University of Glasgow

Department of Physics and Astronomy

Astronomy 1
Problems Handbook
2009–2010
Instructors Manual

Values of astrophysical constants

speed of light	c	$2.998 \times 10^{8}\ \mathrm{m\,s^{-1}}$
gravitational constant	G	$6.673 \times 10^{-11}\ \mathrm{N\,m^{2}\,kg^{-2}}$
Planck constant	h	$6.626 \times 10^{-34}\ \mathrm{J\,s}$
Boltzmann constant	k	$1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ	$5.671 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$
Rydberg constant	R_{∞}	$1.097 \times 10^{7}\ \mathrm{m^{-1}}$
Avogadro constant	N_{A}	$6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
gas constant	R	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
proton mass	m_{p}	$1.673 \times 10^{-27}\ \mathrm{kg}$
electron mass	m_{e}	$9.109 \times 10^{-31}\ \mathrm{kg}$
elementary charge	e	$1.602 \times 10^{-19}\ \mathrm{C}$
electronvolt	eV	$1.602 \times 10^{-19}\ \mathrm{J}$
astronomical unit	AU	$1.496 \times 10^{11}\ \mathrm{m}$
parsec	pc	$3.086 \times 10^{16}\ \mathrm{m}$
light year	ly	$9.461 \times 10^{15}\ \mathrm{m}$
solar mass	$M_{\odot}$	$1.989 \times 10^{30}\ \mathrm{kg}$
solar radius	$R_{\odot}$	$6.960 \times 10^{8}\ \mathrm{m}$
solar luminosity	$L_{\odot}$	$3.826 \times 10^{26}\ \mathrm{W}$
Earth mass	$M_{\oplus}$	$5.976 \times 10^{24}\ \mathrm{kg}$
Earth radius	$R_{\oplus}$	$6.378 \times 10^{6}\ \mathrm{m}$
obliquity of the ecliptic	ϵ	$23^{\circ}\ 26'$

Contents

Notation
In many cases the marks allocated to each question, or section of a question, are shown in square brackets in the right-hand margin. Question numbers followed by a small 'e' are recent exam questions. Answers to some numerical problems are shown in curly brackets. Where available, solutions are shown immediately after their questions (which are given in smaller type), with a horizontal line separating each problem.

Dr M.A. Hendry
September 2009

1 Solar System Physics

1.1e List five main differences between the terrestrial and jovian planets. [5]

Solution: *no solution available*

1.2e State Newton's law of gravitation, defining all the symbols you use. Hence derive an expression for the surface gravity of a spherically symmetrical uniform planet in terms of its mass, radius, and the gravitational constant, G. Calculate the value of the surface gravity of Jupiter, using planetary data from the lecture handout or course textbook. [5]

Solution: Newton's law of gravitation: the magnitude of the gravitational force F between two masses m_1 and m_2 separated by a distance R is

$$F = \frac{G m_1 m_2}{R^2}$$

where G is the gravitational constant, and where the force exerted by m_2 on m_1 is along the line between m_1 and m_2, and vice-versa. If m is the mass of an object near the surface of a body of mass M, radius R, then the gravitational attraction of the mass M on m at the surface is given by

$$F = mg = \frac{GMm}{R^2}$$

where g is the surface gravity of the object of mass M. Rearranging, we have the surface gravity g of an object of mass M and radius R is given by

$$g = \frac{GM}{R^2}$$

Inserting the values given yields $g = 24.9\text{ms}^{-1}$ for Jupiter.

1.3 State Newton's Law of Gravitation and show that the acceleration due to gravity at the surface of a planet of mass M and radius R is given by

$$g = \frac{GM}{R^2},$$

stating clearly any assumptions that you make. [3]

Given that the mass of Venus is 0.817 times that of the Earth and the radius of Venus is 0.97 times that of the Earth, calculate the acceleration due to gravity at the surface of Venus if its value at the surface of the Earth is $9.8\,\text{m}\,\text{s}^{-2}$. $\{8.5\,\text{m}\,\text{s}^{-2}\}$ [3]

Solution: $F_{\text{grav}} = GMm/r^2$ where m and M are masses of bodies separated by r. Force is attractive and acts along line joining bodies, $mg = GMm/R^2 \rightarrow g = GM/R^2$, assuming planet spherical with spherically symmetric density. Thus

$$\frac{g_\text{V}}{g_\text{E}} = \frac{M_\text{V}}{M_\text{E}} \frac{R_\text{E}^2}{R_\text{V}^2} = \frac{0.817}{(0.97)^2},$$

so

$$g_\text{V} = \frac{0.817}{(0.97)^2} \times 9.8 = 8.5\,\text{m}\,\text{s}^{-2}.$$

1.4e Calculate the ratio of the surface gravities of the Earth and Jupiter, given the following data

$$R_{\text{Jupiter}} = 11.2 R_{\text{Earth}} \quad \text{and} \quad M_{\text{Jupiter}} = 317.8 M_{\text{Earth}}$$

where R and M denote respectively the radius and mass of the planet. [2]

Show that, for two planets of equal mean density, the ratio of their surface gravities is equal to the ratio of their radii. [3]

Solution:

$$g = \frac{GM}{R^2}$$

We can write

$$M = \frac{4}{3}\pi R^3 \bar{\rho}$$

$$\Rightarrow g = G\frac{4\pi}{3}\frac{R^3}{R^2}\bar{\rho} = \frac{4}{3}\pi R \bar{\rho}$$

Hence

$$\frac{g_1}{g_2} = \frac{R_1 \bar{\rho}_1}{R_2 \bar{\rho}_2} = \frac{R_1}{R_2} \quad \text{if} \quad \bar{\rho}_1 = \bar{\rho}_2$$

[3]

For Jupiter,

$$\begin{aligned} \frac{g_{Jupiter}}{g_{Earth}} &= \frac{M_J}{M_E} . \left(\frac{R_E}{R_J}\right)^2 \\ &= 317.8 \times \left(\frac{1}{11.2}\right)^2 \\ &= 2.53 \end{aligned}$$

[2]

1.5e Starting from Newton's Law of Gravitation, derive an expression for the gravitational acceleration at the surface of a planet in terms of its radius and average density. Given that Mars and Earth are both terrestrial (rocky) planets, and using planetary data from the lecture handout or textbook, estimate the ratio of the gravitational acceleration at the surface of Mars to that of Earth, stating any assumptions you make. [5]

Solution: Newton's law of gravitation

$$F = \frac{GMm}{r^2}$$

F= gravitational force between bodies of mass M and m separated by distance r.

Surface gravity (gravitational acceleration) = force per unit mass

$$g = \frac{GM}{r^2}$$

where $r = R_{planet}$

$$\text{Express in terms of} \quad \bar{\rho} \qquad \begin{aligned} g &= \frac{G}{r^2}\frac{4}{3}\pi r^3 \bar{\rho} \\ &= \frac{4}{3}\pi G \bar{\rho} r \end{aligned}$$

So ratio of surface gravities of Mars and Earth

$$ratio = \frac{\bar{\rho_{Mars}} r_{Mars}}{\bar{\rho_{Earth}} r_{Earth}}$$

As both are terrestrial planets assume they have same $\bar{\rho} \Rightarrow$

$$\frac{g_{Mars}}{g_{Earth}} \simeq \frac{r_{Mars}}{r_{Earth}} \simeq \frac{3487}{6378} \simeq 0.55$$

[5]

1.6e Give a sketch of the Earth's interior, clearly indicating the different regions, and their approximate dimensions. [5]

Solution: *no solution available*

1.7e Explain the terms

(a) plate tectonics
(b) volcanism
(c) outgassing. [5]

Solution: *no solution available*

1.8e Give a sketch of the atmosphere of the Sun, indicating the approximate dimensions of the various regions and their approximate temperatures. [5]

Solution: *no solution available*

1.9e The Earth and Jupiter's satellite Io both exhibit volcanism. Briefly describe the causes of this volcanism in each case. [5]

Solution: Earth: volcanism is result of radioactive decay within interior heating rocks (radiogenic heating). Rocks melt – lava comes to surface via cracks in crust.

Volcanism found at subduction zones (boundary between two plates where oceanic crust is sliding under continental crust), and also at weak points in plates – 'hot spots' such as at Hawaiian islands.

Io: volcanic activity due to heating by tidal forces exerted primarily by Jupiter, and also by Europa which is in orbital resonance with Io (Europa orbital period = 2 $\times$ Io orbital period). Io is closest of Galilean satellites to Jupiter, Europa next.

1.10e Sketch the *internal* structure of the Sun, indicating the approximate dimensions and temperatures of the various regions. [5]

Solution: *no solution available*

1.11e Explain the mechanism of the greenhouse effect. Give the names of two main greenhouse gases in the Earth's atmosphere. [5]

Solution: *no solution available*

1.12e In what way do the surface features of the Moon and Earth radically differ and why? [5]

Solution: Surface features of Moon Extensive circular impact craters. Smooth dark areas called maria. No water. Old rocks.

Surface features of Earth 3/4 covered in oceans plus ice. Volcanically active. Younger rocks (range of ages). Surface (and atmosphere) modified by life.

Basic differences due to lack of present day volcanic activity and no erosion processes on Moon (no atmosphere and no water), while these activities plus life have modified Earth surface. Moon probably lost its volatile compounds (H_2O, CO_2 etc.) due to heating on impact with Earth of large planetesimal from which it is thought to have formed.

1.13e The surface temperature of Venus is 700 K and that of the Earth is 300 K. Calculate the wavelengths at which the radiative flux from each planet is maximum. You may use Wien's law, $\lambda_{\max} T = 2.9 \times 10^{-3}$ m K. [2]

In what region of the spectrum do these wavelengths lie? [1]

Explain carefully what is meant by the 'greenhouse effect' in planetary atmospheres. Discuss this phenomenon with reference to the atmospheres of the terrestrial planets. [10]

How does the 'runaway' greenhouse effect explain the near absence of water in Venus's atmosphere? [4]

Solution: Using Wien's displacement law, with $T = 700$ K for Venus $\lambda_{\max} = 41.4$ nm and for Earth ($T = 300$ K) $\lambda_{\max} = 97$ nm. Both these wavelengths lie in the infrared region. (Radiation in the infrared is largely absorbed by CO_2 and H_2O, hence greenhouse effect.) Atmosphere transmits light in visible band, but absorbs or reflects light in infrared. Thus light from the Sun is transmitted through atmosphere since it is largely in visible because of high effective temperature (5 800 K) of the Sun. Light from the planet's surface is however emitted at lower temperature (Venus 700 K, Earth 300 K) and so is mainly in infrared, and so is absorbed by atmosphere and reradiated. This effect comes about largely through the presence of CO_2 (H_2O and CO_2 in the case of the Earth), although other molecules will contribute. The effect is particularly strong on Venus, whose atmosphere is dense and made largely of CO_2. On the Earth it is also important, although not so evident. On planets with only thin atmospheres it is fairly unimportant. Venus's surface temperature might be expected to be around 350 K from simple modeling, i.e., assuming surface of planet emits a flux equal to the flux from the Sun, although in reality the surface temperature of Venus is around 700 K. The runaway greenhouse effect is due to positive feedback. Water is evaporated by high surface temperatures on Venus produced by the greenhouse effect due to CO_2. The increase in water vapour increases the greenhouse effect, which in turn increases the surface temperature and water evaporation. Temperature continues to rise, and scale height of H_2O increases until water is vulnerable to dissociation from UV and X-ray radiation from the Sun.

1.14e Show that the surface temperature, T_p, of a planet is given by

$$T_p = (1 - A)^{1/4} \left(\frac{R_\odot}{2r} \right)^{1/2} T_e ,$$

where r is the distance of the planet from the Sun, and $R_\odot$ is the solar radius, T_e is the effective temperature of the Sun, and A is the albedo of the planet. State clearly any assumptions that you make. [10]

Assuming that Mars has an albedo of 0.2, calculate its surface temperature given that the orbital period of Mars is 1.9 years. [5]

What evidence is there for the statement that the surface temperature was considerably higher than this in the past? [2]

(For the Sun, $T_e = 5\,800\,\mathrm{K}$.)

Solution: *no solution available*

1.15 State the main differences between the properties of the terrestrial planets and the jovian planets. [6]

Show that the escape velocity is given by $v_e = (2GM/R)^{1/2}$. (You may assume that the gravitational potential energy of body of mass m on the surface of a sphere of mass M and radius R is given by $-GMm/R$ and its kinetic energy is given by $mv^2/2$.) Calculate the ratio of the escape velocities for the two planets Venus and Jupiter, using the planetary data given in the lectures. [10]

How does the difference in escape velocities explain the difference between the chemical composition of the atmospheres of the two planets? [4]

Solution:

Terrestrial planets	Jovian planets
lower mass, small radii	high mass, large radii
near the Sun	distant from Sun
(higher surface temperature	lower surface temperature)
solid surface	gaseous throughout (except for possible solid core)
depleted in H and He	rich in H and He (same composition as the Sun)
few satellites	many satellites
no rings	many rings

Escape velocity: A particle will escape if its total energy (potential (PE) + kinetic (KE)) is greater than zero i.e., $\mathrm{PE} + \mathrm{KE} > 0$. Now $\mathrm{KE} = mv^2/2$ and $\mathrm{PE} = -GMm/r$ on the surface of the planet. The escape velocity is given by putting E$= 0$. This yields

$$v_e = \left(\frac{2GM}{R}\right)^{1/2}.$$

Ratio of escape speeds from Jupiter and Venus: Using the above formula we have

$$\frac{v_{eJ}}{v_{eV}} = \sqrt{\frac{M_J}{M_V}}\sqrt{\frac{R_V}{R_J}} = 5.85.$$

Composition of atmospheres: The escape speed on Jupiter is greater than on Venus. At a given temperature T, the average speed of a particle depends on the temperature. (In fact the average kinetic energy $\frac{1}{2}mv^2 = \frac{3}{2}kT$, where k is Boltzmann's constant.) Thus at a given temperature, less massive particles will move faster (in proportion to the inverse square root of their mass). Hydrogen and Helium are very light particles, and so will be moving fast enough to escape Venus, say. Heavy particles like CO_2 etc. cannot escape. On Jupiter the situation is different. Firstly the escape velocity is greater and secondly the temperature is lower, so the H and He particles do not have on average enough speed to escape (and of course the heavier particles even less so). So terrestrial planets with lower escape speeds and higher surface temperatures tend to lose H and He, whilst gaseous giants retain them.

1.16 Sketch the monochromatic energy flux, F_λ, against wavelength, λ, at the Sun's surface, clearly indicating the wavelength at which the flux is a maximum. You may assume Wien's law. [4]

Show that the total radiative flux from the sun falling on a planet's surface is given by

$$\left(\frac{R}{r}\right)^2 \sigma T_e^4,$$

where r is the distance of the planet from the Sun, R the radius of the Sun, and T_e the effective surface temperature of the Sun.

Assuming that a planet is in radiative equilibrium, and that a fraction A of the Sun's radiation is reflected by the planetary surface, show that the surface temperature of the planet is given by

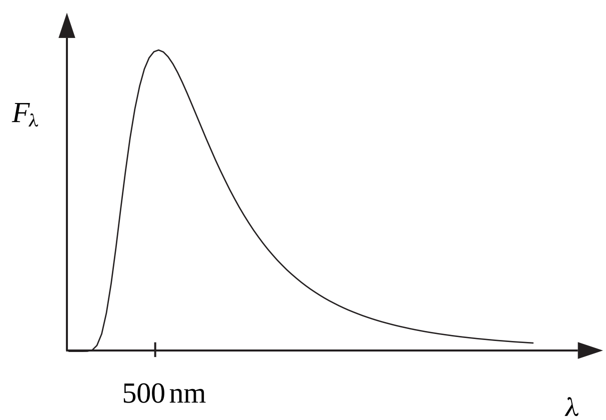
$$T_p = (1 - A)^{1/4}\left(\frac{R}{2r}\right)^{1/2} T_e. \qquad [12]$$

Calculate the surface temperature for Mars given that its albedo, A, is 0.16 and that it is at a mean distance of 1.52 AU from the Sun. {217.2 K} [3]

At what wavelength does Mars emit most of its radiation? [3]

(You may assume that $T_e = 5\,800$ K.)

Solution: Effective temperature of the Sun is 5 800 K. From Wien's law $\lambda_{max} T = 2.9 \times 10^{-3}$ m K. Hence $\lambda_{max} = \frac{2.9\times10^{-3}}{5800} = 5 \times 10^{-7}$ m $= 500$ nm.



Luminosity of the Sun =(surface area)×(flux at surface) $= 4\pi R^2 \sigma T_e^4$. Flux at the surface of Mars is therefore

$$\frac{4\pi R^2 \sigma T_e^4}{4\pi r^2} = \left(\frac{R}{r}\right)^2 \sigma T_e^4.$$

Power falling on Mars' surface = (flux at surface of Mars)×(cross-sectional area of Mars)

$$= \left(\frac{R}{r}\right)^2 \sigma T_e^4 \times \pi r_p^2.$$

Fraction of power falling on Mars' surface that is reflected is A. Therefore fraction that is absorbed is $(1 - A)$. Therefore power absorbed by Mars is

$$(1 - A)\left(\frac{R}{r}\right)^2 \sigma T_e^4 \pi r_p^2.$$

If the planet is in radiative equilibrium, the planet must radiate the same amount of power as it receives. Assuming Mars also behaves as a blackbody, the power it radiates is $4\pi r_{\mathrm{p}}^2 \sigma T_{\mathrm{p}}^4$. (N.B. This radiation will be in the infrared.) In radiative equilibrium we have

$$4\pi r_{\mathrm{p}}^2 \sigma T_{\mathrm{p}}^4 = (1 - A)\left(\frac{R}{r}\right)^2 \sigma T_{\mathrm{e}}^4 \pi r_{\mathrm{p}}^2.$$

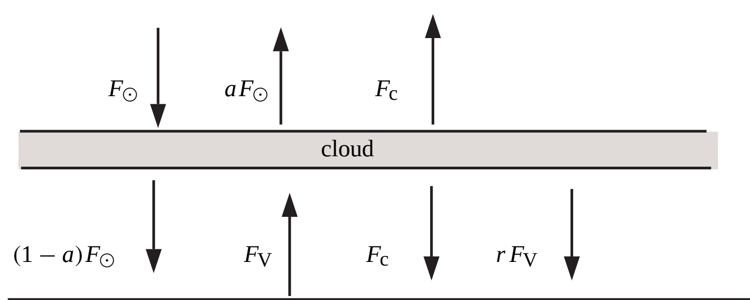
$$\text{Thus} \quad T_{\mathrm{p}} = (1 - A)^{1/4}\left(\frac{R}{2r}\right)^{1/2} T_{\mathrm{e}}.$$

Substituting values for $A = 0.16$, $R = 6.96 \times 10^8$ m, $r = 1.496 \times 10^{11}$ m and $T_{\mathrm{e}} = 5\,800$ K, we obtain $T_{\mathrm{p}} = 217.2$ K.

Using Wien's law again, $\lambda_{\max} = 13\,\mu$m. This is in the infrared. Most radiation will be emitted in the infrared.

1.17 If one ignores the effect of Venus' atmosphere the predicted surface temperature would be around 240 K, which is the temperature above the highly reflective clouds. The measured surface temperature however is about 700 K. Explain the reasons for this difference. [6]

A simple model of the atmosphere of Venus is described by the figure below.



Here $F_\odot$ is the radiative flux from the Sun arriving at the top of Venus's atmosphere. The radiative flux from the Sun reflected by the cloud is given by $aF_\odot$ where a is the albedo, and the radiative flux transmitted through the cloud is $(1-a)F_\odot$.

The radiative flux emitted by the upper and lower surface of the cloud is given by F_{c}. A fraction r of the radiative flux, F_{V}, emitted from Venus's surface is reflected by the cloud and the remaining fraction $(1-r)$ is absorbed by the cloud. Assuming the radiative flux is in overall balance, show that $F_{\mathrm{c}} = (1-a)F_\odot$ and $F_{\mathrm{V}} = F_\odot(1-a) + rF_{\mathrm{V}} + F_{\mathrm{c}}$. Hence show that

$$\frac{F_{\mathrm{V}}}{F_{\mathrm{c}}} = \frac{2}{1-r}$$ [8]

Assuming that both cloud and Venus's surface radiate as black bodies, and that the cloud is at a temperature of 240 K, calculate the value of r necessary to yield the measured surface temperature of Venus of 720 K. {0.975} [6]

Solution: Refer to the diagram. Flux in from the Sun must be equal to the total flux out from Venus. The former is $F_\odot$. The latter is $aF_\odot + F_{\mathrm{c}}$. Thus we have $F_\odot = aF_\odot + F_{\mathrm{c}}$, hence $F_{\mathrm{c}} = (1-a)F_\odot$. Now consider the radiation at surface of Venus. The flux absorbed by the surface is $(1-a)F_\odot +$

$rF_V + F_c$. This must equal the flux emitted by the surface, thus $F_V = (1-a)F_\odot + rF_V + F_c$. Substituting for $(1-a)F_\odot = F_c$ in the above equation we have $F_V = F_c + rF_V + F_c = 2F_c + rF_V$, and so $F_V - rF_V = (1-r)F_V = 2F_c$, or

$$\frac{F_V}{F_c} = \frac{2}{1-r}.$$

For a blackbody at temperature T, the flux is given by $F = \sigma T^4$. Thus

$$\frac{F_V}{F_c} = \frac{\sigma T_V^4}{\sigma T_c^4} = \frac{720^4}{240^4} = 81.$$

Now $F_V/F_c = 2/(1-r)$, so $r = 79/81 \simeq 0.975$.

1.18 Using the data given below, show why Earth has lost all of its atmospheric molecular hydrogen, whereas Jupiter maintains a hydrogen-rich atmosphere. You may use the fact that a planet loses a component of its atmosphere if the thermal speed of that component is greater than $1/6^{th}$ of the escape speed. [6]

[Temperature of Earth's lower atmosphere = 290 K,
Temperature of Jupiter's lower atmosphere = 160 K,
Mass of molecular hydrogen = 3.34×10^{-27} kg]

Solution: Thermal speed of H_2 molecule given by $\frac{3}{2}kT = \frac{1}{2}mv_{th}^2$

So, on Earth, $v_{th,E} = \sqrt{\frac{3kT_E}{m}} = 1.90 km/s$
on Jupiter, $v_{th,J} = 1.86 \times \sqrt{\frac{T_J}{T_E}} km/s$

So $6 \times v_{th,E} = 11.4 km/s > v_E$
$6 \times v_{th,J} = 8.27 km/s < v_J$

i.e. thermal speed $>$ $6\times$ escape speed on Earth but not on Jupiter. Therefore Earth loses H_2, and Jupiter retains it. [6]

1.19 Explain the terms

(a) igneous,

(b) sedimentary,

(c) metamorphic, and

(d) primitive rocks.

Where might these rocks be found? [8]

Calculate the age of a rock sample which contains 46 parts per billion of ^{40}K and 12 parts per billion of ^{40}Ar. Assume that all of the ^{40}Ar found is the direct result of decay from ^{40}K, with a half-life of 1.3×10^9 years. $\{4.3 \times 10^8$ yr$\}$ [8]

Explain why this rock would be considered relatively ancient if found on the Earth's surface but not if it were found on the Moon. [4]

Solution: Igneous refers to rocks which have been *melted* from their primitive state. They are found on Earth at solidified lava flows of various ages. Sedimentary rocks have been formed by *compression* of rock debris and sediment They are found on the Earth's surface and in brecca on the

Moon. Metamorphic rocks are recycled rocks, i.e., sedimentary or igneous rocks that have been reprocessed by tectonic activity (recirculated – e.g., marble). They are also found on the Earth's surface. Primitive rocks have been formed from primitive material in the solar system, without having undergone melting or reprocessing. They are typically found in asteroids and meteorites. The number of undecayed ^{40}K atoms is given by

$$N(t) = N_0 \exp(-\lambda t).$$

λ can be expressed in terms of the half-life, τ, since $\frac{N(\tau)}{N_0} = \frac{1}{2} = \exp(-\lambda\tau)$. This implies that $\ln(e^{-\lambda\tau}) = -\frac{\tau}{\lambda} = -\ln 2$, thus $\lambda = (\ln 2)\tau$.

Now substitute values in the expression above. Since 46 parts remain out of $46 + 12$, we can write

$$\frac{N(t)}{N_0} = \frac{46}{58} = e^{-\lambda t}.$$

Thus taking logarithms to the base e again $-\lambda t = -0.231\,8$. Hence $t = 0.231\,8\lambda \simeq 4.3 \times 10^8$ years.

Most rocks on the Earth are only a hundred million years old because of recycling in the mantle and other tectonic activity and weathering. So this rock would be old. On the Moon most rocks would be of this age, however, because there has been no tectonic or volcanic activity for thousands of millions of years.

1.20 Discuss briefly the internal structure and composition of the *terrestrial planets*, pointing out the similarities and differences. [8]

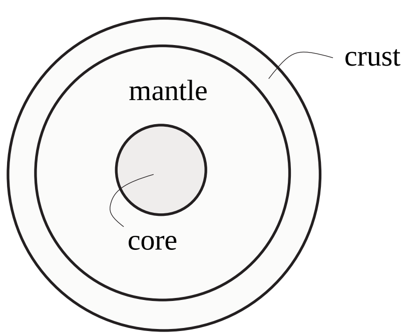
Explain the term *radiogenic heating*. [2]

The decay of ^{40}K into ^{40}Ar is responsible for about 50 % of the radiogenic heating of the Earth today. ^{232}Th(orium), ^{238}U(ranium) and ^{235}U provide the rest. 1 kg of Earth contains about 10^{-8} kg of ^{40}K. The atomic mass of ^{40}K is 39.97 amu and the atomic mass of ^{40}Ar is 39.96 amu. The half life of ^{40}K is about 109 years. *Estimate* the amount of heat thus generated per year per kg of Earth material, and hence *estimate* the power in watts due to radiogenic heating for the whole Earth. $\{\sim 2 \times 10^{-4}\ \mathrm{J\,yr^{-1}}; \sim 3 \times 10^{13}\ \mathrm{W}\}$ [8]

How does this compare with the rate of heating due to the Sun? $\{\sim 10^{17}\ \mathrm{W}\}$ [2]

[You may assume that the solar constant is $1.4 \times 10^3\ \mathrm{W\,m^{-2}}$.]

Solution:



The overall structure is as above. The Earth's core is partly liquid (thus magnetic field), although there is no evidence that the cores of the other planets are. Mantle is molten lava. Mercury has a solid core and a thick crust (small mass so small heat capacity) and it cooled down quickly. Venus shows signs of volcanism. Mars probably was volcanic in past, but no longer (thickness of crust) – again owing to smaller size. All planets are of similar composition (differentiated) although Mercury has the largest uncompressed density. Density of planets around $5\,000\ \mathrm{kg\,m^{-3}}$, indicating high proportion of dense material (iron and nickel). *Radiogenic heating* is heating through the decay of radioactive elements, mainly ^{235}U (uranium 235) and ^{238}U (uranium 238) and ^{40}K (potassium 40), and ^{232}Th (thorium 232). *Heating due to* ^{40}K: Fraction of mass of ^{40}K converted into energy will be about

$$\frac{39.97 - 39.36}{39.97} \simeq 2.5 \times 10^{-4}.$$

In 1 kg terrestrial material there is 10^{-8} kg of ^{40}K. Thus in radiogenic heat provided by 1 kg of terrestrial matter from decay of ^{40}K in half-life of ^{40}K will be

$$\frac{1}{2} \times 10^{-8} \times 2.5 \times 10^{-4} \times c^2 \simeq 1.1 \times 10^5\ \mathrm{J}.$$

Thus rate of heating by 1 kg will be $\sim \frac{1.1\times10^5}{10^9} \simeq 10^{-4}$ joules per year. For the whole Earth, the rate of heating will be $6 \times 10^{-4} = 6 \times 10^{20}\ \mathrm{J\,yr^{-1}}$. This is due to ^{40}K, so for total radiogenic heating we get twice this amount, or about $10^{21}\ \mathrm{J\,yr^{-1}}$. Converting to watts (one year is about 3×10^7 s) this gives 3×10^{13} W. *Rate of Solar heating*: Area of cross section of the Earth = πR^2. (R is about 6300×10^3 m.) Total rate of heating is therefore $\pi(6300 \times 10^3)^2 \times (1.4 \times 10^3) \simeq 1.7 \times 10^{17}$ W. Some of this radiant energy is reflected, Earth's albedo being about 0.4, so solar heating is around 10^{17} W, which is about 3×10^3 times radiogenic heating. Still radiogenic heating is considerable, and enough to account for volcanism on the Earth.

1.21 Explain what is meant by a tidal force and explain in qualitative terms the origin and meaning of the Roche stability limit. [8]

The Roche stability limit of a planet is approximately given by 2.5 times the planetary radius. Estimate the likely maximum distance that a ring could be found from the centre of Jupiter and Saturn respectively. {1.8×10^5 km; 1.5×10^5 km} [4]

Give a possible explanation for the formation and the structure of these rings. [4]

Where is the asteroid belt located? Give a plausible explanation for the Kirkwood gaps. [4]

Solution: Tidal force is the differential gravitational force between two neighbouring points. Thus consider a 'planet' of mass M. Two particles of mass m are placed at a radial distance r and $r + \Delta$ from the planet. The nearer particle experiences a force

$$-\frac{GMm}{r^2}\hat{r}$$

and the further one a force

$$-\frac{GMm}{(r+\Delta)^2}\hat{r}.$$

The difference between these two forces is approximately

$$-\frac{2GMm\Delta}{r^3}\hat{r}.$$

So the nearer particle accelerates with respect to the more distant particle. The Roche stability limit is obtained by equating the tidal force, tending to separate the two particles, to the mutual gravitational attraction of the two particles. Close into the planet the tidal force increases, and will dominate the mutual attraction between the two particles. The limit at which this takes place is called the Roche stability limit. Rings exist where the planetesimals are unable to fuse together because of tidal forces. Hence we would expect rings to form for $r < 2.5R_{\mathrm{p}}$ or within $2.5 \times 71\,000 = 177\,500$ and $2.5 \times 60\,000 = 150\,000$ km for Jupiter and Saturn respectively. Condensation of gases and ice during the formation of the planet, as well as break up of satellites. Composition of rings seems to be dirty ice. Shepherding satellites appear to play a role in the formation of structure within the rings. Asteroid belt lies between Mars and Jupiter. Kirkwood gaps correspond to periods in which the asteroids are not found. Resonance effects with Jupiter probably causes the ejection of the asteroids with these periods from their orbits.

1.22 From the data given in your lecture notes, calculate the average density of each of the planets compared to the Earth. Group the results according to any broad features you observe.

Solution: Assume planets are spherical – volume $4\pi r_{\mathrm{P}}^3/3$, mass m_{P} gives mass density as

$$\rho_{\mathrm{P}} = \frac{3m_{\mathrm{P}}}{4\pi r_{\mathrm{P}}^3}.$$

For the Earth,

$$\rho_{\mathrm{E}} = \frac{3m_{\mathrm{E}}}{4\pi r_{\mathrm{E}}^3}.$$

Hence

$$\frac{\rho_{\mathrm{P}}}{\rho_{\mathrm{E}}} = \frac{m_{\mathrm{P}}}{m_{\mathrm{E}}}\left(\frac{r_{\mathrm{E}}}{r_{\mathrm{P}}}\right)^3 = \frac{m_{\mathrm{P}}}{m_{\mathrm{E}}}\left(\frac{D_{\mathrm{E}}}{D_{\mathrm{P}}}\right)^3.$$

Using data given in the table in notes:

	$m_{\mathrm{P}}/m_{\mathrm{E}}$	$D_{\mathrm{P}}/D_{\mathrm{E}}$	$\rho_{\mathrm{P}}/\rho_{\mathrm{E}}$
Mercury	0.558	0.381	1.009
Venus	0.815	0.951	0.948
Mars	0.107	0.531	0.715
Jupiter	317.9	10.86	0.248
Saturn	95.2	8.99	0.131
Uranus	14.54	3.97	0.232
Neptune	17.15	3.86	0.298
Pluto	0.0022	0.18	0.377

Low density – gaseous giants.
High density ($\simeq$ Earth density) – terrestrials.

1.23 Show that the tidal force acting on a body, mass m, radius R, when at a distance r from an object of mass M, has magnitude

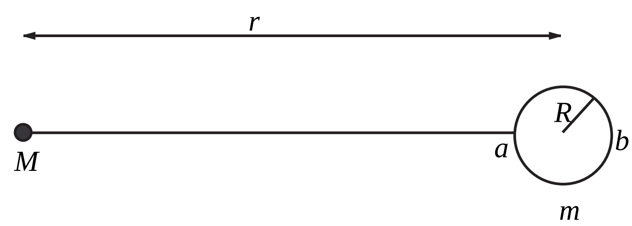
$$F_{\mathrm{tidal}} = \frac{4GMmR}{r^3},$$

by considering the difference in gravitational forces exerted on the far and near side of the affected body. Assume $r \gg R$.

Calculate the ratio of the tidal forces exerted by Jupiter on its satellites *Io* and *Callisto*. Explain how the difference in tidal forces between these two satellites is reflected in their physical properties.

[$r_{\rm Io} = 4.22 \times 10^5$ km, $r_{\rm Callisto} = 1.88 \times 10^6$ km,
$m_{\rm Io} = 8.92 \times 10^{22}$ kg, $m_{\rm Call} = 1.08 \times 10^{23}$ kg,
$R_{\rm Io} = 1815$ km, $R_{\rm Callisto} = 2400$ km]

Solution:



Gravitational force at $a = \frac{GMm}{(r-R)^2}$.
Gravitational force at $b = \frac{GMm}{(r+R)^2}$.

$$\frac{1}{(r \pm R)^2} \simeq \frac{1}{r^2}\left(1 \mp \frac{2R}{r}\right),$$

so differences in forces is

$$F_{\rm tidal} = \frac{GMm}{r^2}\left[1 + \frac{2R}{r} - \left(1 - \frac{2R}{r}\right)\right] = \frac{4GMmR}{r^3},$$

ratio

$$\frac{\rm Io}{\rm Callisto} = \frac{m_{\rm Io}}{m_{\rm Call}}\frac{R_{\rm Io}}{R_{\rm Call}}\frac{r^3_{\rm Call}}{r^3_{\rm Io}} = \left(\frac{8.92 \times 10^{22}}{1.08 \times 10^{23}}\right)\left(\frac{1815}{2400}\right)\left(\frac{1.88 \times 10^6}{4.22 \times 10^5}\right)^3 = 55.2.$$

Io is volcanically active, with hot molten interior as a result of tidal stresses, whereas Callisto has cold cratered surface, showing little volcanic activity consistent with the relative lack of tidal stresses.

1.24 Calculate the ratio of the surface gravities of the Moon and Mercury. Explain the consequences of your answer in terms of the impact cratering on each body. ($M_{\rm Moon} = 0.012 M_{\rm Earth}$, $M_{\rm Mercury} = 0.0558 M_{\rm Earth}$, $D_{\rm Moon} = 0.27 D_{\rm Earth}$, $D_{\rm Mercury} = 0.381 D_{\rm Earth}$) {0.43}

Solution: Surface gravity of body of mass M, radius R is just

$$g_{\rm surface} = \frac{GM}{R^2},$$

$$\text{so} \quad \frac{GM_{\rm Moon}}{R^2_{\rm Moon}}\frac{R^2_{\rm Mercury}}{GM_{\rm Mercury}} = \left(\frac{M_{\rm Moon}}{M_{\rm Mercury}}\right) \times \left(\frac{R_{\rm Mercury}}{R_{\rm Moon}}\right)^2.$$

Substitute the values given to get $\frac{g_{\rm Moon}}{g_{\rm Mercury}} = 0.43$.

Impact craters on Mercury have lower rim walls, and eject a shorter travel distance than on the Moon because Mercury has greater surface gravity.

1.25e Show that the tidal force per unit mass between the centre and the surface of a satellite of radius R_S, at a distance r from the centre of a planet of mass M_P and radius R_P, has magnitude given approximately by

$$F_{\text{Tidal}} \approx \frac{2GM_P R_S}{r^3}.$$ [3]

Hence, or otherwise, show that a crude estimate of the Roche Limit - i.e. the distance within which the satellite will be tidally disrupted - is given by

$$r_{\text{disrupt}} = 2^{\frac{1}{3}} \left(\frac{\overline{\rho}_P}{\bar{\rho}_S} \right)^{\frac{1}{3}} R_P,$$

where $\overline{\rho_P}$ and $\overline{\rho_S}$ denote the mean density of the planet and satellite respectively. [7]

Given that a more precise estimate of the Roche limit is the distance

$$r_{\text{Roche}} = 2.456 \left(\frac{\overline{\rho}_P}{\overline{\rho}_S} \right)^{\frac{1}{3}} R_P,$$

calculate the Roche limit for the planet Saturn, assuming that the mean density of Saturn's moons is $\overline{\rho}_S = 1200\,\mathrm{kg\,m^{-3}}$. [4]

Comment on the significance of your answer for observations, and formation theories, of Saturn's ring system. [3]

Solution:

$$\begin{aligned} |F_{\text{TIDAL}}| &= \frac{GM_P}{(r-R_S)^2} - \frac{GM_P}{r^2} \\ &= \frac{GM_P}{r^2}\left(\frac{1}{(1-R_S/r)^2} - 1\right) \qquad \text{(for a unit mass on the satellite)} \\ &\simeq \frac{2GM_P R_S}{r^3} \qquad \text{(since } R_S << r\text{)} \end{aligned}$$

[3]

Force on a unit mass on satellite's surface, due to gravity of satellite alone

$$F_S = \frac{GM_S}{{R_S}^2}$$

Tidal force on a unit mass due to planet:

$$F_T = \frac{2GM_P R_S}{r^3}$$

Moon is disrupted if $F_T > F_S$ i.e. $\frac{2GM_p R_S}{r^3} > \frac{GM_S}{{R_S}^2}$

Putting $M_S = \frac{4}{3}\pi R_S^3 \bar{\rho}_S$, $M_p = \frac{4}{3}\pi R_S^3 \bar{\rho}_p$

$\Rightarrow$ moon disrupted if $r < 2^{1/3} \left(\frac{\bar{\rho}_p}{\bar{\rho}_S}\right)^{1/3} R_p$ [7]

For Saturn, $\bar{\rho}_p = \frac{\bar{M}_p}{\frac{4}{3}\pi R_p^3} = 620\,\mathrm{kg\,m^{-3}}$

$\Rightarrow r_{Roche} = 2.456 \times \left(\frac{620}{1200}\right)^{1/3} R_p$

$= 1.97 R_p = 1.19 \times 10^8\,\mathrm{m}$ [4]

All of Saturn's moons and most of Saturn's rings lie outside/inside this radius respectively. This is consistent with the theory that dusing planetary formation, tidal forces prevented any material clumping together too close to the planet. [3]

1.26 Calculate the age of a rock sample which contains 46 parts per billion of ^{40}K and 12 parts per billion of ^{40}Ar. Assume that all of the ^{40}Ar found is the direct result of decay from ^{40}K, with a half-life of 1.3 billion years.

Would this rock be considered relatively ancient if found on the Earth's surface? What if it were a Moon rock? $\{4.35 \times 10^8$ yr$\}$

Solution: We use the half-life formula of the form

$$N = N_0 \mathrm{e}^{-\lambda t}$$

where λ is the decay constant, N_0 is the number of radioactive nuclei at $t = 0$ and N is the number after a time t.

The relationship between λ and half-life, $t_{1/2}$, is

$$N = \frac{1}{2} N_0 \quad \text{at} \quad t = t_{1/2}. \tag{1}$$

$$\therefore \frac{1}{2} N_0 = N_0 \mathrm{e}^{-\lambda t_{1/2}} \tag{2}$$

$$\log_{\mathrm{e}} 2 = \lambda t_{1/2} \tag{3}$$

$$\therefore \lambda = \frac{\log_{\mathrm{e}} 2}{t_{1/2}} = \frac{0.693}{t_{1/2}}. \tag{4}$$

Since all the ^{40}Ar is a result of the decay from ^{40}K, we must have

$$N_0 = 46 + 12 \quad \text{parts per billion} \tag{5}$$

$$n = 46 \quad \text{parts per billion} \tag{6}$$

$$\therefore \frac{58}{46} = \mathrm{e}^{\lambda t} \tag{7}$$

$$\Rightarrow t = \log_{\mathrm{e}} \left(\frac{58}{46} \right) \cdot \frac{1}{\lambda} = \log_{\mathrm{e}} \left(\frac{58}{46} \right) \cdot \frac{t_{1/2}}{0.693} \tag{8}$$

$$\text{where} \quad t_{1/2} = 1.3 \times 10^9 \, \text{yr} \tag{9}$$

$$\Rightarrow t = 4.35 \times 10^8 \, \text{yr} = 435 \text{ million years.} \tag{10}$$

$$\tag{11}$$

Typical surface rocks on the earth are 100–200 million years old.Therefore this is a relatively old rock on Earth. Moon rocks are much older however, near to 4.5×10^9 years ($\sim 3 \times 10^9$ years in maria.)

1.27 The angular diameter of the Sun as seen from Earth is 32 arcmin. Use this measured value to calculate the radius of the Sun. Assuming the Sun emits its luminosity like a perfect black body, calculate its effective temperature, and compare this with Earth's surface temperature.

Solution: *no solution available*

1.28 The equation for hydrostatic equilibrium is $dP/dz = -\rho g$ where z is the height above the surface of a planet, g the surface gravitational acceleration, P the pressure and ρ the gas density. Using the ideal gas law $P = \rho kT/\mu$, where T is the temperature, k is Boltzmann's constant and μ the average mass of a gas particle, show that in an isothermal atmosphere (i.e. where T is constant) $P = P_0 \exp(-z/H)$, where $H = kT_0/(\mu g)$. We think of H as the scale-height for variations in the atmosphere. What could the pressure profile be if the atmosphere was not isothermal?

Solution: *no solution available*

1.29 Suppose the surface of a sphere the same size as the Sun was covered in electric light bulbs. What would the wattage of each bulb have to be in order to match the Sun's luminosity? Assume the surface area of a light bulb is 30 cm^2.

Solution: Surface area of Sun $4\pi r_\odot^2 = 6 \times 10^{18}\,\mathrm{m}^2$. Surface area of bulb $30\,\mathrm{cm}^2 = 3 \times 10^{-3}\,\mathrm{m}^2$,

Number of bulbs $= 2 \times 10^{21}$. Luminosity of Sun $= 4 \times 10^{26}\,\mathrm{W}$. Wattage of bulbs $= \frac{4\times 10^{26}}{2\times 10^{21}} = 2 \times 10^5\,\mathrm{W}$.

1.30 Compare the tidal force exerted by the Sun on the Earth with that exerted by the Moon on the Earth. Explain your answer in terms of the spring and neap tides.

Solution: *no solution available*

1.31 State Wien's Law, and describe briefly how it may be used. What is the effective temperature of an object? Radiation from Mars is observed predominantly at a wavelength of $\lambda = 1.45 \times 10^{-5}$ m. Calculate the surface temperature and hence the implied luminosity of Mars. The effective temperature of Mars is 217 K. Calculate the intrinsic luminosity of Mars using this temperature, and compare it with your earlier answer. Can you explain the difference? Calculate Mars' albedo.

[$W = 2.9 \times 10^{-3}$ K m]

Solution: Wien's law states that for a black body $\lambda_{max} T = W$ where $W = 2.9 \times 10^{-3}\ m\ K$ is Wien's constant. By measuring the wavelength λ_{max} at which the spectrum of a body peaks we can find its temperature T, assuming it radiates like a black body. This is called the *colour temperature* T_c.

The effective temperature T_e is the temperature a body would need to have to produce its actual luminosity L if it radiated like a black body. For a body of area A, Stefan's law gives $L = A\sigma T_e^4 = 4\pi R^2 \sigma T_e^4$ for a sphere of radius R

Using Wien's Law, $\lambda_{max} 1.45 \times 10^{-3}\ m$ implies $T = 200\ K$

Then for Mars

$$L_M = 4\pi R^2 \sigma T_e^4 = 4 \times 3.14 \times (3.4 \times 10^6)^2 \times 5.7 \times 10^{-8} \times (200)^4 = 1.32 \times 10^{16} W$$

But using T_e we get

$$L_M = 4\pi R^2 \sigma T_e^4 = 4 \times 3.14 \times (3.4 \times 10^6)^2 \times 5.7 \times 10^{-8} \times (217)^4 = 1.83 \times 10^{16} W$$

The difference shows that Mars does not radiate exactly like a black body.

1.32 Give a sketch of the main regions of the solar atmosphere, indicating the temperature and dimensions.

Using Wien's displacement law calculate the typical wavelength and energy in eV of the photons emitted from these regions, and state the region of the spectrum in which they lie.

The ionisation potential of atomic hydrogen is 13.6 eV. What would the frequency of a photon have to be in order to ionise atomic hydrogen? What would you expect the ionisation potential of $^4_2$He and $^{56}_{26}$Fe to be? (Hint: the ionisation potential of the last electron in an atom is proportional to Z^2, where Z is the atomic number.) What would the corresponding frequency/energy of a photon have to be?

The Sun's luminosity comes from hydrogen fusion in the core of the Sun, which essentially converts 4 protons into $^4_2$He. Given the mass of a proton is 1.0073 amu, and the mass of $^4_2$He is 4.0026 amu, calculate the *fraction* of the mass of a proton that gets converted into energy.

Estimate the relative efficiency of a hydrogen bomb to TNT or some other chemical explosive.

How many kg of H need to be converted into $^4_2$He per second to provide the luminosity of the Sun?

(Subversive question: how much hydrogen would be necessary to blow the Houses of Parliament 100 metres into the sky?)

(Hysterical question: How much Hydrogen would be necessary to deviate a 1 000 m diameter asteroid from collision course with Earth?)

Solution: See lecture notes – photosphere $\sim$100 - 500 km (effective temperature 6 000 K), chromosphere $\sim$ 2 000 km (from 4 000 K – 100 000 K near corona), corona $\sim$ several solar radii, (10^6 K).

Use these temperatures in Wien's law $\lambda_{\max}T = 2.8 \times 10^{-3}$ m K.
For 6 000 K this gives $\lambda_{\max} = 470$ nm. This will be typical wavelength. In energy units, typical energy will be given by $E = h\nu = \frac{hc}{\lambda}$. With $h = 6.63 \times 10^{-34}$ J s, and $c = 3.0 \times 10^8$ m s^{-1} we obtain

$$E = h\nu = \frac{hc}{\lambda} = \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{1.7 \times 10^{-7}} \approx 5 \times 10^{-19}\,\mathrm{J},$$

Convert to eV: 1 eV = 1.602×10^{-19} J, so energy in eV is $\frac{5\times10^{-19}}{1.6\times10^{-19}} \approx 3$ eV. (This is in the visible part of the spectrum.)
Do the same for 10^5 K and 10^6 K. (These will be in the X-ray part of the spectrum.)
Ionisation frequency: use $E = h\nu$ with $E = 13.6\,\mathrm{eV} = 13.5 \times 5 \times 10^{-19}$ J.
Fraction of mass converted to energy

$$\frac{(4 \times 1.0073) - 4.0026}{4 \times 1.0073} \approx \frac{0.026}{4} \approx 0.007,$$

or about 1%.

In chemical reactions we can only get a few eV per atomic mass. Mass of H is about 1 000 MeV. So efficiency is around 10^{-8} or 10^{-9}.

1.33 The gravitational force produced by a body of mass M on another of mass m is given by $-GMm/r^2$.

(a) (i) Use this result to show that the surface gravity at the Earth's surface is $-GM_\oplus/R_\oplus^2$.
The Earth is made up of a metallic core of iron and nickel (about 95% iron). The density of iron at atmospheric pressure and 300 K is about 8 000 kg m^{-3}, and nickel a bit more, about 9 000 kg m^{-3}. The mantle is molten lava with density about 5 200 kg m^{-3}, and the crust is a thin layer (about 100 km or so) of rock with a somewhat lower density.

(ii) Assuming that the core has a radius of 2 000 km, estimate the total mass of the Earth, and its mean density assuming that the radius is 6 400 km.

(iii) Given that the measured surface gravity of the Earth is 9.8 m s^{-2}, estimate the value of G.

(iv) If the core radius was 2 500 km, what value of G would you obtain?

(b) (i) Calculate the relative size of the gravitational forces exerted by the Earth and the Sun on the Moon.

(ii) Calculate the relative size of the gravitational forces exerted by the Moon and the Sun on the Earth.

(iii) Tidal forces are proportional to M/r^3. Estimate the relative size of the tidal force of the Sun and the Moon at the surface of the Earth.

(iv) Discuss the relevance of this result for spring and neap tides.

Solution:

(a) (i) The gravitational force on a particle of mass m at the surface of the Earth is $-GM_\oplus m/R_\oplus^2$. Use force = Mass × Acceleration or $\mathbf{F} = m\mathbf{a}$ so that radial component gives $-GM_\oplus m/R_\oplus^2 = ma$. Evidently the m cancels and we have all bodies accelerating at the same rate towards the centre of the Earth. This acceleration is called the surface gravity. Thus $g_\oplus = GM_\oplus/R_\oplus^2$.

(ii) Mass and Mean density

Take core density to be $8\,500\,\mathrm{kg\,m^{-3}}$, say.
Volume in the core, $V_\mathrm{c} = \frac{4}{3}\pi r_\mathrm{c}^3 = \frac{4}{3} \times 3.14 \times (2.0 \times 10^6)^3 = 3.4 \times 10^{19}\,\mathrm{m}^3$.
Mass in the core is $\rho_c V_C = 8500 \times 3.35 \times 10^{19} = 2.9 \times 10^{23}\,\mathrm{kg}$.
Volume in the mantle, $V_\mathrm{m} = \frac{4}{3}\pi r_\mathrm{m}^3 - V_\mathrm{c} = 1.1 \times 10^{21}\,\mathrm{m}^{-3}$.
Mass in the mantle is $\rho_\mathrm{m} V_\mathrm{m} = 5200 \times 1.1 \times 10^{21} = 5.5 \times 10^{24}\,\mathrm{kg}$.
Volume in the crust, $V_\mathrm{cr} = V - V_\mathrm{m} - V_\mathrm{C} = 3.9 \times 10^{19}\,\mathrm{m}^3$.
Mass in the crust is $\rho_\mathrm{cr} V_\mathrm{cr} = 1.2 \times 10^{23}\,\mathrm{kg}$.
The total mass is $M\rho_\mathrm{cr} V_\mathrm{cr} + \rho_\mathrm{m} V_\mathrm{m} + \rho_\mathrm{C} V_\mathrm{C}$. Substituting in gives a total mass of $5.9 \times 10^{24}\,\mathrm{kg}$.
Mean density is given by Mass/Volume $= 5\,400\,\mathrm{kg\,m^{-3}}$.

(iii) Gravitational constant

$g_\oplus = GM_\oplus/R_\oplus^2$ therefore $G = g_\oplus R_\oplus^2/M_\oplus$. Substituting for mass and radius of the Earth, and for the surface gravity of $9.8\,\mathrm{m\,s^{-2}}$, we obtain
$G = 6.8 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}$.

(iv) Changing the core radius only changes G by a small amount, since most of the mass is in the mantle. For these figures the new value of G is $6.6 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}$.

(b) (i) Force on Moon due to Earth is $-\frac{GM_\oplus M_\mathrm{moon}}{r_\mathrm{em}{}^2}$. Force on moon due to Sun is $-\frac{GM_\odot M_\mathrm{moon}}{r_\mathrm{sm}{}^2}$. Relative strength of Earth to sun is gravitational force is

$$\frac{-\frac{GM_\oplus M_\mathrm{moon}}{r_\mathrm{em}{}^2}}{-\frac{GM_\odot M_\mathrm{moon}}{r_\mathrm{sm}{}^2}} = \frac{M_\oplus}{M_\odot}\left(\frac{r_\mathrm{sm}{}^2}{r_\mathrm{em}{}^2}\right) = \frac{M_\oplus}{M_\odot}\left(\frac{r_\mathrm{sm}}{r_\mathrm{em}}\right)^2.$$

Substituting values in we get the ratio to be

$$\frac{6 \times 10^{24}}{2 \times 10^{30}} \times \left(\frac{8 \times 60}{1.2}\right)^2 \approx 3 \times 10^{-6} \times (400)^2 \approx 0.6.$$

(ii) Gravitational force of the Sun on the Earth must be factor of $\frac{M_\oplus}{M_\odot} \approx 100$ greater than the Sun's gravitational force on the Moon. The Moon's force on the Sun must be the same as the Sun's force on the Moon, which is 1/0.6 of the Earth's force on the Moon. Therefore the Sun's force on the Earth must be $100 \times 1/0.6$ or about 160 times the force of the Moon on the Earth.

(iii) The tidal forces however go as

$$\frac{M_\odot}{M_{\text{moon}}}\left(\frac{r_{\text{m}-\text{e}}}{r_{\text{s}-\text{e}}}\right)^3 = \frac{2\times10^{30}}{7\times10^{22}}\left(\frac{1.2}{8\times60}\right)^3 \approx 0.5.$$

Thus tidal force due to the Sun and the Moon are of the same order of magnitude.

(iv) Neap tides will take place at half-Moon, spring (high) tides at full and new Moon.

1.34e Briefly discuss the *runaway greenhouse effect*, and how it can account for the lack of water on Venus. [5]

Solution: *no solution available*

1.35e Rocks on Earth may be classified into three main types according to how they were formed. Name two of these types and describe the processes by which they have been formed.

A further class of rocks are the *primitive* rocks. Where are these typically found? [5]

Solution: *no solution available*

1.36e Briefly describe with the aid of a diagram, the interior structure of Jupiter. What are the two dominant elements in Jupiter's atmosphere? [5]

Solution: *no solution available*

1.37e Using Newton's Law of Gravitation, write down the force acting on a point mass m at position X which is a distance r from a body of mass M, and give its direction. Also write down the force on such a mass m if it were at position Y, a distance $r+a$ from mass M, where X and Y lie along the same radius vector. Assuming $a \ll r$, show that the magnitude of the force on m at position Y can be approximated to

$$F_{\text{Y}} = \frac{GMm}{r^2}\left(1-\frac{2a}{r}\right).$$

[You may use the binomial theorem: $(1+x)^n \approx 1+nx$, for $|x| \ll 1$.]

Hence show that the magnitude of the tidal force between two masses, each of mass m, at these positions is given by $2GMma/r^3$. [5]

Explain with the aid of a diagram how the Moon produces tides on the Earth, and why there are two high tides per day. [4]

Calculate the *ratio* of the tidal force exerted by the Sun at the Earth to that exerted by the Moon at the Earth. [4]

Explain with reference to the positions of the Sun and the Moon why spring tides are stronger than neap tides. [4]

[mass of Moon $= 7.35\times10^{22}$ kg, mean distance of Moon from Earth $= 3.84\times10^{8}$ m.]

