# Analytics of First Three Orders of The Velocity Moments

We consider weak two–body gravitational (Coulomb–like) scatterings between a test particle of mass $m$ and field particles of mass $m_a$. Let $\mathbf{V}$ be the test speed relative to the bath, and let $v_a$ denote the speed of a field particle drawn from an isotropic speed distribution $f_a(v_a)$, normalized so that the number density is $n_a = 4\pi\int_0^\infty v_a^2 f_a(v_a)\,dv_a$. We decompose velocity increments into components parallel and perpendicular to $\mathbf{V}$: $\Delta v_\parallel$ and $\Delta v_\perp$. The Coulomb logarithm is $\ln\Lambda = \ln(b_{\max}/b_{\min})$ with $b_{\min} = b_{90}(V)$ the 90° deflection scale and $b_{\max}$ a large-scale cutoff.

For an isotropic bath one obtains, first (drift) and second (diffusion) order terms, first derived from Henon (1958):

$$D[\Delta v_\parallel] = -\frac{16\pi^2 G^2 m_a (m+m_a)\ln\Lambda}{v^2}\int_0^v dv_a\, v_a^2 f_a(v_a) \tag{1}$$

$$D[(\Delta v_\parallel)^2] = \frac{32\pi^2 G^2 m_a^2 \ln\Lambda}{3}\left[\int_0^v dv_a\, \frac{v_a^4}{v^3} f_a(v_a) + \int_v^\infty dv_a\, v_a f_a(v_a)\right] \tag{2}$$

$$D[(\Delta v_\perp)^2] = \frac{32\pi^2 G^2 m_a^2 \ln\Lambda}{3}\left[\int_0^v dv_a \left(\frac{3v_a^2}{v} - \frac{v_a^4}{v^3}\right) f_a(v_a) + 2\int_v^\infty dv_a\, v_a f_a(v_a)\right] \tag{3}$$

Following Hénon (1960), the third-order diffusion coefficients can be expressed in the same notation as above:

$$D[(\Delta v_\parallel)^3] = -\frac{32\pi^2 G^2 m_a^3 \ln\Lambda}{m+m_a}\left[\int_0^v dv_a\, v_a^2 f_a(v_a)\left(\tfrac{1}{2} - \frac{v_a^4}{10v^4}\right) + \int_v^\infty dv_a\, v_a^2 f_a(v_a)\,\frac{2v}{5v_a}\right] \tag{4}$$

$$D[\Delta v_\parallel (\Delta v_\perp)^2] = -\frac{16\pi^2 G^2 m_a^3 \ln\Lambda}{m+m_a}\left[\int_0^v dv_a\, v_a^2 f_a(v_a)\left(\tfrac{1}{2} - \frac{v_a^2}{3v^2} + \frac{v_a^4}{10v^4}\right) + \int_v^\infty dv_a\, v_a^2 f_a(v_a)\,\frac{4v}{15v_a}\right]. \tag{5}$$

The analytical behavior of the first five moments is shown in Fig. 1.

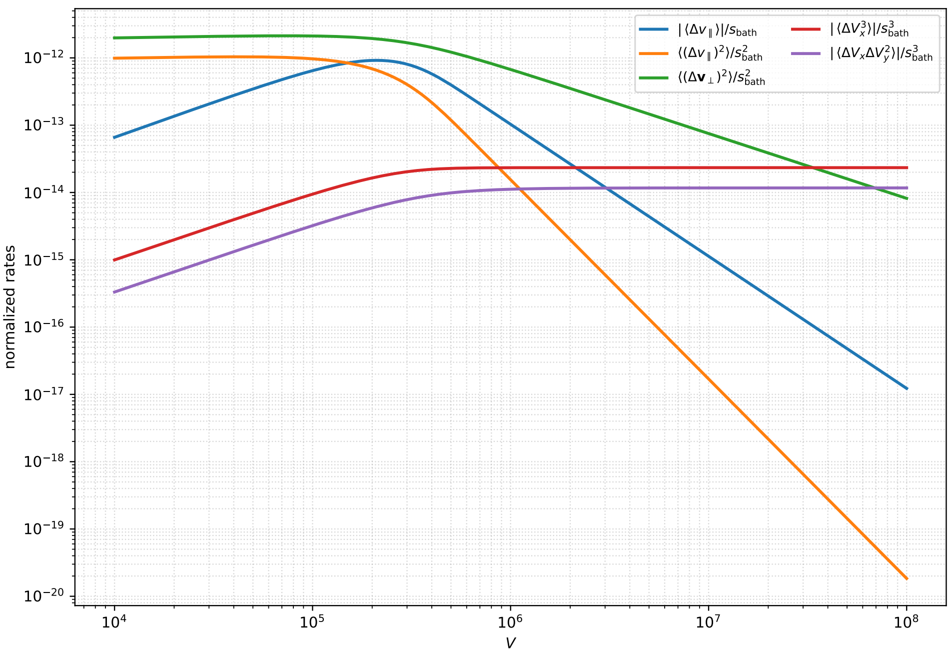


**Figure 1:** Analytical plot of the dependences from the literature. Third order terms are both go near parallel to each other, and initially are less than both first order terms. Later, on $v = 6.7 s_{\mathrm{bath}}$, $\langle \Delta v_\parallel^2 \rangle$ becomes less than both third order terms Then, on $v = 7.6 s_{\mathrm{bath}}$, $\langle \Delta v_\parallel \rangle$ becomes less than both third order terms. Finally, at $v = 600 s_{\mathrm{bath}}$, $\langle \Delta v_\perp^2 \rangle$ becomes less than both third order terms, thus after $v = 600 s_{\mathrm{bath}}$, third order terms normalized by $s_{\mathrm{bath}}$, become less than all first and second order components.

