(W)

Starting from: $r = \dfrac{p}{1+e\cos\theta} = \dfrac{h^2/\mu}{1+e\cos\theta}$.

$$h^2 = \mu r(1+e\cos\theta)$$

if perturbations are not considered h is a "first integral of motion" while in presence of perturbative effects:

$$h^2 = \mu r(1+e\cos\theta) \rightarrow 2h\dot{h} = \mu\dot{r}(1+e\cos\theta) + \mu r\dot{e}\cos\theta - \mu r e\sin\theta\dot{\theta}$$

$\dot{\theta} = \dot{\theta}_K + \dot{\theta}_P$

$\dot{\theta}_K \rightarrow$ Keplerian motion

$\dot{\theta}_P \rightarrow$ induced by perturbative effects

$$h^2 = \mu r(1+e\cos\theta) \xrightarrow{\text{KEPLERIAN}} 2h\dot{h} = \mu\dot{r}(1+e\cos\theta) - \mu r e\sin\theta\dot{\theta} = 0$$

$$h^2 = \mu r(1+e\cos\theta) \xrightarrow{\text{PERTURBED}} 2h\dot{h} = \mu\dot{r}(1+e\cos\theta) + \mu r\dot{e}\cos\theta - \mu r e\sin\theta\dot{\theta}$$

BETTER:

$$2h\dot{h} = \underbrace{\mu\dot{r}(1+e\cos\theta) + \mu r\dot{e}\cos\theta - \mu r e\sin\theta(\dot{\theta}_K+\dot{\theta}_P)}_{\mu\dot{r}(1+e\cos\theta) - \mu r e\sin\theta\dot{\theta}_K = 0}$$

$$\Rightarrow 2h\dot{h} = \mu r\dot{e}\cos\theta - \mu r e\sin\theta\dot{\theta}_P$$

$$\dot{\theta}_P = -\frac{2h\dot{h}}{\mu r e\sin\theta} + \frac{\mu r\dot{e}\cos\theta}{\mu r e\sin\theta}$$

$$\rightarrow \dot{\theta}_P = \frac{\dot{e}\cos\theta}{e\sin\theta} - \frac{2h\dot{h}}{\mu r e\sin\theta}.$$

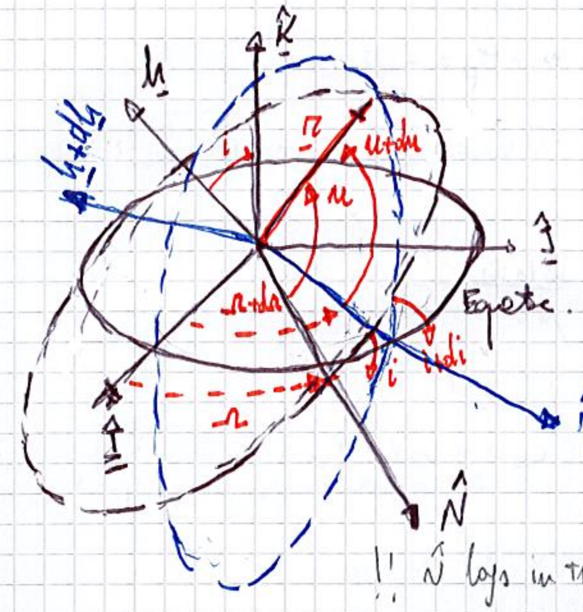
$u \triangleq \theta_p + w \rightarrow u = \hat{N}\,\widehat{\ }\,\hat{r}$

(R)

!! MAINTAINING SAME RADIUS $\Rightarrow$ VARIATIONS IN θ ARE ONLY DUE TO PERTURBATIVE EFFECTS !! (θ_P)

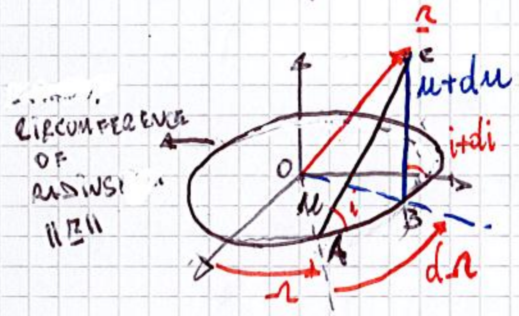
- - - $\rightarrow$: perturbed orbit

- - - $\rightarrow$ unperturbed orbit.



!! $\hat{N}$ lays in the intersection between equatorial plane and orbital plane !!

$\Downarrow$

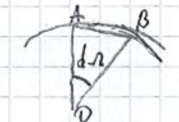


Considering the distances instead of angles

$\overline{AB} \simeq r\,d\Omega$

$\overline{AC} \triangleq a$

$\overline{BC} \triangleq a + da$



(a arbitrary length.)

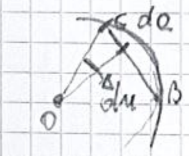
Thanks to Carnot's Theorem: $(a+da)^2 = a^2 + r^2 d\Omega^2 - 2ar\,d\Omega\cos i$

$\cancel{a^2} + da^2 + 2a\,da = \cancel{a^2} + r^2 d\Omega^2 - 2ar\,d\Omega\cos i$

$\rightarrow$ Neglecting 2nd order derivative: $da^2 \rightarrow 0$, $d\Omega^2 \rightarrow 0$

$\cancel{2a}\,da = -\cancel{2a}\,r\,d\Omega\cos i$

But $da = r\,du$



$\Rightarrow \cancel{r}\,du = -\cancel{r}\,d\Omega\cos i$

$du = -d\Omega\cos i$

$\rightarrow$ $u \triangleq \hat{N}\,\widehat{\ }\,\hat{r} \Rightarrow u = w + \theta_p$

$du = -d\Omega\cos i$

Through relations: $\dot\theta_p = \frac{\dot e \cos\theta}{e\sin\theta} - \frac{2h\dot h}{\mu r e \sin\theta}$ (i)

$du = -d\Omega\cos i \Rightarrow \dot u = -\dot\Omega\cos i$

$\dot\omega + \dot\theta_p = -\dot\Omega\cos i$

$\dot\omega = -\dot\Omega\cos i - \dot\theta_p$ (ii)

* Recalled in Maillan approach

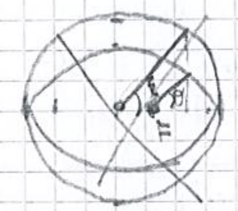
(i) in (ii) $\Rightarrow$ $\dot\omega = -\dot\Omega\cos i - \dot e\,\frac{\cot\theta}{e} + \frac{2h\dot h}{\mu r e\sin\theta}$

where:

- $\dot e = \sqrt{\frac{a(1-e^2)}{\mu}}\,[\sin\theta\, f_r + (\cos\theta + \cos E) f_\theta]$
- $\dot h = r f_\theta$
- $\dot\Omega = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)}\,\frac{\sin(\theta+\omega)}{\sin i}\, f_h$.

$$\frac{2h\dot h}{\mu r e\sin\theta} = 2\frac{\sqrt{\mu p}}{\mu}\cdot\frac{\cancel{r} f_\theta}{\cancel{r} e\sin\theta} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}}\,\frac{2f_\theta}{e\sin\theta}$$

$$\cancel{\frac{d\omega}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}}\Big[-\frac{\sin(\theta+\omega)}{(1+e\cos\theta)}\cot i\, f_h - \frac{\sin\theta\cos\theta}{e\sin\theta}f_r + f_\theta\Big(-\cos\theta\frac{\cot\theta}{e} - \cos E\frac{\cot\theta}{e} + \frac{2}{e\sin\theta}\Big)\Big]}$$



$ae + r\cos\theta = a\cos E$

$\cos E = e + \frac{\cancel{a}(1-e^2)}{\cancel{a}(1+e\cos\theta)}\cos\theta = \cancel{e + \frac{(1-e^2)}{(1+e\cos\theta)}\cos\theta}$

$\Rightarrow \cancel{-\frac{\cos^2\theta}{e\sin\theta}} - \cancel{\frac{e\cos\theta}{e\sin\theta}} + \cancel{\frac{(e^2-1)\cos\theta}{(1+e\cos\theta)e}}$

$$\Rightarrow \dot\omega = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}}\Big[-\frac{\sin(\theta+\omega)}{(1+e\cos\theta)}\,\frac{\cos i}{\sin i}\, f_h - (\sin\theta\, f_r + (\cos\theta+\cos E) f_\theta)\frac{\cot\theta}{e} + \frac{2f_\theta}{e\sin\theta}\Big]$$

$\sin\theta\,\frac{\cot\theta}{e} = \sin\theta\,\frac{\cos\theta}{\sin\theta}\,\frac{1}{e} = \frac{\cos\theta}{e}$

$$= \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}}\Big[-\frac{\sin(\theta+\omega)}{1+e\cos\theta}\cot i\, f_h - \frac{\cos\theta}{e} f_r + f_\theta\Big(\frac{2}{e\sin\theta} - \cos\theta\,\frac{\cot\theta}{e} - \cos E\,\frac{\cot\theta}{e}\Big)\Big]$$

$\cancel{ae + a\cos E = r\cos\theta}$ $\quad a\cos E = ae + r\cos\theta$

$\cos E = -e + \frac{r}{a}\cos\theta = -e + \frac{\cancel{a}(1-e^2)}{\cancel{a}}\,\frac{\cos\theta}{1+e\cos\theta}$

$\cos E = \frac{-1-e^2\cos\theta+\cos\theta-e^2\cos\theta}{(1+e\cos\theta)} \Rightarrow \cos E = \frac{e+\cos\theta}{1+e\cos\theta}$

$$\dot\omega = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}}\Big[-\frac{\sin(\theta+\omega)}{1+e\cos\theta}\cot i\, f_h - \frac{\cos\theta}{e} + f_\theta\Big(\frac{2}{e\sin\theta} - \frac{\cos^2\theta}{e\sin\theta} - \frac{e\cos\theta+\cos^2\theta}{e\sin\theta\,(1+e\cos\theta)}\Big)$$

Only term in f_θ

$$\frac{f_\theta}{e\sin\theta}\Big(\frac{2+2e\cos\theta-\cos^2\theta-e\cos^3\theta-e\cos\theta-\cos^2\theta}{1+e\cos\theta}\Big)$$

$$\frac{f_\theta}{e\sin\theta}\Big(\frac{2-2\cos^2\theta+e\cos\theta-e\cos^3\theta}{(1+e\cos\theta)}\Big)$$

$$= \frac{f_\theta}{e\sin\theta}\Big(\frac{2(1-\cos^2\theta)+e\cos\theta\,(1-\cos^2\theta)}{1+e\cos\theta}\Big) \qquad 1-\cos^2\theta = \sin^2\theta$$

$$= \frac{f_\theta}{e\sin\theta}\Big(\frac{2\sin^2\theta + e\cos\theta\sin^2\theta}{(1+e\cos\theta)}\Big)$$

$$\Rightarrow \frac{f_\theta}{e\sin\theta}\cdot\sin^2\theta\,\frac{(2+e\cos\theta)}{1+e\cos\theta}$$

$$f_\theta\sin\theta\,\frac{(2+e\cos\theta)}{1+e\cos\theta}$$

$$\rightarrow \frac{d\omega}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}}\Big[-\frac{\sin u\cot i}{1+e\cos\theta}f_h + \frac{2+e\cos\theta}{e(1+e\cos\theta)}\sin\theta\, f_\theta - \frac{\cos\theta}{e}f_r\Big]$$

$(M \leftrightarrow \theta)$

Kepler equation:

$$\begin{cases} M = \sqrt{\dfrac{\mu}{a^3}}\,\Delta t \\ M = E - e\sin E \end{cases}$$

$$\Rightarrow \dot M = \dot E - e\dot E\cos E - \dot e\sin E$$

$$\dot M = \dot E(1 - e\cos E) - \dot e\sin E \quad (i)$$

$$r = a(1 - e\cos E).$$

$$ae + r\cos\theta = \cos E\, a \longrightarrow r\cos\theta = a\cos E - ae$$

$$e + \frac{r}{a}\cos\theta = \cos E \longrightarrow \cos E = \frac{e+\cos\theta}{1+e\cos\theta}$$

$$\frac{a(1-e^2)}{1+e\cos\theta}\cos\theta = a\cos E \quad \ldots\ldots \quad r = a(1-e\cos E)$$

$$\longrightarrow \dot r = \dot a(1-e\cos E) + a(-\dot e\cos E + \sin E\,\dot E\,e)$$

$$\dot r = \dot a - \dot a e\cos E - a\dot e\cos E + ae\sin E\,\dot E$$

$$\dot E = \frac{\dot r}{ae\sin E} + \frac{\dot a e\cos E}{ae\sin E} - \frac{\dot a}{ae\sin E} + \frac{a\dot e\cos E}{ae\sin E}.$$

$$\begin{cases} \underline{v} = \sqrt{\dfrac{\mu}{p}}\,[e\sin\theta\,\hat{\underline{r}} + (1+e\cos\theta)\,\hat{\underline{\theta}}] \\ \underline{r} = r\cdot\hat{\underline{r}} \longrightarrow \underline{v} = \dot r\,\hat{\underline{r}} + r\dfrac{d\hat{\underline{r}}}{dt} = \dot r\,\hat{\underline{r}} + r\dot\theta\,\hat{\underline{\theta}} \end{cases}$$

$$\Rightarrow \dot r = \sqrt{\frac{\mu}{p}}\,e\sin\theta.$$

$$h = \sqrt{\mu p} = \sqrt{\frac{\mu}{p}}\,p$$

$$\Rightarrow \sqrt{\frac{\mu}{p}} = \frac{h}{p}$$

$$\dot r = \frac{\mu}{h}\,e\sin\theta.$$

$$\Rightarrow \dot E = \frac{\mu e\sin\theta}{hae\sin E} + \frac{\dot a e\cos E}{ae\sin E} - \frac{\dot a}{ae\sin E} + \frac{a\dot e\cos E}{ae\sin E}. \quad (ii)$$

By substituting $\begin{Bmatrix}\dot e \\ \dot a\end{Bmatrix}$ into (ii) and then everything into (i) is possible to obtain $\dot M$

$$\frac{dM}{dt} = n + f_M(a,e,\theta)$$

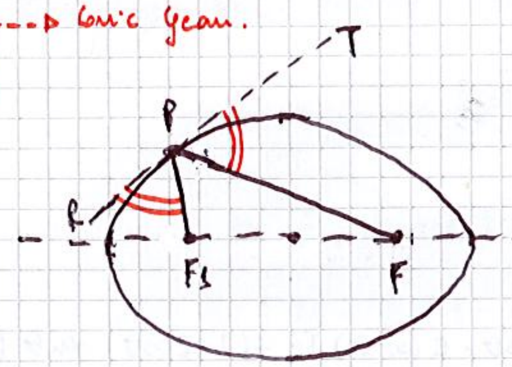
$$n = \frac{\dot r}{ae\sin E}(1-e\cos E) = \sqrt{\frac{\mu}{a^3}}$$

$$f_M = -\sqrt{\frac{a}{\mu}}\,\frac{1-e^2}{e(1+e\cos\theta)}\cdot[(2e-\cos\theta-e\cos^2\theta)f_r + (2+e\cos\theta)\sin\theta\, f_t]$$

→ Procedure to obtain VOP. (SYNTESIS)

II^{nd} Approach: MOULTON's METHOD

---> conic geom.