Solution: (a) Force acting on mass m at X, distance r from M is

$$F_X = \frac{GmM}{r^2}\hat{r}$$

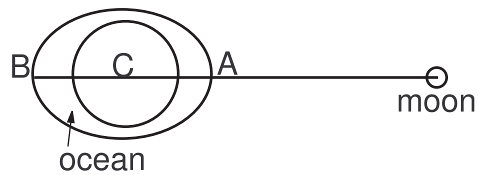
where $\hat{r}$ is in the direction from m to M. Force acting on mass m at Y, distance $r+a$ from M, is

$$\begin{aligned}
F_Y &= \frac{GmM}{(r+a)^2}\hat{r} \\
&= \frac{GmM}{r^2(1+a/r)^2}\hat{r} \\
&= \frac{GmM}{r^2}(1+a/r)^{-2}\hat{r} \\
&\approx \frac{GmM}{r^2}(1-2a/r)\hat{r} \quad \text{by binomial theorem for } a \ll r
\end{aligned}$$

The tidal force F_T is then

$$\begin{aligned} F_T &= F_Y - F_X \\ &\approx -2\frac{GmMa}{r^3}\hat{r} \end{aligned}$$

(b) The moon exerts a tidal force on the earth as shown, such that $F_A > F_C > F_B$. The water at A experiences a maximum pull towards the moon, and at B a minimum pull. Thus 2 tidal bulges



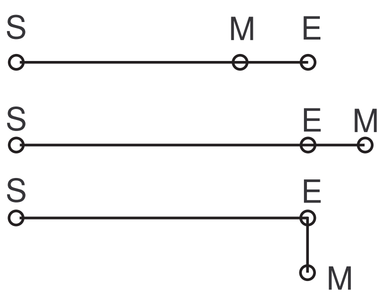
appear, and there are 2 tides a day as the earth rotates.
(c) The tidal force exerted by the Sun at Earth is

$$F_{\odot,E} \approx -\frac{2GM_{/odot}m_{\oplus}r_{\oplus}}{R^3_{\odot,E}}$$

where $M_\odot$ is the solar mass, $m_\oplus$ is the Earth mass, $r_\oplus$ is the Earth's radius, and $R_{\odot,E}$ is the Sun - Earth distance.

The ratio of the tidal force exerted by the sun at the earth to that exerted by the moon at the earth is given by

$$\begin{aligned} \frac{F_{\odot,E}}{F_{M,E}} &\approx \frac{M_\odot}{M_m}\left(\frac{R_{M,E}}{R_{\odot,E}}\right)^3 \\ &= \frac{1.99\times10^{30}}{7.35\times10^{22}}\left(\frac{3.84\times10^{8}}{1.50\times10^{11}}\right)^3 \\ &= 0.454 \end{aligned}$$



(d) Tidal force due to the Sun is a significant fraction of that due to the moon, so the Sun's influence cannot be neglected. Spring Tides occur when the Sun, Moon and Earth are aligned (ie at full moon and new moon - see top two diagrams). Neap tides occur when the Sun, Moon and Earth are in quadrature (see last diagram for an example).

1.38e The dating of rocks can be carried out using the law of radioactive decay of unstable isotopes, given by

$$N = N_0 \exp(-\lambda t)\,,$$

where λ is the decay constant. What are N and N_0 in this equation? What is meant by the half-life, $t_{1/2}$, for a radioactive isotope? Sketch a graph of N versus t, clearly marking the position of $t_{1/2}$.

Show that $t_{1/2} = (\log_e 2)/\lambda$. [8]

A rock sample is found to contain 5 parts per billion of the radioactive isotope of potassium, ^{40}K, and 35 parts per billion of argon, ^{40}Ar. Assuming all of the ^{40}Ar found is the direct result of decay from ^{40}K, calculate the age of the rock. The half-life for decay of ^{40}K is 1.3×10^9 years.

If this rock were found on Earth comment on whether its age would be typical of Earth rocks. [5]

Heating by radioactive decay of naturally occurring unstable isotopes is believed to be the main source of energy within the Earth which drives plate tectonic activity. Give the name of one other radioactive element, apart from potassium, which contributes to this heating, and briefly explain what the term *plate tectonics* means. [4]

Solution: *no solution available*

1.39 (a) What is meant by the effective temperature of a stellar body? [1]

The effective temperature of the Sun is 5 800 K. Calculate the flux at the surface of the Sun, and give the units of flux. [3]

Hence calculate the luminosity of the Sun. [2]

The monochromatic flux F_λ from the Sun peaks in the visible region. At what colour is the peak? [1]

Saturn is approximately 10 times further away from the Sun than the Earth is from the Sun. Give the approximate value for the ratio of the Sun's flux falling on Saturn to that falling on the Earth. Justify your answer. [3]

(b) The ideal gas law may be expressed as $P = \frac{\rho k T}{\mu m_{\mathrm{H}}}$. Give the meaning of all the symbols used in this equation. [3]

The atmospheric pressure at Venus's surface is $90 \times 10^5\,\mathrm{N\,m^{-2}}$. Comment on how this compares to the atmospheric pressure at the Earth's surface. [1]

Assuming that the surface temperature of Venus$=$ 740 K, and that the atmosphere is entirely made of carbon dioxide (CO_2), calculate the surface density of Venus's atmosphere. [2]

Calculate the scale height for CO_2 on Venus, and the pressure at this height, assuming an isothermal atmosphere. [4]

[Assume $\mu = 12 + (2 \times 16)$ for CO_2, and surface gravity of Venus $= 8.88\,\mathrm{m\,s^{-2}}$.]

Solution: *no solution available*

1.40 (a) Give the equation for the magnitude of the gravitational force produced by a body of mass M on another mass m separated by a distance r. In what direction does the force act?

Hence show that the surface gravity on a planet of mass M and radius R is given by $g = GM/R^2$. [4]

A measurement of surface gravity on the Moon gives a value of $g_{\mathrm{Moon}} = 1.6\,\mathrm{m\,s^{-2}}$.

Calculate the mass of the Moon. [3]

Calculate the ratio of the surface gravity on Jupiter to that on Pluto. [3]

[radius of Moon$= 1.74 \times 10^6$ m]

(b) Explain with the aid of a diagram how the Moon produces tides on the Earth and why there are two high tides per day. [4]

Tidal forces are proportional to M/r^3 where M is the mass of the body causing the force and r is the average distance from M. Calculate the relative size of the tidal force exerted by the Sun at the Earth to that exerted by the Moon at the Earth. [3]

Explain with reference to the positions of the Sun and the Moon why spring tides are particularly strong. [3]

[mass of Moon $= 7.35 \times 10^{22}$ kg, mean Earth-Moon distance $= 3.84 \times 10^8$ m.]

Solution: *no solution available*

1.41 The Earth has an average density of $5.5 \times 10^3\ \mathrm{kg\,m^{-3}}$ and a radius of 6 400 km. Calculate its total mass. [3]

The interior of the Earth consists of a dense core, surrounded by a mantle and a crust. If the core has a radius of 3500 km and an average density of $12 \times 10^3\ \mathrm{kg\,m^{-3}}$, what is the mass of the core? Hence calculate the percentage of the mass of the Earth which is contained in its core. [3]

All the terrestrial planets are believed to contain dense cores. Name the process by which such dense cores have been formed. [1]

The core of the Earth is partially in liquid state. Which other terrestrial planet is believed to have a partially liquid core and what evidence do we use to infer this? [2]

Solution: *no solution available*

1.42 Write down expressions for:

(a) The theoretical temperature T of a planet of albedo A at a distance D from the sun, the latter having radius $R_\odot$ and temperature $T_\odot$.

(b) The scale height H of the atmosphere of a planet of gravity g and temperature T for particles of mass m.

Hence show that the value of H for planets with the same values of A, m and g should vary with D according to $H \propto D^{-1/2}$.

One such planet at distance D_1 has $A_1 = 0$, while a second has $A_2 = 0.4$. At what distance D_2 must the second planet lie in order to have the same H as the first?

Solution:

(a) $T = T_\odot (1 - A)^{1/4} \left[R_\odot/(2D)\right]^{1/2}$

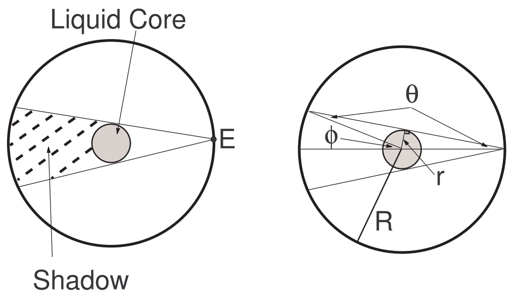
(b) $H = k_B T/(mg)$

Since m, g are the same, $H \propto T$. Since A is the same, $T \propto D^{-1/2}$. Hence $H \propto D^{-1/2}$.

$$\begin{aligned}
T &= T_\odot (1-A)^{1/4} \left(\frac{R_\odot}{2D}\right)^{1/2} \\
A_1 = 0 \Rightarrow T_1 &= T_\odot \left(\frac{R_\odot}{2D_1}\right)^{1/2} \\
A_2 = 0.4 \Rightarrow T_2 &= (0.6)^{1/4} T_\odot \left(\frac{R_\odot}{2D_2}\right)^{1/2} \\
T_1 = T_2 \Rightarrow D_1^{-1/2} &= (0.6)^{1/4} D_2^{-1/2} \\
\therefore D_2 &= \sqrt{0.6} D_1
\end{aligned}$$

1.43 Explain how the earth is known to have a liquid core. If this core has radius 0.5 of the earth's radius, find the semi-angle ϕ of the zone over which no seismic S-waves are detected.

Solution: Earthquake at E is detected by P (pressure) and S (shear) seismic waves. S-waves cannot pass through liquid, so presence of S-wave shadow cone implies liquid core.



$\phi = 180^0 - 2 \times (90^0 - \theta) = 2\theta$. But $r/R = \sin\theta = \sin\phi/2$. If $r/R = 0.5$ then $\phi = 60^0$.

1.44 (a) Show that for planets of the same density, the surface gravity is proportional to radius, i.e. $g \propto R$.

(b) If gas giants have a density one quarter of the rocky terrestrials, show that a gas giant of 4 times the earth's radius would have the same g.

Solution:

(a) $g = GM/R^2 = \frac{4}{3}\pi R^3 \rho G/R^2 = \frac{4}{3}\pi G\rho R$. Hence $g \propto \rho R$. If ρ is identical for the planets in question, then $g \propto R$.

(b) Let the earth be denoted by subscript $\oplus$, and the gas giant by subscript J. Then

$$\frac{g_\oplus}{g_J} = \frac{M_\oplus}{M_J}\frac{R_J^2}{R_\oplus^2} = \frac{\rho_\oplus}{\rho_J}\frac{R_\oplus}{R_J}$$

Hence if $\rho_j = \rho_\oplus/4$ and $R_J = 4R_\oplus$, then $g_J = g_\oplus$.

1.45 (a) Show that a planet of mass M comprised of constituent particles of mass m, formed by gravitational shrinkage to a radius R from a structure of much greater initial radius, would have a formation temperature T_0 given approximately by

$$T_0 \approx \frac{GMm}{k_B R}$$

if cooling processes are neglected. [6]

(b) Show also that such a young planet would then cool by radiation on a timescale τ given approximately by

$$\tau \approx \frac{k_B M}{4\pi\sigma R^2 m T_0^3} = \frac{k^4 R}{4\pi G^3 \sigma M^2 m^4}$$

if heating by the sun is relatively small in comparison. [8]

(c) Using $M = 4\pi R^3\rho/3$, express T_0 and τ in the form

$$T_0 \approx \frac{4\pi G\rho R^2 m}{3k_B} \qquad \tau \approx \frac{9k_B^4}{64\sigma\pi^3 G^3 R^5 \rho^2 m^4}$$

[2,2]

(d) If all the terrestrial planets are taken to have the same m and ρ this would imply $T_0 \propto R^2$ and $\tau \propto R^{-5}$. Use the data on R in the handout to rank the inner planets and the moon in increasing order of: (a) T_0; and (b) τ in this approximation. [1,1]

Solution:

(a) Gravitational energy lost per particle goes into heat, so

$$\left[-\frac{GMm}{r}\right]_{r=R}^{r=\infty}=\frac{GMm}{R}$$

The thermal energy per particle can be approximated as $k_B T_0$, hence $k_B T_0 = GMm/R$. So if there is no cooling, then

$$T_0 \approx \frac{GMm}{k_B R}$$

(b) Let E be the total thermal energy, so that $E \approx k_B T_0 \times$ number of particles. Hence $E \approx M/m \times k_B T_0$. The rate of loss of heat by radiation is $L = 4\pi R^2 \sigma T_0^4$. Ignoring solar heating, a crude approximation for the characteristic time for E to decline is

$$\begin{aligned}
\tau &\approx \frac{E}{L} \\
&\approx \frac{M}{m} k_B T_0 \times \frac{1}{4\pi R^2 \sigma T_0^4} \\
&= \frac{M k_B}{m 4\pi \sigma R^2 T_0^3} \\
&\approx \frac{M k_B}{m 4\pi \sigma R^2} \times \left(\frac{GMm}{k_B R}\right)^3 \\
&= \frac{k_B^4 R}{4\pi \sigma G^3 M^2 m^4}
\end{aligned}$$

(c) Since $M = 4\pi R^3 \rho/3$, then

$$T_0 = \frac{G 4\pi R^3 \rho m/3}{k_B R} = \tfrac{4}{3}\pi G \frac{m\rho R^2}{k_B}$$

and also

$$\tau = \frac{k_B^4 R}{4\pi \sigma G^3 m^4} \times \frac{9}{16\pi^2 R^6 \rho^2} = \frac{9}{64\pi^3}\frac{k_B^4}{\sigma G^3 m^4 \rho^2 R^5}$$

(d) Ranked in order of $T_0 (\propto R^2)$: moon, mercury, mars, venus, earth

Ranked in order of $\tau (\propto R^{-5})$: reverse of above.

1.46 A meteoroid has mass m and speed v on entering the atmosphere. If it stops in a distance L and all its kinetic energy goes into light show that its luminosity is about

$$L = \frac{mv^3}{2L}$$

and calculate L for $m = 0.01$ kg, $v = 30\text{kms}^{-1}$ taking $L = 10$ km.

Solution: Energy lost in stopping is $E = mv^2/2$. Time taken to stop is $t = L/v$ so mean rate of energy loss = luminosity is $L = E/t = mv^3/2L$

Numerically we have $m = 0.01$ kg, $v = 3 \times 10^4$ m s^{-1}, $L = 10^4$ m so $L = 0.01 \times (3 \times 10^4)^3/10^4 = 2.7 \times 10^6 = 2.7$ MegaWatt.

1.47 If the earth had no atmosphere and a very dark surface ($A = 0$) its theoretical temperature would be 280 K.

To what value would A have to increase to result in this temperature falling to the freezing point 273 K of water?

If such an albedo change persisted and resulted in the whole surface of the earth becoming covered in snow ($A = 0.7$) to what temperature would the earth's surface fall?

Why is the earth warmer than these calculations suggest?

Solution: Let the earth's temperature $T = T_0$ when $A = 0$. Then $T = T_0(1-A)^{1/4} \propto (1-A)^{1/4}$ To reduce T from 280 K to 273 K we need

$$(1-A)^{1/4} = \frac{T}{T_0} = \frac{273}{280} = 0.975$$

So $A = 1 - 0.975^4 = 1 - 0.9 = 0.1$

If the resulting snow raised A to 0.7 we would get $T = T_0(1-0.7)^{1/4} = 280 \times 0.74 = 207\,\text{K}$.

Actual earth temperature is higher because of the atmospheric blanketing ('greenhouse') effect.

1.48e By considering the forces acting on a small cylinder of gas in a gravitational field, derive the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\rho g,$$

where P is the gas pressure, r is the distance above the planetary surface, ρ is the gas density and g is the local gravitational acceleration. [8]

What is meant by the *"pressure scale height"* of a planetary atmosphere? [2]

Sketch a graph showing how the pressure of a planetary atmosphere varies with height above the planet's surface. [2]

The pressure scale height of an atmosphere at temperature T is given by

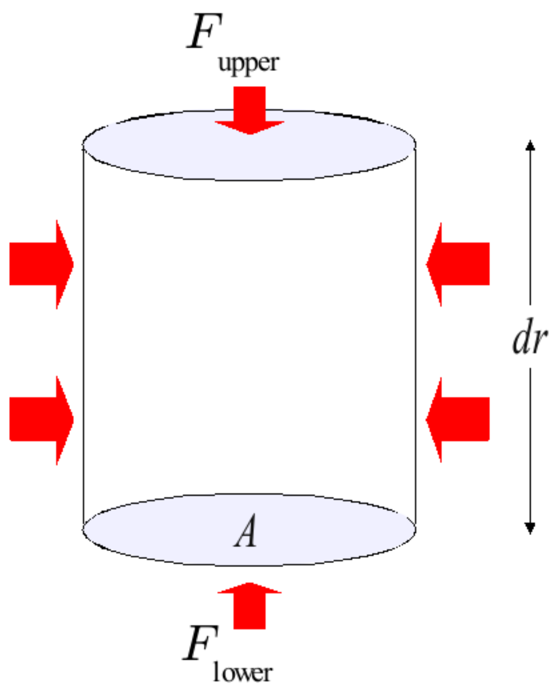
$$H_P = \frac{kT}{\overline{m}g},$$

where $\overline{m}$ is the average mass per particle in the atmosphere and g is the gravitational acceleration at the surface. Given the data below, and any information you may need from the table of astrophysical constants, calculate the ratio of the pressure scale heights of the Martian and Terrestrial atmospheres. [5]

Data:

Planet	*Mass*	*Radius*	*Atmospheric Temperature*	*Average mass per particle*
Earth	5.97×10^{24} kg	6.38×10^3 km	20^oC	$28.9m_p$
Mars	6.42×10^{23} kg	3.39×10^3 km	-50^oC	$44.0m_p$

Solution: We assume that the density of gas in the atmosphere is spherically symmetric. The bottom of the cylinder is a distance r from the centre of the planet. Let the area of the cylinder be a and its height be dr. Suppose the cylinder contains a mass m of gas.



In the following we take upwards as positive. The horizontal forces on the walls of the cylinder cancel out. On the upper face, there is a downward force, due to pressure exerted by gas above the cylinder:

$$F_{\text{upper}} = -AP(r + dr) \ .$$

There will also be a downward force due to the weight of atmosphere above the cylinder, but we don't consider that here since it will also apply to the lower face. On the lower face, there is a upward force due to pressure of the gas below the cylinder, and downward force, due to the weight of the gas in the cylinder:

$$F_{\text{lower}} = AP(r) - \frac{GM(r)m}{r^2} \ ,$$

where $M(r)$ is the mass contained within radius r from the planet's centre. In writing the above equation we take into account the fact that $dr << r$. Using

$$g = \frac{GM(r)}{r^2}$$

and

$$m = \rho A dr \ ,$$

we have:

$$F_{\text{lower}} = AP(r) - A\rho g dr \ .$$

Hydrostatic equilibrium implies that there is no net force on the cylinder, so $F_{\text{upper}} + F_{\text{lower}} = 0$. Therefore,

$$AP(r + dr) - AP(r) + A\rho g dr = 0 \ ,$$

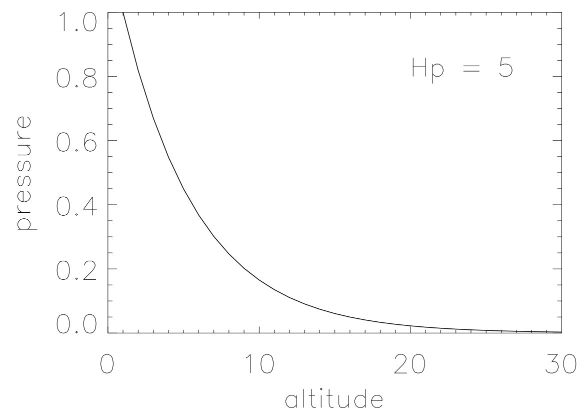
and thus,

$$\frac{P(r + dr) - P(r)}{dr} = -\rho g \ .$$

By taking the limit $dr \to 0$, we obtain the required equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\rho g \ .$$

The pressure scale height is the characteristic length over which the pressure changes significantly (by a factor e).



Pressure scale height ratio of Martian (M) and Terrestrial (E) atmospheres:

$$\begin{aligned} \frac{H_P^{\mathrm{M}}}{H_P^{\mathrm{E}}} &= \frac{T_{\mathrm{M}}\bar{m}_{\mathrm{E}}g_{\mathrm{E}}}{T_{\mathrm{E}}\bar{m}_{\mathrm{M}}g_{\mathrm{M}}} \\ &= \frac{T_{\mathrm{M}}\bar{m}_{\mathrm{E}}}{T_{\mathrm{E}}\bar{m}_{\mathrm{M}}}\frac{M_{\mathrm{E}}R_{\mathrm{M}}^2}{M_{\mathrm{M}}R_{\mathrm{E}}^2} \end{aligned}$$

Using the data above and those in the notes, we have (not forgetting to convert the temperatures to degrees Kelvin):

$$\frac{H_P^{\mathrm{M}}}{H_P^{\mathrm{E}}} = \frac{223 \times 28.9 \times 0.533^2}{293 \times 44 \times 0.107} = 1.33$$

The pressure scale height on Mars is slightly larger than on Earth, which means that the pressure there decreases slightly more slowly with altitude than on Earth. (A more accurate calculation gives a ratio even larger than the one we have just obtained, but still below 2).

1.49e Describe the four basic geological processes which affect the surfaces of the terrestrial planets. In each case detail how they affect the surfaces of each planet and Earth's moon. [10]

What consequences do these processes have on the age of the Earth's surface compared to those of the other terrestrial planets? [1]

Describe what is meant by the term *"Greenhouse effect"*. [2]

Explain, with the aid of a suitable diagram, why Venus has a runaway greenhouse effect. [4]

Solution: *no solution available*

1.50e Sketch the internal structure of the planet Jupiter, showing the different layers in its interior and indicating what each layer is made of. [3]

Jovian planets are rapid rotators (for their size) and are composed mostly of fluid (gases, liquids). Describe two consequences of these properties for the planet Jupiter. [2]

Solution: *no solution available*

1.51e Following earthquakes, waves of two forms propagate through the interior of the Earth: s-waves and p-waves. Describe the main features of these types of waves. How does the behaviour of s-waves tell us that the Earth's core is liquid? How can the presence of a liquid core be inferred from the magnetic field of the Earth? [5]

Solution: *no solution available*

1.52e What are the oldest know objects in the solar system and by what two methods is their age normally estimated? [2]

What factors influence, or have influenced the Solar Rotation rate? [2]

What is the relevance of this to theories of the origin of the planets? [1]

Solution: Meteor(ite)s and related debris. Ages normally estimated by radioactive dating and cosmic ray exposure. [2]

Loss of angular momentum by transfer to centrifugal disk via magnetic field - mainly to Jupiter orbital angular momentum. [2]

Centrifuging removed mass from protosun into a disk which condensed as planets etc. [1]

1.53e Why are the atmospheres of terrestrial planets devoid of free hydrogen and helium while the giant planets are composed mainly of these? [3]

How and why is the composition of the Earth's atmosphere completely different from that of the other terrestrial planets? [2]

Which terrestrial planet, besides the Earth, Has liquid iron in its core? How do we know this? [2]

The core of the Earth comprises 5% of the Earth's total mass and has mean density 1.6 times higher than the overlying layers.
Find:

(a) the core radius as a function of the Earth's radius; and [5]

(b) the angle subtended at the Earth's centre by the seismic S-wave shadow. [5]

Solution:

$$v_{rms} \propto \left(\frac{kT}{m}\right)^{1/2} \propto \frac{T^{1/2}}{m^{1/2}}$$

$$v_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \propto g^{1/2} R^{1/2}$$

Escape of atmosphere $\uparrow$ as $v_{rms}/v_{esc} \uparrow$
terrestrial have high T, low g, low R
giants have low T, large g, large R
H, He have low m so high v_{rms} [3]

In contrast with Venus and Mars, mainly CO_2, the Earth has little CO_2 and a lot of O_2 though O_2 highly reactive. This is because of (plant) life converting $CO_2 \rightarrow O_2$. [2]

Mercury. It has a magnetic field which needs a liquid metal core to carry current. [2]

a) Fraction of mass $f = \frac{\rho_c \frac{4}{3}\pi r_c^3}{\frac{4}{3}\pi R_c^3 + \rho_M \frac{4}{3}\pi (R^3 - R_c^3)}$

So $f = \frac{1}{1+\frac{\rho_M}{\rho_c}\left(\frac{R^3}{R_c}-1\right)}$

$f = 0.05$; $\frac{\rho_M}{\rho_c} = \frac{1}{1.6}$; $\frac{R_c}{R} = n$

So $1 + \frac{1}{1.6}\left(\frac{1}{n^3} - 1\right) = 20$

$\Rightarrow 19 \times 1.6 = \frac{1}{n^3} - 1 \Rightarrow n^3 = \frac{1}{1+1.6\times 19}$

$\Rightarrow n = \underline{0.31}$ [5]

b) $\theta = 2\phi = \tan^{-1}\frac{R_c}{R} = \tan^{-1} n = 17.6$

$\Rightarrow$ angle $= 2\theta = \underline{35^o.2}$ [5]

1.54e Name and describe the four basic geological processes that shape the surface of the terrestrial planets. [4]

Why is the lunar surface more heavily cratered than the surface of the Earth? [1]

Solution: The four geological processes are

- Impact cratering: impact craters formed by asteroids or comets striking planets surface.
- Volcanism: eruption of molten rock (lava) from below planets crust onto surface.
- Tectonics: disruption of planets surface by internal stresses.
- Erosion: wearing down/building up of surface features by action of wind/water/ice-'weather'. [4]

Ongoing volcanism/tectonics/erosion on Earth result in continual creation/destruction of crust - significantly altered since periods of 'heavy bombardment'. In contrast most of lunar surface unchanged since period of heavy bombardment - also no erosion (no atmosphere). [1]

1.55e Derive the expression for the half-life of a radioactive decay process. [3]

Some unusual meteorites thought to originate from Mars contain small amounts of radioactive thorium-232 and its daughter decay product lead-208. The half-life for this decay process is 14 billion years. Analysis of one such meteorite shows that 94% of the original thorium remains. How old is this meteorite? [2]

Solution: The number of atoms decaying dN in time interval dt is proportional to the number of original atoms N

$dN/dt = -\lambda N \quad \Rightarrow N = N_0 e^{-\lambda t}$ where $N_0 =$ original number of radioactive atoms.

When half the original N_0 atoms have decayed, $N = N_0/2$

$\Rightarrow e^{-\lambda t} = 1/2$

so half-life $\quad t_{1/2} = -\log_e(1/2)/\lambda = 0.693/\lambda$ [3]

The half-life for this decay process is 14×10^9 years so

$$\begin{aligned} \lambda &= 0.693/14 \times 10^9 \times 365 \times 24 \times 60 \times 60 \\ &= 1.57 \times 10^{-18} \end{aligned}$$

94% of the original thorium remains, so

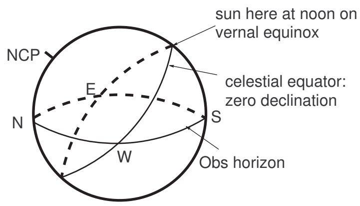
$$0.94 = \frac{N}{n_0} = e^{-\lambda t}$$

$$\log_e(0.94) = -1.57 \times 10^{-18} t$$

so $t = -0.06/-1.57 \times 10^{-18} = 3.94 \times 10^{16}$ seconds
meteorite is 1.2×10^9 years old. [2]

2 Positional Astronomy

2.1e Draw a celestial sphere for north latitude 45°, indicating clearly the zenith point, the north celestial pole and the horizon. Mark in the cardinal points of the horizon and draw in the celestial equator. The date is March 21; mark in the Sun's position at noon. [5]



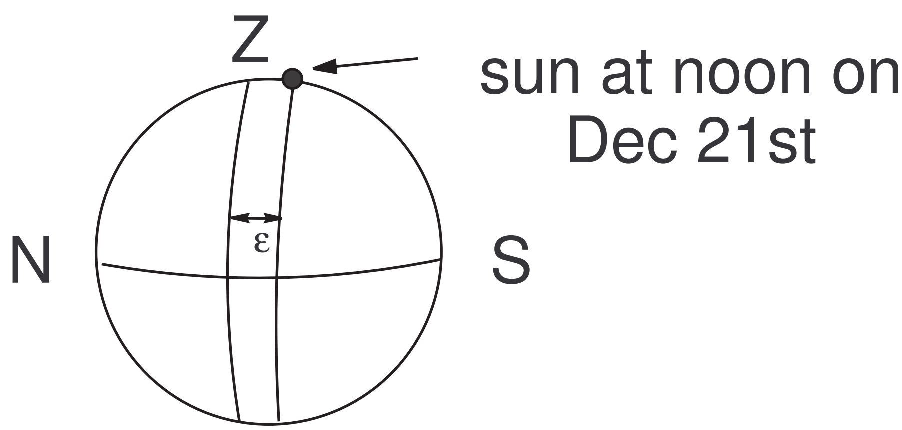
Solution:

2.2e Draw a celestial sphere to indicate clearly the relationship between the equatorial coordinate system of right ascension and declination and the coordinate system of ecliptic latitude and longitude. Indicate the Sun's position on 21 June. What is its declination at this time? [5]

Solution: *no solution available*

2.3e Draw a celestial sphere for an observer on the Earth's equator. Mark in the north and south celestial poles and the celestial equator. Mark in the Sun's approximate position at noon on 21st December. [5]

Solution:



2.4e Indicate clearly on a diagram the position of an inferior planet with respect to the Earth and the Sun for the following configurations:

(a) Inferior conjunction

(b) Superior conjunction

(c) Maximum elongation east

(d) Maximum elongation west. [5]

Solution: *no solution available*

2.5e Define a great circle on the surface of a sphere. Explain why this type of curve is particularly important. What is the length of the great circle arc joining the two points on the surface of the Earth with geographical latitude and longitude (60° N, 90° E) and (60° N, 90° W)? [5]

Solution: A great circle is the intersection of a plane through the centre of the sphere with the surface of the sphere. Its radius is the same as the radius of the sphere itself.
This type of curve is important since the shortest distance between two points on the sphere is the great circle arc joining them. This great circle arc is unique unless the two points are diametrically opposite, in which case any great circle through one point also passes through the other.
The meridian of longitude 90°E is the great circle joining the first point to the north pole. When produced beyond the pole it continues as the meridian of longitude 90°W, also a great circle arc. Hence the great circle joining the two points passes through the north pole. Since each point is 30° from the north pole, the great circle arc joining them is clearly of length 60°.

2.6 Two places A and B on the same parallel of latitude $38^\circ 33'$ N are $123^\circ 19'$ apart in longitude. Calculate, in nautical miles,

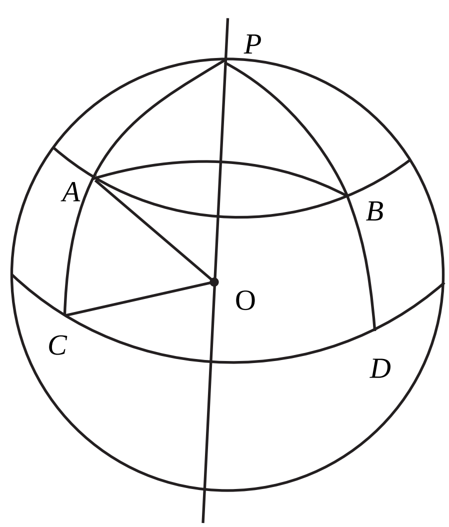
(a) their distance apart along the parallel

(b) the great circle distance AB.

{5786.5 nmi; 5219.7 nmi}

Solution: Distance along the parallel of latitude AB

$$= \text{difference in longitude} \times \cos\text{(latitude)}$$



Distance along the great circle arc *AB* is obtained from the spherical triangle *PAB*, by applying the cosine formula, *viz*

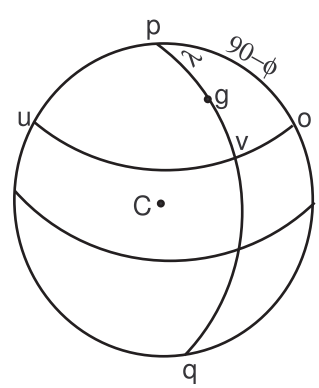
$$\begin{aligned}
\cos AB &= \cos PA \cos PB + \sin PB \cos APB \\
&= \cos(90^\circ - 38^\circ 33') \cos(90^\circ - 38^\circ 33') \\
&\quad + \sin(90^\circ - 38^\circ 33') \sin(90^\circ - 38^\circ 33') \cos(123^\circ 19') \\
&= \sin^2(38^\circ.55) + \cos^2(38^\circ.55)\cos(123^\circ.3167) \\
&= 0.05243 \\
\therefore \quad AB &= 86^\circ.9946 \\
&= 5219'.67 \\
\therefore \quad AB &= 5219.7 \quad \text{nautical miles.}
\end{aligned}$$

2.7e Define carefully the coordinate system of terrestrial latitude and longitude and compare it with the celestial coordinate system of right ascension and declination. [6]

Derive a formula to give the shortest distance between two points on the same parallel of latitude ϕ which differ in longitude by an amount $\Delta\lambda$. Explain carefully how this distance may be expressed in nautical miles. [5]

A star is observed at the zenith at Glasgow University Observatory ($55^\circ 54'$ N, $4^\circ 18'$ W). At the same instant of time a second star is at the zenith at the University of Helsinki Observatory ($60^\circ 09'$ N, $24^\circ 57'$ E). Calculate the right ascension and the declination of each star if the Greenwich sidereal time at this instant is $16^{\mathrm{h}}41^{\mathrm{m}}$. [6]

Solution: Treat the earth as a sphere. Rotation axis intersects surface in geographical north and south poles P and Q. The great circle with poles p or q is the equator. Meridians of (constant) longitude are semi-great circles terminating in p and q. The one through Greenwich (g) is arbitrarily chosen as the prime meridian of zero longitude. Parallels of (constant) latitude are small circles with poles p and q, and are therefore parallel to the equator.

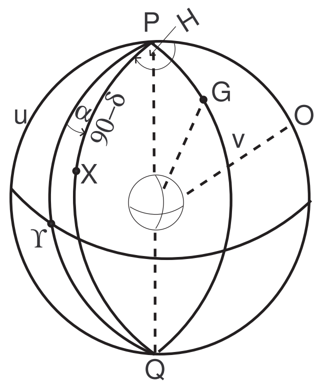


Through any observer o there is a unique meridian of longitude poq and parallel of latitude uvo, and the latitude ϕ and longitude λ of this observer are defined as

$$\begin{aligned}
\phi &= 90^0 - po \\
\lambda &= Gpo
\end{aligned}$$

λ is the longitude east, and is measured anticlockwise. North latitude is positive, south negative. Longitude west may be regarded as negative.

Let C be the earth's centre. Consider a geocentric celestial sphere with the earth inside it. Let Cp, Cq, Co and Cg be produced, intersecting the celestial sphere in P, Q, O and G. Then P and Q are the north and south celestial poles, O is the zenith for the observer at o, and G is the zenith for the observer at g.



Parallels of latitude project into parallels of declination, and the meridians of longitude into meridians of right ascension.

An arbitrary meridian of RA must be chosen as the zero for RA. This is chosen to be PΥQ, where Υ is the sun's position at the (northern) vernal equinox. The RA and dec of a star X are then defined as

$$\begin{aligned}\delta &= 90^0 - PX\\ \alpha &= \Upsilon PX\end{aligned}$$

α is measured eastwards, like longitude. The hour angle of X is H, where H=OPA, measured westwards.

For any object we have α + H = HAΥ=LST (Local Sidereal Time), and LST = GST + λ, where GST is Greenwich Sidereal Time.

Let the two points be X and Y, and let P be the geographical north pole. Then PX=PY=$90^0 - \phi$, and XPY=$\Delta\lambda$. Apply the cosine formula to get

$$\cos XY = \sin^2\phi + \cos^2\phi\cos\Delta\lambda$$

To convert XY into nautical miles, express it in degrees and multiply by 60. Note that the shortest path does not follow the parallel of latitude.

The declination of the zenith is ϕ, the observer's latitude. Let 1st star have $(\alpha_1,\ \delta_1)$, and the second $(\alpha_2,\ \delta_2)$. Let Glasgow be $(\phi_G,\ \lambda_G)$ and Helsinki, $(\phi_H,\ \lambda_H)$. Then $\delta_1 = \phi_G = 55^0 54'$, and $\delta_2 = \phi_H = 60^0 09'$. Now GST = $16^h 41^m$ and LST = GST + λ.

$\lambda_1 = 4^0 18'$W$= -0^h 17^m 12^s$E. Hence at Glasgow, LST = $16^h 23^m 48^s$. Since zenith is on the observer's meridian, H = 0. Therefore $\alpha_1 = \text{LST} = 16^h 23^m 48^s$.

For the second star, $\lambda_2 = 24^o 57'\text{E} = 1^h 39^m 48^s$, $\therefore$ LST $= 16^h 41^m + 1^h 39^m 48^s = 18^h 20^m 48^s$. Again, the star is at the zenith, so H = 0, and $\alpha_2 = \text{LST} = 18^h 20^m 48^s$.

2.8e Define carefully the terms conjunction, opposition and quadrature as they are applied to a superior planet. Define further the sidereal period and the synodic period for such a planet and explain how they are related. [5]

The planet Mars moves round the Sun in an orbit of semi-major axis 1.52 astronomical units. Calculate its sidereal period of revolution in years. [2]

Making the approximation that the orbits of Mars and the Earth are circular and coplanar, calculate the synodic period of Mars in years and the interval of time in days between opposition and quadrature for this planet. [7]

Mars is observed at quadrature on March 21. Calculate its right ascension and declination at this time. [3]

Solution: *no solution available*

2.9e Define carefully what is meant by a *parallel of latitude* and a *meridian of longitude*, illustrating your definition with a diagram. Explain the conventions that are used in assigning a latitude and a longitude to a position on the surface of the Earth. Give a formal definition of the *nautical mile*. [6]

State, without proof, the *cosine formula* of spherical trigonometry. [2]

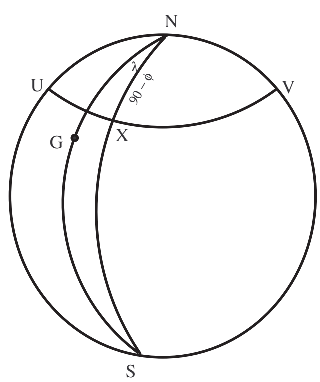
Prove that the shortest distance between the two points P_1 and P_2 on the Earth's surface with geographical coordinates (ϕ_1, λ_1) and (ϕ_2, λ_2), latitude and longitude respectively, is given by

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2).$$

[4]

Find the length in nautical miles of the shortest air route from Dunedin (45°51′ S, 170°30′ E) to Los Angeles (33°51′ N, 118°21′ W). [5]

Solution: A parallel of latitude is a small circle parallel to the equator and therefore having the north and south geographical polse as its poles. A meridian of longitude is a semi-great circle terminating at teh geographical poles, eg NXS.

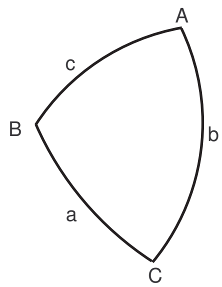


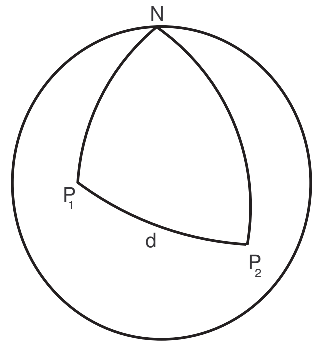
The prime meridian of longitude is arbitrarily chosen as the meridian of longitude through Greenwich, G. The point X has latitude and longitude ϕ, λ defined by

$$\phi = 90^\circ - NX$$
$$\lambda = GNX$$

Longitude (east) is measured in an anti-clockwise direction as seen from above N. Longitude west can be treated as negative. North latitude is positive; south negative.
The nautical mile is the distance on the earth surface that subtends one arc minute at the earth's centre.
In spherical triangle ABC, the cosine formula is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$





The shortest distance from P_1 to P_2 is the great circle arc joining them, which is part of the spherical triangle NP_1P_2. In this triangle, $NP_1 = 90^\circ - \phi_1$, $NP_2 = 90^\circ - \phi_2$ and P_1NP_2=GNP_2-$GNP_1 = \lambda_2 - \lambda_1$. So the cosine formula gives

$$\begin{aligned}\cos d &= \cos(90^\circ - \phi_1)\cos(90^\circ - \phi_2) + \sin(90^\circ - \phi_1)\sin(90^\circ - \phi_2)\cos(\lambda_2 - \lambda_1)\\ &= \sin\phi_1 \sin\phi_2 + \cos\phi_1\cos\phi_2\cos(\lambda_2 - \lambda_1)\end{aligned}$$

Taking P_1 to be Los Angeles, and P_2 to be Dunedin, we have

$$\begin{aligned}\phi_1 &= 33^\circ.85\\ \phi_2 &= -45^\circ.85\\ \lambda_1 &= -118^\circ.35 = 241^\circ.65\\ \lambda_2 &= 170^\circ.5\\ \lambda_1 - \lambda_2 &= 71^\circ.15\end{aligned}$$

Applying the formula:

$$\begin{aligned}\cos d &= -0.21277\\ \Rightarrow d &= 102^\circ.285\end{aligned}$$

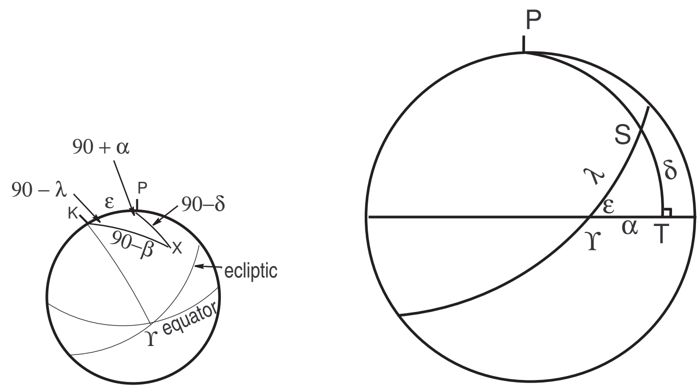
Multiply by 60 to convert to nautical miles: $d = 6137.1$ nautical miles.

2.10e Define carefully the coordinate system of *ecliptic longitude* and *ecliptic latitude*. Indicate how a star's ecliptic coordinates (λ, β) are related to its right ascension and declination (α, δ) by identifying the appropriate spherical triangle on the celestial sphere. In particular prove that

$$\cos\lambda\cos\beta = \cos\alpha\cos\delta. \qquad [8]$$

Estimate the date when the Sun's ecliptic longitude is 45° and calculate the Sun's right ascension and declination for that date. [12]

Solution: Consider the geocentric celestial sphere as shown:



P is North Celestial Pole, K is pole of the ecliptic, Υ is the vernal equinox, ε is the obliquity of the ecliptic, λ is ecliptic longitude, β ecliptic latitude, α right-ascension, and δ declination of star X. Then $KX = 90^0 - \beta$, $\Upsilon KX = \lambda$, $PX = 90^0 - \delta$, $\Upsilon PX = \alpha$. Hence $PKX = 90^0 - \lambda$ and $KPX = 90^0 + \alpha$.

Applying the sine formula to ΔKPX we have

$$\frac{\sin(90^0 - \delta)}{\sin(90^0 - \lambda)} = \frac{\sin(90^0 - \beta)}{\sin(90^0 + \alpha)}$$

that is,

$$\cos\lambda\cos\beta = \cos\alpha\cos\delta$$

Sun is on the ecliptic, and therefore $\beta = 0$; treating earth's orbit as circular, it moves round the ecliptic uniformly. From the diagram, S is the sun; join PS and produce to cut the equator at T.

Sun is at Υ on March 21, hence it is at S a time t later where $t = 365.25 \times 45/360 = 45.65$ days. Hence the date is 46 days later than March 21, namely May 6th.

From the first part of the question,

$$\cos\lambda_\odot = \cos\alpha_\odot\cos\delta_\odot$$

Applying the sine rule to the triangle SΥT yields

$$\sin\lambda_\odot = \sin\delta_\odot/\sin\varepsilon$$

Inserting the values $\varepsilon = 23.5^0$, $\quad \lambda_\odot = 45^0$ yields $\delta_\odot = 16^0.377 = 16^0 22'.6$.

Using this value of $\delta_\odot$ we get the right ascension from

$$\cos\alpha_\odot = \cos\lambda_\odot/\cos\delta_\odot = 0.737$$

yielding $\alpha_\odot = 42^0.523 = 2^h.8348 = 2^h 50^m 05^s$

2.11 An aircraft leaves Lima (12°10′ S, 77°05′ W) and flies directly to Rome (41°53′ N, 12°33′ E). Draw a diagram marking the position of Lima and Rome, and the great circle joining them.

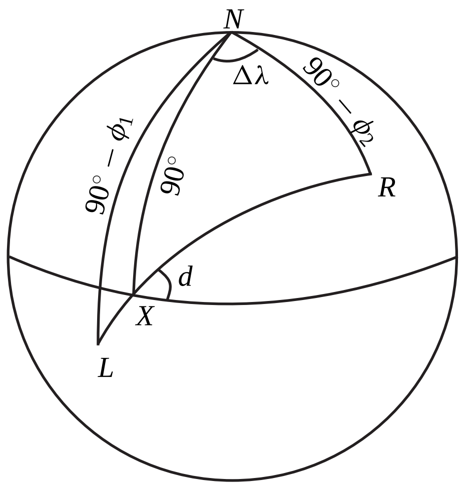
(a) Calculate the distance travelled in nautical miles.

(b) Calculate the aircraft's bearing as it approaches Rome.

(c) Determine the longitude of the point on the flight where the aircraft crosses the Equator.

{5869 nmi, $99^\circ.357\,69$, $63^\circ 35'$}

Solution:



Let (ϕ_1, λ_1) denote the geographical coordinates (latitude and longitude) of Lima, and (ϕ_2, λ_2) those of Rome. Then the great circle distance d is given by the formula

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_2 - \lambda_1),$$

The values that we have for the coordinates are

$$\phi_1 = -12^\circ 10' = -12^\circ.166\,67 \quad \phi_2 = 41^\circ 53' = 41^\circ.883\,33,$$

$$\lambda_1 = -77^\circ 05' \qquad \lambda_2 = 12^\circ 33'$$
$$\text{So,} \quad \lambda_2 - \lambda_1 = 89^\circ 38' = 89^\circ.633\,33.$$

Inserting these values in the equation gives

$$\cos d = -0.136\,05,$$

$$d = 97^\circ.819\,15 = 5869.15\,\text{nautical miles}.$$

On the diagram T represents Lima, R Rome and N is the geographical North Pole. In spherical triangle NLR we were effectively given NL, NR and LNR and we have worked out LR.
Let X be the point where the flight path crosses the equator. Then $NX = 90^\circ$. Let us call angle XNR $\Delta\lambda$. If we can work out this angle the problem will be solved. However, in spherical triangle NRX only two parts are known, namely NR and NX. A third one is required if we are to determine $\Delta\lambda$. The obvious part is to use is angle NRX, since it is also a part of the original triangle NRL. Let's call it θ. It is the bearing required in (b).
From the sine formula in spherical triangle NRL we have

$$\sin\theta \equiv \sin NRL = \sin NL \sin LNR / \sin LR = \cos\phi_1 \sin(\lambda_2 - \lambda_1)/\sin d$$

Inserting the values we find that $\sin\theta = 0.986\,69$. The sine formula is always ambiguous and this gives two possible values for θ, namely $80^\circ.642\,31$ or $99^\circ.357\,69$.
In this case this is not a problem for the plane obviously approaches Rome from the south and so $\theta = 99^\circ.357\,69$. The ambiguity could have been avoided by using the cosine formula instead, *viz*

$$\sin\phi_1 = \sin\phi_2\cos d + \ cos\phi_2 \sin d\cos\theta,$$

thus could provide a useful check.
Reverting to spherical triangle *NRX*, we use the four parts formula

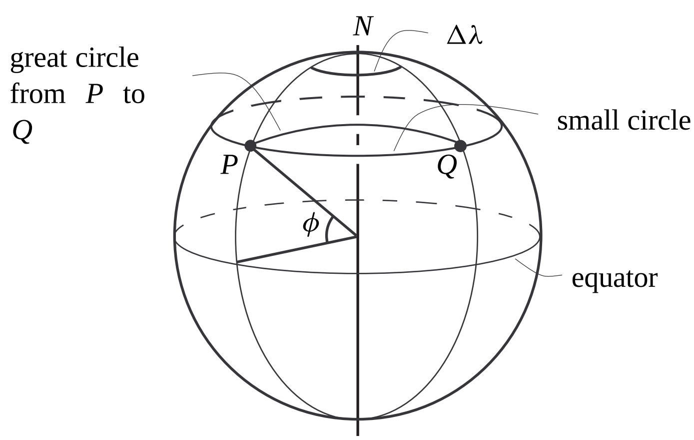
$$\cos\Delta\lambda\sin\phi_2 = \cos\phi_2\cot 90^\circ - \sin\Delta\lambda\cot\theta,$$

i.e. $\tan\Delta\lambda = -\sin\phi_2\tan\theta$,
This gives $\Delta\lambda = \tan^{-1}(4.051\,30) = 76^\circ.134\,61 = 76^\circ 08'$,
So the required longitude is $(12^\circ 33' - 76^\circ 08')$ E $= 63^\circ 35'$ W.

2.12 Two seaports are on the same parallel of latitude $42^\circ 27'$ N. Their difference in longitude is $137^\circ 36'$. Ship *A* sails along the parallel of latitude from one port to the other, while ship *B* follows the most direct great circle route. Ship *B* sails at a constant speed of 20 knots. How long will it take to complete the voyage? $\{10^{\mathrm{d}}20^{\mathrm{h}}48^{\mathrm{m}}\}$ [12]

What average speed will ship *A* have to maintain to complete its voyage in the same time as ship *B*? {23.36 knots} [8]

Solution:



Let ϕ be the latitude of the two seaports and let $\Delta\lambda$ be their difference in longitude. Then

$$\phi = 42^\circ 27' = 42.45^\circ$$
$$\Delta\lambda = 137^\circ 36' = 137.6^\circ.$$

We calculate the length of the great circle arc PQ. This is best considered as a part of the spherical triangle NPQ, where N is the geographical North Pole. In this triangle we can identify

$$NP = NQ = 90^\circ - \phi,$$
$$PNQ = \Delta\lambda.$$

So using the cosine formula yields $\cos PQ = \sin^2\phi + \cos^2\phi \cos\Delta\lambda$.
Inserting the values gives $PQ = 86.9330^\circ = 5215.98$ nautical miles.
Since ship B covers this distance at 20 knots, it is easily worked out that the duration of its voyage is 260.8 hours, i.e., 10 days 20 hours 48 minutes.

Ship A is following the longer parallel of latitude ϕ, which is the small circle arc PQ with pole N. We find the length of this arc as

$$\text{small circle arc } PQ = \Delta\lambda \sin NP = \Delta\lambda \cos\phi.$$

This length is readily calculated as $101.5304^\circ = 6091.8$ nautical miles. Since this distance is to be covered in 260.8 hours, the required average speed is found to be 23.36 knots.

2.13 Calculate the length in degrees of the great circle arc joining San Francisco ($37^\circ 40'$ N, $122^\circ 25'$ W) and Tokyo ($35^\circ 48'$ N, $139^\circ 45'$ E). {76.95°} [9]

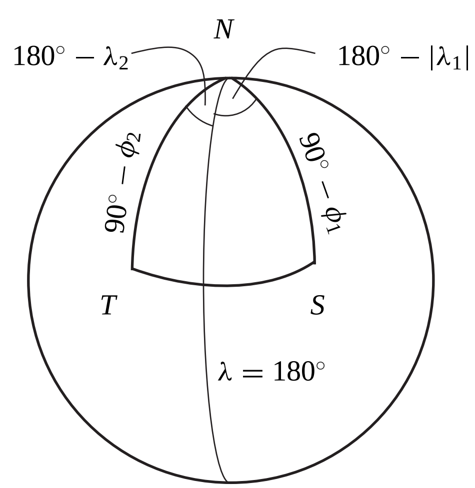
What is the shortest distance between these two cities in nautical miles? {4617.03 nmi} [2]

An aircraft leaves San Francisco at 10 pm local time on August 21 and flies directly to Tokyo maintaining an average speed of 480 knots. Calculate

(a) the duration of the flight {$9^{\mathrm{h}}18^{\mathrm{m}}$} [3]

(b) the local time and date of arrival in Tokyo. {00:18} [6]

Solution:



On the diagram let the points S and T denote San Francisco and Tokyo respectively and let N denote the geographical North Pole. Let S and T have geographical coordinates (ϕ_1, λ_1) and (ϕ_2, λ_2) respectively. So from the data we have

$$\phi_1 = 37^\circ 40' = 37.6667^\circ, \qquad \lambda_1 = -122^\circ 25' = -122.4167^\circ$$
$$\phi_2 = 35^\circ 48' = 35.8^\circ, \qquad \lambda_2 = 139^\circ 45' = 139.75^\circ$$

using the convention that north latitudes and east longitudes are treated as positive.

We consider the spherical triangle NST in which we can identify the following parts: $NS = 90^\circ - \phi_1$, $NT = 90^\circ - \phi_2$, and, since the plane is flying in the direction of *decreasing* longitude (east) $SNT = \lambda_1 - \lambda_2$. This gives the surprising answer $SNT = -262.1667^\circ$: but that is the same thing as 97.8333°, since we are always at liberty to add or subtract 360°. Applying the cosine formula to spherical triangle NST gives the result

$$\cos ST = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2).$$

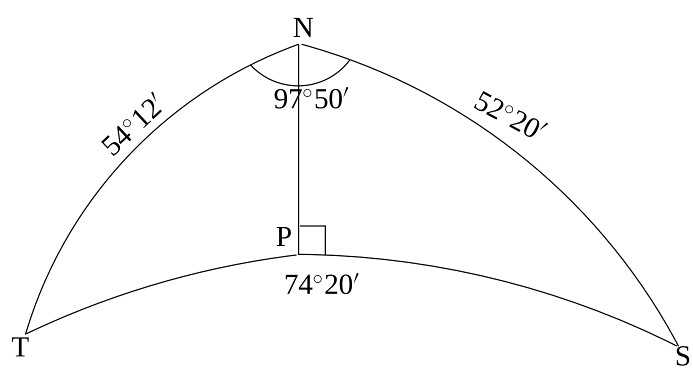
Inserting values yields $\cos ST = 0.269946$, i.e., $ST = 74.3390^\circ$. Since one degree is 60 nautical miles, the distance from San Francisco to Tokyo is found to be 4460.34 nautical miles.

The flight time in hours is then found to be 4460.34/480, which is 9.2924 hours, or 9 hours 18 minutes.

Since the plane leaves at 10 pm on 21st August, it will arrive at 7.37 am on 22nd August. This, however, will be in San Francisco time. Now Tokyo is 9 hours ahead of Greenwich due to the longitude difference, and San Francisco is similarly 8 hours behind Greenwich. Consequently Tokyo is 17 hours ahead of San Francisco. The local time of arrival in Tokyo is, therefore, 7.18 am + 17 hours, which will be 12.18 am on 23rd August. The additional day arises because the International Date Line has been crossed during the flight.

2.14 Using the result of the previous question, determine the direction in which an aircraft should depart from San Francisco to follow the most direct route to Tokyo. Calculate further the latitude and longitude of the most northerly point that will be reached on this direct route. {$56^\circ 34'$ W of N; $48^\circ 39'$ N; $169^\circ 38'$ W}

Solution: We will use the same notation as in the previous question. The known data are as indicated in the diagram.



Let us use the analogue formula in spherical triangle NST to work out the spherical angle NST, yielding,

$$\begin{aligned} \sin 74^\circ 20' \cos NST &= \cos 54^\circ 12' \sin 52^\circ 20' \\ &\quad - \sin 54^\circ 12' \cos 52^\circ 20' \cos 97^\circ 50' \\ \text{i.e.} \qquad \cos NST &= 0.55105 \\ NST &= 56^\circ 34' \end{aligned}$$

So the plane departs from San Francisco on a bearing of $56^\circ 34'$ west of north.

