

# Radio Interferometry -- II

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Fourteenth NRAO Synthesis Imaging Summer School  
Socorro, NM



# Topics

- Practical Extensions to the Theory:
  - Finite bandwidth
  - Rotating reference frames (source motion)
  - Finite time averaging
  - Local Oscillators and Frequency Downconversion
- Coordinate systems
  - Direction cosines
  - 2-D ('planar') interferometers
  - 3-D ('volume') interferometers
- U-V Coverage, Visibilities, and Simple Structures.

# Review

- In the previous lecture, I set down the principles of Fourier synthesis imaging.

- I showed:

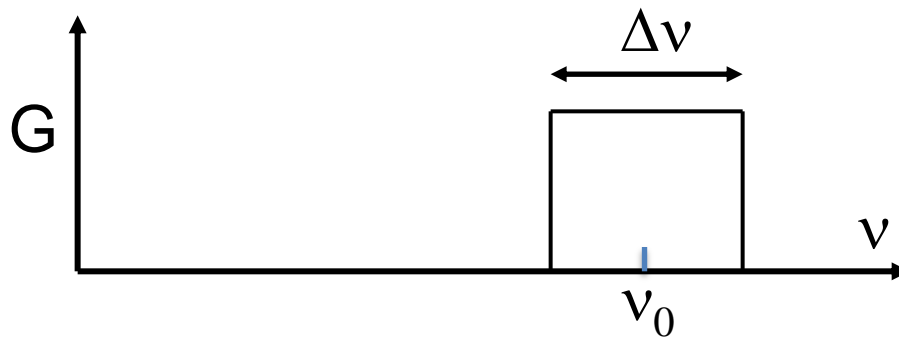
$$V_v(\mathbf{b}) = R_C - iR_S = \iint I_v(s) e^{-2\pi i v \mathbf{b} \cdot \mathbf{s} / c} d\Omega$$

Where the intensity  $I_v$  is a real function, and the visibility  $V(\mathbf{b})$  is complex and Hermitian.

- The model used for the derivation was idealistic – not met in practice:
  - Monochromatic
  - Stationary reference frame.
  - RF throughout
- We now relax, in turn, these restrictions.

# The Effect of Bandwidth.

- Real interferometers must accept a range of frequencies. So we now consider the response of our interferometer over frequency.
- Define the frequency response functions,  $G(\nu)$ , as the amplitude and phase variation of the signal over frequency.



- The function  $G(\nu)$  is primarily due to the gain and phase characteristics of the electronics, but can also contain propagation path effects.
- In principle,  $G(\nu)$  is a complex function.

# The Effect of Bandwidth.

- To find the finite-bandwidth response, we integrate our fundamental response over a frequency width  $\Delta\nu$ , centered at  $\nu_0$ :

$$V = \int \left( \frac{1}{\Delta\nu} \int_{\nu_0 - \Delta\nu/2}^{\nu_0 + \Delta\nu/2} I(\mathbf{s}, \nu) G_1(\nu) G_2^*(\nu) e^{-i2\pi\nu\tau_g} d\nu \right) d\Omega$$

- If the source intensity does not vary over the bandwidth, and the instrumental gain parameters  $G_1$  and  $G_2$  are square and identical, then

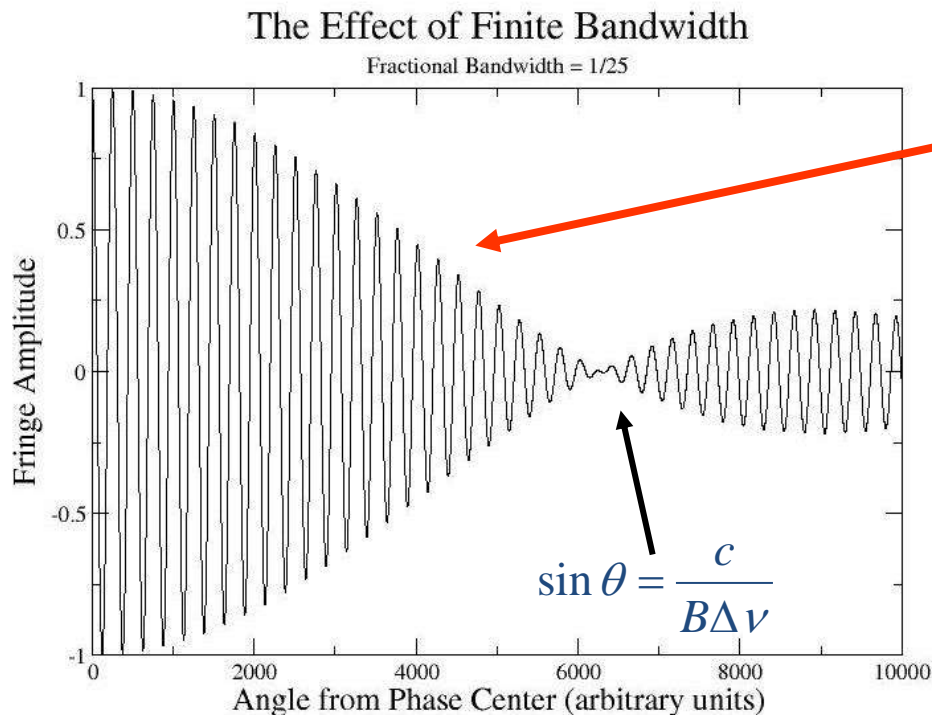
$$V = \iint I_\nu(\mathbf{s}) \frac{\sin(\pi\tau_g\Delta\nu)}{\pi\tau_g\Delta\nu} e^{-2i\pi\nu_0\tau_g} d\Omega = \iint I_\nu(\mathbf{s}) \text{sinc}(\tau_g\Delta\nu) e^{-2i\pi\nu_0\tau_g} d\Omega$$

where the **fringe attenuation function**,  $\text{sinc}(x)$ , is defined as:

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

# Bandwidth Effect Example

- For a square bandpass, the bandwidth attenuation reaches a null when  $\tau_g \Delta \nu = 1$ , or
 
$$\sin \theta = \lambda / B \Delta \nu / \nu_0 = \frac{c}{B \Delta \nu}$$
- For the old VLA, and its 50 MHz bandwidth, and for the 'A' configuration, the null was ~1.3 degrees away.
- For the EVLA,  $\Delta \nu = 2$  MHz, and  $B = 35$  km, then the null occurs at about 27 degrees off the meridian.



**Fringe Attenuation function:**

$$\text{sinc}\left(\frac{B \Delta \nu}{\lambda \nu} \sin \theta\right) = \text{sinc}\left(\frac{B \Delta \nu}{c} \sin \theta\right)$$

**Note:** The fringe-attenuation function depends only on bandwidth and baseline length – not on frequency.

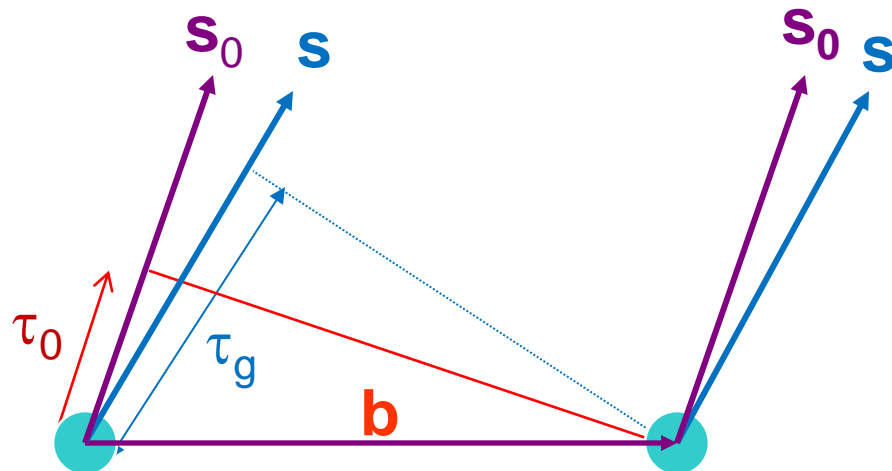
# Observations off the Meridian

- In our basic scenario -- stationary source, stationary interferometer -- the effect of finite bandwidth will strongly attenuate the visibility from sources far from the meridional plane.
- Since each baseline has its own plane, the only point on the sky free of attenuation for all baselines is a small angle around the zenith (presuming all baselines are coplanar).
- Suppose we wish to observe an object far from the zenith?
- One solution is to use a very narrow bandwidth – this loses sensitivity, which can only be made up by utilizing many channels – feasible, but computationally expensive.
- Better answer: Shift the fringe-attenuation function to the center of the source of interest.
- How? By adding time delay.

# Adding Time Delay

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$

$$\tau_0 = \mathbf{b} \cdot \mathbf{s}_0 / c$$



$\mathbf{s}_0$  = reference  
(delay)  
direction  
 $\mathbf{s}$  = general  
direction

$$V_1 = E e^{-i\omega(t-\tau_g)}$$

$$V_2 = E e^{-i\omega t}$$

The entire fringe  
pattern has been  
shifted over by  
angle

$$\sin \theta = c\tau_0/b$$

$$V_2 = E e^{-i\omega(t-\tau_0)}$$

$$\begin{aligned} V &= \langle V_1 V_2^* \rangle = E^2 e^{-i[\omega(\tau_0 - \tau_g)]} \\ &= E^2 e^{i2\pi[\nu \mathbf{b} \cdot (\mathbf{s} - \mathbf{s}_0) / c]} \end{aligned}$$