Considering the extreme asymptotic regime $V \gg s_{\rm bath}$, the diffusion coefficients scale as

$$D[\Delta v_{\|}] \sim -\frac{1}{V^2}, \tag{6}$$

$$D[(\Delta v_{\|})^2] \sim \frac{1}{V^3}, \tag{7}$$

$$D[(\Delta v_{\perp})^2] \sim \frac{1}{V}. \tag{8}$$

In the asymptotic regime $V \gg s_{\rm bath}$, the diffusion coefficients scale as

$$D[\Delta v_{\|}] \sim -V^{-2}, \qquad D[(\Delta v_{\|})^2] \sim V^{-3}, \qquad D[(\Delta \mathbf{v}_{\perp})^2] \sim V^{-1}. \tag{9}$$

The higher-order moments behave as

$$\langle \Delta V_x^3 \rangle \approx \tfrac{1}{2} - \tfrac{v_a^4}{10V^4}, \qquad \langle \Delta V_x \Delta V_y^2 \rangle \approx \tfrac{1}{2} - \tfrac{v_a^2}{3V^2} + \tfrac{v_a^4}{10V^4}, \tag{10}$$

and both approach constant values as $V \to \infty$.

### Charecterististic velocity

Let us examine more closely the intersection between $\langle \frac{\Delta v_\perp^2}{dt} \rangle$ and $\langle \frac{\Delta v_\|^3}{dt} \rangle$. Define

$$A_2 = \frac{32\pi^2 G^2 m_{\rm f}^2}{3}, \qquad P_{13} = -\frac{32\pi^2 G^2 m_{\rm f}^3}{M + m_{\rm f}}, \tag{11}$$

for equal masses, and we could write the equation

$$\frac{|P_{13}|(\frac{1}{2}I_2)}{s^3} = \frac{A_2(3I_2)\ln\Lambda}{s^2 V} \Rightarrow V_\perp^\star = 4s\ln\Lambda \tag{12}$$

Using our model, where $\ln\Lambda(V) = \ln[1 + 2w^3V^2]$, we obtain

$$e^{V_\perp^\star/4s} = 1 + 2w^3(V_\perp^\star)^2, \tag{13}$$

which is transcendental. Numerically, in our case, this gives

$$V_\perp^\star = 2.2 \cdot 10^2\, s. \tag{14}$$

## Can higher-order moments be neglected?

From Hénon (1960), it was all also stated: Comparing (4) and (5) with first/second-order results (1), (2) and (3), the $\log\Lambda$ factor disappears for higher-order moments. This recovers the known result: higher-order moments can be neglected, and the stellar velocity distribution evolves according to a **Fokker–Planck equation** limited to second-order terms Gasiorowicz et al. (1956).

$$D^{(q)} = \frac{d}{dt}\langle |\Delta \boldsymbol{v}|^q \rangle = \int_{b_{\rm min}}^{b_{\rm max}} (2\pi b\, db)\, n_a V \left[\frac{2G(m+m_a)}{bV}\right]^q \tag{15}$$

$$\int_{b_{\rm min}}^{b_{\rm max}} b^{1-q}\, db = \begin{cases} \ln\left(\dfrac{b_{\rm max}}{b_{\rm min}}\right) = \ln\Lambda, & q = 2 \\ \dfrac{b_{\rm min}^{2-q} - b_{\rm max}^{2-q}}{2-q} = \dfrac{b_{\rm min}^{2-q}}{2-q}, & q > 2 \end{cases} \tag{16}$$

With the followinng assymptotics for large $V$:

$$D^{(1)} \propto \frac{n_a}{V^2} \ln\Lambda,$$
$$D^{(2)} \propto \frac{n_a}{V} \ln\Lambda, \tag{17}$$
$$D^{(q)} \propto \frac{n_a}{V^{q-1}} b_{\rm min}^{2-q}, \quad \text{for } q \geq 3.$$

Thus for $q \geq 3$:

$$\frac{D^{(q)}}{D^{(2)}} = \frac{\rm const}{\ln\Lambda}\left(\frac{b_{\rm min}}{V}\right)^{2-q} \tag{18}$$

Thus, with $\Lambda \to \infty$:

$$\frac{D^{(q)}}{D^{(2)}} = 0 \tag{19}$$

originally,

$$\Lambda = \frac{b_{\max}}{b_{90}} \tag{20}$$

Setting $\theta = \pi/2$ gives the strong–deflection scale

$$b_{90}(v) = \frac{\kappa}{\mu v^2}, \tag{21}$$

with the large–scale cutoff is set by the force range or geometry; with a mediator mass:

$$b_{\max} \simeq \omega \tag{22}$$

$$\Lambda = \frac{\omega \mu v^2}{\kappa} = C\omega v^2, \tag{23}$$

# Angle-independent cross-section $\sigma(v)$

The velocity distribution we used to form the velocties of the bath particles, that our target particle $v_{rm}$ collide with:

$$f(v_f) = \frac{1}{(2\pi s_{\text{bath}}^2)^{3/2}} \exp\left(-\frac{|v_f|^2}{2s_{\text{bath}}^2}\right),$$