Let P be the northerly point of the aircraft's route. At this point it is flying due west and so *NPS* is a right angle. We now consider th espherical triangle *NPS* in order to determine the latitude and longitude of P. Note that *NP* is the colatitude of P and that *PNS* is the longitude difference of P from San Francisco. By the sine formula,

$$\sin PN = \sin NS \sin PSN$$

since *NPS* is a right angle. It is then found that $NP = 41^\circ 21'$ and so the latitude of P is $48^\circ 39'$N.

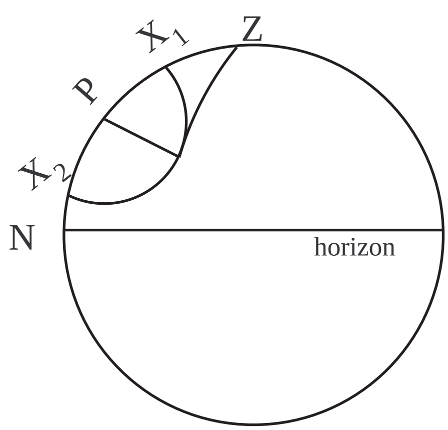
Finally let us use the four-parts formula to derive the angle *PNS*. We have

$$\begin{aligned} \cos PNS &= \tan NP \cot NS \\ &= 0.67919 \\ PNS &= 47^\circ 13' \end{aligned}$$

Adding this to the longitude of San Francisco gives the longitude of P as $169^\circ 38'$W.

2.15 The altitudes of a star at upper and lower transits (both north of the zenith) are $65^\circ 23'$ and $14^\circ 01'$. Find the latitude of the observer and the star's declination and calculate the star's altitude, azimuth and hour angle when it is at its maximum azimuth west. $\{\delta = 64^\circ 19'; \phi = 39^\circ 42'; a = 45^\circ 08'; A = 37^\circ 17'\}$

Solution: Let Z be the zenith, and P the pole in the diagram.



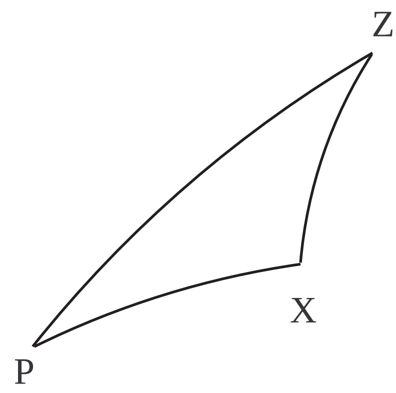
The diurnal path is the small circle with pole P. Let X_1, X_2 be the star at upper and lower transit. Then $PX_1 = PX_2 = 90^\circ - \delta$, $NP = \phi$ where N is the north point of the horizon.

$$\begin{aligned} 65/deg23' &= NX_1 = NP + PX_1 = \phi + 90^\circ - \delta \\ 14^\circ 01' &= NX_2 = NP - PX_2 = \phi - 90^\circ + delta \end{aligned}$$

that is,

$$\phi + \delta = 104^\circ 01', \ -\phi + \delta = 24^\circ 37'$$

Hence $\phi = 39^\circ 42'$, $\delta = 64^\circ 19'$. Let X be star's position at maximum azimuth. Then PXZ=90°. Also, PZ=$90^\circ - \phi$, PX=$90^\circ - \phi$, ZX=90°−a, ZPX=H, PZX = A (W of N). Solve for a, A, H from the triangle PZX.
From the cosine rule,



$$\sin\phi = \sin\delta\sin a + \cos\delta\cos a\cos 90^\circ$$

So

$$\begin{aligned} \sin a &= \sin\phi\csc\delta \\ &= 0.70879 \\ a &= 45^\circ 08' \end{aligned}$$

From the sine rule,

$$\frac{\sin A}{\cos\delta} = \frac{\sin 90^\circ}{\cos\phi}$$

yielding

$$\sin A = \cos\delta\sec\phi = -0.56329$$

so that $A = 34^\circ 17'$.
The four-parts formula can be used to get H:

$$\sin\delta\cos H = \cos\delta\tan\phi - \sin H\cot 90^\circ$$

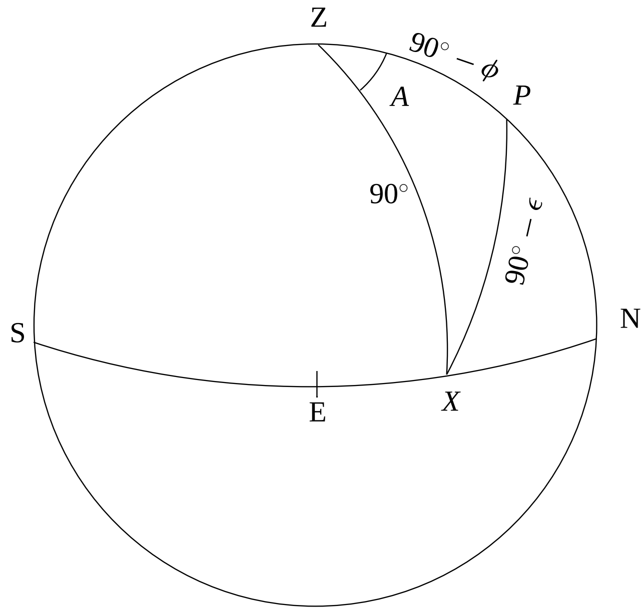
implying $\cos H = \tan\phi\cot\delta = 0.39926 \Rightarrow H = 66^\circ 47' = 4^h 25^m.9$
Alternatively the sine or cosine rule can be re-applied to get the same result:

$$\frac{\sin H}{\cos a} = \frac{\sin 90^\circ}{\cos\phi}$$

$$\sin a = \sin\delta\sin\phi + \cos\delta\cos\phi\cos H$$

2.16 Calculate the azimuth of the Sun at rising on Midsummer's day at Stonehenge (latitude $50^\circ 10'$ N) at a time when the obliquity of the ecliptic was $23^\circ 48'$.

Solution: On midsummer's day the Sun reaches its greatest northerly declination, namely $\delta = +\epsilon$, where ϵ is the obliquity of the ecliptic. The required spherical triangle will be ZPX (zenith - pole - Sun), in which $PS = 90^\circ - \epsilon$.



Since the Sun is rising $ZX = 90^\circ$, and $ZP = 90^\circ - \phi$, where ϕ is the observer's latitude.

The angle PZX is the azimuth east of north or minus the azimuth west of north.

By the cosine formula

$$\begin{aligned}
\cos(90^\circ - \epsilon) &= \cos 90^\circ \cos(90^\circ - \phi) + \sin(90^\circ - \phi)\cos A \\
\text{i.e} \qquad \sin\epsilon &= \cos\phi\cos A. \\
\text{so} \qquad \cos \mathrm{A} &= \frac{\sin 23^\circ 48'}{\cos 51^\circ 10'} = 0.6436 \\
A &= 49^\circ 57' \mathrm{E\ of\ N} \\
\text{or} \qquad \mathrm{A} &= 310^\circ 03' \mathrm{W\ of\ N}
\end{aligned}$$

2.17 Describe the celestial sphere and the diurnal motions of the stars as they would appear for an observer at

(a) the Earth's North Pole

(b) a point on the Earth's equator.

Solution:

(a) The north celestial pole is at zenith for an observer at the earth's north pole. The stars will describe small circles about the zenith due to the diurnal rotation of the earth. Each star therefore, has a fixed altitude equal to its declination and its azimuth continually changes with reference to a fixed point on the horizon. Strictly speaking this point is arbitrary as the north point of the horizon is undefined at the pole itself. All stars of positive declination are circumpolar, and all those of negative declination are permanently below the horizon, that is the celestial equator coincides with the horizon.

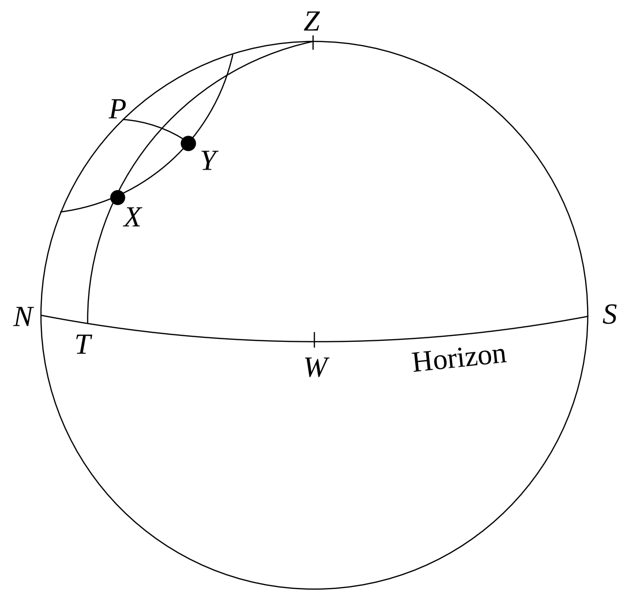
(b) For an observer on the earth's equator, the two celestial poles are at the north and south points of the horizon. The celestial equator coincides with the prime verticals, that is the great circle EZW. All stars rise and set, no star is circumpolar or permanently invisible. A star of declination ϵ rises at an azimuth $90^\circ - \epsilon$ east of north and sets 12 hours later at the same azimuth west of north. Its greatest altitude at transit is again $90^\circ - \epsilon$.

2.18 Prove that the maximum azimuth (east or west of north) of a circumpolar star is

$$A = \sin^{-1}(\cos\delta \sec\phi).$$

The right-hand side of this equation is undefined when $\delta < \phi$. How do you account for this?

Solution: The diurnal path of a circumpolar star X is a small circle about the north celestial pole P as shown in the diagram.



Join X to the zenith point Z by the great circle arc ZX and produce it to cut the horizon in T. The azimuth of X is spherical angle PZX or equivalently arc NT. In general the arc ZT cuts the small circle in two points rather than one – there are two points of the diurnal path with the same azimuth. Maximum azimuth occurs at Y where ZT is tangential to the small circle. It is easily seen that PYZ is a right angle.

In spherical triangle PZY, we have

$$\begin{aligned} PZ &= 90^\circ - \phi \\ PY &= 90^\circ - \delta \\ PZY &= A_{\max} \\ ZYP &= 90^\circ. \end{aligned}$$

By the sine formula

$$\frac{\sin A}{\cos\delta} = \frac{\sin 90^\circ}{\cos\phi}$$

Hence result.

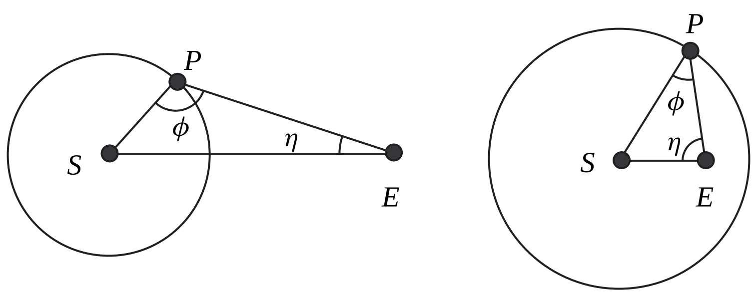
When $\delta < \phi$ the circumpolar star transits south of zenith and all values of the azimuth are possible. There is then no maximum.

2.19 Define the *phase angle*, ϕ, and the *elongation*, η, of a planet illustrating your definition with diagrams for an inferior and a superior planet. [5]

Why is the phase angle so called? [3]

Assuming that the Moon is at a distance of 3.84×10^5 km from the Earth, calculate its elongation to the nearest arcminute when it is observed to be exactly half illuminated. [12]

Solution: For simplicity the approximation is made that the planets, including the earth, are moving in circular coplanar orbits. Then any planet would be observed to lie in the ecliptic.



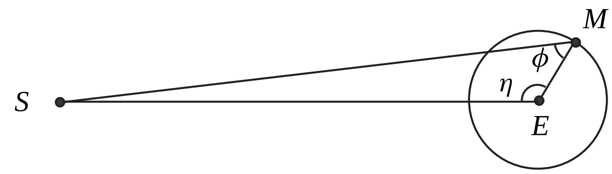
In the two diagrams above S represents the Sun, E the Earth and P the planet under consideration. If we measure distances in astronomical units then $SE = 1$. The diagrams show the possible relative configurations for an inferior planet and a superior planet. If we denote the length of SP by a in each diagram, then $a < 1$ in the first case and $a > 1$ in the second. The elongation η is the angular separation of the planet from the Sun as seen from the Earth, i.e., angle SEP. The phase angle ϕ is the angular separation of the Earth from the Sun as seen by the planet, i.e., angle SPE. These definitions apply in either diagram.

For an inferior planet the elongation is restricted to $\eta \leq \sin^{-1} a$, but the phase angle can take any value. For a superior planet there is no restriction on the elongation but $\phi \leq \sin^{-1} 1/a$.

The phase angle takes its name from the fact that it can be used to calculate the observed phase of the planet. The phase q is defined as the fraction of the planet's disc that is illuminated and is given by

$$q = \frac{1}{2}(1 + \cos\phi).$$

The same considerations can be applied to the Moon instead of a planet, although, since the Moon is in orbit round the earth, the geometry is different. In the diagram below M now represents the Moon, S and E the Sun and the Earth as before.



We are given that $EM = 3.84 \times 10^5$ km, and SE is one astronomical unit. When the Moon is exactly half illuminated the phase angle SME is 90°. The Moon's elongation, η, is the angle SEM, and from the right-angled triangle we deduce that $\cos\eta = \frac{EM}{SE}$.

Putting in values it is found that $\cos\eta = 2.5668 \times 10^{-3}$, giving an elongation of $89^\circ 51'$.

2.20 An asteroid is moving in an orbit of semi-major axis 2.87 astronomical units. Calculate its sidereal period of revolution. {4.86 yr} [3]

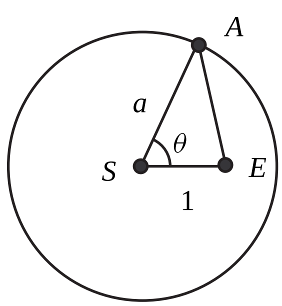
Assuming that the asteroid is moving in a circular orbit in the ecliptic plane, calculate the synodic period in years. {1.26 yr} [3]

What is the maximum phase angle and the minimum phase for this asteroid? {20.39°; 0.969} [5]

Work out the interval of time in days between opposition and the occurrence of this minimum phase. {0.24 yr} [9]

Solution: Let us call the semi-major axis of the asteroid's orbit a (= 2.87 AU). By Kepler's third law the sidereal period, T, is easily calculated in years from the formula $T^2 = a^3$. It is found that $T = 4.86$ years.

The synodic period, S, is, for a superior planet, related to the sidereal period by the formula $S^{-1} = 1 - T^{-1}$, which yields $S = 1.26$ years.



Consider the triangle SEA (Sun Earth asteroid). The maximum phase angle, ϕ, or SAE, occurs at quadrature when angle $SEA = 90^\circ$. Hence $\sin\phi = a^{-1}$, giving $\phi = 20.3914^\circ$. The minimum phase $q = (1 + \cos\phi)/2 = 0.9687$.

Now the angle that increases uniformly with time is not the phase angle or the elongation but the third angle ASE, labelled θ in the diagram. This increases by 360° in one synodic period. Clearly $\theta = 90^\circ - 20.3914^\circ = 69.6086^\circ$. The required interval of time is $\theta S/360 = 0.2434$ years (= 88.91 days).

2.21 Draw a celestial sphere for an observer at latitude 56° N indicating the *horizon* and the *celestial equator*. Mark in the north, south and west points of the horizon and also the *north celestial pole* and the *zenith point*. [6]

A star has declination 18°. Indicate its approximate position on the diagram when it is setting and calculate its azimuth at that instant. {303°33′} [7]

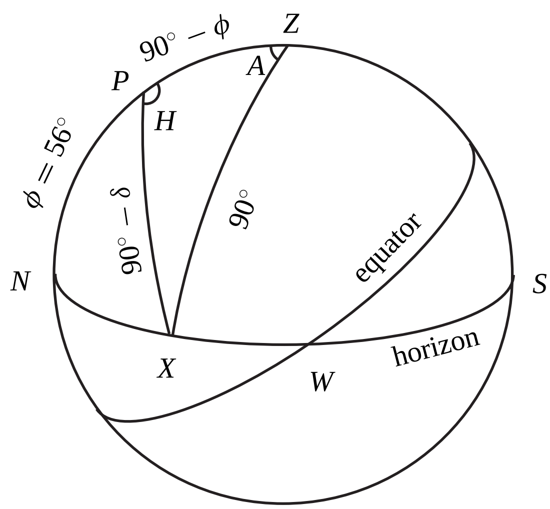
What is the hour angle of the star when it sets? $\{7^{\mathrm{h}}55^{\mathrm{m}}\}$ [5]

How long is the star above the horizon in

(a) sidereal time? $\{15^{\mathrm{h}}50^{\mathrm{m}}\}$ [1]

(b) solar time? $\{15^{\mathrm{h}}48^{\mathrm{m}}\}$ [1]

Solution:



The zenith point is Z, the north celestial pole is P and the north, south and west points of the horizon are respectively N, S and W. Notice that NP is the latitude ϕ (= 56°), which means that $PZ = 90^\circ - \phi$.

The star is at X, which is on the horizon since it is setting. The great circle arc $PX = 90^\circ - \delta$, where δ is its declination of 18°. The star's azimuth is the spherical angle PZX indicated by A in the diagram. In the spherical triangle PZX we may identify the following parts:

$$\begin{aligned} ZX &= 90^\circ, \\ PX &= 90^\circ - \delta, \\ PZ &= 90^\circ - \phi, \\ PZX &= A. \end{aligned}$$

Applying the cosine formula gives

$$\cos PX = \cos PZ \cos ZX + \sin PZ \sin ZX \cos PZX$$

i.e., $\sin\delta = \cos\phi \cos A$,
yielding $A = \cos^{-1}(\sin\delta \sec\phi)$.
Inserting values for the declination and the latitude gives $A = 303^\circ 33'$. The hour angle of the star at setting is spherical angle ZPX, labelled H in the diagram. A further application of the cosine formula yields

$$\cos 90^\circ = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H.$$

Hence $H = \cos^{-1}(-\tan\delta\tan\phi)$, and inserting values gives $H = 118.80^\circ = 7.92\,\mathrm{h} = 7^{\mathrm{h}}55^{\mathrm{m}}$.

The sidereal time that the star is above the horizon is simply $2H$ or 15 hours 50 minutes. To obtain the solar time interval we multiply this by the factor $365.25/366.25 = 0.99727$ getting 15 hours 48 minutes.

2.22 (a) What is the ecliptic latitude and longitude of the Sun on May 1st?

(b) Calculate the Sun's right ascension and declination for this date, assuming that the obliquity of the ecliptic $\epsilon = 23^\circ.5$.

(c) Calculate the hour angle of the Sun at sunset for this date for an observer at Glasgow, latitude 56°N, longitude 4°15′W.

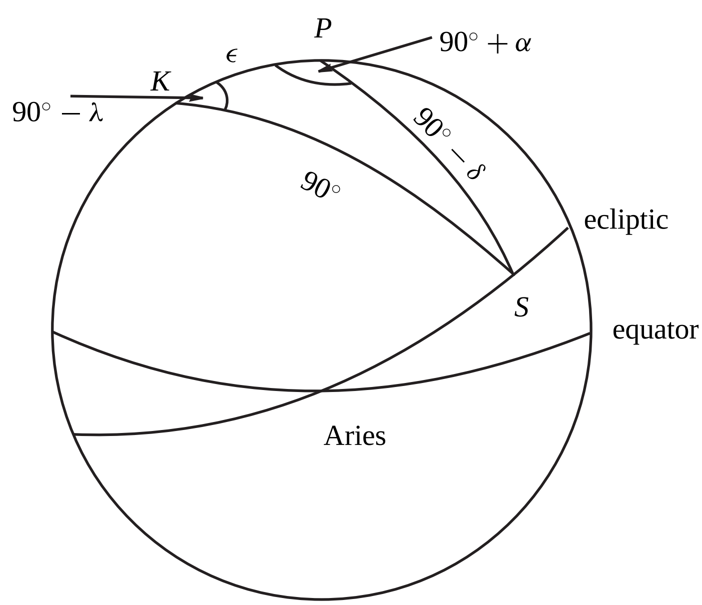
(d) What is the interval (in solar time) between sunrise and sunset for this observer on this date? Determine the Greenwich mean times of sunset and sunrise.

(e) What is the local sidereal time and what is the Greenwich sidereal time of sunset for the Glasgow observer on May 1st?

Solution: This question attempts to draw together a number of different ideas from the lecture notes. As such it is long and involved and considerably more difficult than any question that would be set under exam conditions!

(a) The Sun always lies on the ecliptic and so its ecliptic latitude is always zero. The ecliptic longitude (λ) is zero on March 21st when the Sun is at Υ. It increases uniformly by 360° in one year. May 1st is 41 days after March 21st and so we may calculate $\lambda = 41 \times 360^\circ/365.25 = 40^\circ.4$.

(b) Consider the diagram as shown, in which S is the Sun (on May 1st), P is the north celestial pole, K is the North Pole of the ecliptic and Aries is the equinox (Υ).



Let the Sun's right ascension and declination be (α, δ). Then $PS = 90^\circ - \lambda$ and spherical angle $KPS = 90^\circ + \alpha$. Moreover, $PKS = 90^\circ - \lambda$. Finally $KS = 90^\circ$, since the Sun is on the ecliptic. The spherical triangle we must consider is clearly KPS. By the cosine formula,

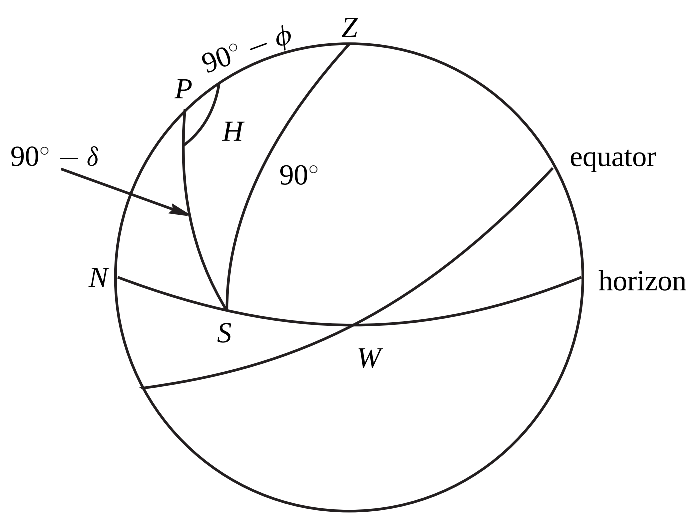
$$\begin{aligned}
\cos(90^\circ - \delta) &= \cos(\epsilon)\cos(90^\circ) \\
&\quad + \sin(\epsilon)\sin(90^\circ)\cos(90^\circ - \lambda), \\
\text{i.e.}\quad \sin(\delta) &= \sin(\epsilon)\sin(\lambda) \\
&= \sin(23^\circ.5)\sin(40^\circ.4) = 0.2584 \\
\text{Hence}\quad \delta &= 15^\circ.0.
\end{aligned}$$

Again by the four parts formula,

$$\begin{aligned}
\cos\epsilon\cos(90^\circ - \lambda) &= \sin\epsilon\cot 90^\circ - \sin(90^\circ - \lambda)\cot(90^\circ + \alpha), \\
\text{i.e.}\quad \cos\epsilon\sin\lambda &= -\cos\lambda(-\tan\alpha), \\
\text{giving}\quad \tan\alpha &= \cos\epsilon\tan\lambda \\
&= \cos 23^\circ.5\tan 40^\circ.4 = 0.7805
\end{aligned}$$

$$\text{Hence}\quad \alpha = 38^\circ.0 = 2^{\mathrm{h}}32^{\mathrm{m}}.$$

(c) To calculate the hour angle of the Sun we need a new diagram. Let S be the position of the Sun at sunset, P the north celestial pole and Z the zenith point for the Glasgow observer. Then $ZS = 90^\circ$. Further $PZ = 90^\circ - \phi$, where ϕ is the latitude of the observer, namely 56°. PS is still $90^\circ - \delta$ and the spherical angle ZPS is the required hour angle, H say.



Clearly we must consider the spherical triangle PZS. By the cosine formula

$$\begin{aligned}
\cos 90^\circ &= \cos(90^\circ - \phi)\cos(90^\circ - \delta) + \sin(90^\circ - \phi)\sin(90^\circ - \delta)\cos H \\
\text{giving}\quad \cos H &= -\tan\phi\tan\delta \\
&= -\tan 56^\circ\tan 15^\circ = -0.3973.
\end{aligned}$$

Hence the hour angle is found to be $113^\circ.4$ or $7^{\mathrm{h}}34^{\mathrm{m}}$.

(d) The interval between sunrise and sunset is clearly $2H$ or $15^{\mathrm{h}}08^{\mathrm{m}}$. Since the observer's longitude is $4^\circ 15'$ W, which is 17^{m} in time measure, both sunrise and sunset will occur 17 minutes later than for a point at the same latitude on the Greenwich meridian. So sunrise is $7^{\mathrm{h}}17^{\mathrm{m}}$ before noon (GMT) and sunset is $7^{\mathrm{h}}51^{\mathrm{m}}$ after noon. The times are, therefore, 4.43 am and 7.51 pm.

(e) For any celestial object X we have the relationship LST = HA(X) + RA(X). This can be applied to the sun at sunset giving LST$= 7^{\mathrm{h}}34^{\mathrm{m}} + 2^{\mathrm{h}}32^{\mathrm{m}} = 10^{\mathrm{h}}06^{\mathrm{m}}$.

Finally GST =LST −longitude(East)$= 10^{\mathrm{h}}23^{\mathrm{m}}$.

2.23 The planet Jupiter is moving in an orbit of semimajor axis 5.202 astronomical units. Calculate its *sidereal* period in years.

Calculate the *synodic* period of Jupiter in days making the approximation that both Jupiter and the Earth move round the sun in circular coplanar orbits.

Calculate the interval of time in days during which Jupiter's sidereal motion will be *retrograde*.

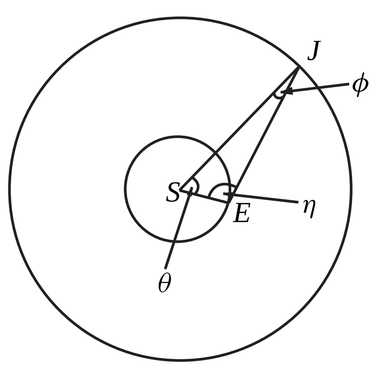
Solution: Let us denote the semimajor axis of Jupiter's orbit by a and its sidereal period by T, where the units are astronomical units and years. Then by Kepler's third law, $a^3 = T^2$.
Hence $T = (5.202)^{3/2} = 11.86\,\text{years}$.
The synodic period S is given, for a superior planet, by the formula

$$\frac{1}{S} = 1 - \frac{1}{T},$$

Substituting $T = 11.86\,\text{years}$, we obtain

$$\frac{1}{S} = 1 - \frac{1}{T} = 1 - \frac{1}{11.86} = 1 - 0.6417 = 0.9157\,\text{yr}^{-1},$$

Thus $S = 1.092\,\text{years or } 398.87\,\text{days}$.
In the diagram S represents the Sun, E the Earth and J the position of Jupiter when it is at a stationary point.



The condition for this was shown in the lecture notes to be that

$$\cos\phi + a^{-1/2}\cos\eta = 0$$

where ϕ is the phase angle and η is the elongation at this time. Using this result and the sine formula in triangle SEJ, it was shown that the value of the elongation at a stationary point is given by the formula

$$\tan\eta = -\frac{a}{(a+1)^{1/2}},$$

Inserting the value of a, we can calculate the elongation as $\eta = 115^\circ.58$. Now the phase angle may be calculated from the condition for the stationary point since

$$\cos\phi = -a^{-\frac{1}{2}}\cos\eta,$$

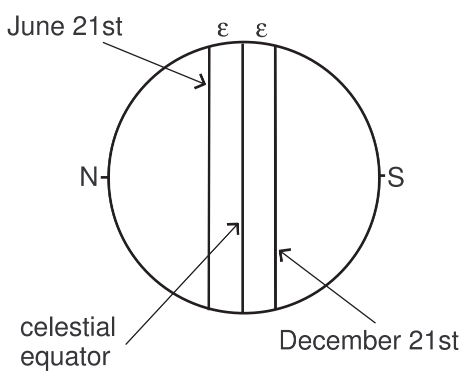
Hence $\phi = 9^\circ.98$.
It follows that the third angle

$$\theta = 180^\circ - \eta - \theta = 54^\circ.44.$$

Now the importance of this angle is that it increases uniformly with time going from 0 to 360° in one synodic period. since Jupiter will retrograding between the two stationary points at $\theta = \pm 54^\circ.44$, the required time interval in days will be

$$54.44 \times S/180 = 120.64\,\text{days}.$$

2.24e Draw a celestial sphere for an observer on the equator. Indicate the positions of the north and south celestial poles and draw in the Sun's diurnal path for June 21 and December 21. [5]



Solution: See diagram

2.25e Show that the sidereal motion of a superior planet must be retrograde at opposition and direct at quadrature. [5]

Solution: *no solution available*

2.26e Draw a geocentric celestial sphere, marking clearly the celestial equator and ecliptic. State the Sun's right ascension, declination, ecliptic longitude and ecliptic latitude for each of the following dates: [1]

(a) March 21 [1]

(b) June 21 [1]

(c) September 21 [1]

(d) December 21. [1]

Solution: It is conventional to quote RA in hours, rather than degrees, so that $360^0 = 24^h$. Let the sun's right ascension, declination, ecliptic longitude and ecliptic latitude be designated $RA_\odot$, $\delta_\odot$, $\lambda_\odot$ and $\beta_\odot$ respectively. Note that the sun always lies on the ecliptic, and so $\beta_\odot = 0^0$.

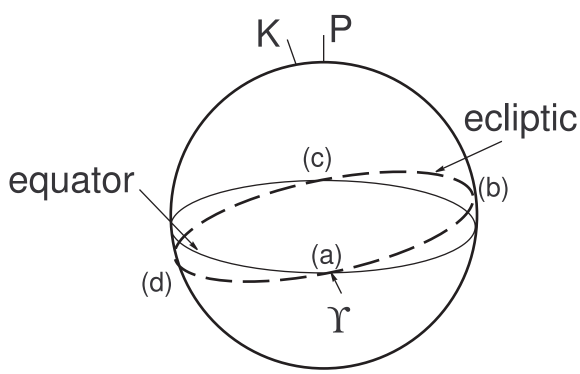
(a) March 21 is the vernal equinox, so $RA_\odot = 0^h$, $\delta_\odot = 0^0$, $\lambda_\odot = 0^0$.

(b) June 21 is midsummer, so $RA_\odot = 6^h$, $\delta_\odot = \epsilon$, $\lambda_\odot = 90^0$.

(c) September 21 is the autumnal equinox: $RA_\odot = 12^h$, $\delta_\odot = 0^0$, $\lambda_\odot = 180^0$.

(d) December 21 is midwinter: $RA_\odot = 18^h$, $\delta_\odot = -\epsilon$, $\lambda_\odot = 270^0$.

The corresponding solar positions are marked on the diagram below.



2.27e Explain carefully how the *latitude* and *longitude* of a point on the Earth's surface are defined. [4]

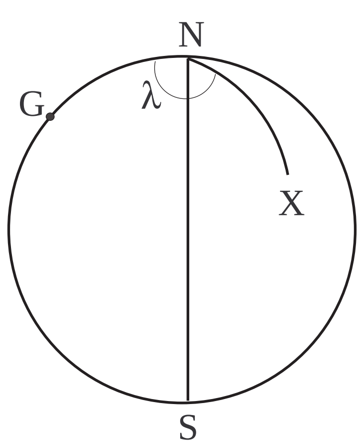
Two points on the Earth's surface have latitude and longitude (ϕ_1, λ_1) and (ϕ_2, λ_2) respectively. Prove that the shortest distance between them is d, where

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2).$$

How would you express this distance in nautical miles? [5]

On March 21 the planet Mercury had right ascension and declination $(0^{\rm h}45^{\rm m}, 5^\circ 04')$. Calculate its elongation from the Sun at that time. [8]

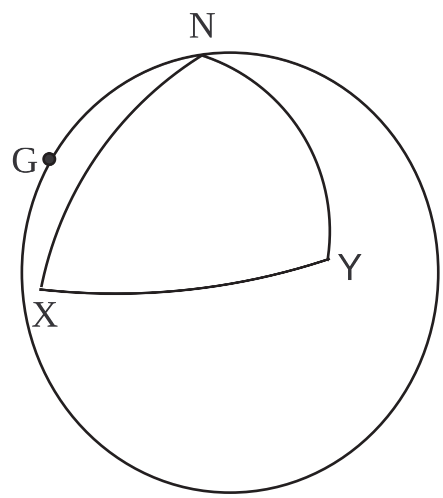
Solution:



Rotation axis is NS. Greenwich meridian NGS is arbitrarily chosen as the meridian of zero longitude. Point X has latitude ϕ and longitude λ such that

$$\phi = 90^\circ - NX$$
$$\lambda = GNX$$

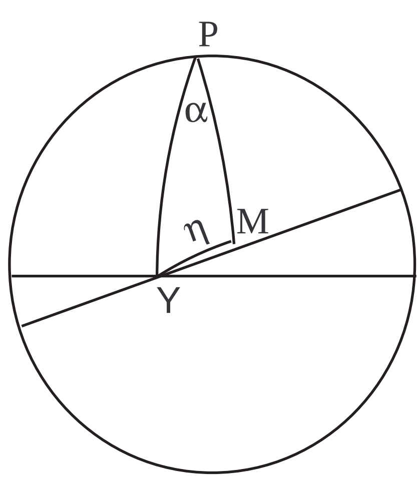
Convention: N latitudes and E longitudes are positive.



Two points have co-ordinates X(ϕ_1, λ_1), Y(ϕ_2, λ_2). Spherical triangle NXY has XY= d, NX= $90^\circ - \phi_1$,NY= $90^\circ - \phi_2$, GNX= λ_1, GNY= λ_2, and so XNY= $\lambda_2 - \lambda_1$.
Apply the cosine formula:

$$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_2 - \lambda_1)$$

Obtain d in degrees, and multiply by 60 to get the distance in nautical miles.



In the diagram, P is the north celestial pole and Υ is the vernal equinox. The sun is at Υ, and Mercury has co-ordinates (α, δ) say. $P\Upsilon = 90°$; $PM = 90° - \delta$; $\Upsilon PM = \alpha = 0^h 45^m = 11°15'$. $\Upsilon M = \eta$, the required elongation. By the cosine formula applied to spherical triangle ΥPM,

$$\begin{aligned}\cos\eta &= \cos 90° \sin\delta + \sin 90° \cos\delta\cos\alpha \\ &= \cos\delta\cos\alpha \\ &= 0.97695 \\ \eta &= 12°.325 = 12°19'30"\end{aligned}$$

2.28e Define precisely what is meant by a *circumpolar* star. Show that the condition for a star to be circumpolar is that its declination δ satisfies the inequality

$$\delta > 90° - \phi$$

where ϕ is the latitude of the observing site. Show further that if $\phi > 45°$ than all stars that transit north of the zenith are circumpolar. [5]

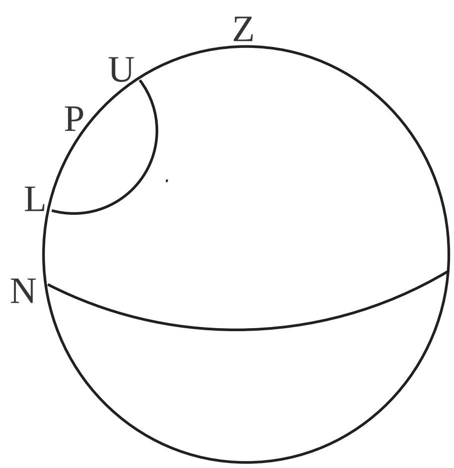
Show that if the Sun becomes circumpolar for part of the year for all points within the arctic circle, i.e. $\phi > (90° - \epsilon)$, where ϵ is the obliquity of the ecliptic. Show further that this period of continuous daylight (midnight Sun) persists while the ecliptic longitude λ of the Sun satisfies the equation

$$\sin\lambda > \cos\phi \operatorname{cosec}\epsilon$$

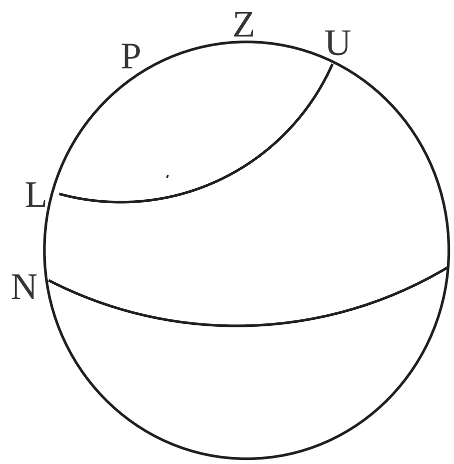
[6]

Calculate the length of the period of continuous daylight for an observer at latitude 72°, on the assumption that the ecliptic longitude of the Sun increases strictly uniformly during the year. [6]

Solution: For a star to be circumpolar, the lower transit must be above the observer's horizon.



Now $PL = 90° - \delta$, where δ is the declination, and $PN = \phi$, the observer's latitude. Hence the star is circumpolar if $90° - \delta < \phi$, i.e. $\phi + \delta > 90°$. Here, $\phi + \delta = 48° + 60° = 108° > 90°$ and so the star is circumpolar.
The altitude at lower transit is NL, but NL+LP=NP=ϕ, and LP=$90° - \delta$. Hence NL=$\phi - (90° - \delta) = 18°$.
Altitude at upper transit U = NU=NP+NU=$\phi + (90° - \delta) = 78°$.



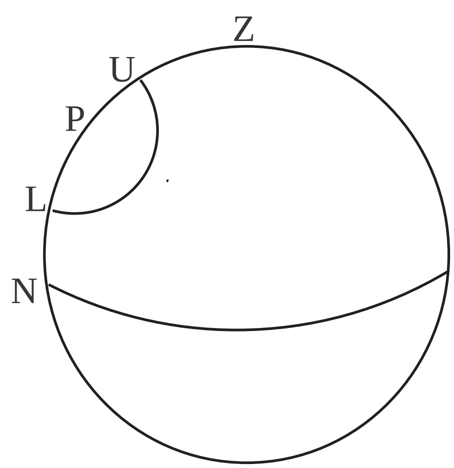
PN=ϕ, PL=90° $-\ \delta$, PZ=90° $-\ \phi$, PU=PL=90° $-\ \delta$
PU=Pz+ZU, that is, ZU=PU-PZ=$\phi - \delta$>0. Given that $\phi + \delta > 90°$, $\phi > 45°$.

2.29 Define what is meant by a circumpolar star using a diagram to illustrate your definition. [4]

The altitude of such a star is $75°23'$ at upper transit and $18°09'$ at lower transit. Both of these transits occur north of the zenith. Find the latitude of the observing site and the declination of the star. [7]

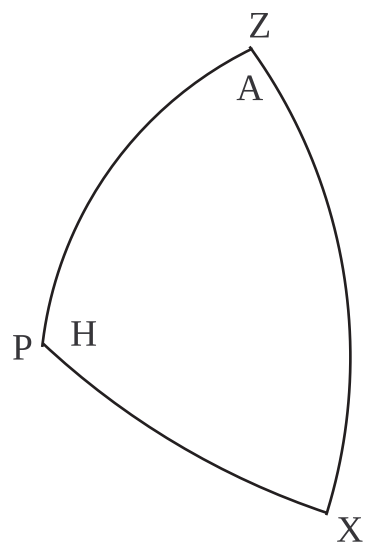
Calculate the maximum azimuth that the star can have and its altitude and hour angle at that time. [9]

Solution: In the diagram, Z is zenith, P the pole, and upper and lower transits are denoted U,L respectively.



A circumpolar star is one which has a diurnal path totally above the horizon, and so never sets.
Altitude of upper transit is NU, so $75°23' = NU = NP + PU = \phi + 90° - \delta$, given that PN=$\phi$, the observer's latitude, and PU=PL=90° $-\ \delta$, where δ is the star's declination. Hence $\delta - \phi = 90° - 75°23' = 14°37'$.
Altitude of lower transit is NL, so that $18°09' = NL = NP - PL = \phi - (90° - \delta)$. Hence $\delta + \phi = 90° + 18°09' = 108°09'$. Using these relations, $\delta = 61°23'$, and $\phi = 46°46'$.

At U and L, the star's azimuth is zero. Clearly, as the star follows its diurnal path it is restricted to a range in azimuth east and west of north. It reaches maximum azimuth when ZX is tangential to the diurnal path, i.e. PXZ = 90°. Consider the spherical triangle PZX when the star is at its maximum azimuth west (which is easier to draw).



PX=$90^\circ - \delta$, PZ=$90^\circ - \phi$, PZX = A (required azimuth west), ZPX = H (required hour angle), PXZ = 90°. By the sine formula,

$$\sin A = \sin 90^\circ \cos\delta / \cos\phi$$

yielding $A = \sin^{-1} 0.6992 = 44^\circ 22'$.
Now use the cosine formula twice:

$$\cos ZX = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H$$
$$\sin\phi = \sin\delta \cos ZX + \cos\delta \sin ZX \cos 90^\circ$$

. Eliminating $\cos ZX$ (and remembering $\cos 90^\circ = 0$) yields eventually

$$\cos H = \tan\phi \cot\delta$$
$$= 0.5803 \Rightarrow H \qquad = 54^\circ.5267 = 3^h 38^m.1$$

The maximum azimuth east is $44^\circ 22'$ at an hour angle of $24^h - H = 20^h 21^m.9$

2.30 The Sun is observed to set at 8.00 pm (British Summer Time) at the location of Glasgow ($55^\circ 52'$N, $4^\circ 15'$W).

(a) What is the Sun's hour angle at this time?

(b) Calculate the Sun's declination.

(c) On the assumption that the obliquity of the ecliptic is $23^\circ 26'$, calculate the Sun's ecliptic longitude at this time.

(d) What are the two possible dates for this observation?

Solution: The time of sunset is 7pm Greenwich Mean Time, or Universal Time to use the astronomical term. The Greenwich hour angle (GHA) of the Sun is therefore, 7^{h} or 105°. The hour angle for the Glasgow observer is given by the formula $\mathrm{HA} = \mathrm{GHA} + \lambda$. In time measure, λ for Glasgow is -17^{m}, since west longitude is regarded as negative. So the hour angle of the Sun at setting is $6^{\mathrm{h}}43^{\mathrm{m}}$, which is $100^\circ.75$.

Consider now the celestial sphere for the Glasgow observer, the left hand diagram. Here Z is zenith, P the north celestial pole and S the Sun at sunset. In triangle *PZS* we have the following parts;
$ZS = 90^\circ$, $PZ = 90^\circ - \phi$, $PS = 90^\circ - \delta$, $ZPS = H = 100^\circ.75$, where δ is the declination of the Sun. Apply the cosine formula to get

$$\cos 90^\circ = \sin\delta \sin\phi + \cos\delta \cos\phi \cos H,$$

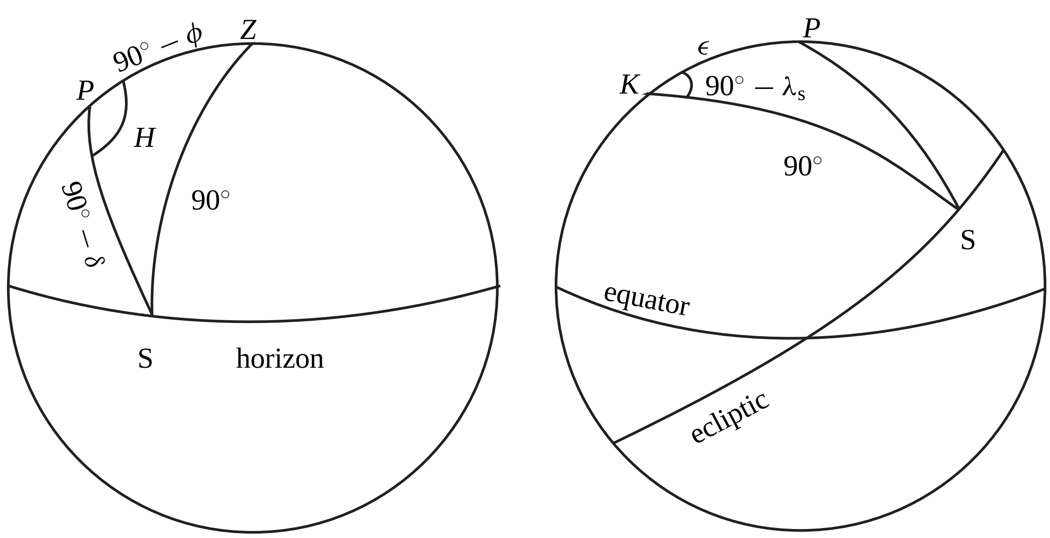
giving $\tan\delta = -\cos H \cot\phi$.
Inserting values, namely $\phi = 55^\circ.867$, $H = 100^\circ.75$, will yield $\delta = 7^\circ.206 = 7^\circ 12'.4$.

Consider now the geocentric celestial sphere, the right hand diagram. Here P and S are again the north celestial pole and the position of the Sun, and K is the north pole of the ecliptic. We now consider spherical triangle KPS in which we can identify the following parts;
$KS = 90^\circ$, $PS = 90^\circ - \delta$, $KP = \epsilon$, and $PKS = 90^\circ - \lambda_s$, λ_s being the ecliptic longitude of the Sun. Apply the cosine formula to get

$$\sin\delta = \cos 90^\circ \cos\epsilon + \sin 90^\circ \sin\epsilon \sin\lambda_s.$$

Hence, $\sin\lambda_s = \sin\delta \csc\epsilon = 0.31544$.
So we find that the ecliptic longitude of the Sun is $18^\circ.388$ or $161^\circ.612$.

We make the assumption that the ecliptic longitude of the Sun increases uniformly during the year and is zero on March 21. Th is equivalent to assuming that the earth's orbit around the Sun is circular. The the number of days after March 21 will be $365.25 \times \lambda_s \div 360^\circ$. The two dates are then found to be approximately April 9th and September 2nd.



2.31 In each answer, draw a suitably labelled diagram showing the positions of the transits, the celestial pole and the zenith.

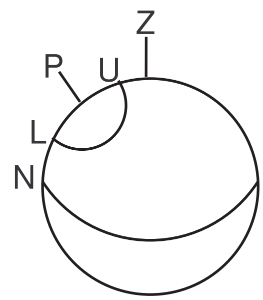
(a) An observer at latitude 48^0N observes a star of declination $\delta = 60^0$N. Prove that this star is circumpolar, and calculate its altitude at upper and lower transit. [8]

(b) Prove that for a circumpolar star of declination δ to have its upper transit south of the zenith for an observer at latitude ϕ, $\phi > \delta$. [5]

(c) A star has zenith distances at lower and upper transits of 24°N of zenith, and 74°S of zenith, respectively. Calculate ϕ and δ. [7]

Solution:

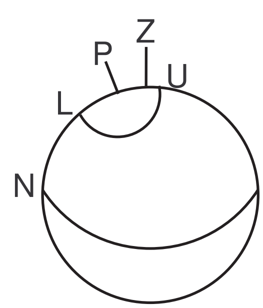
(a) For a star to be circumpolar, the lower transit must be above the observer's horizon. Now $PL = 90^o - \delta$, where δ is the declination. $PN = \phi$, the observer's latitude. Hence circumpolar means $90^0 - \delta < \phi$, i.e. $\phi + \delta > 90^0$. Here $\phi = 48^o$, $\delta = 60^0$ and so $\phi + \delta = 108^0$, meaning that the star is circumpolar.



Altitude of lower transit = NL. But $NL + LP = NP = \phi$, and $LP = 90^0 - \delta$. Hence $NL = \phi$-$(90^0 - \delta) = \phi + \delta - 90^0 = 18^0$

Altitude at upper transit $U = NU = NP + PU = \phi + (90^0 - \delta) = 78^0$.

(b) PN $= \phi$, $PL = 90^0 - \delta$, $PZ = 90^0 - \phi$, $PU = PL = 90^0 - \delta$. Want $ZU > 0$.



$PU = PZ + ZU$, i.e. $90^0 - \delta = 90^0 - \phi + ZU$. Hence $ZU = \phi - \delta > 0$, so $\phi > \delta$.

(c) We know that the altitude of the North Celestial Pole is equal to the latitude of the observer. From the diagram (see below) we see that $NL + LP = \phi$ (where L marks the position of the lower transit).

We also know that the angle (along the Observer's Meridian) between the North Celestial Pole and the position of the upper and lower transits is in both cases $90^\circ - \delta$.

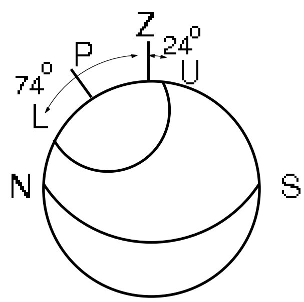
In particular, then, for the lower transit $LP = 90^\circ - \delta$. Substituting for LP (and using the fact that $NL = 90^\circ - ZL = 16^\circ$) gives the equation

$$\phi + \delta = NL + 90^\circ = 106^\circ.$$

Similarly, $PU = PZ + ZU = 90^\circ - \delta$ (where U marks the position of the upper transit). But we know that $PZ = 90^\circ - \phi$ and we are given that $ZU = 24^\circ$. Re-arranging gives the equation

$$\phi - \delta = 24^\circ.$$

Adding these two equations gives $2\phi = 130^\circ$, so that $\phi = 65^\circ$. Substituting this into the second equation gives $\delta = \phi - 24^\circ = 41^\circ$.



2.32 Show that the number of hours of daylight on a given day can be estimated as

$$2\cos^{-1}\left(-\tan\delta_\odot \tan\phi_\odot\right).$$

where $\delta_\odot$ is the solar declination, and ϕ is the observer's latitude.

Calculate the hour angle of the setting of the sun for an observer at latitude $54^\circ 55'$N, given that the sun's ecliptic longitude is $49^\circ 49'$. What date does this correspond to?

Solution: *no solution available*

2.33 The star γ Dra has declination $51^\circ 29' 27''$. When observed from Strasbourg, it's azimuth ranges between $289^\circ 44' 32''$ and $70^\circ 15' 28''$. What is the latitude of the observatory at Strasbourg?

Solution: *no solution available*

2.34e Define precisely what is meant by a circumpolar star: use a clearly annotated diagram to assist with the definition. [4]

Show that the condition for a star to be circumpolar for an observer at latitude ϕ is that its declination δ satisfies

$$\delta > 90^o - \phi.$$

Show further that if $\phi > \delta$ then the upper transit is south of the zenith. [5]

Using the sine formula,

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = \frac{\sin d}{\sin D}$$

for the arbitrary spherical triangle BCD with sides b,c,d show that the maximum azimuth A east or west of north of a circumpolar star is given by

$$A = \sin^{-1}(\cos\delta \sec\phi).$$

[4]

The star γ-Dra has declination $289^o 29' 27"$. When observed from Strasbourg, its azimuth ranges between $289^o 44' 32"$ and $70^o 15' 28"$. Taking the star to be circumpolar, what is the latitude of the observatory at Strasbourg? [4]

Solution: *no solution available*

2.35e Draw an annotated celestial sphere showing the following: the zenith point Z; the celestial pole P; the observer's horizon and meridian; the celestial equator; the cardinal points NSEW and the observer's latitude ϕ. Mark the position of an arbitrary star X on your diagram, and show the associated hour angle H, altitude a and declination δ of X. [6]

Using the cosine formula for the spherical triangle DEF with sides def,

$$\cos d = \cos e \cos f + \sin e \sin f \cos D,$$

show that, for the star X,

$$\sin a = \sin\phi \sin\delta + \cos\phi \cos\delta \cos H.$$

[4]

When the star sets, show that its hour angle H_s is given by

$$H_s = \cos^{-1}(-\tan\delta \tan\phi).$$

[3]

For the particular case of the Sun, prove that:

(a) there are 12 hours of daylight at the equinoxes;

(b) the Sun is Circumpolar if $\tan\phi = \pm 1/\tan\varepsilon$, where ε is the obliquity of the ecliptic. [4]

Solution: *no solution available*

2.36e Draw a celestial sphere showing the co-ordinate system of ecliptic latitude and longitude, annotating your diagram clearly.
What is the Sun's ecliptic latitude and longitude at the autumnal equinox? [5]

Solution: *no solution available*

2.37e Discuss briefly the differences between

(a) apparent solar time, and

(b) Universal Time (UT). [3]

[Your answer should mention the Equation of Time.]

Solution: *no solution available*

2.38e What is meant by a circumpolar star? [2]

What is the zenith distance of Vega (declination $+33^o 42'$) at its upper transit at Moscow (latitude $55^o 45'$)? [3]

Solution: A circumpolar star is one which does not set, i.e. it is always above the observer's horizon [2]

$$\begin{aligned} PZ &= 90 - \phi \\ PU &= 90 - \delta \end{aligned}$$

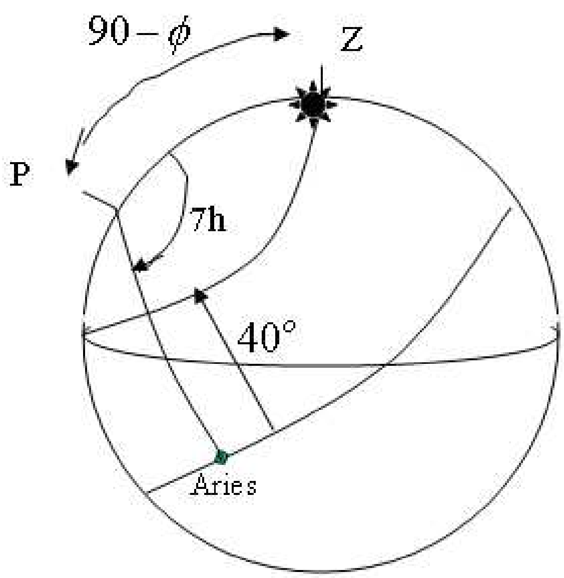
For Vega, $PU = 90 - 38^o41' = 51^o18'$
For Moscow, $PZ = -55^o45' + 90 = 34^o15'$
$\therefore \quad ZU = 51^o18' - 34^o15' = 17^o03'$ [3]

2.39e What would be the position in the sky of a star of Right Ascension 7hr, declination 40^o, one hour after sunset at the vernal equinox, for an observer at latitude 40^o? [5]

[Hint: draw a diagram]

Solution:

1 hour after sunset at vernal equinox - Sun is at ♈ , and has hour angle 7h.
RA of star is 7h, so star is on observers meridian.
Stars declination is 40^o which is same as zenith distance from equator. Hence star is at zenith.



