$LO_6 \to$ "perturbation analysis"

R2BP must include perturbative term:

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3}\underline{r} + \underline{a}_p$$

LIFT/DRAG (ATMOSPHERE)

NON UNIFORM MASS DISTRIBUTION OF THE MAIN ATTRACTOR.

SOLAR RADIATION PRESSURE

CONTROL FORCE.

PRESENCE OF A THIRD BODY

Perturbation analysis take place considering the unperturbed orbit:

OBSCOLUTATING ORBIT $\Rightarrow$ Keplerian orbit

solution of: $\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3}\underline{r}$

IST Approach: VOP

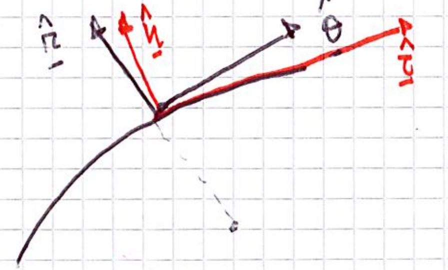
" A first approach in perturbations consists in looking for equations expressing time-dependence of Keplerian parameters. "

* Keplerian model: $\left.\begin{matrix}\underline{h}\\ \underline{e}\\ \varepsilon\end{matrix}\right| \Rightarrow \text{const}|_t$

* Perturbed model: $\frac{d\underline{h}}{dt} \neq \underline{0}$; $\frac{d\underline{e}}{dt} \neq 0$

$\frac{d\varepsilon}{dt} \neq 0$.

$\hookrightarrow$ In order to express all perturbative effects is convenient to consider 2 different reference frames:



$\underline{a}_p^T = [\,f_r\,;\ f_\theta\,;\ f_h\,]$

$\underline{a}_p^{n} = [\,f_t\,;\ f_n\,;\ f_h\,]$

!! $\hat{h}$ is time dependent $\Rightarrow \hat{h} = \hat{h}(t)$ anyway condition is still valid:

$\forall t$: $\hat{h}(t) \perp [\hat{n}, \hat{t}]$; $\hat{h}(t) \perp [\hat{r}, \hat{\theta}]$!!

* POWER EQUIVALENCE.

POWER OF EXTERNAL FORCES (PERTURBATIVE) = POWER: VARIATION OF ENERGY IN TIME. $\longrightarrow \Pi_{a_p} = \Pi \longrightarrow \underline{f} \cdot \underline{v} = \frac{d\varepsilon}{dt}.$

$$\begin{cases} \Pi = \frac{d\varepsilon}{dt} \\ \varepsilon = -\frac{\mu}{2a} \end{cases} \longrightarrow \Pi = \frac{d}{dt}\left(-\frac{\mu}{2a}\right) = -\frac{\mu}{2}\left(-\frac{1}{a^2}\right)\frac{da}{dt} = \frac{\mu}{2a^2}\frac{da}{dt}$$

$$\longrightarrow \boxed{\frac{\mu}{2a^2}\frac{da}{dt} = \underline{f}\cdot\underline{v}} \quad (i)$$

$\underline{f} \triangleq \underline{a}_p = [f_r ; f_\theta ; f_h]$ $\longrightarrow \frac{\mu}{2a^2}\frac{da}{dt} = f_r \cdot v.$

* ANGULAR MOMENTUM is NO MORE CONSERVED.

$$\underline{h} = \underline{r} \times \underline{v} \Rightarrow \frac{d\underline{h}}{dt} = \underline{r} \times \frac{d\underline{v}}{dt} + \frac{d\underline{r}}{dt} \times \underline{v} = \underline{r} \times \frac{d^2\underline{r}}{dt^2} + \underbrace{\cancel{\underline{v} \times \underline{v}}}_{=0}$$

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3}\underline{r} + \underline{a}_p.$$

$$\frac{d\underline{h}}{dt} = \underline{r} \times \left[-\frac{\mu}{\|\underline{r}\|^3}\underline{r} + \underline{a}_p\right]$$

$$= \underbrace{\underline{r} \times \underline{r}}_{=0}\left(-\frac{\mu}{\|\underline{r}\|^3}\right) + \underline{r} \times \underline{a}_p \longrightarrow \boxed{\frac{d\underline{h}}{dt} = \underline{r} \times \underline{a}_p} \quad (ii)$$

Computing with vector product:

$$\dot{\underline{h}} = \frac{d\underline{h}}{dt} = \underline{r} \times \underline{a}_p = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{h} \\ r & 0 & 0 \\ f_r & f_\theta & f_h \end{vmatrix} = -r f_h \hat{\theta} + r f_\theta \hat{h}$$