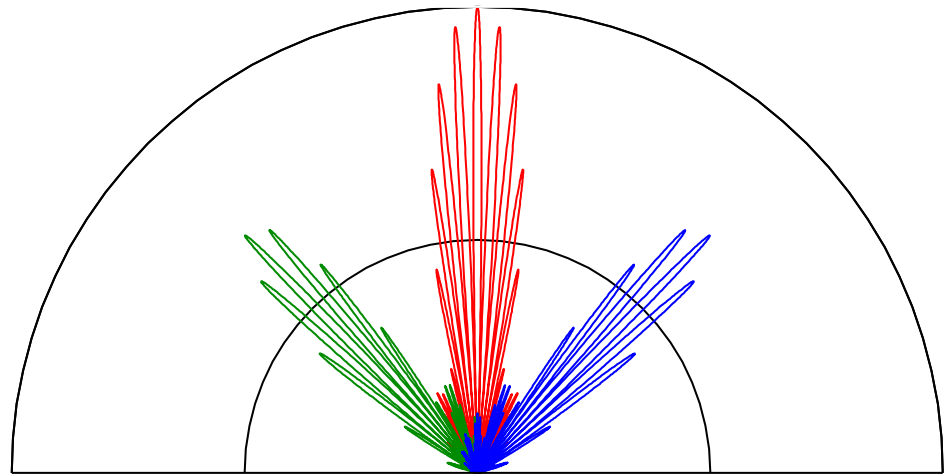
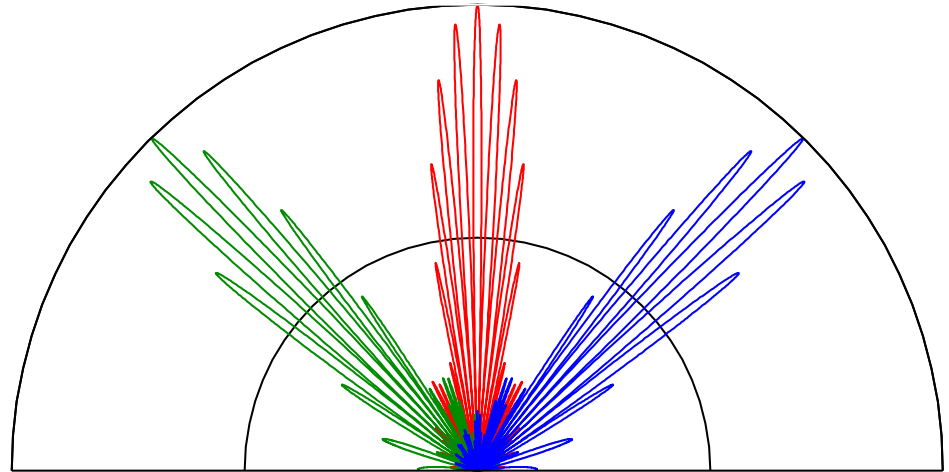


# Observations from a Rotating Platform

- Real interferometers are built on the surface of the earth – a rotating platform. From the observer's perspective, sources move across the sky.
- Since we know how to adjust the interferometer to move its coherence pattern to the direction of interest, it is a simple step to continuously move the pattern to follow a moving source.
- All that is necessary is to continuously slip the inserted time delay, with an accuracy  $\delta\tau \ll 1/\Delta\nu$  to minimize bandwidth loss.
- For the 'radio-frequency' interferometer we are discussing here, this will automatically track both the fringe pattern and the fringe-washing function with the source.
- Hence, a point source, at the reference position, will give uniform amplitude and zero phase throughout time (provided real-life things like the atmosphere, ionosphere, or geometry errors don't mess things up ...  
☺ )

# Illustrating Delay Tracking

- Top Panel:  
Delay has been added and subtracted to move the delay pattern to the source location.
- Bottom Panel:  
A cosinusoidal sensor pattern is added, to illustrate losses from a fixed sensor.



# Another Justification for Delay Tracking

- There is another very good reason to ‘track’ the fringe pattern by adding time delay.
- The ‘natural fringe rate’ – due to earth’s rotation, is given by

$$\nu_f = u \omega_e \cos \delta \quad \text{Hz}$$

- Where  $u = B/\lambda$ , the (E-W) baseline in wavelengths, and  $\omega_e = 7.3 \times 10^{-5}$  rad/s is the angular rotation rate of the earth.
- For a million-wavelength baseline,  $\nu_f \sim 70$  Hz – that’s fast.
- There is \*no\* useful information in this fringe rate – it’s simply a manifestation of the platform rotation (indeed, it’s a Doppler shift).
- Tracking, or ‘stopping’ the fringes greatly slows down the \*post-correlation\* data processing/archiving needs.



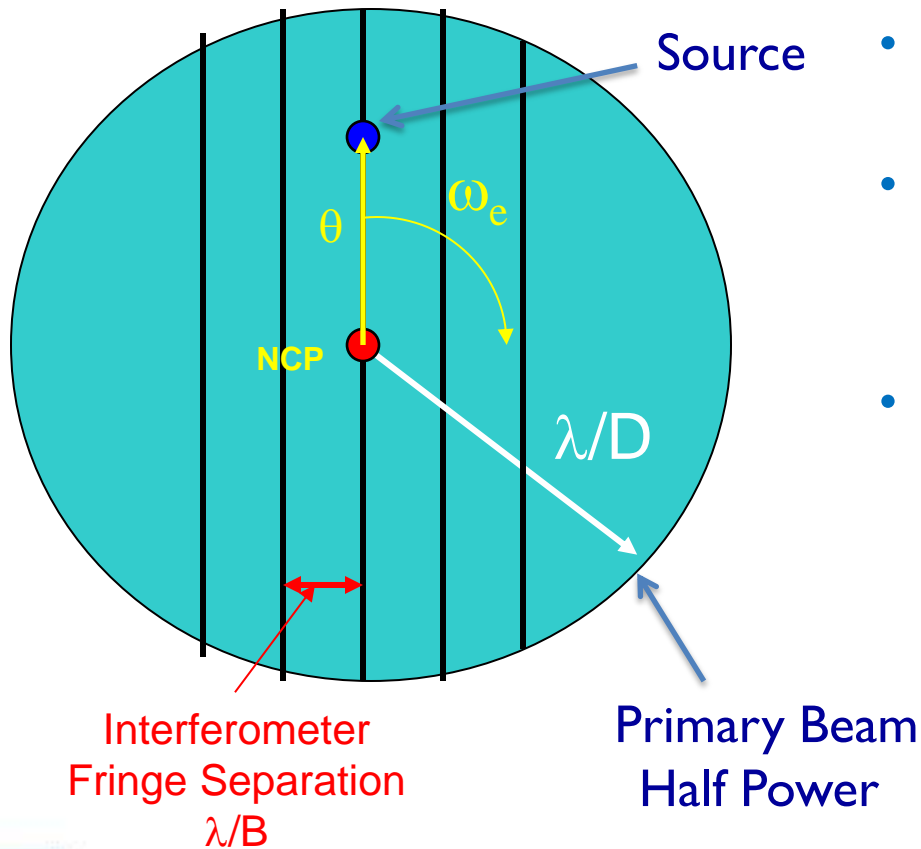
# Time Averaging Loss

- So – we can track a moving source, continuously adjusting the delay to move the fringe pattern with the source.
- This does two good things:
  - Slows down the data recording needs
  - Prevents bandwidth delay losses.
- From this, you might think that you can increase the time averaging for as long as you please.
- But you can't – because the convenient tracking only works perfectly for the object 'in the center' – the point for which the delays have been pre-set.
- All other sources are moving w.r.t. the fringe pattern – and this is where the essential information lies...



# Time-Smearing Loss Timescale

Simple derivation of fringe period, from observation at the NCP.



- Turquoise area is antenna primary beam on the sky – radius =  $\lambda/D$
- Interferometer coherence pattern has spacing =  $\lambda/B$
- Sources in sky rotate about NCP at angular rate:

$$\omega_e = 7.3 \times 10^{-5} \text{ rad/sec.}$$

- Minimum time taken for a source to move by  $\lambda/B$  at angular distance  $\theta$  is:

$$t = \frac{\lambda}{B \omega_e \sin \theta}$$

$$\approx \frac{D}{\omega_e B}$$

For sources at the half power distance

# Time-Averaging Loss

- So, what kind of time-scales are we talking about now?
- How long can you integrate before the differential motion rotates the source through the fringe pattern?
- Case A: A 25-meter paraboloid, and 35-km baseline:
  - $t = D/(B\omega_e) = 10$  seconds. (independent of wavelength)
- Case B: Whole Hemisphere for a 35-km baseline:
  - $t = \lambda/(B\omega_e) \text{ sec} = 83 \text{ msec}$  at 21 cm.
- Averaging for durations longer than these will cause severe attenuation of the visibility amplitudes.
- To prevent ‘delay losses’, your averaging time must be much less than this.
  - Averaging time 1/10 of this value normally sufficient to prevent time loss.

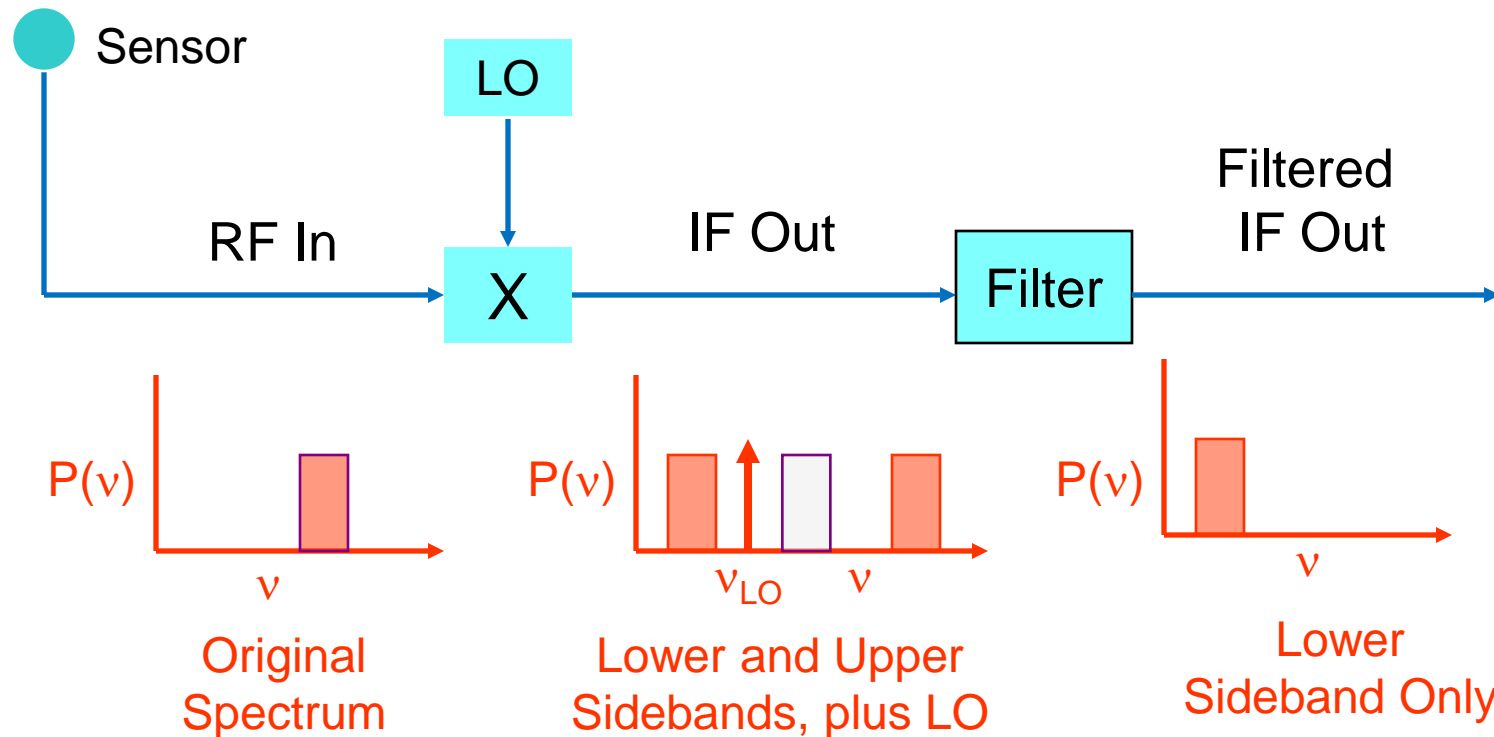


# The Heterodyne Interferometer: LOs, IFs, and Downconversion

- This would be the end of the story (so far as the fundamentals are concerned) if all the internal electronics of an interferometer would work at the observing frequency (often called the 'radio frequency', or RF).
- Unfortunately, this cannot be done in general, as high frequency components are much more expensive, and generally perform more poorly than low frequency components.
- Thus, most radio interferometers use 'down-conversion' to translate the radio frequency information from the 'RF' to a lower frequency band, called the 'IF' in the jargon of our trade.
- For signals in the radio-frequency part of the spectrum, this can be done with almost no loss of information.
- But there is an important side-effect from this operation in interferometry which we now review.