2.40e Draw a celestial sphere showing the definitions of RA (right ascension), declination (δ) and HA (hour angle) for an arbitrary star X. [4]

Define the local sidereal time (LST) and demonstrate that

$$HA\ X + RA\ X = LST.$$

[3]

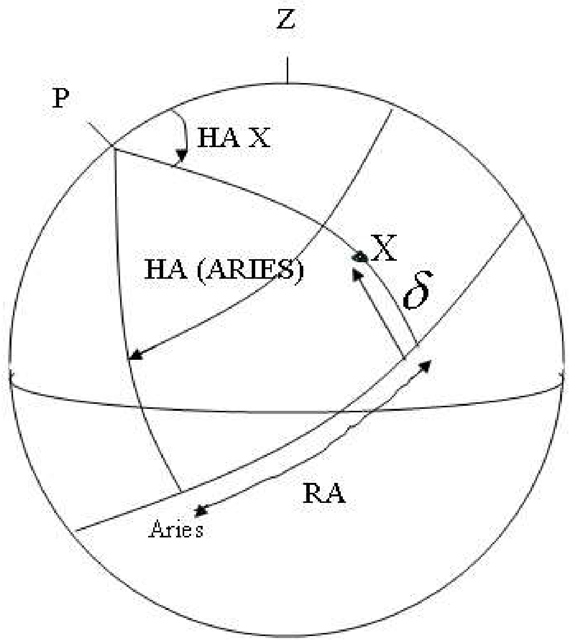
Show that the hour angle of X at setting is given by

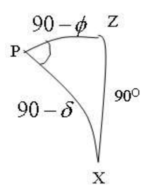
$$cos(HA\ X) = -\tan\phi\tan\delta,$$

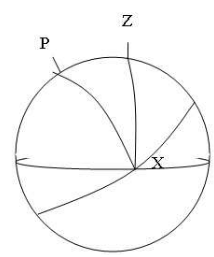
where ϕ is the latitude of the observer. [6]

The star Sirius has right ascension $6^h45^m8.9^s$, and declination $-16^o43'$. Calculate the local sidereal time of its setting, as observed from Glasgow (latitude $55^o52'$). [4]

Solution:







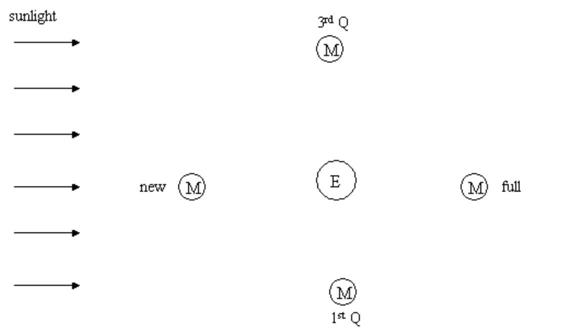
$LST = HA\Upsilon$
It is clear from the diagram that $HA\Upsilon = HAX + RAX$
i.e. $HAX + RAX = LST$.
For a setting star, X must be on the horizon

$$\begin{aligned}
\cos 90^o &= \sin\phi\sin\delta + \cos\phi\cos HAX \\
\therefore 0 &= \sin\phi\sin\delta + \cos\phi\cos HAX \\
\therefore \cos HAX &= -\frac{\sin\phi\sin\delta}{\cos\phi\cos\delta} \\
&= -\tan\phi\tan\delta
\end{aligned}$$

For Sirius, $\cos(HAS) = -\tan\phi\tan\delta = -\tan(55.867^o)\tan(-16.717^o) = 0.443$
$\therefore HAS = 63.705^o = 4^h14^m49.2^s$
$\therefore LST = 6^h45^m8.9^s + 4^h14^m49.2^s = 10^h59^m58.1^s$

2.41e Draw a diagram showing the Moon in orbit around the Earth, marking clearly the positions of the new and full moon, and first and third quarters. The Moon is observed to have exactly one half of its disk brightly lit early one morning; estimate the interval to the next new Moon. [5]

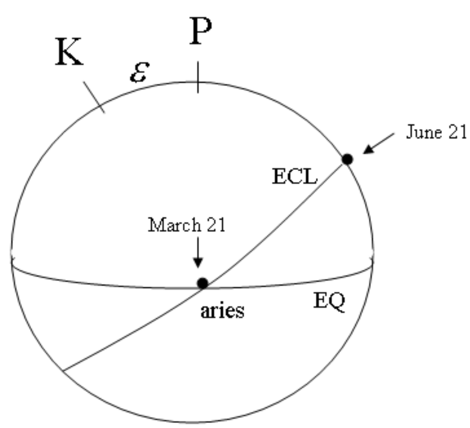
Solution:



½ disk lit in early morning – 3rd quarter moon. Therefore approx. 1 week to net new moon.

2.42e Draw a celestial sphere on which are marked the equator and ecliptic. Show clearly the position of the Sun at March 21st and June 21st, and estimate the Sun's right ascension and declination on these dates. [5]

Solution:



March 21st	**RA(Sun) = 0^h**	**dec(Sun) = 0**
June 21st	**RA(Sun) = 6^h**	**dec(Sun) = epsilon**

2.43e Define the following co-ordinate systems:

(a) alt-azimuth

(b) hour angle and declination

Ensure that you supply a clearly labelled diagram in your answer. [6]

Prove that for a star of right ascension α and declination δ, its altitude a and azimuth A at local sidereal time lst for an observer at latitude ϕ may be calculated from

$$\sin a = \sin\phi \sin\delta + \cos\phi \cos\delta \cos(LST - \alpha)$$

[6]

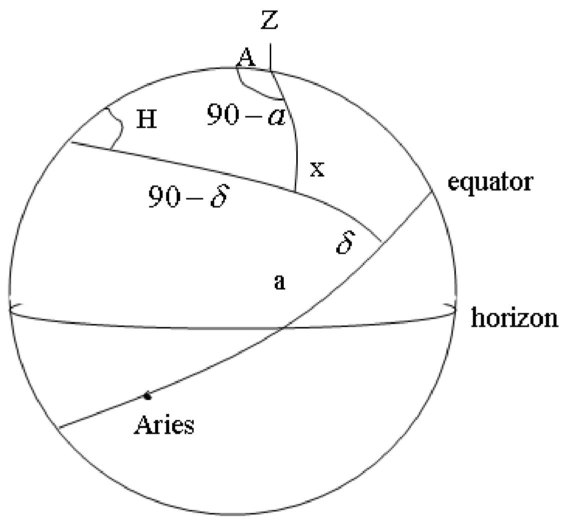
$$\cos A = \frac{\sin\delta - \sin\phi\sin a}{\cos\phi\cos a}$$

Prove further that the azimuth of a star at setting is given by

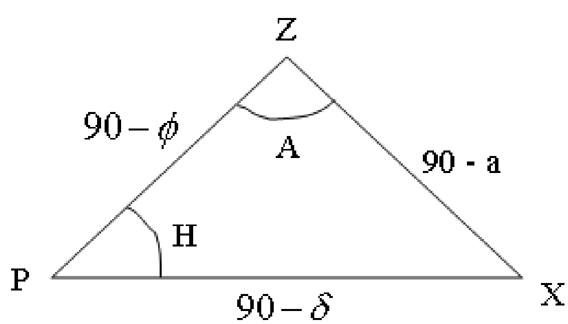
$$A = \cos^{-1}\left(\frac{\sin\delta}{\cos\phi}\right)$$

What do you conclude if $\sin\delta/\cos\phi > 1$? [5]

Solution:



Given (α, δ) then $H = LST - \alpha$ consider the spherical triangle



Apply cosine formula:

$$\cos(90^o - a) = \cos(90^o - \delta)\cos(90^o - \phi) + \sin(90^o - \delta)\sin(90^o - \phi)\cos H$$

i.e. $\sin a = \sin\delta\sin\phi + \cos\delta\cos\phi\cos(LST - \alpha)$ $\rightarrow$ gives altitude

To get azimuth, apply cosine formula again.

$$\cos(90^o - \delta) = \cos(90^o - \phi) cos(90^o - a) + \sin(90^o - \phi)\sin(90^o - a)\cos A$$

i.e.$\sin\delta = \sin\phi\sin a + \cos\phi\cos a\cos A$, so that $\cos A = \frac{\sin\delta - \sin\phi\sin a}{\cos\phi\cos a}$

At setting, $a = 0^o$ and so $\cos A = \frac{\sin\delta}{\cos\phi}$, i.e. $A = \cos^{-1}(\frac{\sin\delta}{\cos\phi})$

If $\frac{\sin\delta}{\cos\phi} > 1$ then the star must be circumpolar, and there is no solution to the setting azimuth.

2.44e Define the co-ordinate system of terrestrial latitude (ϕ) and longitude (λ). [4]

Given the cosine formula,

$$\cos a = \cos b\cos c + \sin b\sin c\cos A,$$

for a spherical triangle with sides a, b and c, and corresponding angles A, B and C, show that the shortest distance between the two points (ϕ_1, λ_1) and (ϕ_2, λ_2) is given by

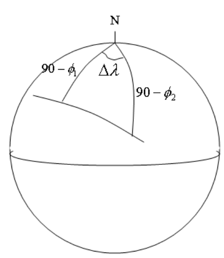
$$\cos d = \sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\cos(\lambda_1 - \lambda_2).$$ [5]

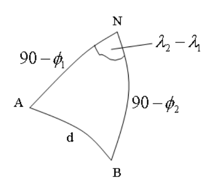
An aircraft flies from a point A at latitude 52^oN and follows a great circle path, initially due East, without deviation for 800 nautical miles and lands at site B. What is the latitude of its destination?
What is the bearing of the aircraft from B just before landing? [8]

Solution:

Latitude - small circles parallel to the equator. Longitude - great circles through N-S poles. Zero for latitude is equator; N +ve, S -ve. Zero for longitude is Greenwich meridian. [4]

Shortest distance is arc segment of great circle which connects them.

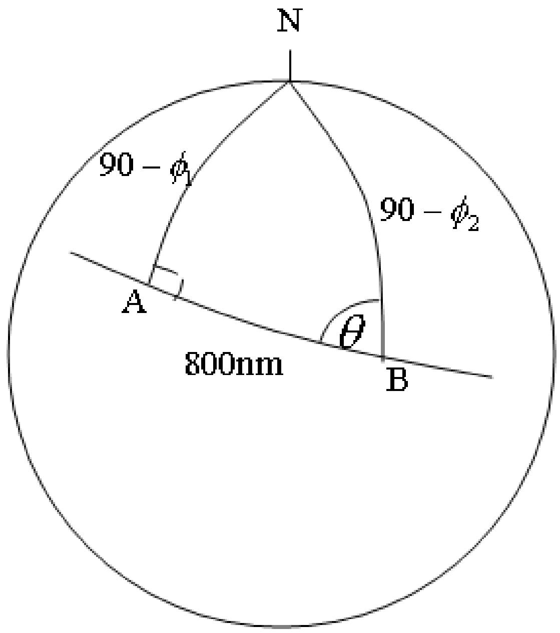




Apply cosine formula

$$\cos d = \cos(90-\phi_1)\cos(90-\phi_2) + \sin(90-\phi_1)\sin(90-\phi_2)\cos(\lambda_2 - \lambda_1)$$

i.e. $\cos d = \sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\cos(\lambda_2 - \lambda_1)$ [5]



arc $AB = 800\,\text{nm} = 800$ minutes of arc $= 13.33^o$.

Apply cosine formula:

$$\begin{aligned}\cos(90-\phi_2) &= \cos(90-\phi_1)\cos(13.33)+\sin(90-\phi_1)\sin(13.33)\cos 90\\ \sin\phi_2 &= \sin\phi_1\cos(13.33) = 0.76676\\ \text{so}\quad \phi_2 &= 50.06^o\end{aligned}$$

To get bearing θ, use sine formula:

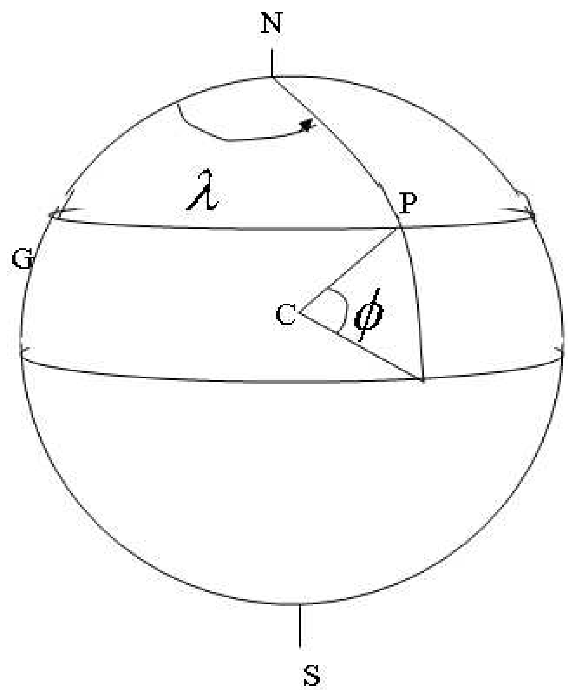
$$\frac{\sin\theta}{\sin(90-\phi_1)} = \frac{\sin 90}{\sin(90-\phi_2)} \quad \text{i.e.} \quad \sin\theta = \frac{\cos\phi_1}{\cos\phi_2} = 0.9595 \Rightarrow \theta = 73.64^o$$

[8]

2.45e Define latitude and longitude as a co-ordinate system for describing a point on the Earth's surface, and define what is meant by zenith and altitude in celestial co-ordinates. Prove that the altitude of the North Celestial Pole is equal to the latitude of the observing site. [8]

Two observatories are at the same latitude of 55^oN, but differ in longitude by 40^o. A star is observed at the zenith point of the more westerly of the two observatories. Calculate the star's altitude at the other observatory at this same instant. [9]

Solution:



Latitude is ϕ measured up/down from equator. +ve N, -ve S. Longitude is measured ϵ from Greenwich meridian. W is -ve.

Meridians of longitude are semi great circles. Parallels of latitude are small circles.

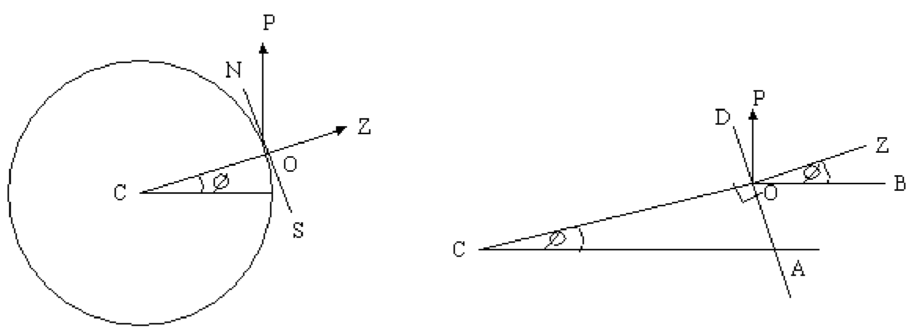
Observer O at latitude ϕ. NOS is observer's horizon N-S plane, P is NCP.

$$ACO = \phi$$

$$\therefore BOP = \phi$$

$$\therefore POZ = 90 - \phi$$

$$\therefore DOP = \phi$$



$\phi = 55^o$ and $\Delta\lambda = 40^o = 2^h 40^m$. At West observatory, star is at zenith so hour angle is 0. At same instant, at East observatory, $H + 2^h 40^m = 40^o$

Star is at X on diagram. We want altitude a. Use cosine formula

$$\cos(90-a) = \cos(90-\delta)\cos(90-\phi) + \sin(90-\delta)\sin(90-\phi)\cos H$$

Now since star is at zenith for other observatory, $\delta = \phi = 55^o$. Hence

$$\begin{aligned} \sin a &= \sin^2\delta + \cos^2\delta\cos H = \sin^2 55 + cos^2 55\cos 40 \\ &= 0.671 + 0.329 \times 0.766 = 0.923 \\ \Rightarrow a &= 67.27^o \end{aligned}$$

3 Dynamical Astronomy

3.1e Show, assuming Newton's law of gravitation, that the gravitational potential energy, $V(r)$, of a small body of mass m at a distance r from a planet of mass M is given by

$$V(r) = -\frac{GmM}{r},$$

where G is the constant of gravitation. [5]

Solution: *no solution available*

3.2e State Kepler's Laws of Planetary Motion.

Define the term *angular momentum* and explain, using the case of a circular orbit, the significance of Kepler's Second Law in terms of angular momentum. [5]

Solution: Kepler 1: A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse. Kepler 2: A line connecting a planet to the sun sweeps out equal areas in equal times. Kepler 3: The square of the periods of planetary orbits are proportional to the cubes of their semi-major axes.

The angular momentum of an object of mass m moving in an orbit of radius r about a point, and having speed v, is mrv. Angular momentum is constant in the absence of external forces. If the orbit is circular, so that the radius is a constant in time, then the orbital speed must also be constant. Hence the object will sweep out equal areas in equal time as it orbits, which is consistent with Kepler's second law.

3.3e A body of mass m moves under gravity in a circular orbit of radius R about a body of much larger mass M. Given that the centripetal force on a body of mass m moving with velocity v in a circular orbit of radius R is mv^2/R show that the period of the orbit is given by the equation

$$T^2 = \frac{4\pi^2 R^3}{GM},$$ [5]

where G is the constant of gravitation.

Solution: The force of attraction of M upon m is $F = GMm/R^2$, but this must be balanced by the centripetal force to keep the body in its circular orbit. Centripetal force $= mv^2/R$, hence

$$\frac{mv^2}{R} = F = \frac{GMm}{R^2},$$

so that $v^2 = GM/R$. But $v = 2\pi R/T$, hence

$$\left(\frac{2\pi R}{T}\right)^2 = \frac{GM}{R}, \text{ giving } \frac{4\pi^2 R^2}{T^2} = \frac{GM}{R} \text{ then } T^2 = \frac{4\pi^2 R^3}{GM}.$$

3.4 State Newton's law of gravitation. Demonstrate, for a circular orbit, how it is consistent with Kepler's second and third laws of planetary motion. [9]

Show that the orbital period of a mass in a circular orbit of radius r about a much larger mass M is

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM}.$$

What is the corresponding expression for an elliptical orbit? [5]

The comet Faye is in an orbit with a period of 7.38 years, an eccentricity of 0.576 and a perihelion distance of 1.608 AU. The comet Wolf has a period of 8.43 years, and a perihelion distance of 2.507 AU. Calculate the eccentricity of its orbit. {0.395} [6]

Solution: The force (attractive) between two bodies of mass m and M separated by a distance r acts along the line joining the two bodies and is given by

$$F = \frac{GMm}{r^2}.$$

Assume gravitational force provides centripetal force for orbit. Kepler's 2nd Law implies vr is constant, so angular momentum is constant (i.e., no torques) so forces are central and act along the line of centres of the two bodies.
Kepler's 3rd states T^2/a^3 is a constant. Choose a circular orbit of radius r. Central force needed for acceleration of v^2/r is mv^2/r. But $v = 2\pi r/T$, so $F = 4\pi^2 mr/T^2$. Again by Kepler's 3rd Law, and substituting for T we have

$$F \propto \frac{m}{r^2}.$$

By symmetry, (action and reaction) we also have $F \propto M/r^2$. This implies $F \propto Mm/r^2$, or $F = GMm/r^2$ where G is a constant.

Second part: The force equation $mv^2/r = GMm/r^2$ implies that $v^2 = GM/r$. But for circular orbit, $v = 2\pi r/T$, so $4\pi^2 r^2/T^2 = GM/r$ so that $T^2/r^3 = 4\pi^2/GM$.
For elliptical orbit $T^2/a^3 = 4\pi^2/GM$ where a is the semi-major axis.

For comet Faye, $T_\mathrm{F} = 7.38$ years, $e_\mathrm{F} = 0.576$. Perihelion distance $= a_\mathrm{F}(1 - e_\mathrm{F}) = 1.608$ AU so $a_\mathrm{F} = 3.792$ AU. For comet Wolf, $T_\mathrm{W} = 8.43$ years and $a_\mathrm{W}(1 - e_\mathrm{W}) = 2.507$ AU. We need to find e_W. Using

$$\left.\frac{T^2}{a^3}\right|_\mathrm{F} = \left.\frac{T^2}{a^3}\right|_\mathrm{W}$$

$$\text{we have} \quad a_\mathrm{W} = a_\mathrm{F}\left(\frac{T_\mathrm{W}^2}{T_\mathrm{F}^2}\right)^{1/3}.$$

Hence $a_\mathrm{W} = 1.093$. $a_\mathrm{F} = 4.144$. Thus $4.144(1 - e_\mathrm{W}) = 2.507$ or $e_\mathrm{W} = 0.395$.

3.5e A body of negligible mass m is in an elliptical orbit around the Sun. Write down an expression for the total energy, E, of the body and show that when it is at a distance r from the Sun its velocity, v, can be written as

$$v^2 = GM\left(\frac{2}{r} + C\right),$$

where the constant $C = 2E/(GMm)$ and G is the gravitational constant. [6]

Given that the perihelion and aphelion distances of the orbit are $a(1-e)$ and $a(1+e)$, where a is the semi-major axis and e is the eccentricity of the orbit, use the above expression to find the ratio of the velocity of the body at perihelion to that at aphelion. [2]

From conservation of angular momentum derive a second expression for this ratio of velocities. [2]

Hence show that

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right). \quad [3]$$

The comet Wolf has an eccentricity of 0.395 and a perihelion distance of 2.507 AU. Calculate the velocity of the comet at perihelion. [4]

Solution: This is largely from the notes.

$$E = \tfrac{1}{2}mv^2 - GMm/r$$

$$\Rightarrow v^2 = GM\left(\frac{2}{r} + \frac{2E}{GMm}\right) \qquad (1)$$

Conservation of angular momentum gives

$$v_1 a(1-e) = v_2 a(1+e)$$

, where v_1, v_2 are the perihelion and aphelion speeds, respectively. So

$$\frac{v_1}{v_2} = \frac{1+e}{1-e} \qquad (2)$$

From (1):

$$v_1^2 = GM\left(\frac{2}{a(1-e)} + C\right); \quad v_2^2 = GM\left(\frac{2}{a(1+e)} + C\right)$$

That is,

$$\frac{v_1^2}{v_2^2} = \frac{1+e}{1-e}\frac{2+Ca(1-e)}{2+Ca(1+e)}$$

Combine now with (2) to eliminate the vs:

$$(afrac1+e1-e)^2 = (afrac1+e1-e)\,\frac{2+Ca(1-e)}{2+Ca(1+e)}$$

which reduces to $C = -1/a$. Hence

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

. For Wolf: $2.507 = a(1-e) \Rightarrow a = 4.144$. Hence when $r = 2.507$AU, $v = 2\pi(2/2.507 - 1/4.144)^{1/2} = 4.687$AU/d, using natural units.

3.6e If a body of negligible mass is in orbit around a mass M, its speed, v, is given by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

where r is the distance between the centres of the two masses, a is the semi-major axis of the orbit and G is the gravitational constant. An artificial satellite is in a circular orbit about the Earth at a height of 1 000 km above the surface. Use the above expression to calculate the speed of the satellite. [3]

The satellite is to be transferred to an orbit of radius 30 000 km. Describe an economical way of carrying out this transfer and calculate the change in speed required to inject it into its transfer orbit. [6]

Solution: The satellite orbit is circular, so $a = r$ and therefore $v^2 = GM/r$. Now

$$r = r_\oplus + 1000\,\text{km} = 6379\,\text{km} + 1000\,\text{km} = 7379\,\text{km}$$

Hence

$$v^2 = \frac{3.989 \times 10^{14}\,\text{m}^2\text{s}^{-2}}{7.379 \times 10^6\,\text{m}} = 5.406 \times 10^7\,\text{m}^2\text{s}^{-2}$$

Thus $v = 7.353\,\text{km s}^-1$

New orbit has radius $r = 30,000$ km. Most efficient way to achieve this is via a Hohmann transfer orbit.

To calculate the speed boost, compare the original orbit speed with the transfer orbit speed at perigee. For the transfer orbit,

$$2a = r_{\oplus} + 10^6\,\mathrm{m} + 3 \times 10^7\,\mathrm{m} = 3.7379\,\mathrm{m}$$

Therefore $a = 1.869 \times 10^7$ m.

At perigee,

$$v_p^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{GM}{10^6}\left(\frac{2}{7.379} - \frac{1}{18.69}\right) = 8.677 \times 10^7\,\mathrm{m}^2\,\mathrm{s}^{-2}$$

i.e. $v_p = 9.315 \times 10^3\,\mathrm{m\,s}^{-1}$. Hence $\Delta v = 9.315 \times 10^3 - 7.353 \times 10^3 = 1.962\,\mathrm{km\,s}^{-1}$.

3.7e Explain what is meant by the *escape velocity* from the surface of a planet. Given that the gravitational potential energy, V, of a mass m at the surface of a planet of mass M and radius R is $V = -GmM/R$, derive an expression for its escape velocity from the planet's surface. [4]

The orbit of the Moon around the Earth has a semi-major axis of 384 400 km and a period of 27.32 days. Given that the radius of the Earth is 6 378 km, calculate the acceleration due to gravity and the escape velocity at the Earth's surface. [8]

Solution: *no solution available*

3.8 Derive an expression for the period, T, of a body of negligible mass in a circular orbit with radius R about the Earth. [9]

Derive the *height* above the equator for a geostationary communications satellite, given that the radius of the Earth is 6.38×10^6 m and the (surface) acceleration due to gravity is $9.8\,\mathrm{m\,s}^{-2}$. [8]

How many satellites are needed to give complete coverage of the Earth (excluding polar regions)? [2]

Solution: Period of circular orbit:

$$m\omega^2 R = Gm_e m/R^2$$

Therefore

$$\frac{4\pi^2}{T^2} = \frac{Gm_e}{R^3}$$

Hence

$$T = \left(\frac{4\pi^2 R^3}{Gm_e}\right)^{1/2}$$

Height of Geostationary orbit:
Told that $g = 9.8\mathrm{m\,s}^{-1}$. Know that $g = Gm_e R_e^{-2}$ where R_e is earth radius. Hence $Gm_e = gR_e^2$. Also,

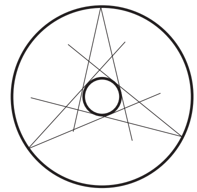
$$R^3 = \frac{Gm_e T^2}{4\pi^2} = \frac{gR_e^2 T^2}{4\pi^2}$$

and so

$$\left(\frac{R}{R_e}\right)^3 = \frac{gT^2}{4\pi^2 R_e}$$

If $T = 1\,\text{day} = 8.62 \times 10^4\,\text{s}$, and $R_e = 6.38 \times 10^6\,\text{m}$ then $R = 6.61 R_e$. So the height is $5.61 R_e = 35\,800\,\text{km}$.

Note that 3 satellites are required to give complete coverage, to ensure that at least in the equatorial



plane, no part of the earth is hidden from the satellites.

3.9 Show that the gravitational potential energy of a mass, m, a distance r from the centre of a planet of mass M ($\gg m$) is

$$V(r) = \frac{-GMm}{r},$$

and write down an expression for the total energy. $\{\frac{1}{2}mv^2 - GMm/r\}$ [8]

Explain the meaning of the phrase 'escape velocity from the surface of a planet' and derive an expression for it in terms of the radius of the planet and the acceleration due to gravity at its surface. Given that the radius of the Earth is 6.38×10^6 m and using the data given and derived earlier calculate the escape velocity from the surface of Venus. $\{10.3\,\text{km}\,\text{s}^{-1}\}$ [6]

Solution: Since Force is attractive and acts along the line joining bodies, work done to separate the bodies by an amount δr is $\delta V = \frac{GMm}{r^2}\delta r$, so the potential energy is

$$V(r) = \int \frac{GMm}{r^2}\,dr = -\frac{GMm}{r} + \quad \text{const.}$$

If we take $V(r) \to 0$ as $r \to \infty$ then

$$V(r) = -\frac{GMm}{r}.$$

Now, the total energy E is the sum of the potential and kinetic energies, i.e.,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}.$$

The escape velocity is the velocity a body must have to escape from the planet's surface. This happens when $E \geq 0$, or when

$$\frac{1}{2}mv^2 \geq \frac{GM_\text{p}m}{R_\text{p}},$$

where R_p is the radius of the planet and M_p its mass. The escape velocity is therefore

$$v_\text{escape} = \sqrt{\frac{2GM_\text{p}}{R_\text{p}}}.$$

For a body at the planet surface, $mg_\text{p} = GM_\text{p}m/R_\text{p}^2$, so $v_\text{escape} = \sqrt{2g_\text{p}R_\text{p}}$. For Venus, $R_\text{V} = 0.97R_\text{E} = 6.2 \times 10^6\,\text{m}$ so $v_\text{escape} = 10.3\,\text{km}\,\text{s}^{-1}$.

3.10 By conservation of energy and angular momentum for a small body in an orbit about the Sun we can get the relations

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}, \quad \text{and} \quad h = rv\sin\theta,$$

where M and m are the masses of the Sun and the body respectively, r is the distance of the body from the Sun, G is the gravitational constant, v is the speed of the body, θ is the angle its direction of motion makes with a line to the body from the Sun, and h is a constant.

Explain the origin of the terms in the above equations. [3]

If the orbit is an ellipse of semi-major axis a and eccentricity e, show (by considering the perihelion and aphelion of the orbit) that

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \qquad [10]$$

$$\text{and} \quad h^2 = GMa(1-e^2).$$

An asteroid is detected at a distance of 1.26 AU from the Sun. Its velocity is $33.3\,\mathrm{km\,s^{-1}}$ with directed at 74.7° to the line joining the two. Calculate the semi-major axis and period of its orbit. Does the orbit cross that of the Earth? [7]

(Use solar units for this problem; note that $1\,\mathrm{km\,s^{-1}} = 0.211\,\mathrm{AU\,yr^{-1}}$.)

Solution: The kinetic energy of a body is $\frac{1}{2}mv^2$ and its gravitational potential energy is $-GMm/r$. The angular momentum per unit mass of the body is $rv\sin\theta$, which is conserved for the motion. The total energy is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r},$$

so that

$$v^2 = GM\left(\frac{2}{r} + C\right), \qquad \text{where} \quad C = \frac{2E}{GMm}.$$

Let v_1 be the velocity at perihelion and v_2 be the velocity at aphelion. Then

$$h = v_1 a(1-e) = v_2 a(1+e), \qquad (12)$$

and

$$v_1^2 = GM\left[\frac{2}{a(1-e)} + C\right], \qquad (13)$$

$$v_2^2 = GM\left[\frac{2}{a(1+e)} + C\right]. \qquad (14)$$

Equations (2) and (3) give

$$\frac{v_1^2}{v_2^2} = \frac{\left[\frac{2}{a(1-e)} + C\right]}{\left[\frac{2}{a(1+e)} + C\right]} = \frac{(1+e)}{(1-e)}\frac{2+Ca(1-e)}{2+Ca(1+e)}.$$

But from Equation (1)

$$\frac{v_1}{v_2} = \frac{1+e}{1-e},$$

so that

$$\left(\frac{1+e}{1-e}\right)^2 = \frac{(1+e)}{(1-e)}\frac{2+Ca(1-e)}{2+Ca(1+e)},$$

and therefore

$$\frac{1+e}{1-e} = \frac{2+Ca(1-e)}{2+Ca(1+e)}.$$

This implies that $2 + 2e + Ca(1 + 2e + e^2) = 2 - 2e + Ca(1 - 2e + e^2)$, so $Ca(4e) = -4e$, and $C = -1/a$. Thus

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

and so

$$v_1^2 = GM\left(\frac{2}{a(1-e)} - \frac{1}{a}\right) = \frac{GM}{a}\frac{(1+e)}{(1-e)}.$$

But

$$h = v_1 a(1-e) = \sqrt{\frac{GM}{a}}\frac{a\sqrt{1+e}}{\sqrt{1-e}}(1-e) = \sqrt{GMa}\sqrt{1-e^2}$$

$$\text{or} \quad h^2 = GMa(1-e^2). \tag{15}$$

Note that $1\,\mathrm{km\,s^{-1}} = 0.211\,\mathrm{AU\,yr^{-1}}$ and $33.3\,\mathrm{km\,s^{-1}} = 7.03\,\mathrm{AU\,yr^{-1}}$. Also

$$v^2 = 4\pi^2\left(\frac{2}{r} - \frac{1}{a}\right),$$

since $G = 4\pi^2$. Substituting, we have

$$(7.03)^2 = 4\pi^2\left(\frac{2}{1.26} - \frac{1}{a}\right),$$

which gives $a = 2.96\,\mathrm{AU}$. Now $T^2 = a^3 = (2.96)^3$, therefore $T = 5.09\,\mathrm{yr}$. $h^2 = GMa(1-e^2) = 4\pi^2 a(1-e^2)$. On the other hand $h = vr\sin\theta = 7.03 \times 1.26 \times \sin(74.7^\circ) = 8.54$. i.e., $(8.54)^2 = 4\pi^2 \times 2.96(1-e^2)$, so $e = 0.613$. The perihelion radius is $R_\mathrm{p} = a(1-e) = 2.96(1-0.613) = 1.14\,\mathrm{AU}$, so it does not cross Earth's orbit.

3.11 Show that the laboratory value of the gravitational constant can, with other measurements given below, give the masses of the Sun and the Earth.

semi-major axis of Earth's orbit	1.496×10^8 km
radius of Earth	6371 km
surface gravitational acceleration on Earth	$9.81\,\mathrm{m\,s^{-2}}$
Earth year	365.25 d

Solution: *no solution available*

3.12 The satellite *Europa* describes an orbit about Jupiter of semi-major axis 6.71×10^5 km in a period of 3.552 days. Neptune's satellite *Triton* has semi-major axis 3.55×10^5 km and orbital period 5.877 days. Calculate the ratio of the masses of the two planets. {18.49}

Solution: We must use Newton's amended form of Kepler's third law, namely

$$M + m = \frac{4\pi^2}{G}\frac{a^3}{T^2}.$$

Let the suffix 1 refer to the Jupiter-Europa system and the suffix 2 to that of Neptune and Triton. Then

$$\frac{M_1 + m_1}{M_2 + m_2} = \frac{a_1^3}{a_2^3}\frac{T_2^2}{T_1^2}.$$

Inserting values, namely $a_1 = 6.71 \times 10^5$ km, $a_2 = 3.55 \times 10^5$ km, $T_1 = 3.552$ days, $T_2 = 5.877$ days, yields

$$\frac{M_1 + m_1}{M_2 + m_2} = 18.49.$$

Neglecting the masses of the satellites in relation to the masses of their planets, this is the ratio of the mass of Jupiter to that of Neptune.

3.13 Halley's comet has an orbital period of 76 years and its perihelion distance is 0.59 AU. Calculate its semi-major axis and its greatest distance from the Sun in AU. Use your lecture notes to calculate its velocity at perihelion and aphelion in AU yr^{-1}. Convert these to km s^{-1}, given that the Earth's orbital velocity is 29.8 km s^{-1}. What is the ratio of its greatest and least orbital velocities? {17.942 AU; 35.294 AU; 11.47 AU yr^{-1}; 0.192 AU yr^{-1}; 59.7}

Solution: In the usual notation $T = 76$ years and $a(1 - e) = 0.59$ AU. Working in these units Kepler's third law yields

$$a^3 = T^2, \qquad \text{so} \quad a = 76^{\frac{2}{3}} = 17.942\ \mathrm{AU},$$

$$(1 - e) = \frac{0.59}{17.942} = 0.033 \qquad \therefore \quad e = 0.967.$$

The greatest distance from the Sun is $a(1 + e) = 17.942 \times 1.967 = 35.294$ AU. The comet's velocity is obtained from the formula $V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$. In units of AU year^{-1}, the Earth's orbital velocity is 2π. So inserting $r = 1 = a$ yields $\mu = 4\pi^2$ in these units. The comet's perihelion and aphelion velocities may then be calculated. for example, at perihelion

$$V = 2\pi\left(\frac{2}{0.59} - \frac{1}{17.942}\right)^{\frac{1}{2}} = 11.47\ \mathrm{AU\ year}^{-1}.$$

Similarly at aphelion

$$V = 2\pi\left(\frac{2}{35.294} - \frac{1}{17.942}\right)^{\frac{1}{2}} = 0.192\ \mathrm{AU\ year}^{-1}.$$

To convert to km s^{-1} we must replace 2π by 29.8, getting 54.4 km s^{-1} at perihelion and 0.91 km s^{-1} at aphelion. The required ratio is $11.47/0.192 = 59.7$.

3.14 A comet, moving towards perihelion, has a velocity of 31.53 km s^{-1} when 1.70 AU from the Sun, directed at 143.16° from the direction of the radius vector. Use this information to calculate a and h. Also obtain the eccentricity, perihelion and aphelion distances and period of the orbit. {$a = 17.84$ AU; $e = 0.966$; 0.607 AU; 35.07 AU; 75.35 yr}

Solution: Using units with mass of Sun = 1, unit of length is AU and unit of time is one year, speed of the comet is

$$V = \frac{31.531}{4.741} = 6.651\ \mathrm{AU\ yr}^{-1}.$$

Using $V^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$ with $GM = 4\pi^2$ and $r = 1.70$ AU, $a = 17.84$ AU.
$h = rV_n = 1.70 \times 6.651 \times \sin 143.16^\circ\ \mathrm{AU}^2\,\mathrm{yr}^{-1} = GMa(1 - e^2) = 4\pi^2(1 - e^2)a.$

This gives $e = 0.966$.
Period of orbit $= a^{3/2}$ years $= 75.35$ years.
Perihelion distance $= a(1 - e) = 0.607$ AU.
Aphelion distance $= a(1 + e) = 35.07$ AU.

3.15 The satellite *Phobos* of Mars has an orbit with a period of 0.3189 days and a semi-major axis of 9.38×10^3 km. The diameter of Mars is 6 762 km. Show that, without a knowledge of the gravitational constant, this is sufficient to calculate the surface gravity of Mars and its surface escape velocity. Obtain values for these quantities.

Solution: *no solution available*

3.16 Explain what is meant by a Hohmann transfer orbit.

A manned spacecraft leaves the vicinity of the Earth to go to Venus using the most economical transfer orbit. Calculate the minimum change in velocity required to reach the orbit of Venus and the time taken for the transfer. If the spacecraft is to rendezvous with Venus, what should be the position of Venus relative to the Earth in order to achieve this? Calculate the change in velocity required once the spacecraft reaches Venus. How long is required before the spacecraft can return to Earth using a Hohmann transfer orbit?

Assume that the orbits of Earth and Venus are circular and coplanar with radii of 1.0 and 0.723 AU respectively. Ignore the gravitational effects on the spacecraft by Earth and Venus.

Solution:

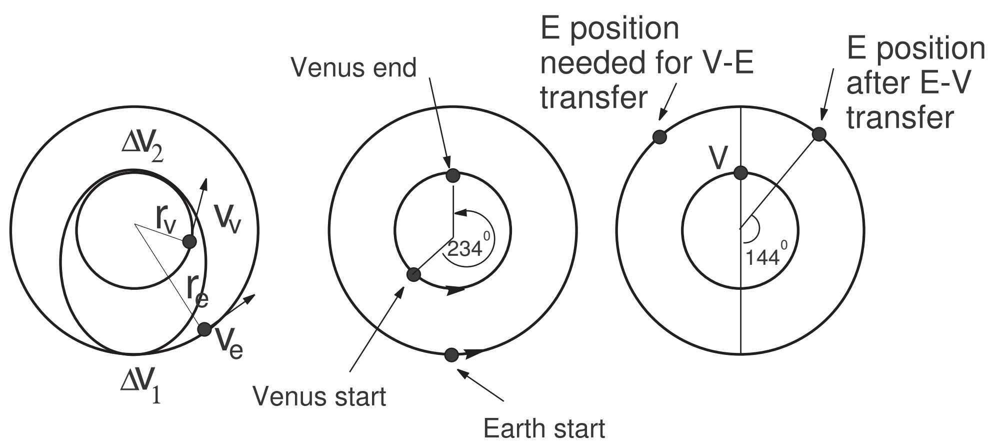


Fig 1 Fig 2 Fig 3

$r_e = 1$ AU, $r_v = 0.723$ AU. Earth's orbital velocity $v_e = 2\pi$ AU yr_{-1}, hence in units of AU and yr, $GM = 4\pi^2$. For the transfer orbit $a_T = (r_v + r_e)/2 = 0.8615$ AU. Transfer time = period/2 $= a_T^{3/2}/2 = 0.400$ yr. If v_a, v_p are the speeds at aphelion and perihelion, then

$$v_a^2 = GM\left(\frac{2}{r_e} - \frac{1}{a_T}\right)$$

which yields $v_a = 5.756\,\mathrm{AU\,yr}_{-1}$. Also, $v_e = 2\pi/r_e^{1/2} = 6.283\,\mathrm{AU\,yr}^{-1}$. So

$$\triangle v_1 = v_a - v_e = -0.527\,\mathrm{AU\,yr}^{-1}$$

. Similarly,

$$v_p = 2\pi \left(\frac{2}{r_v} - \frac{1}{a_r}\right)^{1/2} = 7.961\,\text{AU}\,\text{yr}^{-1}$$

,

$$v_v = 2\pi / r_v^{1/2} = 7.389\,\text{AU}\,\text{yr}^{-1}$$

and so

$$\triangle v_2 = v_v - v_p = -0.572\,\text{AU}\,\text{yr}^{-1}$$

Period of Venus $= r_v^{3/2} = 0.615\,\text{yr}$. In transfer time it completes $0.4/0.615 \times 360^0 = 234^0$ of its orbit, so starting configuration is given in fig 2. When craft reaches Venus, Earth has travelled $0.4 \times 360^0 = 144^0$, so at arrival, the configuration is as shown in fig 3. For the return trip, Earth must be 36^0 ahead of (the moving) Venus, so Venus has to catch up $360^0 - 72^0 = 288^0$ with respect to Earth, which will occur at a rate

$$\triangle\omega = \omega_v - \omega_e = 2\pi \left(\frac{1}{T_v} - \frac{1}{T_e}\right) = 3.933\,\text{rad}\,\text{yr}^{-1}$$

Now $288^0 = 5.03$ radians, so time taken is $5.03/3.933 = 1.278\,\text{yr}$

3.17 The semi-major axis of the orbit of Mars is 1.524 AU and the orbital eccentricity is 0.093. Assuming the Earth's orbit to be circular and coplanar with that of Mars, calculate:

(a) the distance of Mars from the Earth at closest approach

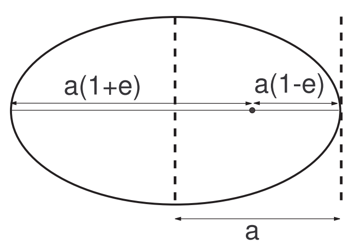
(b) the ratio of the speeds of Mars in its orbit at perihelion and aphelion

(c) the speed of Mars at aphelion in AU per year.

Solution:

(a) Perihelion distance from Mars is

$$a(1-e) = 1.524 \times (1 - 0,093) = 1.38\,\text{AU}$$



So the closest distance to earth is 0.38 AU.

(b) Conservation of angular momentum requires

$$m v_p r_p = m v_a r_a$$

$$\frac{v_p}{v_a} = \frac{r_a}{r_p} = \frac{1+e}{1-e} = 1.205$$

(c) Use

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right), \text{ i.e. } v^2 = C\left(\frac{2}{r} - \frac{1}{a}\right)$$

where C is a constant. For earth,

$$\left(\frac{2\pi \times 1}{1}\right)^2 = C\left(\frac{2}{1} - \frac{1}{1}\right) \Rightarrow C = 4\pi^2$$

if v is expressed in AU/yr, and r and a are given in AU. Hence

$$v_a^2 = \frac{4\pi^2}{a}\left(\frac{2}{1+e} - 1\right) = 21.5$$

and so $v_a = 4.64\,\text{AU(yr)}^{-1}$.

3.18 Two artificial satellites are in elliptical orbits about the Earth and both have the same period. The ratio of the velocities at perigee is 1.5 and the eccentricity of the satellite with the greater perigee velocity is 0.5. Calculate the eccentricity of the orbit of the other satellite and the ratio of the apogee velocities of the two satellites.

Solution: *no solution available*

3.19 Halley's Comet moves in an elliptical orbit with an eccentricity of 0.9673. Calculate the ratios of

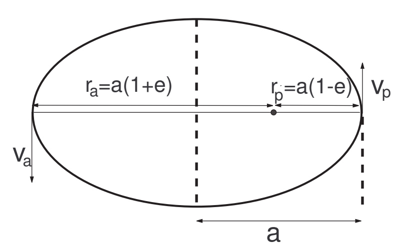
(a) the linear velocities

(b) the angular velocities at aphelion and perihelion.

Solution:

(a) $r_a = a(1+e), r_p = a(1-e)$. Conservation of angular momentum requires $mv_pr_p = mv_ar_a$, and so

$$\frac{v_a}{v_p} = \frac{r_p}{r_a} = \frac{1-e}{1+e} = 0.0167$$



(b) $\omega = v/r$, so that at aphelion and perihelion,

$$\frac{\omega_a}{\omega_p} = \frac{v_a r_p}{r_a v_p} = \left(\frac{1-e}{1+e}\right)^2 = 2.76 \times 10^{-4}$$

3.20 The period of Jupiter is 11.86 years and the masses of Jupiter and the Sun are respectively 3.3×10^5 and 318 times that of the Earth. Calculate the change in Jupiter's orbital period if the semi-major axis was the same but its mass was the same as the Earth.

Solution: *no solution available*

3.21 A lunar probe is put into an elliptical transfer orbit from a circular parking orbit (radius 6 878 km) about the Earth. It is intended that the apogee of the transfer orbit should touch the Moon's orbit (assumed circular with a radius of 3.844×10^5 km). If the velocity in the parking orbit is 7.613 km s^{-1}, calculate:

(a) the semi-major axis and eccentricity of the transfer orbit
(b) the time the probe takes to reach apogee
(c) the required velocity increment to give the transfer orbit.

Solution: *no solution available*

3.22 Explain what is meant by the linear momentum of a body and describe how it changes when a force is applied to the body.

(a) Explain what is meant by the angular momentum of a body about an axis. What quantity (analogous to force) changes the angular momentum of a body?
(b) Write down Kepler's laws of planetary motion. Explain, using a circular orbit, the physical basis of the second law.
(c) Show how a cone can be sliced to give a circle, an ellipse or a hyperbola. Where does a parabola fit in here?
(d) Consider a satellite in a circular orbit about the Earth. At a particular point in the orbit the spacecraft engines are fired to increase the tangential velocity of the craft. Sketch the possible trajectories of the spacecraft.
(e) Sketch an ellipse and mark clearly the positions of the foci. If the Sun is at one focus of an elliptical orbit, mark the positions of perihelion and aphelion and write down expressions for the perihelion and aphelion distances in terms of the semi-major axis and eccentricity of the ellipse.

The comet Encke has an orbit with a period of 3.3 years, a perihelion distance of 0.339 AU and an eccentricity of 0.847. Comet Halley has an orbit with perihelion distance of 0.587 AU and eccentricity of 0.967 AU. Calculate the length of the semi-major axis for each of the above orbits and using Kepler's third law find the period of comet Halley.

Solution: Linear Momentum = product of mass and velocity.
Newton's second law states that

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv) \rightarrow mv = \int_0^t F\,\mathrm{d}t + c,$$

so linear momentum changes as indicated.

Angular momentum of a body = '*first moment of momentum*', e.g., about the origin $L = mvr\sin\theta$ (Angular equivalent of linear momentum). The quantity analogous to Force is *Torque*. Torque tends to twist or rotate a body (cf. Torque spanner for tightening nuts or bolts). Torque about the origin $= rF\sin\theta$.

$$\underset{\substack{\uparrow\\ \text{action on body}}}{\text{Torque about origin}} = \underset{\substack{\uparrow\\ \text{action of body}}}{\frac{\mathrm{d}}{\mathrm{d}t}(\text{Angular Momentum about 0})}$$

Kepler's Laws

First Law The orbit of each planet is an ellipse with the Sun at one focus.

Second Law For all planets, the line joining the planet to the Sun sweeps out equal areas in equal times.

Third Law The cubes of the semi-major axes of the planetary orbits are proportional to the square of the planets' periods of revolution about the Sun.

2nd Law: Area swept out in unit time $\sim \frac{2}{2}rv$. So $\frac{1}{2}rv = \text{const} \rightarrow mvr = \text{const}$. i.e., angular momentum is conserved. So no torque on planet. So all forces act through the centre of the motion ($\sim$ the Sun) $\rightarrow$ Force of Gravitation is a central Force.

<u>Cone</u> PIC!
Parabola – limiting case between ellipse and hyperbola – section parallel to the surface of the cone.
Rockets fired during orbit PIC!
PIC!
Perihelion distance $= a(1-e)$
Aphelion distance $= a(1+e)$

$$\begin{array}{rll} \text{Encke} & a(1-e) = 0.339, & e = 0.847 \rightarrow a_{\mathrm{E}} = 2.215 \\ \text{Halley} & a(1-e) = 0.587, & e = 0.967 \rightarrow a_{\mathrm{H}} = 17.78 \end{array}$$

$$\text{Kepler's 3rd Law} \quad \rightarrow \frac{T_E^2}{a_E^3} = \frac{T_H^2}{a_H^3} \rightarrow \frac{(3.3)^2}{(2.215)^3} = \frac{T_H^2}{(17.78)^3} \rightarrow T_{\mathrm{H}} = 75\,\text{years}.$$

3.23e State Kepler's three laws of planetary motion.

Given that the semi-major axis of the orbit of Venus is 0.7233 AU, calculate its sidereal period in years. [5]

Solution: *no solution available*

3.24e The orbital period, T, of a body of mass m in an elliptic orbit about another body of mass M is

$$T = 2\pi \left[\frac{a^3}{G(M+m)}\right]^{1/2} = 2\pi \left(\frac{a^3}{\mu}\right)^{1/2},$$

where a is the orbital semi-major axis and G the gravitaional constant. Also if v is the speed of the body of mass m when its radius vector is r, then

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

Show that, if at any time the direction of motion of m is changed without changing the magnitude of the velocity and without changing the length of the radius vector r, the semi-major axis a and the period T of the resulting orbit remain unaltered. [5]

Solution: *no solution available*

3.25e A planet of mass m is in orbit about the Sun, of mass M. The sum of the kinetic and potential energy of the planet in its orbit is a constant, C, given by

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C,$$

where v is the magnitude of the velocity of the planet when its radius vector is r, $\mu = G(M+m)$ and G is the gravitational constant. Show that

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

where a is the planet's orbital semi-major axis. [11]

Halley's Comet moves in an elliptical orbit of eccentricity 0.9673. Calculate the ratio of the magnitudes of (a) its linear velocities, (b) its angular velocities, at perihelion and aphelion. [6]

Solution: *no solution available*

3.26e Explain the importance of conservations laws for dynamical astronomy calculations, and give two examples of quantities that are conserved during simple gravitational interactions. [5]

A body of mass m is in orbit about a planet of mass $M(\gg m)$ taken to be at rest. Write down an expression of the total energy of this system in terms of the body's speed, v, and distance from the centre of the planet, r, and show that

$$v^2 = GM\left(\frac{2}{r} + C\right)$$

where C is a constant for the orbit. [4]

It has been suggested that liquid oxygen fuel pods, mined from the Moon's South Pole, could be launched to waiting spacecraft in orbit at a radius of 2 lunar radii about the Moon. Given that $C = -1/a$, where a is the semi-major axis of the orbit, use the above equation to determine the initial velocity required

(a) launching the pods vertically from the surface, [3]

(b) launching the pods parallel to the surface. [3]

What other major factor would be important to complete the transfer? Explain why (b) might be preferred overall. [2]

Solution: Conservation laws are important because using them we can determine properties of the orbits without performing a full dynamical analysis. The three most important conserved quantities in orbital dynamics are momentum, energy and angular momentum. There are conserved if there are no external forces and the masses can be regarded as point-like. [5]

The kinetic energy of the body is $mv^2/2$ and the potential energy of the system is $-GMm/r$. As the sum of these is a constant, E, we can write

$$v^2 = GM\left(\frac{2}{r} + \frac{2E}{GM^2}\right),$$

and as energy is conserved, $2E/(GM^2)$ is a constant, C. [4]

(a) A vertically-launched pod will reach a maximum altitude at which its speed is zero. In orbital terms it will be in a *very* thin orbit, with $b = 0$ and $2a = 2r_\mathrm{m}$, where r_m is the radius of the Moon. Its velocity in this orbit when close to the lunar surface satisfies

$$v_0^2 = GM\left(\frac{2}{r_m} - \frac{1}{r_\mathrm{m}}\right) = \frac{GM}{r_\mathrm{m}}.$$

(b) The orbit of the pod in this case will be such that the pod will reach maximum altitude on the opposite side of the Moon to the launch, so that $2a = r_\mathrm{m} + 2r_\mathrm{m} = 3r_\mathrm{m}$. Its initial velocity (when $r = r_\mathrm{m}$) must therefore satisfy

$$v_0^2 = GM\left(\frac{2}{r_m} - \frac{2}{3r_\mathrm{m}}\right) = \frac{4}{3}\frac{GM}{r_\mathrm{m}}.$$

So the initial speed in case (b) is $\sqrt{4/3}$ times higher than in case (a). It therefore takes more energy to launch into the (b) orbit and this may make it appear less attractive. However this is only half the story. The fuel is to service a spacecraft travelling around the Moon in a circular orbit so some kind of engine is needed to get the fuel up to the speed of the spacecraft once it has reached the correct altitude. In case (a) the fuel has zero speed when at the correct orbital radius, so its speed would need to be boosted significantly to inject it into a circular orbit for the spacecraft to pick it up easily. In case (b) the fuel already has an orbital speed satisfying

$$v_0^2 = GM\left(\frac{2}{2r_m} - \frac{2}{3r_\mathrm{m}}\right)$$

i.e.,

$$v = \sqrt{\frac{GM}{3r_\mathrm{m}}},$$

so needs less 'Δv' to inject it into the correct orbit.

3.27e The *Venus Express* probe set off from Earth on 9 November 2005, following a Hohmann transfer orbit to the planet Venus. Explain the basic features of a Hohmann transfer orbit, and draw a diagram showing the orbits of Earth, Venus and the probe during this transfer. [5]

Given that Venus has an orbital period of 225 days, use Kepler's Third Law to estimate the day of the year on which you would expect the probe to arrive at Venus. [8]
[You may assume the planetary orbits are circular.]

If we use the astronomical unit (AU) as the unit of distance, and AU/year as the unit of speed, any orbit of semimajor axis a around the Sun obeys the relation

$$v^2 = 4\pi^2\left(\frac{2}{r} - \frac{1}{a}\right),$$

where v is the speed of the orbiting body when a distance r from the Sun. Use this relationship to determine the speeds of Venus and *Venus Express* when the probe reaches the planet. [4]

Solution: *no solution available*

3.28e Carefully draw a labelled diagram showing the elliptical orbit of a satellite around the Earth. Identify on your diagram:

(a) the semimajor axis length, a

(b) the semiminor axis length, b

(c) the eccentricity, e, (as a length ae)

(d) the semi-latus rectum length, r_0

(e) the perigee point

(f) the apogee point. [6]

Show that the sum of the distances from a point on the ellipse to the two foci is $2a$ (given it is the same for any choice of point). Using Pythagoras's Theorem, or otherwise, go on to show that the semi-latus rectum is of length [7]

$$r_0 = a(1 - e^2).$$

A small satellite, in a circular orbit or radius $3R_E$ around the Earth (radius R_E), is given a small radial kick by a laser beam, shot vertically from the Earth's surface, boosting it into an elliptical orbit of eccentricity 0.01. What is the maximum altitude of the satellite above the Earth's surface in its new orbit? [4]

Solution: *no solution available*

3.29e From Kepler's laws of motion and gravitation, derive the equation for Kepler's Third Law of Planetary Motion for a planet in a circular orbit about a much more massive star. [3]

How is this equation modified to account for

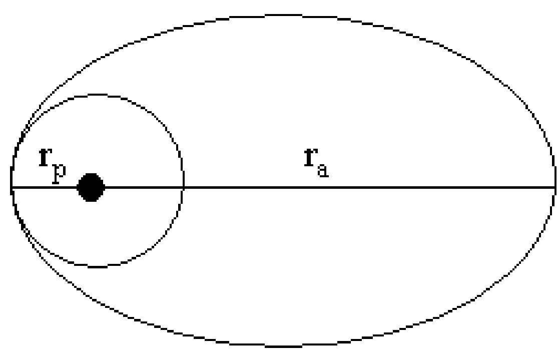
(a) an elliptical orbit,

(b) a binary star system? [2]

Solution: *no solution available*

3.30e A satellite, presently in a low Earth orbit of radius $1.1R_{\mathrm{Earth}}$, needs to be boosted into a Hohmann transfer orbit so that it can reach its final geostationary position at $6.6R_{\mathrm{Earth}}$. Calculate the semimajor axis and the eccentricity of the transfer orbit. [5]

Solution:



$r_p = 1.1R_E$ and $r_a = 6.6R_E$

For transfer orbit: $2a = r_a + r_p = 7.7R_E \Rightarrow \underline{a = 3.85R_E}$.