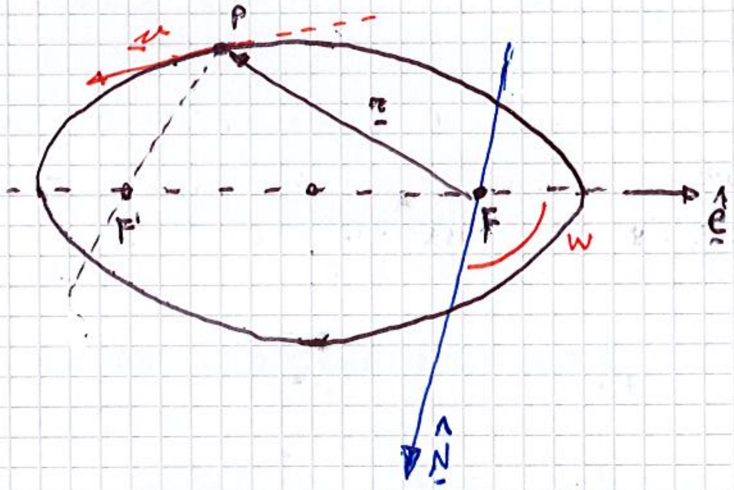
PROPERTIES:

(i) $\overline{PF} + \overline{PF_1} = 2a \quad \forall P$

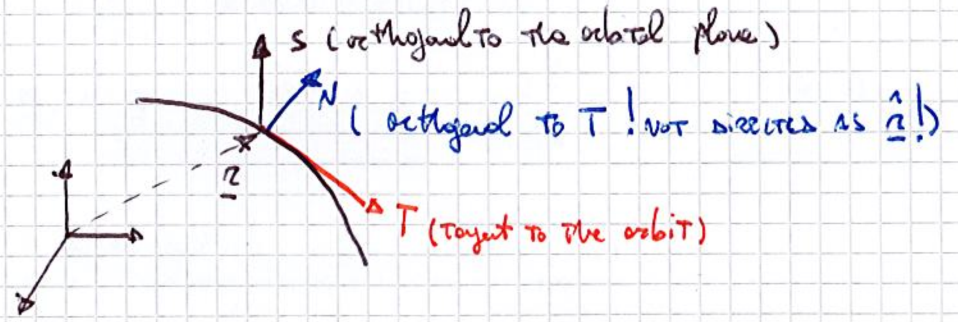
(ii) $\widehat{\overline{PF}\,\overline{TR}} = \widehat{\overline{PF_1}\,\overline{TR}} \quad \forall P$

($\overline{TR}$ → tangent direction in P to the conic.)

⇒ considering a trajectory:



Indeed moulton's approach is convenient to consider the $\{R, T, N\}$ reference frame



a_S → ORTHOGONAL COMPONENT (S)

≃ obvious: ORTHOGONAL COMPONENT DON'T AFFECT $\begin{Bmatrix} a \\ e \\ \omega \end{Bmatrix}$* ACTING OUT OF ORBITAL PLANE.

* an orthogonal pert. don't affect ω if is the only effect present.

in fact as in VOP approach: * $\dot{\omega} = -\dot{\Omega} \cos i - \dot{e} \frac{\cot\theta}{e} + \frac{2h\dot{h}}{\mu r e \sin\theta}$.

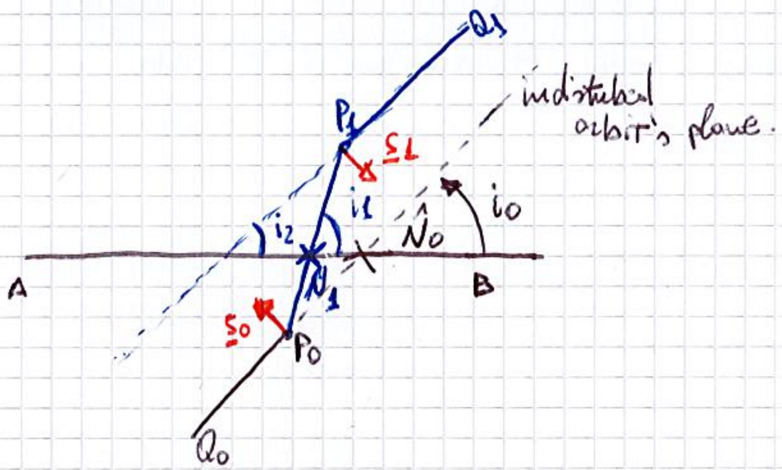
$$\frac{d\Omega}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)} \frac{\sin(\theta+\omega)}{\sin i} f_h = f(f_h)$$

$$\frac{dh}{dt} = r f_\theta$$

$$\frac{de}{dt} = f(f_\theta, f_r)$$

⇒ IF $\dot{\Omega} = 0$ ⇒ $f_h = 0$ THEN ω IS NOT AFFECTED BY f_h.

IN PAROLE POVERE: $\frac{d\omega}{dt}$ VARIA CON f_h (S) COME CONSEGUENZA DELLA VARIAZIONE DI Ω E DEL CONSEGUENTE SPOSTAMENTO DELL'ASSE NODALE. ⇒ L'EFFETTO DI f_h È SUFFICIENTEMENTE DESCRITTO COME VARIAZIONE DEI PARAMETRI "i" E "Ω"



$\overline{AB}$ → equatorial plane.

Q_0 → undisturbed orbit.

Q_1 → perturbed orbit.

P_0 → initial position

P_1 → position after the perturbation

When the satellite is in P_0 receives an impulse S_0 moving it in direction $\overline{P_0P_1}$. Once the sat. arrives again in P_1 ⇒ another impulse S_1 occurs

($\underline{a}$) ($\underline{e}$) ($\underline{\omega}$) $\Rightarrow$ NOT DIRECTLY AFFECTED BY ORTHOGONAL COMPONENT.

b) $\Rightarrow$ TANGENTIAL COMPONENT (T)

($\underline{a}$)

from VOP-like approach:

$$\begin{cases} \frac{d\varepsilon}{dt} = \frac{d}{dt}\left(-\frac{\mu}{2a}\right) = +\frac{\mu}{2a^2}\frac{da}{dt} \\ \frac{d\varepsilon}{dt} = \underline{T}\cdot\underline{v} \end{cases} \Rightarrow \frac{da}{dt} = \frac{2a^2}{\mu}\underline{T}\cdot\underline{v}$$

$$da = \frac{2a^2}{\mu}\underline{T}\cdot\underline{v}\,dt = \frac{2a^2}{\mu}T\,v\,dt \; *$$

obs:

force $\underline{T}$ acting on the body is most effective if acting when $\|\underline{v}\|$ is higher (e.g. at the pericenter)

$\Rightarrow$ dependence of a on T is: $\frac{\partial a}{\partial T} = \frac{\partial a}{\partial v}\cdot\frac{\partial v}{\partial T}$.

$*$ $F_t\,dt = dv$
$\Downarrow$
$da = \frac{2a^2}{\mu} v\,dv$

in fact:

$$\varepsilon = -\frac{\mu}{2a} = \frac{1}{2}v^2 - \frac{\mu}{r} \longrightarrow a = -\frac{\mu/2}{\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right)} \Rightarrow \frac{\partial a}{\partial v} = -\frac{\mu}{2}\,\frac{0 - \frac{1}{2}2v}{\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right)^2}$$

opp: SUPPOSING THAT $\underline{T}$ IS NOT ABLE TO CHANGE INSTANTANEOUSLY THE POSITION (r) OF THE SATELLITE $\Rightarrow$ $\frac{\partial r}{\partial T} = 0$

$$\frac{\partial a}{\partial v} = +\frac{\mu}{2}\frac{v}{\underbrace{\left(\frac{1}{2}v^2 - \frac{\mu}{r}\right)^2}_{\varepsilon = -\frac{\mu}{2a} \to \varepsilon^2 = \frac{\mu^2}{4a^2}}} = +\frac{\cancel{\mu}}{2}\frac{v}{+\frac{\mu^{\cancel{2}}}{4a^2}} = \frac{1}{\cancel{2}}\frac{2\cancel{4}a^2 v}{\mu} = \frac{2a^2}{\mu}v$$

$$\frac{\partial a}{\partial T} = \frac{2a^2}{\mu}v\cdot\frac{\partial v}{\partial T}$$

$\longrightarrow$ $\frac{\partial v}{\partial T} > 0$ ($\Rightarrow \underline{T}\cdot\underline{v} > 0$) "a" increases

$\searrow$ $\frac{\partial v}{\partial T} < 0$ ($\Rightarrow \underline{T}\cdot\underline{v} < 0$) "a" decreases.

obs:

Since $\frac{da}{dt} = \frac{2a^2}{\mu}\underline{T}\cdot\underline{v}$; $\underline{N}\cdot\underline{v} = 0$

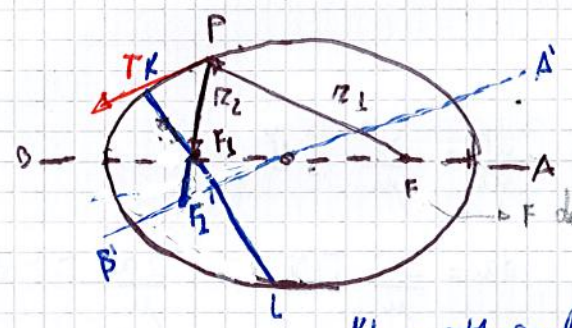
$\Rightarrow$ a is <u>not</u> affected by a normal component of force.

$\underline{N}$ $\Rightarrow$ NO EFFECT ON "a"

$\underline{T}$ $\Rightarrow$ EFFECT ON "a" MAXIMUM AT THE PERICENTER.

(e)

In studying e is possible to observe geometrically how the apsidal line will change



— F obviously don't move.

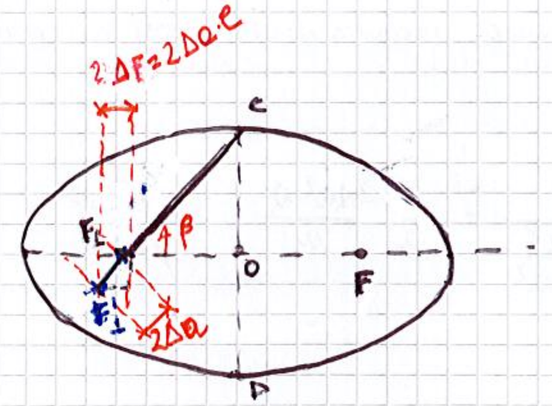
KL → orthogonal to $\underline{e}'$ but passing through F_1

eccentricity definition:

$$e = \frac{\overline{FF_1}}{2a}$$

Since orbits are perturbed $\begin{cases} 2a \neq 2a' \\ \overline{FF_1} \neq \overline{F_1F_1'} \end{cases} \Rightarrow e = \frac{\overline{FF_1}}{2a}$

$$e' = \frac{\overline{FF_1'}}{2a'}$$



$\Delta F \stackrel{\Delta}{=} \overline{FF_1'} - \overline{FF_1}$

$\Delta a \stackrel{\Delta}{=} a' - a$

obs:
- SINCE at the pericenter $\Delta a = \Delta F$
- THEN eccentricity must be increased at that point.

$\Rightarrow \quad e' = \frac{\overline{FF_1'}}{2a'} = \frac{\overline{FF_1} + \Delta F}{2a + 2\Delta a}$

$\overline{OC} = b = a\sqrt{1-e^2}$, $\overline{OF_1} = ae \Rightarrow \tan\beta = \frac{\cancel{a}\sqrt{1-e^2}}{\cancel{a}e} \stackrel{\Delta}{=} \frac{\sin\beta}{\cos\beta}$

MINOR semi-axis

β: $\cos(\beta) = e$ $\Rightarrow \cos\beta = \frac{\overline{FF_1}}{a} = \frac{2\overline{FF_1}}{2a}$

$\sin(\beta) = \sqrt{1-e^2}$

$\Rightarrow \Delta F = 2\Delta a \cdot \cos\beta \Rightarrow \Delta F = 2e\Delta a$

" Eccentricity is unchanged by tangential component when the satellite is at the end of the minor axis of its orbit. "

Anyway:

$$\begin{cases} e' = \frac{\overline{FF_1'}}{2a'} = \frac{\overline{FF_1} + \Delta F}{2a + 2\Delta a} \\ \Delta F = 2e\cdot\Delta a \;;\; \overline{FF_1} = 2ae \end{cases} \Rightarrow e' = \frac{ea + 2e\Delta a}{2a + 2\Delta a}.$$

$$e' = \frac{ea + 2e\Delta a}{2a + 2\Delta a}$$

To compute Δa: $\frac{\partial a}{\partial T} = \frac{2a^2}{\mu} v \frac{\partial W}{\partial T}$

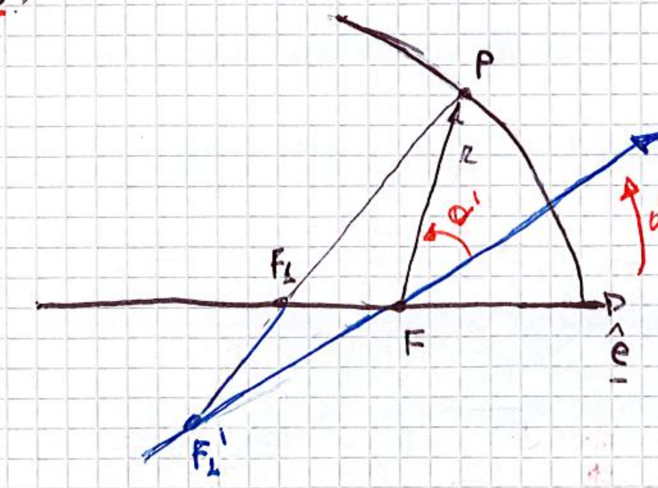
($\underline{i}$, $\underline{\Omega}$)

!! ARE AFFECTED ONLY BY COMPONENT NORMAL TO ORBITAL PLANE (S or f_h) !!

As confirmed also by VOP approach: $\frac{di}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)} \cos(\theta+\omega) f_h$

$$\frac{d\Omega}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)} \frac{\sin(\theta+\omega)}{\sin i} f_h.$$

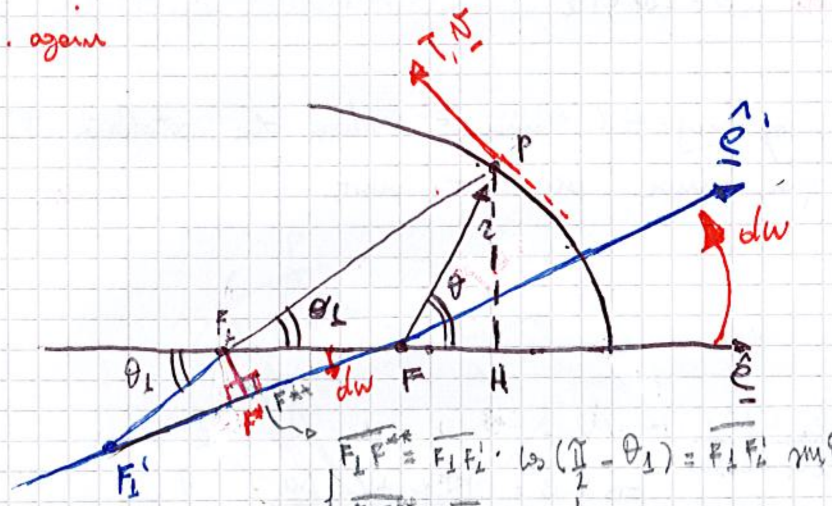
($\underline{\omega}$)



$\text{prop}(ii) \Rightarrow \overline{PF} + \overline{PF_1} = 2a$

$\overline{PF} + \overline{PF_1'} = 2a'$

. . . again



$$\begin{cases} \overline{F_1F^{**}} = \overline{F_1F_1'}\cdot\cos\left(\frac{\pi}{2}-\theta_1\right) = \overline{F_1F_1'}\sin\theta_1 \\ \overline{F_1F^{**}} = \overline{FF_1}\sin dw \end{cases}$$

• prop (i) → $\overline{PF_1} + \overline{FP} = 2a$

$\overline{FP} + \overline{F_1'P} = 2a'$

$\underbrace{\overline{PF} + \overline{PF_1}}_{2a} + \underbrace{\overline{F_1F_1'}}_{2\Delta a} = 2a'$

!! così si dimostra che $\overline{F_1F_1'} = 2\Delta a$!!

|| $\overline{F_1F_1'} = 2a' - 2a = 2\Delta a$.