# Downconversion

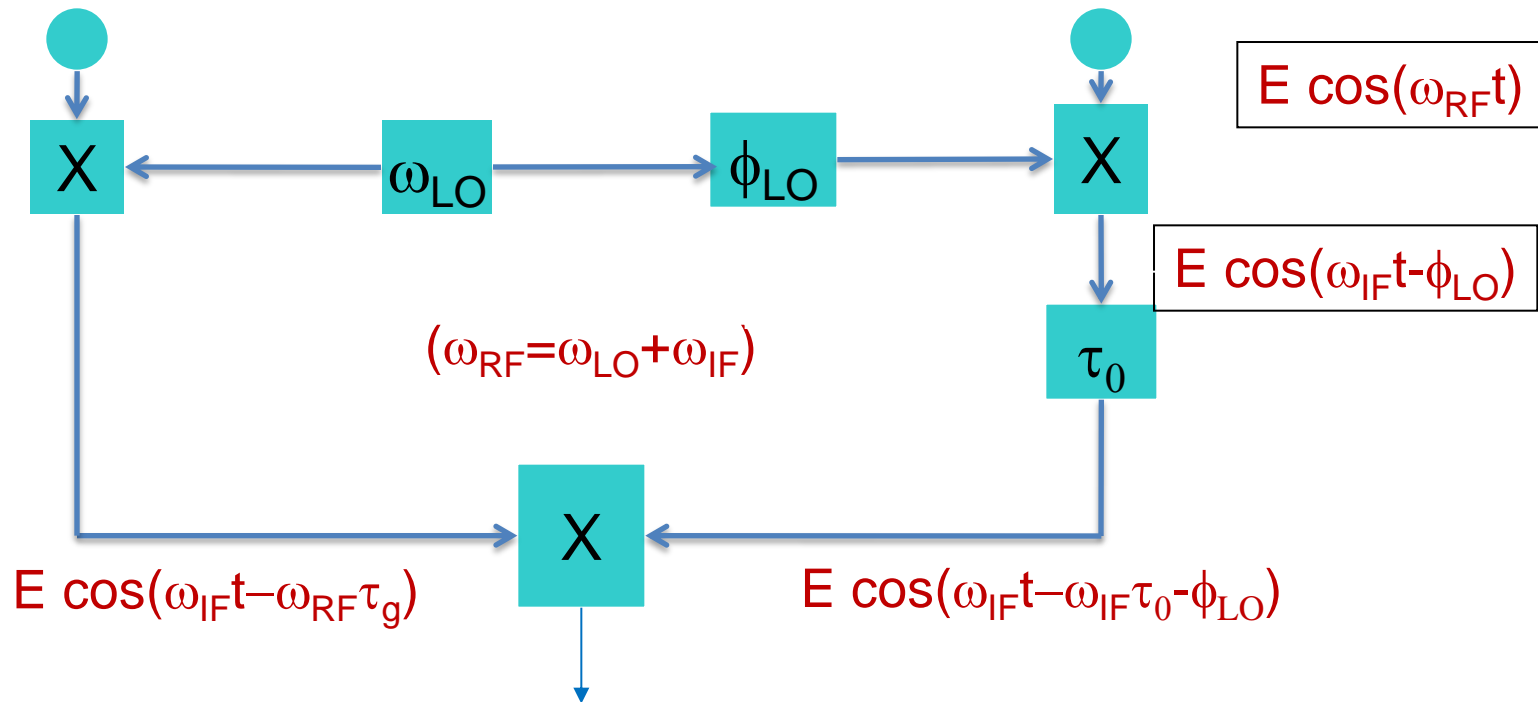
At radio frequencies, the spectral content within a passband can be shifted – with almost no loss in information, to a lower frequency through multiplication by a ‘LO’ signal.





# Signal Relations, with LO Downconversion

- The RF signals are multiplied by a pure sinusoid, at frequency  $\nu_{LO}$
- We can add arbitrary phase  $\phi_{LO}$  on one side.



$$V = E^2 e^{-i(\omega_{RF} \tau_g - \omega_{IF} \tau_0 - \phi_{LO})}$$

# Recovering the Correct Visibility Phase

- The correct phase (RF interferometer) is:  $\omega_{RF} (\tau_g - \tau_0)$
- The observed phase (with frequency downconversion) is:

$$\omega_{RF} \tau_g - \omega_{IF} \tau_0 - \phi_{LO}$$

- These will be the same when the LO phase is set to:

$$\phi_{LO} = \omega_{LO} \tau_0$$

- This is necessary because the delay,  $\tau_0$ , has been added in the IF portion of the signal path, rather than at the frequency at which the delay actually occurs.
- The phase adjustment of the LO compensates for the delay having been inserted at the IF, rather than at the RF.

# The Three ‘Centers’ in Interferometry

- You are forgiven if you’re confused by all these ‘centers’.
- So let’s review:
  1. **Beam Tracking (Pointing) Center:** Where the antennas are pointing to. (Or, for phased arrays, the phased array center position).
  2. **Delay Tracking Center:** The location for which the delays are being set for maximum wide-band coherence.
  3. **Phase Tracking Center:** The location for which the LO phase is slipping in order to track the coherence pattern.
- Note: Generally, we make all three the same. #2 and #3 are the same for an ‘RF’ interferometer. They are separable in a LO downconversion system.



# Interferometer Geometry

- We have not defined any geometric system for our relations.
- The response functions we defined were generalized in terms of the scalar product between two fundamental vectors:
- The baseline '**B**', defining the direction and separation of the antennas, and
- The unit vector '**s**', specifying the direction of the source.
- At this time, we define the geometric coordinate frame for the interferometer.
- We begin with a special case: An interferometer whose antennas all lie on a single plane.

# The 2-Dimensional Interferometer

To give better understanding, we now specify the geometry.

## Case A: A 2-dimensional measurement plane.

- Let us imagine the measurements of  $V_v(\mathbf{b})$  to be taken entirely on a plane.
- Then a considerable simplification occurs if we arrange the coordinate system so one axis is normal to this plane.
- Let  $(u,v,w)$  be the coordinate axes, with  $w$  normal to this plane. Then:

$$\mathbf{b} = (\lambda u, \lambda v, \lambda w) = (\lambda u, \lambda v, 0)$$

$u, v$ , and  $w$  are always measured in wavelengths.

- The components of the unit direction vector,  $\mathbf{s}$ , are:

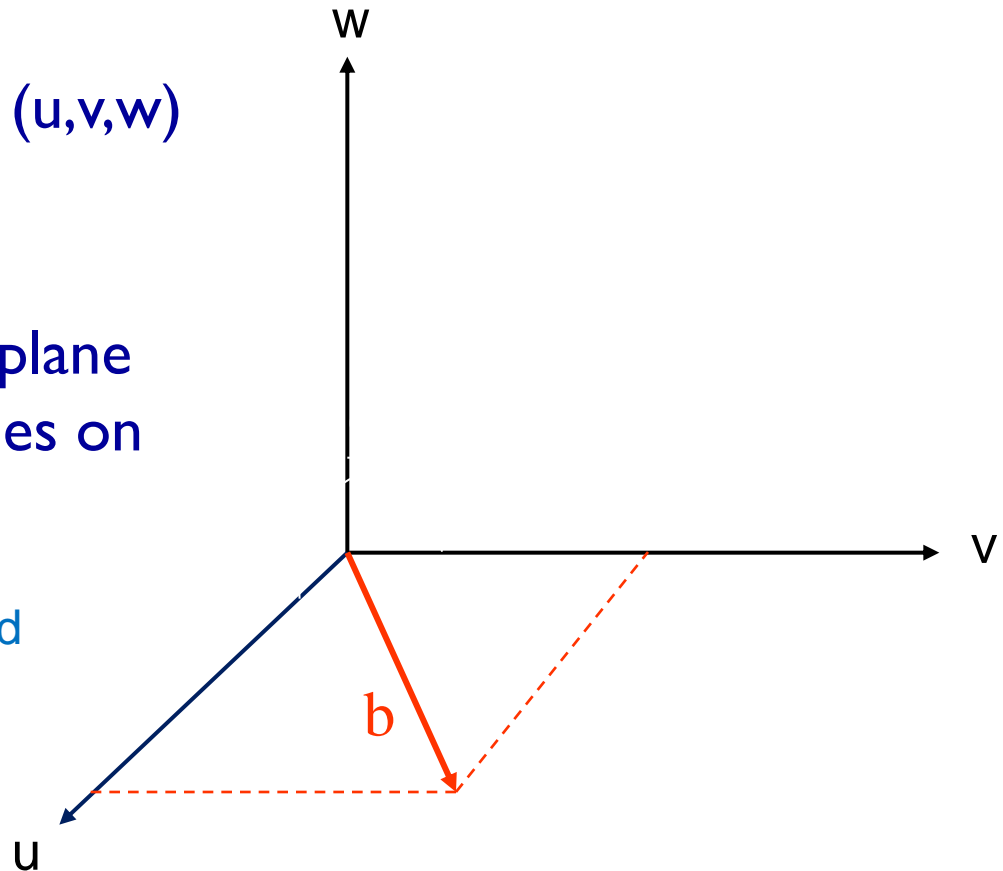
$$\mathbf{s} = (l, m, n) = \left( l, m, \sqrt{1 - l^2 - m^2} \right)$$

# The (u,v,w) Coordinate System.

- Pick a coordinate system (u,v,w) to describe the antenna positions and baselines.
- Orient this frame so the plane containing the antennas lies on  $w = 0$ .

