}{2}}\right)^2 (P_g)^n_{j-\frac{1}{2}} V^n_{j-\frac{1}{2}}}{G M^n_{j-\frac{1}{2}}} \tag{40}$$

Shocks are treated as in Stellingwerf (1975) through the artificial viscous pressure given by

$$(P_{av})^n_{j-\frac{1}{2}} = c_1^2\left[v^n_j - v^n_{j-1} - c_2\ (c_s)^n_{j-\frac{1}{2}}\right]^2 / V^n_{j-\frac{1}{2}} \tag{41}$$

for $v^n_j - v^n_{j-1} > c_2\ (c_s)^n_{j-\frac{1}{2}}$, else zero if otherwise. Here c_s is the local adiabatic sound speed and c_1 and c_2 are free parameters (usually set to $c_1 = 4$ and $c_2 = 0.1$). The convective and turbulent luminosity terms are expressed as:

$$(L_c)^n_{j-1} = 4\pi \frac{\left(r^n_{j-1}\right)^2}{V^n_{j-1}} \Pi^n_{j-1} \tag{42}$$

$$(L_t)^n_{j-1} = -\frac{8}{3}\pi\alpha_t \Lambda^n_{j-1}\left(\frac{r^n_{j-1}}{V^n_{j-1}}\right)^2 \frac{\left(\varpi^n_{j-\frac{1}{2}}\right)^3 - \left(\varpi^n_{j-\frac{3}{2}}\right)^3}{\Delta M_{j-1}} \tag{43}$$

The remaining terms in the equations 29 and 27 are expressed as:

$$(C_r)^n_{j-\frac{1}{2}} = c\kappa^n_{j-\frac{1}{2}}\left(a\left(T^n_{j-\frac{1}{2}}\right)^4 - J^n_{j-\frac{1}{2}}\right) \tag{44}$$

$$(C_\varpi)^n_{j-\frac{1}{2}} = (S_\varpi)^n_{j-\frac{1}{2}} - (D_\varpi)^n_{j-\frac{1}{2}} \tag{45}$$

with

$$(S_\varpi)^n_{j-\frac{1}{2}} = \frac{GM^n_{j-\frac{1}{2}}\left(\nabla_{ad}\right)^n_{j-\frac{1}{2}}}{\left(r^n_{j-\frac{1}{2}}\right)^2\left((P_g)^n_{j-\frac{1}{2}} + (P_r)^n_{j-\frac{1}{2}}\right)V^n_{j-\frac{1}{2}}}\left(\frac{\Pi^n_j + \Pi^n_{j-1}}{2}\right) \tag{46}$$

$$(D_\varpi)^n_{j-\frac{1}{2}} = \frac{\alpha_d}{\Lambda^n_{j-\frac{1}{2}}}\left(\left(\varpi^n_{j-\frac{1}{2}}\right)^3 - \varpi^3_F\right) \tag{47}$$

where ϖ_F is a small positive velocity (see Olivier & Wood 2005 for details). Finally it is assumed all the remaining thermodynamic variables, the nuclear energy generation rate and the opacity are functions of the density and temperature, and is thus expressed as

$$(e_g)^n_{j-\frac{1}{2}} = e_g\left(V^n_{j-\frac{1}{2}}, T^n_{j-\frac{1}{2}}\right) \tag{48}$$

$$(\nabla_{ad})^n_{j-\frac{1}{2}} = \nabla_{ad}\left(V^n_{j-\frac{1}{2}}, T^n_{j-\frac{1}{2}}\right) \tag{49}$$

$$(c_s)^n_{j-\frac{1}{2}} = c_s\left(V^n_{j-\frac{1}{2}}, T^n_{j-\frac{1}{2}}\right) \tag{50}$$

$$(\epsilon_{nuc})^n_{j-\frac{1}{2}} = \epsilon_{nuc}\left(V^n_{j-\frac{1}{2}}, T^n_{j-\frac{1}{2}}\right) \tag{51}$$

$$\kappa^n_{j-\frac{1}{2}} = \kappa\left(V^n_{j-\frac{1}{2}}, T^n_{j-\frac{1}{2}}\right) \tag{52}$$

We have a total of $8\,N$ unknowns (r^n_j, v^n_j, $V^n_{j-\frac{1}{2}}$, $T^n_{j-\frac{1}{2}}$, $\varpi^n_{j-\frac{1}{2}}$, $(e_r)^n_{j-\frac{1}{2}}$, $(L_r)^n_j$ and Π^n_j). The equations 23 to 30, each couple some or all the variables associated across different mass zones. In total there are $8\,(N-1)$ equations of this form. An additional 8 equations are obtained from the imposed boundary conditions.