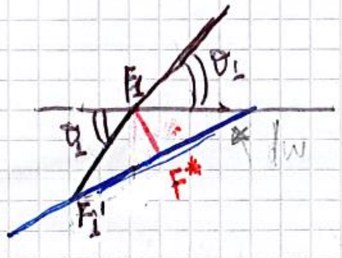
• Now: $\overline{F_1F^*} = \underbrace{\overline{FF_1}}_{=2ae}\cdot\sin dw = 2ae\sin dw \simeq 2ae\,dw$.

• $\begin{cases} \overline{PH} = \overline{PF}\sin\theta \xrightarrow{\overline{PF}=r} \overline{PH} = r\sin\theta \\ \overline{PH} = \overline{PF_1}\sin\theta_1 \xrightarrow{\text{prop(i)}} \overline{PH} = (2a-\overline{PF})\sin\theta_1 = (2a-r)\sin\theta_1 \end{cases}$

Through equivalence: $r\sin\theta = (2a-r)\sin\theta_1$

$$\sin\theta_1 = \frac{r}{2a-r}\sin\theta$$

• looking at triangle: $\overset{\triangle}{F_1F^*F_1'}$



$\Rightarrow \overline{F_1F^*} = \overline{F_1F_1'}\cdot\cos\left(\frac{\pi}{2}-\theta_1\right)$

$= \overline{F_1F_1'}\sin\theta_1$

$\Rightarrow \overline{F_1F^*} = 2\Delta a\frac{r}{2a-r}\sin\theta.$

• Through equivalence between the 2 previous expression of $\overline{F_1F^*}$

$$\cancel{2}ae\,dw = \cancel{2}\Delta a\frac{r}{2a-r}\sin\theta$$

↳ for small changes: $\Delta a = da$, $da = \frac{2a^2}{\mu}T\cdot v\,dt$

$\frac{d}{dt}\left(-\frac{\mu}{2a}\right) = T\cdot v \Rightarrow +\frac{\mu}{2}\frac{1}{a^2}\frac{da}{dt} = T\cdot v \to da = \frac{2a^2}{\mu}T\cdot v\,dt$

$$a\,e\,dw = \frac{2a^2}{\mu}T\cdot v\frac{r}{2a-r}\sin\theta\,dt$$

↳ $\varepsilon = -\frac{\mu}{2a} = \frac{1}{2}v^2 - \frac{\mu}{r} \to v^2 = 2\mu\left(\frac{1}{r}-\frac{1}{2a}\right) = \cancel{2}\mu\left(\frac{2a-r}{\cancel{2}a\cdot r}\right)$

$v^2 = \frac{2a-r}{a\cdot r}\cdot\mu$

$\frac{2a-r}{r} = \frac{v^2 a\cdot r}{\mu}$

finally: $\cancel{a}\,e\,dw = \frac{2a^{\cancel{2}}}{\cancel{\mu}}T\cdot\cancel{v}\frac{\cancel{\mu}}{v^{\cancel{2}}\cancel{a}r}\sin\theta\,dt$

$e\frac{dw}{dt} = \frac{T}{v}\sin\theta$ → $\boxed{\frac{dw}{dt} = \frac{1}{e}\frac{\sin\theta}{v}\cdot T}$

!! ONCE dw IS KNOWN (⇒ IS KNOWN THE ROTATION OF ECCENTRICITY VECTOR) IS THEN POSSIBLE TO COMPUTE THE $\frac{de}{dt}$ FORMULATION. !!

Unperturbed orbit: $\overline{FF_1} = 2ae$

Perturbed orbit: $\overline{FF_1'} = \overline{FF^*} + \overline{F^*F_1'} = 2a'e'$

~~Now: $\overline{F^*F_1'} = 2\Delta(ae) \xrightarrow{\text{small variations}} \overline{F^*F_1'} = 2d(ae) = 2e\,da + 2a\,de$~~

~~Projection of F_1 on $\hat{e}$ is equal to e ...~~

$\Rightarrow \cos\beta = e$

(... $\underline{e}$)

$$\begin{cases} \overline{FF_1'} = 2a'e' \\ \overline{FF_1'} = \overline{FF^*} + \overline{F^*F_1'} \longrightarrow \quad \overline{F^*F_1'} \simeq \overline{F_1F_1'} \cdot \cos\theta_1 \simeq 2\Delta a \cos\theta_1 \end{cases}$$

• from the previous picture, along x-axis:

$$\overline{PF_1}\cos\theta_1 = \overline{FH} + \overline{FF_1}$$

$$\overline{PF_1}\cos\theta_1 = r\cos\theta + 2ae$$

prop (i)

$r_1 + r = 2a \longrightarrow \quad (2a - r)\cos\theta_1 = r\cos\theta + 2ae$

$\overline{PF_1} + \overline{PF} = 2a$

$$\Rightarrow \cos\theta_1 = \frac{r}{2a-r}\cos\theta + \frac{2ae}{2a-r} \qquad \Big|\ \frac{r}{2a-r} = \frac{\mu}{v^2 a}$$

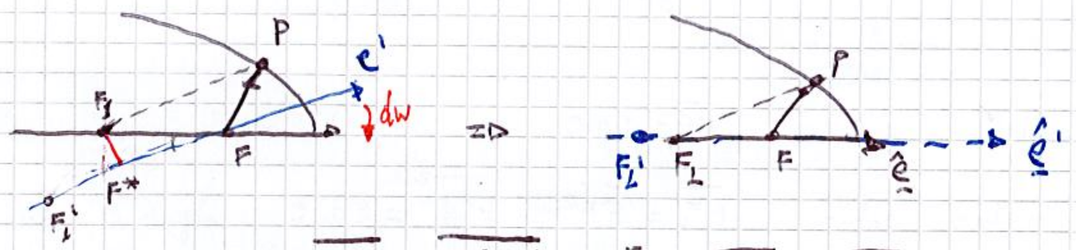
$$\cos\theta_1 = \frac{\mu}{v^2 a}\cos\theta + \frac{2ae}{2a-r}$$

$$\Rightarrow \overline{F^*F_1'} \simeq 2\Delta a \left[\frac{\mu}{v^2 a}\cos\theta + \frac{2ae}{2a-r}\right] \Rightarrow \overline{F^*F_1'} = 2da\,\frac{2ae + r\cos\theta}{(2a-r)}$$

• considering small perturbations

$$\overline{FF_1'} = 2a'e' = 2(a+da)(e+de) = \underbrace{2ae}_{\overline{FF_1}} + 2a\,de + 2e\,da + \underbrace{2\,da\,de}_{\to 0}$$

if there's no w perturbation. (OR IF IT'S REALLY SMALL)



$$\overline{F_1'F_1} \simeq \overline{F_1'F^*} \Rightarrow \overline{F_1'F^*} \simeq \overline{F_1'F_1} = \overline{F_1'F} - \overline{F_1F}$$

$$\Rightarrow \overline{F_1'F^*} \simeq \overline{FF_1'} - \overline{F_1F} = \overline{FF_1'} - 2ae = 2a\,de + 2e\,da.$$

$\overline{FF^*} = \overline{FF_1}\cdot\cos(dw) \simeq \overline{FF_1} \Rightarrow$ if considering repeated effect of perturbation is possible to return $dw \neq 0$

$\Rightarrow$ Imposing the equivalence.

$$2a\,de + 2e\,da = 2\,da\,\frac{2ae + r\cos\theta}{(2a-r)}$$

$$a\,de = da\left[\frac{2ae + r\cos\theta}{2a-r} - e\right]$$

$$= de\left[\frac{\cancel{2ae} + r\cos\theta - \cancel{2ae} + re}{2a-r}\right]$$

$$\Rightarrow a\,de = da\,\frac{r}{2a-r}(e+\cos\theta) \qquad \Big\|\ \frac{r}{2a-r} = \frac{\mu}{v^2 a}$$

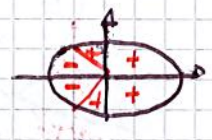
$$de = \frac{da}{a}\,\frac{\mu}{v^2 a}(e+\cos\theta) \qquad \Big\|\ da = \frac{2a^2}{\mu}\,Tv\,dt$$

$$de = \frac{2a^{\cancel{2}}}{\cancel{\mu}}\cdot\frac{1}{a}\,Tv\,dt\cdot\frac{\cancel{\mu}}{v^2\cancel{a}}(e+\cos\theta)$$

$$\boxed{\frac{de}{dt} = \frac{2(e+\cos\theta)}{v}\,T}$$

Tangential component.

$\dfrac{di}{dt} = \dfrac{d\Omega}{dt} = 0$; $\dfrac{da}{dt} = \dfrac{2a^2}{\mu}\,T\cdot v$  $(T>0)$

$\dfrac{de}{dt} = 2\,\dfrac{(e+\cos\theta)}{v}\,T$  $(T>0)$

$\dfrac{dw}{dt} = \dfrac{1}{e}\,\dfrac{\sin\theta}{v}\,T$  $(T>0)$

c) → NORMAL COMPONENT

(a) obviously $\frac{da}{dt} = 0$ since:

$$\frac{d\varepsilon}{dt} = \frac{d}{dt}\left(-\frac{\mu}{2a}\right) = \underline{a}_p \cdot \underline{v}$$

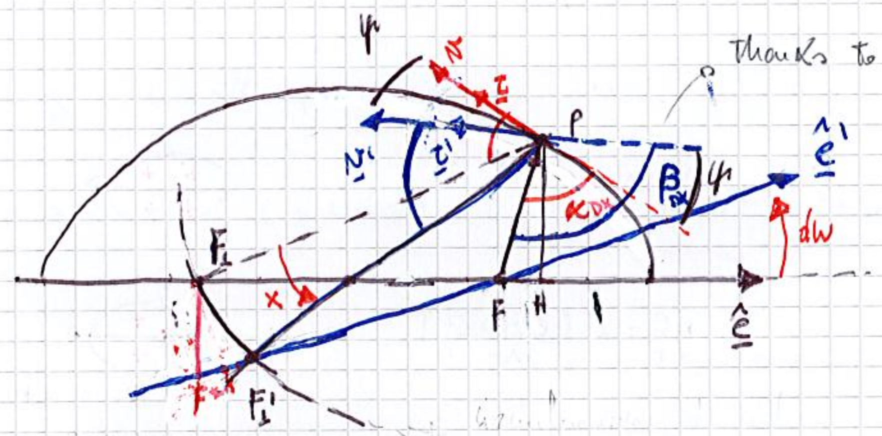
$$+ \frac{\mu}{2a^2}\frac{da}{dt} = [T\,\hat{t} + N\,\hat{n} + S\,\hat{h}] \cdot v\,\hat{t}$$

$$\Downarrow \quad \hat{n}\cdot\hat{t} = 0 \,;\; \hat{h}\cdot\hat{t} = 0$$

$$\frac{\mu}{2a^2}\frac{da}{dt} = T \cdot v \quad \rightarrow \boxed{\frac{da}{dt} = 0}$$

!! Semiaxis is not affected by normal component of perturbative effect !!

(w)



Thanks to conic property (ii)

$\alpha_{DX} = \alpha_{SX}$ (RED)

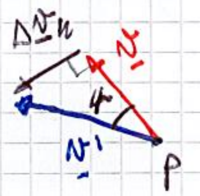
$\beta_{DX} = \beta_{SX}$ (BLUE)

$\Downarrow$

$\psi_{DX} = \psi_{SX}$

$$\psi \triangleq \widehat{v\,v'} = \widehat{t\,t'} = \beta - \alpha$$

• acting on VELOCITIES TRIANGLE:



$$\Rightarrow \tan\psi = \frac{\|\Delta \underline{v}_n\|}{\|\underline{v}\|}$$

⇒ for small displacements: $\Delta \underline{v}_n \simeq d\underline{v}_n = \underline{N} \cdot dt \leftarrow a = \frac{dv}{dt} \,;\; dv = a\,dt$

$$\tan\psi \simeq \psi = \frac{\Delta v_n}{\|v\|}$$

$$\Rightarrow \begin{cases} \Delta v_N = N\,dt \\ \psi \simeq \frac{\Delta v_N}{v} \end{cases} \Rightarrow \text{(i)} \quad \psi \simeq \frac{N}{v}\,dt$$

• Thanks to prop (ii) ⇒ $\beta_{DX} = \beta_{SX}$

$$\begin{cases} \beta_{DX} = \alpha + \psi \\ \beta_{SX} = x + \alpha - \psi \end{cases} \Rightarrow x + \cancel{\alpha} - \psi = \cancel{\alpha} + \psi \rightarrow \text{(ii)}\; x = 2\psi$$

• Since "e" remains constant.

$$\text{prop(i)} \rightarrow \begin{cases} \overline{PF} + \overline{PF_1} = 2a \\ \overline{PF} + \overline{PF_1'} = 2a \end{cases} \rightarrow \overline{PF_1} = \overline{PF_1'} \;;\; \begin{cases} \overline{FF_1} = 2ea \\ \overline{FF_1'} = 2e'a \end{cases}$$

e changes.