Also $r_p = a(1-e) \Rightarrow e = 1 - \frac{r_p}{a} = 1 - \frac{1.1R_E}{3.85R_E} = 0.714$ [5]

3.31e Using energy arguments, explain what is meant by the *escape velocity* of a body when applied to orbital motion. [4]

Assuming the planets presently orbit the Sun in circles, show that if the Sun rapidly lost up to half its mass the planets would still move in closed orbits. [5]

The speed (v) of a planet in orbit around a star of mass M is related to its orbital distance (r) and orbitalsemimajor axis (a) by

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right).$$

Use this equation to show that if the Sun instantly lost a fraction f of its mass, reducing its mass from $M_\odot$ to $M_\odot(1-f)$, then the Earth's (originally circular) orbit would have a period of

$$T = \left(\frac{1-f}{1-2f}\right)^{\frac{3}{2}} \text{ years}$$ [8]

[You may assume Kepler's Third Law: orbital period is proportional to orbital semimajor axis to the power 3/2.]

Solution:
Total energy $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$ is conserved, i.e. $\frac{1}{2}mv^2 = E + \frac{GMm}{r}$

If $E > 0$, then $v^2 > 0$ for all r, i.e. the masses are not bound in an orbit. If $E < 0$ then no real solutions exist for v when r is sufficiently large, i.e. the masses are bound together. At the escape velocity $E = 0$: the mass can **just** escape, i.e.

$$\frac{1}{2}mv_{\rm esc}^2 = \frac{GMm}{r} \Rightarrow v_{\rm esc} = \sqrt{\frac{2GM}{r}}$$

[4]

In orbit about the Sun,

$$\frac{mv^2}{r} = \frac{GM_\odot m}{r^2} \quad \text{i.e.} \quad v^2 = \frac{GM_\odot}{r^2}$$

If no escape $v^2 < v_{\rm esc}^2$, i.e. $\frac{GM_\odot}{r} < \frac{2GM}{r}$

where M is the new mass of the Sun i.e. $M > M_\odot/2$ maintains orbits. [5]

If $v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$ and originally $v^2 = \frac{GM_\odot}{R}$ (R= Earth's orbital radius)
Then after the loss of mass

$$\begin{aligned}
\frac{GM_\odot}{R} &= GM_\odot(1-f)\left(\frac{2}{R} - \frac{1}{a}\right) \\
\text{i.e.} \quad 1 &= (1-f)\left(2 - \frac{R}{a}\right) \\
\frac{1}{1-f} &= 2 - \frac{R}{a} \\
\frac{R}{a} &= 2 - \frac{1}{1-f} = \frac{2-2f-1}{1-f} = \frac{1-2f}{1-f} \\
\text{i.e.} \quad \frac{a}{R} &= \frac{1-f}{1-2f}
\end{aligned}$$

Put period $\propto a^{3/2}$, so

$$\frac{T}{\rm yr} = \left(\frac{1-f}{1-2f}\right)^{3/2}$$

[8]

3.32e State Kepler's three laws of planetary motion. [3]

Saturn is about 9.6 times further away from the Sun than is the Earth. Estimate Saturn's orbital period, in years. [2]

Solution: 1. Planets orbit the Sun in ellipses with the Sun at one focus of the ellipse.

2. A line joining a planet to the Sun sweeps out equal areas in equal time.

3. The square of the period of a planetary orbit is proportional to the cube of its orbital semi-major axis. [3]

From (3) $\quad T^2 \propto a^3$, so $\quad \frac{T_S}{T_E} = \left(\frac{a_S}{a_E}\right)^{3/2}$

If $T_E = 1yr$, and $a_E = 1\text{AU} \quad \Rightarrow \quad T_S = (9.6)^{3/2}\text{yr} = 29.7\,\text{AU}$ [2]

3.33e Define the gravitational potential energy, U, in a two-body system. [2]

By considering the work done in changing the radial distance of a test mass in the gravitational field of a planet, show that the gravitational potential energy of a mass, m, at radius r from the centre of a planet of mass M is given by

$$U = -\frac{GMm}{r},$$

where G is the universal constant of gravitation. [4]

What is meant by the escape speed(v_{esc}) from the surface of a planet? Show that the escape speed from a planet of mass M and radius R is given by

$$v_{\text{esc}} = \left(\frac{2GM}{R}\right)^{\frac{1}{2}}.$$

Calculate the escape speeds of the Earth and of Jupiter. [5]

Solution: Gravitation Potential Energy of a particle at distance r from a gravitating body defined as work done in moving particle from $r = \infty$ to the point in question. [2]

$$\begin{aligned}
\text{W.D. moving from} \quad r_1 \quad \text{to} \quad r_2 \quad &= -\int_{r_1}^{r_2} \underline{F}_g d\underline{r} \qquad \underline{f}_g = -\frac{GMm}{r^2}\underline{\hat{r}}_{12} \\
&= GMm \int_{r_1}^{r_2} \frac{1}{r^2} dr \\
&= GMm \left[-\frac{1}{r}\right]_{r_1}^{r_2} \\
&= -GMm \left[\frac{1}{r_2} - \frac{1}{r_1}\right] \\
\text{if} \quad r_1 = \infty, r_2 = r \quad \text{then} \quad U \quad &= -\frac{GMm}{r}
\end{aligned}$$

[4]

v_{esc} = velocity that an object at the surface of a planet must have to escape its gravitational field, i.e. to have $v = 0$ at $r = \infty$.

Use fact that $K_0 + U_0 = \text{const} = K_\infty + U_\infty$.

Hence $\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{R} = 0 \quad \Rightarrow \quad v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$

Escape speed from Earth's surface

$$v_E = \sqrt{\frac{2GM_E}{R_E}} = 11.2\,\text{km/s}$$

From Jupiter

$$v_J = \sqrt{\frac{2GM_J}{R_J}} = 59.5\,\text{km/s}$$

[5]

3.34e Explain what is meant by a 'conserved' quantity in orbital dynamics, and give two examples of conserved quantities. [3]

Write down an expression for the total (kinetic + potential) energy, E, of a body in orbit about a massive body, mass M. Show that at any point in the orbit its speed, v, satisfies

$$v^2 = GM\left(\frac{2}{r} + C\right),$$

where r is the separation of the bodies, G is Newton's constant of gravitation and C is a constant that depends on the energy. [5]

Show that, for a circular orbit of radius a, this constant must equal $-1/a$. [2]

Assuming that $C = -1/a$ for a general elliptical orbit of semimajor axis a, explain why the total energy and the period of an orbit are related. [2]

A satellite orbits the Earth in a circular orbit of radius a_0. Its engines are fired to rapidly increase its speed to the point where its total energy is halved (i.e. is closer to zero). What is the eccentricity of the new orbit? [5]

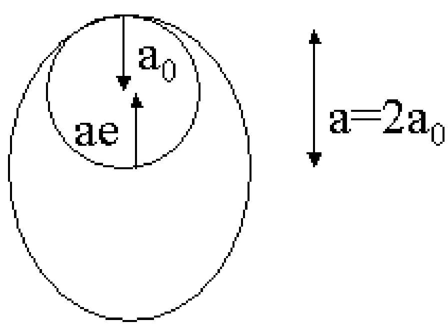
Solution:

Conserved = not changing during motion. E.g. energy, angular momentum, momentum, mass (total for system) [3]

$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \Rightarrow v^2 = GM\left(\frac{2}{r} + \frac{2E}{GMm}\right) = GM\left(\frac{2}{r} + c\right)$ [5]

For a circular orbit $\frac{mv^2}{a} = \frac{GMm}{a^2} \Rightarrow v^2 = \frac{GM}{a}$, so $c = -\frac{1}{a}$ [2]

If $c = -\frac{1}{a}$ then $E = -\frac{GMm}{2a}$ and KIII says that $T^2 \propto a^3$ so $\underline{T \propto E^{-3}}$ [2]



$E_0 = -\frac{GMm}{2a}$; $\quad E = \frac{E_0}{2} \Rightarrow a = 2a_0$

so $ae = a_0 \quad = \frac{a}{2}$ i.e. $e = \frac{1}{2}$ [5]

3.35e Draw an elliptical orbit, identifying the eccentricity, e, semimajor axis, a, and semi latus rectum, r_0, on your diagram. [3]

Explain what is meant by the 'string' definition of an ellipse, and use it to prove that

$$r_0 = a(1 - e^2).$$

[5]

Given that

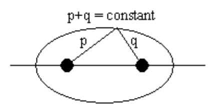
$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

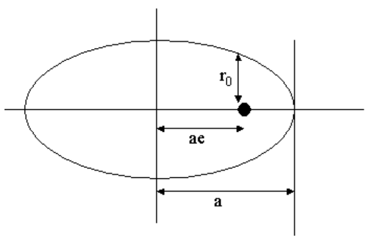
where v is the speed of an orbiting body of mass m a distance r from an attracting mass M, and G is the gravitational constant show, by considering the angular momentum (L) at periastron, that

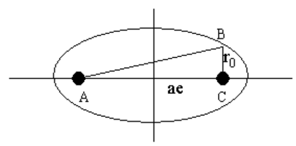
$$L^2 = GMm^2 r_0.$$

Explain the significance of this result for orbital manoeuvres. [9]

Solution:







[3]

String$= 2a$
$AB^2 = r_0^2 + (2ae)2$
also $\quad AB = 2a - r_0$
so $\quad r_0^2 + 4a^2e^2 = 4a^2 - 4ar_0 + r_0^2$
$r_0 = a(1 - e^2)$ [5]

At periastron $\quad r_p = a(1 - e)$
$L = mv_pr_p; \quad v_p^2 = GM\left(\frac{2}{r_p} - \frac{1}{a}\right) \Rightarrow L^2 = m^2r_p^2GM\left(\frac{2}{r_p} - \frac{1}{a}\right)$
$\Rightarrow L^2 = GMm^2(2a(1 - e) - a(1 - e)^2) = GMm^2a(1 - e^2)$
but $\quad a(1 - e^2) = r_0$, so $\quad L^2 = GMm^2r_0$

Any manoeuvre that does not change L will not change r_0, e.g. radial kick. [9]

3.36e Draw a diagram to show the elliptical orbit of a planet around a star. Identify the semimajor and semiminor axes of the orbit. Also identify its semi-latus rectum and describe the significance of this dimension in orbital dynamics. [5]

Given that the sum of the distances from a point on an ellipse to its two foci is constant (the 'string definition'), show that the semi-latus rectum has length

$$r_o = a(1 - e^2),$$

where a is the semimajor axis and e the eccentricity of the ellipse. [5]

A freely floating astronaut of mass m is in a circular orbit of radius R around Earth. Show that the astronaut's angular momentum is

$$L = m\sqrt{GMR},$$

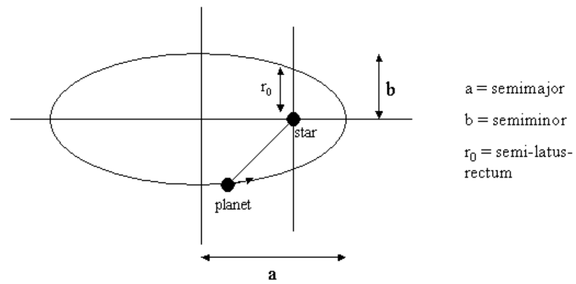
where G is the constant of gravitation and M the mass of the Earth. [3]

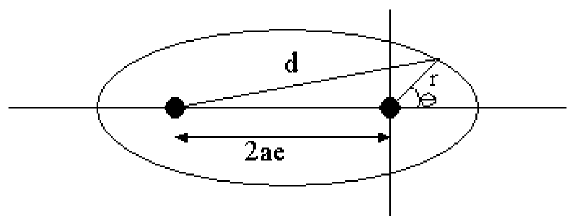
The astronaut throws a screwdriver towards the Earth, giving him a radial kick outwards. Given that his new orbit has an eccentricity e, show he gets no closer to the Earth than $R/(1+e)$ and that his speed at this point is

$$v = (1+e)\sqrt{\frac{GM}{R}}.$$

Solution:

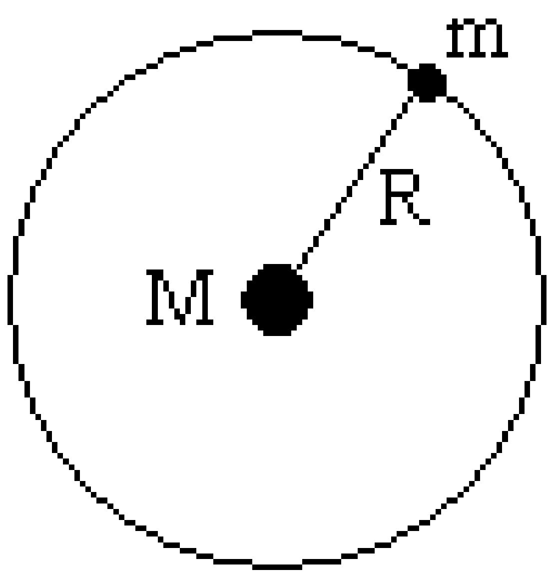
SLR is important because its size depends on the orbital angular momentum. Orbital changes that do not affect L leave r_0 unchanged too. [5]





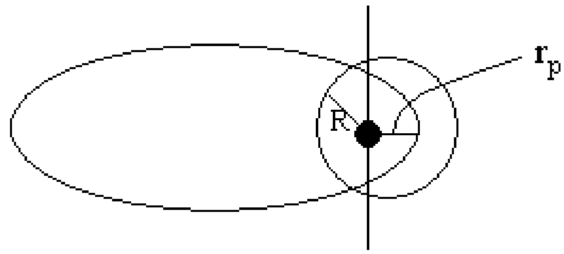
When $\theta = 0$, string length=$ae + a + (a - ae) = 2a$
so $d + r = 2a$

When $\theta = \pi/2$, $4a^2e^2 + r_0^2 = d^2 = (2a - r_0)^2 = 4a^2 - 4ar_0 + r_0^2$
i.e. $\div 4a \quad ae^2 = a - r_0$
$\therefore \quad r_0 = a(1 - e^2)$ [5]



By definition, $L = mvR$, but $\frac{mv^2}{R} = \frac{GMm}{R^2}$

so $m^2v^2R^2 = GMm^2R$, and $L = m\sqrt{GMR}$ [3]



Radial kick means L is unchanged, so $r_0 = R = a(1-e^2)$, i.e.

$$a = \frac{R}{(1-e^2)}$$

$$\begin{aligned} \text{perigee distance} = r_p &= a(1-e) = \frac{R}{1-e^2}(1-e) \\ &= R/(1+e) \end{aligned}$$

Conserving L: $mv_pr_p = m\sqrt{GMR} \Rightarrow v_p = \frac{1}{r_p}\sqrt{GMR} = (1+e)\sqrt{\frac{GM}{R}}$ [4]

4 Stellar Astrophysics part I

4.1e What are the main differences between population I stars and population II stars? Why should we expect most of the population I stars to be in the galactic plane? [5]

Solution:

Population I **stars** have a ratio (H+He)/metals similar to that in the Sun. The stars are young, 'second generation' stars, largely made from material associated with ejecta from older stars. They formed late in the evolution of the galaxy, and are therefore predominantly in the galactic plane.

Population II **stars** have a ratio (H+He)/metals about 100 times greater than is seen in the Sun (they are 'metal poor'). These stars were formed before the galaxy was a disc and can be found in the galactic halo.

4.2e Write down Rydberg's equation, giving the wavelengths of spectral lines from atomic hydrogen. Calculate the wavelengths of the first three spectral lines in the hydrogen Balmer series (i.e., Hα, Hβ and Hγ). [5]

Solution: Rydberg's equation is (for atomic hydrogen)

$$\frac{1}{\lambda} = R_\infty \left(\frac{1}{m^2} - \frac{1}{n^2} \right) ,$$

where λ is the wavelength of the radiation associated with the transition, R_∞ is Rydberg's constant and m and n are integers.

The Balmer series has $m = 2$, so

$$\frac{1}{\lambda_n} = R_\infty \left(\frac{1}{2^2} - \frac{1}{n^2} \right) .$$

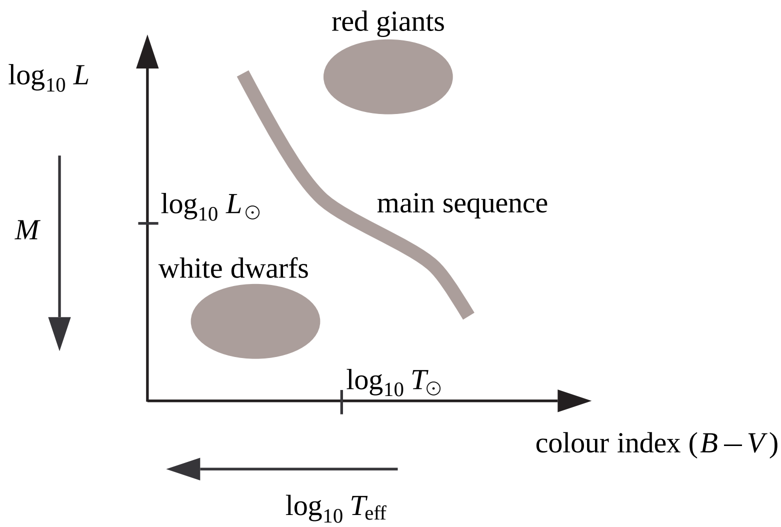
For Hα, we set $n = 3$, giving $\lambda_3 = 36/(5R_\infty) = 656.3\,\text{nm}$.

For Hβ, we set $n = 4$, giving $\lambda_4 = 16/(3R_\infty) = 486.2\,\text{nm}$.

For Hγ, we set $n = 5$, giving $\lambda_5 = 100/(21R_\infty) = 434.1\,\text{nm}$.

4.3e Draw a sketch of the general Hertzsprung-Russell diagram for stars. Label the axes suitably and clearly mark those regions corresponding to main sequence stars, red giants and white dwarfs. [5]

Solution: The Hertzsprung-Russell diagram:



4.4e Briefly, explain what are:

(a) Cepheid variables
(b) T Tauri stars
(c) eclipsing binary stars
(d) planetary nebulae? [5]

Solution:

Cepheid variables are giant, variable stars whose luminosity varies regularly on timescales of 1∼100 days. The period of the variability depends critically on the star's average luminosity.

T Tauri stars are irregular variables. They are pre-main sequence and of low mass (0.5 to $3\,M_\odot$) which eject mass as they flare in luminosity, often forming bipolar outflows. The stars are usually embedded in the gas and dust from which they formed.

Eclipsing binary stars are stars in a binary system which we see nearly 'edge on', so that the stars regularly eclipse each other. They are identified by their characteristic light curves.

Planetary nebulae are gas shells, blown off low-mass stars during their dying phases. about half the mass of the star can be separated, and the shell is illuminated by the exposed hot core of the star.

4.5e If we assume that a star radiates as a blackbody, what is the relationship between its luminosity, radius and temperature? [5]

Solution: A blackbody is an object that absorbs all wavelengths of light falling on it. A blackbody at temperature T obeys the Stefan-Boltzmann law:

$$\text{Power emitted per unit area} = \sigma T^4 \ ,$$

where σ is the Stefan-Boltzmann constant. A star of radius R has a surface area of $4\pi R^2$, so the total luminosity of the star is

$$L = 4\pi R^2 \sigma T^4 \; .$$

Hence the radius, temperature and luminosity of a star (if it is assumed to radiate as a blackbody) are related by this equation.

4.6e Explain how measurements of a star's colour index and spectral type can each be used to determine its effective temperature. [5]

Solution:

- The colour index of a star is a measure of the power flux from the star in adjacent spectral bands. Hence
$$B - V = m_B - m_V = -2.5 \log_{10} \left(\frac{F_B}{F_V} \right) \; ,$$
where F_B is the flux in the B-band and F_V is the flux in the V-band. Hot stars are bluer than cool stars so F_B/F_V increases with temperature. Therefore the $B-V$ colour index decreases as the temperature increases. As a rule-of-thumb for an average star of temperature $T_{\rm eff}$ kelvin, $B - V = -0.56 + 7000/T_{\rm eff}$.

- The spectral type of a star is defined by the ratio of absorption line strengths in the stellar spectrum. These lines are very sensitive to the gas temperature in the outer regions of the star, and the classification sequence is a sequence of effective temperature. Hence O stars are hot and M stars are cool.

4.7e Write brief notes on any three of the following topics, highlighting their context within stellar astrophysics:

(a) The determination of stellar mass
(b) The determination of the effective surface temperature of stars
(c) Line production and the spectral classification of stars
(d) Stellar evolution off the main sequence
(e) Variable stars. [17]

Solution:

Stellar mass Notes should include: Importance of binary systems. Use of Kepler's law, $R^3/T^2 = G(m_1 + m_2)/(4\pi^2)$, and the position of centre of mass ($m_1 r_1 = m_2 r_2$). Measurement of the orbital elements by direct observation, spectral analysis (spectroscopic binaries) and eclipsing binaries. Complications when orbits are elliptical and observations are not made in the orbital plane.

Effective surface temperature of stars Notes to include: blackbody approach using the Stefan Boltzmann equation, $L = 4\pi R^2 \sigma T^4$, spectral fitting to Planck curves, the use of colour index as a measure of temperature and the use of spectral classification. Some mention of why line strengths are affected by temperature. The realisation that a stellar surface is not all at the same temperature. The idea of effective temperature as a measure of L/R^2.

Line production Ideas include:

- Spectra as chemical 'fingerprints' identifying the element species present in the stellar atmosphere.
- Line strengths dependent on the number of atoms present and the gas temperature.
- Wavelengths and widths of lines as measures of stellar rotation, large scale expansion/contraction, temperature and pressure.
- Bohr Model, $1/\lambda = R_\infty(1/m^2 - 1/n^2)$, for hydrogen.
- Importance of absorption spectra from chromosphere.
- Importance of an 'activation energy' to prepare atoms in the right state for absorption (e.g., $n = 2$ for Balmer series) leading to an absence of Balmer lines in cool stars.
- Effects of ionisation at high temperatures suppressing absorption lines. Spectral classification (OBAFGKM) as a series of descending T. Molecular lines at low T, He ii lines at high T.

Stellar evolution Low mass stars: End of H $\rightarrow$ He reactions, triple-α process and the helium flash. Increase in core temperature and luminosity, inflation of outer envelope … red giant. Shell He burning leading to Red supergiants. Planetary nebulae. White dwarf.

High mass stars: More nuclear reactions due to greater pressures and temperatures in the core. Creation of variable stars in the instability strip. Shell burning and the 'onion skin' model. Endothermic fusion for species more massive than Fe leading to a drop in core pressure and supernova.

Variable stars Including Real variables: Novae (cataclysmic variables) T-Tauri stars (irregular), Cepheid variables. Use of Cepheids as distance indicators. Eclipsing binary stars (apparent variables) and their light curves, etc.

4.8e The relationship between the mass, M, and luminosity, L, of a lower-main sequence star is roughly $L \propto M^{3.5}$. If a star begins to leave the main sequence when 20 % of its hydrogen has been converted into helium, estimate the time spent on the main sequence by a star of $10M_\odot$ and one of $0.1M_\odot$, given that the proton-proton reaction converts 0.7 % of the mass of the hydrogen fuel actually used into energy. You may assume that the star is initially 100 % hydrogen. [10]

Discuss the significance of your result and use it to describe how the Hertzsprung-Russell diagram for a cluster of stars (all of the same age, but differing in mass) evolves over time. [7]

Solution: The energy produced is

$$E = \Delta mc^2 = \underbrace{0.2}_{20\%} \times \underbrace{M}_{\text{stellar mass}} \times c^2 \times \underbrace{0.007}_{\text{yield}} . \qquad [2]$$

Write this as

$$E = fMc^2 ,$$

where $f = 1.4 \times 10^{-3}$. We are told that $L/L_\odot = (M/M_\odot)^{3.5}$ (the mass luminosity relation), and we know that if the lifetime of the star is τ and its luminosity, L, is take as constant for this period, then

$$L \simeq \frac{E}{\tau} = \frac{fMc^2}{\tau} . \qquad [2]$$

Therefore

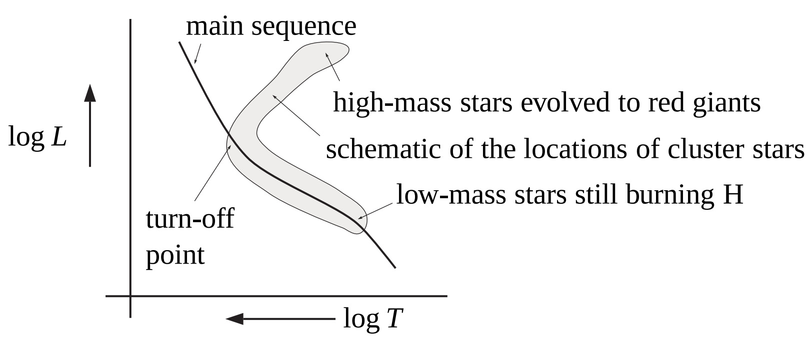
$$\tau = \frac{fMc^2}{L} = \frac{fMc^2}{L_\odot}\left(\frac{M}{M_\odot}\right)^{-3.5} = fc^2\frac{M_\odot}{L_\odot}\left(\frac{M}{M_\odot}\right)^{-2.5}. \qquad [2]$$

Using the numbers given:

$$\tau = 6.3\times10^{17}(M/M_\odot)^{-2.5}\,\text{sec} \quad \simeq 2\times10^{10}(M/M_\odot)^{-2.5}\,\text{yr} \qquad [2]$$

Therefore, if $M = 10M_\odot$, $\tau = 6.3\times10^7$ years, and if $M = 0.1M_\odot$, $\tau = 6.3\times10^{12}$ years. [2]

Low-mass stars spend a great deal longer on the main sequence than do high-mass stars (10^5 times longer in this case). Stars in a cluster have similar ages, but different masses. We would therefore expect the more massive stars (at the top of the Hertzsprung-Russell diagram) to evolve off the main sequence well before the low-mass stars at the bottom of the main sequence.



The location of the turn-off point identifies the age of the cluster, as calculated above. A star will stay on the main sequence for considerably longer than its life as either a protostar or a giant. [7]

4.9 (a) Given that the Sun-Earth distance is 1.5×10^8 km, the angular diameter of the Sun is 32 arcmin, and the solar constant is 1.36×10^3 W m^{-2}, determine the Sun's luminosity, $L_\odot$, and effective surface temperature T_e. $\{3.85\times10^{26}$ W; 5 771 K$\}$ [7]

(b) The luminosity of a star is calculated, from its apparent magnitude and distance, to be $16L_\odot$. Its surface temperature is estimated to be 5 000 K. Estimate the radius of the star. [6]

(c) A $0.8M_\odot$ white dwarf star has an effective temperature of 5 000 K and luminosity of 10^{24} W. Calculate the radius and mean density of the star. $\{4.7\times10^4$ km; 3.68×10^6 kg m$^{-3}\}$ [7]

Solution:

(a) If the flux of solar energy at a radius r is F, then the luminosity of the Sun is $L_\odot = 4\pi r^2 F$. Setting $F = 1.36\times10^3$ W m^{-2} and $r = 1.5\times10^{11}$ m gives $L_\odot = 3.85\times10^{26}$ W.

The radius of the Sun is given by $R_\odot = r\theta/2$, where θ is the angular diameter of the Sun in radians, i.e.,

$$\theta = \frac{32}{60}\times\frac{1}{360}\times2\pi.$$

Therefore $R_\odot = 6.98\times10^8$ m. Using Stefan's Law, $L_\odot = (4\pi R_\odot^2)\sigma T_\odot^4$ we get

$$T_\odot = \left(\frac{L_\odot}{4\pi R_\odot^2\sigma}\right)^{1/4} \simeq 5771\,\text{K}.$$

(b) For the star

$$L = 16L_{\odot} = 4\pi R^2 \sigma T^4.$$

For the Sun, $L_{\odot} = 4\pi R_{\odot}^2 \sigma T_{\odot}^4$. Divide the first equation by the second to get

$$16 = \left(\frac{R}{R_{\odot}}\right)^2 \left(\frac{T}{T_{\odot}}\right)^4,$$

$$\text{or,} \qquad R = 4\left(\frac{T_{\odot}}{T}\right)^2 R_{\odot},$$

i.e., $R = 4 \times \left(\frac{5\,771}{5\,000}\right)^2 R_{\odot} = 5.38 R_{\odot}$.

(c) As above,

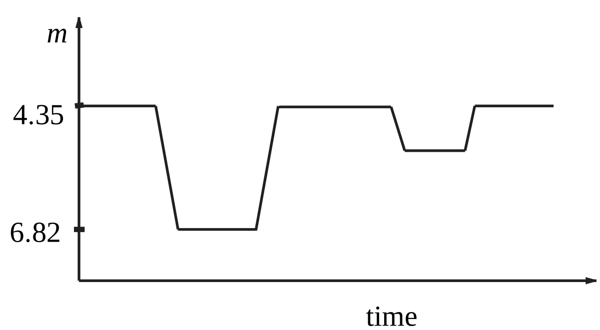
$$R = \left(\frac{L}{4\pi \sigma T^4}\right)^{1/2},$$

where T is the effective temperature of the white dwarf. Inserting the numbers gives $R = 4.7 \times 10^4$ km. Taking the mass of the Sun to be $M_{\odot} = 2 \times 10^{30}$ kg, the mean density of the white dwarf is

$$\rho_{\text{mean}} = \frac{0.8 \times 2 \times 10^{30}}{4\pi R^3/3} = 3.68 \times 10^6 \,\text{kg}\,\text{m}^{-3}.$$

4.10 An eclipsing binary has a constant apparent magnitude 4.35 between minima and apparent magnitude 6.82 at primary minimum. Assuming that the eclipse is total at primary minimum, calculate the apparent magnitudes and the relative fluxes received from the two components. (Hint: remember that the flux, F, received from a star is related to its apparent magnitude, m, by $m = -2.5 \log_{10}(F/F_0)$, where F_0 is the flux from a zero magnitude star.) {0.116}

Solution:



Let m_A and m_B, F_A and F_B be the apparent magnitudes and fluxes of the two components A and B of the binary. Then $F_A = F_0 \times 10^{-0.4 m_A}$, $F_B = F_0 \times 10^{-0.4 m_B}$, and $F = F_0 \times 10^{-0.4m}$, where m and F are the apparent magnitude and flux of the binary between minima.

$$\text{Then} \quad F = F_A + F_B \quad \text{also} \quad F = F_0 \times 10^{-0.4 \times 4.35} = F_0 \times 10^{-1.740}. \tag{16}$$

But at primary minimum star B is eclipsed $\therefore \quad m_A = 6.82$.

Then

$$F_{\mathrm{A}} = F_0 \times 10^{-0.4\times 6.82} = F_0 \times 10^{-2.728}$$

Thus, from $F = F_{\mathrm{A}} + F_{\mathrm{B}}$, we have

$$F_0 \times 10^{-1.740} = F_0 \times 10^{-2.728} + F_0 \times 10^{-0.4 m_{\mathrm{B}}},$$

$$\text{or} \quad 10^{-0.4 m_{\mathrm{B}}} = 10^{-1.740} - 10^{-2.728} = 0.018 - 0.002 = 0.016.$$

Hence $m_{\mathrm{B}} = 4.48$, $\therefore$ the magnitudes of the components are 6.82 and 4.48. The ratio of their fluxes is $\frac{F_{\mathrm{A}}}{F_{\mathrm{B}}} = 10^{-0.4(6.82-4.48)} = 10^{-2.012} = 0.116$.

4.11 Two stars are seen to be in a circular binary system that is eclipsing, indicating that we are seeing them almost in the plane of their orbit. An astronomer observes the Doppler shifts of spectral lines from the stars and deduces the following radial (i.e., line-of-sight) velocities for the stars as a function of time:

Time (years)	v_{r} of star 1 ($\mathrm{km\,s^{-1}}$)	v_{r} of star 2 ($\mathrm{km\,s^{-1}}$)
0.0000	40	40
0.0005	130	15
0.0010	160	0
0.0015	130	15
0.0020	40	40
0.0025	−50	65
0.0030	−80	80
0.0035	−50	65
0.0040	40	40
0.0045	130	15
0.0050	160	0
0.0055	130	15
0.0060	40	40

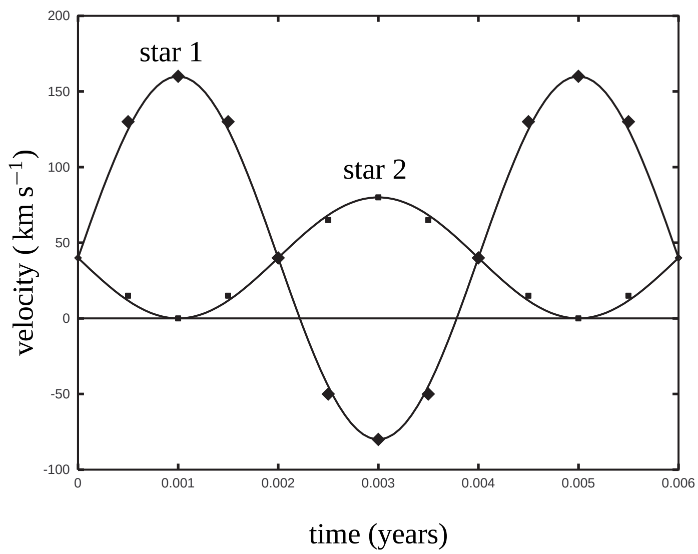
(a) On one graph, plot the velocity curves of the two stars as a function of time. Hence determine the period of the system, P, and from the mean velocities of the stars the recession velocity of the whole system, v_0. {1.461 days; 40 $\mathrm{km\,s^{-1}}$} [5]

(b) Show that the separation of two stars in such a system can be written as $R = P(v_1 + v_2)/(2\pi)$, where v_1 and v_2 are the orbital speeds of the stars, and determine this separation. (Remember to take account of v_0.) {0.02 AU} [5]

(c) Show also that the separation of the stars is related to the sum of their masses, m_1 and m_2 by $R^3/P^2 = G(m_1 + m_2)/(4\pi^2)$, where G is the gravitational constant. Hence determine the total mass of the system. [5]

(d) Finally, show that the ratio of the masses equals the reciprocal ratio of their orbital speeds. From this and the result of part (c) above, calculate the individual masses of the two stars. [5]

Solution: Graph as below (sinusoids fitted)



(a) The period is clearly 0.004 years (= 1.461 days). The whole system has a recession velocity of $v_0 = 40\,\mathrm{km\,s^{-1}}$, and the radial velocities of the two stars vary about this mean value.

(b) If the angular velocity of the system is ω, then $v_1 = r_1\omega$ and $v_2 = r_2\omega$ where r_1 and r_2 are the distances of the two stars from their common centre of mass. Summing these we get $(v_1 + v_2) = (r_1 + r_2)2\pi/P$, so $(r_1 + r_2) = R = P(v_1 + v_2)/(2\pi)$.

If $P = 1.26 \times 10^5$ s, $v_1 = 1.2 \times 10^5\,\mathrm{m\,s^{-1}}$, and $v_2 = 4 \times 10^4\,\mathrm{m\,s^{-1}}$, then $R = 3.21 \times 10^9$ m = 0.02 AU. Note that the orbital speeds of the stars equals their maximum radial velocity along our line of sight.

(c) Using Newton's Second Law:

$$\omega^2 r_1 = \frac{Gm_2}{R^2}, \quad \text{and} \quad \omega^2 r_2 = \frac{Gm_1}{R^2}.$$

Adding these we get

$$\omega^2 = \frac{G(m_1 + m_2)}{R^3},$$

i.e.,

$$\frac{R^3}{P^2} = \frac{G(m_1 + m_2)}{4\pi^2}.$$

Using the figures above we get $m_1 + m_2 = 1.23 \times 10^{30}\,\mathrm{kg} = 0.62 M_\odot$.

(d) From (b) we can see that

$$\frac{v_1}{v_2} = \frac{r_1}{v_2} = \frac{m_2}{m_1},$$

using the facts that the stars share the same angular frequency and that the centre of mass is at $m_1 r_1 = m_2 r_2$. From the graph $m_2/m_1 = 3$. If we set $a = m_1/m_2 = 1/3$ and $b = m_1 + m_2 = 1.23 \times 10^{30}$ kg then:

$$m_1 = \frac{ab}{(1 + a)} = 3.08 \times 10^{29}\,\mathrm{kg},$$

and
$$m_2 = \frac{b}{(1+a)} = 9.22 \times 10^{29}\ \text{kg}.$$

4.12e What is the Stefan-Boltzmann equation, relating the power emitted from a blackbody per unit surface area to its temperature, T? Explain how the relationship can be used to relate the temperature, radius and luminosity of a star. [6]

Explain how an eclipsing binary system can be identified from its characteristic light curve, distinguishing between the *primary* and *secondary* eclipse. [6]

An eclipsing binary system comprises two stars of temperatures $T_1 > T_2$ and radii $R_1 < R_2$. Assuming that the stars can be modelled as uniformly bright flat discs (area πR^2) in the plane of the sky, show that the temperatures of the stars are related by
$$\frac{F - F_\text{p}}{F - F_\text{s}} = \frac{T_1^4}{T_2^4},$$
where F_p and F_s are the fluxes received from the whole system during the primary and secondary eclipses, and F is the flux received when there is no eclipse. [5]

Solution: The Stefan Boltzmann equation states that the power emitted per unit area from a blackbody of temperature T is σT^4, where σ is the Stefan Boltzmann constant. This power per unit area has dimensions of flux, and is sometimes called the *exitance* to distinguish it from the flux received from the blackbody.

Assuming the star is a uniform spherical blackbody the luminosity of the star, L, is simply the exitance times the star's surface area, i.e.
$$L = 4\pi r^2 \sigma T^4,$$
where r is the stellar radius. In practice stars are not perfect blackbodies, but we can still define an effective temperature $T_\text{e} = L/(4\pi r^2 \sigma)^{1/4}$ which is closely related to the average photospheric temperature.

Eclipsing binaries: Take two gravitationally bound stars, star 1 which is small and hot and star 2 that is large and cool, and observe them in the plane of their orbit. The received flux from this star system will vary over time as the two stars eclipse each other. When the larger star (2) is in front of the smaller star (1), 1 is totally hidden and the total flux drops to a minimum – the *primary eclipse*. When 1 is in front of 2, some of 2's surface is hidden and the flux also drops. However in this *secondary eclipse* the drop is not as great. Some flux is lost from the hidden section, but the total contribution from 1 is greater than what is lost (as 1 is hotter than 2) so the drop is not as great as in the primary eclipse. Between eclipses we see the sum of the fluxes of the individual stars.

The two stars are very nearly the same distance away from us, so their fluxes are simply proportional to their effective luminosities, once eclipsing is taken into account. The flux of each will therefore be proportional to σT^4 times their unobstructed surface area. Therefore if the stars are modelled as flat discs in the sky of radius r_1 and r_2, the unobstructed flux from each will be $F_1 = ar_1^2 T_1^4$ and $F_2 = ar_2^2 T_2^4$ where a is a constant. The total unobstructed flux is then
$$F = a(r_1^2 T_1^4 + r_2^2 T_2^4).$$
At primary eclipse, only star 2 is in view, so
$$F_\text{p} = a(r_2^2 T_2^4).$$

At the secondary eclipse we see all of star 1 and all but a star-1-sized bit of star 2, so

$$F_\mathrm{s} = ar_1^2T_1^4 + (r_2^2 - r_1^2)T_2^4.$$

Therefore

$$\frac{F - F_\mathrm{p}}{F - F_\mathrm{s}} = \frac{r_1^2T_1^4}{r_1^2T_2^4} = \frac{T_1^4}{T_2^4}.$$

4.13 (a) Distinguish between the spectrum of a typical O star and a typical red giant. What information about the nature of these stars would you be able to deduce from a study of their spectra? [7]

(b) Helium shows an absorption line at a wavelength of 447.1 nm due to the excitation of an electron in the atom already at an energy level of −3.5 eV (i.e., 3.5 eV below the ionisation level). Calculate the energy of the electron after absorption. {−0.73 eV} [7]

(c) The ionisation potential of He is 24.47 eV. At what temperature would you expect to observe the appearance of singly ionised helium (He ii) lines, and why? {15 800 K} [6]

Solution:

(a) O stars are hot so spectrum peaks in blue wavelengths. Few lines – strong He ii absorption (and sometimes emission) lines. Red giants are cool (spectral type K or M) so spectrum peaks in red wavelengths. Strong neutral metal absorption lines (e.g. Ca, H, K). Coolest (M type) also show molecular absorption bands (e.g. TiO).

Spectra can tell us about stars'

- chemical composition (from line species present),
- temperature (from relative line strengths),
- luminosity and/or pressure (from line widths),
- bulk mass motion (from Doppler shifts).

Any 7 correct statements for 7 marks, but no more than 4 from each half of the section.

(b) The He atom absorbs a photon of wavelength 447.1 nm, with an energy of $E = h\nu = \frac{hc}{\lambda} = \frac{6.625\times10^{-34}\times3\times10^{8}}{447.1\times10^{-9}} = 4.44\times10^{-19}$ J $= 2.77$ eV, so the final energy of the electron is $-3.5 + 2.77$ eV $= -0.73$ eV.

7 marks

(c) Singly ionised He lines (He ii) will be observed when there are a significant number of singly ionised helium ions present. A 'rule of thumb' for the Saha Equation, as presented in the lectures, is that a species with an ionisation energy of χ will be about 50 % ionised at a temperature satisfying $kT \simeq \frac{\chi}{18}$. For He ii this is a temperature of $T = \frac{24.47\times1.602\times10^{-19}}{18\times1.38\times10^{-23}} =$ 15 800 K. So a reasonable approximation for the temperature above which we might expect to observe singly ionised helium lines is 15 800 K.

(The lines will increase in strength with increasing temperature, until the temperature becomes so high that doubly ionised – He iii – helium ions start to become dominant.
Give credit for a statement like this, but not required for full marks from this section.)

6 marks

4.14 A star has a mass of $6M_\odot$.

(a) Show, by a sketch, how energy is transported in the interior of this star during its main sequence phase. [3]

(b) Describe briefly the nuclear reaction that powers a star of this mass. Why do stars of a much lower mass fuse hydrogen via a different chain reaction? [4]

(c) Describe in outline the formation of such a star from a dense cloud. Your description should finish when the star joins the main sequence. How will its luminosity vary during this process? [10]

(d) The reactions of the CNO cycle are:

$$\begin{aligned}
{}^{12}\mathrm{C} + \mathrm{p}^+ &\to {}^{13}\mathrm{N} + \gamma \\
{}^{13}\mathrm{N} &\to {}^{13}\mathrm{C} + \mathrm{e}^+ + \nu \\
{}^{13}\mathrm{C} + \mathrm{p}^+ &\to {}^{14}\mathrm{N} + \gamma \\
{}^{14}\mathrm{N} + \mathrm{p}^+ &\to {}^{15}\mathrm{O} + \gamma \\
{}^{15}\mathrm{O} &\to {}^{15}\mathrm{N} + \mathrm{e}^+ + \nu \\
{}^{15}\mathrm{N} + \mathrm{p}^+ &\to {}^{12}\mathrm{C} + {}^{4}\mathrm{He}
\end{aligned}$$

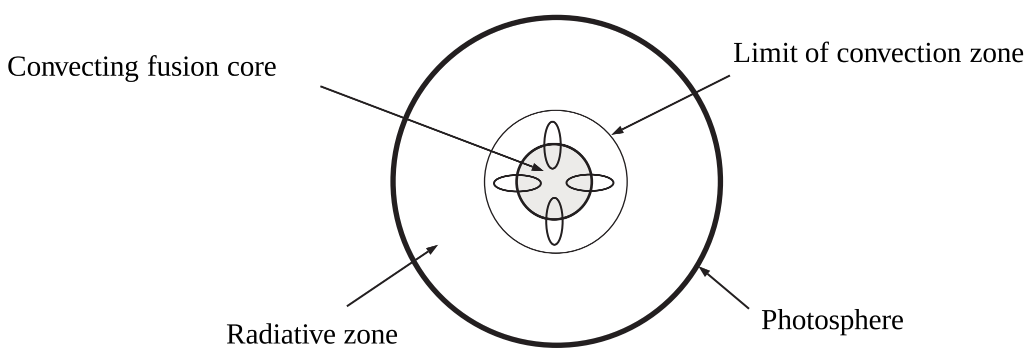
Show that the overall reaction is

$$4\mathrm{p}^+ \to {}^{4}\mathrm{He} + 2\mathrm{e}^+ + 3\gamma + 2\nu,$$

and that ${}^{12}\mathrm{C}$ acts as a catalyst. [3]

Solution:

(a) Sketch of the interior of a $6M_\odot$ star:



(b) In a main sequence star of mass $6M_\odot$ the CNO cycle will predominate. This produces ^{4}He from ^{1}H ($= \mathrm{p}^+$) via changing abundances of various isotopes of C, N and O which collectively act as catalysts. If the mass of the star is less than about $1.5M_\odot$, the CNO cycle cannot proceed because the core temperature is not high enough. In these stars fusion occurs via the proton-proton (p-p) chain. This chain is less efficient than the CNO cycle, but can occur at temperatures as low as 4×10^6 K. Both chains have a strong temperature dependence. In particular, the rate of reaction of the CNO chain is proportional to something like T^{17}. This leads to convection in the core, as the heat generation in the inner core far exceeds the heat generation in the outer core.

(c) Answers should include the following points:

- Trigger, e.g., supernova, spiral density wave, close approach star-starts a dense cloud contracting.
- Dense cloud fragments, and each fragment contrives to contract. As it contracts, temperature tends to rise.
- Initially, energy escapes.
- Then escaping radiation trapped, and the temperature, T, rises steeply. After a few thousand years T rises to 2-3 000 K.
- Fragment is now protostar.
- Core temperature increases further, until nuclear reactions begin. This is now a main sequence star.

The change in luminosity will be similar to that of the $5M_\odot$ in the lecture handout showing Hayashi tracks i.e., initial small drop and small sharp rise followed by gradual rise then a small drop on to the main sequence.

(d) This can be done either by adding up all the left hand sides and all the right hand sides and cancelling all nuclei common to both, or by noting that each nuclide of N and O produced in one reaction is consumed in a subsequent reaction.

^{12}C takes part in the reaction but is not consumed; hence it is a catalyst.

4.15 One hundred years ago astronomers accepted the idea that the origin of the Sun's luminosity was gravitational contraction. Assume that the gravitational potential energy of the Sun is given by

$$\Omega = -\frac{\alpha G M_\odot^2}{R_\odot},$$

where α is a numerical constant. Hence, using the virial theorem, show that the luminosity would be given by

$$L_\odot = \frac{\Omega}{2R_\odot}\frac{\mathrm{d}R_\odot}{\mathrm{d}t}.$$

Taking $\alpha = 1$, and the given values for $L_\odot$, $M_\odot$ and $R_\odot$, work out the change in radius over 1 year.

Solution: If the Sun's radius changed by $\Delta R_\odot$, its gravitational potential energy would change by approximately

$$\Delta\Omega \simeq \frac{\mathrm{d}\Omega}{\mathrm{d}R_\odot}\Delta R_\odot = \frac{\alpha G M_\odot^2}{R_\odot^2}\Delta R_\odot = -\frac{\Omega\Delta R_\odot}{R_\odot}. \quad (1)$$

According to the virial theorem the change in kinetic (thermal) energy of the Sun would be

$$\Delta K = -\frac{\Delta\Omega}{2}$$

Therefore the change in total energy of the Sun is given by

$$\Delta E_{\text{tot}} = \Delta K + \Delta\Omega = \frac{\Delta\Omega}{2}.$$

Thus $-\frac{\Delta\Omega}{2}$ must be radiated away.
If this takes time Δt we have

$$L_\odot \simeq -\frac{\Delta\Omega}{2\Delta t}$$

and as $\Delta t \to 0$ using (1)

$$L_\odot = -\frac{1}{2}\frac{d\Omega}{dt} = \frac{\Omega}{2R_\odot}\frac{dR_\odot}{dt}. \tag{2}$$

To estimate decrease in radius over one year we use equation (2)

$$dR_\odot \simeq \frac{dR_\odot}{dt}\Delta t = \frac{2L_\odot R_\odot}{\Omega}\Delta t = -\frac{2L_\odot R_\odot^2}{GM_\odot^2}\Delta t \qquad (\alpha = 1)$$

Substituting $\Delta t = 365 \times 24 \times 60 \times 60\,\mathrm{s}$; $L_\odot = 4 \times 10^{26}\,\mathrm{W}$; $R_\odot = 7 \times 10^{8}\,\mathrm{m}$; $G = 6.7 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$; $M_\odot = 2 \times 10^{30}\,\mathrm{kg}$ we have $\Delta R_\odot = -46\,\mathrm{m}$ i.e. radius has to shrink by 46 m, which is not perceptible.

4.16 Describe the main features of the Hertzsprung-Russell diagram, and show on your diagram the position of

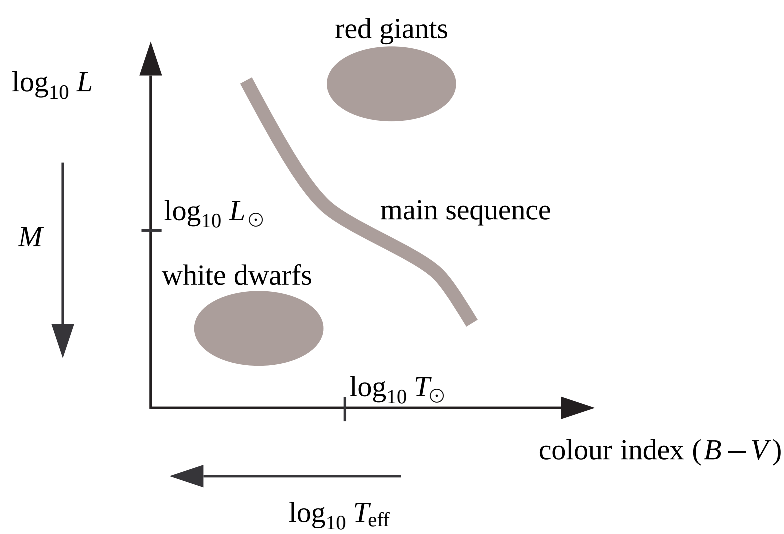
(a) Main Sequence stars

(b) Red Giant stars

(c) White Dwarf stars.

White dwarf stars have absolute magnitude of between 6 and 8. What will be the range of apparent magnitude of such stars in a globular cluster at a distance of 10 kpc?

Solution: The Hertzsprung-Russell diagram:



Using the distance modulus equation (with distance, r, in parsecs)

$$m = M + 5\log r - 5 \tag{1}$$

For $M = 6$, $m = 6 + 5 \times 4 - 5 = 21$
For $M = 8$, $m = 8 + 5 \times 4 - 5 = 23$

4.17 Many astrophysical objects (e.g., solar and stellar flares) attain such high temperatures that even heavy elements like iron (atomic number 26) are almost completely ionised. Using the Bohr theory, calculate the wavelengths of the first four spectral lines and the series limit for electrons undergoing transitions to the ground state of 25 times ionised iron (i.e., iron which has lost 25 of its 26 electrons). What part of the electromagnetic spectrum are these lines in?

The series limit of a set of lines from 25 times ionised iron occurs at 2.16 nm. To which level are the transitions in this series taking place?

Solution: Rydberg formula for 25 times ionised iron is

$$\frac{1}{\lambda_{mn}} = R_\infty Z^2 \left(\frac{1}{n^2} - \frac{1}{m^2}\right) \tag{1}$$

For the ground state, $n = 1$, first four lines correspond to $m = 2, 3, 4, 5$. Taking $Z = 26$, $R_\infty = 1.097 \times 10^7\ \mathrm{m}^{-1}$. Plugging into the formula, this gives

$$\frac{1}{\lambda_{21}} = R_\infty Z^2 \left(\frac{1}{1} - \frac{1}{2^2}\right) \to \lambda_{21} = 1.798 \times 10^{-10}\,\mathrm{m} \tag{1}$$

$$\frac{1}{\lambda_{31}} = R_\infty Z^2 \left(\frac{1}{1} - \frac{1}{3^2}\right) \to \lambda_{31} = 1.517 \times 10^{-10}\,\mathrm{m} \tag{1}$$

$$\frac{1}{\lambda_{41}} = R_\infty Z^2 \left(\frac{1}{1} - \frac{1}{4^2}\right) \to \lambda_{41} = 1.438 \times 10^{-10}\,\mathrm{m} \tag{1}$$

$$\frac{1}{\lambda_{51}} = R_\infty Z^2 \left(\frac{1}{1} - \frac{1}{2^2}\right) \to \lambda_{51} = 1.405 \times 10^{-10}\,\mathrm{m} \tag{1}$$

These lines are in the X-ray part of the electromagnetic spectrum.

Series limit at 2.16 nm = 2.16×10^{-9} m. Hence

$$\frac{1}{2.16 \times 10^{-9}} = R_\infty Z^2 \left(\frac{1}{n^2}\right) \to n = 4 \tag{2}$$

4.18 A spherical star of mass M and radius R is in hydrostatic equilibrium.

(a) If the density is given by
$$\rho = \rho_{\mathrm{c}} \left(1 - \frac{r}{R}\right),$$
where ρ_{c} is the central density, show (by determining the mass, M, of the star) that
$$\rho_{\mathrm{c}} = \frac{3M}{\pi R^3}.$$

(b) Use the equation of hydrostatic equilibrium to find the central pressure, p_{c}, of the star.

(c) Assuming the star is made entirely of hydrogen, use the ideal gas law to determine the central temperature of this star, T_{c}, and evaluate T_{c} for a star with the same mass and radius as the Sun. (hint: remember electrons are particles too!)

(d) Why is Jupiter not a star? ($M_{\mathrm{Jupiter}} = 0.001 M_\odot$, $R_{\mathrm{Jupiter}} = 0.1 R_\odot$)

(e) Would you expect to get a higher or lower central temperature if you considered a more realistic chemical composition for the star?