Boundary conditions

At the interior boundary we set:

$$r^n_N - r_{core} = 0 \tag{53}$$

$$v^n_N = 0 \tag{54}$$

i.e. that is assuming a rigid core in the case where $r_{core} \neq 0$. The radiative luminosity at the interior boundary must satisfy:

$$(L_r)^n_N + (L_c)^n_N + (L_t)^n_N - L_{core} = 0 \tag{55}$$

where we set $(L_c)^n_N = 4\pi \frac{(r^n_N)^2}{V^n_{N-\frac{1}{2}}} \Pi^n_N$ and $(L_t)^n_N = 0$. We also make the following assumption:

$$\Pi^n_{N-1} - \Pi^n_N = 0 \tag{56}$$

At the suface we apply a (small) constant gas pressure P_{ext} in the surface "buffer" zone and also assume the atmosphere is isothermal near the stellar surface. This then gives the following two boundary conditions:

$$(P_g)^n_{\frac{1}{2}} - P_{ext} = 0 \tag{57}$$

$$T^n_{\frac{1}{2}} - T^n_{\frac{3}{2}} = 0 \tag{58}$$

In the surface "buffer" zone we also ignore the contributions due to $(P_{tv})^{n-\frac{1}{2}}_{\frac{1}{2}}$ and $(P_{av})^{n-\frac{1}{2}}_{\frac{1}{2}}$ in the relevant momentum conservation equation. We then require a further two boundary conditions. For the turbulent velocity we assume that:

$$\varpi^n_{\frac{1}{2}} - \varpi^n_{\frac{3}{2}} = 0 \tag{59}$$

whilst the radiative luminosity should satisfy the equation of radiative transfer in the optically thin limit (assuming a plane parallel atmosphere):

$$(L_r)^n_1 - 2\pi c\,(r^n_1)^2\,J^n_1 = 0 \tag{60}$$

where $J^n_1 = \left[\frac{(e_r)^n_{\frac{1}{2}}\,(e_r)^n_{\frac{3}{2}}}{V^n_{\frac{1}{2}} V^n_{\frac{3}{2}}}\right]^{\frac{1}{2}}$. With the above 8 boundary conditions (equations 53 to 60) we now have a closed system of $8N$ difference equations and $8N$ unknowns.

Solution of the difference equations

The time-dependent equations

In principle the set of $M = 8N$ difference equations and $M = 8N$ unknowns can be solved at time t_n given a model at time t_{n-1} and a time-step Δt_n. The pulsation is thus treated as an initial value problem. The set of nonlinear difference equations are solved with the well-known Henyey method which uses a generalized Newton-Raphson iteration procedure.

This iteration procedure can be understood as follows. Given a set of M nonlinear equations ($f_j = 0$ $j = 1, M$), and an approximate solution $(x_1, \ldots, x_M)$, one would like to find a set of M corrections, $(\delta x_1, \ldots, \delta x_M)$ such that:

$$f_j\left(x_1 + \delta x_1, \ldots, x_M + \delta x_M\right) = 0 \quad j = 1, \ldots, M \tag{61}$$

Linearization (i.e truncating the Taylor series expansion at first order) of the left-hand sides of the above set of equations one have:

$$f_j\left(x_1, \ldots, x_M\right) + \sum_{i=1}^{M} \frac{\partial f_j}{\partial x_i} \delta x_i = 0 \quad j = 1, \ldots, M \tag{62}$$

or

$$\mathbf{J}\, \delta \vec{\mathbf{x}} = -\vec{\mathbf{f}} \tag{63}$$

where $\mathbf{J}$ is the Jacobian matrix (size $M \times M$) with components $\left(J_{ij} = \frac{\partial f_j}{\partial x_i},\, i, j = 1, M\right)$, $\delta \vec{\mathbf{x}}$ is the one dimensional vector of corrections with components $(\delta x_i,\, i = 1, M)$ and $\vec{\mathbf{f}}$ is the one dimensional vector of function values with components $\left(f_j\left(x_1, \ldots, x_M\right),\, j = 1, M\right)$. In general the set of corrections obtained by solving this linear system will not take us exactly to the true solution from an initial approximate solution, since we have ignored higher order terms in the linearization process. This is particularly true if the approximate solution is far from the true solution. So in general the process has to be repeated, updating the Jacobian matrix $\mathbf{J}$ and the vectors $\delta \vec{\mathbf{x}}$ and $\vec{\mathbf{f}}$ each time. In summary, given an approximate solution $\vec{\mathbf{x}}^{k-1}$ with components $\left(x_1^{k-1}, \ldots, x_M^{k-1}\right)$, calculate the Jacobian matrix $\mathbf{J}^{k-1}$ and the vector $\vec{\mathbf{f}}^{k-1}$, solve for the corrections $\delta \vec{\mathbf{x}}^k$ and calculate a new approximation to the solution:

$$\vec{\mathbf{x}}^k = \vec{\mathbf{x}}^{k-1} + \beta \delta \vec{\mathbf{x}}^k \tag{64}$$

where $\beta = 1$ normally. Repeat the process several times until the solution has been reached to desired precision, e.g. when $|\delta x_i^k / x_i^k|_{max} < \epsilon_1$ where ϵ_1 is some small tolerance value. Here $|\delta x_i^k / x_i^k|_{max}$ denotes the maximum fractional correction of all the variables. During the initial iteration steps, when still far from true solution, the full corrections $\vec{\delta \mathbf{x}}^{\,k}$ are not used but are rather scaled by a factor $\beta = \epsilon_2 / |\delta x_i^k / x_i^k|_{max}$ first. Here ϵ_2 is some user estimated tolerance. Once the corrections become sufficiently small, i.e. when close enough to the true solution (taken when $|\delta x_i^k / x_i^k|_{max} < \epsilon_2$), during the iteration process, the full corrections ($\beta = 1$) are applied. In the current implementation of the code the solution at the current time ($t = t_n$) is found by such a Newton-Raphson iteration scheme using the solution at the previous time-step ($t = t_{n-1}$) as a starting point.

In general the matrix $\mathbf{J}$ is a sparse matrix, since the individual difference equations depend only on variables defined in a small number of neighbouring zones, as opposed to the full set of $M = 8N$ variables. Careful inspection of the current system here will show only a maximum of 4 zones are coupled by any given difference equation. Proper ordering (as opposed random ordering) of the variables and equations will lead to a special band matrix form for $\mathbf{J}$.

In the current implementation of the code, the following ordering of the 8N variables and 8N equations is implemented:

$$\vec{\mathbf{x}} = \left(\varpi_{\frac{1}{2}}^n, V_{\frac{1}{2}}^n, T_{\frac{1}{2}}^n, (e_r)_{\frac{1}{2}}^n, r_1^n, v_1^n, \Pi_1^n, (L_r)_1^n, \ldots\ldots, \varpi_{N-\frac{1}{2}}^n, V_{N-\frac{1}{2}}^n, T_{N-\frac{1}{2}}^n, (e_r)_{N-\frac{1}{2}}^n, r_N^n, v_N^n, \Pi_N^n, (L_r)_N^n\right)$$

$$\vec{f} = \left(f_1^{\varpi}, f_1^V, f_1^T, f_1^{e_r}, f_1^r, f_1^v, f_1^{\Pi}, f_1^{L_r}, , \ldots\ldots, f_N^{\varpi}, f_N^V, f_N^T, f_N^{e_r}, f_N^r, f_N^v, f_N^{\Pi}, f_N^{L_r}, \right)$$

The set of functions, f_j^x, are given by the left hand sides of the following difference equations (denoted F^K

where K is the relevant equation number)

$$for\ \ j=1\ \ldots \begin{cases} f_j^{\varpi} = F^{59} \\ f_j^{V} = F^{57} \\ f_j^{T} = F^{58} \\ f_j^{e_r} = F^{60} \\ f_j^{r} = F^{23} \\ f_j^{v} = F^{24} \\ f_j^{\Pi} = F^{30} \\ f_j^{L_r} = F^{28} \end{cases} \qquad for\ \ 1<j<N\ \ldots \begin{cases} f_j^{\varpi} = F^{29} \\ f_j^{V} = F^{25} \\ f_j^{T} = F^{26} \\ f_j^{e_r} = F^{27} \\ f_j^{r} = F^{23} \\ f_j^{v} = F^{24} \\ f_j^{\Pi} = F^{30} \\ f_j^{L_r} = F^{28} \end{cases} \qquad for\ \ j=N\ \ldots \begin{cases} f_j^{\varpi} = F^{29} \\ f_j^{V} = F^{25} \\ f_j^{T} = F^{26} \\ f_j^{e_r} = F^{27} \\ f_j^{r} = F^{53} \\ f_j^{v} = F^{54} \\ f_j^{\Pi} = F^{56} \\ f_j^{L_r} = F^{55} \end{cases}$$