Obs

Since "a" remains the same and prop (i) must be satisfied for both orbits (pert. and unperturbed)

$$\Rightarrow \begin{cases} \overline{PF_1} + \overline{PF} = 2a \\ \overline{PF_1'} + \overline{PF} = 2a \end{cases} \quad (\overline{PF} = r)$$

the foci F_1' of the perturbed orbit must be at distance $2a - r$ from P and belong to $\hat{e}'$

⇒ F_1' IS INDIVIDUATED BY THE INTERCEPTION BETWEEN $\hat{e}'$ AND A CIRCUMFERENCE OF RADIUS $(2a-r)$ CENTERED IN P (as sketched)

• Considering triangle: $\overset{\triangle}{P F_1 F_1'}$

→ for small displacement $\widehat{F_1F_1'} \simeq \overline{F_1F_1'}$ the arch of circumference can be retained equal to the line.

for SMALL DISPLACEMENT:

$$\overline{F_1F_1'} \simeq 2\,\overline{PF_1}\,\sin\left(\frac{x}{2}\right) \simeq 2\,\overline{PF_1}\,\frac{x}{2}$$

but: prop(i) → $\overline{PF} + \overline{PF_1} = 2a$ ⇒ $PF_1 = 2a - r$

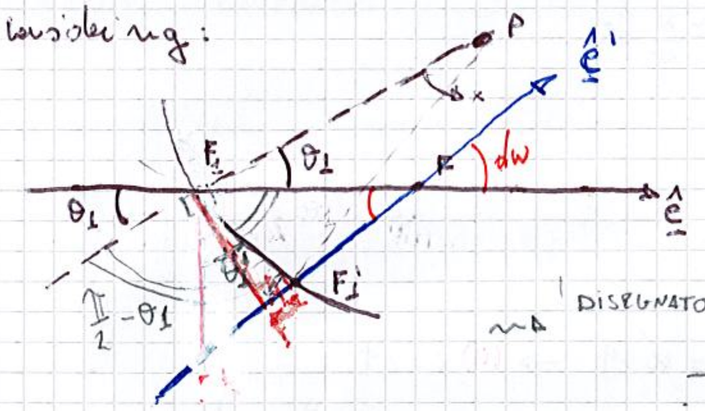
$$\begin{cases} x = 2\psi \\ \psi \simeq \frac{N}{v}\,dt \end{cases}$$

$$\Rightarrow \text{(iii)} \quad \overline{F_1F_1'} = 2\,(2a - r) \cdot \frac{N}{v}\,dt$$

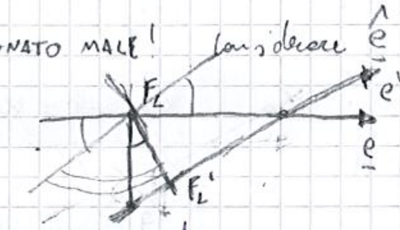
• As did before:

$$\overline{F_1H} = \overline{F_1F} + FH \Rightarrow (2a - r)\cos\theta_1 = 2ea + r\cos\theta \quad \text{(iv)}\; \cos\theta_1 = \frac{ea + r\cos\theta}{2a - r}$$

• considering:



~> DISEGNATO MALE! considerare ê' non inclinato



$\overset{\triangle}{F F_1 F_1^*} \Rightarrow \quad \overline{F_1 F^*} = 2ea \tan dw \simeq 2ea\, dw$

$\overset{\triangle}{F_1 F^* F_1'} \Rightarrow \quad \overline{F_1 F^*} = \overline{F_1 F_1'} \cos\theta_1 \xrightarrow[(iv)]{(iii)}$ ~~$\overline{F_1F^*} = 2(2a-r) \ldots$~~

$(\overline{F_1 F^*} = F_1$

$$\overline{F_1 F^*} = 2(\cancel{2a-r}) \frac{N}{v} \frac{ea + r\cos\theta}{(\cancel{2a-r})} dt$$

$$\overline{F_1 F^*} = 2 \frac{N}{v} (ea + r\cos\theta)\, dt$$

⇒ Through equivalence:

$$\cancel{2} \frac{N}{v} (ea + r\cos\theta)\, dt = \cancel{2} ea\, dw$$

$$\frac{dw}{dt} = \frac{N}{v} \frac{(ea + r\cos\theta)}{ea}$$

$$\boxed{\frac{dw}{dt} = \frac{1}{v} \frac{ea + r\cos\theta}{ea} N}$$

(e)

$\overline{FF_1} = 2ae$

$$\begin{cases} \overline{FF_1'} = 2ae' = 2a(e+de) \\ \overline{FF_1'} = \overline{FF^*} - \overline{F_1'F^*} = \overline{FF^*} - \overline{F_1F_1'} \sin\theta_1 \end{cases} \qquad (\overline{F_1F_1'} = 2(2a-r)\frac{N}{v}dt)$$

As did before: $\begin{cases} PH = r\sin\theta \\ PH = \overline{PF_1} \cdot \sin\theta_1 = (2a-r)\sin\theta_1 \end{cases} \rightarrow \sin\theta_1 = \frac{r}{(2a-r)}\sin\theta.$

$$\Rightarrow \begin{cases} \overline{FF_1'} = 2a(e+de) \\ \overline{FF_1'} = \overline{FF^*} - 2(\cancel{2a-r})\frac{N}{v} dt \cdot \frac{r}{(\cancel{2a-r})} \sin\theta \end{cases}$$

↳ As before: since $dw \to 0 \Rightarrow \hat{e}' \simeq \hat{e}$ so F^* dot coincides with F1

$\Rightarrow \overline{FF_1} \simeq \overline{FF^*} = 2ae$

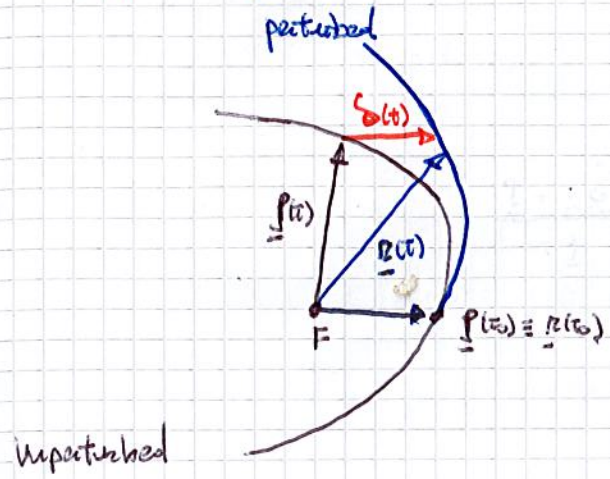
$$\Rightarrow \begin{cases} \overline{FF_1'} = 2a(e+de) \\ \overline{FF_1'} = \overline{FF^*} - 2\frac{N}{v} r\sin\theta\, dt \simeq 2ae - 2\frac{N}{v} r\sin\theta\, dt \end{cases}$$

Through equivalence: $\cancel{2}ae + \cancel{2}ade = \cancel{2}ae - \cancel{2}\frac{N}{v} r\sin\theta\, dt$

$$\cancel{ae} + ade = \cancel{ae} - \frac{N}{v} r\sin\theta\, dt$$

$$\boxed{\frac{de}{dt} = -\frac{r\sin\theta}{av} N}$$

IIIrd Approach: ENKE'S METHOD



$\underline{\rho}(t) \Rightarrow$ position on osculating orbit at time t

$\underline{r}(t) \Rightarrow$ position on the REAL (perturbed) orbit

$\Downarrow$

$$\underline{\delta}(t) \triangleq \underline{r}(t) - \underline{\rho}(t)$$

At time instant "t_0" the perturbed/unperturbed orbits coincides $\Rightarrow \underline{\rho}(t_0) = \underline{r}(t_0)$

$$\underline{\delta}(t) = \underline{r}(t) - \underline{\rho}(t) \Rightarrow \ddot{\underline{\delta}}(t) = \underbrace{\ddot{\underline{r}}(t)}_{\text{PERTURBED}} - \underbrace{\ddot{\underline{\rho}}(t)}_{\text{KEPLERIAN}}$$

$$\Rightarrow \ddot{\underline{\delta}}(t) = -\frac{\mu}{\|\underline{r}\|^3}\underline{r} + \underline{a}_p + \frac{\mu}{\|\underline{\rho}\|^3}\underline{\rho} \qquad \| \ \underline{\rho} = \underline{r} - \underline{\delta}$$

$$= -\frac{\mu}{\|\underline{r}\|^3}\underline{r} + \underline{a}_p + \frac{\mu}{\|\underline{\rho}\|^3}\underline{r} - \frac{\mu}{\|\underline{\rho}\|^3}\underline{\delta}$$

$$= \left[\frac{\mu}{\rho^3} - \frac{\mu}{r^3}\right]\underline{r} + \underline{a}_p - \frac{\mu}{\rho^3}\underline{\delta}$$

$$= \frac{\mu}{\rho^3}\left[\left(1 - \frac{\rho^3}{r^3}\right)\underline{r} - \underline{\delta}\right] + \underline{a}_p$$

$$\Rightarrow \ddot{\underline{\delta}}(t) = \underline{a}_p + \frac{\mu}{\rho^3}\left[\left(1 - \frac{r^3}{\rho^3}\right)\underline{r} - \underline{\delta}\right]$$

$\Rightarrow$ computing:

$$\frac{r^2}{\rho^2} = \frac{(\underline{\rho}+\underline{\delta})\cdot(\underline{\rho}+\underline{\delta})}{\rho^2} = \frac{\rho^2 + 2\underline{\rho}\,\underline{\delta} + \delta^2}{\rho^2} = 1 + \frac{2\underline{\rho}\,\underline{\delta} + \delta^2}{\rho^2}$$

and defining

$$q \triangleq \frac{\underline{\delta}\cdot(\underline{\rho} + \underline{\delta}/2)}{\rho^2} = \frac{\underline{\rho}\cdot\underline{\delta} + \delta^2/2}{\rho^2} \Rightarrow \frac{r^2}{\rho^2} = 1 + 2q$$

" IF PERTURBATIVE EFFECTS ARE SMALL: $\delta^2 \to 0$; $\underline{\rho}\cdot\underline{\delta} \simeq \rho \Rightarrow \rho/\rho^2 \ll 1$ "

from the 3 relations obtained:

$$\ddot{\underline{\delta}} = \underline{a}_p + \frac{\mu}{\rho^3}\left[\left(1 - \frac{r^3}{\rho^3}\right)\underline{r} - \underline{\delta}\right]$$

$$\frac{r^2}{\rho^2} = 1 + 2q \quad \text{where:} \quad q = \underline{\delta} \cdot \frac{\underline{\delta}/2 + \underline{\rho}}{\rho^2}$$

$$\Rightarrow \quad 1 - \frac{r^3}{\rho^3} = 1 - (1+2q)^{3/2}$$

binomial series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$\text{e.g.: } (1+x)^2 = 1 + 2x + \frac{2(2-1)}{2\cdot 1}x^2 + 2\cdot(2-1)\overset{=0}{(2-2)}x^3$$

$$\Rightarrow \quad 1 - \frac{r^3}{\rho^3} = 1 - (1+2q)^{3/2} = 1 - \left(1 + \frac{3}{2}\cdot 2q + \frac{1}{2}\cdot\frac{3}{2}\left(\frac{3}{2}-1\right)4q^2 + \dots\right)$$

$$= \underbrace{3q + \frac{3}{2}q^2 + \dots}_{f(q)}$$

Since q is really small: $q \ll 1 \Rightarrow f(q) = q\,(\underbrace{3 + \frac{3}{2}q^2 + \dots}_{\simeq 3}) \simeq 3q$

$$\Rightarrow \quad \ddot{\underline{\delta}} = \underline{a}_p + \frac{\mu}{\rho^3}\left[3q\cdot\underline{r} - \underline{\delta}\right]$$

everything is a function of ($\underline{\rho}$ and $\underline{\delta}$)
- $\underline{\rho}$ → KNOWN FROM KEPLERIAN SOLUTION
- $\underline{\delta}$ → VARIABLE OF THE DYNAMIC PROBLEM.

So:

$$\begin{cases} \ddot{\underline{\delta}} = \underline{a}_p + \dfrac{\mu}{\rho^3}\left[3q\,(\underline{\rho}+\underline{\delta}) - \underline{\delta}\right] \\ q = \underline{\delta}\cdot\dfrac{\underline{\rho}+\underline{\delta}/2}{\rho^2} \end{cases}$$

→ ITERATIVE PROCEDURE: Encke's method.

1. $\{\underline{r}(t_0);\ \underline{v}(t_0)\} \xrightarrow{\text{RV2ORBPAR}} \{a, e, i, \Omega, \omega, \theta\}$

2. $\{a, e, i, \Omega, \omega, \theta\} \xrightarrow{\text{KEPLER}} \underline{\rho}(t)$ KNOWN.

3. COMPUTE: $q \rightarrow f(q) = 3q \quad ; \quad q = \underline{\delta}\cdot\dfrac{\underline{\rho}+\underline{\delta}/2}{\rho^2}$
   - ↳ INITIAL GUESS: $q = 1;\ f(q) = 3q$.

4. NUMERICAL INTEGRATION OF:
$$\begin{cases} \ddot{\underline{\delta}} = \underline{a}_p + \dfrac{\mu}{\rho^3}\left[3q\,(\underline{\rho}+\underline{\delta}) - \underline{\delta}\right] \\ \text{c.c.} \begin{cases} \underline{\delta}(t) \\ \dot{\underline{\delta}}(t) \end{cases} \end{cases}$$
   - ↳ INITIAL GUESS: $\begin{cases} \dot{\underline{\delta}}(t_0) = 0 \\ \underline{\delta}(t_0) = 0 \end{cases}$
   - → OBTAIN $\begin{cases} \underline{\delta}(t+\Delta t) \\ \dot{\underline{\delta}}(t+\Delta t) \end{cases}$

5. $\{a, e, i, \Omega, \omega, \theta(t)\} \xrightarrow{\text{KEPLER}} \{a, e, i, \Omega, \omega, \theta(t+\Delta t)\}$
$\Rightarrow \underline{\rho}(t+\Delta t)$

11) → " Geopotential function":

a) → NON UNIFORMITY OF MASS.

--- conservative field.

GRAVITY: $\exists\, U$ ($U \to$ potential) : $\underline{F} = \nabla U$

⇒ if $\exists\, U : \underline{F} = -\nabla(U)$ ⇒ $\begin{cases} \Delta U = U_2 - U_1 = L \\ L = \int_\gamma \underline{F}\cdot d\underline{s} \quad \forall \gamma \text{ connecting } 1\text{-}2 \end{cases}$

* UNIFORM MASS (PUNCTIFORM) → $U = \frac{\mu}{r}$

$$\underline{F} = -\frac{\mu}{\|\underline{r}\|^3}\vec{r} \; ; \; \nabla U = \nabla\left(\frac{\mu}{r}\right) = -\frac{\mu}{r^2}\hat{\underline{r}} = -\frac{\mu}{r^3}\underline{r}$$

* NON UNIFORM MASS DISTRIBUTION → $U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\delta)\right]$

P_n → Legendre polinomia

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

J_n → zonal armonic
SPERIMENTAL DATA.

the first 3 Legendre's polinomia are:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$P_1(x) = x$

$$P_2(x) = \frac{1}{4\cdot 2}\frac{d^2}{dx^2}\left[(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)\,2x\right] = \frac{1}{2}\left[2x^2 - 1 + 4x^2\right]$$
$$= 3x^2 - 1$$

$P_3(x) = \dots = \frac{1}{2}(5x^3 - 3x)$

↳ To understand how geo-potential function is obtained needs to think about another property of conservative field

→ GEO-POTENTIAL FUNCTION

$$\underline{F} = \nabla U \; ; \; U = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\delta)\right]$$

WHERE
- $\{J_2 \dots J_n\}$ → ZONAL HARMONIC (sperimental coefficients)
- $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$ → LEGENDRE POLINOMIA.