Solution:

Central density In general we can calculate $m(r)$, the mass enclosed within a radial distance r of the star's centre from

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\,\mathrm{d}r\,,$$

where $\rho(r)$ is the mass density at r. Substituting $\rho(r) = \rho_\mathrm{c}(1 - r/R)$ we get

$$\begin{aligned} m(r) &= 4\pi\rho_\mathrm{c} \int_0^r r^2 \left(1 - \frac{r}{R}\right)\,\mathrm{d}r \\ &= 4\pi\rho_\mathrm{c} \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)\,. \end{aligned}$$

Therefore setting $r = R$, the mass of the star is $M = m(R) = \pi\rho_\mathrm{c} R^3/3$. Hence

$$\rho_\mathrm{c} = \frac{3M}{\pi R^3}\,.$$

Central pressure Using the equation of hydrostatic equilibrium

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}\,,$$

we have (from our expression for $m(r)$ above)

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}r} &= -\frac{G4\pi\rho_\mathrm{c}^2}{r^2}\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)\left(1 - \frac{r}{R}\right) \\ &= -4\pi G\rho_\mathrm{c}^2\left(\frac{r}{3} - \frac{7}{12}\frac{r^3}{R} + \frac{r^3}{4R^2}\right)\,. \end{aligned}$$

Integrating with respect to r we obtain

$$p(r) = 4\pi G\rho_\mathrm{c}^2\left(\frac{r^2}{6} - \frac{7}{36}\frac{r^3}{R} + \frac{r^4}{16r^2}\right) + C\,,$$

where C is a constant of integration. Clearly, this constant is also the pressure at $r = 0$, i.e., $p_\mathrm{c} = C$. We know that the pressure is zero at the surface of the star, i.e., $p(r = R) = 0$. Therefore

$$\begin{aligned} C = p_\mathrm{c} &= 4\pi G\rho_\mathrm{c}^2\left(\frac{R^2}{6} - \frac{7}{36}R^2 + \frac{R^2}{16}\right) \\ &= \frac{5}{36}\pi G\rho_\mathrm{c}^2 R^2\,. \end{aligned}$$

Substituting for $\rho_\mathrm{c} = 3M/(\pi R^3)$ we have

$$p_\mathrm{c} = \frac{5GM^2}{4\pi R^4}\,.$$

Central temperature The equation of state for an ideal gas states that

$$p = \frac{\rho kT}{\mu}\,,$$

where k is Boltzmann's constant and μ is the mean mass of the particles in the gas. Rearranging this, and substituting the expressions for p_c and ρ_c derived above we get

$$T_c = \frac{5\mu GM}{12Rk}$$

for the central temperature of the star. If we assume the star is entirely composed of hydrogen, and that it is totally ionised, there are as many electrons as there are protons in the gas. The mean particle mass is therefore approximately half the mass of a proton, i.e., $m_p/2$. Substituting in the values of the mass and radius of the Sun we get

$$T_c = 5 \times 10^6\,\mathrm{K}\,.$$

(realistic calculations of the central temperature using all the equations of stellar structure, including energy transport and power production, yield a somewhat higher temperature of 1.5×10^7 K.)

Jupiter The central temperature scales as M/R. This ratio is 100 times smaller for Jupiter than for the Sun, so we can expect a central temperature in Jupiter of something like 5×10^4 K. This would be insufficient for appreciable hydrogen fusion in the core of the Sun, and even less favourable in the core of Jupiter where the density and pressure are lower too.

Chemical composition Adding helium (and heavier elements) will tend to increase the mean particle mass, and hence increase the central temperature.

4.19 The brightest main sequence star of a certain cluster has absolute magnitude 0, and a mass of $4M_\odot$. Given that a star leaves the main sequence when its helium core constitutes 1/5 of the total mass of the star, initially of pure hydrogen, calculate the age of the cluster from the data given below.

Sun's luminosity	4×10^{26} W
Sun's absolute magnitude	$+4.6$
Sun's mass	2×10^{30} kg
Energy released per unit mass H$\rightarrow$He	6×10^{14} J kg^{-1}

Solution: Using Pogson's formula $M_S - M_\odot = -2.5 \log_{10} \frac{L_S}{L_\odot} \rightarrow L_S = 4 \times 10^{26} \times 69.1 = 2.77 \times 10^{28}$ W
Mass of hydrogen converted to helium in this star $= 0.2 \times 4 \times 2 \times 10^{30} = 1.6 \times 10^{30}$ kg
Energy released $= 6 \times 10^{14} \times 1.6 \times 10^{30} = 9.6 \times 10^{44}$ J
Age of cluster = Energy / luminosity $= 9.6 \times 10^{44}/2.77 \times 10^{28} = 3.47 \times 10^{16}$ sec =1.1 billion years.

4.20 Taking the mass luminosity (M, L) relationship for main sequence stars to be $L/L_\odot = (M/M_\odot)^{3.5}$ and assuming that a star leaves the main sequence after burning 20 % of its hydrogen, show that the main sequence lifetime of the Sun is 1.93 times longer than that of a star of mass equal to the limiting mass of a white dwarf star. (Take this limit to be $1.3M_\odot$). [4]

Solution: Let L_{CH} and M_{CH} be the luminosity and mass of the star of limiting mass (CH for Chandrasekhar). Then

$$\frac{L_{CH}}{L_\odot} = \left(\frac{M_{CH}}{M_\odot}\right)^{3.5}$$

But lifetime of the Sun $= \tau_\odot = \frac{1}{5} \times M_\odot \times 6 \times \frac{10^{14}}{L_\odot}$. Similiarly, lifetime of the star $= \tau_{\rm CH} = \frac{1}{5} \times M_{\rm CH} \times \frac{6\times10^{14}}{L_{\rm CH}}$

$$\therefore \frac{\tau_\odot}{\tau_{\rm CH}} = \frac{L_{\rm CH}}{L_\odot}\frac{M_\odot}{M_{\rm CH}} = \left(\frac{M_{\rm CH}}{M_\odot}\right)^{2.5} = 1.3^{2.5} = 1.93$$

4.21 Calculate the fraction of the mass of ^{4}He that can be released through the nucleosynthesis of ^{12}C through the triple α reaction

$$3\,^4\mathrm{He} \to \,^{12}\mathrm{C},$$

given that the mass of ^{4}He is 4.003 amu and the mass of ^{12}C is 12 amu. [2]

The core of a $2M_\odot$ star consists of pure ^{4}He and constitutes $1/5$ of the total mass of the star. Assuming the star has a luminosity $L = 10L_\odot$, estimate the maximum continued lifetime of the star. [4]

Solution: ^{12}C has lower potential energy than three ^{4}He nuclei.

Fraction of mass that can be released $= \frac{3\times4.003-12}{3\times4.003} \simeq \frac{0.009}{12} = 0.000\,75$.

Continued lifetime of star (assuming luminosity is constant), $\tau = \frac{\text{remaining available energy}}{\text{luminosity}}$

$$= \frac{\left(\frac{1}{5} \times 2M_\odot \times 0.000\,75\right) \times c^2}{10L_\odot}\ \text{sec.}$$

Substituting values

$$\tau = \frac{\frac{1}{5} \times 2 \times (2 \times 10^{30}) \times 0.000\,75 \times 9 \times 10^{16}}{10 \times 4 \times 10^{26}} = 1.35 \times 10^{16}\,\text{s} = 4.3 \times 10^{8}\,\text{yr}.$$

(We have ignored further burning of ^{12}C to form heavier elements, as this provides very little energy. Not all the core is necessarily consumed, so the figure above is an upper limit, although approximate.)

4.22 A star 5 times the mass of the Sun explodes as a supernova at a distance of 2 000 parsec. It is postulated that in the rapid progression of the conversion of hydrogen to iron, 80 % of the hydrogen is first converted to helium via the proton-proton cycle mechanism. Estimate the number of neutrinos that would pass through your body as a result of this initial phase of the explosion. $\{1.4 \times 10^{17}\,\mathrm{m}^{-2}\}$

Assuming the explosive event takes just a few seconds, how does this rate of arrival compare with the regular dosage provided by the Sun (see your notes for this figure)? [10]

(Assume that the original star comprises 100 % hydrogen.)

Solution: Number of protons in the star $= \frac{5\times2\times10^{30}}{1.67\times10^{-27}}$. Each pair of protons produces 1 neutrino and 80 % of the protons are used in this way. Hence number of neutrinos produced is $\frac{5\times2\times10^{30}\times0.8}{1.67\times10^{-27}\times2} = 6.68 \times 10^{57}$. The numbers are diluted according to the inverse square law and the number passing through each square metre on the Earth is given by

$$\frac{6.68 \times 10^{57}}{4\pi(2000 \times 3.08 \times 10^{16})^2} = 1.4 \times 10^{17}\,\mathrm{m}^{-2}.$$

(Approximately equivalent to a human cross-section). From your notes, the Sun provides about 6.45×10^{14} neutrinos $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$. Hence for a brief time the neutrino arrival rate is about 100 times greater and the event should be detectable.

4.23 Assuming that stars radiate as blackbodies, show how the Stefan-Boltzmann law relates the radius, luminosity and effective temperature of a star. [5]

Stars towards the middle of the main sequence on the Hertzsprung-Russel diagram approximately obey the *mass-luminosity relation*, $L \propto M^3$, where L is a star's luminosity and M its mass. With the simplifying assumption that these stars all have approximately the same mean densities, show that their luminosities and temperatures would be related by

$$\log L = \frac{36}{7} \log T + \text{constant} \quad ,$$

i.e., that the main sequence would have a slope of about -5.1 on the Hertzsprung-Russell diagram. [10]

(Note that in reality, the mean densities of low-mass stars are significantly greater than the mean densities of high-mass stars.)

Draw a schematic H-R diagram, labelling the axes in terms of $\log(L/L_\odot)$ and $\log(T/T_\odot)$, and include a line for the main sequence with the slope you have just derived. Show also lines of constant stellar radius. [5]

Solution: Stefan-Boltzmann law: power radiated by a blackbody at temperature T is σT^4 per unit area. Surface area of a star is $4\pi R^2$, so luminosity is

$$L = 4\pi R^2 \sigma T_e^4,$$

where T_e is the stars 'effective temperature'.

$$\text{If} \quad L \propto M^3 \tag{1}$$

$$\text{and} \quad L \propto R^2 T^4 \tag{2}$$

$$\text{and} \quad M \propto R^3 \quad \text{(constant mean density)} \tag{3}$$

$$\text{from (3)} \;\; R^2 \propto M^{\frac{2}{3}}$$

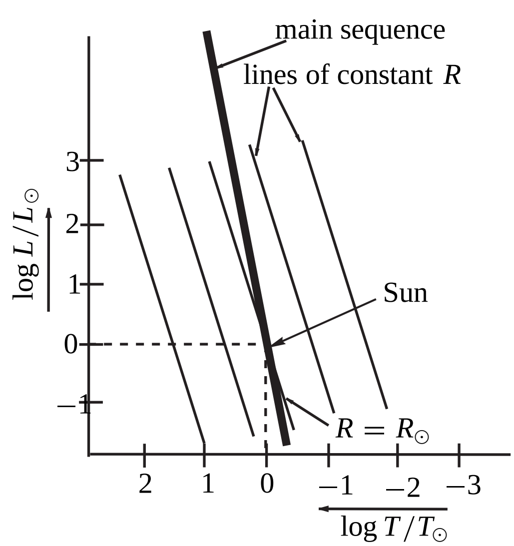
$$\text{from (1)} \;\; M \propto L^{\frac{1}{3}}$$

$$\Rightarrow R^2 \propto L^{\frac{2}{9}}$$

$$\text{from (2)}, \;\; L \propto L^{2/9} T^4 \quad \text{i.e.,} \;\; L^{7/9} \propto T^4$$

$$\text{so,} \;\; \frac{7}{9} \log L = 4 \log T + \text{constant}$$

$$\text{then,} \;\; \log L = \frac{36}{7} \log T + \text{constant}$$



$\log L = 2\log R + 4\log T$, so lines of constant R have a slope of -4.

4.24 Use the lecture handout giving data on stellar spectral types to answer the following:

Rigel A is the prominent bluish-white star in the constellation of Orion. Its spectrum is such that it is classified as a B8 star.

(a) Estimate its effective temperature.

(b) How would you expect the relative strengths of the Hydrogen Balmer, He i and Ca ii lines to compare in this star? (This is one of the criteria by which spectral type is identified.)

(c) Spectroscopic studies lead to an estimate of $1.4 \times 10^5\, L_\odot$ for its luminosity. How would you expect its spectral lines to differ from those of a far less luminous star of the same surface temperature?

(d) Calculate the radius of Rigel A in solar radii.

Solution:

(a) From the chart showing line strength vs. temperature or spectral type we can see that a B8 star has an effective temperature of about 13 000 K.

(b) From the same graph we can see that the Balmer lines should be quite strong (the strongest Balmer lines are seen in the adjacent A0 stars) and that absorption from helium atoms (He i) is just starting. The star is not hot enough to show strong He i absorption though. The star is too hot to show much absorption from Ca ii. So in summary we see strengths in the order H i>He i>Ca ii.

(c) A far less luminous star of the same temperature would be expected to have broader spectral lines and weaker lines from certain ionized atoms. The broadening reflects the greater density of gas in the photospheres of these smaller, less luminous stars. This increases the collision rate between atoms and hence broadens the lines.

(d) We know that $L = 4\pi R^2 \sigma T^4$, assuming a blackbody spectrum. We can therefore write $R/R_\odot = (L/L_\odot)^{\frac{1}{2}} (T_\odot/T)^2 = (1.4 \times 10^5)^{\frac{1}{2}} (5\,770/13\,000)^2 \approx 74$.

4.25e Assuming that all stars obey the Stefan-Boltzmann law, show how lines of constant stellar radius appear on the Hertzsprung-Russell diagram. Explain how red giants and white dwarfs fit into this diagram. [5]

Solution: Stefan-Boltzmann law: power radiated by a blackbody at temperature T is σT^4 per unit area. Surface area of a star is $4\pi R^2$, so luminosity is

$$L = 4\pi R^2 \sigma T_e^4,$$

where T_e is the stars ‘effective temperature’.
$\log L = 2\log R + 4\log T$, so lines of constant R have a slope of -4. Red giants and white dwarfs populate the upper right and lower left of the H-R diagram.

4.26e A spectroscopic double star emits a spectral line at 441 nm that splits into two components of maximum separation 0.2 nm every 10.25 days. What is the minimum spatial separation of the two stars, assuming they are in circular orbits? [5]

Solution: From the Doppler formula, $\Delta\lambda/\lambda_0 \simeq v/c \rightarrow v = 3\times10^5\times0.2/441 = 136$ km s^{-1}.
$v = 2\pi R/P$ (assuming circular orbit, perpendicular to line of sight).
Thus, $R = P\times v/2\pi = 10.25\times24\times3600\times136/2\pi = 19.2\times10^6$ km.

4.27e Explain why the Hα absorption line (the first line in the Balmer series) is weak for both very hot and very cool stars. Which stellar spectral type has the strongest hydrogen absorption lines? [5]

Solution: The Hα absorption line is weak for very hot stars because there are too few electrons in the low energy level ($m = 2$) required for this line; electrons are either in much higher energy levels, or indeed atoms may be fully ionised. For very cool stars, conversely, most electrons will be in the ground state, and thus not enough electrons are in the $m = 2$ state required for Balmer absorption. A type stars have the strongest H lines.

4.28e Stars on the main sequence have masses of between about 0.1 and 50 $M_\odot$.

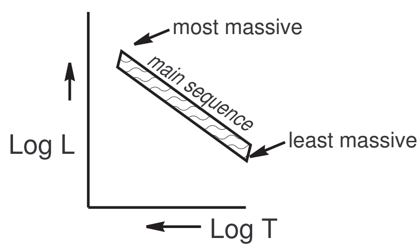
(a) Show where the least and most massive of these stars are found on the Hertzsprung-Russell diagram. [4]

(b) What factors influence the cut-offs at the two extremes? [3]

(c) How does the position of a star change on the main sequence as it ages? [2]

(d) Approximately what range in luminosity would youu expect for these stars? [4]

(e) If the Sun has an expected lifetime of about 10^{10} years, estimate the lifetimes of stars at the extremes of the main sequence. [4]



Solution: (a) diagram: [4] marks

(b) Least massive: not hot enough to begin nuclear reactions [1]; most massive: also extremely luminous, blowing off outer layers to lose mass (Eddington Limit) [2].

(c)*Small* movement left to right as star ages, otherwise star remains at same location on M-S throughout its H-burning lifetime. [2]

(d) using mass-luminosity relation $M^\alpha \propto L$, and taking $\alpha = 3.0$ (though any α in range $3.0-3.5$ acceptable)

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^\alpha \qquad [1]$$

Plugging in numbers:

$$\begin{aligned}\text{For } M &= 0.1M_\odot \Longrightarrow L/L_\odot = 0.1^3 = 10^{-3} \qquad [1]\\ \text{For } M &= 50M_\odot \Longrightarrow L/L_\odot = (50)^3 \approx 10^5\end{aligned}$$

(e) Lifetime τ = energy available $\div$ luminosity $\propto M/M^\alpha$. Hence $\tau \propto M^{-2}$ (see notes) [2] So

$$\frac{\tau}{\tau_\odot} = \left(\frac{M_\odot}{M}\right)^2$$

$$\text{For } M = 0.1M_\odot \Longrightarrow \tau = 100\tau_\odot = 10^{12} \quad \text{years} \quad [1]$$
$$\text{For } M = 0.1M_\odot \Longrightarrow \tau = \tau_\odot/2500 = 4 \times 10^{6} \quad \text{years} \quad [1]$$

4.29e How are stellar surface temperatures, T_e, measured? What is the sequence of stellar spectral types from hot to cool? In what part of this sequence would you see spectral lines of

(a) helium,

(b) molecules? [5]

A spherical star of ionised hydrogen has a mass M, radius R, and uniform density ρ. Given that gas pressure is $P = 2nkT$, where n is the proton number density, show by considering the pressure needed to balance the gravitational attraction of two halves of the star that its core temperature T_c depends approximately on M and R according to

$$T_c = \frac{GMm_p}{6kR}. \quad [7]$$

Give an expression for the luminosity L of a star in terms of R and T_e, and given also the mass-luminosity relation $L \sim M^{7/2}$ on the Main Sequence (MS), show that the surface temperature of a MS star depends on M and R as

$$T_e \sim \frac{M^{7/8}}{R^{1/2}}. \quad [5]$$

Solution: *no solution available*

4.30e Summarise how a large gas cloud turns into a star, stating at what point it becomes s Main Sequence star. Why is a protoplanetary disk produced? [3]
Estimate the temperature reached when a cloud of one solar mass has collapsed to the size of the Sun. [2]

Solution: *no solution available*

4.31e Sketch two Hertzprung-Russell diagrams, one for a young star cluster, and one for an old star cluster, labelling the axes, the regions of main stellar types, and the locations of stars of 0.1, 1, and 10 solar masses. [4]
State why the two diagrams differ. [1]

Solution: *no solution available*

4.32e Using the Stefan-Boltzmann law $L = 4\pi R^2 \sigma T^4$ relating luminosity L with temperature T and stellar radius R, sketch a model H-R diagram, showing clearly how lines of constant radius arise, and why they have gradient of minus 4. [4]

Using a second drawing, sketch an observed H-R diagram, showing the position of the Sun, and marking clearly the main sequence, white dwarf and red giant regions. [3]

The proton-proton chain is the main nuclear fusion reaction for main sequence stars. The reaction can be summarised as

$$4\,{}^1_1\text{H} \rightarrow {}^4_2\text{He} + \text{energy}$$

in which the mass of a helium nucleus is 3.972 times the mass of a proton. Assuming that the Sun initially is composed entirely of hydrogen, and that only 10% of its mass is available for hydrogen fusion, estimate the main sequence lifetime of the Sun. [6]
How will the Sun evolve after its period on the main sequence? [4]

Solution: *no solution available*

4.33e List the spectral classes in the Harvard Scheme for stellar classification, and state the basis on which the scheme is founded. In which class are:

(a) molecular lines strong?

(b) hydrogen lines most prominent? [5]

Solution: *no solution available*

4.34e The Stefan-Boltzmann law can be expressed in the form $L \propto ST^4$ where S is the surface area of the emitter at temperature T, and L is its luminosity. Explain how this equation can be used to link the size and effective temperature of stars. Ensure that you explain the meaning of effective temperature. State further how the concept of radiation flux arises, by considering the inverse square law. [4]

Given that the fluxes of radiation from two objects may be characterised in terms of a difference in apparent magnitude, according to the equation

$$m_1 - m_2 = -2.5 \log(F_1/F_2),$$

demonstrate that for two stars at the same distance from an observer, with equally perfect viewing conditions,

$$m_1 - m_2 = -5 \log(R_1/R_2) - 10 \log(T_1/T_2),$$

where R_1, R_2 are the radii of the stars, and T_1, T_2 are the corresponding effective temperatures. [3]

An eclipsing binary star system comprising of a larger, brighter star with a smaller, dimmer companion can be identified from its characteristic light curve. Sketch this light curve, identifying the primary and secondary minima, and explaining briefly how they arise. [4]

For such a system, show that the ratio of the stellar temperatures can be given by

$$\frac{T_1}{T_2} = \left(\frac{F - F_P}{F - FS}\right)^{\frac{1}{4}},$$

where F, F_P and F_S are respectively the flux received from the binary system when no eclipse is present, the flux received during a primary eclipse, and the flux received during a secondary eclipse. [6]

Solution: *no solution available*

4.35e What nuclear process sets in after H to He burning runs out in main-sequence (MS) stars? [1]
What readjustments in the stellar structure cause this to happen? [1]
What is the heaviest element reached in the fusion sequence of massive MS stars? [1]

Estimate the energy released when the 2 solar mass core of a massive star collapses to a radius of 10km and name the resulting observational phenomenon. [2]

[$G = 6.7 \times 10^{-11}$ N m^2 kg^{-2}, solar mass = 2×10^{30} kg]

Solution: *no solution available*

4.36e In the Harvard spectral classification scheme, state the spectral type that is characterised by the following line emissions:

(a) strong molecular bands

(b) strong ionised Helium lines in emission or absorption

(c) dominant metallic lines.

Describe very briefly how such spectral lines form, and why they are useful for stellar classification. [5]

Solution: *no solution available*

4.37e Name and briefly describe the three outer layers of the Sun. [5]

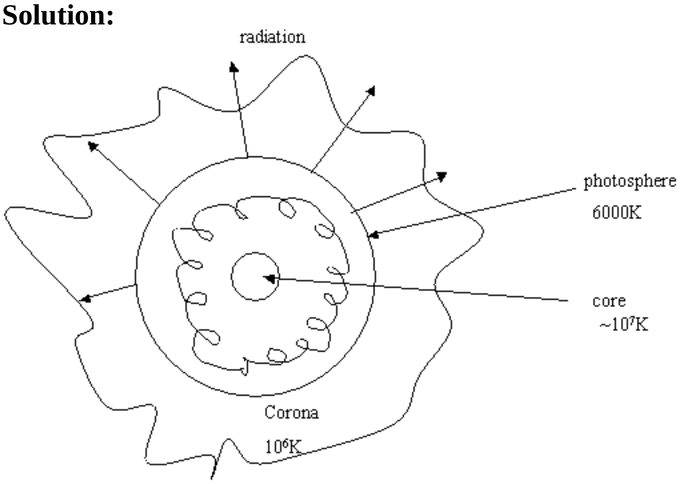
Solution: *no solution available*

4.38e Sketch the structure of the Sun, indicating the approximate temperatures of the distinct radial regions. [2]

What process is believed to heat the corona? [1]

Estimate the temperature at the centre of the Sun given that it has mass 2.0×10^{30} kg and radius 7.0×10^{8} m. [2]

Solution:



[2]

Corona heated by dissipation of magnetic energy [1]

T_c roughly given by the balance of thermal (pressure) and gravitational energy

$$\text{i.e.}\quad kT_c \simeq \frac{GMm_p}{R}$$

so $T_c \simeq \frac{6.7\times10^{-11}\times2\times10^{30}\times1.7\times10^{-27}}{7\times10^{8}\times1.4\times10^{-23}}$
$\simeq 2.3 \times 10^{7}$ K [2]

4.39e The temperature of the surface of the Sun is 5800 K. If the Sun radiates like a blackbody and the albedo of the Earth in sunlight is 0.37, estimate the temperature at the surface of the Earth. [7]

What assumptions did the calculation above include? [3]

Based on the above result estimate, using Wien's Displacement Law, the wavelength at which the Earth's radiated spectrum peaks. [2]

Comment on how your calculated value compares with the actual temperature at the Earth's surface, discussing possible reasons for any discrepancy. [4]

What planet has a large discrepancy between surface temperature calculated as above, and actual surface temperature? [1]

[Wien's constant = 2.9×10^{-3} Km]

Solution:
Can calculate the luminosity of the Sun using

$$\begin{aligned} L &= 4\pi R_\odot^2 \sigma T_\odot^4 = 4\pi \times (6.97 \times 10^8)^2 \times 5.7 \times 10^{-8} \times (5800)^4 \\ &= 3.94 \times 10^{26}\,\mathrm{W} \end{aligned} \quad [1]$$

Now, the power per unit area at a distance D from the Sun is

$$F(D) = \frac{L}{4\pi D^2} = \frac{3.94\times10^{26}}{4\pi(1.5\times10^{11})^2} = 1.39 \times 10^{13}\,\mathrm{W\,m^{-2}} \quad \text{(the solar constant)} \quad [1]$$

Power absorbed is

$$P_{in} = (1 - A)P \Rightarrow \quad P_{in}(1 - A)\pi a^2 F(D)$$

At temperature T_p planet emits: $P_{out} = a\pi a^2 \sigma T_p^4$

$$\begin{aligned} P_{in} &= P_{out} \\ &\Rightarrow (1 - A)\pi a^2 F(D) = 4\pi a^2 \sigma T_p^4 \\ &\Rightarrow (1 - A)F(D) = 4\sigma T_p^4 \\ &\Rightarrow T_p^4 = \frac{(1 - A)F(D)}{4\sigma} \\ T_p^4 &= \frac{(1 - 0.37)1.39 \times 10^3}{4 \times 5.7 \times 10^{-8}} = 3.85 \times 10^9 \end{aligned}$$

[4]

hence $T_p = 249K$ [1]

Assumptions made include:
T_p uniform over lanet
Earth behaves like a perfect 'blackbody' in emission!
All latitudes receive equal amounts of incoming solar radiation [3]

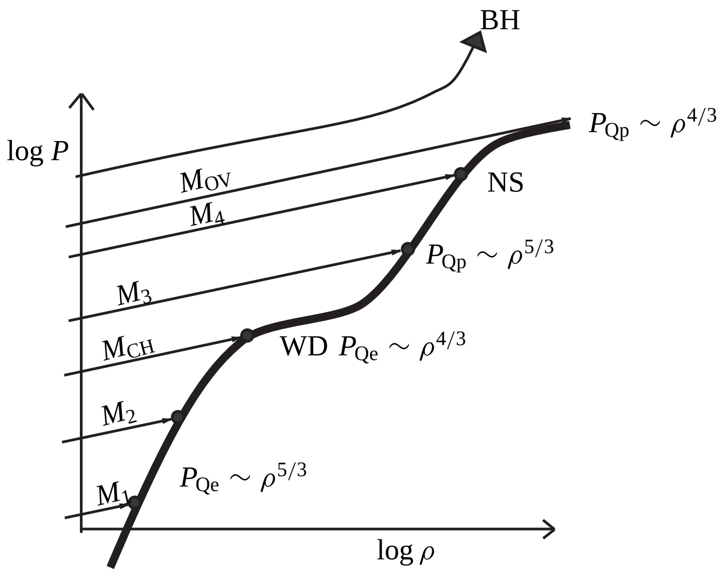
Wien's Displacement Law states:
$\lambda_{max} T = \text{constant} = 2.9 \times 10^{-3}\,\mathrm{Km}$
$\Rightarrow \lambda_{max} = 2.9 \times 10^{-3}/249 = 11.6$ microns [2]

Calculated value T_p is ~ 30 to 40 degrees lower than actual average value [1]
Reason is moderate 'greenhouse effect' on Earth [1]
Solar radiation easily penetrates atmosphere - absorbed at Earth and re-radiated at longer IR wavelengths [1]
Absorbed by $C0_2$ and H_2O in atmosphere - trapped- increases temperature [1]
Venus has a large discrepancy between surface temperature calculated as above, and actual surface temperature [1]

5 Stellar Astrophysics part II

5.1e Draw a (log P, log ρ) diagram for cold dense matter, labelled with the main sources of pressure involved in each part. Show the tracks of contracting stellar remnants and indicate the locations of white dwarf stars, neutron stars, and black holes. [5]

Solution:



5.2e Explain briefly why a star can never reach a final static equilibrium state unless it develops an internal pressure which does not depend on temperature. Name the two classes of such cold stellar remnants, identify the range of masses of each. What do stellar remnants heavier than these become? [5]

Solution: Thermal pressure support at finite T $\Rightarrow$ energy loss by radiation. Once there is no fusion energy left shrinkage is required $\Rightarrow$ no static equilibrium, $\therefore$ 'cold' pressure required.

Equilibrium states are white dwarf (electron quantum pressure) $\mathrm{M} < \mathrm{M_{Ch}} = 1.3\mathrm{M}_\odot$, and neutron star (nucleon quantum pressure) $\mathrm{M} < \mathrm{M_{OV}} = 2.7\mathrm{M}_\odot$ and $\mathrm{M} > \mathrm{M_{OV}} \rightarrow$ Black Holes.

5.3e What observational features characterise pulsars? Derive an expression for the minimum rotation period of a mass M, radius R and express this in terms of density ρ. [5]

Solution: Regular ($\lesssim 1\,\mathrm{s}$) pulses (radio or optical usually), very constant period ($\mathrm{P}/\dot{\mathrm{P}} \sim$ centuries or more) and high pulse luminosity.

Minimum rotation period $\tau = \frac{2\pi}{\omega}$ set by centrifugal limit $\frac{GM}{R^2} \geq \omega^2 R \Rightarrow \tau \geq 2\pi \left(\frac{R^3}{GM}\right)^{\frac{1}{2}} = \left(\frac{3\pi}{Gp}\right)^{1/2}$.

5.4e What is meant by a compact accretion X-ray source ? Write down an expression for the luminosity, L_{acc}, due to accretion and derive an expression for the limit set by radiation pressure. [5]

Solution: Source powered by infall of $\dot{M}$ kg s^{-1} (from comparison star) in gravitational potential $\frac{GM}{R}$ of compact object,

$$L = \frac{GM}{R}\dot{M}$$

Radiation pressure force on area σ_p is $\frac{L\sigma_p}{4\pi r^2 c} = F_R$

Gravitational force $= \frac{GMm_p}{r^2} = F_G$, $F_G \geq F_R \Rightarrow L \leq \frac{4\pi GMm_p c}{\sigma_p}$.

5.5e Explain briefly why pulsars pulse. The period of the Crab pulsar is 33 ms. If the number of pulses received on successive days drops by 3 per day calculate its slowdown rate, $\dot{P}$, in seconds per day. [5]

Solution: *no solution available*

5.6e Explain why pulsars must be small massive objects and why white dwarf stars are ruled out in favour of neutron stars. [5]

Show how neutron stars acquire such high rotation rates and strong magnetic fields and obtain an expression for the minimum rotation period. [6]

Due to internal restructuring the radius R of a neutron star is reduced to $0.9R$. Assuming mass, angular momentum and total magnetic flux are conserved, obtain the factors by which the following change:

(a) rotation rate [2]

(b) magnetic field, B. [2]

Solution: High luminosity and slow change of period $\Rightarrow$ massive object (large energy) and small pulse period $\Delta t \Rightarrow$ size $\leq c\Delta t$. White dwarfs cannot rotate fast enough because centrifugal limit $\sim \rho^{1/2}$ is not fast enough for White Dwarf ρ values.

Angular momentum conservation

$$MR^2\omega = MR_0{}^2\omega_0 \Rightarrow \omega \propto \frac{1}{R^2}.$$

R decrease enormously in collapse to NS $\Rightarrow$ very high ω. Magnetic flux conservation

$$4\pi R^2 B = 4\pi R_0{}^2 B_0 \Rightarrow B \propto \frac{1}{R^2}$$

$$\omega^2 R \leq \frac{GM}{R^2} \Rightarrow \tau \geq 2\pi\sqrt{\frac{R^3}{GM}}$$

$$\frac{\omega_2}{\omega_1} = \left(\frac{R_1}{R_2}\right)^2 = \left(\frac{1}{0.9}\right)^2,$$

$$\frac{B_2}{B_1} = \left(\frac{1}{0.9}\right)^2.$$

5.7e Given the expressions $p = \hbar/\Delta x$, $P = \beta nE$ and $E_e = p^2/(2m_e)$, where the symbols have their usual meanings, derive the quantum electron pressure in a hydrogen gas at density ρ:

$$P_{Qe} = \frac{\beta\hbar^2}{2m_e}\left(\frac{\rho}{m_p}\right)^{5/3}.$$

[6]

Given also that the central pressure needed to balance gravity for a spherical mass M is

$$P_c = \left(\frac{\pi}{6}\right)^{1/3} G M^{2/3} \rho^{4/3},$$

derive the density of a white dwarf star:

$$\rho_{\mathrm{wd}} = \frac{4\pi}{3} \frac{G^3 m_p^5 m_e^3}{\beta^3 \hbar^6} M^2.$$

[5]

Write down the corresponding expression for a neutron star and state the source of pressure. [3]

Hence show that $R_{\mathrm{ns}}/R_{\mathrm{wd}} \simeq m_e/m_p$. [3]

Solution:

$$\Delta x = n^{-1/3} = \left(\frac{\rho}{m_p}\right)^{-1/3} \Rightarrow p = \hbar \left(\frac{\rho}{m_p}\right)^{1/3}$$

$$P_{\mathrm{Qe}} = \beta n E = \beta \left(\frac{\rho}{m_p}\right) \frac{\hbar^2}{2m_e} \left(\frac{\rho}{m_p}\right)^{2/3} = \beta \frac{\hbar^2}{2m_e} \left(\frac{\rho}{m_p}\right)^{5/3}$$

$$P_c(\rho) = P_{\mathrm{Qe}}(\rho)$$

leads to

$$\rho_{\mathrm{WD}} = \frac{4\pi}{3} \frac{G^3 m_p^5 m_e^3}{\beta^3 \hbar^6} M^2$$

For neutron star, corresponding expression is

$$\rho_{\mathrm{NS}} = \frac{4\pi}{3} \frac{G^3 m_p^8}{\beta^3 \hbar^6} M^2$$

Pressure source is neutron degeneracy pressure.

For same M

$$\frac{R_{\mathrm{NS}}}{R_{\mathrm{WD}}} = \frac{\left(\frac{M}{\rho_{\mathrm{NS}}}\right)^{1/3}}{\left(\frac{M}{\rho_{\mathrm{WD}}}\right)^{1/3}} = \left(\frac{\rho_{\mathrm{WD}}}{\rho_{\mathrm{NS}}}\right)^{1/3} = \left(\frac{m_e^3}{m_p^3}\right)^{1/3} = \frac{m_e}{m_p}.$$

5.8 Two degenerate white dwarf stars have masses $\mathcal{M}_1$, $\mathcal{M}_2$ and surface temperatures T_1, T_2. Assuming them to radiate like blackbodies (i.e., bolometric luminosity $L_{\mathrm{bol}} = 4\pi R^2 \sigma T^4$), show that their absolute bolometric magnitudes M_1, M_2 satisfy

$$M_1 - M_2 = \frac{5}{3} \log_{10} \frac{\mathcal{M}_1}{\mathcal{M}_2} - 10 \log_{10} \frac{T_1}{T_2}.$$

Solution: By Stefan's law

$$\frac{L_1}{L_2} = \frac{4\pi R_1^2 \sigma T_1^4}{4\pi R_2^2 \sigma T_2^4} = \left(\frac{R_1}{R_2}\right)^2 \left(\frac{T_1}{T_2}\right)^4.$$

Therefore

$$M_1 - M_2 = -2.5 \log_{10} \frac{L_1}{L_2} = 5 \log_{10} \frac{R_2}{R_1} - 10 \log_{10} \frac{T_1}{T_2}.$$

But for white dwarf stars radius $\propto$ mass$^{-1/3}$, therefore

$$\frac{R_2}{R_1} = \left(\frac{\mathcal{M}_1}{\mathcal{M}_2}\right)^{1/3},$$

and so to the answer.

5.9e Briefly describe the observational evidence for the existence of white dwarfs and neutron stars. [6]

By estimating the gravitational attraction between two halves of a uniform sphere, show that the pressure within a star of mass M and radius R is approximately

$$P \simeq \frac{GM^2/(4R^2)}{\pi R^2},$$

and hence that $P \propto M^{2/3}\rho^{4/3}$, where ρ is the stellar mass density. [4]

In white dwarfs, this pressure is balanced by an outward pressure that increases as $\rho^{5/3}$. Describe the source of this pressure and use the relation to show that, in equilibrium, the radius of white dwarfs varies as $R \propto M^{-1/3}$. [4]

Compare this with the corresponding result for stars of constant density. [3]

Solution:

White Dwarfs: We can measure the effective temperature of a star from, e.g., $\lambda_{\max} T_{\rm eff} =$ const., and determine its luminosity from measurements of its flux, F, and distance, D, using $L = 4\pi D^2 F$. The radius of the star, R, can then be determined from the Stefan-Boltzmann law, using $L = 4\pi R^2 \sigma T^4$. When this analysis is performed for many stars, there is a class of object that clearly shows $T_{\rm eff} \geq 20\,000$ K and $R \leq 300$ km, consistent with the parameters predicted for white dwarfs. The analysis of the orbit of these objects in binary systems confirm the $R \propto M^{-1/3}$ rule predicted for white dwarf mass, M.

Neutron Stars: These are not usually seen via their thermal radiation as there surface area is too small to give significant luminosity. Instead, they are seen indirectly as pulsars (rapidly rotating neutron stars that emit non-thermal radiation at a very high luminosity). The pulse width gives some indication of the size of the pulsar ($\Delta t \sim R/c$) implying that $R \sim 10$ km. The rapid rotation rate also implies a small radius, to prevent centrifugal breakup. The regularity of the pulse, and the dynamics of pulsars in binary systems indicate a massive object, always more massive than the Chandrasekhar limit. The non-thermal radiation indicates that pulsars possess a strong magnetic field.

Take two equal hemispheres of mass, each of mass $M/2$ separated by a distance R. Approximating each hemisphere as a sphere of mass $M/2$, the force of attraction between the two halves is

$$F = \frac{GM^2/4}{R^2},$$

so the pressure between the surfaces is

$$P = \frac{\text{force}}{\text{area}} = \frac{GM^2/(4R^2)}{\pi R^2} \propto M^2 R^{-4}.$$

The density of the material is

$$\rho = \frac{\text{mass}}{\text{volume}} \propto M R^{-3},$$

so $R \propto \rho^{-1/3} M^{1/3}$, i.e.,

$$P \propto \rho^{4/3} M^{-4/3} M^2 \propto \rho^{4/3} M^{2/3}.$$

This gravitational pressure is balanced by the 'degeneracy pressure' of the electrons in the white dwarf. If an electron is spatially confined to Δx, the x-component of its momentum must be at least $p_x \simeq \hbar/\Delta x$. The Pauli exclusion principle means that N electrons in a volume V only get about V/N of the volume each.

If the degeneracy pressure is $\propto \rho^{5/3}$ then in equilibrium,

$$\rho^{5/3} \propto \rho^{4/3} M^{2/3}, \quad \text{i.e.,} \quad \rho^{1/3} \propto M^{2/3}.$$

By definition of density, $\rho \propto M R^{-3}$, so

$$M^{1/3} R^{-1} \propto M^{2/3}, \quad \text{i.e.,} \quad R \propto M^{-1/3}.$$

A star of constant density would have $M \propto R^3$, i.e., $R \propto M^{+1/3}$ (cf. above). A star of constant density would get bigger with greater mass, whereas white dwarfs get smaller.

5.10 Sirius B is a white dwarf companion to the brightest star in the night sky, Sirius A. It has an effective temperature of 29 000 K and a mass of 1 $M_\odot$.

Use your lecture notes and the physical constants at the front of the problems handbook to estimate:

(a) the radius of Sirius B, R {4.8×10^6 m}

(b) its mean density, ρ {4.3×10^9 kg m^{-3}}

(c) its luminosity, L. Express this as a multiple of the solar luminosity, $L_\odot$, and comment on the value {$0.03\, L_\odot$}

(d) The electron degeneracy pressure at its centre, P_Q, expressed in pascals (1 Pa = 1 N m^{-2}). {1.96×10^{22} Pa}

The thermal gas pressure that supports main-sequence stars against gravitational collapse will also contribute slightly to supporting white dwarfs. Taking the thermal pressure at the centre of a white dwarf to be $P_\mathrm{T} = 2kT\rho/m_\mathrm{p}$, where T is the temperature k is Boltzmann's constant and m_p the mass of a proton, estimate its value and compare it to the electron degeneracy pressure calculated above. {2×10^{18} Pa}

At what temperature would these pressures be equal? {2.75×10^8 K}

Assuming Sirius B consists of a soup of electrons and protons, what is the mean separation of the protons? How does this compare with the separation of atoms in a 'normal' solid? {$\sim 7 \times 10^{-13}$ m}

[$\hbar = 1.054 \times 10^{-34}$ J s. Note that any answer close to those quoted is acceptable. In fact the inside of a white dwarf is thought to be considerable hotter than the surface.]

Solution: Note that only *estimates* are required, so that any order-of-magnitude answer is acceptable. In the lectures we develop a simple model of a white dwarf and derive a value for its radius that is about a factor of three smaller than the accepted value. Here, we will use the standard numerical expression for radius:

$$R = 4\,800 \left(\frac{M}{M_\odot} \right)^{-1/3} .$$

(a) Taking $M = M_\odot$, we get $R = 4.8 \times 10^6$ m

(b) $\rho = M/V$. The volume of the white dwarf is $V = 4\pi R^3/3 = 4.6 \times 10^{20}$ m^3. Using a mass of $M = 2 \times 10^{30}$ kg we get $\rho = 4.3 \times 10^9$ kg m^{-3}.

(c) Use the Stefan-Boltzmann law: $L = 4\pi R^2 \sigma T^4$. Inserting the numbers gives $L = 1.16 \times 10^{25}\,\mathrm{W} = 0.03\,L_\odot$.

(d) The model derived in the lectures gives

$$P_\mathrm{Q} \simeq \frac{\hbar^2}{3m_\mathrm{e}} \left(\frac{\rho}{m_\mathrm{p}}\right)^{5/3} = 1.96 \times 10^{22}\,\mathrm{Pa}$$

for the electron degeneracy pressure in the white dwarf.

We are told that the thermal pressure is $P_\mathrm{T} = 2kT\rho/m_\mathrm{p}$. The factor of two is there because both the protons and electrons contribute equally to the thermal pressure, and ρ/m_p just equals the number density of protons alone. Inserting the numerical values we get a thermal pressure of $P_\mathrm{T} = 2 \times 10^{18}$ Pa which is about 10^4 times smaller than the electron degeneracy pressure calculated above. The thermal pressure does not make a significant contribution to the the support of a white dwarf, despite their high temperatures. For the pressures to be equal we would need a temperature about 10^4 times higher, i.e., about 275 million degrees.

Finally, we determine the number density of protons in the white dwarf, n_p:

$$\rho = \frac{M}{V} \simeq n_\mathrm{p} m_\mathrm{p}\,,$$

so $n_\mathrm{p} = 2.57 \times 10^{36}\,\mathrm{m}^{-3}$. The volume occupied by one proton is $1/n_\mathrm{p}$, and the distance between then is about $1/n_\mathrm{p}^{1/3}$. Inserting our value for n_p we get a proton separation of about 7×10^{-13} m. The diameter of a hydrogen atom is about 10^{-10} m, so the protons/electrons in our white dwarf are about 100 times closer together than are the atoms in normal matter. This is why the densities of white dwarfs are about $(100)^3$ times higher than the densities of normal cold matter, such as ice cream.

5.11 Show that the density of a neutron star of M just above M_Ch is about $(m_\mathrm{p}/m_\mathrm{e})^3$ times larger than that of a white dwarf of M just below M_Ch. Hence show that the minimum rotation period of a white dwarf is $(m_\mathrm{p}/m_\mathrm{e})^{3/2}$ times larger than that of a neutron star of comparable mass. [12]

The angular velocity ω of rotation of a self-gravitating sphere of mass M, radius R, is limited by centrifugal break-up to $\omega < \omega_1$ as described in your notes. In addition, relativity forbids any part of the rotating body to move faster than the speed of light, c, so setting another limit $\omega < \omega_2$. Show that the centrifugal limit, ω_1, always applies first (i.e., show that $\omega_1 < \omega_2$) since, for the reverse to be true would imply $R < R_\mathrm{S}/2$, where R_S is the Schwarzschild radius, – i.e., the body would be a black hole. [8]

Solution: Minimum rotation period $\tau = 2\pi/\omega$ (centrifugal limit) is given by

$$\frac{4\pi^2}{\tau^2} = \frac{GM}{R^2},$$

so that

$$\tau = 2\pi \left(\frac{R^3}{M}\right)^{1/2} \propto \frac{1}{\rho^{1/2}}.$$

But since the objects have the same mass they have densities in the ratio

$$\frac{\rho_1}{\rho_2} = \frac{\rho_\mathrm{wd}(M = M_\mathrm{Ch})}{\rho_\mathrm{wd}(M = M_\mathrm{Ch})} = \frac{\frac{4\pi}{3\beta^3}\frac{G^3 m_\mathrm{e}^3 m_\mathrm{p}^5}{\hbar^6}}{\frac{4\pi}{3\beta^3}\frac{G^3 m_\mathrm{e}^8}{\hbar^6}} = \left(\frac{m_\mathrm{e}}{m_\mathrm{p}}\right)^3.$$

Hence

$$\frac{\tau_1}{\tau_2} = \left(\frac{\rho_2}{\rho_1}\right)^{1/2} = \left(\frac{m_p}{m_e}\right)^{3/2}.$$

The centrifugal limit ω_1 is $\omega_1 = (GM/R^3)^{1/2}$. The relativistic limit is $\omega_2 = c/R$. So

$$\frac{\omega_1}{\omega_2} = \left(\frac{GM}{R^3}\frac{R^2}{c^2}\right)^{1/2} = \left(\frac{GM}{Rc^2}\right)^{1/2} = \left(\frac{R_S}{2R}\right)^{1/2}.$$

Hence $\omega_1 < \omega_2$ only if $R < R_S/2$, in which case object has become a black hole.

5.12 The period of the Crab pulsar is 33 ms and the luminosity associated with it is 2×10^{31} W. Assuming the neutron star to have about one solar mass and a radius of 5 km, find the time scale on which the Crab pulse period is increasing. (l year $= 3.1 \times 10^7$ s)

Solution: Rotational energy

$$E_{\rm rot} = \frac{1}{2}I\omega^2 = \frac{1}{5}MR^2\omega^2 = \frac{4\pi^2}{5}\frac{MR^2}{\tau^2}$$

where τ = rotation period.

Power radiated = L

$$\therefore \text{spin} - \text{down time } \tau_{SD} = E_{\rm rot}/L = \frac{4\pi^2}{5}\frac{MR^2}{L\tau^2}$$

$$\therefore \tau_{sd} = \frac{4\pi}{5} \times \frac{2 \times 10^{30} \times (5 \times 10^3)^2}{2 \times 10^{31} \times (33 \times 10^{-3})^2} = 1.81 \times 10^{10}\,\text{secs} = 585\,\text{years}.$$

5.13 A binary star system comprises an ordinary star which shows periodic Doppler shifts of amplitude $\Delta\lambda = 0.168$ nm in an optical line of $\lambda_0 = 600$ nm, and a pulsar the apparent period of which varies with an amplitude $\Delta t = 1.26$ ms about a mean of $t_0 = 0.9$ s. If the primary star has a spectral type indicating a mass of $10M_\odot$, find the mass of the companion in units of $M_\odot$.

Given also that the orbital period is 2.78 hours, find the inclination of the system.

Solution: *no solution available*

5.14 Show that if matter accretes *directly* at a rate $\dot{M}$ on to a spherical mass and if its energy is radiated off as blackbody radiation, the temperature resulting is

$$T = \left(\frac{G\dot{M}\rho}{3\sigma}\right)^{1/4},$$

where σ is the Stefan-Boltzmann constant and ρ the mean density of the object. Calculate T for a white dwarf star of $1M_\odot$ and for a neutron star of $2M_\odot$ using the relevant values or expressions for G, σ and ρ from your notes. Assume mass increases by accretion on a timescale of 10^8 years.

Solution: Accretion luminosity

$$L_{acc} = GM\dot{M}/R$$

Blackbody luminosity of sphere

$$= 4\pi R^2\sigma T^4$$

Steady state implies that these two quantities are balanced:

$$4\pi R^2 \sigma T^4 = GM\dot{M}/R$$

so that

$$T = \left(\frac{G\dot{M}}{\sigma} \times \frac{M}{4\pi R^3}\right)^{1/4} = \left(\frac{G\dot{M}\rho}{3\sigma}\right)^{1/4}$$

5.15 Show that the mass M of a spherical mass accreting matter at the Eddington limit is increasing on a timescale

$$\tau = \frac{\sigma_{\rm p}}{4\pi m_{\rm p} c} \frac{M}{R},$$

where $\sigma_{\rm p}$ is the photo absorption cross-section for the accreting protons.

Solution: *no solution available*

5.16 It was at one time proposed that the Sun was kept luminous by the potential energy released due to infall ('accretion') of meteors. Calculate approximately how long it would take the mass of the Sun to double due to this process. (l year $= 3 \times 10^7$ s).

Can you suggest at least two observations which would refute this idea?

Solution: Potential energy released in infall of 1 kg onto radius R in gravitational field of mass M is

$$E = \frac{GM}{R} (\,\mathrm{J\,kg^{-1}})$$

If mass (meteors) falls in at a rate $\dot{M}$ kg s^{-1} then the rate of energy supply is

$$\dot{M}E = \frac{GM\dot{M}}{R} (\,\mathrm{W\ or\ \ J\,s^{-1}})$$

If this is to supply the luminosity we must have $\dot{M}E = L$ or

$$\dot{M} = \frac{RL}{GM}.$$

At this rate, the (approx.) time taken for the Sun's mass to double is the time to accumulate a further mass Mat a rate $\dot{M}$ or

$$\tau = \frac{M}{\dot{M}} = \frac{GM^2}{RL}.$$

Numerically,

$$\tau = \frac{6.7 \times 10^{-11} \times 4 \times 10^{60}}{7 \times 10^8 \times 4 \times 10^{26}} = 9.6 \times 10^{14}\,\mathrm{sec} = 3 \times 10^7\,\mathrm{year}.$$

Contradictory Observations

(1) If the Sun's mass doubled in 30 million years, the Earth's orbit would shrink (due to increased gravity). Thus 30 million years ago the earth would have been further from the sun. In addition the sun would have been cooler then. The existence of fossils older than 10^9 years rules this out.

(2) Even in 1 000 years the Sun's mass would change by 3×10^{-4} – enough to change the orbits of planets at a detectable level.

(3) The necessary $\dot{M}$ is quite large namely $M_\odot/\tau \simeq 2 \times 10^{15}\,\mathrm{kg\,s^{-1}}$. If this comprised micrometers of say $m = 1$ microgram each then the meteor inflow rate would be $\dot{M}/m = 10^{24}$ per second. At the Earth's orbit this inflow would occur over a sphere of radius $R = 1\,\mathrm{AU} = 1.5 \times 10^{11}\,\mathrm{m}$ so the flux would be $10^{24}/4\pi R^2 = 10\,\mathrm{m^{-2}\,s^{-1}}$. This is far higher than the meteor rate seen in space.

(4) There is no evidence (spectral) of heavy elements, of which meteors are made, being present in the Sun's atmosphere.

Notes:

(a) The calculation given is only a rough approximation. To be exact we should write

$$\tau = \int_M^{2M} \frac{GM'\mathrm{d}M'}{R(M')L(M')}.$$

The approximation used is effectively to say $R(M') = R(M)$, $L(M') = L(M)$ – the first terms in a Taylor series expansion. Usually such rough estimates are correct within factors of about 2.

(b) You might notice that this '*meteor*' timescale is actually the same as the Kelvin-Helmholtz time. That is because the K-H time is really the timescale to supply L by letting the Sun's R shrink so that the mass $M$of the sun releases energy by felling under the gravity of M itself. In the case of meteors, we are allowing M to increase by infall. But in both cases we are dealing with supply of L by changing the total gravitational energy

$$\Omega \simeq -\frac{GM^2}{R} \quad \text{so} \quad L = \frac{\mathrm{d}\Omega}{\mathrm{d}t}.$$

In one case varying R and in the other M.

5.17 In your notes, degenerate objects like white dwarfs and neutron stars have been described in terms of 'cold' degenerate pressure, P_{Qe}. In practice, such stars are quite hot when they form but at high densities, the thermal gas pressure, $P = \rho kT/(\mu m_\mathrm{p})$, is assumed to be negligible. Taking the expression for the degenerate electron pressure P_{Qe} etc. from your notes, show that the ratio of thermal to degenerate pressure at temperature T for a white dwarf of mass M is

$$\frac{P}{P_{\mathrm{Qe}}} = \frac{2\beta}{\mu}\left(\frac{3}{4\pi}\right)^{2/3} \frac{\hbar^2 kT}{G^2 m_\mathrm{e} m_\mathrm{p}^{8/3}} \frac{1}{M^{4/3}}.$$

Using the values of the constants from your notes, calculate what temperature T a white dwarf of $M = M_\odot$ would have to have before P was roughly equal to P_{Qe}. $\{3.2 \times 10^9\,\mathrm{K}\}$

(For this purpose take $\mu = \beta = 1$.)

Solution:

$$P = \frac{\rho kT}{\mu m_\mathrm{p}} \quad \text{and} \quad P_{\mathrm{Qe}} = \frac{\beta \hbar^2 \rho^{5/3}}{2 m_\mathrm{e} m_\mathrm{p}^{5/3}}.$$

Substitute for ρ for white dwarf stars, i.e.,

$$\rho = \frac{4\pi}{3} \frac{G^3 m_\mathrm{e}^3 m_\mathrm{p}^5 M^2}{\beta^3 \hbar^6},$$

to give

$$\frac{P}{P_{\mathrm{Qe}}} = \frac{2\beta}{\mu}\left(\frac{3}{4\pi}\right)^{2/3} \frac{\hbar^2 kT}{G^2 m_{\mathrm{e}} m_{\mathrm{p}}^{8/3}} \frac{1}{M^{4/3}}.$$

Numerically, with $\mu = \beta = 1$ this gives for $M = 2 \times 10^{30}$ kg:

$$\frac{P}{P_{\mathrm{Qe}}} = 3.1 \times 10^{-10} T,$$

so that P reaches the value P_{Qe} only for $T = 3.2 \times 10^9$ K.

5.18 Taking the gravitational energy of a uniformly dense sphere of mass M, radius R to be $\Omega = -3GM^2/(5R)$, show using the expression for R_{ns} from your notes that the gravitational energy of a neutron star of mass M is

$$\Omega = \frac{-3}{5\beta}\left(\frac{4\pi}{3}\right)^{2/3} \frac{G^2 M^{7/3} m_{\mathrm{p}}^{8/3}}{\hbar^2},$$

and calculate Ω for a neutron star of $M = 2M_{\odot}$ (taking $\beta = 1$). Compare this with the Sun's output $L_{\odot}$ over 10^{10} years.

A planet of the same mass m and radius r as Jupiter orbits a giant star at a distance $D = 8.2 \times 10^{11}$ m. This star turns supernova and releases energy $|\Omega|$ by the collapse of a $2M_{\odot}$ core into a neutron star. If 1 % of $|\Omega|$ goes into radiation, calculate the planet's rise in temperature. [To do this, assume that the planet intercepts a fraction $\pi r^2/(4\pi D^2) = (r/2D)^2$ of the total radiation and that the rise in temperature ΔT is given by $(m/m_{\mathrm{p}})kT = U$ where U is the total energy absorbed, k is Boltzmann's constant, and m_{p} is the proton mass (so that m/m_{p} is the number of 'atoms' in the planet)].

What would be the effect on the planet?

The other 99 % of energy goes into neutrinos. If each of these has energy 1 MeV, calculate the total number of neutrinos which would pass through you (assume an area of 1 m^2) if the explosion took place in the Larger Magellanic Cloud (distance 50 kpc).