As mentioned above, at most 4 zones are coupled by any given difference equation, this means that any given row in the Jacobian matrix **J** will have at most $4 \times 8 = 32$ non-zero entries. Taking as an example a stellar model divided up into 6 zones, inspection of the relevant equations, given the ordering above, will show that **J** will have the structure shown in figure 2. For a given number of zones N the ratio of possible non-zero matrix elements ($4 \times 8^2 N$) to total number of entries ($8N \times 8N$) is $\frac{4}{N}$. For $N = 400$, as a typical example, this ratio is just 1%. The matrix can be efficiently stored (by storing only the x's in figure 2) and solution of this system can also be very efficiently done, using standard gaussian elimination techniques only on the stored non-zero matrix elements. Since an implicit hydrodynamical scheme is used (i.e. variables at $t = t_n$ depend on variables at $t = t_{n-1}$ AND $t = t_n$), there is no restriction on the time-step size based on stability considerations such as the Courant-Friedrichs-Lewy condition. However, one usually restricts the time step by not letting some of the variables at any given mesh point (usually the pressure, temperature and specific turbulent energy) change by more than a specified upper limit. Also the time-step is not allowed to increase by more than a given fraction of the previous time-step (usually ~ 35 percent) and is forced to remain smaller than a specified upper limit, usually a tenth or less of the pulsation period. To start the pulsation of the stellar model, a static model ($\frac{1}{\Delta t} = 0$ and $\theta = 1$) is constructed by the same Newton-Raphson procedure from an initial approximate static solution, and then given a small velocity perturbation. This perturbed model is then used as the model at time t_0.

An initial approximate static solution is contructed in STaRPuLS by solving a simplified version of the static equations where L_t and P_t is ignored and the equation for radiative transfer replaced by $e_r = \frac{aT^4}{\rho}$. This simplified set of static equations is then solved using a shooting method, integrating from the surface inwards, iterating on the surface radius and luminosity in the case where nuclear energy generation is included. If nuclear energy generation is ignored (see section on installation and operation) the surface luminosity is set equal to the core luminosity, and remains fixed in the shooting shooting algorithm.

Linear Stability Analysis

Consider the difference equations 23 to 30 and the associated boundary conditions 53 to 59. Letting $\Delta t_n \to 0$ analytically, one will have $x^n \to x^{n-1}$ and the equations will become (dropping the time index n):

$$\frac{r_j^3 - r_{j-1}^3}{\Delta M_{j-\frac{1}{2}}} - \frac{3}{4\pi} V_{j-\frac{1}{2}} = 0 \tag{65}$$

$$\frac{d}{dt} r_{j-1} - v_{j-1} = 0 \tag{66}$$

$$\frac{d}{dt} v_{j-1} + 4\,\pi\, r_{j-1}^2 \frac{(P_{tot})_{j-\frac{1}{2}} - (P_{tot})_{j-\frac{3}{2}}}{\Delta M_{j-1}} + \frac{G\, M_{j-1}}{r_{j-1}^2} = 0 \tag{67}$$

$$\frac{d}{dt}(e_{tot})_{j-\frac{1}{2}} + (P_{tot})_{j-\frac{1}{2}} \frac{d}{dt} V_{j-\frac{1}{2}} + \frac{L_j - L_{j-1}}{\Delta M_{j-\frac{1}{2}}} - (\epsilon_{nuc})_{j-\frac{1}{2}} = 0 \tag{68}$$

$$\frac{d}{dt}(e_r)_{j-\frac{1}{2}} + (P_r)_{j-\frac{1}{2}} \frac{d}{dt} V_{j-\frac{1}{2}} + \frac{(L_r)_j - (L_r)_{j-1}}{\Delta M_{j-\frac{1}{2}}} - (C_r)_{j-\frac{1}{2}} = 0 \tag{69}$$

$$\frac{1}{c} \kappa_{j-1} (L_r)_{j-1} + 16\pi^2 \left(r_{j-1}\right)^4 \frac{(P_r)_{j-\frac{1}{2}} - (P_r)_{j-\frac{3}{2}}}{\Delta M_{j-1}} = 0 \tag{70}$$