CONSEQUENCE OF A CONSERVATIVE FIELD:

- $\underline{F} = \nabla U$ (on a closed trajectory work is null.)
- $\Delta U = L = \int_\gamma \underline{F}\cdot d\underline{s}$ ⇒ $\oint_\gamma \underline{F}\cdot d\underline{s} = 0$

⇒ Thanks to Stokes Theorem: $\oint_{\gamma=\partial S} \underline{F}\cdot d\underline{s} = \int_S (\nabla\times\underline{F})\cdot\underline{n}\, dS = 0$

So what we obtain is:

$$\int_S (\nabla\times\underline{F})\cdot\underline{n}\, dS = 0 \;\to\; \begin{cases}\nabla\times\underline{F} = \underline{0} \\ \underline{F} = \nabla U\end{cases} \;\to\; \nabla\times(\nabla U) = \underline{0} \quad \text{(QED)}$$

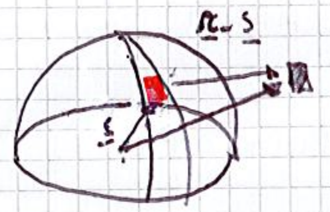
CONSERVATIVE FIELD ⇔ IRROTATIONAL

⇒ IF A FIELD IS IRROTATIONAL POTENTIAL SATISFY LAPLACE EQUATION in fact:

$$\nabla\times\underline{F} = 0 \;\to\; \exists\, U : \underline{F} = \nabla U$$

$$\nabla\cdot(\nabla U) = \nabla^2 U = 0$$

To build geopotential function:



$$\begin{cases} \mu = G \cdot M_\oplus \\ M_\oplus = \int_V \rho(x,y,z)\, dV = \int_0^{M_\oplus} dm. \end{cases}$$

$\Rightarrow$ $$U = G \int_{body} \frac{dm}{\|\underline{r}-\underline{s}\|}$$

if density is constant: $dm = \bar{\rho}\, dV$ (PERFECT SPHERE)

$$U = G \int \frac{\bar{\rho}\, dV}{r} = G\bar{\rho} \int \frac{dV}{r}$$

$$= G\bar{\rho}\int_0^{R_\oplus} \frac{4\pi R^2}{r} dR = G\overbrace{\bar{\rho}\,\frac{4}{3}\pi R^3}^{M_\oplus}\frac{1}{r}$$

$$= \frac{G M_\oplus}{r} = \frac{\mu}{r}$$

$$U = G \int_{body} \frac{dm}{\left((\underline{r}-\underline{s})\cdot(\underline{r}-\underline{s})\right)^{\frac{1}{2}}}$$ (*)

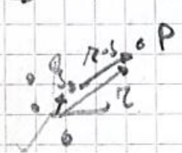
$$= G \int_{body} \frac{dm}{(r^2+s^2-2\underline{r}\cdot\underline{s})^{\frac{1}{2}}} = G \int_{body} \frac{dm}{(r^2+s^2-2rs\cos\beta)^{\frac{1}{2}}}$$

through defining $\rho \triangleq \dfrac{\|\underline{s}\|}{\|\underline{r}\|}$

$$U = G \int_{body} \frac{dm}{\left[r^2\left(\frac{s^2}{r^2}+1-2\frac{s}{r}\cos\beta\right)\right]^{\frac{1}{2}}} = G \int_{body} \frac{dm}{\left[r^2(1+\rho^2-2\rho\cos\beta)\right]^{\frac{1}{2}}}$$

$$= G \int_{body} \frac{1}{r}\frac{dm}{(1+\rho^2-2\rho\cos\beta)^{1/2}} = \frac{G}{r}\int_{body}\frac{dm}{(1+\rho^2-2\rho\cos\beta)^{\frac{1}{2}}}$$

(*) $\to$ FOR A DISCRETE SYSTEM OF MASSES IN R2BP HYP.



$\Rightarrow U_i = \dfrac{G\, m_i}{\|\underline{r}-\underline{s}_i\|} \xrightarrow{\text{OVERALL}} U = \sum_i \dfrac{G m_i}{\|\underline{r}-\underline{s}_i\|} \xrightarrow{\text{HOMOGENEOUS}} \begin{cases} U = \int_{body} \dfrac{G\, dm}{\|\underline{r}-\underline{s}\|}. \\ dm = \rho dV \end{cases}$

$\Rightarrow$ $$U = \frac{G}{r}\int_{body}\frac{dm}{(1+\rho^2-2\rho\cos\beta)^{1/2}} = \frac{G}{r}\int_{body}(1+\rho^2-2\rho\cos\beta)^{-\frac{1}{2}}\, dm.$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots+\frac{n(n-1)\dots(n-k+1)}{k!}x^k+\dots$$

So

$$(1+\underbrace{\rho^2-2\rho\cos\beta}_{x})^{-\frac{1}{2}} = 1-\frac{1}{2}(\rho^2-2\rho\cos\beta) - \frac{1}{2}\frac{1}{2}\left(-\frac{1}{2}-1\right)(\rho^2-2\rho\cos\beta)^2+\dots$$

$$= 1-\frac{1}{2}\rho^2+\rho\cos\beta+\frac{3}{8}(\rho^4+4\rho^2\cos^2\beta-4\rho^3\cos\beta)$$

In other writing $\to$ Binomial expansion

$\to$ binomial expansion.

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots = \sum_{m=1}^{\infty}\frac{n(n-1)\dots(n-m+1)}{m!}x^m$$

e.g. $n=3$ $\quad \dfrac{n(n-1)(n-2)}{3!}x^3$

So:

$$(1+\rho^2-2\rho\cos\beta)^{-\frac{1}{2}} = 1+\sum_{m=1}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-m+1\right)}{m!}(\rho^2-2\rho\cos\beta)^m$$

CAN BE DEMONSTRATED (**) $\to$

$$\boxed{(1+\rho^2-2\rho\cos\beta)^{-\frac{1}{2}} = \sum_{l=0}^{+\infty} P_l(\cos\beta)\,\rho^l}$$

" can be demonstrated that each term of the binomial expansion converges to Legendre polinomia as a function of $(\cos\beta)$ multiplied for ρ^l. "

(**) $1 \to l=0 \to$ in fact: $\quad P_l(x) = \dfrac{1}{2^l\, l!}\dfrac{d^l}{dx^l}\left[(x^2-1)^l\right] = \dfrac{1}{2^0\, 0!}\dfrac{d^0}{dx^0}1 = 1$

$(0! = 1)$

$m=1 \to l=1 \to$ in fact: $\quad P_1(\cos\beta) = \dfrac{1}{2\, 1!}\dfrac{d}{d\cos\beta}\left[(\cos^2\beta-1)\right] = \dfrac{1}{2}\,2\cos\beta = \cos\beta$

$m=2 \to \begin{cases} l=2 \\ l=3 \end{cases}$ $\quad P_2(\cos\beta) = \dfrac{1}{2}(3\cos^2\beta-1)\ ; \quad P_3(\cos\beta) = \dfrac{1}{2}(5\cos^3\beta-3\cos\beta)$

$$\sum_{l=0}^{+\infty} P_l(\cos\beta)\,\rho^l = 1+\rho\cos\beta+\frac{3}{2}\rho^2\cos^2\beta-\frac{1}{2}\rho^2+\dots$$

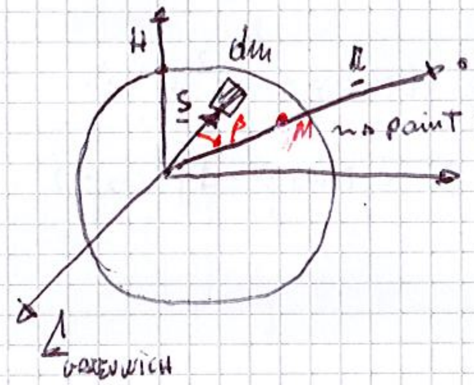
$$1+\sum_{m=1}^{\infty}\frac{n(n-1)\cdot(n-m+1)}{m!}(\rho^2-2\rho\cos\beta)^m = 1+\rho\cos\beta-\frac{1}{2}\rho^2+\frac{3}{2}\rho^2\cos^2\beta+\dots = (1+\rho^2-2\rho\cos\beta)^{-\frac{1}{2}}.$$

THE 2 SERIES ARE COINCIDENT

$\Rightarrow \quad U = + \frac{G}{r} \int_{body} \left(\sum_{l=0}^{\infty} P_l(\cos\beta) \cdot \rho^l \right) dm$

↳ IT'S IMPOSSIBLE TO INTEGRATE OVER β SINCE IS AN UNDEFINED ANGLE ON THE BODY

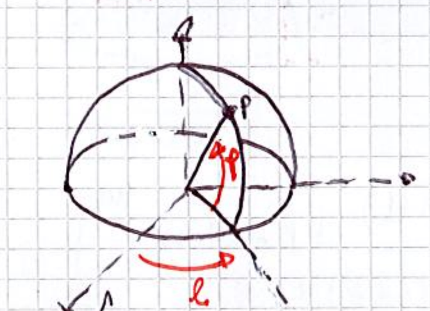
Hp*: CONSIDERING AN ALMOST SPHERICAL BODY



→ point of interception of r with the sphere

(Special enough to use spherical Triangles properties)

2 ANGLES TO INDIVIDUATE A POINT ON A SPHERE

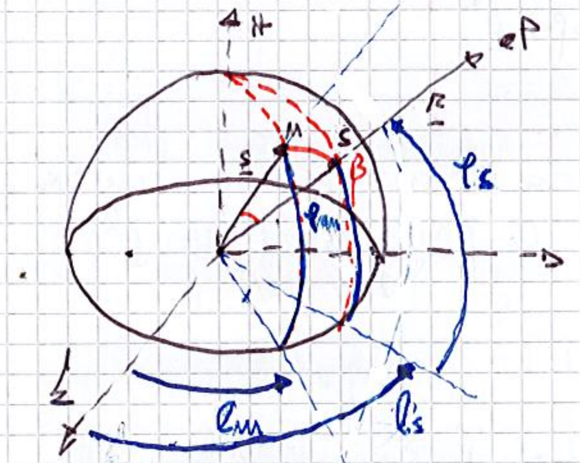


$\Rightarrow$ $\{\varphi_s, l_s\}$ → individuates the spacecraft (THE INTERCEPTION OF $\vec{r}$ WITH THE SPHERE)

$\{\varphi_m, l_m\}$ → individuate the mass

--- Spherical triangle

$\frac{\sin\Lambda}{\sin\lambda} = \frac{\sin\Theta}{\sin\theta} = \frac{\sin\Phi}{\sin\varphi}$

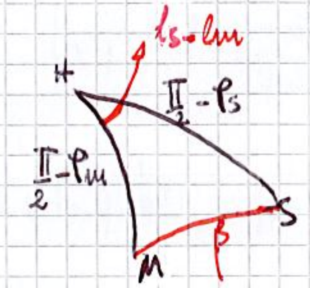


$\Rightarrow$

$\cos\Lambda = \cos\Phi \cos\Theta + \sin\Theta \sin\Phi \cos\lambda$

$\cos\lambda = -\cos\theta \cos\varphi + \sin\theta \sin\varphi \cos\Lambda$

in $\overset{\Delta}{HMS}$ Triangle:



$\Rightarrow \cos\beta = \cos\left(\frac{\pi}{2} - \varphi_m\right) \cos\left(\frac{\pi}{2} - \varphi_s\right) + \sin\left(\frac{\pi}{2} - \varphi_m\right) \sin\left(\frac{\pi}{2} - \varphi_s\right) \cdot \cos(l_s - l_m)$

So: $\cos\beta = \sin(\varphi_m)\sin(\varphi_s) + \cos(\varphi_m)\cos(\varphi_s) \cdot \cos(l_s - l_m)$

THANKS TO ADDITIONAL THEOREM →

This passage is PURE MAGIC.

$\cos\beta = \sin(\varphi_m)\sin(\varphi_s) + \cos(\varphi_m)\cos(\varphi_s)\cos(l_s - l_m)$

$P_l[\cos\beta] = P_l(\sin\varphi_s) P_l(\sin\varphi_m) + \sum_{K=1}^{l} \frac{2(l-K)!}{(l+K)!} [AA' - BB']$

where:

$A = P_{l,K}[(\sin\varphi_s)]\cos(K l_s)$

$A' = P_{l,K}[(\sin\varphi_m)]\cos(K l_m)$

$B = P_{l,K}[(\sin\varphi_s)]\sin(K l_s)$

$B' = P_{l,K}[(\sin\varphi_m)]\sin(K l_m)$

where $P_{l,K}$ represent the "ASSOCIATED LEGENDRE POLINOMIA"

$P_{l,K}(x) = \frac{(1-x^2)^{K/2}}{2^l\, l!} \frac{d^{l+K}}{dx^{l+K}} (x^2 - 1)^l$

$\Rightarrow U = \frac{G}{r} \int_{body} \sum_{l=0}^{\infty} \rho^l [P_l(\sin\varphi_s) P_l(\sin\varphi_m)] + \sum_{l=1}^{\infty} \rho^l \sum_{K=1}^{l} \frac{2(l-K)!}{2(l+K)!} [AA' - BB']$

$\rho = \frac{s}{r}$; r don't depends on body locations

$= \frac{G}{r} \left\{ \sum_{l=0}^{\infty} \frac{1}{r^l} \int_{body} [P_l(\sin\varphi_s) P_l(\sin\varphi_m)]\, s^l \right. +$

$+ \sum_{l=1}^{\infty} \frac{1}{r^l} \sum_{K=1}^{l} \frac{2(l-K)!}{2(l+K)!} \left[\int_{body} s^l \{ P_{l,K}[(\sin\varphi_s)]\cos(K l_s)\cos(K l_m)\, P_{lK}[\sin(\varphi_m)] \} dm \right.$

$+ \sum_{l=1}^{\infty} \frac{1}{r^l} \sum_{K=1}^{l} \frac{2(l-K)!}{2(l+K)!} \left[\int_{body} s^l \{ P_{l,K}[(\sin\varphi_m)] \sin(K l_s)\sin(K l_m)\, P_{lK}[\sin(\varphi_m)] \} dm \right\}$

~ obviously l_s, φ_s don't depends on the body → Just individuate the spacecraft position.

$\Rightarrow$ CAN BE EXTRACTED FROM THE INTEGRAL $\int_{body} \dots dm$.

$$\Rightarrow U = \frac{G}{r}\left[\sum_{l=0}^{+\infty} \frac{1}{r^l} P_l(\sin\varphi_s) \underbrace{\int_{body} P_l[\sin\varphi_m]\, s^l\, dm}_{\triangleq\, C'_{l,0}}\right] +$$

$$+ \frac{G}{r}\left[\sum_{l=1}^{+\infty} \frac{1}{r^l} \sum_{K=1}^{l} P_{l,K}[\sin(\varphi_s)] \cos(K l_s) \cdot \underbrace{\int_{body} s^l \cos(K l_m) P_{l,K}[\sin(\varphi_m)]\, dm}_{\triangleq\, C'_{l,K}}\right] + \quad (C \to \cos\ldots)$$

$$+ \frac{G}{r}\left[\sum_{l=1}^{+\infty} \frac{1}{r^l} \sum_{K=1}^{l} P_{l,K}[\sin(\varphi_s)] \sin(K l_s) \underbrace{\int s^l \sin(K l_m) P_{l,K}[\sin(\varphi_m)]\, dm}_{\triangleq\, S'_{l,K}}\right] \quad (S \to \sin\ldots)$$

Ooops missing: $\sum_l \sum_K \frac{1}{r^l} \frac{2(l-K)!}{2(l+K)!} \ldots$

So in a more compact form the potential can be written as:

$$U = \frac{G}{r}\left\{\sum_{l=0}^{+\infty} \frac{1}{r^l} P_l[\sin(\varphi_s)] \cdot C'_{l,0} + \sum_{l=1}^{+\infty} \frac{1}{r^l} \sum_{K=1}^{l} C'_{l,K}\, P_{l,K}[\sin(\varphi_s)] \cos(K l_s) + \sum_{l=1}^{+\infty} \frac{1}{r^l} \sum_{K=1}^{l} S'_{l,K}\, P_{l,K}[\sin(\varphi_s)] \sin(K l_s)\right\}$$

↳ Through an adimensionalization of the 3 coefficients:

$$C_{l,0} = \frac{C'_{l,0}}{R_\oplus^l M_\oplus} \quad ; \quad C_{l,K} = \frac{C'_{l,K}}{R_\oplus^l M_\oplus} \quad ; \quad S_{l,K} = \frac{S'_{l,K}}{R_\oplus^l M_\oplus}$$

$$U = \frac{\overbrace{G M_\oplus}^{=\mu}}{r}\left\{\sum_{l=0}^{+\infty} \frac{R_\oplus^l}{r^l} P_l[\sin(\varphi_s)]\, C_{l,0} + \sum_{l=1}^{+\infty} \frac{R_\oplus^l}{r^l} \sum_{K=1}^{l} P_{l,K}[\sin(\varphi_s)] \sin(K l_s)\, C_{l,K} + \sum_{l=1}^{+\infty} \frac{R_\oplus^l}{r^l} \sum_{K=1}^{l} P_{l,K}[\sin(\varphi_s)] \cos(K l_s)\, S_{l,K}\right\}$$

Correct!!

$$U = \frac{\mu}{r}\left\{\sum_{l=0}^{+\infty} \frac{R_\oplus^l}{r^l} P_l[\sin(\varphi_s)] \cdot C_{l,0} + \sum_{l=0}^{+\infty} \frac{R_\oplus^l}{r^l} \sum_{K=1}^{l} \frac{2(l-K)!}{(l+K)!} P_{l,K}[\sin(\varphi_s)] \sin(K l_s)\, C_{l,K} + \sum^{+\infty} \frac{R_\oplus^l}{r^l} \sum^{l} \frac{2(l-K)!}{\ldots} \ldots\right.$$

$$\Rightarrow U = \frac{\mu}{r}\left\{\sum_{l=0}^{+\infty} \frac{R_\oplus^l}{r^l} P_l[\sin(\varphi_s)] \cdot C_{l,0} + \sum_{l=1}^{+\infty} \frac{R_\oplus^l}{r^l} \sum_{K=1}^{l} P_{l,K}[\sin(\varphi_s)] \sin(K l_s)\, C_{l,K} + \sum_{l=1}^{+\infty} \frac{R_\oplus^l}{r^l} \sum_{K=1}^{l} P_{l,K}[\sin(\varphi_s)] \cos(K l_s)\, S_{l,K}\right\}.$$

↳ Since the first Legendre polinomial: $P_0 = 1$

$$U = \frac{\mu}{r}\left\{C_{0,0} + \sum_{l=1}^{+\infty} \frac{R_\oplus}{r^l} P_l[\sin(\varphi_s)] + \ldots\ldots\right\}$$

it's still possible to recognize the contribution of a perfect sphere $\Rightarrow C_{0,0} = 1$

!! $U_{SPHERE} = \frac{\mu}{r} \Rightarrow C_{0,0} = 1$!!