Solution: We know that the gravitational potential energy of the neutron star, Ω, is

$$\Omega = -\frac{3GM^2}{5R_{\mathrm{ns}}},$$

and that its radius is

$$R_{\mathrm{ns}} = \left(\frac{3}{4\pi}\right)^{2/3} \frac{\beta\hbar^2}{G m_{\mathrm{p}}^{8/3} M^{1/3}}.$$

Therefore

$$\Omega = \frac{-3}{5\beta}\left(\frac{4\pi}{3}\right)^{2/3} \frac{G^2 M^{7/3} m_{\mathrm{p}}^{8/3}}{h^2}.$$

So with $M = 4 \times 10^{30}$ kg and $\beta = 1$ we get $\Omega = -6.4 \times 10^{47}$ J.

Comparison with the Sun: The luminosity of the Sun is $L_{\odot} = 4 \times 10^{26}$ J s^{-1}. In 10^{10} years, the Sun therefore emits $(4 \times 10^{26}) \times (3.1 \times 10^7) \times (10^{10}) = 1.2 \times 10^{44}$ J of energy.

Energy radiated is $E = 0.01 \times |\Omega| = 6.4 \times 10^{45}$ J. Energy intercepted is

$$\begin{aligned}
E' &= E \times \text{(fraction of sphere covered by cross sectional area of planet)} \\
&= E\left(\frac{r}{2D}\right)^2 \\
&= (6.4 \times 10^{45}\,\mathrm{J}) \times \left(\frac{7.2 \times 10^7}{2 \times 8.2 \times 10^{11}}\right)^2 \\
&= 1.2 \times 10^{37}\,\mathrm{J}
\end{aligned}$$

Hence

$$\Delta T = \frac{1.2 \times 10^{37}}{k} \frac{m}{m_{\rm p}} = 7.5 \times 10^5\,{\rm K}\,,$$

which is sufficient to vaporise all solid material.

Total neutrino energy $E_\nu = 0.99|\Omega| \simeq |\Omega|$. So the number of neutrinos emitted is $N = \frac{|\Omega|}{1\,{\rm MeV}} = \frac{6.4\times10^{47}}{1.6\times10^{-19}\times10^6} = 4 \times 10^{60}$.

Number intercepted by $1\,{\rm m}^2$ at $R = 50\,{\rm kpc}$ is

$$\begin{aligned} N_{\rm intercept} &= N \times \frac{1}{4\pi R^2} \\ &= \frac{4 \times 10^{60}}{4\pi \times (50\,000)^2 \times (3.1 \times 10^{16})^2} \\ &= 1.3 \times 10^{17}\,. \end{aligned}$$

NOTE - these simply pass through a human body because the absorption cross section is so tiny.

5.19 The angular velocity of rotation, Ω, of a self-gravitating sphere of mass M, radius R, is limited by centrifugal break-up to $\Omega = \Omega_1$ as described in your notes. In addition, relativity forbids any part of the rotating body to move faster than the speed of light c, so setting another limit $\Omega = \Omega_2$. Show that the centrifugal limit, Ω_1, always applies first (i.e., show that $\Omega_1 < \Omega_2$) since, for the reverse to be true would imply $R < R_{\rm S}/2$, where $R_{\rm S}$ is the Schwarzschild radius, so that the body would be a black hole.

Solution:

$$\Omega \le \Omega_1$$

where

$$\Omega_1^2 R = GM/R^2 \text{ centrifugal limit}$$

i.e.

$$\Omega \le \Omega_1 = \sqrt{GM/R^2}.$$

$$\Omega \le \Omega_2$$

where $\Omega_2 R =$ tangential speed $= c$. So

$$\Omega \le \Omega_2 = c/R$$

$$\frac{\Omega_1}{\Omega_2} = \sqrt{GM/R^2} \times \frac{R}{c} = \left(\frac{GM}{Rc^2}\right) = \sqrt{R_s/(2R)}$$

If we had $\Omega_1 > \Omega_2$ then $R < R_2/2$ and the object would be a black hole. Thus the Ω_1 limit is encountered first.

5.20 A star of mass M and radius R, made of ionised hydrogen, collapses without mass loss to become a white dwarf of radius $R_{\rm wd} \ll R$, and in doing so loses gravitational energy Ω. Taking the gravitational energy at radius R to be $\Omega = -3GM^2/(5R)$, and assuming that half of the energy lost goes into heat, show that the star's temperature increases during collapse by an amount

$$T = \frac{GMm_{\rm p}}{10kR_{\rm wd}},$$

where k is Boltzmann's constant, G the constant of gravitation, and $m_{\rm p}$ the proton mass.

Hence calculate the mean temperature at formation of a white dwarf of 1 solar mass, and radius 2.3×10^6 m, neglecting the initial temperature. If the surface of the star had this temperature in what wavelength range would you expect it to emit?

Solution: The gravitational energy lost as heat is

$$0.5\Delta\Omega = 0.5(\Omega_1 - \Omega_2) = 0.5\left(\frac{-3GM^2}{5R_1} - \frac{-3GM^2}{5R_2}\right) \approx \frac{3GM^2}{10R_2}\,,$$

(since $R_1 \gg R_2$), where R_2 is the radius of the white dwarf, R_{wd}, and R_1 is the radius of the original star.

The thermal energy of electrons and protons is $3kT/2$ per particle or $3kT$ per (p,e) pair. There are $N = M/m_{\mathrm{p}}$ such pairs, so $U = 3MkT/m_{\mathrm{p}}$. Therefore

$$\Delta U = 3Mk\Delta T/m_{\mathrm{p}}\,.$$

Equating $\Delta U = 0.5\Delta\Omega$, and assuming $\Delta T \gg T_{\mathrm{initial}}$, gives a final temperature of

$$T = \frac{GMm_{\mathrm{p}}}{10kR_{\mathrm{wd}}}$$

as required. Substituting numbers gives

$$T = \frac{6.6\times10^{-11}\times2\times10^{30}\times1.7\times10^{-27}}{10\times1.4\times10^{-23}\times2.3\times10^{6}} = 7\times10^{8}\,\mathrm{K}\,.$$

From Wien's displacement law ($\lambda_{\max}T =$ constant) the emission would be in the X-ray wavelength range ($\lambda_{\max} \approx 4.3\times10^{-3}$ nm).

5.21 (a) Each of two white dwarf stars has mass M_1 and radius R_1 and is rotating at angular velocity ω_1. If these stars coalesce into a single white dwarf of mass $2M_1$, and corresponding white dwarf radius, and if rotational angular momentum ($\sim MR^2\omega$) is conserved, show that the angular velocity of the new single star is

$$\omega_2 = \omega_1(M_2/M_1)^{2/3} = 2^{2/3}\omega_1.$$

If the initial stars are at their critical centrifugal break-up angular speed of

$$\omega_1 = (GM_1/R_1^3)^{1/2},$$

show that the new star rotates below its critical break- up ω value by a factor of $2^{1/3}$.

(b) A rotating neutron star gradually increases in mass due to accretion of matter. Angular momentum is conserved in the process and the accreted matter initially has no angular momentum. Use the $R(M)$ relation for neutron stars to show that the angular speed ω, and rotational kinetic energy E, vary with M according to

$$\frac{\omega}{\omega_0} = \frac{E}{E_0} = \left(\frac{M_0}{M}\right)^{1/3}$$

while the (negative) gravitational binding energy Ω varies as

$$\frac{\Omega}{\Omega_0} = \left(\frac{M}{M_0}\right)^{7/3}$$

where subscript $_0$ refers to initial values.

Is the total energy $E + \Omega$ decreasing or increasing? Where is the energy going to, or coming from?

Solution:

(a) Angular momentum $AM = I\omega \sim MR^2\omega$.

$$\begin{aligned}
\text{Conservation of } AM \quad &\Rightarrow M_2 R_2^2 \omega_2 = 2M_1 R_1^2 \omega_1 \\
&\Rightarrow \frac{\omega_2}{\omega_1} = 2\frac{M_1}{M_2}\frac{R_1^2}{R_2^2} \\
\text{but} \quad R \sim M^{-1/3} \quad \text{so} \quad & \frac{\omega_2}{\omega_1} = 2\left(\frac{M_1}{M_2}\right)^{1/3} = 2(1/2)^{1/3} = 2^{2/3}
\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad \omega_{\text{crit}} &= \sqrt{GM/R^3} \\
\text{and} \quad \omega_1 = \omega_{\text{crit}_1} &= \sqrt{GM_1/R_1^3} \\
\text{and} \quad \omega_{\text{crit}_2} &= \sqrt{GM_2/R_2^3}
\end{aligned}$$

$$\begin{aligned}
\text{so,} \quad \frac{\omega}{\omega_{\text{crit}_2}} &= \frac{2^{2/3}\omega_1}{\sqrt{GM_2/R_2^3}} = \frac{2^{2/3}\sqrt{GM_1/R_1^3}}{\sqrt{GM_2/R_2^3}} \\
&= 2^{2/3}\sqrt{\frac{M_1 R_2^3}{M_2 R_1^3}} = 2^{2/3}\sqrt{\frac{M_1^2}{M_2^2}} = \frac{2^{2/3}}{2} = 2^{-1/3}.
\end{aligned}$$

(b) Angular momentum $AM = I\omega \sim MR^2\omega \sim M^{1/3}\omega$ (since $R \sim M^{-1/3}$). So, if AM is constant, $\omega \sim M^{-1/3}$ or

$$\frac{\omega}{\omega_0} = \left(\frac{M_0}{M}\right)^{1/3}$$

Rotational kinetic energy $E = I\omega^2 \sim MR^2\omega^2 \sim M^{1/3}\omega^2$ so

$$\frac{E}{E_0} = \left[\frac{\omega}{\omega_0}\right]^2 \left[\frac{M}{M_0}\right]^{1/3} = \left[\frac{M_0}{M}\right]^{2/3} \left[\frac{M}{M_0}\right]^{1/3} = \left[\frac{M_0}{M}\right]^{1/3}$$

$$\begin{aligned}
\text{Gravitational energy} \quad & \Omega = -\frac{2}{5}\frac{GM^2}{R} \propto \frac{M^2}{M^{-1/3}} \propto M^{7/3}. \\
\text{So} \quad & \frac{\Omega}{\Omega_0} = \left(\frac{M}{M_0}\right)^{7/3} \\
\text{Total energy} \quad & E + \Omega = E_0\left(\frac{M_0}{M}\right)^{1/3} + \Omega_0\left(\frac{M_0}{M}\right)^{7/3}
\end{aligned}$$

For large M the second term dominates and the total energy increases in size but is negative since Ω is negative. This excess loss of gravitational energy will go into heat and radiation.

5.22 A neutron star radiates by direct accretion onto its surface (i.e., no accretion disk is formed) at a rate set by the Eddington limit. Show that the temperature T of the resulting blackbody radiation from the neutron star surfaces varies as its mass M increases, according to

$$\frac{T}{T_0} = \left(\frac{M}{M_0}\right)^{5/12}.$$

Solution: *no solution available*

5.23 (a) White dwarf material is almost perfectly conducting so that such stars can be taken as having a uniform temperature T. If a white dwarf has mass M and radius R and is taken to be made up of protons of mass m_{p}, and electrons of very small mass, each with thermal energy $\frac{3}{2}kT$, use Stefan's law to show that radiation will cool a white dwarf on a timescale of about

$$\tau_{\mathrm{cool}} \simeq \frac{3Mk}{4\pi m_{\mathrm{p}} \sigma T^3 R^2},$$

where k and s are the Boltzmann and Stefan-Boltzmann constants. Hence

(i) Calculate the cooling time for a white dwarf of mass 4×10^{30} kg and radius 2×10^6 m, at temperature $T = 10^7$ K.

(ii) Show that two white dwarfs of masses M_1, M_2 and of temperatures T_1, T_2 have cooling times in the ratio

$$\frac{\tau_{\mathrm{cool1}}}{\tau_{\mathrm{cool2}}} = \left(\frac{M_1}{M_2}\right)^{5/3} \left(\frac{T_2}{T_1}\right)^3,$$

and luminosity L_1, L_2 in the ratio

$$\frac{L_1}{L_2} = \left(\frac{M_2}{M_1}\right)^{2/3} \left(\frac{T_1}{T_2}\right)^4.$$

(b) A more exact description of the cooling described above is to consider the differential equation which says that rate of charge of thermal energy equals the rate of radiative output. That is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3MkT}{m_{\mathrm{p}}}\right) = \frac{3Mk}{m_{\mathrm{p}}}\frac{\mathrm{d}T}{\mathrm{d}t} = -4\pi R^2 \sigma T^4,$$

where M, R are constant. Show that the resulting time evolution of the temperature $T(t)$ is then given by

$$\frac{T(t)}{T(0)} = \frac{1}{(1 + 3t/\tau_{\mathrm{cool}})^{1/3}},$$

where τ_{cool} is as given in the first part of the question for $T = T_0$.

Solution:

(a)

$$\tau_{\mathrm{cool}} \simeq \frac{\text{Total thermal energy content, } E}{\text{Rate of energy loss by radiation, } L}$$

$$\text{with} \qquad E = \text{Total number of particles} \times \frac{3}{2}kT = 2 \times \frac{M}{m_{\mathrm{p}}} \times \frac{3}{2}kT$$

,where 2 allows for electrons and protons, and

$$L = 4\pi R^2 \sigma T^4,$$

$$\text{thus} \qquad \tau_{\mathrm{cool}} = \frac{M}{m_{\mathrm{p}}} \times 3kT \times \frac{1}{4\pi R^2 \sigma T^4} = \frac{3Mk}{4\pi m_{\mathrm{p}} \sigma T^3 R^2}.$$

(i)

$$\tau_{\mathrm{cool}} = \frac{4 \times 10^{30}}{1.7 \times 10^{-27}} \times \frac{3}{4\pi} \times \frac{1.4 \times 10^{-23}}{5.7 \times 10^{-8}} \times \frac{1}{(10^7)^3} \times \frac{1}{(2 \times 10^6)^2} = 3.45 \times 10^7 \text{ s} = 1.11\text{years}$$

(ii)

$$\frac{\tau_{\text{cool}_1}}{\tau_{\text{cool}_2}} = \frac{3M_1k}{4\pi m_{\text{p}}\sigma T_1^3 R_1^2} \times \frac{4m_{\text{p}}\sigma T_2^3 R_2^2}{3M_2k} = \frac{M_1}{M_2}\left(\frac{R_2}{R_1}\right)^2\left(\frac{T_2}{T_1}\right)^3$$

For White Dwarf stars

$$\frac{R_2}{R_1} = \left(\frac{M_1}{M_2}\right)^{\frac{1}{3}}$$

$$\therefore \frac{\tau_{\text{cool}_1}}{\tau_{\text{cool}_2}} = \left(\frac{M_1}{M_2}\right)^{\frac{5}{3}}\left(\frac{T_2}{T_1}\right)^3$$

$$\text{and} \qquad \frac{L_1}{L_2} = \frac{R_1^2}{R_2^2}\frac{T_1^4}{T_2^4} = \left[\left(\frac{M_2}{M_1}\right)^{\frac{1}{3}}\right]^2\frac{T_1^4}{T_2^4} = \left(\frac{M_2}{M_1}\right)^{\frac{2}{3}}\left(\frac{T_1}{T_2}\right)^4.$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3MkT}{m_{\text{p}}}\right) = \frac{3Mk}{m_{\text{p}}}\frac{\mathrm{d}T}{\mathrm{d}t} = -4\pi R^2\sigma T^4$$

$$\therefore \frac{\mathrm{d}T}{T^4} = -\left(\frac{4\pi m_{\text{p}}\sigma R^2}{3Mk}\right)\mathrm{d}t = -\mathrm{d}\left(\frac{1}{3T^3}\right)$$

$$\therefore \frac{1}{T^3(t)} = \left(\frac{4\pi m_{\text{p}}R^2\sigma}{Mk}\right)t + \text{constant}$$

and constant = $\frac{1}{T_0^3}$ from initial condition

$$\therefore \frac{T_0^3}{T^3} = \frac{4\pi m_{\text{p}}R^2\sigma}{Mk}T_0^3 t + 1 = \frac{3t}{\tau_{\text{cool}}} + 1$$

Hence result.

Note that τ_{cool} decreases as T^{-3} so that at $T = 10^6$ K the cooling time will have increased from $\simeq 1$ year to $\simeq 10^4$ years and at $T = 10^5$ K to 10^7 years. This shows that τ_{cool} is an estimate of the cooling time based on the local gradient $\frac{\mathrm{d}T}{\mathrm{d}t}$

5.24e What are the main observational features of *pulsars*? [2]

Describe briefly the accepted model of pulsars and explain the origin of their high magnetic fields and spin rates. Why do they produce high energy particles ? [3]

Solution: *no solution available*

5.25e What is meant by the term *black hole*? Name, and obtain an expression for, the effective radius of a black hole of mass M. [2]

Name, and give the value of, the mass above which cold stellar remnants are expected to form black holes. What evidence is there for the existence of black holes? [3]

Solution: *no solution available*

5.26e Why are gravitationally accreting neutron stars proposed to explain some powerful compact X-ray sources? What provides the accreting mass and by what two mechanisms can this mass be transferred to the neutron star? [6]

Write down an expression for the accretion luminosity for a neutron star of mass M and radius R, and derive the Eddington radiation pressure limit to this luminosity. [4]

If the neutron star were to accrete the mass directly onto its surface (no accretion disk) show, using Stefan's law, that the steady surface temperature, T, would be

$$T = \left[\frac{GM\dot{M}}{4\pi R^3 \sigma}\right]^{1/4}$$

[4]

and evaluate T for $M = 4 \times 10^{30}$ kg, $R = 10^4$ m, $\dot{M} = 10^{-6} \mathrm{M}_\odot \, \mathrm{yr}^{-1}$. [3]

[$1\mathrm{yr} = 3.2 \times 10^7$ sec]

Solution: *no solution available*

5.27e Describe the main observational and physical properties of white dwarf and neutron stars, and explain how these two classes of object arise in models of stellar evolution. [8]

Show that for any self-gravitating spherical object of radius R and mass M there is a minimum spin period given by

$$P_{\mathrm{min}} \approx 2\pi \left(\frac{R^3}{GM}\right)^{1/2}$$

where G is the gravitational constant, such that if the period is shorter than P_{min} the object will disintegrate. [5]

Estimate the ratio of P_{min} for a white dwarf compared to a neutron star, if the white dwarf has a radius 500 times larger than the neutron star. Assume that both objects are close to the Chandrasekhar limit.

By evaluating the minimum periods for such objects using typical values, how does this information support the idea that pulsars are rapidly rotating neutron stars? [4]

Solution: *no solution available*

5.28e Describe briefly what happens in a type-II supernova from the instant that the stellar core begins to collapse after nuclear fusion ceases. [5]

Solution: *no solution available*

5.29e Summarise the observational evidence for the existence of white dwarfs and neutron stars. [5]

By estimating the gravitational attraction between two halves of a uniform sphere, show that the pressure P_c at the core of a star of mass M and radius R is approximately

$$P_c = \frac{GM^2}{4\pi R^4}$$ [4]

and hence that $P_c \approx M^{2/3} \rho^{4/3}$, where ρ is the stellar mass density. [3]

In white dwarfs, this pressure is balanced by an outward pressure that increases as $\rho^{5/3}$. Describe the source of this pressure and show that white dwarf star radii vary as $R \approx M^{-1/3}$. [3]

Compare this with the corresponding result for $R(M)$ for stars of constant density. [2]

Solution: *no solution available*

5.30e What are the observational characteristics of pulsars? Obtain an expression for the minimum rotation period of a gravitationally bound spherical object of density ρ. [5]

Solution: *no solution available*

5.31e Draw a ($\log P$, $\log \rho$) diagram for cold dense matter, labelling the main sources of pressure in each part. Show the tracks of contracting stellar remnants and the locations of white dwarf stars, neutron stars and black holes. [5]

Solution: *no solution available*

5.32e Starting from the hydrostatic equation $DdD/dr = -\rho g$ show that the central pressure $P - c$ in a spherical mass M of uniform density density ρ is

$$P_c = \left(\frac{\pi}{6}\right)^{\frac{1}{3}} GM^{\frac{2}{3}}\rho^{\frac{4}{3}}.$$ [4]

Given the expressions (1) $\Delta p \Delta x \geq \hbar$ where Δx and Δp are the uncertanties in particle position and momentum, and (2) $P = \beta n E$, which gives the quantum degeneracy pressure P for particles of energy E and number density n, and where β is a number, show that for electrons in hydrogen of matter density ρ,

$$P = \frac{\beta\hbar^2}{2m_e}\left(\frac{\rho}{m_p}\right)^{\frac{5}{3}},$$

where m_e and m_p are the electron and proton mass, respectively. [3]

[You may use $E = p^2/(2m)$.]

Hence show that the radius R_{WD} and density ρ_{WD} of a white dwarf star are

$$R_{WD} = \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \frac{\beta\hbar^2}{G m_e m_p^{5/3}} M^{-\frac{1}{3}},$$

$$\rho_{WD} = \frac{4\pi}{3\beta^3}\frac{G^3 m_e^3 m_p^5}{\hbar^6} M^2.$$ [4]

Write down approximate corresponding expressions for a neutron star, explaining why P_{Qe} is not relevant and what replaces it. [2]

If the surface temperatures T of white dwarfs at the time of formation satisfy $T \propto M/R$ show that their blackbody luminosities L then would obey $L \propto M^{\frac{14}{3}}$. [4]

Solution: *no solution available*

5.33e State the relationship between radius R and mass M for white dwarf stars and for neutron stars. [1]
Roughly what is the ratio of the radii of white dwarf stars to those of neutron stars? [1]
Write down an expression for the luminosity, L, of a compact object accreting mass. [1]
Show that, if the accretion rate is fixed, $L \propto M^{4/3}$. [2]

Solution: *no solution available*

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6 Observational methods

6.1e A galaxy is imaged by optical, infra-red, radio and X-ray telescopes. Describe briefly the likely differences in the features of the four maps. [5]

Solution:

Optical - optical emission and the effects of absorption

IR - thermal emission from dust

radio - hydrogen emission

X-ray - individual high energy 'point' sources

Where dark bands appear in the optical, bright bands appear in the infra-red – the halo distribution and the dust follow the galactic arms. The X-ray sources have some concentration in the galactic plane – but there are some objects in the halo.

6.2e A comet displays a tail of 15° and it is planned to take photograph of it with a regular 35 mm camera with the picture format having dimensions 24 mm × 36 mm. Calculate the focal length of the lens that would be appropriate. [5]

Solution: Assume tail to be along one of the sides of the format – for example, 36 mm
Now,

$$s = F\theta \Rightarrow F = \frac{s}{\theta} = \frac{36 \times 360}{15 \times \pi} = 166\,\mathrm{mm}.$$

(other starting assumptions would be acceptable but the resulting value should be between 100 and 200 mm.)

6.3e Describe the physical reasoning behind the criterion that the minimum angle resolvable by a telescope may be represented by

$$\alpha = \frac{1.22\lambda}{D},$$

where λ is the wavelength of the radiation and D the diameter of the telescope. [5]

Solution: Point images produce diffraction patterns in circular symmetry. Overlap of patterns occurs if objects are close to each other. From the form of the pattern a useful criterion emerges whereby point images can be considered to be resolved if the centre of the second image coincides with the first dark ring of the first image $\Rightarrow$

$$\alpha = \frac{1.22\lambda}{D}.$$

A diagram would be useful.

6.4e Define the term *parallactic angle*. Given that the smallest annual parallax that can be measured by ground based telescopes is 0.001 arcsec, calculate the maximum stellar distance that can be measured in parsecs and in metres. [5]

Solution: 'Parallactic angle' is the shift in angular position brought about by a change in the viewpoint; the amount of shift depends on the baseline between the two viewpoints and inversely on the distance of the object.

$$0.001\,\text{arcsec} = 1\,000\,\text{pc}$$

$$\frac{1\,\text{AU}}{d} = \frac{0.001}{206\,265}$$

$$\therefore d = \frac{206\,265}{0.001} \times 1.496 \times 1011\,\text{m} = 3.08 \times 10^{19}\,\text{m}.$$

6.5e Explain why large objective telescope designs are no longer considered in preference to reflector systems. [5]

Solution:

- Large lenses are heavy and can only be supported around the rim – optical distortions caused by changes in orientation.
- Chromatic problems.
- Large lenses are thick – absorption is serious.
- Chances of encountering a defect (bubble) are greater with thicker materials.
- Large lenses $\Rightarrow$ long focal lengths which usually require matching tube length.

All of the above are not apparent or are of reduced significance if the system is a reflector.

6.6e Describe the optical layout of a Cassegrain telescope. [6]

Explain the use of a Fabry or field lens in the design of a photoelectric photometer. [5]

Such an instrument is designed for use with a 2.2 m telescope with a focal ratio of $f/12$. Calculate the diameter in the focal plane of the field stop which limits the field of view to 15 arcsec. [6]

Solution: Diagram showing primary mirror with central hole, spherical primary; secondary mirror (convex) altering the focal ratio and directing the radiation to a focus behind the primary rear plane.

Non-uniformity of the photocathode would produce spurious signals if the star image position moves during measurement. Seeing and telescope tracking make this inevitable. A simple ray diagram shows that by spreading out the light such that the telescope aperture is focused on the photocathode, the light patch remains fixed for small image displacements.

$$\text{Focal length} = 2.2 \times 12\,\text{m},$$

$$s = F\theta = \frac{2.2 \times 12 \times 15}{206\,265}\,\text{m} = 0.019\,\text{m} = 1.9\,\text{mm}.$$

6.7e Explain why a dish type radio telescope provides poorer angular resolution than its optical counterpart. [7]

Describe the operation of a simple two-element radio interferometer. [5]

An interferometer with baseline of 1 km operates at a frequency of 3 000 MHz. Estimate the angular resolution of the system. [5]

Solution: Compare typical wavelengths and telescope dimensions in the resolution equation $\theta \sim \frac{\lambda}{D}$ optical $\frac{500\,\mathrm{nm}}{1\,\mathrm{m}} \Rightarrow 5 \times 10^{-7}$ radians, radio $\frac{0.1\,\mathrm{m}}{20\,\mathrm{m}} \Rightarrow 5^{-3}$ radians. Radio is worse off by a factor of $\sim 10^4$.

The radio antenna along an EW direction (PIC!)

Constructive interference when $D \sin \alpha = n\lambda$ the receiver displays interference fringes as the object drifts across the meridian. Its position can be 'highlighted' from the record of the central fringe, resolution $\propto \frac{\lambda}{D}$,

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{3 \times 10^9} = 0.1\,\mathrm{m} \Rightarrow \propto \sim \frac{0.1}{10^3} \sim 10^{-4}\,\text{radians}.$$

6.8e Describe the basic photographic process for recording images of galactic clusters. [7]

Explain how magnitude values might be obtained by laboratory analysis of the plate. [7]

What is meant by the phrase: *the dynamic range of a photographic star plate is 5 magnitudes*? [4]

Solution:

- Exposures of plate in focal plane of telescope/camera.
- Development to convert ionised silver bromide crystals to silver grains.
- Fixing process to work out remaining Ag Br crystals.
- Laboratory data reduction using a machine like COSMOS.

Because of seeing and scattering, the diameters of stellar images depend on the stellar magnitude. If a threshold level of image strength is used to measure the image diameter there is a linear relationship between m and D.

The linearity only operates over a limited range of magnitude values: for faint stars, no record; for bright stars, over exposure. As a plate is exposed, the bright stars are over exposed and fainter and fainter stars are recorded. At the end of the exposure, there is a range of stellar magnitudes which are recorded in the linear response region: linearity covers a range ~ 5 mags.

6.9e Describe the layout of a simple X-ray telescope. [6]

Describe the principle of a proportional counter for use as an X-ray detector. [6]

An X-ray source is at 50 pc and has a luminosity of 6×10^{31} W, chiefly liberated in the X-ray region. Estimate the detector pulse count production rate if the telescope has a collection area of $840\,\mathrm{cm}^2$. [5]

[Assume X-ray photon energy $= 4\,\mathrm{keV} \equiv 6.4 \times 10^{-16}$ J.]

Solution: One of these designs might be presented

(a) 'Box of tubes'.

(b) Fourier grid design.

(c) Grazing incidence focussing system.

Expect to see labelled diagram of detector system with comments about location of passage of X-ray by resistive anode and analysis of pulse shape.

Flux of X-ray radiation at Earth

$$= \frac{6 \times 10^{31}}{4\pi(50 \times 3.086 \times 10^{16})^2}\,\mathrm{m}^{-2}.$$

Flux entering telescope

$$= \frac{6 \times 10^{31}}{4\pi(50 \times 3.086 \times 10^{16})^2}\frac{840}{10^4}.$$

A 4 keV X-ray photon $\rightarrow 4 \times 1.6 \times 10^{-16}\,\mathrm{J} = 6.4 \times 10^{-16}\,\mathrm{J}$. Hence event rate

$$= \frac{6 \times 10^{31} \times 840}{4\pi(50 \times 3.086 \times 10^{16})^2}\frac{1}{10^4 \times 6.4 \times 10^{-16}}$$
$$\simeq 2.6 \times 10^8\,\mathrm{s}^{-1}.$$

6.10 (a) The CO molecule produces a spectral line at 230 GHz. What is the wavelength of the line? {1.3 mm}

(b) Using the identity $\lambda_{\mathrm{nm}} = 1\,240/E_{\mathrm{eV}}$, where λ_{nm} is the wavelength in nanometres and E_{eV} is the energy in electronvolts, calculate the energy of a photon associated with Hβ emission with wavelength of 486.1 nm. {2.55 eV}

(c) A band of X-ray radiation has a wavelength of l nm. What is the photon energy expressed in keV? {1.24 keV}

[1 GHz = 10^9 Hz]

Solution:

(a) $\lambda = \frac{c}{\nu} = \frac{3\times10^8}{230\times10^9}$ m or 1.30 mm.

(b) $E_{\mathrm{eV}} = \frac{1\,240}{486.1} = 2.55\,\mathrm{eV}$.

(c) The energy of a photon is $E = h\nu = \frac{hc}{\lambda} = \frac{6.6\times10^{-34}\times(3\times10^8)}{1\times10^{-9}}$ J. Expressed in eV this is 1.24 keV.

6.11 A star provides $4 \times 10^{-17}\,\mathrm{W\,m^{-2}}$ of Hα radiation at the bottom of the Earth's atmosphere. Calculate the number of Hα photons per second entering a telescope of 500 mm diameter.

Assuming that the main source of noise is the randomness associated with photon counting, estimate the signal-to-noise ratio of an Hα brightness measurement of the star using an integration time of 50 s.

Solution:

$$\mathrm{H}_\alpha \text{ radiation entering the telescope} = 4 \times 10^{-17} \times \frac{\pi}{4}(0.5)^2\,\mathrm{W}$$
$$= 7.85 \times 10^{-18}\,\mathrm{W}.$$

Wavelength of $\mathrm{H}_\alpha = 656.2$ nm.

$$\text{Energy of } \mathrm{H}_\alpha \text{ photon} = h\nu = \frac{hc}{\lambda} = \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{6.562} \times 10^{-7}\,\mathrm{J}$$
$$= 3.02 \times 10^{-19}\,\mathrm{J}.$$

$$\text{Hence the number of photons per sec entering the telescope} = \frac{7.85 \times 10^{-18}}{3.02 \times 10^{-19}}$$
$$= 26 \quad \text{photons per sec.}$$

In 50 second integration, expected photon count $= 50 \times 26 = 1300$.

For photon counting, s/n $\simeq N/\sqrt{N} = \sqrt{N} = 36$.

(In reality, the situation is worse than this because of transmission losses in the telescope and the quantum efficiency of the detector. Recorded count would be $\simeq$ a factor of 10 less, giving a signal to noise ratio which is a factor $\simeq \sqrt{3}$ lower.)

6.12 (a) The 2.5 m Isaac Newton Telescope is used at $f/15$ with a 5 mm square CCD chip at its focus. Calculate the field of view that would be recorded within the picture format. [5]

(b) (i) An $f/10$ telescope with a focal length of 3 000 mm is used with an eyepiece of 20 mm focal length. What is the magnifying power of the system? Would this magnification allow all the collected light to enter the pupil of the eye? [6]

(ii) A double star has a separation of 3″ (3 arcsec) on the sky. What would be the physical separation of the images in the telescope focal plane? [4]

(iii) The limiting magnitude for this telescope is 13.6. Based on this figure, what limiting magnitude might be expected for a telescope of 500 mm diameter? [5]

Solution:

(a) In order to determine the size of images in the telescope focal plane, the focal length value is required. Now $F = fD = 15 \times 2.5 = 37.5$ m. The angle subtended by a dimension l in the focal plane is given by $\theta = l/F = 5 \times 10^{-3}/37.5$ radians, or $(5 \times 10^{-3}/37.5) \times 206\,265 =$ 27.5 arcsec.

(b) (i) Magnifying power is $F_t/F_e = \frac{3\,000}{20} = 150$. Magnifying power is also defined by D_t/D_e. Hence $D_e = D_t/M$. Now the diameter of the telescope is given by $F/f = \frac{3\,000}{10} = 300$. Therefore, $D_e = \frac{300}{150} = 2$ mm. This is smaller than the eye's pupil and all the light can be accepted by the eye.

(ii) $l = F\theta = \frac{3\,000 \times 3}{206\,265} = 0.044$ mm.

(iii) The limiting magnitude of the telescope depends on the amount of energy collected from the star, which in turn depends on the area of the telescope's aperture, or the square of its diameter, D. A telescope with an aperture of 500 mm is therefore $500^2/300^2 = 2.78$ times more sensitive than one of aperture 300 mm. Two stars that differ in flux density by a factor of 2.78 will differ in magnitude by $\Delta m = -2.5 \log_{10} 2.78 = -1.1$. The new limiting magnitude is therefore 13.6 +1.1 = 14.7. (Larger number means fainter.)

6.13 (a) A 3-mirror coudé telescope feeds an all-reflection spectrometer comprising a collimator, a diffraction grating and a camera. The entrance slit of the instrument allows through only 50 % of the light contained in star images. Assuming the reflectivity of all the mirrors to be 80 % and the efficiency of the reflection grating to be 50 % , estimate the overall optical transmittance of the system. {8.2%} [4]

(b) An astronaut observes a star while in a launch vehicle and records its magnitude as 4.5 when it is at a zenith distance of 25°. Given that the zenith extinction is 0.5 mag, what magnitude would the star appear to have when viewed from the module in orbit? {3.95} [4]

(Bouger's law for atmospheric extinction may be written as $m(\zeta) = m_0 + \Delta m \sec \zeta$.)

Solution:

(a) Transmittance of telescope $= (0.8)^{0.3} = 0.512$.
Effect of slit cut-off $= 0.5$.
Transmittance of spectrometer $= 0.8 \times 0.5 \times 0.8 = 0.32$.
Total transmittance efficiency $= 0.512 \times 0.5 \times 0.32 = 0.082$ or 8.2 % .

(b) Above the Earth's atmosphere, the astronaut would see the star without extinction $\equiv m_0$. Now $m(\zeta) = m_0 + \Delta m \sec \zeta$, therefore $m_0 = m(\zeta) - \Delta m \sec \zeta = 4.5 - 0.5 \sec(25^\circ) = 3.95$.

6.14 A 3-mirror coudé, telescope has a primary mirror of 2 m diameter and is used in the infra-red, the pass band of the detector having a wavelength of 1.2 μm. What is the theoretical angular resolution of the telescope?

Solution:

$$\alpha = 1.22\lambda/D = 1.22 \times 1.2 \times 10^{-6}/2 = 7.32 \times 10^{-7}$$

6.15 Explain why the sizes of in-focus images of stars photographed by a telescope reflect their flux densities. [6]

From a calibration study using standard stars in the field, it is found that a measured image diameter, d (in μm), can be related to a magnitude by

$$m = 10 - \log_{10} d.$$

Two stars are measured to have image diameters of 100 and 150 μm respectively. What is their magnitude difference? {0.176 mag.} [4]

Solution:

$$\begin{aligned}\Delta m = m_1 - m_2 &= -\log_{10} 100 + \log_{10} 150 \\ &= -2 + 2.176 = 0.176 \quad \text{magnitudes}.\end{aligned}$$

6.16 For a photographic plate with a gamma of $\gamma = 0.6$ it is found that reasonable exposures cover a density range of 2. If a star of 8^{th} magnitude is at the limit of over exposure, calculate the brightness of the faintest star that can be recorded satisfactorily so allowing a good measurement of its brightness.

Solution: *no solution available*

6.17 Compare and contrast the use of photographic plates and photomultiplier tubes as a means of performing stellar observations.

It is planned to observe photometrically 25 stars within a stellar cluster to an accuracy of ± 0.05 magnitudes. Estimate whether it is better in terms of efficiency of telescope time to perform the measurements using photography or whether a photoelectric photometer might be used.

Solution: *no solution available*

6.18 Sketch out the design of an imaging X-ray telescope.

Describe the principle of a Proportional Counter as used in X-ray Astronomy.

An X-ray collimating telescope comprises a honeycomb of tubes with a total collecting area of $0.5\,\mathrm{m}^2$. A source has an emission line close to 10 nm and provides a flux of $10^{-18}\,\mathrm{W\,m^{-2}}$ at this wavelength. Estimate the count rate from a proportional counter when this telescope system is directed to the source.

Solution: *no solution available*

6.19 The Lovell radio dish at Jodrell Bank has a diameter 80 m and the receivers can be tuned to 1 420.4 MHz, with a band-pass of 10 MHz. A point radio source of 2.5 janskys (Jy) is observed. What is the theoretical angular resolution of this system? What is the power available for detection?

$(1\,\text{Jy} = 10^{-26}\,\text{W}\,\text{m}^{-2}\,\text{Hz}^{-1})$

Solution:

$$\alpha = \frac{1.22\lambda}{D}$$

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8}{1.42 \times 10^9} = 0.21\,\text{m}$$

$$\text{Hence} \quad \alpha = \frac{1.22(0.21)}{80}\ \text{radians}$$

$$= 0.0032\ \text{radians}$$

$$\text{so} \quad \alpha = 11\ \text{arcmin.}$$

1 Jansky $= 10^{-26}\text{W m}^{-2}\ \text{Hz}^{-1}$.

Assuming perfect reflection from the telescope dish, the power collected is

$$= 2.5 \times 10^{-26}\frac{\pi(80)^2}{4} \times 10^7\,\text{W}$$

$$= 1.26 \times 10^{-15}\,\text{W}.$$

6.20 Describe the principle of a microchannel plate as might be used in deep ultraviolet astronomy. Also describe the principle of a proportional counter as used in X-ray astronomy.

An X-ray collimating telescope comprises a honeycomb of tubes with a total collecting area of $0.5\,\text{m}^2$. A source has an emission line close to 10 nm and provides a flux of $10^{-18}\,\text{W}\,\text{m}^{-2}$ at this wavelength. Estimate the count rate from a proportional counter when used with the telescope which is directed to the source.

Solution: Microchannel Plate

– Full description in Lecture notes.

  Small holes at entrances to curved cylinders.

– photosensitive material on the edge of the holes releases electrons. Because of the potential gradient along the length of the cylinder, electrons are accelerated. On impact with the cylinder walls, further electrons are ejected, so providing *'gain'*. For each photo-electron, a shower of $\sim 10^4$ electrons occurs at the output.

Proportional counter – Full description in Lecture notes.

Alcohol/argon mixture in a tube with outer sheath (transparent, e.g., mica) acting as cathode and inner central rod acting as anode. When an X-ray photon passes through, ionisation occurs and a pulse is registered. A resistive anode allows the position of the photon to be determined. The strength of the pulse indicates the energy of the photon – hence the device's name.

Energy received by telescope $= 10^{-18} \times 0.5\,\text{W}$.

$$\begin{aligned}
\text{Energy of X-ray photon} &= h\nu \\
&= \frac{hc}{\lambda} = \frac{6.6 \times 10^{-34} \times 3 \times 10^{8}}{10 \times 10^{-9}}\,\text{J} \\
&= 2 \times 10^{-17}\,\text{J.} \\
\text{Hence count rate} &= \frac{10^{-18} \times 0.5}{2 \times 10^{-17}} = 0.025\,\text{s}^{-1}.
\end{aligned}$$

6.21e The Hα line (656 nm) of a star is observed in emission and the radiation from a narrow bandpass centered on the line provide a flux of 10^{-17} Wm^{-2} at the bottom of the Earth's atmosphere. A monochromator tuned to this wavelength is attached to a 2.5 m telescope. Calculate the photon collection rate of the system assuming that there are no losses in the instrumentation. [5]

Solution: *no solution available*

6.22e A CCD chip comprises an array of 2 048 $\times$ 2 048 elements each of 15 μm square. It is to be used on a telescope of 10m focal length. What is the width of the field of view (arcsec) that the system would capture? [5]

[1 radian = 206 265 arcsec]

Solution: focal length = 10m

size of image in focal plane given by $s = f \times \theta$ where θ is angular extent of object being viewed.

FOV width must be less than width of CCD chip (=2048 x 15 microns = 0.031m)

Therefore $\theta 3.1x10^{-}3$ radians $= 0.18$ degrees $= 630''$

6.23e Sketch out the optical layout of a grating stellar spectrometer indicating the important parts of the system. [7]

A spectrometer with spectral resolving power of 5×10^5 is used to determine the radial velocities of stars. What is the typical minimum stellar velocity that could be detected with the system? [4]

What is meant by the RLD (reciprocal linear dispersion) of a spectrometer? [2]

A CCD with 300 pixels per row, each element being 15 μm wide, is attached to a spectrometer with an RLD of 25 Å mm^{-1}. What is the spectral range covered in a single exposure? [4]

Solution: *no solution available*

6.24e Describe the configuration of an X-ray telescope suitable for imaging. [5]

Such a system with a collection aperture of 30 cm diameter is designed to take X-ray images at energies of 2 keV. Using the Rayleigh criterion, estimate the angular resolution of the instrument in arcsec. [7]

Describe the principle of a detector system suitable for operation with the above instrument. [5]

[1 radian= 206 265 arcsec.]

Solution: *no solution available*

6.25e Two stars radiate as black bodies with maximum brightness at wavelengths, λ_{max}, of 500 nm and 700 nm. It is also known that the first star has a radius three times larger than the second. What is the ratio of their luminosities? [5]

Solution: *no solution available*

6.26 A star provides flux of $6 \times 10^{-17}\,\mathrm{W\,m^{-2}}$ of optical radiation in a wavelength passband centered on 500nm at the bottom of the Earth's atmosphere. A 2.5 metre telescope is used to perform photometry.

(a) Determine the theoretical angular resolution of the telescope.

(b) Calculate the flux entering the telescope.

(c) Estimate the number of photons per second collected by the aperture.

(d) Given that the transmittance of the optical system is 60% and the quantum efficiency of the photo-electric detector is 15%, determine the signal in terms of recorded photo-electrons per second.

(e) Calculate the observing time required to obtain a signal-to-noise ratio of a measurement of 10.

(f) Prior to the display of the analogue signal, the electronic system provides a gain of 10^6. Calculate the current flow to be converted as the final display.

Solution: In all areas of observational astronomy (e.g. Optical, X-ray, etc.) it is useful to know the fundamental limit that the photon *flow* imposes on the detection of sources. To do this, flux needs to be converted to photons per second. The signal can be converted again later to whatever units are appropriate to the exercise. This Tutorial provides a simple exercise on conversion procedures.

(a) Theoretical angular resolution is given by $\alpha = 1.22\frac{\lambda}{D}$. Inserting the values given, $\alpha = 1.22\frac{500\times10^{-9}}{2.5} \times 206\,265 = 0.05\,\mathrm{arcsec}$.

(b) Flux entering the telescope is

$$6 \times 10^{-17} \times \frac{\pi D^2}{4} = 6 \times 10^{-17} \times \frac{\pi 2.5^2}{4} = 2.94 \times 10^{-16}\,\mathrm{W}.$$

(c) The above is the power in the captured signal and this can be converted into a photon count rate, according to the energy carried by each photon.

$$\text{Energy of each photon is } h\nu = \frac{hc}{\lambda} = \frac{6.6 \times 10^{-34} \times 3 \times 10^8}{500 \times 10^{-9}} = 3.96 \times 10^{-19}\,\mathrm{J}.$$

$$\text{Hence photon arrival rate } = \frac{2.94 \times 10^{-16}}{3.96 \times 10^{-19}} = 7.42 \times 10^2\,\mathrm{s^{-1}}.$$

(d) After passing through the optics and being converted into photoelectrons, the event rate is $7.42 \times 10^2 \times 60/100 \times 15/100 = 66.8\,\mathrm{s^{-1}}$.

(e) In order to obtain a S/N ratio of 10, the number of photoelectrons required for detection is given by $10 = \sqrt{N}$. Hence $N = 100$. Time required is $100/66.8 = 1.50\,\mathrm{s}$.

(f) Current flow=electron flow $\times$ charge on each electron. So, current flow $= 66.8 \times 1.6 \times 10^{-19}$ amp $= 1.10 \times 10^{-11}$ amp.

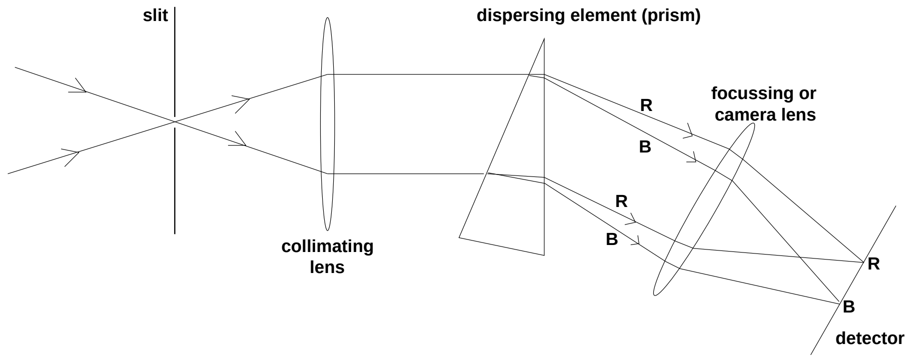
6.27 Sketch out the optical elements and their layout for a basic spectrometer. [9]

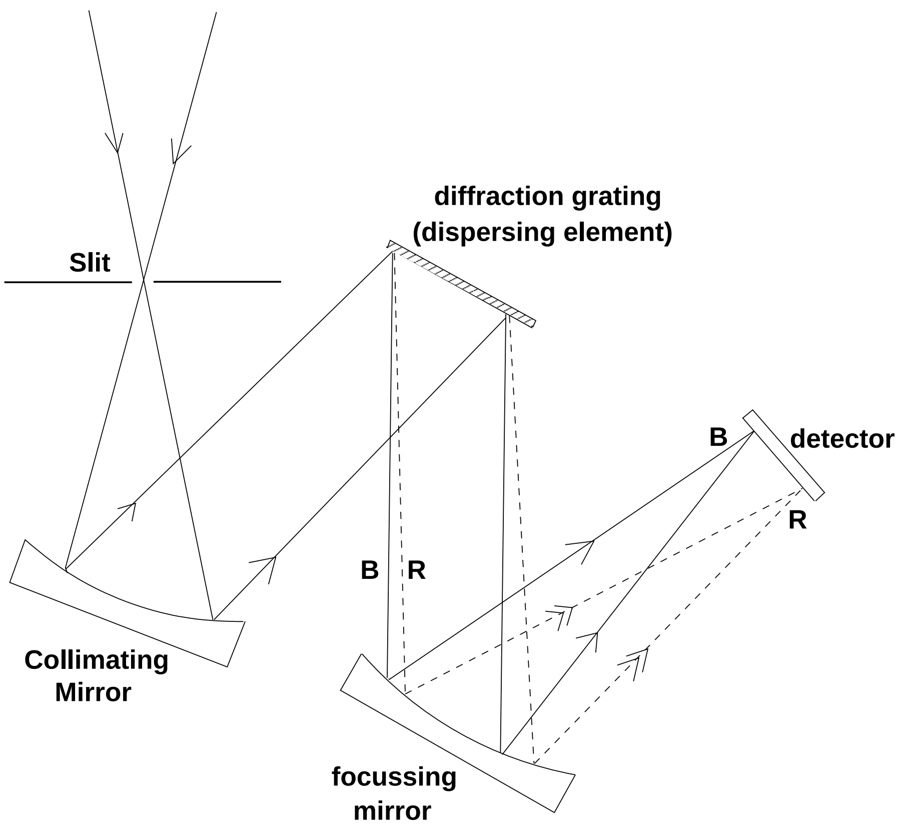
Estimate the typical transmission efficiency of an optical spectrometer. [3]

Define the term *Spectral Resolving Power*. [3]

A spectrometer has a 25 mm diffraction grating with a ruling of 1 800 lines per mm and is used in first order. Estimate the smallest stellar radial velocity that might be detected with this instrument. [5]

Solution: (a) See below for sketches of a spectrometer based on refraction (LHS) and reflection (RHS)





Marking scheme:

slit, collimator, dispersing element, mirror, detector

1/2 mark for each element included, 1/2 each for correct name = [5]

Elements in correct order = [1]

Light is parallel after reflecting from collimator = [1]

Light is dispersed by prism/grating = [1]

Light is focussed onto detector with different colours at different positions = [1]

Total = [9]

(b) Transmission Efficiency:

Grating spectrometer has 3 reflecting elements (collimator, grating, camera). So $T \sim (0.65)^3 = 0.27$ (n.b. anything from 0.6 to 0.8 OK for reflection efficiency of a single element).

Prism spectrometer has 3 refracting elements, so $T \sim (0.9)^3 = 0.73$ (n.b. transmission efficiency for a lens is higher than reflection efficiency for a reflecting surface).

(c) Spectral Resolving Power: R

$$R = \frac{\lambda}{\Delta\lambda}$$

where λ = wavelength at which observation made and $\Delta\lambda$ = minimum difference in wavelength that can be resolved (or distance in nm between 2 spectral features that can just be separated)

(d) Spectral resolving power R given by

$$R = Nm$$

where N = total number of lines on illuminated part of grating and m = order of diffraction. N = 25mm $\times$ 1800 lines/mm = 45000. Observation made in first order so $m = 1$. Therefore **R = 45000**.

So

$$R = \frac{\lambda}{\Delta\lambda} = \frac{c}{v_{min}}$$

giving $v_{min} = 6.67 \times 10^3$m s^{-1}

6.28 A 4 m Cassegrain telescope works at $f/15$.

(a) Determine the theoretical resolving power of the telescope. [3]

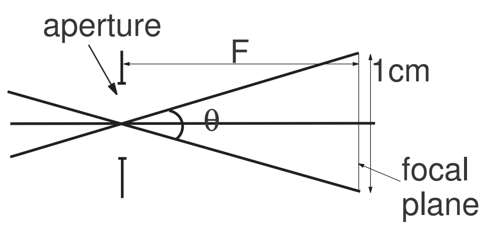
(b) What is the size of the recordable field by a 1 cm×1 cm square CCD chip in the telescopes focal plane? [4]

(c) Given that the quantum efficiency of the CCD is 60%, estimate the overall telesope/detector efficiency. [4]

(d) If the telescope were to be used visually, estimate the limiting magnitude. [3]

Solution: (a) $\alpha = 1.22\lambda/D$, the Rayleigh criterion. Take $\lambda = 550$nm (though anywhere from 450-600nm would do) to get

$$\alpha = 1.22 \times 500 \times 10^{-9}/4 = 1.68 \times 10^{-7}$$



(b)

size of CCD chip: $1cm$

size of image in focal plane: $S = F\theta$, where F is telescope focal length, and θ is angular size of object.

$\Rightarrow \theta =$ angular field of view $= S/F$

$$f/15 \Rightarrow F/D = 15 \Rightarrow F = 15D = 60m$$

Hence $\theta = 0.01m/60m = 1.66 \times 10^{-4}$radians

(c) Cassegrain telescope has 2 mirrors, primary and secondary. Typical mirror efficiency is approx 0.65 (from notes). Hence Telescope efficiency $= T_{primary} \times T_{secondary} \approx 0.65^2 = 0.42$.

Telescope + Detector efficiency is then $0.42 \times$ detector quantum efficiency $= 0.42 \times 0.6$ Hence overall efficiency ≈ 25, i.e. 25%.

NB: since none of the numerical values for T were given, reasonable values (between say 50% and 100%) for telescope reflectivity are acceptable, though one should be able to work out T using the examples given in lecture notes.

(d)

$$m_{lim} = 6 - 5\log_{10} 8 + 5\log_{10} D(\text{mm})$$

assuming fully adapted pupil to have $d_e = 8$mm. So

$$m_{lim} = 6 - 5\log_{10} 8 + 5\log_{10} 4000 = 19.5$$

6.29 A 50 cm space telescope in near Earth orbit is equipped with a filter, centred at 200 nm, with a passband which is 2 nm wide. The combined transmission efficiency of telescope and filter is 1%. It is used to observe a star at a distance of 15pc. At a wavelength of 200nm, the star's specific luminosity (luminosity per unit wavelength) is approximately 7×10^{22} Watts nm^{-1}. How many photons per second from the star will be transmitted by the telescope and filter? [8]

Solution: Total power in 2 nm passband at 200 nm = $L_{200} = 7 \times 10^{22} \times 2$ nm = 1.4×10^{23} W.

Energy Flux received at Earth/Telescope = $L_{200}/(4\pi d^2) = 5.2 \times 10^{-14}$W m^{-2}

Power at telescope aperture $= 5.2 \times 10^{-14} \times \pi \times (D/2)^2 = 1.0 \times 10^{-14}$ W

Photon energy $\epsilon = hc/\lambda = 9.9 \times 10^{-19}$ Joules

Number of *photons* per second at telescope aperture = 1.0×10^{-14}Watts$/9.9 \times 10^{-19}$Joules $=$ 1.0×10^4 photons sec^{-1}

So total number of photons per second transmitted $= 0.01 \times 1.0 \times 10^4 = 100$

6.30 A Newtonian telescope has a primary mirror of diameter 12 cm and a secondary flat. Each of these has a reflection efficiency of 0.70. Attached to the telescope is an eyepiece with transmission efficiency of 30%. The telescope is pointed at a star which gives a flux of visible light at Earth of 10^{-14}W m^{-2}.

(a) what is the combined efficiency of this system?

(b) estimate how many photons per second will be (i) collected by the telescope primary and (ii) transmitted by the telescope and eyepiece combined.

Clearly state any assumptions that you make [6]

Solution: (a) combined efficiency = $(0.70)^2 \times 0.30 = 0.15$

(b) Energy/s at primary = Flux at Earth $\times$ area of primary

$$1.0 \times 10^{-14} \times \pi \times (d/2)^2 = 1.13 \times 10^{-16} \text{ Watts.}$$

Assume that typical wavelength of a visible photon is 550nm (note, anything between 500 and 600nm is acceptable). Then energy per photon = $hc/\lambda = 3.62 \times 10^{-19}$Joules. Hence, number of photons/s at primary given by

$$1.13 \times 10^{-16}/3.62 \times 10^{-19} = 312 \text{ photons/s}$$

So, transmitted photon rate = $312 \times 0.15 = 46.8$ photons/s.

6.31 The globular cluster M13 is at a distance of 7.20 kpc, and has a measured angular size at Earth of 16.6 arcseconds. The energy flux received from M13 at the top of the Earth's atmosphere is 2.45×10^{-10} W m^{-2}.

(a) What is the diameter of M13 in light years?

(b) Assuming that M13 is spherical, what is the *solid angle* it subtends at Earth?

(c) Assuming that M13 is made up of stars all of approximately solar luminosity, calculate how many stars M13 contains.

