Now, this is a long and difficult way to get back the old answer. Your reaction may be, "We don't need these complex numbers. We have enough problems in life; we're doing well with sines and cosines, thank you." But now I'm going to give you a problem where you cannot talk your way out by just turning it into a word problem.

17.4 Exponential functions as generic solutions

Here is the problem: a mass m, connected to a spring of force constant k, is moving on a surface with friction. The minute there is friction, you have an extra force. We know that if you're moving to the right, the force of friction is to the left, and, if you are moving to the left, the force is to the right, that is, the frictional force is velocity dependent. The equation that crudely models this velocity dependence is

$$m\ddot{x} = -kx - \gamma\, m\dot{x} \tag{17.56}$$

where I include a factor m in the frictional coefficient to simplify subsequent algebra. Dividing by m, our equation becomes

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2 x = 0. \tag{17.57}$$

Can you solve this as a word problem? It's going to be difficult, because you want a function that, when you take two derivatives, add some amount of its own derivative, and then some of itself, gives zero. It is not clear a trigonometric function can do that. However, an exponential has to work because it reproduces itself no matter how many derivatives you take. Thus we make the ansatz

$$x(t) = Ae^{\alpha t}. \tag{17.58}$$

Note that I do not explicitly use a complex exponential. If α is meant to be complex, it will come out that way; we are not forcing it to be real in making this ansatz. When we feed it into Eqn. 17.57 we find, because every derivative brings a factor of α,

$$A(\alpha^2 + \gamma\,\alpha + \omega_0^2)e^{\alpha t} = 0. \tag{17.59}$$

Once again, A cannot be the cause of the zero, because if A vanishes you've killed the whole solution and $e^{\alpha t}$ is not going to vanish, so the only way is for the stuff in brackets to vanish:

$$(\alpha^2 + \gamma\alpha + \omega_0^2) = 0. \tag{17.60}$$

That means the α that you put into this guess must be one of the roots

$$\alpha_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}. \tag{17.61}$$

The general solution is

$$x(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t} \tag{17.62}$$

$$= A\exp\left[\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right) t\right]$$
$$+ B\exp\left[\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right) t\right]. \tag{17.63}$$

The motion described by the solution depends on the value of $\frac{\gamma}{2\omega_0}$.

17.5 Damped oscillations: a classification

Let us classify the different kinds of behavior that emerge as we vary $\frac{\gamma}{2\omega_0}$.

17.5.1 Over-damped oscillations

We first consider the *over-damped case*

$$\frac{\gamma}{2} > \omega_0. \tag{17.64}$$

In this case both roots $\alpha_{\pm}$ are real and both are negative: α_- is negative being a sum of two negative numbers, while α_+ is negative because the positive square root is smaller than $\gamma/2$. This means that $x(t \to \infty) \to 0$, which is in accord with our expectation that friction will eventually bring the oscillations to an end.

How about A and B? First of all, they are both real as can be seen by equating $x(t)$ to its conjugate. Because the exponentials are real they do not respond to conjugation and we require $A = A^*$ and $B = B^*$.

To find A and B, we need two pieces of data, which I will take to be initial position, $x(0)$, and the initial velocity, $v(0)$. If we put $t = 0$ in Eqn. 17.62 we find

$$x(0) = A + B. \tag{17.65}$$

Next I take the derivative of Eqn. 17.63 *and then set* $t = 0$ to find

$$v(0) = A\alpha_+ + B\alpha_-. \tag{17.66}$$

Solving these simultaneous equations will yield A and B. To test yourself, try showing that if the oscillator is displaced to some $x(0) > 0$ and released from rest, that is, $v(0) = 0$, then $x(t)$ never becomes 0 and hence cannot become negative. This means the mass will simply relax to its equilibrium position without any oscillations.

17.5.2 Under-damped oscillations

In turning on friction we got carried away: from being 0 in the very first example, γ jumped to a value greater than $2\omega_0$. Consider now the intermediate case when $0 < \gamma < 2\omega_0$. What do the solutions look like now? We should be able to guess that, at least for very tiny values of γ, the oscillator will oscillate as before, but with a slowly diminishing amplitude. Let us verify and quantify this expectation.