Before the collision, the vector of relative velocity was $\vec{g}$ with some unit vector $\hat{g} = \frac{\vec{g}}{|\vec{g}|}$. After the collision the $|\vec{g}|$ saved its absolute value, but rotated by some angle $\theta$ with the direction along unit vector $\hat{n}$, thus:

$$\hat{n} \cdot \hat{g} = \cos\theta$$

In monte carlo (MC), the $\hat{n}$ was generated from the sampled $\cos\theta$ and random azimuthal angle $\phi$ around $\hat{g}$. Since $\vec{g}$ before and after collision remains constant $|\vec{g}| = |\vec{g'}|$, the relative speed $g'$ after the collision could be found as:

$$\Delta\vec{v} = \frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}} g(\hat{n} - \hat{g})$$

Resolve this along the $\hat{g}$–axis ("$\|$") and its perpendicular plane ("$\perp$"):

$$\frac{\langle \Delta v_\| \rangle}{\Delta t} = \int d^3v_f\ f(v_f)\, n_f\, \sigma(V)\, V\, \langle \Delta v_\| \rangle_{\hat{n}}, = \Big\langle n_f\, \sigma(V)\, V\ \big(-\tfrac{\mu}{m_t} g_x\big) \Big\rangle_{v_f}$$

$$\frac{\langle (\Delta v_\|)^2 \rangle}{\Delta t} = \int d^3v_f\ f(v_f)\, n_f\, \sigma(V)\, V\, \langle (\Delta v_\|)^2 \rangle_{\hat{n}}, = \Big\langle n_f\, \sigma(V)\, V\ \big(\tfrac{\mu}{m_t}\big)^2 \big(\tfrac{V^2}{3} + g_x^2\big) \Big\rangle_{v_f}$$

$$\frac{\langle \Delta v_\perp^2 \rangle}{\Delta t} = \int d^3v_f\ f(v_f)\, n_f\, \sigma(V)\, V\, \langle \Delta v_\perp^2 \rangle_{\hat{n}}, = \Big\langle n_f\, \sigma(V)\, V\ \big(\tfrac{\mu}{m_t}\big)^2 \big(\tfrac{5}{3}V^2 - g_x^2\big) \Big\rangle_{v_f}$$

$$\frac{\langle (\Delta v_\|)^3 \rangle}{\Delta t} = \int d^3v_f\ f(v_f)\, n_f\, \sigma(V)\, V\, \langle (\Delta v_\|)^3 \rangle_{\hat{n}}, = \Big\langle n_f\, \sigma(V)\, V\ \big(\tfrac{\mu}{m_t}\big)^3 \big(-V^2 g_x - g_x^3\big) \Big\rangle_{v_f}$$

$$\frac{\langle \Delta v_\|\, \Delta v_\perp^2 \rangle}{\Delta t} = \int d^3v_f\ f(v_f)\, n_f\, \sigma(V)\, V\, \langle \Delta v_\|\, \Delta v_\perp^2 \rangle_{\hat{n}} = \Big\langle n_f\, \sigma(V)\, V\ \big(\tfrac{\mu}{m_t}\big)^3 \big(g_x^3 - \tfrac{5}{3}V^2 g_x\big) \Big\rangle_{v_f}$$

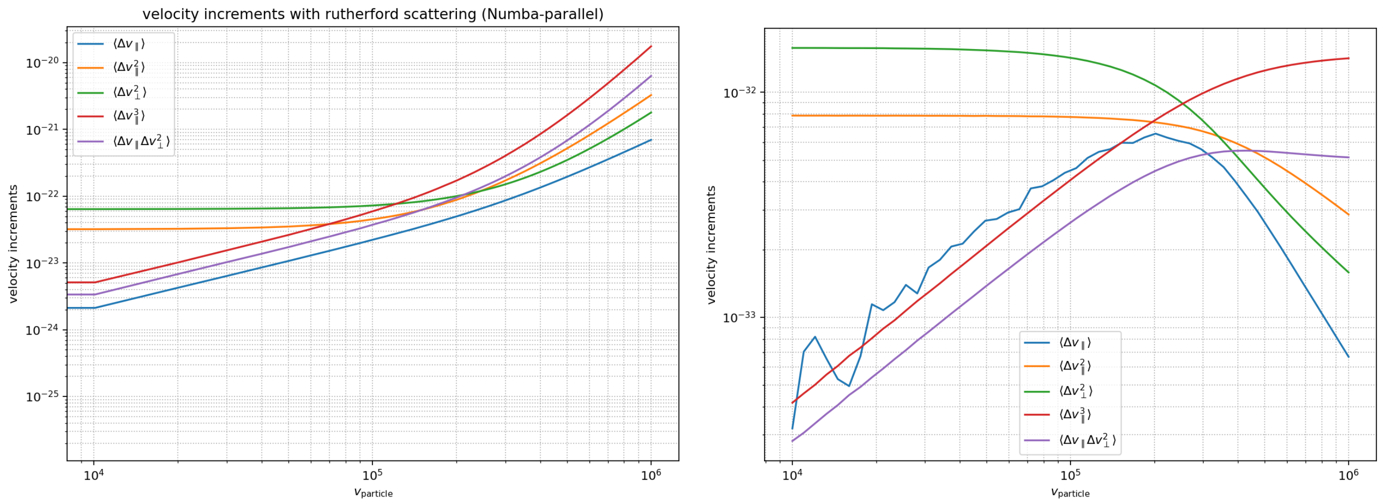


**Figure 2:** Left: Dependence of normalized first three-order average particle velocity increments on particle speed, from numerical calculations with constant cross-section $\sigma(v) = \sigma_0$. Right: The same dependence with Yukawa cross-section $\sigma(v) = \sigma_0 / \left[1 + (v/v_0)^4\right]$.