The baseline vector **b** is specified by its coordinates (u,v,w) (measured in wavelengths). In the case shown,  $w = 0$ , and

$$\mathbf{b} = (\lambda u, \lambda v, 0)$$



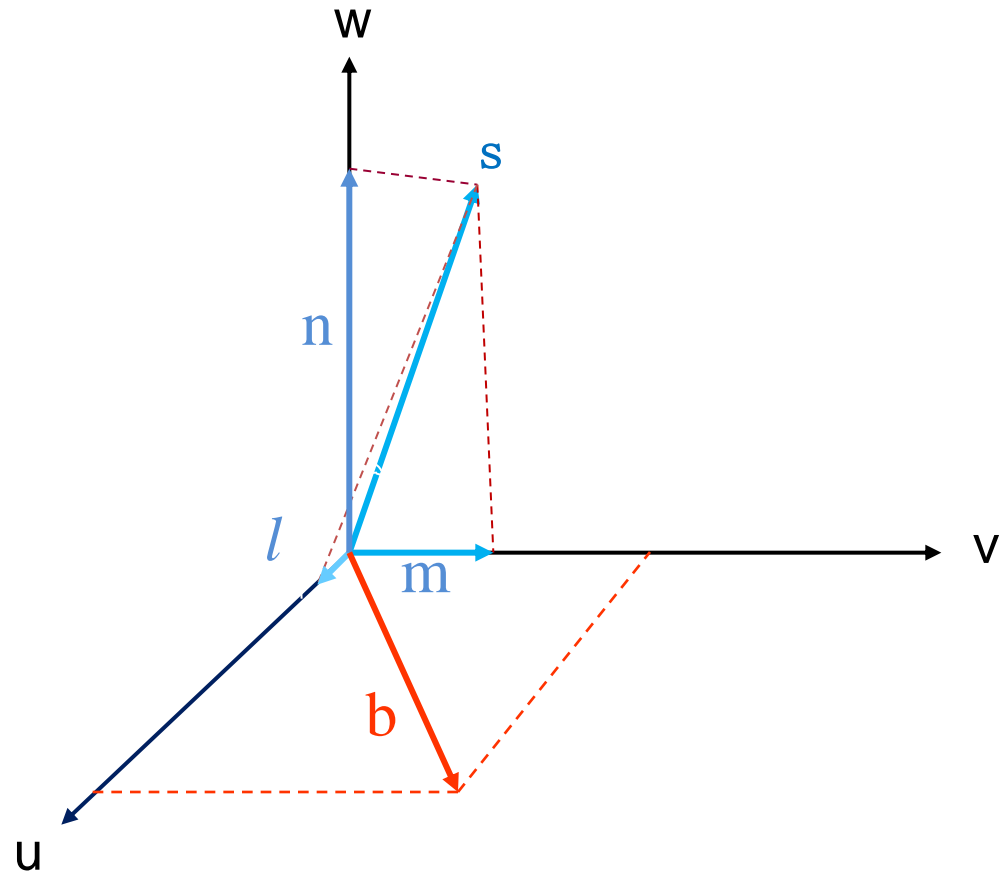
# Direction Cosines – describing the source

The unit direction vector  $\mathbf{s}$  is defined by its projections  $(l, m, n)$  on the  $(u, v, w)$  axes. These components are called the **Direction Cosines**.

$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\theta) = \sqrt{1 - l^2 - m^2}$$



The angles,  $\alpha$ ,  $\beta$ , and  $\theta$  are between the direction vector and the three axes.

# The 2-d Fourier Transform Relation

Then,  $\sqrt{b \cdot s/c} = u/l + v/m + w/n = u/l + v/m$ , from which we find,

$$V_v(u, v) = \iint \frac{I_v(l, m)}{\sqrt{1 - l^2 - m^2}} e^{-i2\pi(ul + vm)} dl dm$$

which is a **2-dimensional Fourier transform** between the projected brightness and the spatial coherence function (visibility):

$$I_v(l, m) / \cos(\theta) \Leftrightarrow V(u, v)$$

And we can now rely on two centuries of effort by mathematicians on how to invert this equation, and how much information we need to obtain an image of sufficient quality.

Formally,

$$I_v(l, m) = \cos(\theta) \iint V_v(u, v) e^{i2\pi(ul + vm)} du dv$$

In physical optics, this is known as the ‘Van Cittert-Zernicke Theorem’.



# Interferometers with 2-d Geometry

- **Which interferometers can use this special geometry?**

- a) Those whose baselines, over time, lie on a plane (any plane).

- All E-W interferometers are in this group. For these, the w-coordinate points to the NCP.

- WSRT (Westerbork Synthesis Radio Telescope)
      - ATCA (Australia Telescope Compact Array) (before the third arm)
      - Cambridge 5km (Ryle) telescope (almost).

- b) Any coplanar 2-dimensional array, at a single instance of time.

- In this case, the 'w' coordinate points to the zenith.

- VLA or GMRT in snapshot (single short observation) mode.

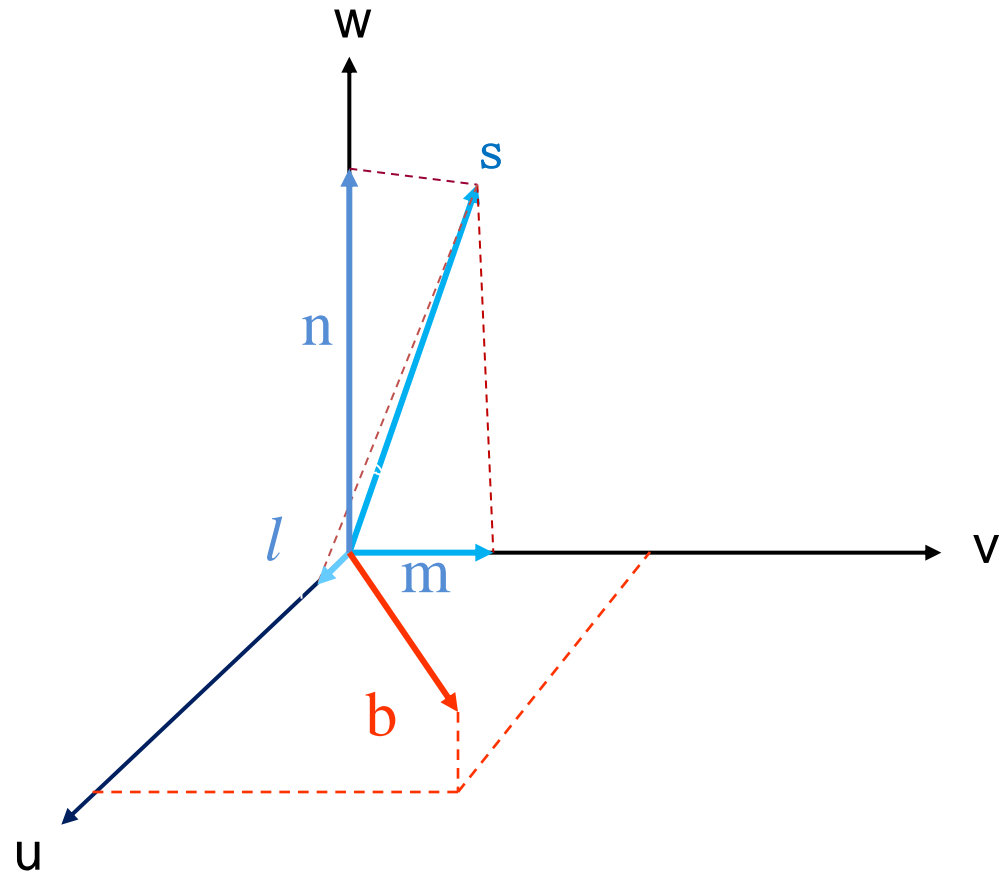
- **What's the 'downside' of 2-d (u,v) coverage?**

- Resolution degrades for observations that are not in the w-direction.

- E-W interferometers have no N-S resolution for observations at the celestial equator.
    - A VLA snapshot of a source will have no 'vertical' resolution for objects on the horizon.

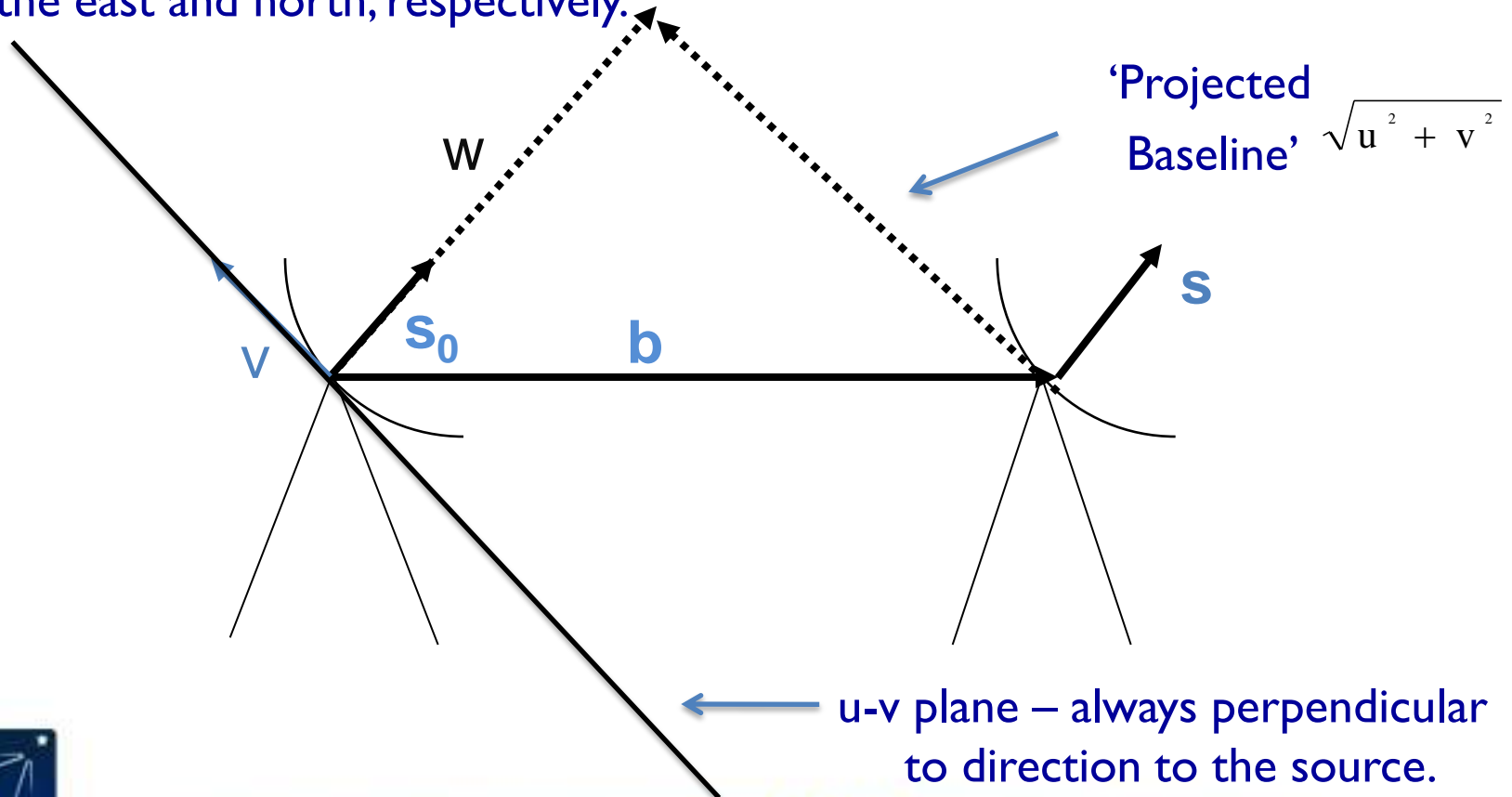
# Generalized Baseline Geometry

- Coplanar arrays (like the VLA) cannot use the 2-d geometry, since the plane of the array is rotating w.r.t. the source.
- (Or, the zenith distance of the source keeps changing w.r.t. the array plane).
- In this case, we must adopt a more general geometry, where all three baseline components are to be considered.