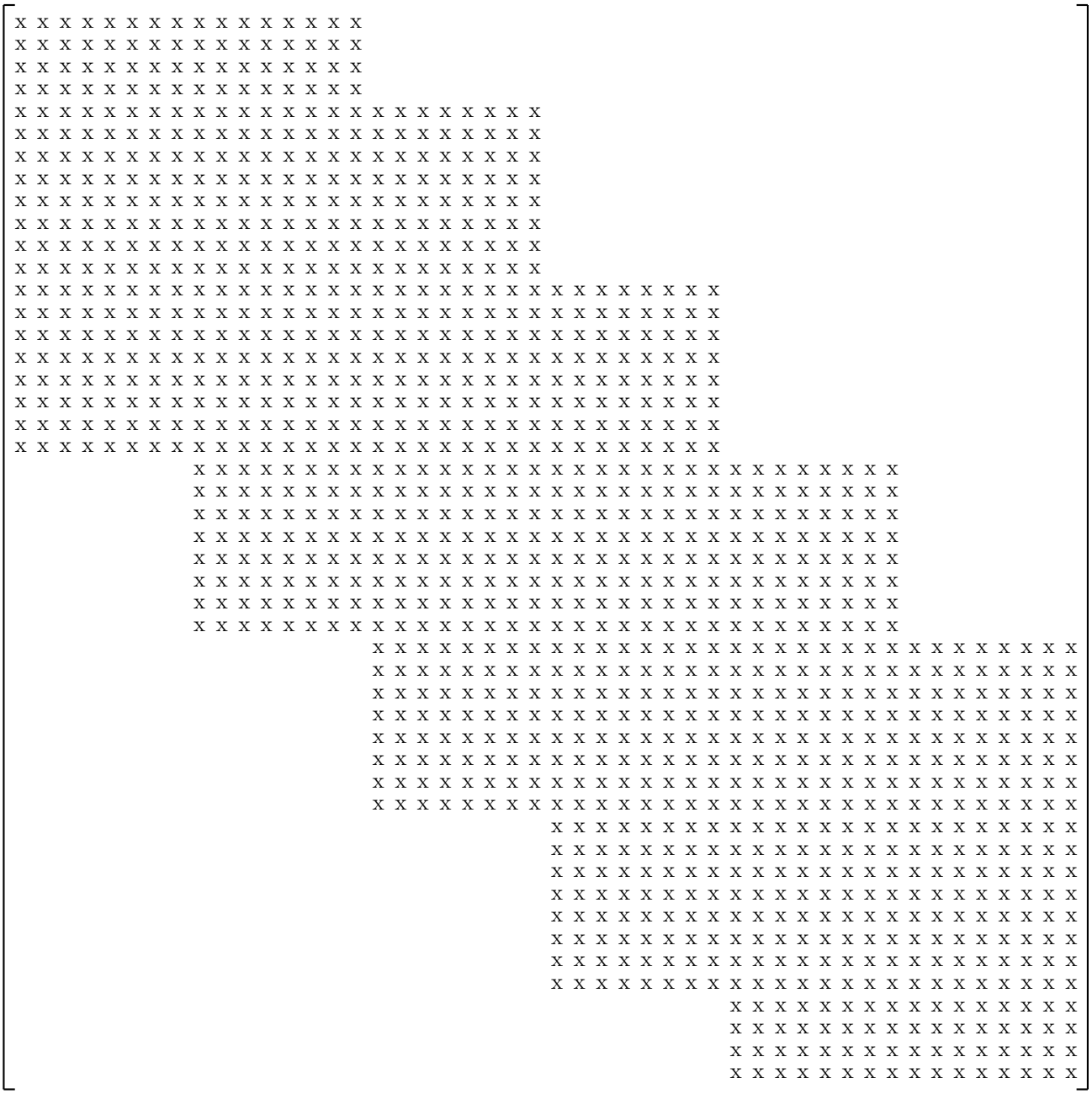

Figure 2: Structure of the Jacobian matrix J for a 6 zone stellar model. Here x denotes a possible non-zero matrix element and a blank entry denotes a zero value matrix element.

$$\frac{d}{dt}\left(\varpi_{j-\frac{1}{2}}\right)^2 + \left(P_t + P_{tv}\right)_{j-\frac{1}{2}} \frac{d}{dt} V_{j-\frac{1}{2}} + \frac{\left(L_t\right)_j - \left(L_t\right)_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \left(C_\varpi\right)_{j-\frac{1}{2}} = 0 \tag{71}$$

$$\Pi_{j-1} + 4\pi\alpha_s\varpi_{j-1}\left(r_{j-1}\right)^2 \frac{\left(c_P\right)_{j-1}}{V_{j-1}}\Lambda_{j-1}\left[\frac{T_{j-\frac{1}{2}} - T_{j-\frac{3}{2}}}{\Delta M_{j-1}} - \left(\left(\frac{\partial T}{\partial M_r}\right)_{ad}\right)_{j-1}\right]\left(f_{pec}\right)_{j-1} = 0 \tag{72}$$