↳ If we consider the "ASSOCIATED LEGENDRE POLINOMIA."

$$P_{l,K}(x) = \frac{1}{2^l\, l!}(1-x^2)^{K/2} \frac{d^{l+K}}{dx^{l+K}}(x^2-1)^l \Rightarrow \text{for } K=0 \quad P_{l,K} = P_l.$$

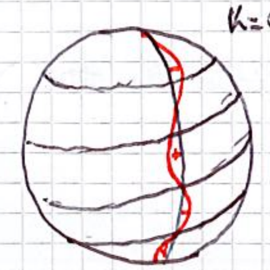
↱ J definition.

$C_{l,0} \triangleq -J_l$. ┘ in fact for $K=0$:

- $C_{l,0} = \int_{body} s^l P_l[\sin(\varphi_m)]\, dm$; $C_{l,K} = \int_{body} s^l \underbrace{\cos(0 \cdot l_m)}_{=1} P_{l,0}[\sin(\varphi_m)]\, dm. \triangleq C_{l,0}$
- $S_{l,0} = \int s^l \underbrace{\sin(K l_m)}_{=0} P_{lK}[\sin(\varphi_m)]\, dm = 0$

↳ BY IMPOSING $K=0$ IT FOLLOWS THAT WE ARE CONSIDERING ONLY ZONAL-ARMONICS

$K=0$



Zonal Armonic

} → Higher l is considered ⇒ higher n° of meridians are used to approximate potential.

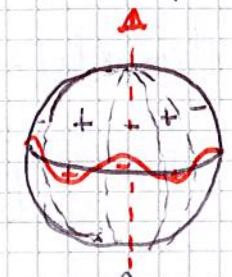
$K=l\neq 0$

Settorial Armic

Zonal Armonic

$K=0$ ↓

supposing constant density along each meridian.



Settorial armonic

$K=\ell\neq 0$

Tesseral armonic.

$K\neq \ell \neq 0$

↓ FOUNDAMENTAL APPROXIMATIONS/ ASSUMPTIONS

1) SUPPOSING EARTH SYMMETRIC WITH RESPECT TO NORTH POLE AXIS WE ARE ALLOWED TO CONSIDER ZONAL ARMONIC ONLY. ⇒ opp: $K=0 \Rightarrow S_{\ell,K}=0$

$$\Rightarrow \begin{cases} U=\dfrac{\mu}{r}\left\{1-\displaystyle\sum_{\ell=1}^{+\infty} J_\ell \left(\dfrac{R_\oplus}{r}\right)^{\ell} P_\ell[\sin(\varphi_s)]\right\} \\ J_\ell = C_{\ell,0}=\left[\displaystyle\int_{body} P_\ell[\sin(\varphi_m)]\, s^\ell\right.\end{cases}$$

but: $C_{1,0}=-J_1=\int_{body} s\ P_\ell[\sin(\varphi_m)]\,dm=0$

in fact for a symmetric body with respect to the ~~…~~ symmetry axis

2) Then considering ONLY THE 2^{nd} ZONAL ARMONIC (much more high than the others)

$$\begin{cases} U=\dfrac{\mu}{r}\left[1-J_2\left(\dfrac{R_\oplus}{r}\right)^2 P_2[\sin(\varphi_s)]\right] \\ P_2[\sin(\varphi_s)]=\dfrac{1}{2}(3\sin^2\varphi_s-1)\end{cases}$$

$P_2(x)=\frac{1}{2}(3x^2-1)$

$$\Rightarrow \boxed{U=\frac{\mu_\oplus}{r}\left[1-J_2\left(\frac{R_\oplus}{r}\right)^2\left(\frac{3}{2}\sin^2(\varphi_s)-\frac{1}{2}\right)\right]}$$

obs 1

WELL: in imposing $K=0$

$$S_{\ell,0}=\int_{body} s^\ell \underbrace{\sin(0\cdot\lambda_m)}_{=0} P_{\ell,}[\sin(\varphi_m)]=0$$

$$C_{\ell,0}=\int_{body} s^\ell \underbrace{\cos(0\cdot\lambda_m)}_{=1} P_\ell[\sin(\varphi_m)] \stackrel{\Delta}{=} -J_\ell$$

!! it's completly neglected inside the body-integral the contribution due to latitude ⇒ $0\cdot\lambda_m$ ⇒ I'm assuming AXIAL SIMMETRY !!

obs 2

in defining and considering the situation with $K=0$.

$$U=\frac{\mu}{r}\left\{\sum_{\ell=0}^{+\infty}\frac{R_\oplus^\ell}{r^\ell}P_\ell[\sin(\varphi_s)]+\underbrace{\sum_{\ell=1}^{+\infty}\sum_{K=1}^{\ell}C_{\ell,K}\,P_{\ell,K}[\sin(\varphi_s)]\cos(K\lambda_s)+\sum_{\ell=1}^{+\infty}\sum_{K=1}^{\ell}S_{\ell,K}\,P_{\ell,K}[\sin(\varphi_s)]\sin(K\lambda_s)}_{=0}\right\}$$

in fact in imposing $K=0$ this 2 terms must not be considered.

what remains is only

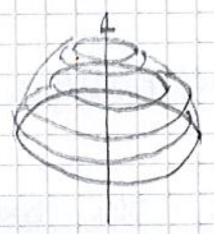
$$U=\frac{\mu}{r}\left\{\sum_{\ell=0}^{+\infty}\frac{R_\oplus^\ell}{r^\ell}P_\ell[\sin(\varphi_s)](-J_\ell)\right\}$$

$$U=\frac{\mu}{r}\left\{1-\sum_{\ell=1}^{+\infty}\frac{R_\oplus^\ell}{r^\ell}P_\ell[\sin\varphi_s]J_\ell\right\}$$

12) → "Gravity Turn and flight optimization"

Hp:
- 2D problem
- $\underline{W}_\oplus = \underline{0}$ (neglect earth rotation)
- $\underline{I}, \underline{J}$ → REPRESENT EQUATORIAL PLANE.
  - ↳ launching from equator is possible to reach the higher orbit.

earth.



$\underline{V}_I = \underline{W}_\oplus \times \underline{r} \Rightarrow$ $r_{MAX} = R_{eq}$.

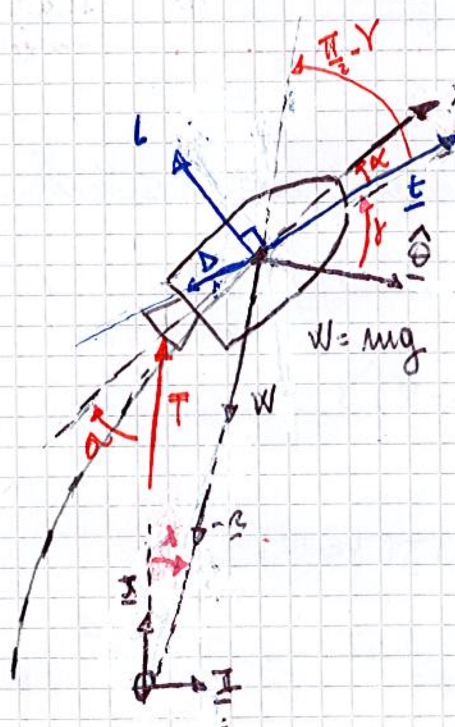
→ inertial: $[\underline{I}, \underline{J}, \underline{K}]_\oplus$

→ local: $[\underline{x}, \underline{y}, \underline{z}]_{body}$

→ intrinsic: $[\underline{t}, \underline{n}, \underline{b}]_{body}$ (TANGENT TO TRAJECTORY)

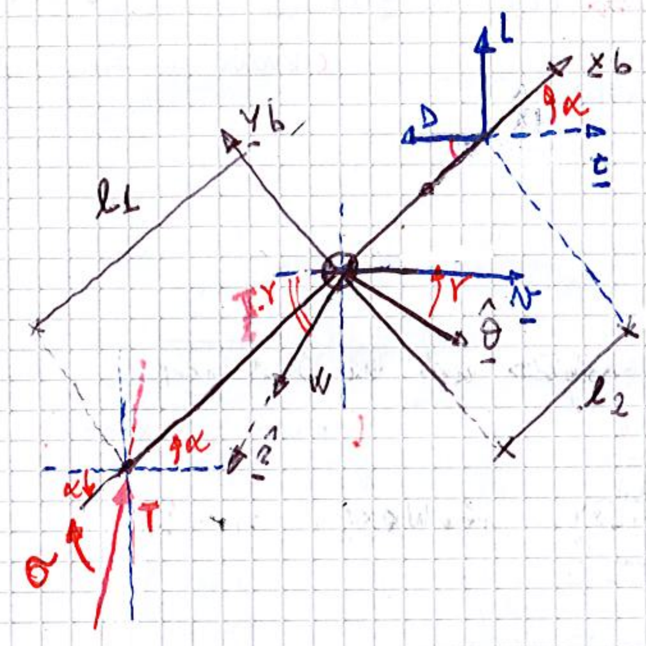
→ body: $[x_b, y_b, z_b]$

→ This reference frame are not inertial since depends on time.



- Newton law: $\frac{d\underline{q}}{dt} = \sum_i \underline{F}_i$
- Usefull angles:
  - γ) FLIGHT PATH ANGLE: $\gamma: \hat{t} \wedge \hat{\theta}$
  - α) BODY FLIGHT ANGLE: $\alpha: \hat{t} \wedge \hat{x}_b$ (ANGOLO DI INCIDENZA)
  - σ) THRUST ANGLE: $\sigma: \underline{T} \wedge \hat{x}_b$
  - λ) LONGITUDE ANGLE: $\lambda: \hat{I} \wedge \hat{r}$
- Moment equation: $\frac{d\underline{M}_O}{dt} = \sum_i \underline{F}_i \times (P_i - O)$

$\underline{M}_O = I_{zb} \underline{W} = I_{zb} \ddot{\alpha}$



$$\Rightarrow \begin{cases} M_{zb} = -T \sin\sigma\, l_1 + [D \sin\alpha + L \cos\alpha]\, l_2 & \text{(moment equation)} \\ M_{zb} = I_z \cdot \dot{W} \end{cases}$$

1) $m \frac{dV}{dt} = -W \cos(\frac{\pi}{2} - \gamma) + T\cos(\sigma + \alpha) - D$ → $\frac{dV}{dt} = \frac{T}{m}\cos(\sigma + \alpha) - \frac{D}{m} - g \sin\gamma$

2) $I_{zb} \frac{dW}{dt} = -T \sin\sigma\, l_1 + [D \sin\alpha + L\cos\alpha]\, l_2$ → $\frac{dW}{dt} = \frac{1}{I_{zb}} [-T \sin\sigma\, l_1 + (D\sin\alpha + L \cos\alpha) l_2]$

$\frac{dW}{dt} = \dot{W}$ → IS AN ANGULAR ACCELERATION $\ddot{}$ (angle)

3) $\frac{dr}{dt} = -V\cos(\frac{\pi}{2} + \gamma)$ $\quad \underline{V} = \frac{d\underline{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ (projection of $\underline{V}$ on $\hat{r}$) → $\frac{dr}{dt} = V \sin\gamma$

4) $r \frac{d\lambda}{dt} = V\cos\gamma$ $\quad V_{\sin\gamma}$ $\quad r\dot{\lambda} = V_{tang}$ → $\frac{d\lambda}{dt} = \frac{V}{r}\cos\gamma$

5) $m \frac{dn}{dt} = T \sin(\sigma + \alpha) - W\cos\gamma + L$

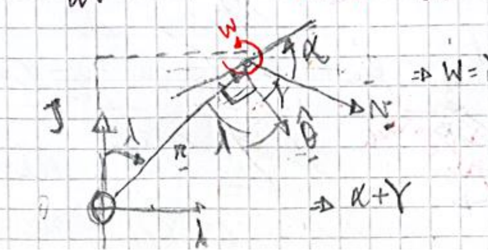
$V_n = V \sin\gamma \xrightarrow{\text{SMALL ANGLES}} V_n \simeq V\gamma \Rightarrow \frac{d(V\gamma)}{dt} = \frac{dV}{dt}\gamma + V\frac{d\gamma}{dt}$

NOT DUE TO ⇑ FORCES

in $\hat{t}, \hat{n}$ reference frame

→ $\frac{d\gamma}{dt} = [\frac{T}{m}\sin(\sigma + \alpha) - g\cos\gamma + \frac{L}{m}]\frac{1}{V}$

6) $\frac{d\alpha}{dt} = W + \cancel{\frac{d\lambda}{dt}} - \frac{d\gamma}{dt}$ → $\frac{d\alpha}{dt} = W - \frac{d\gamma}{dt}$



+ $T = T(t)$
$\sigma = \sigma(t)$
$m = m(t)$
$I = I(t)$

a) TURNOVER → Hp^a: $\frac{dw}{dt} = 0$ ⇒ MOMENT ARE COMPLETELY BALANCED

$\left(\sum_i \underline{F}_i \times (P_i - O) = \underline{0} \right)$

α changes because $\underline{v}$ is changing not because the rocket is rotating

⇒ $W = \dot{\alpha} + \dot{\gamma}$ ⇒ $W = \text{const.}$ ⇔ $(\dot{\alpha} + \dot{\gamma}) = \overline{(\gamma + \alpha)}$

$W = 0$

(FIXED AND KNOWN)

(ROCKET IS NOT SPINNING)

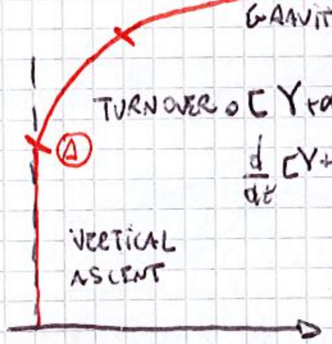
Under the possibility of controlling thrust angle:

$$\frac{dW}{dt} = 0 = \frac{1}{I_{zb}} \left[-T \sin\sigma\, l_1 + (D \sin\alpha + L\cos\alpha)\, l_2 \right]$$

$$T \sin\sigma\, l_1 = (D\sin\alpha + L\cos\alpha)\, l_2$$

$$\rightarrow \sin\sigma = \frac{1}{T l_1} \left[D\sin\alpha + L\cos\alpha \right] l_2$$

↳ THE OVERALL MANEUVER CAN BE DIVIDED IN 3 PHASES:



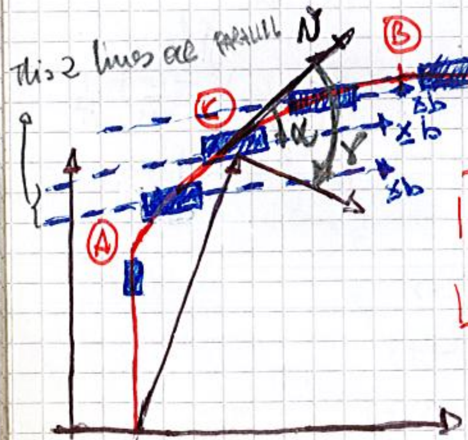
in (A) the rocket turns ($w \neq 0$) putting itself with a certain α (!! α is negative $\alpha < 0$!!)