# General Coordinate System

- This is the coordinate system in most general use for synthesis imaging.
- $\mathbf{w}$  points to, and follows the source,  $\mathbf{u}$  towards the east, and  $\mathbf{v}$  towards the north celestial pole. The direction cosines  $l$  and  $m$  then increase to the east and north, respectively.



# 3-d Interferometers

## Case B: A 3-dimensional measurement volume:

- What if the interferometer does not measure the coherence function on a plane, but rather does it through a volume? In this case, we adopt a different coordinate system. First we write out the full expression:

$$V_v(u, v, w) = \iint \frac{I_v(l, m)}{\sqrt{1 - l^2 - m^2}} e^{-2i\pi(ul + vm + wn)} dl dm$$

(Note that this is not a 3-D Fourier Transform).

- We orient the w-axis of the coordinate system to point to the region of interest. The u-axis point east, and the v-axis to the north celestial pole.
- We introduce phase tracking, so the fringes are 'stopped' for the direction  $l=m=0$ . This means we adjust the phases by  $e^{2i\pi w}$
- Then, remembering that  $n^2 = 1 - l^2 - m^2$  we get:

$$V_v(u, v, w) = \iint \frac{I_v(l, m)}{\sqrt{1 - l^2 - m^2}} e^{-2i\pi[ul + vm + w(\sqrt{1 - l^2 - m^2} - 1)]} dl dm$$

## 3-d to 2-d

- The expression is still not a proper Fourier transform.
- We can get a 2-d FT if the third term in the phase factor is sufficient small.
- The third term in the phase can be neglected if it is much less than unity:

$$w \left| 1 - \sqrt{1 - l^2 - m^2} \right| = w(1 - \cos \theta) \sim w\theta^2 / 2 \ll 1$$

- This condition holds when:  
(angles in radians!)

$$\theta_{\max} < \sqrt{\frac{1}{2w}} \sim \sqrt{\frac{\lambda}{B}} \sim \sqrt{\theta_{\text{syn}}}$$

- If this condition is met, then the relation between the Intensity and the Visibility again becomes a 2-dimensional Fourier transform:

$$V_v(u, v) = \iint I_v(l, m) e^{-2i\pi(ul+vm)} dl dm$$

# The Problem with Non-coplanar Baselines

- Use of the 2-D transform for non-coplanar interferometer arrays (like the VLA, when used over time) always results in an error in the images.
- The ‘Clark Condition’ for trouble is:  $\frac{\lambda B}{D^2} > 1$
- Hence, the problem is most acute for small-diameter antennas and long wavelengths.
- The problems are not in the principles, but in the cost of the solutions. Full 3-D imaging works, but isn’t cheap.
- Steve Myers will be giving a later lecture on this interesting issue.

# Coverage of the U-V Plane

- Obtaining a good image of a source requires adequate sampling ('coverage') of the (u,v) plane.
- Adopt an earth-based coordinate grid to describe the antenna positions:
  - X points to  $H=0, \delta=0$  (intersection of meridian and celestial equator)
  - Y points to  $H = -6, \delta = 0$  (to east, on celestial equator)
  - Z points to  $\delta = 90$  (to NCP).
- Then denote by  $(B_x, B_y, B_z)$  the coordinates, measured in wavelengths, of a baseline in this earth-based frame.
- $(B_x, B_y)$  are the projected coordinates of the baseline (in wavelengths) on the equatorial plane of the earth.
- $B_y$  is the East-West component
- $B_z$  is the baseline component up the Earth's rotational axis.

# (u,v,w) Coordinates

- Then, it can be shown that

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \sin H_0 & \cos H_0 & 0 \\ -\sin \delta_0 \cos H_0 & \sin \delta_0 \sin H_0 & \cos \delta_0 \\ \cos \delta_0 \cos H_0 & -\cos \delta_0 \sin H_0 & \sin \delta_0 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

- The u and v coordinates describe E-W and N-S components of the projected interferometer baseline.
- The w coordinate is the delay distance in wavelengths between the two antennas. The geometric delay,  $\tau_g$  is given by

$$\tau_g = \frac{\lambda}{c} w = \frac{w}{\nu}$$

- Its derivative, called the fringe frequency  $\nu_F$  is

$$\nu_F = \frac{dw}{dt} = -\omega_E u \cos \delta_0$$

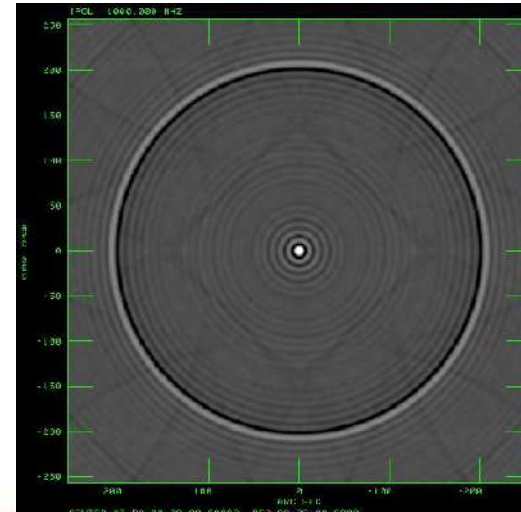
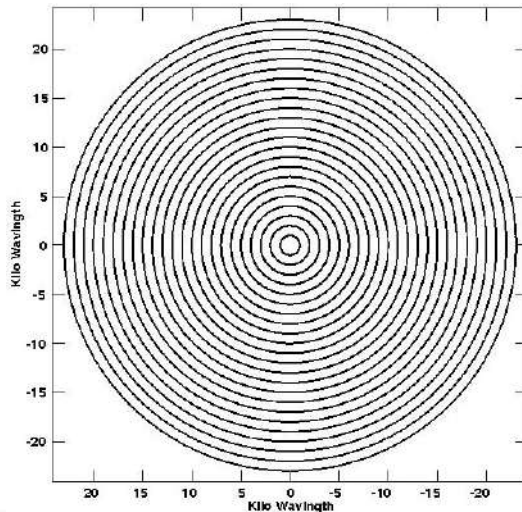


# E-W Array Coverage and Beams

- The simplest case is for E-W arrays, which give coplanar coverage.
- Then,  $B_x = B_z = 0$
- Consider a 'minimum redundancy array', with eight antennas located at 0, 1, 2, 11, 15, 18, 21 and 23 km along an E-W arm.

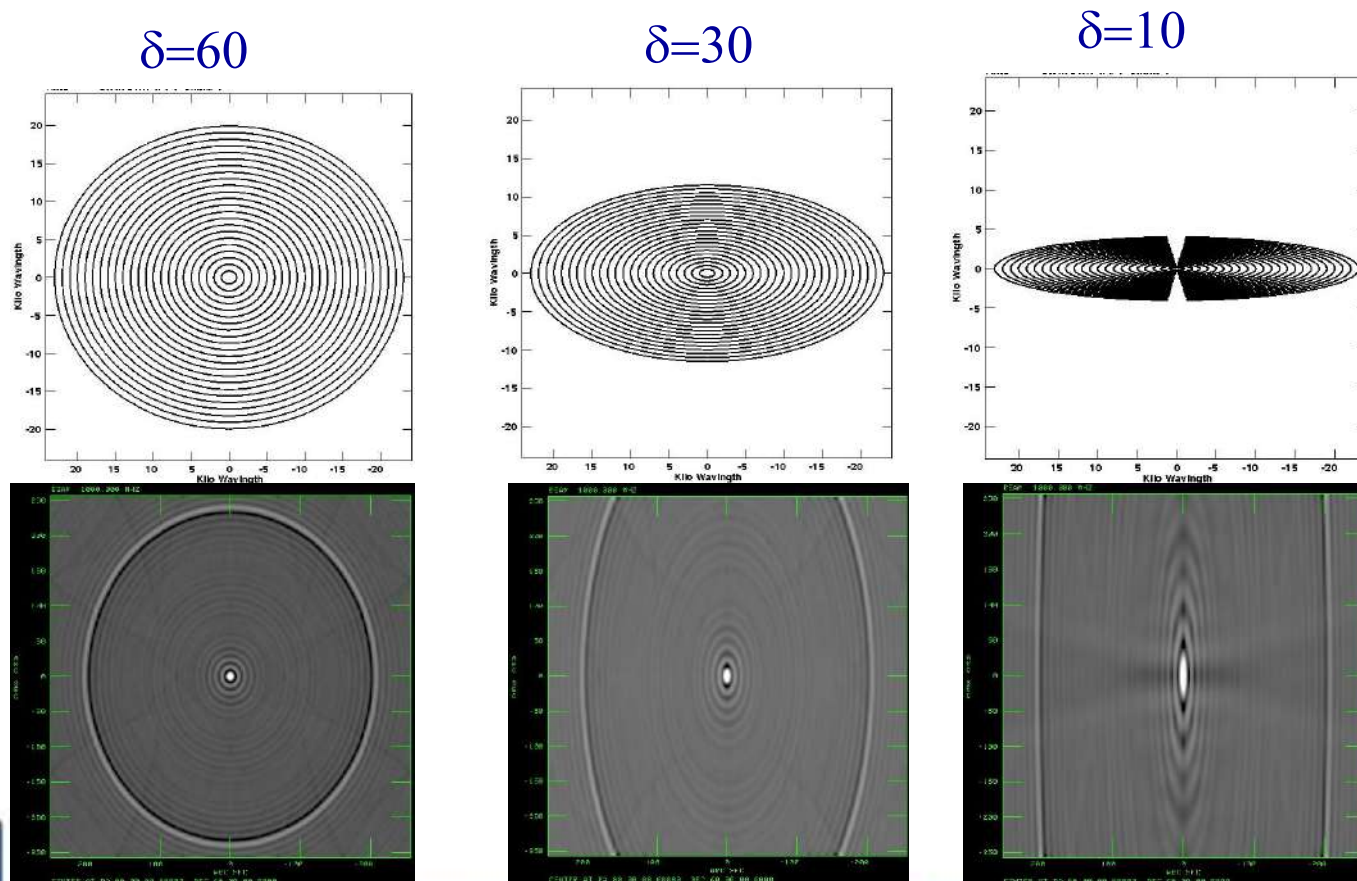


- Of the 28 simultaneous spacings, 23 are of a unique separation.
- The U-V coverage (over 12 hours) at  $\delta = 90$ , and the synthesized beam are shown below, for a wavelength of 1 m.