with the following boundary conditions

$$r_N - r_{core} = 0 \tag{73}$$

$$v_N = 0 \tag{74}$$

$$\left(L_r\right)_N + \left(L_c\right)_N + \left(L_t\right)_N - L_{core} = 0 \tag{75}$$

$$\Pi_{N-1} - \Pi_N = 0 \tag{76}$$

$$\left(P_g\right)_{\frac{1}{2}} - P_{ext} = 0 \tag{77}$$

$$T_{\frac{1}{2}} - T_{\frac{3}{2}} = 0 \tag{78}$$

$$\varpi_{\frac{1}{2}} - \varpi_{\frac{3}{2}} = 0 \tag{79}$$

$$\left(L_r\right)_1 - 2\pi c\left(r_1\right)^2 J_1 = 0 \tag{80}$$

where $\left(L_c\right)_N = 4\pi\left(r_N\right)^2 V_{N-\frac{1}{2}}\Pi_N$, $\left(L_t\right)_N = 0$, $J_1 = \left[\frac{\left(e_r\right)_{\frac{1}{2}}\left(e_r\right)_{\frac{3}{2}}}{V_{\frac{1}{2}}V_{\frac{3}{2}}}\right]^{\frac{1}{2}}$ and the contributions due to $\left(P_{tv}\right)_{\frac{1}{2}}$ and $\left(P_{av}\right)_{\frac{1}{2}}$ are ignored in the relevant momentum conservation equation.

The equations above represent a discretization in space only of the original partial differential equations. It is on this set of equations that a linear stability analysis is performed. First we write each time-dependent variable in the above set of equations as $x(t) = x_0 + \delta x(t)$, where x_0 is value of the variable in the hydrostatic solution. We do a formal Taylor expansion for each the above set $8N$ of nonlinear equations in terms of the perturbations $\delta x(t)$ to first order. Subtracting each equation in the hydrostatic limit from the corresponding first order Taylor expansion of the nonlinear time-dependent equation results in the following set of *linearized* difference equations:

$$\frac{r_j^2\delta r_j - r_{j-1}^2\delta r_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \frac{1}{4\pi}\delta V_{j-\frac{1}{2}} = 0 \tag{81}$$

$$\frac{d}{dt}\delta r_{j-1} - \delta v_{j-1} = 0 \tag{82}$$

$$\frac{d}{dt}\delta v_{j-1} + 4\,\pi\, r_{j-1}^2\frac{\left(\delta P_{tot}\right)_{j-\frac{1}{2}} - \left(\delta P_{tot}\right)_{j-\frac{3}{2}}}{\Delta M_{j-1}} - 4\frac{G\,M_{j-1}}{r_{j-1}^3}\delta r_{j-1} = 0 \tag{83}$$

$$\frac{d}{dt}\left(\delta e_{tot}\right)_{j-\frac{1}{2}} + \left(P_{tot}\right)_{j-\frac{1}{2}}\frac{d}{dt}\delta V_{j-\frac{1}{2}} + \frac{\delta L_j - \delta L_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \left(\delta\epsilon_{nuc}\right)_{j-\frac{1}{2}} = 0 \tag{84}$$

$$\frac{d}{dt}\left(\delta e_{r_{j-\frac{1}{2}}}\right) + \left(P_r\right)_{j-\frac{1}{2}}\frac{d}{dt}\delta V_{j-\frac{1}{2}} + \frac{\left(\delta L_r\right)_j - \left(\delta L_r\right)_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \left(\delta C_r\right)_{j-\frac{1}{2}} = 0 \tag{85}$$

$$\frac{1}{c}\left[\delta\kappa_{j-1}(L_r)_{j-1} + \kappa_{j-1}(\delta L_r)_{j-1}\right] + 16\pi^2\left(r_{j-1}\right)^4\left[4\frac{\delta r_{j-1}}{r_{j-1}}\frac{\left(P_r\right)_{j-\frac{1}{2}} - \left(P_r\right)_{j-\frac{3}{2}}}{\Delta M_{j-1}} + \frac{\left(\delta P_r\right)_{j-\frac{1}{2}} - \left(\delta P_r\right)_{j-\frac{3}{2}}}{\Delta M_{j-1}}\right] = 0 \tag{86}$$