!! THEN DURING THE WHOLE TURNOVER ROCKET IS NOT ROTATING AROUND HIS INERTIAL AXIS!

⇓ THIS IS ACHIEVED BY ORIENTING THE NOZZLE ORIENTING THE NOZZLE ACCORDING WITH

$$\sin\sigma = \frac{1}{T(t)\, l_1} \left[D\sin\alpha + L\cos\alpha \right] l_2$$

in (B) BECOMES GRAVITY TURN (∅ LIFT) ⇒ SUCH THAT $\alpha = 0$



$\alpha_A < \alpha_C$

$-10 \quad -5$

IN MODULO $\alpha_A > \alpha_C$

$\gamma_A > \gamma_C$

$\alpha_A + \gamma_A = \alpha_C + \gamma_C$

b) GRAVITY TURN → Hp^b: (i) $\hat{\underline{T}} \equiv \underline{x}_b$ ⇒ $\sigma = 0$ (∅ TORQUE)

(ii) $\hat{\underline{v}} \equiv \hat{\underline{t}} \equiv \underline{x}_b$ ⇒ $\alpha = 0$ ⇒ $L = 0$ (SINCE $C_L = 0$)

(iii) ∅ λ (FLAT HORIZON)

(iv) MOMENT EQUILIBRIUM ($\underline{W} = \underline{0}$)

Previous equilibrium equations becomes:

1) $\frac{dv}{dt} = \frac{T}{m}\cos(\sigma + \alpha) - g\sin\gamma - \frac{D}{m}$ → $\frac{dv}{dt} = \frac{T}{m} - g\sin\gamma - \frac{D}{m}$

5) $\frac{d\gamma}{dt} = \left[\frac{T}{m}\sin(\sigma + \alpha) - g\cos\gamma + \frac{L}{m} \right] \frac{1}{v}$ → $\frac{d\gamma}{dt} = \frac{1}{v}\left(\overset{=0}{\frac{L}{m}} - g\cos\gamma \right)$ → $\frac{d\gamma}{dt} = -\frac{g}{v}\cos\gamma$

3) $\frac{dr}{dt} = v\sin\gamma$

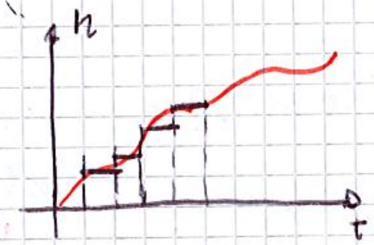
4) NOT CONSIDERED (Hp^b(iii))

6), 2)

So system reduces to 2 equation (FULLY DEFINED WITH RESPECT TO THE REMAINED DEGREES OF FREEDOM)

$$\begin{cases} \frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g\sin\gamma \\ \frac{d\gamma}{dt} = -\frac{g}{v}\cos\gamma \end{cases}$$

Hp^*



; $h \triangleq \frac{T-D}{W}$

$$\begin{cases} \frac{dv}{dt} = g \cdot \frac{(T-D)}{gm} - g\sin\gamma \\ \dots \end{cases}$$

!! SUPPOSE CONSTANT h IN EACH TIME INTERVAL !!

$$\begin{cases} \frac{1}{g}\frac{dv}{dt} = (h - \sin\gamma) \\ \frac{d\gamma}{dt} = -\frac{g}{v}\cos\gamma \end{cases}$$

" The aim of the control law is to obtain $\gamma = \gamma(t)$ to satisfy this PDE system "

FULL CONTROL LAW OBTAINED AT $*^2$

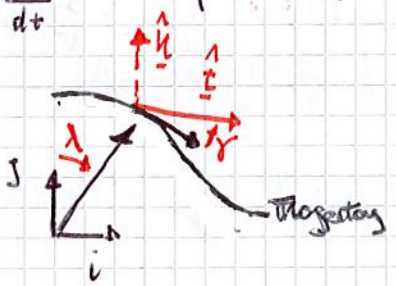
→ RIGOROUS DERIVATION OF THE ROCKET'S DYNAMIC. (SAME 6 eqn OF BEFORE)

3eq) MOMENTUM) $\frac{dQ}{dt} = \sum_{i=1}^{N} F_i \xrightarrow[m = \bar{m}]{m \neq t} \frac{dv}{dt} = \frac{T}{m} + g + \frac{D}{m} + \frac{L}{m}$.

3eq) MOMENT BALANCE) $\frac{d\Gamma_{cu}}{dt} = \sum_{j=1}^{N} M_j \longrightarrow \frac{d\omega}{dt} = \frac{1}{I}[-T\sin\theta\, l_1 + (D\sin\alpha + L\cos\alpha)\, l_2]$

but: $\underline{v} = \|\underline{v}\|\, \hat{\underline{v}} = v\,\hat{t} \Rightarrow \frac{d\underline{v}}{dt} = \frac{dv}{dt}\cdot\hat{t} + v\frac{d\hat{t}}{dt}$

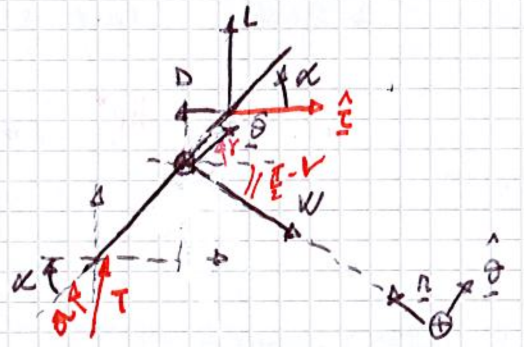
!! $\frac{d\hat{t}}{dt}$ is expressed in $\{\hat{i}, \hat{j}, \hat{k}\}$ reference frame (inertial) !!



$\Rightarrow \frac{d\hat{t}}{dt} = [\frac{d\gamma}{dt} - \frac{d\lambda}{dt}](\hat{k}\times\hat{t}) = [\frac{d\gamma}{dt} - \frac{d\lambda}{dt}]\,\hat{n}$ (*)

So expressing the whole dynamic in $\{\hat{k}, \hat{t}, \hat{n}\}$ reference frame:

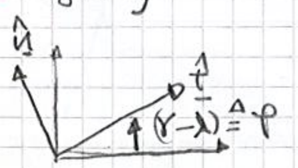
- $\frac{d\underline{v}}{dt} = \frac{dv}{dt}\hat{t} + [\frac{d\gamma}{dt} - \frac{d\lambda}{dt}]\, v\,\hat{n}$
- $\frac{T}{m} = \frac{T}{m}[\cos(\theta+\alpha)\,\hat{t} + \sin(\theta+\alpha)\,\hat{n}]$
- $\frac{g}{m} = -\frac{g}{m}\sin\gamma\,\hat{t} - \frac{g}{m}\sin\gamma\,\hat{n}$
- $\frac{D}{m} + \frac{L}{m} = -\frac{D}{m}\hat{t} + \frac{L}{m}\hat{n}$



⇒ From the Newton law we have:

$$\frac{dv}{dt}\hat{t} + [\frac{d\gamma}{dt} - \frac{d\lambda}{dt}]\, v\cdot\hat{n} = [\frac{T}{m}\cos(\theta+\alpha) - \frac{g}{m}\sin\gamma - \frac{D}{m}]\,\hat{t} + [\frac{T}{m}\sin(\theta+\alpha) - \frac{g}{m}\cos\gamma + \frac{L}{m}]\,\hat{n}$$

(*) rigorously



$\hat{t} = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j} \rightarrow \frac{d\hat{t}}{dt} = [-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}]\dot{\varphi} = \dot{\varphi}\,\hat{n}$

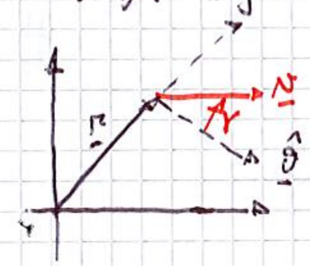
$\hat{n} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$ $\Rightarrow \frac{d\hat{t}}{dt} = (\dot{\gamma} - \dot{\lambda})\,\hat{n}$.

⇒ 1) $\frac{dv}{dt} = \frac{T}{m}\cos(\theta+\alpha) - \frac{g}{m}\sin\gamma - \frac{D}{m}$

2) $\frac{d\gamma}{dt} = \frac{T}{m}\sin(\theta+\alpha) - \frac{g}{m}\cos\gamma + \frac{L}{m} + \frac{d\lambda}{dt}$

+ KINEMATIC (λ, γ are linked only to trajectory)

$\frac{d\underline{r}}{dt} = \dot{r}\,\hat{r} + r\,\cancel{\dot{\theta}}\,\hat{\theta}$
$= \dot{r}\,\hat{r} + r\,\dot{\lambda}\,\hat{\theta}$



$\Rightarrow \begin{cases} v\cos\gamma = r\dot{\lambda} \\ v\sin\gamma = \dot{r} \end{cases}$

3) $\frac{d\lambda}{dt} = \frac{v}{r}\cos\gamma$ 4) $\frac{dr}{dt} = v\sin\gamma$.

↳ While acting on the 2^{nd} cardinal equation:

5) $\frac{d\omega}{dt} = \frac{1}{I_{cu}}[-T\sin\theta\, l_1 + (D\sin\alpha + L\cos\alpha)\, l_2]$

But with respect to $\{\underline{i}, \underline{j}, \underline{k}\}$ reference frame:

BODY) $\omega = \dot{\alpha}$ → $\{\underline{i}, \underline{j}, \underline{k}\}$) $\omega = \dot{\alpha} + \dot{\gamma} - \dot{\lambda}$ ⇒ 6) $\frac{d\alpha}{dt} = \omega + \frac{d\lambda}{dt} - \frac{d\gamma}{dt}$

Full Rocket dynamic

- $\frac{d\underline{v}}{dt} = \sum_i F_i$ → $\frac{dv}{dt} = \frac{T}{m}\cos(\theta+\alpha) - \frac{g}{m}\sin\gamma - \frac{D}{m}$

$\frac{d\gamma}{dt} = [\frac{T}{m}\sin(\theta+\alpha) - \frac{g}{m}\cos\gamma + \frac{L}{m}]\cdot v + \frac{d\lambda}{dt}$

- KINEMATIC → $\frac{d\lambda}{dt} = \frac{v}{r}\cos\gamma$

$\frac{dr}{dt} = v\sin\gamma$

- $I_{cu}\frac{d\omega}{dt} = \sum_i M_i$ → $\frac{d\omega}{dt} = [-T\sin\theta\, l_1 + (D\sin\alpha + L\cos\alpha)\, l_2]$

→ $\frac{d\alpha}{dt} = \omega + \frac{d\lambda}{dt} - \frac{d\gamma}{dt}$.

$*^2 \rightarrow$ " Gravity Turn's control-law "

Hp: (i) $\frac{dW}{dt} = 0$; $W=0$

(ii) $\nexists\ \lambda$ (FLAT horizon)

(iii) $\frac{d\alpha}{dt} = 0$; $\alpha = 0 \Rightarrow L = 0$

(iv) $\hat{T} \equiv x_b \Rightarrow \delta = 0$

(*) $n = \frac{T-D}{W}$ constant in each time interval.

$$\rightarrow \begin{cases} \frac{dv}{dt} = \frac{T}{m} - \frac{D}{m} - g \sin\gamma \\ \frac{d\gamma}{dt} = -g/v \cos\gamma \end{cases}$$

$$\frac{dv}{dt} = g \cdot \left[\frac{T-D}{mg} - \sin\gamma \right]$$

So:

$$\begin{cases} \frac{1}{g}\frac{dv}{dt} = \frac{T-D}{W} - \sin\gamma = n - \sin\gamma \\ \frac{d\gamma}{dt} = -g/v \cos\gamma . \end{cases}$$

$$\varphi \triangleq \frac{\pi}{2} - \gamma$$

$$\begin{cases} \frac{1}{g}\frac{dv}{dt} = n - \cos\varphi \\ -\frac{d\varphi}{dt} = -g/v \sin\varphi \end{cases} \rightarrow \begin{cases} \frac{1}{g}\frac{dv}{dt} = n - \cos\varphi \\ \frac{d\varphi}{dt} = g/v \sin\varphi \quad (*) \end{cases}$$

since v is a function only of γ:

$$\frac{dv}{dt} = \frac{dv}{d\varphi} \cdot \frac{d\varphi}{dt}$$

So:

$$\begin{cases} \frac{dv}{d\varphi}\frac{d\varphi}{dt} = [n - \cos\varphi] g \\ \frac{d\varphi}{dt} = g/v \sin\varphi \end{cases} \rightarrow \frac{dv}{d\varphi} = \frac{n - \cos\varphi}{(g/v)\sin\varphi} \cancel{g} \rightarrow \frac{dv}{d\varphi} = \frac{n - \cos\varphi}{\sin\varphi} \cdot v .$$

$\Rightarrow$ We can proceed with the integration of this differential equation:

$$\frac{dv}{d\varphi} = \frac{n - \cos\varphi}{\sin\varphi} \cdot v \rightarrow \frac{1}{v}\frac{dv}{d\varphi} = \frac{n - \cos\varphi}{\sin\varphi}$$

$$\rightarrow \frac{1}{v} dv = \frac{n - \cos\varphi}{\sin\varphi} d\varphi$$

$$\int_{v_0}^{v} \frac{1}{v} dv = \int_{\varphi_0}^{\varphi} \frac{n - \cos\varphi}{\sin\varphi} d\varphi$$