# E-W Arrays and Low-Dec sources.

- But the trouble with E-W arrays is that they are not suited for low-declination observing.
- At  $\delta=0$ , coverage degenerates to a line.

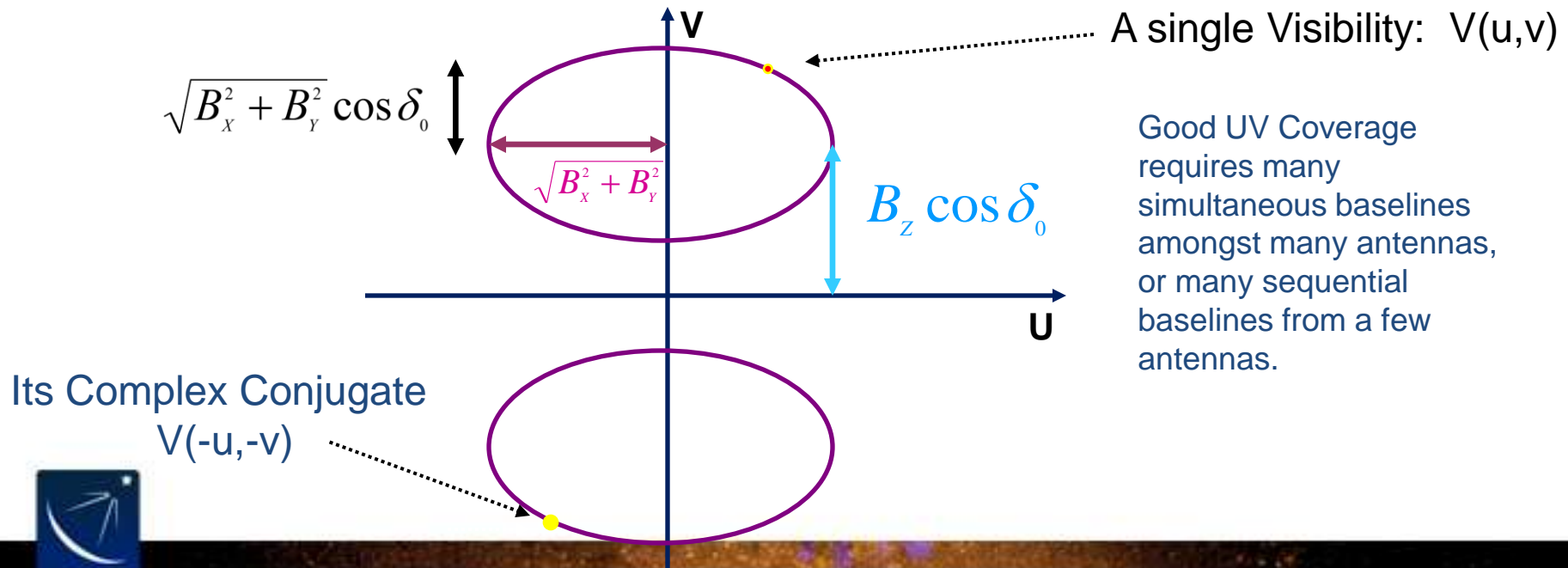


# Baseline Locus – the General Case

- Each baseline, over 24 hours, traces out an ellipse in the (u,v) plane:

$$u^2 + \left( \frac{v - B_z \cos \delta_0}{\sin \delta_0} \right)^2 = \frac{B^2}{\sin^2 \delta_0}$$

- Because brightness is real, each observation provides us a second point, where:  $V(-u, -v) = V^*(u, v)$
- E-W baselines ( $B_x = B_z = 0$ ) have no 'v' offset in the ellipses.



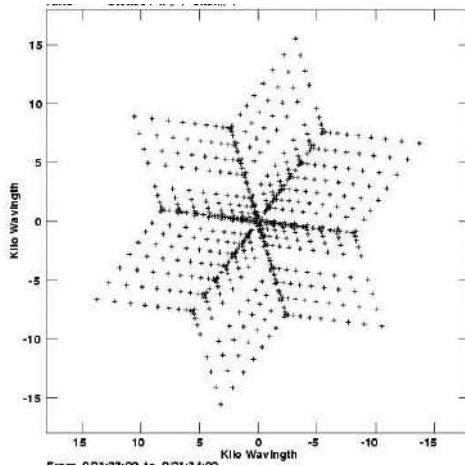
# Getting Good Coverage near $\delta = 0$

- The only means of getting good 2-d angular resolution at all declinations is to build an array with N-S spacings.
- Many more antennas are needed to provide good coverage for such geometries.
- The VLA was designed to do this, using 9 antennas on each of three equiangular arms.
- Built in the 1970s, commissioned in 1980, the VLA vastly improved radio synthesis imaging at all declinations.
- Each of the 351 spacings traces an elliptical locus on the (u,v) plane.
- Every baseline has some (N-S) component, so none of the ellipses is centered on the origin.

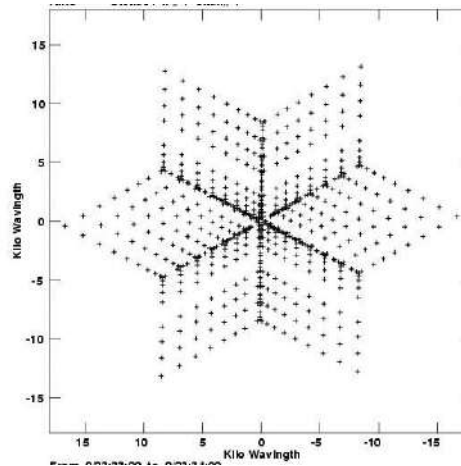


# Sample VLA (U,V) plots for 3C147 ( $\delta = 50$ )

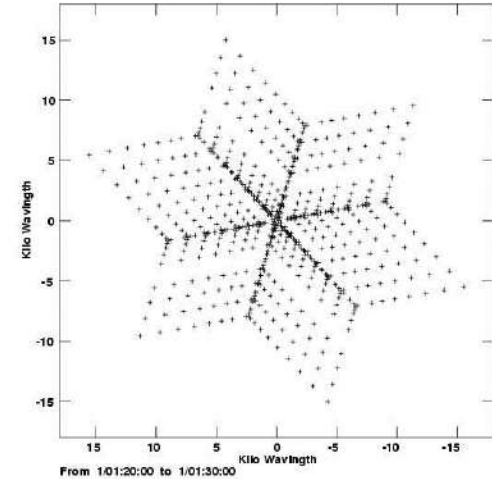
- Snapshot (u,v) coverage for HA = -2, 0, +2 (with 26 antennas).



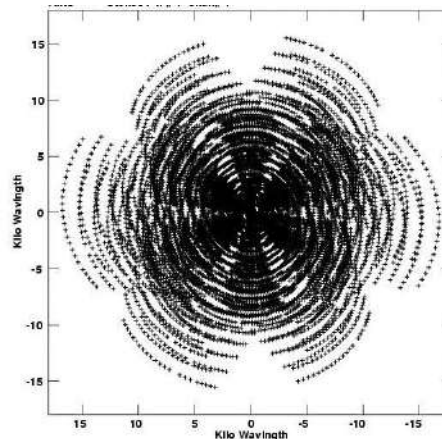
HA = -2h



HA = 0h



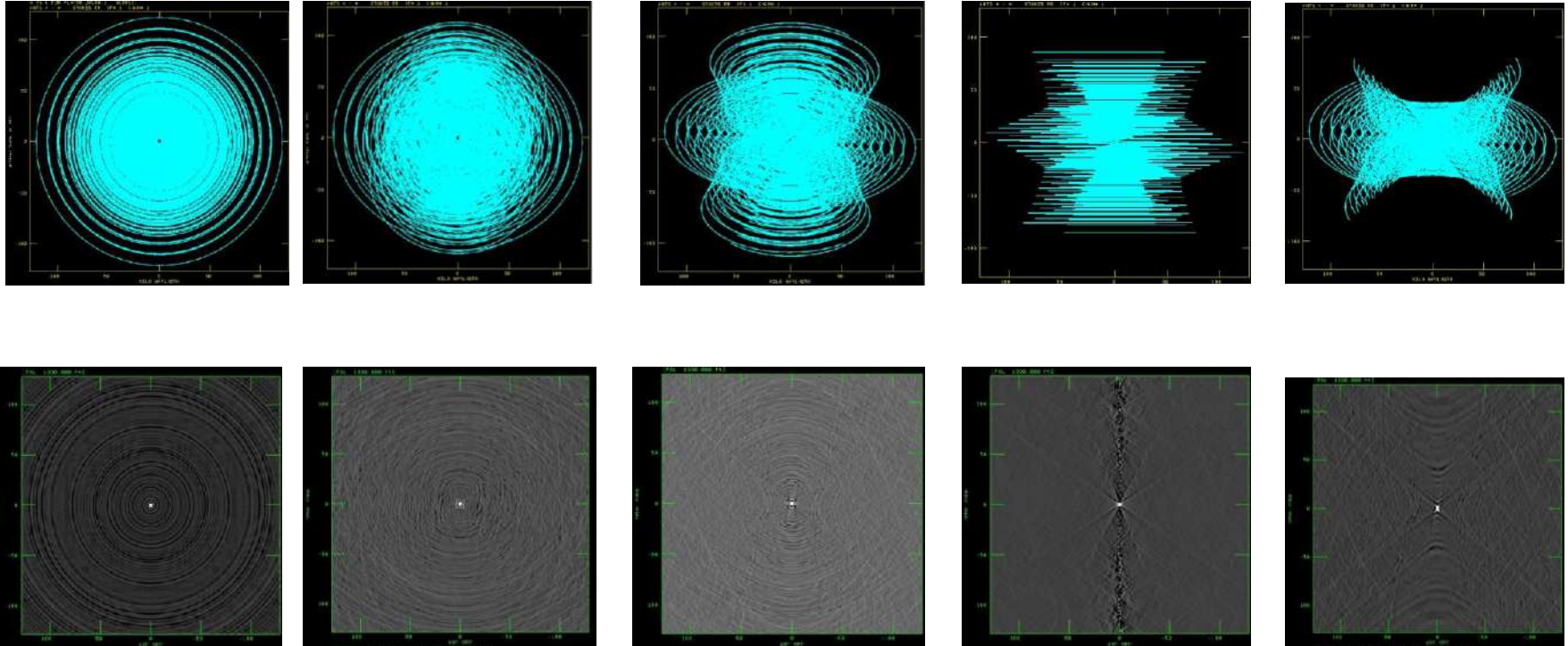
HA = 2h



Coverage over  
all four hours.