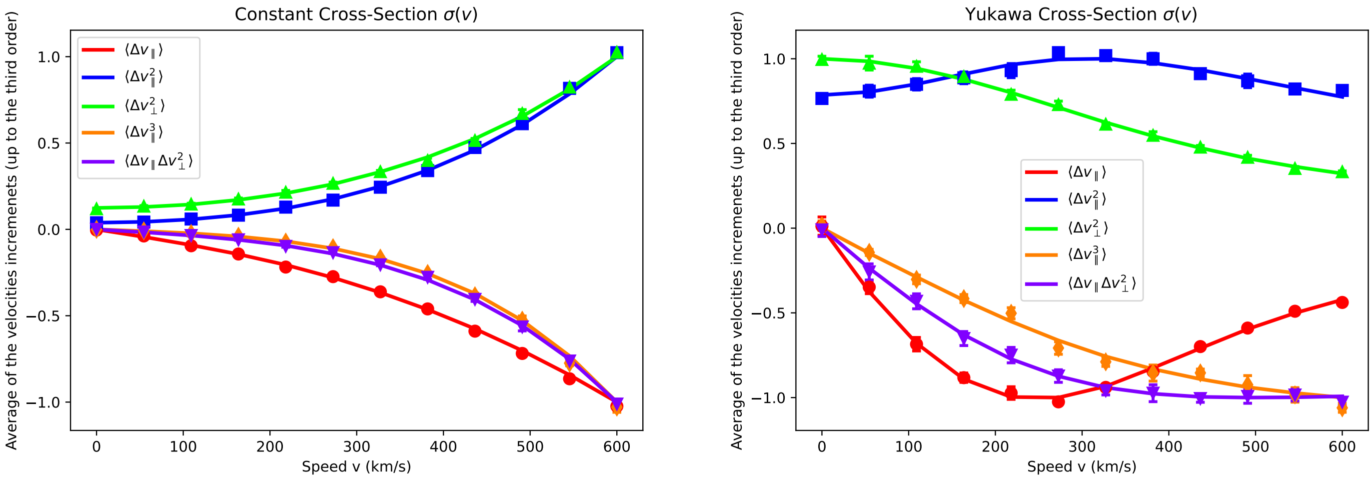


**Figure 3:** Analytical dependence of the velocity change for both constant and Yukawa cross-section cases.

## Rutherford scattering

For our theoretical work we will be using the Rotherford scattering Yang & Yu (2022) equation:

$$\frac{d\sigma}{d\cos\theta} = \frac{\sigma_0 w^4}{2\left[w^2 + v^2\sin^2(\theta/2)\right]^2} \tag{24}$$

Let introduce the averaged $\sigma_{\text{total}}$ variable:

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\cos\theta} d\cos\theta = \frac{\sigma_0}{1+\left(\frac{v}{\omega}\right)^2} \tag{25}$$

In the lab reference frame, the test particle with the mast $m_t$ move with velocity $\mathbf{v}$ surrounded by the bath particles with the velocities $\mathbf{v}$ distributed by Maxwell-Bolzman distribution. After the colission with one of them, with bath particle velocity $v_f$, the resulted relative velocity $g = v - v_f$, and the change of test velocity is

$$\Delta v = \frac{m_f}{m_t + m_f} v(\hat{n} - \hat{g}) \tag{26}$$

where $\hat{g} = \frac{\vec{g}}{|g|}$ and $\vec{n}$ post-collision relative direction. Also the scattering angle $\theta$ is defined as $\cos\theta = \hat{n} \cdot \hat{g}$. The axial symmetry arround $\hat{g}$:

$$\langle \hat{n} \rangle_\theta = C_1 \cdot \hat{g} \tag{27}$$

$$\langle (\hat{n} \cdot \hat{g})^2 \rangle_\theta = a_\parallel = C_2 \tag{28}$$

Perpendicular to $\hat{g}$ introduce unit vector $\hat{e}_\perp$:

$$\langle (\hat{n} \cdot \hat{e}_\perp)^2 \rangle_\theta = a_\perp = \tfrac{1}{2}(1 - C_2) \tag{29}$$

Average $\Delta v$ over the scattering angles with $V = |\vec{g}|$

$$\langle \Delta v \rangle_\theta = \frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}} V\left(\langle \hat{n} \rangle_\theta - \hat{g}\right) \rightarrow \langle \Delta v \rangle_\theta = \frac{m_f}{m_t + m_f} V\left(C_1 - 1\right)\hat{g} \tag{30}$$

$\hat{e} = \frac{\vec{v}}{|\vec{v}|}$ be the unit vector along the particle's velocity. The velocity change along $\hat{e}$ in one collision is:

$$\Delta v_\parallel = \hat{e} \cdot \Delta v = \frac{m_f}{m_t + m_f} V\left(C_1 - 1\right)\left(\hat{e} \cdot \hat{g}\right) \tag{31}$$

and after multiplying by the probability rate $n_f \sigma_{\text{tot}}(V) V$:

$$\frac{\langle \Delta v_\parallel \rangle}{dt}(v) = \left\langle n_f \sigma_{\text{tot}}(V) V \cdot \frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}} (C_1 - 1)(V \mu_e) \right\rangle_{v_f} \tag{32}$$

let the squared change along $\hat{e}$ from one collision is:

$$(\Delta v_\parallel)^2 = \left[\frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}} V\right]^2 \left[a_\perp(1-\mu_e^2) + b_\parallel \mu_e^2\right] \tag{33}$$

where we defined $a_\perp = \frac{1}{2}(1-C_2)$ is the mean squared change perpendicular to $\hat{g}$ and along $\hat{g}$: $b_\parallel = C_2 - 2C_1 + 1$, which is obvious to proof. Then

$$\frac{\langle(\Delta v_\parallel)^2\rangle}{dt} = \left\langle n_f \sigma_{\text{tot}}(V) V \cdot \left(\frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}}\right)^2 V^2 \left[a_\perp(1-\mu_e^2) + b_\parallel \mu_e^2\right] \right\rangle_{v_f} \tag{34}$$

total mean squared velocity change in one collision is $\langle \Delta v^2 \rangle_\theta = 2\left(\frac{m_{\text{bath}}}{m_{\text{bath}}+m_{\text{particle}}}\right)^2 V^2(1-C_1)$, and the perpendicular component is total minus the parallel piece

$$\langle \Delta v_\perp^2 \rangle_\theta = 2\left(\frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}}\right)^2 V^2(1-C_1) - \left(\frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}}\right)^2 V^2 \left[a_\perp(1-\mu_e^2) + b_\parallel \mu_e^2\right] \tag{35}$$

$$\frac{\langle \Delta v_\perp^2 \rangle}{dt} = \left\langle n_f \sigma_{\text{tot}}(V) V \cdot \left(\frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}}\right)^2 \left[2V^2(1-C_1) - V^2\left(a_\perp(1-\mu_e^2) + b_\parallel \mu_e^2\right)\right] \right\rangle_{v_f}. \tag{36}$$

$$= \mu_e^3 C_3 + \tfrac{3}{2}\mu_e(1-\mu_e^2)(C_1 - C_3) \tag{37}$$

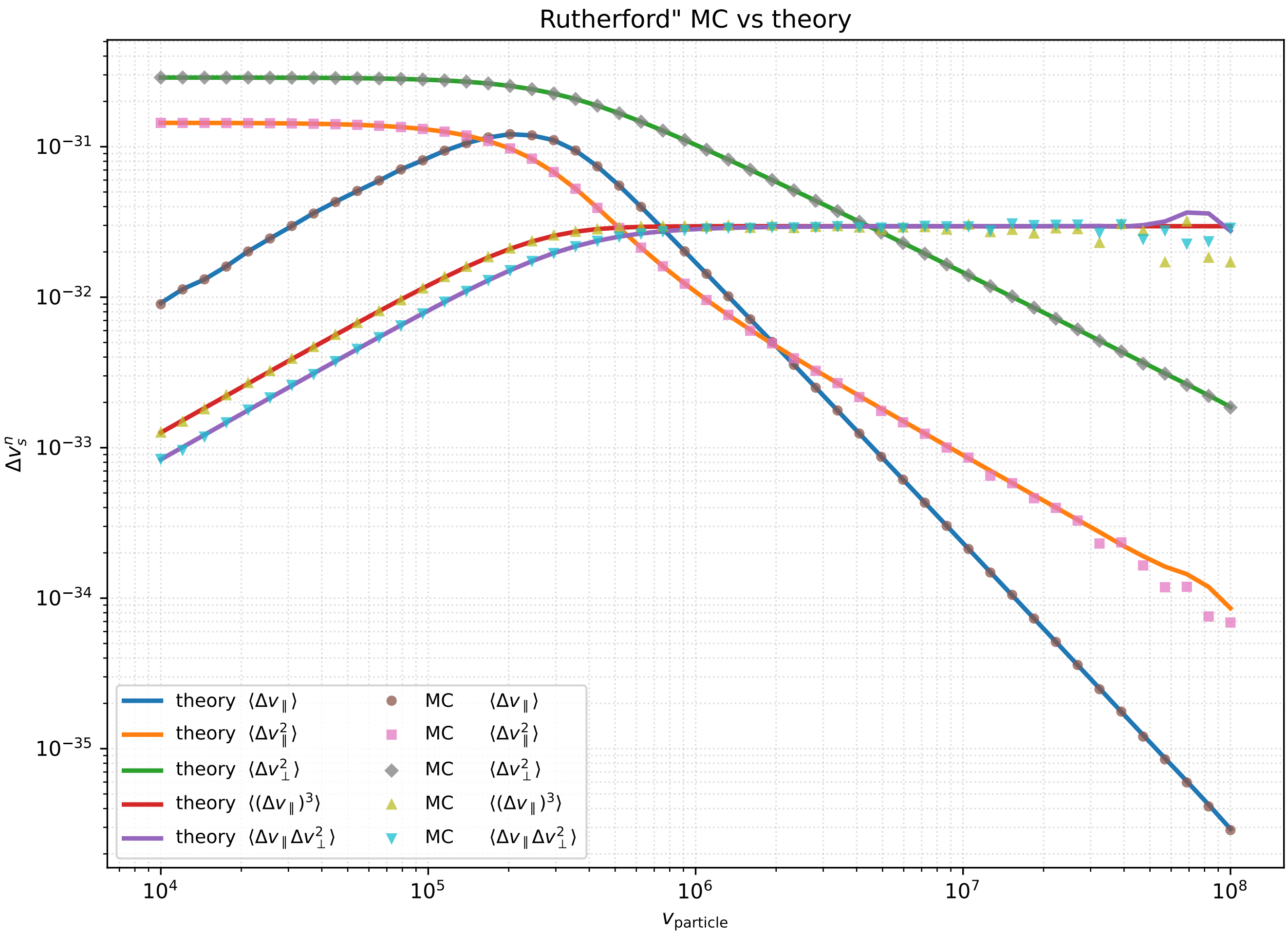


**Figure 4:** Comparison of the Analytical and Simulational data points: velocities changes for the $\sigma(\theta)$ following Rutherford scattering. Left: analytics. Right: simulations. Because when calculating $1 - 2C_1 + C_2$ and $C_3 - 3C_2 + 3C_1 - 1$ this causes catastrophic cancellation at large $V/\omega$ which then feeds into the third–order rates and creates a false plateau and a false dip.