$z = \tan\frac{\varphi}{2}$

$\tan\frac{x}{2} = \frac{1 - \cos x}{\sin x}$

$z = \frac{1 - \cos\varphi}{\sin\varphi}$

$\sqrt{1 - \cos\varphi} = \frac{1 - \cos\varphi}{2}$

$$z = \tan\frac{\varphi}{2} \rightarrow z = \frac{1 - \cos\varphi}{\sin\varphi} \rightarrow z^2 = \frac{(1 - \cos\varphi)^2}{(1 - \cos^2\varphi)} = \frac{(1-\cos\varphi)(1-\cos\varphi)}{\cancel{(1-\cos\varphi)}(1+\cos\varphi)}$$

$$(1 + \cos\varphi) z^2 = 1 - \cos\varphi \rightarrow \cos\varphi \cdot (z^2 + 1) = 1 - z^2 \rightarrow \cos\varphi = \frac{1 - z^2}{1 + z^2}$$

$$\rightarrow \sin\varphi = \sqrt{1 - \cos^2\varphi} = \sqrt{\frac{1 + z^4 + 2z^2 - 1 - z^4 + 2z^2}{(1+z^2)^2}} = \frac{2z}{1+z^2}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\rightarrow dz = \frac{1}{2}\frac{1}{\cos^2(\varphi/2)} d\varphi \rightarrow d\varphi = 2 \frac{1+\cos\varphi}{2} dz \rightarrow d\varphi = \frac{1+z^2+1-z^2}{1+z^2} = \frac{2}{1+z^2} dz$$

$$\hookrightarrow \ln\left(\frac{v}{v_0}\right) = \int_{\varphi_0}^{\varphi} \frac{n - \cos\varphi}{\sin\varphi} d\varphi$$

$$\ln\left(\frac{v}{v_0}\right) = \int_{z_0}^{z} \frac{n - \frac{1-z^2}{1+z^2}}{\frac{2z}{1+z^2}} \cdot \frac{2}{1+z^2} dz = \int_{z_0}^{z} \frac{n}{z} - \frac{1-z^2}{z(1+z^2)} dz$$

$$= \int_{z_0}^{z} \frac{n}{z} - \frac{1 - z^2 + z^2 - z^2}{z(1+z^2)} dz = \int_{z_0}^{z} \frac{n}{z} - \frac{1}{z} - \frac{2z}{(1+z^2)} dz .$$

$$\hookrightarrow \ln\left(\frac{v}{v_0}\right) = (n-1)\ln\left(\frac{z}{z_0}\right) + \ln(1+z^2)\Big|_{z_0}^{z} = (n-1)\ln\left(\frac{z}{z_0}\right) + \ln\left(\frac{1+z^2}{1+z_0^2}\right)$$

$\Rightarrow$ imposing $\begin{cases} v_0 = 0 \\ z_0 = 0 \end{cases}$ in order to solve the indefinite integration.

$$\ln v = \ln[z^{n-1} \cdot (1+z^2)] \qquad v = z^{n-1} + z^{n-1+2} = [z^{n-1} + z^{n+1}] \cdot c$$

By substituting such result in equation (*)

$$d\varphi = \frac{2}{1+z^2} dz . \qquad \sin\varphi = \frac{2z}{1+z^2}$$

$$\frac{d\varphi}{dt} = g/v \,\sin\varphi \;; \qquad v = [z^{n-1} + z^{n+1}]\cdot c \;; \qquad \sin\varphi = \frac{2z}{1+z^2}$$

$$d\varphi = \frac{2}{1+z^2}\,dz$$

↓

$$v \cdot \frac{1}{\sin\varphi}\, d\varphi = g\,dt$$

↓

$$\frac{\cancel{(1+z^2)}}{2z}\cdot c \cdot [z^{n-1} + z^{n+1}]\,\frac{\cancel{2}}{\cancel{(1+z^2)}}\,dz = g\,dt$$

↓

$$c\cdot[z^{n-2} + z^{n}]\,dz = g\,dt$$

→ analitical solution

$$g\,(t-t_0) = \frac{z^{n-1}}{n-1} + \frac{z^{n+1}}{n+1} + c'$$

$$n = \frac{T-D}{W}$$

↓ supposed constant inside integration interval.

$$z = \tan \varphi/2 \;, \qquad \varphi = \frac{\pi}{2} - \gamma(t)$$

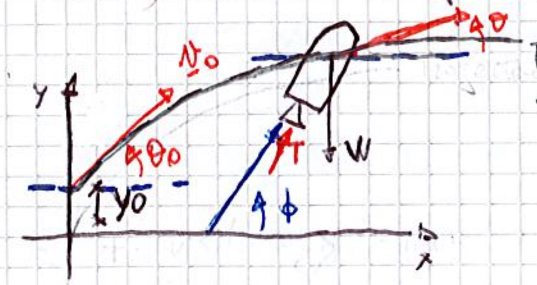
and

$$v(t) = c\cdot[z^{n+1} + z^{n-1}]$$

→ for a given t is possible to compute γ, v, during the gravity turn.

→ "Flight optimization"

"How to establish the optimum flight path to place a satellite in orbit"



KINEMATIC

$$\underline{v} = \begin{Bmatrix} u \\ w \end{Bmatrix} ; \quad \underline{r} = \begin{Bmatrix} x \\ y \end{Bmatrix} \Rightarrow \begin{cases} u = \dfrac{dx}{dt} \\ w = \dfrac{dy}{dt} \end{cases}$$

ϕ angle with respect to the horizon.

$\boxed{t=0}$
$$\begin{cases} x = 0 \\ y = y_0 \\ u = v_0 \cos\theta_0 \\ w = v_0 \sin\theta_0 \end{cases}$$

$\boxed{t=T^*}$
$$\begin{cases} x = x^* \\ y = y^* \\ u = v\cos\theta \\ w = v\sin\theta \end{cases}$$

ROCKET DYNAMIC
$$\begin{cases} \dfrac{du}{dt} = \dfrac{T}{m}\cos\phi \\ \dfrac{dw}{dt} = \dfrac{T}{m}\sin\phi - g \end{cases}$$

at a given time T horizontal velocity will be U

so:
$$U = \int_0^T \frac{T}{m}\cos\phi\,dt + u_0$$

Aim of optimization

$$U = u_0 + \int_0^T \frac{T}{m}\cos\phi\,dt$$

⇒ OBTAIN THE MAXIMUM VALUE OF HORIZONTAL VELOCITY AT A GIVEN TIME ($u(T) = U$) FOR AN ASSIGNED HEIGHT (y) UNDER THE CONDITION $w(t=T) = 0$

So:

$$J \triangleq U = \int_0^T \frac{T}{m}\cos\phi\,dt + u_0 \quad \text{(cost function)}$$

* $\dot{w} = \frac{T}{m}\sin\phi - g = 0$ → $\dot{w} - \frac{T}{m}\sin\phi + g = 0$ (I^{st} costrain)

$\dot{y} = w$ → $\dot{y} - w = 0$ (II^{nd} costrain)

$$\Rightarrow J^* = \int_0^T \left[\frac{T}{m}\cos\phi + \lambda_1\left(\dot{w} - \frac{T}{m}\sin\phi + g\right) + \lambda_2(\dot{y} - w)\right] dt + u_0$$

$$(P) \quad J^* = u_0 + \int_0^T \left[\frac{F}{m}\cos\phi + \lambda_1 \left(\dot w - \frac{F}{m}\sin\phi + g\right) + \lambda_2 (\dot y - w) \right] dt$$

C.C. at $t=0$ $\quad \begin{cases} y = y_0 ; \\ x = 0 \end{cases} \quad \begin{cases} u_0 = v_0 \cos\theta_0 \\ w_0 = v_0 \sin\theta_0 \end{cases}$

at $t=T$ $\quad y = Y \; ; \; w = 0$

!! Also θ_0 is kept as a degree of freedom capable to maximize U !!

$\Downarrow$

$$\delta J = \delta u_0 + \delta \int_0^T f\left(\phi, \dot w, w, \dot y, \frac{F}{m}\right) dt = -v \sin\theta_0\, \delta\theta_0 + \delta \int_0^T f\left(\phi, \dot w, w, \dot y, \frac{F}{m}\right)$$

$$\frac{\partial f}{\partial \phi} = -\frac{F}{m}\sin\phi - \frac{F}{m}\lambda_1 \cos\phi = -\frac{F}{m}(\sin\phi - \lambda_1 \cos\phi)$$

$$\frac{\partial f}{\partial w} = -\lambda_2$$

$$\frac{\partial f}{\partial \dot w} = \lambda_1$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial \dot y} = \lambda_2$$

$$\frac{\partial f}{\partial \dot \phi} = 0$$

$$\Rightarrow \delta J^* = -v_0 \sin\theta_0\, \delta\theta_0 + \int_0^T \lambda_1\, \delta \dot w\, dt + \int_0^T \lambda_2\, \delta \dot y\, dt - \int_0^T \frac{F}{m}(\sin\phi - \lambda_1\cos\phi)\,\delta\phi\, dt - \int_0^T \lambda_2\, \delta w.$$

$\hookrightarrow$ integrating by parts the 2 terms:

e.g. $\int_0^T \lambda_1 \delta \dot w\, dt = \lambda_1 \delta w - \int_0^T \frac{d\lambda_1}{dt}\delta w\, dt.$

$$\delta J^* = -v_0 \sin\theta_0\, \delta\theta_0 + \int_0^T \lambda_1\, \delta \dot w + \int_0^T \lambda_2\, \delta \dot y\, dt - \int_0^T \frac{F}{m}(\sin\phi - \lambda_1\cos\phi)\,\delta\phi\, dt - \int_0^T \lambda_2\, \delta w$$

$$\delta J^* = -v_0 \sin\theta_0\, \delta\theta_0 + \lambda_1 \delta w \Big|_0^T - \int_0^T \frac{d\lambda_1}{dt}\delta w\, dt + \lambda_2 \delta y\Big|_0^T - \int_0^T \frac{d\lambda_2}{dt}\delta y\, dt - \int_0^T \frac{F}{m}(\sin\phi - \lambda_1\cos\phi)\,\delta\phi + \lambda_2 \delta w\, dt$$

so:

$$\delta J^* = -v_0 \sin\theta_0\, \delta\theta_0 + \lambda_1 \delta w\Big|_0^T + \lambda_2 \delta y\Big|_0^T - \int_0^T \left[\frac{F}{m}(\sin\phi - \lambda_1\cos\phi)\delta\phi + \left(\frac{d\lambda_1}{dt} - \lambda_2\right)\delta w + \frac{d\lambda_2}{dt}\delta y\right] dt$$

$\hookrightarrow$ $\delta w\Big|_0^T = \delta w(T) - \delta w(0)$

$\Rightarrow$ $w(0) = v_0 \sin\theta_0 \rightarrow \delta w(0) = v_0 \cos\theta_0\, \delta\theta_0$

$w(T) = \int_0^T \frac{F}{m}\sin\phi - g\, dt + v_0 \sin\theta_0$

$\uparrow$ obtained from dynamic $\dot w = \frac{T}{m}\sin\phi - g$

$\Rightarrow$ $\delta w(T) = \int_0^T \left[\frac{F}{m}\cos\phi\,\delta\phi - g\right] dt + v_0 \cos\theta_0\, \delta\theta_0$

$\Rightarrow$ OPTIMUM CONDITION $\Leftrightarrow$ $\delta J^* = 0 \quad \forall \{\delta\theta_0, \delta\phi, \delta w, \delta y\}$

$\hookrightarrow$ 4 equations are obtained for the 4 unknowns $\{\phi_{opt}; \theta_{opt}; \lambda_1; \lambda_2\}$

$$\delta J^* = -v_0 \sin\theta_0\, \delta\theta_0 - \lambda_1(0)\cdot v_0 \cos\theta_0 \delta\theta_0 + \lambda_1(T)\int_0^T \left[\frac{F}{m}\cos\phi\,\delta\phi - g\right] dt + \lambda_1(T)\, v_0 \cos\theta_0\, \delta\theta_0$$

$$+ \lambda_2(T)\, \underbrace{\delta y(T)}_{=0} - \lambda_2(0)\, \underbrace{\delta y(0)}_{=0} - \int_0^T \left[\frac{F}{m}(\sin\phi - \lambda_1\cos\phi)\,\delta\phi + \left(\frac{d\lambda_1}{dt} - \lambda_2\right)\delta w + \frac{d\lambda_2}{dt}\delta y\right] dt$$

($y(T) = Y \Rightarrow$ NOT ARBITRARY) . ($y(0) = y_0 \Rightarrow \delta y(0)$ is NOT ARBITRARY)

!! Our aim is to find the optimum $\{\phi, \theta_0, \lambda_1, \lambda_2\}$ for arbitrary variations of $\{\delta\theta_0; \delta\phi; \delta w(T)\ \delta y(T); \delta w; \delta y\}$

$\delta w(T) = 0$ $\quad$ $\delta y(T) = 0$

Not arbitrary cause we assumed to find the maximum v at a given height $(y = Y)$ / at a given $w(Y,T) = 0$ $\Rightarrow \begin{cases} \delta y(T) = 0 \\ \delta w(T) = 0 \end{cases}$!!

Since $\delta w(T) = 0$

$$\Rightarrow w(T) = \int_0^T \left[\frac{F}{m}\sin\phi - g\right] dt + v_0 \sin\theta_0$$

$$\delta w(T) = \int_0^T \frac{F}{m}\cos\phi\, \delta\phi\, dt + v_0 \cos\theta_0\, \delta\theta_0$$

$\frac{\partial f}{\partial x_i} = 0$ $\quad x_i = \{\phi, \theta_0\}$

To have $\delta w(T) = 0$ for an arbitrary variation of $\{\delta\phi, \delta\theta_0\}$ the condition $w = 0$ will be obtained at time $T + \delta t$

$\delta w = \dot{w}\, \delta t$ $\quad$ $\delta w = g\, \delta t$ $\Rightarrow$ $\delta w(T) = g\, \delta T$

vertical acceleration

so:

$$\delta J^* = -v_0(\sin\theta_0 + \lambda_1(0)\cos\theta_0)\,\delta\theta_0 + \lambda_1(T)\, g\, \delta T$$

$$- \int_0^T \left[\frac{F}{m}(\sin\phi + \lambda_1\cos\phi)\,\delta\phi + \frac{d\lambda_2}{dt}\,\delta y + \left(\frac{d\lambda_1}{dt} + \lambda_2\right)\delta w\right] dt$$

$\delta J^* = 0 \quad \forall \{\delta\theta_0, \delta T; \delta w; \delta y\}$

$\forall\, \delta\theta_0$: $\; -v_0(\sin\theta + \lambda_1(0)\cos\theta_0) = 0 \longrightarrow \tan\theta_0 = -\lambda_1(0)$ $\quad (cc_1^*)$

$\forall\, \delta T$: $\; g\,\lambda_1(T) = 0 \longrightarrow \lambda_1(T) = 0$ $\quad (cc_2^*)$

$\forall\, \delta\phi$: $\; \frac{F}{m}(\sin\phi + \lambda_1\cos\phi) = 0 \longrightarrow -\lambda_1 = \tan\phi$

$\forall\, \delta y$: $\; \frac{d\lambda_2}{dt} = 0 \longrightarrow \lambda_2 = -c_2$

$\forall\, \delta w$: $\; \frac{d\lambda_1}{dt} + \lambda_2 = 0 \longrightarrow \lambda_1(t) = c_2 t + c_1$ $\quad (*)$

solving the differential equation in λ_1 containing the boundary conditions $(* + cc_1^* + cc_2^*)$

(P) $\quad \lambda_1(t) = c_2 t + c_1$ $\quad (\tan\phi(t) = -\lambda_1)$

(c.c.) $\quad \begin{cases} \lambda_1(0) = -\tan\theta_0 \\ \lambda_1(T) = 0 \end{cases}$

$$\Rightarrow \begin{cases} 0 = c_2 T + c_1 \\ c_1 = -\tan\theta_0 \end{cases} \rightarrow \begin{cases} c_2 = \frac{1}{T} c_1 = \frac{1}{T}\tan(\theta_0) \\ c_1 = -\tan(\theta_0) \end{cases} \Rightarrow \lambda_1(t) = \frac{\tan(\theta_0)}{T}\, t - \tan\theta_0$$

for an assigned θ_0

$$\tan(\phi_{opt}) = \left(1 - \frac{t}{T}\right)\tan\theta_0$$