# VLA Coverage and Beams



$\delta=90$

$\delta=60$

$\delta=30$

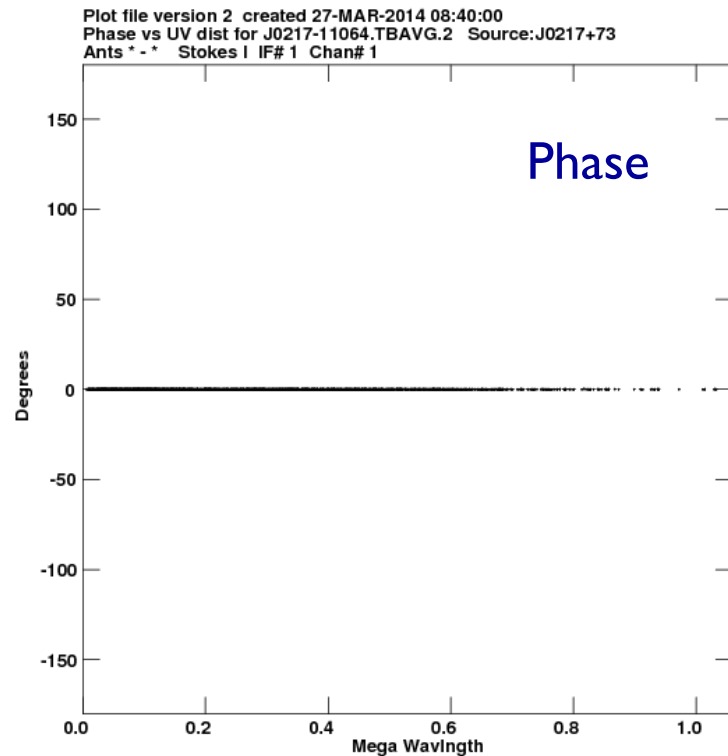
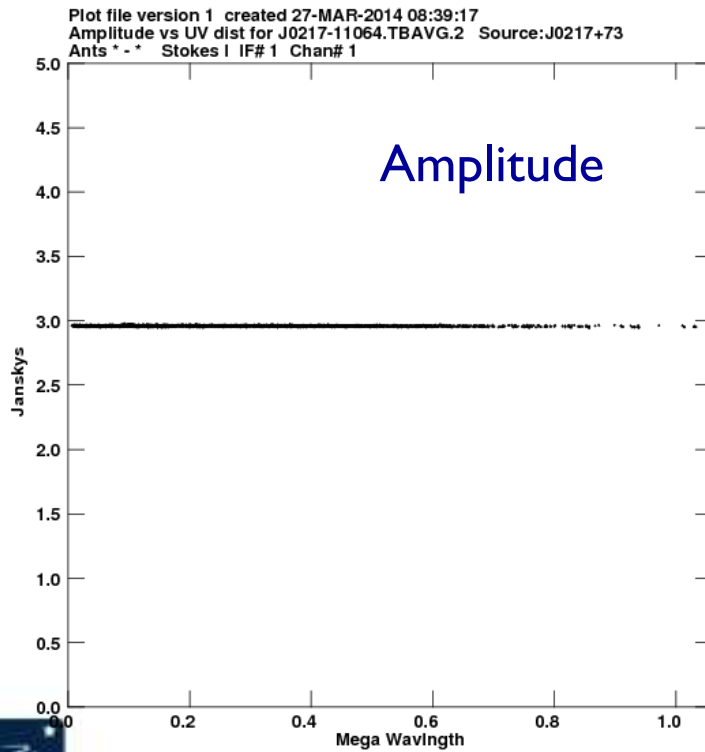
$\delta=0$

$\delta=-30$

- Good coverage at all declinations, but troubles near  $\delta=0$  remain.

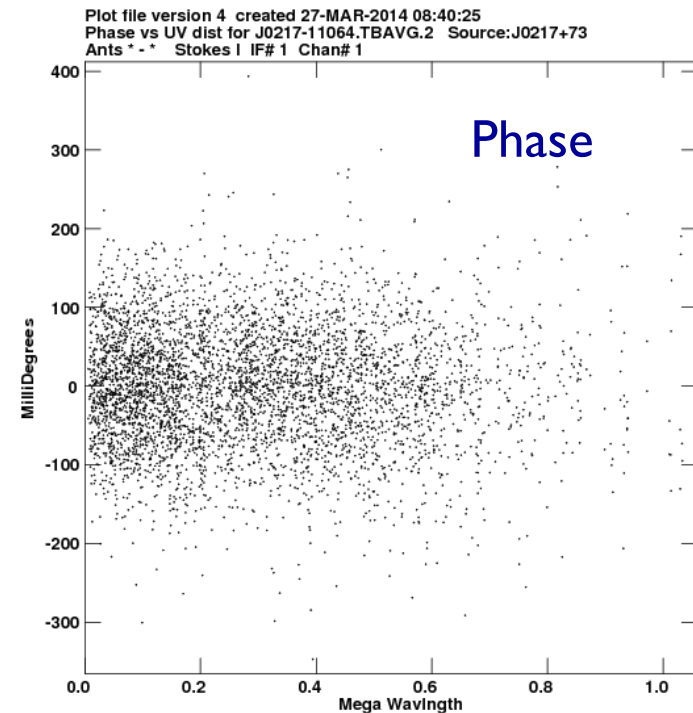
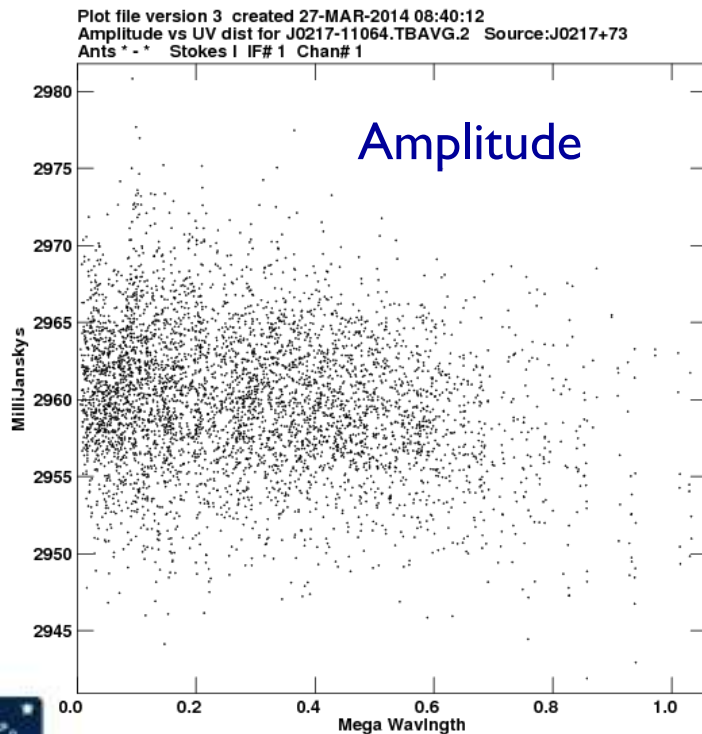
# Examples of Visibilities – A Point Source

- Suppose we observe an unresolved object.
- What is its visibility function?



# Zoom in ...

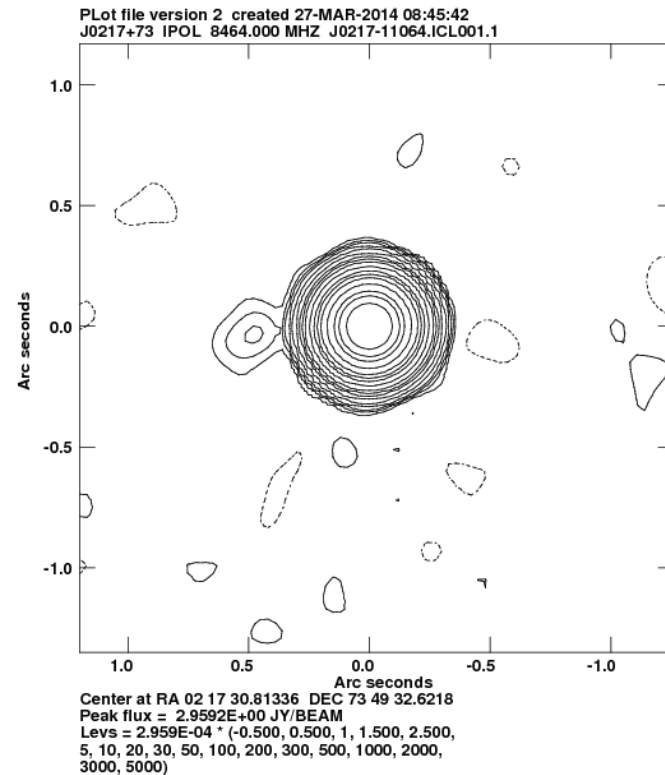
- Suppose we observe an unresolved object.
- What is its visibility function?