## Expanding the velocity diffucion for $\sigma(V)$ following Rutherford Scattering

$$\Delta v = \frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}} V\,(\hat{n} - \hat{g}) \tag{38}$$

$$\Delta v_{\parallel} = \hat{e}\cdot\Delta v = \frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}} V\,\Big(\hat{e}\cdot\hat{n} - \mu_e\Big). \tag{39}$$

$$|\Delta v|^2 = 2\Big(\frac{m_{\text{bath}}}{m_{\text{bath}} + m_{\text{particle}}}\Big)^2 V^2(1-\cos\theta);\;\; \Delta v_{\perp}^2 = |\Delta v|^2 - (\Delta v_{\parallel})^2. \tag{40}$$

Let $\hat{e} = \mu_e\,\hat{g} + \sqrt{1-\mu_e^2}\,\hat{e}_{\perp}$, where $\hat{e}_{\perp} \perp \hat{g}$. When we take averaging over athimutal angle for the first three orders:

$$\begin{cases} \langle \hat{e}\cdot\hat{n}\rangle = \mu_e\cos\theta \\ \langle(\hat{e}\cdot\hat{n})^2\rangle = \mu_e^2\cos^2\theta + \frac{1}{2}(1-\mu_e^2)\sin^2\theta \\ \langle(\hat{e}\cdot\hat{n})^3\rangle = \mu_e^3\cos^3\theta + \frac{3}{2}\mu_e(1-\mu_e^2)\cos\theta\,\sin^2\theta \end{cases} \tag{41}$$

Considering first three orders, letting $F = \frac{m_{\text{bath}}}{m_{\text{bath}}+m_{\text{particle}}}V$:

$$\big\langle \Delta v_{\parallel}\big\rangle = F(C_1 - 1) \tag{42}$$

$$\big\langle (\Delta v_{\parallel})^2\big\rangle = F^2(C_2 - 2C_1 + 1) \tag{43}$$

$$\big\langle \Delta v_{\perp}^2\big\rangle = F^2(1 - C_2) \tag{44}$$

$$\big\langle (\Delta v_{\parallel})^3\big\rangle = F^3(C_3 - 3C_2 + 3C_1 - 1) \tag{45}$$

$$\big\langle \Delta v_{\parallel}\Delta v_{\perp}^2\big\rangle = \tfrac{1}{2}F^3(C_1 + C_2 - C_3 - 1) \tag{46}$$

## Appendix: Calculcating $\langle\cos\theta\rangle$, $\langle\cos^2\theta\rangle$, $\langle\cos^3\theta\rangle$

Let's introduce

$$C_n = \frac{\int_{-1}^{1} \frac{d\sigma}{d\cos\theta}\cos^n\theta\;d\cos\theta}{\int_{-1}^{1} \frac{d\sigma}{d\cos\theta}\;d\cos\theta} \tag{47}$$

for simplisity $A = w^2 + \frac{v^2}{2}$, $B = \frac{v^2}{2}$. By the direct integration

$$C_1(V) = \frac{A^2 - B^2}{2}\left[\frac{A}{B^2}\Big(\frac{1}{A-B} - \frac{1}{A+B}\Big) + \frac{1}{B^2}\ln\frac{A-B}{A+B}\right] \tag{48}$$

$$C_2(V) = \frac{A^2 - B^2}{2}\left[\frac{A^2}{B^3}\Big(\frac{1}{A-B} - \frac{1}{A+B}\Big) + \frac{2A}{B^3}\ln\frac{A-B}{A+B} - \frac{2}{B^2}\right] \tag{49}$$

$$C_3(V) = \frac{A}{B^3}\Big(3A^2 - 2B^2\Big) + \frac{3A^2(A^2-B^2)}{2B^4}\,\ln\frac{A-B}{A+B} \tag{50}$$

**Values of $C_1, C_2, C_3$ needs to be verified by comparison with numerical calculations** Let $\eta = \frac{v}{w}$, $b = \frac{\eta^2}{2+\eta^2} = \frac{B}{A}$

$$\int_{-1}^{1} \frac{x}{\Big(1 + \frac{1}{2}\eta^2(1-x)\Big)^2}\,dx = \ln(1-b) - \ln(1+b) + \frac{1}{1-b} - \frac{1}{1+b}$$

$$C_1 = \ln\left(1 - \frac{B^2}{A^2}\right) + \frac{2AB}{A^2 - B^2}$$

Which matches with the numerical integration, but does matches with (48), assuming that (48) is wrong, higher order terms (49) and (50) are mistaken.