$$2\varpi\frac{d}{dt}\delta\varpi_{j-\frac{1}{2}} + \left(P_t\right)_{j-\frac{1}{2}}\frac{d}{dt}\delta V_{j-\frac{1}{2}} + \frac{\left(\delta L_t\right)_j - \left(\delta L_t\right)_{j-1}}{\Delta M_{j-\frac{1}{2}}} - \left(\delta C_\varpi\right)_{j-\frac{1}{2}} = 0 \tag{87}$$

$$\delta\Pi_{j-1} + \delta A_{j-1}\left[\frac{T_{j-\frac{1}{2}} - T_{j-\frac{3}{2}}}{\Delta M_{j-1}} - \left(\left(\frac{\partial T}{\partial M_r}\right)_{ad}\right)_{j-1}\right] + A_{j-1}\left[\frac{\delta T_{j-\frac{1}{2}} - \delta T_{j-\frac{3}{2}}}{\Delta M_{j-1}} - \left(\delta\left(\frac{\partial T}{\partial M_r}\right)_{ad}\right)_{j-1}\right] = 0 \tag{88}$$

where $A_{j-1} = 4\pi\alpha_s\varpi_{j-1}\left(r_{j-1}\right)^2\frac{\left(c_P\right)_{j-1}}{V_{j-1}}\Lambda_{j-1}\left(f_{pec}\right)_{j-1}$. Linearizing the boundary conditions gives:

$$\delta r_N = 0 \tag{89}$$

$$\delta v_N = 0 \tag{90}$$

$$(\delta L_r)_N + (\delta L_c)_N + (\delta L_t)_N = 0 \tag{91}$$

$$\delta\Pi_{N-1} - \delta\Pi_N = 0 \tag{92}$$

$$(\delta P_g)_{\frac{1}{2}} = 0 \tag{93}$$

$$\delta T_{\frac{1}{2}} - \delta T_{\frac{3}{2}} = 0 \tag{94}$$

$$\delta\varpi_{\frac{1}{2}} - \delta\varpi_{\frac{3}{2}} = 0 \tag{95}$$

$$(\delta L_r)_1 - \frac{1}{2}\left[4\frac{\delta r_1}{r_1} + \frac{(\delta e_r)_{\frac{1}{2}}}{(e_r)_{\frac{1}{2}}} + \frac{(\delta e_r)_{\frac{3}{2}}}{(e_r)_{\frac{3}{2}}} - \frac{\delta V_{\frac{1}{2}}}{V_{\frac{1}{2}}} - \frac{\delta V_{\frac{3}{2}}}{V_{\frac{3}{2}}}\right](L_r)_1 = 0 \tag{96}$$

Expanding all the perturbations in the above equations to first order in terms of the perturbations of the basic set of 8N variables:

$$\varpi_{\frac{1}{2}}, V_{\frac{1}{2}}, T_{\frac{1}{2}}, (e_r)_{\frac{1}{2}}, r_1, v_1, \Pi_1, (L_r)_1 \ \ldots\ldots, \varpi_{N-\frac{1}{2}}, V_{N-\frac{1}{2}}, T_{N-\frac{1}{2}}, (e_r)_{N-\frac{1}{2}}, r_N, v_N, \Pi_N, (L_r)_N$$

the above set of linearized equations can be represented as:

$$\mathbf{J_0}\,\delta\vec{\mathbf{x}} + \mathbf{J_1}\frac{d}{d\,t}\delta\vec{\mathbf{x}} = 0 \tag{97}$$

where the one-dimensional vector $\delta\vec{x}$ is:

$$\left(\delta\varpi_{\frac{1}{2}}, \delta V_{\frac{1}{2}}, \delta T_{\frac{1}{2}}, (\delta e_r)_{\frac{1}{2}}, \delta r_1, \delta v_1, \delta\Pi_1, (\delta L_r)_1, \ldots\ldots, \delta\varpi_{N-\frac{1}{2}}, \delta V_{N-\frac{1}{2}}, \delta T_{N-\frac{1}{2}}, (\delta e_r)_{N-\frac{1}{2}}, \delta r_N, \delta v_N, \delta\Pi_N, (\delta L_r)_N\right)$$

$\mathbf{J_0}$ is the jacobian matrix in the static case defined earlier and $\mathbf{J_1}$ is also a band matrix of the same dimensions as $\mathbf{J_0}$ The above system of coupled linear differential equations can be solved using the usual method of letting $\delta\vec{\mathbf{x}}(t) = \delta\vec{\mathbf{x}}_{\mathbf{sp}}\exp\omega t$, where ω and the components of $\delta\vec{\mathbf{x}}_{sp}$ can be complex. Substituting the above form for $\delta\vec{\mathbf{x}}(\mathbf{t})$ in equation 97 gives:

$$(\mathbf{J_0} + \omega\mathbf{J_1})\,\delta\vec{\mathbf{x}}_{\mathbf{sp}} = 0 \tag{98}$$