The roots now become

$$\alpha_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{17.67}$$

$$= -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{17.68}$$

$$\equiv -\frac{\gamma}{2} \pm i\omega'. \tag{17.69}$$

We have introduced yet another frequency

$$\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} < \omega_0, \tag{17.70}$$

which describes the oscillatory part of the motion. Note that the roots are complex conjugates

$$\alpha_+ = \alpha_-^* \tag{17.71}$$

and the general solution becomes

$$x(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t} \tag{17.72}$$

$$= e^{-\frac{1}{2}\gamma t}\left[Ae^{i\omega' t} + Be^{-i\omega' t}\right]. \tag{17.73}$$

I leave it to you to verify that once again $x = x^*$ implies $A^* = B$ because the A and B terms get exchanged under complex conjugation. Repeating the analysis for the case $\gamma = 0$, this solution may be rewritten as

$$x(t) = Ce^{-\frac{1}{2}\gamma t}\cos\left[\omega' t + \phi\right] \quad \text{where} \tag{17.74}$$

$$C = 2|A| \quad \text{and} \quad A = |A|e^{i\phi}. \tag{17.75}$$

Figure 17.4 shows what the damped oscillation looks like for $A = 2$, $\gamma = 1$, and $\omega' = 2\pi$. This is typically what you will see if you excite any

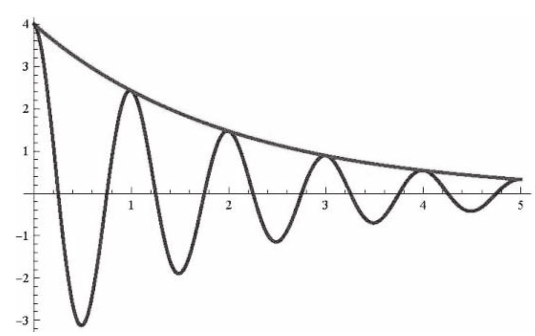

Figure 17.4 Damped oscillations with $x(t) = 4e^{-.5t}\cos(2\pi t)$, i.e., $A = 2$, $\gamma = 1$, and $\omega' = 2\pi$. The falling exponential shows the decay of the amplitude.

system with some modest amount of frictional loss. If γ is very small, you may not realize the oscillations are being damped.

17.5.3 Critically damped oscillations

Having considered the cases $\gamma > 2\omega_0$ (over-damped) and $\gamma < 2\omega_0$ (under-damped), we turn to the *critically damped* case $\gamma = 2\omega_0$. In this case $\alpha_+ = \alpha_- = -\frac{\gamma}{2}$. Where is the second solution to accompany $Ae^{-\frac{\gamma t}{2}}$? We know in every problem there must be two solutions, because we should be able to pick the initial position and velocity at will. That's an area of mathematics I don't want to enter now, but you can verify that the second solution is $Bte^{-\frac{\gamma t}{2}}$, which is not a pure exponential. You will find the derivation of this solution in my math book. The general solution for the critically damped case is thus

$$x(t) = e^{-\frac{\gamma t}{2}}\left[A + Bt\right]. \tag{17.76}$$

Try to show in this case that $A = x(0)$ and $B = v(0) + \frac{\gamma}{2}x(0)$.

17.6 Driven oscillator

Next we turn to a more challenging problem. I have, as before, the mass, the spring, and friction. But now I'm going to apply an extra force, $F_0 \cos\omega t$. This is called a *driven oscillator*. Imagine that I am actively shaking the mass with my hand, exerting the force $F_0 \cos\omega t$. Now there are three ω's: $\omega_0 = \sqrt{\frac{k}{m}}$, the natural frequency of the undamped free oscillator; $\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, which entered the under-damped oscillator; and finally ω, the frequency of the driving force, which is completely up to me to choose. The equation to solve is

$$m\ddot{x} + \gamma\, m\dot{x} + kx = F_0 \cos\omega t, \tag{17.77}$$

which we rewrite as

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t. \tag{17.78}$$

This problem is difficult because you cannot guess the answer to it by turning it into a word problem: neither $x(t) \propto \cos(\omega t)$ nor $x(t) \propto \sin(\omega t)$ is