## Column-like/Gravitational Scattering

Angular momentum is $L = \mu v b$. The deflection angle follows from the orbit integral

$$\theta(b) = \pi - 2\int_{r_{\min}}^{\infty} \frac{b\,dr}{r^2\sqrt{1 - \dfrac{b^2}{r^2} - \dfrac{2U(r)}{\mu v^2}}} \tag{51}$$

where $r_{\min}$:

$$r_{\min} = 1 - \frac{b^2}{r_{\min}^2} - \frac{2U(r_{\min})}{\mu v^2} = 0 \tag{52}$$

from axial simmetry (with solid angle $d\Omega = 2\pi \sin\theta\, d\theta$)

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \Leftrightarrow \frac{d\sigma}{d\theta} = 2\pi\, b\left|\frac{db}{d\theta}\right| \tag{53}$$

Due to the Columb law $(U(r) = \kappa/r)$

$$\tan\frac{\theta}{2} = \frac{\kappa}{\mu v^2\, b} \Leftrightarrow b(\theta) = \frac{\kappa}{\mu v^2}\cot\frac{\theta}{2} \tag{54}$$

substituing from the previous

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^2}{16E^2}\csc^4\frac{\theta}{2} = \frac{\kappa^2}{4\,\mu^2 v^4}\,\frac{1}{\sin^4(\theta/2)} \tag{55}$$

$$\frac{d\sigma}{d\theta} = 2\pi b\left|\frac{db}{d\theta}\right| = \frac{\pi\kappa^2}{\mu^2 v^4}\cot\frac{\theta}{2}\csc^2\frac{\theta}{2} \tag{56}$$

Since $b \le r_{\max}$, the smallest deflection is

$$\theta_{\min} = 2\arctan\!\left(\frac{\kappa}{\mu v^2\, r_{\max}}\right) \tag{57}$$

and the integrated cross section into angles $\theta \ge \theta_0$ equals the area in $b$–space with $b \le b_0 = b(\theta_0)$

$$\sigma(\theta \ge \theta_0) = \pi\, b_0^2 = \pi\left(\frac{\kappa}{\mu v^2}\cot\frac{\theta_0}{2}\right)^2 \tag{58}$$

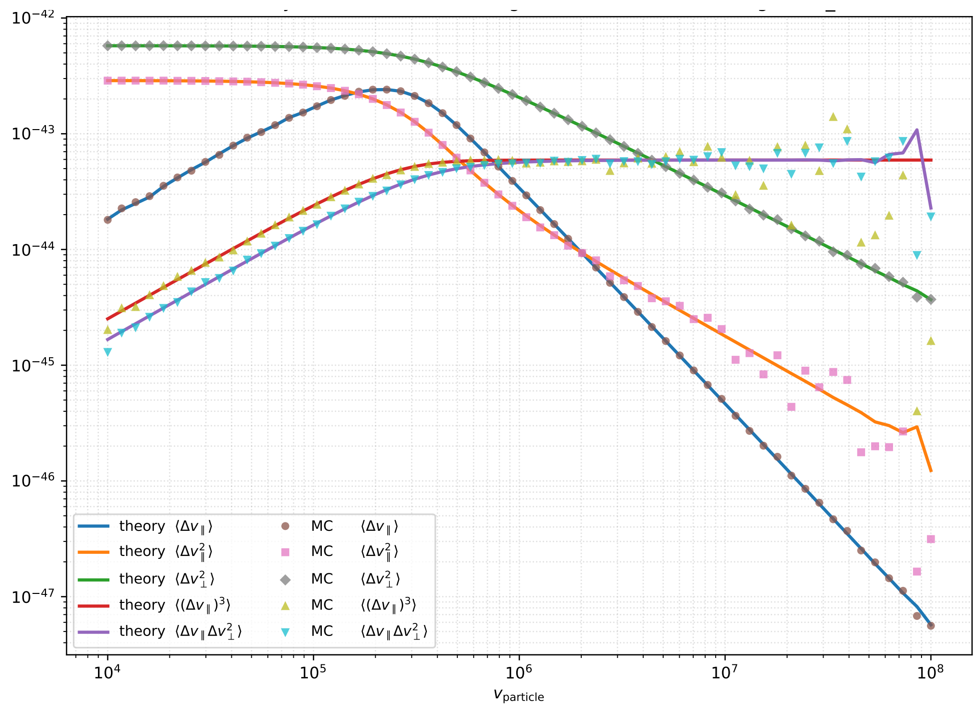


**Figure 5:** Left: Coulomb kernel with screening-angle mapping $\theta_{\min}(V) = 2w/V$ (matches the shapes of Fig. 3). Center: Coulomb kernel with fixed $\theta_{\min}$ (overall level set by $\ln(1/\theta_{\min})$). Right: dependence on $\theta_{\min}$—decreasing $\theta_{\min}$ leads to the expected Coulomb–log growth.