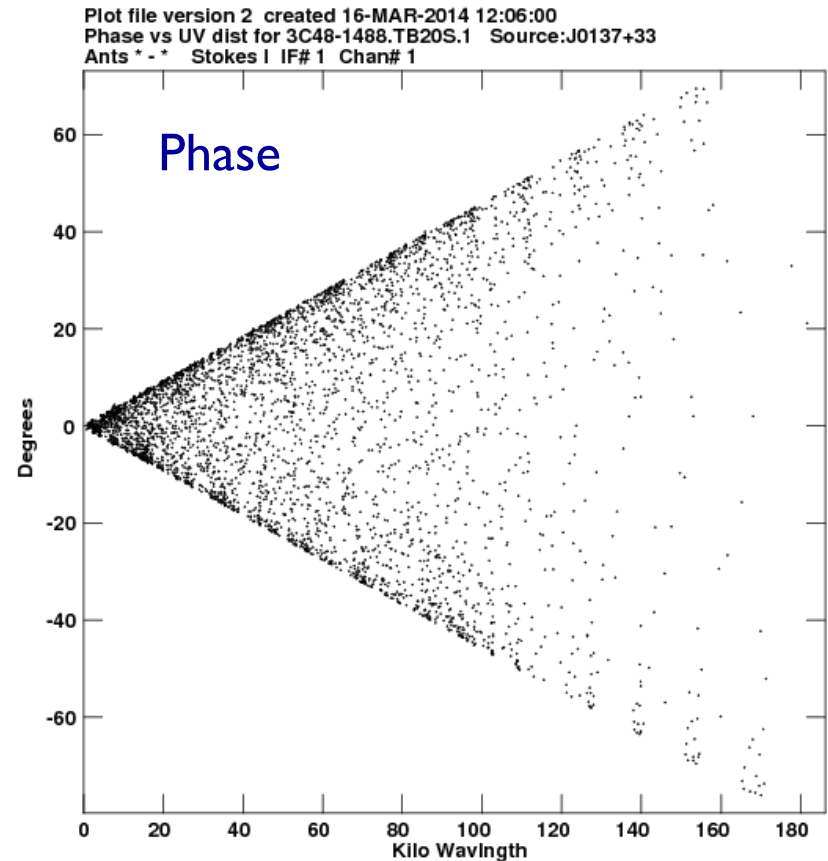
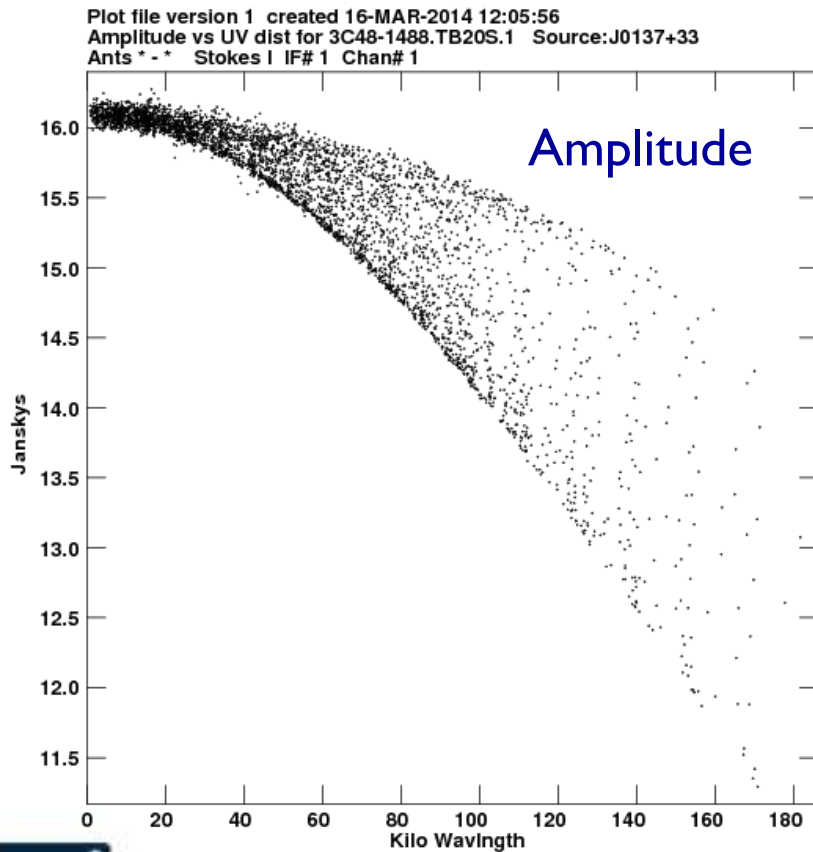


# And the Map ...

- The source is unresolved ... but with a tiny background object.
- Dynamic range: 50,000:1.

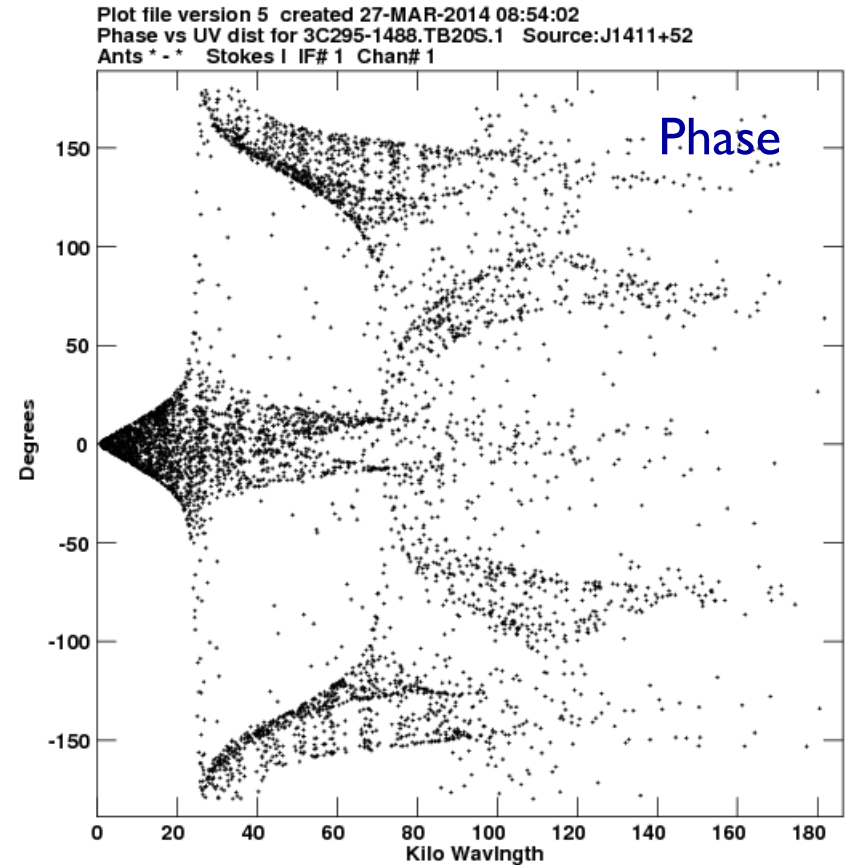
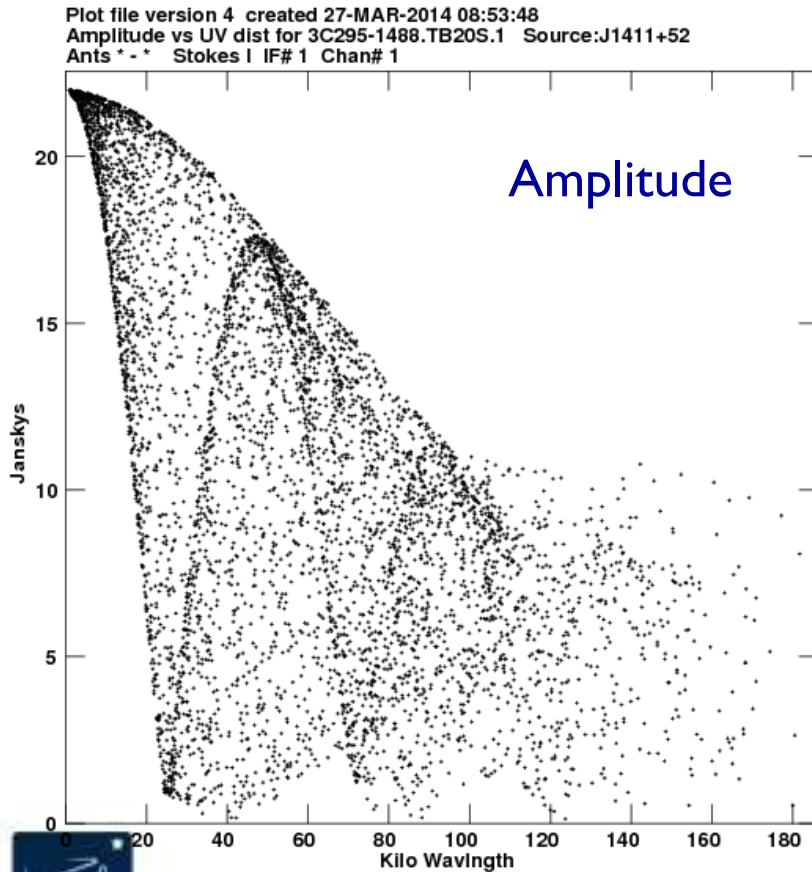


# 3C48 at 21 cm wavelength – a slightly resolved object.



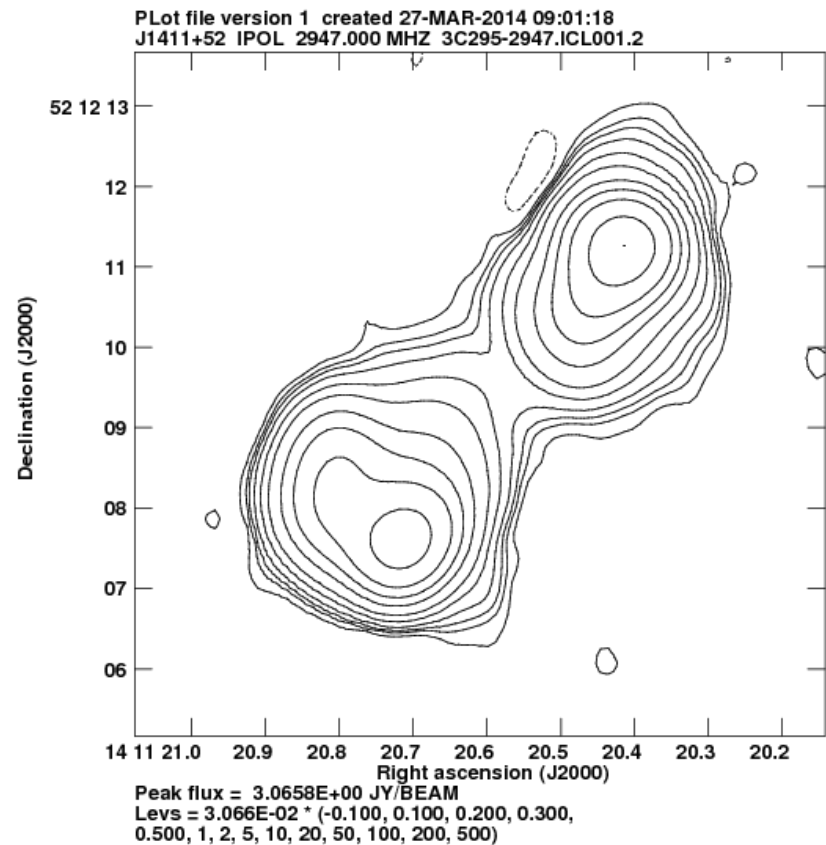
# Examples of Visibilities – a Well Resolved Object

- The flux calibrator 3C295



# 3C295 Image

- A 5-arcsecond double.



# UV Coverage and Imaging Fidelity

- Although the VLA represented a huge advance over what came before, its UV coverage (and imaging fidelity) is far from optimal.
- The high density of samplings along the arms (the 6-armed star in snapshot coverage) results in 'rays' in the images due to small errors.
- A better design is to 'randomize' the location of antennas within the span of the array, to better distribute the errors.
- Of course, more antennas would really help! :) .
- The VLA's wye design was dictated by its 220 ton antennas, and the need to move them. Railway tracks were the only answer.
- Future major arrays will utilize smaller, lighter elements which must not be positioned with any regularity.