The system of equations 98 is a generalized eigenvalue problem, with ω representing the eigenvalues and $\delta\vec{\mathbf{x}}_{\mathbf{sp}}$ the eigenvectors. The eigenvalues and eigenvectors are in general complex, and each pair of eigenvalue and eigenvector describes one mode of vibration. Note that any linear combination of eigensolutions (or modes) is also a solution to linear stability problem 97. The complex part of eigenvalue represents the frequency of the oscillatory motion, and the real part of the eigenvalue determines whether the amplitude of the oscillatory motion will grow or decay with time. The real part of the eigenvector components gives the amplitude (i.e. the amount of deviation from the hydrostatic solution) for that component at $t = 0$. The difference in the arguments of any two components of the eigenvector gives the phase difference between those components.

Note that for a given mode of vibration if all the components of the eigenvector can be scaled by the same arbitary constant, the new resulting vector will still be a solution to the eigensystem 98. We exploit this fact to solve the eigensystem as follows. First we remove equation 89 from the original set of linearized equations and replace it with the surface boundary (or nomalization) condition:

$$\frac{\delta r_1}{r_1} = 1 \tag{99}$$

and update the matrix $\mathbf{J_0}$ appropriately. This then leads to the following set of non-homogenous linear equations:

$$(\mathbf{J_0} + \omega\mathbf{J_1})\,\delta\vec{\mathbf{x}}_{\mathbf{sp}} = \vec{\mathbf{b}} \tag{100}$$

where $\vec{\mathbf{b}} \neq 0$. Given a trail value for ω one can then solve the system 100 for $\delta\vec{\mathbf{x}}_{\mathbf{sp}}$. A solution to the original eigenvalue problem 98 is then found by searching for the value(s) of ω which yields a vector $\delta\vec{\mathbf{x}}_{\mathbf{sp}}$ via system 100 that satisfy condition 89.

The Constitutive Relations

Equation of state (EOS) tables in ASCII format for fixed compostion are provided with the code. The EOS tables were calculated using a code very similar to that discussed in detail by Stellingwerf (1982), which supplies all the needed thermodynamic quantities as functions of temperature, density and composition and assumes thermodynamic equilibrium between the matter and radiation. The ionization of H, He, He^+ and M, where M is some fictional metal, are included and calculated from the usual Saha equations. The EOS tables lists the quantities $\log e_g$, $\log c_V$, $\log P_g$, χ_V and χ_T, on a uniform $\log_{10}\rho$ - $\log_{10}T$ grid. During a stellar model computation bicubic spline interpolation is done on the relevant EOS table and the necessary conversions are done to obtain the required thermodynamic quantities and their derivatives with respect to V and T. The partial derivatives at the grid points needed for the bicubic interpolation are evaluated at code startup using a natural cubic spline interpolation routine and stored in memory. The radiation contribution to pressure, internal energy, and their partial derivatives are ignored in the tables since it is explicitly treated separately from the gas in the pulsation equations.

The contribution due to the photon gas is included in the calculation for c_P and ∇_{ad} though, assuming thermodynamic equilibrium between the gas and radiation (i.e. $e_r = \frac{aT^4}{\rho}$ and $P_r = \frac{1}{3}aT^4$). At layers well below the stellar atmosphere this assumption is true to very high precision. In the stellar atmosphere this is not necessarily the case. However, the error introduced when making this assumption for the stellar atmospheric layers, is expected to be neglible given the relatively low temperature conditions in these layers. One should note however that this is not true for the very outermost stellar layers with extremely low density, where the radiation contribution to c_P and ∇_{ad} is predicted to be significant assuming thermodynamic equilibrium. Fortunately thermal convection in these very outer layers is expected to be negligible, and thus these two quantities will play no role there.

Opacity tables are also provided with the code in ASCII format. These are OPAL Grevesse & Noels (1993) opacity tables, supplemented by the opacity tables from the AESOPUS code (Marigo & Aringer 2009) at low T. The opacity, and its partial derivatives with respect to V and T, are evaluated using the same interpolation method used for the EOS calculation described above.

There is an option in the code to include or neglect nuclear burning. The inclusion of nuclear energy generation in the code is still pretty much under development.

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