## Same bath

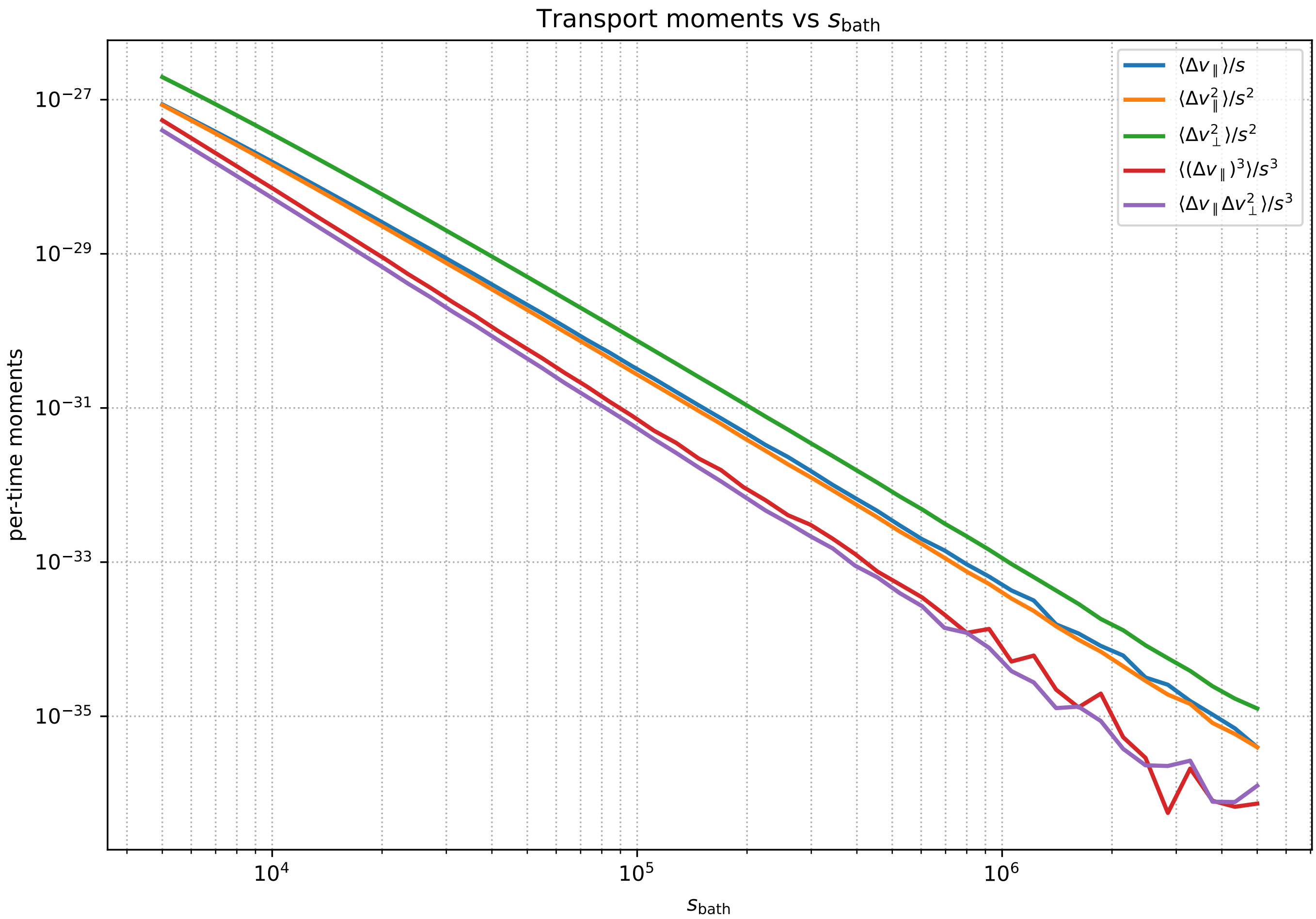


**Figure 6:** Dependence of transport moments per-time for the first three orders from $s_{\rm bath}$. For the case, when both target and field particles velocities are follow Maxwell-Boltzmann distribution with a dispersion $s_{\rm bath}$.

## Orbit-averaging

Let $\hat{v} = v/|v|$, then

$$A_v = D[\Delta v_\parallel]\hat{v}, C_v = D[(\Delta v_\parallel)^2]\hat{v}\hat{v} + \tfrac{1}{2}D[(\Delta v_\perp)^2](I - \hat{v}\hat{v}) \tag{59}$$

Also, first and second order terms of energy:

$$D[\Delta E] = vD[\Delta v_\parallel] + \tfrac{1}{2}\left(D[(\Delta v_\parallel)^2] + D[(\Delta v_\perp)^2]\right) \tag{60}$$

$$D[(\Delta E)^2] = v^2 D[(\Delta v_\parallel)^2]. \tag{61}$$

likewise, for angular momentum:

$$D[\Delta(L^2)] = 2v^\top M A_v + \mathrm{tr}(MC_v), \qquad D[(\Delta(L^2))^2] = 4v^\top M C_v M v, \tag{62}$$

$$D[\Delta E \Delta(L^2)] = 2v^\top C_v M v. \tag{63}$$

$$D[\Delta L] = \frac{1}{2L}D[\Delta(L^2)] - \frac{1}{8L^3}D[(\Delta(L^2))^2], \qquad D[(\Delta L)^2] = \frac{1}{4L^2}D[(\Delta(L^2))^2], \tag{64}$$

$$D[\Delta E \Delta L] = \frac{1}{2L}D[\Delta E \Delta(L^2)]. \tag{65}$$

orbit averaging:

$$\overline{Q}(E, L) = \frac{1}{T_r(E, L)} \oint Q(r(t), v(t))dt, \tag{66}$$

FP equation:

$$\partial_t f(\mathbf{x}, t) = -\sum_{i=1}^{d} \partial_{x_i}(A_i f) + \frac{1}{2}\sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}(B_{ij} f) \tag{67}$$

$$\partial_t f(E, L, t) = -\partial_E(\overline{D[\Delta E]} f) - \partial_L(\overline{D[\Delta L]} f) + \frac{1}{2}\partial_E^2(\overline{D[(\Delta E)^2]} f) + \partial_E\partial_L(\overline{D[\Delta E \Delta L]} f) + \frac{1}{2}\partial_L^2(\overline{D[(\Delta L)^2]} f) \tag{68}$$

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