[10]

Solution: Angular extent of M13 in radians = 8.05×10^{-5} radians

Diameter of M13 in pc is given by $7.20 \times 10^3 \times 8.05 \times 10^{-5} = 0.58$ pc

Diameter of M13 in light years is 1.89 ly

Solid angle subtended by M13 = $\pi \times (8.05 \times 10^{-5})^2 = 2.04 \times 10^{-8}$ sterad.

Total luminosity of M13 is $F \times 4\pi d^2$ where d is distance (m) to M13. So $L = 2.45 \times 10^{-10} \times 4\pi \times (7.20 \times 3.086 \times 10^{19})^2 = 1.52 \times 10^{32}$ W

Solar luminosity = 3.826×10^{26} W, so the luminosity of this globular cluster could be provided by 3.98×10^5 solar-type stars.

6.32 To make a reliable measurement of a faint stellar spectral line, it is necessary to have a detected signal-to-noise ratio (SNR) of 10. Using a particular telescope and detector combination, and an integration time of 10s leads to a SNR of 5. How long an integration time is necessary to achieve the required SNR value?

Updates to the optics in the telescope result in an improved throughput, such that the total transmission efficiency increases by a factor 2. With this new system, how long an integration time, for the same star, will give the required SNR of 10? [6]

Solution: For the 5s integration, let there be N_1 events detected in total. The SNR is $\sqrt{N_1} = 5$. Desired SNR = $\sqrt{N_2} = 10$. The total number of events detected in the integration time is

$$N_1 = F \times A \times T \times \ \Delta t_1 \ \times \eta$$

where F = photon number flux arriving at telescope, A = telescope aperture, T = transmission efficiency and η is detector quantum efficiency, and Δt_1 is the integration time. So, in the case that only the integration time is varied we have

$$\mathrm{SNR}_1/\mathrm{SNR}_2 = \sqrt{N_1}/\sqrt{N_2} = 5/10 = (\Delta t_1/\Delta t_2)^{1/2}$$

Therefore an integration time of $\Delta t_2 = 4\Delta t_1 = 40$ s is required. With the system changed so that the throughput is doubled, we have

$$\mathrm{SNR}_1/\mathrm{SNR}_2 = \sqrt{N_1}/\sqrt{N_2} = 5/10 = (T_1\Delta t_1/T_2\Delta t_2)^{1/2}$$

We know that $T_2 = 2T_1$, which requires $\Delta t_2 = 2\Delta t_1 = 20$ s.

6.33e A spectrometer has a reciprocal linear dispersion (RLD) of 20 Åmm^{-1}. It is used to record the spectrum of a star with an estimated recessional velocity of 50 km s^{-1}. Calculate the displacement of the recorded lines relative to a laboratory reference spectrum in the wavelength region around 5000 Å. [5]

Solution: Using

$$\Delta\lambda/\lambda = v/c$$

we find $\Delta\lambda$ = 0.83Å. With a RLD of 20 Å mm^{-1}, this corresponds to a displacement of $0.83/20 = 0.042$ mm

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6.34e (a) Describe the principal components of a photo-electric photometer (optics and electronics). [6]

(b) A star provides a flux of 10^{-16}W m^{-2} per Å bandpass in the spectral region around 500 nm and is observed with a telescope of 1m diameter. The telescope has a transmission efficiency of 60% and the detector a quantum efficiency of 25%. A filter limits the bandpass to 10 Å. Estimate the photoelectron count rate from the system [7]

(c) Assuming that the limiting noise on photometry is photon shot noise, calculate the integration time necessary to obtain measurements which are reproducible to ± 1%. [4]

Solution:

(a) Diagram from notes, but components which must be included are: photocathode (liberation of photo-electrons), at -ve potential with respect to set of dynodes (for amplification of signal - shower of 3-4 electrons out per electron in); Anode, connected to d.c. amplifier, and electronic data recorder (computer).

(b) Multiplying power per unit area per unit wavelength by telescope primary area and passband gives total power entering telescope aperture

$$10^{-16} \times \pi \times 0.5^2 \times 10 = 7.8 \times 10^{-16}\text{W}$$

Energy per photon from $\epsilon = hc/\lambda = 4.0 \times 10^{-19} J$. Then number of photons/s arriving = $7.8 \times 10^{-16}/4.0 \times 10^{-19} = 2.0 \times 10^3$ photons per second.

Transmission efficiency × quantum efficiency = $0.6 \times 0.25 = 0.15$ therefore photo-electron count rate = $2 \times 10^3 \times 0.15 = 300$ counts/s.

(c) To have results reproducible to ± 1%, need ratio of noise to signal to be ≤ 0.01.

$$N/S = 1/\sqrt{S} \leq 0.01$$

Therefore, $S \geq 100^2 = 10^4$. At 300 counts/s, this would take a minimum of 33 seconds to accumulate.

6.35e Describe, with the aid of diagrams where necessary, the principle and operation of a charge-coupled device (CCD). In your answer, you should address both the way in which photons are recorded by the CCD , and also the process of CCD readout. [8]

With the aid of a diagram, argue that the angular resolving power of a telescope of diameter D is

$$\Delta\alpha = \frac{1.22\lambda}{D}$$

where λ is the wavelength at which the observations are made. [4]

A star is observed with a 50cm diameter space telescope at f/20. In the focal plane is a CCD with pixels that are 5 microns on a side. Estimate the diameter in CCD pixels of the image of the first minimum (dark ring) in the diffraction pattern around the star. Assume that the wavelength of observation is 500nm. [5]

Solution: *no solution available*

6.36e Modern astronomical research telescopes are increasingly implementing adaptive optics systems. Describe the purpose of an adaptive optics system and the way that it works (you may omit details of the wavefront sensor). Your answer should include an appropriate diagram. [9]

A star provides a flux at Earth of $10^{-16}\,\text{W}\,\text{m}^{-2}$. It is observed with a 2m telescope with a total transmission efficiency of 0.5. The output is fed to a photo-electric detector of quantum efficiency 0.5. For how long must the star be observed to obtain a signal-to-noise ratio of 30? Assume that the observation is made at a wavelength of 500nm. [7]

It is decided to try repeating the observation with a photo-electric detector with a higher quantum efficiency and work-function of 5eV. Why is this modification doomed to fail? [1]

Solution: *no solution available*

6.37e A 20 mm-wide diffraction grating has ruling of 1000 lines per mm, and a reciprocal linear dispersion of 0.3 nm per mm. What is the spectral resolving power of this grating used in *second order*? What would be the seperation in mm at the detector plane of a just-resolved spectral line pair centred at 589nm? [5]

Solution: *no solution available*

6.38e Draw a diagram of a grating spectrometer, labelling each component, and giving a brief description of its purpose. [9]

By considering light passing through a transmission grating, show that the reciprocal linear dispersion (RLD) of a grating spectrometer working in second order is given by

$$RLD = \frac{d\cos\theta}{2F}$$

where d is the spacing of the slits on the grating, θ is the angle (with respect to the optic axis) at which the slit image is formed, and F is the focal length of the imaging lens or mirror. [6]

A 1cm-wide diffraction grating is ruled such that the spacing between rulings is 10^{-6} m. What is the spectral resolving power of this grating used in *second order* and what is the minimum velocity signal that could be detected using this grating to view the $H\alpha$ line? [2]

[Data: Wavelength of $H\alpha = 656.3$ nm]

Solution: *no solution available*

6.39e Sketch the optical layout of (i) a Newtonian telescope and (ii) a Cassegrain telescope, drawing also the light path through the system of parallel rays incident at the primary. Discuss one possible advantage and one disadvantage of *either* of these types of telescope over the prime focus design, when used for imaging faint, distant galaxies. [6]

A 1m f/5 telescope is to be used to image the galaxy NGC 4414, which has a diameter of 17 kiloparsecs, and is at a distance of 19 megaparsecs. Imaging in the focal plane is done by means of a CCD with pixels measuring $5\mu m$ on a side. Approximately how many pixels across is the image of the galaxy? [5]

NGC 4414 has an absolute visual magnitude of $M_V = -21.3$. Estimate (i) the total number of counts per second registered by a CCD attached to the telescope described above, and (ii) the signal-to-noise ratio on a 10 second-long observation. Assume that the mean wavelength of light detected is 550 nm, and that the combined transmission efficiency of the telescope optics is 0.3 and the CCD quantum efficiency is 0.75. [6]

[Data: Solar absolute visual magnitude, $M_V = +4.8$.

Solar visual band luminosity, $L_V = 3.62 \times 10^{26}$ W]

Solution: *no solution available*

6.40e Briefly describe the physical reasoning behind Rayleigh's criterion, which states that the angular resolution of a telescope is given by

$$a = \frac{1.22\lambda}{D},$$

where λ is the wavelength of the radiation, and D is the diameter of the telescope used. Calculate the angular resolution, in degrees, of a microwave telescope of diameter 50 cm, operating at frequency of 10^{10} Hz. [5]

Solution: *no solution available*

6.41e Sketch the optical layout of a Cassegrain telescope, showing the light path, labelling the important parts and describing their purpose. [4]

A 50 cm Cassegrain telescope with focal length 4 m is used for planetary imaging, together with a CCD in the focal plane. The CCD pixel size is $10\,\mu m \times 10\mu m$. Calculate the diameter in CCD pixel of the image of Jupiter formed on the CCD, at a time when Jupiter's distance from Earth is 780 million km. Jupiter's diameter is 1.43×10^8 m. [5]

Jupiter's rotation period is 0.414 days. A spectrometer with a reciprocal linear dispersion (RLD) of 15 nm mm^{-1} is used to detect and record a spectral line of CO_2 from Jupiter. The line has a rest wavelength of 10^4 nm. The spectrometer slit is placed first over the '*approaching*' side of the planet, and then over the '*receding*' side. Assuming that the planet is being viewed equator-on, calculate the distance between the two positions of the line on the spectrometer detector. [5]

Low frequency radio emission emitted by Jupiter's atmosphere is to be detected with an interferometer composed of two 10m diameter dishes, each receiving at 300 MHz. If the two antennas in the interferometer are 300m apart, what is the angular resolution in *arcseconds* with which this emission will be observed? [3]

Solution: *no solution available*

6.42e What is meant by an *atmospheric window*? What are the main atmospheric windows, and what prevents good observations from being made outside of atmospheric windows (give at least two specific examples)? [4]

Discuss the effect on optical observations of stars of each of the following:

(a) aperture diffraction;

(b) atmospheric scintillation.

Include in your answer a description of the physical reason for the effects. You may illustrate your answer with a diagram. [5]

Sketch a diagram for a telescope *adaptive* optics system, labelling the important components. Describe what it is used for, the steps in the adaptive optics correction loop, and discuss the important limiting factors in designing such a system for ground-based optical astronomy. [8]

Solution: *no solution available*

6.43e With the aid of a diagram, demonstrate that the magnification, m, of a refracting telescope is given by

$$m = \frac{F_O}{F_E},$$

where F_O and F_E are the focal lengths of objective lens and eyepiece lens respectively. State clearly any approximations that you make. [5]

Solution: *no solution available*

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7 Cosmology

7.1e State Hubble's law for the expansion of the Universe. A galaxy is observed to have a redshift of 0.03. Estimate its distance in Mpc given a Hubble constant of 70 km s^{-1} Mpc^{-1}. [5]

Solution:

$$v = Hd$$

where v is the velocity of the receding object and d is the distance from the object.

$$\begin{aligned} \text{If} \quad z = 0.03 \quad \text{we can say} \quad \frac{v}{c} = 0.03 \\ \text{or} \quad 0.03 = \frac{H}{c}d \quad \text{or} \quad d = \frac{0.03}{H}c \\ \text{with} \quad H = 65\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1} \quad \text{and} \quad c = 3 \times 10^5\,\text{km}\,\text{s}^{-1} \\ \text{we have} \quad d = \frac{0.03}{70} 3 \times 10^5\,\text{Mpc} \simeq 1.3 \times 10^{-3} \times 10^5 = 130\,\text{Mpc}. \end{aligned}$$

7.2e Explain what is meant by the term *cosmological dark matter*. What evidence is there for the existence of such matter? [5]

Solution: Dark matter is matter which is not itself luminous, but which is nevertheless implied to be present in the Universe. Evidence: rotation curves of galaxies indicate more matter is present than just luminous stars; clusters of galaxies appear to be bound together by gravity, but have velocity dispersions which would imply them to be gravitationally unbound if they consisted only of luminous matter.

7.3e Describe what is meant by a *standard candle* method for the determination of a distances in the Universe. Describe how the *period-luminosity* law as obtained from studies of Cepheid variable stars is used to determine distances. [5]

Solution: 'Standard candle' method assumes that certain types of object have equal luminosity. Measurements of apparent brightness lead to distance measures via the inverse square law. Magellanic Cloud studies revealed a relationship between period of variability and apparent brightness for Cepheid variables. All objects in the LMC can be assumed to be at the same distance, which implies a period luminosity law. Period determinations now can be used to estimate luminosity and from the apparent brightness, distance can be inferred.

7.4e State three observational facts that support the hot big bang model. What is meant by the term recombination? [5]

Solution: CMBR, recession of galaxies, abundance of He and H, (and light elements). Recombination took place in the early Universe $z \sim 1\,000$ when T had dropped sufficiently for (re)combination of e^{-} and p from neutral hydrogen.

7.5e State two main features for each of the following:

(a) spiral galaxies

(b) elliptical galaxies

(c) quasars. [5]

Solution:

(a) Spiral arms, stars and dust in disc

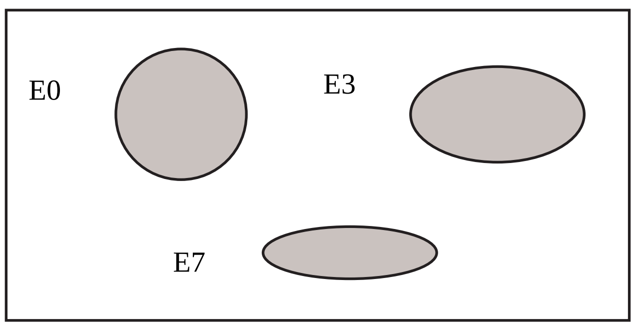
(b) Elliptical, fairly massive, no dust

(c) quasars, high redshift, extremely luminous, broad emission lines, point-like.

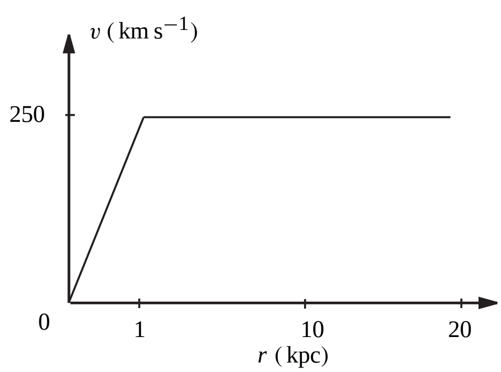
7.6 Elliptical galaxies are characterised as 'En', where $n = 10(1 - b/a)$ and a and b are the major and minor axes of the ellipse as seen on the sky. Calculate the ratio b/a for elliptical galaxies described as E0, E3 and E7, and sketch approximately to scale the appearance of such galaxies.

Solution:

$$\begin{aligned}
\mathrm{E0} &\Rightarrow 1 - b/a = 0 \Leftrightarrow b/a = 1 \\
\mathrm{E3} &\Rightarrow 1 - b/a = 0.3 \Leftrightarrow b/a = 0.7 \\
\mathrm{E7} &\Rightarrow 1 - b/a = 0.7 \Leftrightarrow b/a = 0.3
\end{aligned}$$



7.7 The rotation curve of a typical spiral galaxy can be approximated as shown in the diagram below.



Here v is the speed of rotation of matter about the centre of the galaxy (assuming circular motion), and r is the distance from the centre. From the graph, calculate the period of rotation of stars situated at 1 kpc, 10 kpc and 20 kpc from the centre of this galaxy. Express your answers in years. Suppose that the visible extent of the galaxy is 15 kpc in radius. What type of observations can be used to extend the rotation curve beyond the visible extent of the galaxy? Briefly comment on what conclusions can be drawn about the distribution and extent of matter in the galaxy from the shape of the rotation curve. (l year = 3.16×10^7 s)

Solution: At $r = 1$ kpc the rotation velocity is $v = 250\,\mathrm{km\,s^{-1}}$, so the rotation period, P, is

$$P = \frac{2\pi r}{v} = \frac{2 \times \pi \times (3.086 \times 10^{19}\,\mathrm{m})}{2.5 \times 10^5\,\mathrm{m\,s^{-1}}} = 7.756 \times 10^{14}\,\mathrm{s} = 2.45 \times 10^7\,\mathrm{yrs}$$

$$\text{At } r = 10\,\mathrm{kpc},\ v = 250\,\mathrm{km\,s^{-1}} \to P = 2.45 \times 10^8\,\mathrm{yrs}$$
$$\text{At } r = 20\,\mathrm{kpc},\ v = 250\,\mathrm{km\,s^{-1}} \to P = 4.90 \times 10^8\,\mathrm{yrs}$$

Observations of neutral hydrogen emission (21 cm line) by radio telescopes can extend the rotation curve beyond the visible extent of the galaxy. If the gravitating matter consisted only of the visible stars (i.e., assuming the mass of the exterior hydrogen clouds to be negligible) then we expect to find $v(r) \propto r^{-1/2}$. Since we *don't* see this, but instead we find $v(r) =$ constant to large radii, this is generally explained by supposing that there is a massive halo of dark matter surrounding the galactic disk and extending well beyond the distance of the visible edge of the galaxy. The flat rotation curve would be explained by a halo which was spherical in shape.

7.8 Explain what is meant by a rotation curve for a spiral galaxy and how it is measured. [4]

If the Sun is taken to be moving in a keplerian orbit of radius 8 kpc about the centre of our Galaxy with a velocity of 200 $\mathrm{km\,s^{-1}}$, estimate the mass of the galaxy within the Sun's orbit

(a) in kilograms $\{1.3 \times 10^{41}\,\mathrm{kg}\}$

(b) in solar masses. $\{0.7 \times 10^{11}\,M_\odot\}$

State any assumptions you make. [8]

Sketch the rotation curve of a typical spiral galaxy and explain why this could provide evidence for the existence of dark matter. [8]

Solution: A spiral galaxy rotation curve is plot of rotation speed against radial distance from centre of galaxy. It is measured from e.g. the doppler shift of neutral hydrogen regions, or some other prominent spectral feature, at various distances from centre of the galaxy.

[4 marks]

Assume that the mass in the galaxy is spherically symmetric. Thus if in circular orbit the acceleration will be given by v^2/r.

[2 marks]

This must also equal the gravitational acceleration GM/r^2. Thus we can obtain the mass from

$$\frac{GM}{r^2} = \frac{v^2}{r}, \quad \text{or} \quad M = \frac{rv^2}{G}.$$

[2 marks]

Substitute in for velocity and radius in SI units to obtain mass in kilograms:

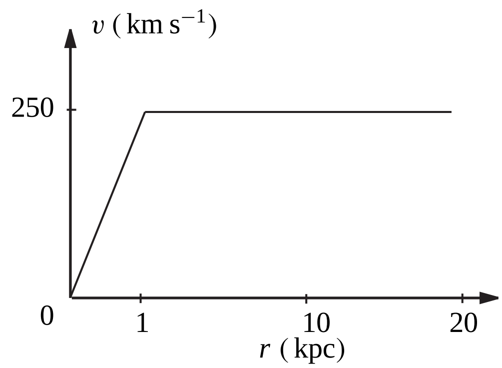
$$M = \frac{rv^2}{G} = \frac{(8 \times 10^3) \times (3.09 \times 10^{16}) \times (200 \times 10^3)}{(6.67 \times 10^{-11})} \simeq 1.3 \times 10^{41}\,\text{kg}.$$

[3 marks]

In solar masses this is $\frac{1.3 \times 10^{41}}{1.99 \times 10^{30}}$, or 0.7×10^{11}.

[1 mark]

Schematically, the rotation curve of a typical spiral galaxy looks like this:



[3 marks]

If only luminous matter were present, and Newtonian gravity applies, we should expect the rotation curve instead to behave as $v^2 = GM/r$.

[2 marks]

The observed rotation curve does not fall off as $r^{-1/2}$; one way to explain this is to suppose the existence of a halo of dark matter outside the luminous galaxy.

[3 marks]

7.9 Broad and strong emission lines at 663 nm, 491 nm and 438 nm are observed in the core of a galaxy. Explain why these could be interpreted as due to Hα, Hβ, and Hγ emission and estimate the redshift of the galaxy. {0.01} [4]

Assuming Hubble's constant to be 70 km s^{-1} Mpc^{-1} calculate the distance to the galaxy in Mpc. {46.2 Mpc} [4]

Calculate the optical luminosity of the galaxy if its optical flux is 10^{-11} W m^{-2}. {2.2×10^{38} W} [4]

Assuming that this luminosity is produced primarily by the accretion of matter onto a central supermassive black hole, find a lower limit on the mass accreted per year. {0.02 $M_\odot$} [4]

(You may assume Hα is observed in laboratory at 656.3 nm)

Solution: Redshift: Hα is observed in laboratory at 656.3 nm. This corresponds to $n = 3$ to $n = 2$ transitions in the hydrogen atom. The energy associated with this transition is $13.6(\frac{1}{2^2} - \frac{1}{3^2})$ eV, where the ionisation potential of hydrogen is 13.6 eV.

The energies associated with the Hβ ($n = 4$ to $n = 2$) and Hγ ($n = 5$ to $n = 2$) are $13.6(\frac{1}{2^2} - \frac{1}{4^2})$ eV and $13.6(\frac{1}{2^2} - \frac{1}{5^2})$ eV respectively. Since $E = h\nu = hc/\lambda$, the wavelengths are in the ratio

$$\frac{\lambda_\alpha}{\lambda_\beta} = \frac{\frac{1}{2^2} - \frac{1}{4^2}}{\frac{1}{2^2} - \frac{1}{3^2}} = 1.35.$$

Similarly, $\lambda_\alpha/\lambda_\gamma = 1.512$. Since all these lines would be redshifted by the same amount we should expect if the emission was from a galaxy for the same ratios to hold. Now the ratios observed are $\frac{663}{491} = 1.35$ and $\frac{663}{438} = 1.514$ which is in agreement, if one allows a small observational error.

Distance: The redshift is given by

$$1 + z = \frac{\lambda_{\mathrm{observed}}}{\lambda_{\mathrm{lab}}} = \frac{663}{656.3} = 1.01.$$

Thus the redshift is $z = 0.01$. Use Hubble's law to obtain distance. Thus $v = Hd$, or $z = Hd/c$, and we obtain a distance of

$$d = \frac{cz}{H} = \frac{3 \times 10^5\,\mathrm{km\,s^{-1}} \times 0.01}{70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}} = 42.9\,\mathrm{Mpc}.$$

Luminosity: This is given by $4\pi d^2 F$, where $F = 10^{-11}\,\mathrm{W\,m^{-2}}$. Convert parsecs to metres by $1\,\mathrm{pc} = 3.1 \times 10^{16}\,\mathrm{m}$, and substitute to obtain $L = 2.2 \times 10^{38}\,\mathrm{W}$.

Mass accreted onto black hole: If all the mass accreted were transformed to energy ($E = mc^2$) mass accretion rate would have to be

$$\frac{2.2 \times 10^{38}\,\mathrm{W}}{(3 \times 10^8\,\mathrm{m\,s^{-1}})^2} = 1.44 \times 10^{21}\,\mathrm{kg\,s^{-1}}.$$

Thus mass accreted during a year would be about $1.44 \times 10^{21} \times 3 \times 10^7 \simeq 4 \times 10^{28}$ kg, or 0.02 solar masses. However the efficiency of conversion is only about 10 % , so this is a lower limit.

7.10 Describe two methods for obtaining a distance estimate for a galaxy whose approximate distance is 20 Mpc. [4]

The period mean luminosity relation for cepheid variable population I stars is given approximately by

$$M = -2.6 - 3.8 \log_{10} P,$$

where M is the absolute magnitude and P the period in days. A cepheid star is observed in a neighbouring galaxy to have a period of 20 days, and an apparent magnitude of 19. What is the distance to the neighbouring galaxy? {2 Mpc} [8]

Assuming that the error bar on the constant term is ± 0.2, but the value of 3.8 is exact, what are the error bars on the distance? $\{\pm 0.2\,\mathrm{Mpc}\}$ [4]

Solution: Use e.g. supernova method, Tully-Fisher relation between rotation velocity dispersion and luminosity (see notes), or even cepheid variable period mean luminosity relation. Substitute into the relation

$$M = -2.6 - 3.8 \log_{10} P.$$

For $P = 20$ we find that the absolute magnitude is $M = -2.6 - 3.8 \log_{10} 20 = -7.5$. Now use Pogson's equation $m - M = 5 \log_{10} r - 5$ to obtain r, where r is in parsecs.

$$\log_{10} r = \frac{m - M + 5}{5} = \frac{19 - (-7.5) + 5}{5} = \frac{31.5}{5} = 6.3.$$

Thus $r = 2.0 \times 10^6$ parsecs or 2 Mpc.

Errors on distance: Upper limit on absolute magnitude is now -7.7, and lower is -7.3. Corresponding distances will be given by

$$\log_{10} r = \frac{m - M + 5}{5} = \frac{19 - (-7.7) + 5}{5} = 6.34,$$

and $r = 2.19\,\text{Mpc}$ and lower limit

$$\log_{10} r = \frac{m - M + 5}{5} = \frac{19 - (-7.3) + 5}{5} = 6.26,$$

and $r = 1.82\,\text{Mpc}$. Thus error bar is $\pm 0.2\,\text{Mpc}$.

7.11 Assuming that a supercluster of galaxies contains around 10^{15} solar masses, and the average separation of superclusters is 100 Mpc, estimate the mean mass density of the Universe, and compare this with $\rho_c = 3H_0^2/(8\pi G)$.

Describe some of the difficulties involved in determining the mean mass density of the Universe. (Take $H = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$)

Solution: Average separation of superclusters $= 100\,\text{Mpc} = 10^2\,\text{Mpc}$. So, typically we have one supercluster in a cubic volume of $10^6\,\text{Mpc}^3$ (imagine a cube of side 100 Mpc centered on each supercluster).

$$\begin{aligned}\text{mean mass density } &= \frac{10^{15} M_\odot}{10^6\,\text{Mpc}^3} = \frac{10^{15} \times 1.989 \times 10^{30}\,\text{kg}}{10^6 \times (3.086 \times 10^{22})^3\,\text{m}^3} \\ &= 6.77 \times 10^{-29}\,\text{kg}\,\text{m}^{-3}\end{aligned}$$

$$\begin{aligned}\text{Compare with } \rho_{\text{crit}} &= \frac{3H_0^2}{8\pi G} \\ &= \frac{3 \times \left[\frac{70\times10^3}{(3.086\times10^{22})}\right]^2}{8\pi \times 6.673 \times 10^{-11}} \\ &= 9.2 \times 10^{-27}\,\text{kg}\,\text{m}^{-3}\end{aligned}$$

i.e. mean supercluster mass density $\ll$ critical density.

Determining the mean mass density is difficult since we must find methods which take account of the dark matter, which we cannot see directly, such as galaxy rotation curves, virial theorem in galaxy clusters, gravitational lensing. We cannot simply add up the mass of the visible galaxies. Many of these indirect methods involve making assumptions — e.g. assuming an isotropic cluster when applying the virial theorem, since we only measure radial velocities — which are difficult to check.

7.12 Verify that the energy equation of Newtonian cosmology, for the case of critical density, can be written in the form

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \left(\frac{2GM}{R}\right)^{1/2}.$$

Here M is the constant mass inside a sphere of galaxies of radius R, which expands with the Universe.

Show by integration, using the big bang condition $R = 0$ at $t = 0$, that the present age of the Universe, for the case of critical density, is

$$t = \frac{2}{3H},$$

where H is the present value of Hubble's constant defined by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = HR.$$

Solution: From Newtonian derivation of Friedmann equation, for a galaxy of mass m,

$$-\frac{1}{2}m\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + \frac{GMm}{R} = k$$

Putting $k = 0$ gives the required result. Rearranging

$$(2GM)^{1/2}\,dt = R^{1/2}dR$$

Integrating,

$$(2GM)^{1/2}\,t = \frac{2}{3}R^{3/2} + C$$

But $R \to 0$ as $t \to 0$, so $C = 0$. Also

$$H = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{(2GM)^{1/2}}{R^{3/2}} \Rightarrow t = \frac{2}{3H}$$

7.13e Describe the origin of the cosmic background radiation (CMB) according to the hot big bang model. [4]

Explain the significance of the anisotropies observed in the cosmic background radiation for the subsequent formation of large scale structures in the Universe. [3]

The scale factor, $R(t)$, in a big bang cosmological model is given as a function of cosmic time, t, by $R(t) = at^{2/3}$, where a is a constant. Explain why in this model the matter density of the early Universe would have been extremely high. [2]

Assuming that the energy density, u, of the CMB radiation varies as $R(t)^{-4}$, calculate the redshift, z, at which recombination took place, taking the present day temperature of the CMB to be 3 K. Taking the present age of the Universe to be 1.5×10^{10} years, calculate the time at which recombination took place. (You may assume that recombination took place at a temperature of 3 000 K.) [8]

Solution: Initial density of the Universe was very high. Initial temperature was very high. Expansion$\longrightarrow$ cooling. CMB formed at the time of recombination (i.e. when the temperature was around 3 000 K.)

Although the CMB is highly isotropic, anisotropic (other than dipole) would have been necessary for the condensation of galaxies and cosmic structure.

Assuming $\rho R^3 =$ constant, it is clear that $\rho \propto R^{-3}$ or $\rho \propto E^{-2}$. As $t \to 0$, $\rho \to \infty$.

Taking $u \propto R(t)^{-4}$ we have $aT^4 \propto R^{-4} \Rightarrow T \propto \frac{1}{R}$. This implies

$$\frac{T_{\mathrm{rec}}}{T_0} = \frac{R_0}{R_{\mathrm{rec}}} = 1 + z_{\mathrm{rec}},$$

$$\text{Thus}\qquad 1 + z_{\mathrm{rec}} = \frac{3000}{3} \text{ or } z \simeq 1000,$$

Now for time of recombination we have

$$\frac{T_{\mathrm{rec}}}{T_0} = \frac{R_0}{R_{\mathrm{rec}}} = \left(\frac{t_0}{t_{\mathrm{rec}}}\right)^{2/3}$$

$$\therefore\ \frac{t_0}{t_{\mathrm{rec}}} = \left(\frac{T_{\mathrm{rec}}}{T_0}\right)^{2/3} \text{ or } t_{\mathrm{rec}} = \left(\frac{3000}{3}\right)^{-3/2} \times t_0$$

$$\text{and}\qquad t_{\mathrm{rec}} = 1.5 \times 10^{10} \times \left(\frac{3}{3000}\right)^{3/2} \simeq 4.7 \times 10^5\ \text{year}.$$

7.14e State Hubble's law for the expansion of the Universe. The value of the Hubble constant is still somewhat uncertain. What reasons are there for this?

The distance, $d(t)$, between any comoving galaxies at any cosmic time, t, is given by $d(t) = R(t)/R_0 \times d_0$, where d_0 is the distance at the present epoch. If $R(t) = R_0\,(t/t_0)^{2/3}$, show that Hubble's constant is given by $H_0 = \frac{2}{3t_0}$.

If the mean distance between superclusters of galaxies today is 20 Mpc, what was the mean distance at redshift time $t = t_0/8$? What is the redshift corresponding to this time?

Solution:

$$v = Hd$$

v is radial velocity, d is distance.

Precise distance measurements of galaxies are very difficult. Hubble's constant is $\dot{R}/R$ with $R = R_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}$

$$\text{we have} \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{R_0}{t_0^{2/3}}\frac{2}{3}t^{-1/3}$$

$$\text{and} \qquad \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{3}\frac{R_o}{t_0^{2/3}}t^{-1/3}\frac{t_0^{2/3}}{R_0 t^{2/3}} = \frac{2}{3t},$$

At the present time

$$\frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{3t_0} = H_0,$$

If the mean distance today is 20 Mpc, $\quad d_0 = 20,$

$$d\left(\frac{t_0}{8}\right) = d_0\left(\frac{t_0}{8t_0}\right)^{2/3} = \frac{d_0}{4} = \frac{20}{4} = 5,$$

$\therefore$ Mean distance was $\frac{20}{4} = 5\,\mathrm{Mpc}$
Redshift of $\frac{t_0}{8}$:

$$1 + z = \frac{R(t_0)}{R\left(\frac{t_0}{8}\right)} = 8^{2/3} = 4$$

$\therefore z = 3.$

7.15e What is meant by the terms

(a) galaxy cluster
(b) supercluster? [4]

Explain what is meant by the critical matter density. [5]

An average supercluster contains about 10^{15} solar masses. Taking the mean separation between superclusters to be about 100 Mpc, estimate the mass density of the Universe in $\mathrm{kg\,m^{-3}}$. Hence *estimate* the value for the ratio of matter density to the critical mass density, Ω. [6]

State two observational facts that indicate the presence of significant amounts of dark matter in the Universe. [2]

[You may assume that $\rho_{\mathrm{crit}} = 3H_0^2/(8\pi G)$, $H_0 = 70\,\mathrm{kms^{-1}Mpc^{-1}}$]

Solution:

(a) A galaxy cluster is a group of galaxies. The Milky Way belongs to the local group.

(b) A supercluster is a cluster of galaxy clusters.

Critical density determines whether the Universe is open, flat or closed. In the first two cases, there is indefinite expansion, in the third case, there is collapse.

Matter in supercluster $\simeq 2 \times 10^{15} \times 10^{30}$ kg

$$\therefore \quad \text{Matter density} \simeq \frac{2 \times 10^{45}}{(100 \times 3.1 \times 10^{22})^3} \simeq 7 \times 10^{-29}\,\mathrm{kg\,m^{-3}}$$

$$\begin{aligned}\rho_{\mathrm{crit}} &= \frac{3H_0^2}{8\pi G} \\ &= \frac{3}{8\pi}\left(\frac{70 \times 10^3}{3.1 \times 10^{22}}\right)^2 \frac{1}{7 \times 10^{-11}} \\ &\simeq 8.69 \times 10^{-27}\,\mathrm{kgm^{-3}}\end{aligned}$$

$$\text{Thus} \quad \frac{\rho}{\rho_{\mathrm{crit}}} = \Omega = \frac{7 \times 10^{-29}}{8.69 \times 10^{-27}} \simeq 0.01,$$

Rotational curves of galaxies: velocity dispersion within clusters, (and globular clusters), would mean these systems were unbound if there were no dark matter.

7.16 In a big bang model the scale factor $R(t)$ is given by $R(t) = R_0 t^{2/3}$.

(a) If the average distance between galaxy clusters is now 20 Mpc , what would it have been when the Universe was half its present age?

(b) show that Hubble's constant in this model is given by $H = \frac{2}{3t}$. What is the age of the Universe with a Hubble constant of $70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$?

(c) Evaluate the critical density at the present epoch, given that

$$\rho_c = \frac{3H_0^2}{8\pi G},$$

.

The matter density in the Universe varies as

$$\frac{\rho}{\rho_0} = \left(\frac{R_0}{R}\right)^3,$$

and the energy density of radiation as

$$\frac{u}{u_0} = \left(\frac{R_0}{R}\right)^4,$$

Sketch ρ/ρ_0, u/u_0 and u/ρ as a function of time.

(d) Will there be a time in the Universe's past when the radiation energy density was greater than the matter density? If so, how would you go about calculating it?

Assume that the present matter density (including dark matter) is 0.2 of the critical density. Hence the present ratio of radiation to matter density is about 10^{-4}.

Hint: The energy density of blackbody radiation is given by $u = aT^4$ where $a = 7.57 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$ hence the energy density of the radiation is given by $u = 7.57 \times 10^{-16}(3\,\mathrm{K})^4\,\mathrm{J\,m^{-3}\,K^{-4}}$ or about $6 \times 10^{-15}\,\mathrm{J\,m^{-3}}$ (You need to convert this to $\mathrm{kg\,m^{-3}}$).

(e) what was the age of the Universe at recombination, if this took place at $T = 3\,000$ K? Was this before radiation became dominant or after in the life of the Universe?

Questions:
How does Ω vary with cosmic time this model?
Do you believe in the big bang?

For the very keen - If the scale factor was given by $R(t) = R_0 t^{\alpha}$, where $\alpha > 0$, how would the above calculation differ?

Solution: In a big bang model the scale factor $R(t)$ is given by $R(t) = R_0 t^{2/3}$.

(a) The scale factor gives the separation of 'comoving' (ie. moving with the universal expansion) galaxies. Thus

$$\frac{\mathrm{d}(t)}{\mathrm{d}(t_o)} = \frac{t^{2/3}}{t_0^{2/3}} = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{1}{2}\right)^{2/3}$$

Hence average distance was 20 Mpc $\times$ 0.629 = 12.57 Mpc.

(b) Hubble's constant is given by $H = \frac{\dot{R}(t)}{R(t)}$. In this model

$$H = \frac{\dot{R}(t)}{R(t)} = \frac{\frac{2}{3} R_0 t^{-1/3}}{R_0 t^{2/3}} = \frac{2}{3t},$$

(Note that Hubble's 'constant' changes with cosmic time.) The age of the Universe in this model is given by $t = \frac{2}{3H}$ with$H_0 = 50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Now expressing H_0 in terms of $\mathrm{s^{-1}}$ we get $H_0 = 70 \times 10^3\,\mathrm{ms^{-1}} \times (3 \times 10^{16} \times 10^6\,\mathrm{m})^{-1} = 2.33 \times 10^{-18}\,\mathrm{s^{-1}}$. Substituting into the equation yields the age of the Universe as 2.9×10^{17} s.

(c) Substitute value $G = 6.7 \times 10^{-11}\,\mathrm{N\,kg^{-2}\,m^{-2}}$ and the above value of $H_0 = 2.21 \times 10^{-18}\,\mathrm{s^{-1}}$into $\rho_c = \frac{3H_0^2}{8\pi G}$ to obtain 3.4×10^{-27}.
Sketch!!!

(d) There will evidently be a time in the Universe's past when the radiation energy density was greater than the matter density since u/ρ varies as $1/R$.

Assume that the present matter density (including dark matter) is 0.2 of the critical density. Hence the present ratio of radiation to matter density is about 10^{-4}. The energy density of blackbody radiation is given by $u = aT^4$ where $a = 7.57 \times 10^{-16}\,\mathrm{J\,m^{-3}\,K^{-4}}$ hence the energy density of the radiation is given by $u = 7.57 \times 10^{-16}(3K)^4\,\mathrm{J\,m^{-3}\,K^{-4}}$ or about $6 \times 10^{-15}\,\mathrm{J\,m^{-3}}$. Convert to $\mathrm{kg\,m^{-3}}$ by dividing by c^2 to yield $6 \times 10^{-15}\,\mathrm{J\,m^{-3}}/(3 \times 10^8\,\mathrm{m\,s^{-1}})^2$ to give about $7 \times 10^{-32}\,\mathrm{kg\,m^{-3}}$.

Compare this with $0.2 \times \rho_c$ to give ratio of about 5×10^{-5}.

(e) Since u behaves as R^{-4} and for blackbody radiation $u = aT^4$, T must behave as $1/R$. Hence scale factor must have changed by factor 3/3 000 and you can work out the time. This must have taken place after matter domination.

7.17$_e$ State how an Sa and Sc spiral galaxy differ in appearance. If the apparent major and minor axes of an elliptical galaxy subtend angles of 0.5 arcmin and 0.2 arcmin respectively, what is the Hubble classification of this galaxy? [5]

Solution: Sa spiral: large nucleus, tightly wound spiral arms. Sc spiral: small nucleus, wide open spiral arms. $b/a = 0.2/0.5 = 0.4 \Rightarrow 10(1 - b/a) = 6$. The galaxy is, therefore, classified as E6

7.18e Explain what is meant by the *cosmological principle*. Give two pieces of observational evidence which support the validity of the cosmological principle. [5]

Solution: The cosmological principle states that the Universe is isotropic – i.e. looks the same in all directions – and homogeneous – i.e. looks the same at all positions. Two pieces of evidence: isotropy of the CMBR; absence of galaxy clustering on very large scales.

7.19e Explain how HST observations of Cepheid variables in other galaxies significantly improved estimates of the Hubble constant, H_0. [5]

Give two reasons why type Ia supernovae (SN Ia) are useful standard candles for measuring extragalactic distances. [2]

A SNIa, SN1998a, is observed to have a B band apparent magnitude at maximum light of $m_{\rm B}(\max) = 10.3$. Cepheid observations in the SNIa host galaxy give an estimated distance modulus for the galaxy of $\mu = 30.0$. What is the estimated distance of the galaxy, in Mpc? What is the estimated B band absolute magnitude of the SNIa at maximum light? [5]

From a sample of distant SNIa the following relation is found between $m_{\rm B}(\max)$ and the recession velocity, v, in $\rm kms^{-1}$, of the host galaxies:

$$m_{\rm B}(\max) = 5\log_{10} v - 3.6$$

By substituting Hubble's Law into the above relation, obtain an estimate of H_0 from the apparent magnitude and estimated distance of SN1998a. [5]

Solution:

- HST Cepheid observations have extended the use of primary distance indicators to about 20 Mpc, i.e. beyond the Local Supercluster. This means that the primary distance scale is now linked directly to secondary indicators, such as the Tully Fisher relation or those based on elliptical galaxies in rich clusters of galaxies (e.g. Virgo). These in turn link to galaxies at sufficient distance that peculiar velocities may be ignored. Previously, three steps on the cosmic distance ladder were required for this: 1) LMC to Local Group; 2) Local Group to nearby clusters and field galaxies; 3) nearby clusters and field galaxies to distant clusters and field galaxies. HST has replaced steps 1) and 2) with a single step.

- SN Ia are useful standard candles since they have a very narrow range of absolute magnitude at maximum light and they are observable to very large distances.

- From distance modulus formula, $\mu = 5\log_{10} r + 25$, where the distance r is in Mpc. Rearranging, $\log_{10} r = 0.2(\mu - 25) = 1$. Thus, $r = 10\,$Mpc. Also, $\mu = m_{\rm B}(\max) - M_{\rm B}(\max) \Rightarrow M_{\rm B}(\max) = 10.3 - 30.0 = -19.7$

- Hubble's law states that $v = H_0\, r$. It follows that

$$m_{\rm B}(\max) = 5\log_{10} H_0 + 5\log_{10} r - 3.6$$

and

$$\log_{10} H_0 = 0.2(m_{\rm B}(\max) - 5\log_{10} r + 3.6) = 1.78 \Rightarrow H_0 = 60.3\,{\rm km\,s^{-1}\,Mpc^{-1}}$$

7.20 A cluster containing 1 000 galaxies has a characteristic radius of 10 Mpc and an r.m.s. radial peculiar velocity of $1\,000\,{\rm km\,s^{-1}}$ at this radius from the centre of the cluster. Using the formula given in the lectures, calculate a virial estimate of the mass of the cluster, expressing your answer in solar masses.

If the average number of stars per galaxy is 10^{11}, and the average stellar mass is approximately one solar mass, calculate the mass of the cluster in the form of stars. Comment on your answer.

Solution: Virial mass, $M = \frac{6<v^2>R}{G}$ and $R = 10\,\mathrm{Mpc}$ and $< v^2 >^{1/2} = 1\,000\,\mathrm{km\,s^{-1}}$, (NB. formula includes an extra factor of 3 because $< v^2 >$ is only radial component).

$$\begin{aligned}
\Rightarrow \quad M &= \frac{6 \times (1\,000 \times 1000)^2 \times 10 \times 3.086 \times 10^{22}}{6.673 \times 10^{-11}}\,\mathrm{kg} \\
&= 2.775 \times 10^{46}\,\mathrm{kg} \\
&= \left(\frac{2.775 \times 10^{46}}{1.989 \times 10^{30}}\right) \mathrm{M}_\odot \\
&= 1.395 \times 10^{16}\,\mathrm{M}_\odot \\
\text{luminous mass} &= (\text{average stellar mass}) \times (\text{average no. of stars per galaxy}) \times (\text{no. of galaxies}) \\
&= 1 \times 10^{19} \times 1\,000 = 10^{14}\,\mathrm{M}_\odot
\end{aligned}$$

Since luminous mass $\ll$ virial mass this provides evidence for dark matter in clusters.

7.21 State two properties which a good *standard candle* distance indicator should possess. Give an example of a standard candle distance indicator used to measure galaxy distances beyond the Local Group. [4]

The period mean luminosity relations for Cepheid variables in the V and I wavelength bands are given approximately by

$$M_V = -2.76 \log_{10} P - 1.40$$
$$\text{and} \quad M_I = -3.06 \log_{10} P - 1.81$$

where P is the period of the Cepheid in days, and M_V and M_I are its V band and I band mean absolute magnitudes respectively. A Cepheid is observed in a distant galaxy to have a period of 35 days, and a mean V band and I band apparent magnitudes of $m_V = 24.9$ and $m_I = 23.6$ respectively. Estimate the distance of the galaxy using first the V band and then the I band Cepheid relation. [12]

Suggest an explanation why the V band and I band distance estimates are different. [4]

Solution: A good standard candle distance indicator should have a small dispersion in luminosity (absolute magnitude) and be easily observable to large distances. [2]

An example of a standard candle used beyond the Local Group is type Ia supernovae or brightest cluster elliptical galaxies. [2]

$P = 35\text{days} \Rightarrow \log_{10} \mathrm{P} = 1.544$

From the Cepheid PL relations :-

$$\begin{aligned}
M_\mathrm{v} &= -2.76 \times 1.544 - 1.40 = -5.66 \qquad [4] \\
M_\mathrm{I} &= -3.06 \times 1.544 - 1.81 = -6.53
\end{aligned}$$

Combining with the observed apparent magnitudes

$$\begin{aligned}
\mu_\mathrm{v} &= m_\mathrm{v} - M_\mathrm{v} \qquad [4] \\
&= 24.9 - (-5.66) \\
&= 30.56 \\
&= 5 \log_{10} d + 25 \\
\Rightarrow \log_{10} d &= 0.2 \times (30.56 - 25) \\
\Leftrightarrow d_\mathrm{v} &= 12.9\,\mathrm{Mpc}
\end{aligned}$$

$$\begin{aligned} \mu_{\mathrm{I}} &= m_{\mathrm{I}} - M_{\mathrm{I}} \\ &= 23.6 - (-6.53) = 30.13 \\ &= 5\log_{10} d + 35 \\ \Rightarrow \log_{10} d &= 0.2 \times (30.13 - 25) \\ \Leftrightarrow d_{\mathrm{I}} &= 10.6\,\mathrm{Mpc} \end{aligned}$$ [4]

The V band distance estimate is larger than the I band estimate. This could be due to the effects of extinction, which will be greater in V than in I, making the Cepheid appear dimmer (and thus apparently more distant) than it actually is. [4]

7.22 Explain what is meant by saying that the Universe is *homogeneous* and *isotropic*. Above what approximate scale do galaxy redshift surveys begin to display homogeneity and isotropy? [4]

Write down the relation between the *proper distance*, $r(t)$, and *comoving separation*, s, of two galaxies at time, t, in terms of the scale factor, $R(t)$. Show how this relation implies a linear Hubble law between recession velocity and proper distance. [4]

In a critical density Universe the time evolution of the scale factor is described by $R(t) = R_0(t/t_0)^{2/3}$ (where the subscript '0' denotes the present day value). Show that this equation implies that the Hubble constant, H_0, and the present age of a critical density Universe, t_0, are related by $H_0 = \frac{2}{3}t_0^{-1}$. [5]

Show that the following relation holds between redshift and time for a critical density Universe $z = (t/t_0)^{-2/3} - 1$. [3]

At what redshift would a galaxy be observed if its light was emitted when the Universe was one tenth of its current age? What was the value of the scale factor at this epoch, in terms of its current value? [4]

Solution:

- Homogeneous means that the Universe appears to be essentially the same everywhere; isotropic means that the Universe appears to be essentially the same in all directions. Galaxy redshift surveys begin to display homogeneity and isotropy above a scale of about 10 000 km s^{-1}
-
$$r(t) = R(t)s$$
Differentiating this equation
$$\frac{\mathrm{d}r}{\mathrm{d}t} = v = \frac{\mathrm{d}R}{\mathrm{d}t}.s = \frac{1}{R}\frac{\mathrm{d}R}{\mathrm{d}t}.(Rs) = Hr$$
which is Hubble's law. (n.b. s is constant $\Rightarrow \mathrm{d}s/\mathrm{d}t = 0$)
- $H = \frac{\dot{R}(t)}{R(t)} = \frac{\frac{2}{3}R_0 t^{-1/3}}{R_0 t^{2/3}} = \frac{2}{3t}$. Hence $H_0 = \frac{2}{3}t_0^{-1}$
- Rearranging, $(t/t_0)^{-2/3} = R_0/R(t) = 1 + z$. Hence $z = (t/t_0)^{-2/3} - 1$
- $t/t_0 = 0.1 \Rightarrow z = 3.64$. $R(t)/R_0 = (4.64)^{-1} = 0.22$

7.23e Explain what is meant by the term *cosmological distance ladder*.

Describe one method of determining the distances of galaxies out to 3 Mpc, and another for distances beyond 50 Mpc. Explain why such measurements cannot be considered to be very accurate. [10]

Solution: Measurement of the distance of progressively more distant objects, e.g. Cepheids, in our galaxy, (use absolute magnitude for Cepheids in other galaxies), identify the type of galaxy and use to estimate distance of galaxy in distant cluster etc.

Out to 3 Mpc use Cepheids. Period-mean luminosity correlation. Measure period $\Rightarrow$ absolute luminosity. Measure apparent magnitude to yield distance modulus.

Beyond 50 Mpc any of, supernova, Tully-Fisher type relation or $D_{\mathrm{n}} - \sigma$

Accuracy limited by selection function and natural dispersion.

7.24e What is meant by the epoch of recombination? [2]

Explain why the Cosmic Microwave Background Radiation (CMBR) is observed at the epoch of recombination. [4]

What type of radiation is the CMBR? Explain why its mean temperature is inversely proportional to the cosmic scale factor. [3]

Assume that the current temperature of the CMBR is 3K. If the CMBR was emitted at redshift 1000, what was its approximate mean temperature when it was emitted? [2]

Explain the significance of intrinsic temperature variations in the CMBR for theories of how the structure in the Universe formed. [6]

Solution: *no solution available*

7.25e The cosmological principle relies on two assumptions: homogeneity and isotropy. Outline the observational evidence which supports these two assumptions. [4]

Suppose that we live in a universe with a Hubble constant $H_0 = 75\,\mathrm{km\,s^{-1}Mpc^{-1}}$. Calculate the value of the Hubble time in years. [5]

In a critical density universe with zero cosmological constant the scale factor is $R(t) = At^{2/3}$, where A is a constant. Show that this equation implies a current age t_0 of the universe given by

$$t_0 = \frac{2}{3H_0} \qquad [6]$$

Solution: *no solution available*

7.26e Define the term *redshift*, as used in astronomy. [1]
The Hydra Cluster, at a distance of 120 Mpc, consists of several hundred galaxies, moving within the cluster, with speeds that are typically 700 km/s (the velocity dispersion of the cluster).
Calculate the redshift of the cluster, using a Hubble constant of 72 km/s/Mpc. [2]
Calculate the additional red or blue shift of a galaxy moving with the typical internal speed within the cluster. [2]

Solution: *no solution available*

7.27e Give two examples of standard candles and explain briefly why they are considered to be good distance indicators. [5]

Solution: *no solution available*

7.28e Explain what is meant by the *cosmological principle* and state two pieces of observational evidence which support it. [4]

In a model universe with zero cosmological constant, Friedmann's equation for the evolution of the cosmic scale factor, $R(t)$, may be written as

$$\frac{\dot{R}^2}{R^2} - \frac{8\pi G\rho}{3} = -\frac{k}{R^2}$$

where ρ is the matter density and k is a constant. Show that the case where $k = 0$ reduces to

$$\left(\frac{dR}{dt}\right)^2 = \frac{A}{R}$$

where A is a constant, stating clearly any assumptions that you make. [5]

Hence show that, in this case, $R(t) \propto t^{2/3}$. [4]

A quasar is observed at redshift $z = 3$. Show that, in this model, the light from the quasar was emitted when the universe was $1/8^{th}$ its present age. [4]

Solution: *no solution available*

7.29e Describe carefully why measurements of HI 21cm emission are an effective tool for mapping the spiral structure of the Milky Way galaxy. [6]

Explain briefly why galaxy rotation curves provide evidence for the existence of dark matter haloes surrounding spiral galaxies. [3]

The rotation curve of spiral galaxy NGC 309-G17 flattens off at $v_{rot} = 210\,\mathrm{km\ s^{-1}}$. The inclination of the galaxy to the line of sight is $i = 53^o$ and its I-band apparent magnitude is $m_I = 12.27$. Use the I-band Tully-Ficher relation

$$M_I = -7.68 \log_{10}\left(\frac{v_{rot}}{\sin i}\right) - 4.79$$

to estimate the absolute magnitude, and thus the distance (in Mpc) of NGC 309-G17. [4]

If the recession velocity of NGC 309-G17 is $5175\,\mathrm{km\ s^{-1}}$, obtain an estimate of the Hubble constant from this galaxy, and suggest two reasons why this estimate may not be reliable. [4]

Solution: *no solution available*

7.30e The proper distance, $r(t)$, between two galaxies with constant comoving separation, s, is given by the equation $r(t) = R(t) \cdot s$, where $R(t)$ is the cosmic scale factor at time, t. Show that this equation implies that an astronomer in either galaxy would observe Hubble's Law. [5]

Solution: *no solution available*

7.31e State two properties which a good *standard candle* distance indicator should possess. Two cD elliptical galaxies, of V-band apparent magnitude $m_V = 13.2$ and $m_V = 16.7$ respectively are observed in two rich galaxy clusters. Estimate the ratio of the distance to the two clusters, stating any assumptions that you make. [5]

Solution: *no solution available*

7.32e What is the *Cosmological Principle?* Why do we only apply it to the Universe on the largest scales? Give one strong piece of evidence that this principle is valid. [6]

Explain the difference between the *comoving separation* and *proper distance* to a galaxy in term of the cosmic scale factor, $R(t)$. Show how a changing scale factor leads to Hubble's Law, and derive an expression for Hubble's constant, H, in terms of R. [6]

In a flat, matter-dominated, universe the scale factor changes with time as

$$R(t) \propto t^{\frac{2}{3}}.$$

What do we mean when we say a universe is 'flat'? [2]

Show that in these circumstances the age of the universe, t_0, is related to Hubble's constant by

$$H_0 = \frac{2}{3t_0}.$$ [3]

Solution: *no solution available*

7.33e What is meant by the *rotation curve* of a spiral galaxy? How might the curve be measured? [4]

How may spiral galaxy rotation curves be used to determine the distance to the galaxy? [4]

Draw a typical rotation curve for a spiral galaxy and explain how it provides evidence for the existence of dark matter in the galaxy. [2]

The Sun moves in an approximately circular orbit of radius 8.3 kpc around the galactic centre, completing an orbit every 2.5×10^8 years. Using this information, estimate the mass of the Galaxy in solar masses. Carefully state all the assumptions you make. [7]

[1 kpc $\approx 2 \times 10^8$ AU]

Solution: *no solution available*

7.34e To within a factor of 10, what is:

(a) the diameter of our Galaxy in parsecs? [1]

(b) the age of the Universe in years? [1]

(c) Hubble's constant, H_o, in km s^{-1} Mpc^{-1}? [1]

(d) temperature of the Cosmic Microwave Background, in kelvin? [1]

(e) the site of the largest galaxy supercluster, in Mpc? [1]

Solution: *no solution available*

7.35e What is characteristic about *Cepheid variable* stars? Explain how these stars can be used to measure distances. [5]

Solution: *no solution available*