Obs: Evaluating $\dot{\underline{h}}$ as a modulus:

$$\|\dot{\underline{h}}\| = \frac{\|\underline{h}(t+dt)\| - \|\underline{h}(t)\|}{dt} \quad ; \quad \underline{h}(t+dt) \simeq \underline{h}(t) + d\underline{h} = \underline{h}(t) + \begin{Bmatrix} dh_r \\ dh_\theta \\ dh_h \end{Bmatrix}$$

variation of $\underline{h}$ along all the three directions.

$$\Rightarrow \|\underline{h}(t+dt)\| = \sqrt{(h + dh_h)^2 + dh_r^2 + dh_\theta^2}$$
$$= \sqrt{h^2 + dh_h^2 + 2h\,dh_h + dh_r^2 + dh_\theta^2}$$

Neglecting 2^{nd} order infinitesimal:

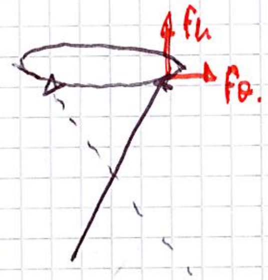
$$\|\underline{h}(t+dt)\| \simeq \sqrt{h^2 + 2h\,dh_h} = h\sqrt{1 + \frac{2dh_h}{h}} \simeq h + dh_h$$

$$\Rightarrow \|\underline{h}(t+dt)\| \simeq \|\underline{h}(t)\| + dh_h \longrightarrow \frac{\|\underline{h}(t+dt)\| - \|\underline{h}(t)\|}{dt} \simeq \frac{dh_h}{dt}.$$

$$\Rightarrow \boxed{(ii^{MOD}) \quad \|\dot{\underline{h}}\| \simeq r f_\theta \Rightarrow \frac{dh}{dt} \simeq r f_\theta}$$

In other words:

$\dot{\underline{h}} = -r f_h \hat{\theta} + r f_\theta \hat{h} \Rightarrow$ variation along $\hat{\theta}$ are negligible in computing $\|\dot{\underline{h}}\|$ but highly affect direction $\hat{h}$ of angular momentum.



F_θ modifies $\hat{h}$ direction
$F_h \to$ modifies $\|\underline{h}\|$ modulus.

* DIFFERENT APPROACH IN COMPUTING $\dot{\underline{h}}$

Instead of substituting the newton's equation:

$$\underline{h} = \underline{r} \times \underline{v} = \underline{r} \times [\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}]$$

$$r\dot{\theta}\hat{\theta} = \dot{\theta}\hat{h} \times \underline{r} \triangleq \underline{w} \times \underline{r}$$

$$\Rightarrow \boxed{\underline{h} = \underline{r} \times [\dot{r}\hat{r} + \underline{w} \times \underline{r}] \qquad \underline{w} = \dot{\theta}\hat{h}}$$

$$\underline{h} = \underline{r} \times [\dot r \hat{\underline{r}} + \underline{W} \times \underline{r}]$$

$$a\times(b\times c) = \underline{b}\,(\underline{a}\cdot\underline{c}) - \underline{c}\,(\underline{a}\cdot\underline{b})$$

$$\underline{h} = \underbrace{\underline{r}\times(\dot r\hat{\underline{r}})}_{=0} + \underline{r}\times(\underline{W}\times\underline{r}) = \underline{W}\,(\underline{r}\cdot\underline{r}) - \underline{r}\,(\underline{W}\cdot\underline{r})$$

$$\underline{h} = r^2\underline{W} - (\underline{W}\cdot\underline{r})\,\underline{r} \Rightarrow \underline{h} = r^2\underline{W} - \dot\Upsilon r\,\underline{r}$$

$$\dot\Upsilon \triangleq \underline{W}\cdot\hat{\underline{r}}$$

!! Since problem is perturbed condition $\underline{h}\perp\underline{r}$ is no more guaranteed. !!
$\Rightarrow \dot\Upsilon \neq 0$

Computing $\underline{W}$ from previous equation:

$$\underline{h} = r^2\underline{W} - \dot\Upsilon\,\underline{r} \longrightarrow \underline{W} = \frac{\underline{h}}{r^2} + \dot\Upsilon\frac{r}{r^2}\,\underline{r} \longrightarrow \underline{W} = \frac{\underline{h}}{r^2} + \dot\Upsilon\hat{\underline{r}}$$

↳ Computing derivative:

$$\frac{d\underline{h}}{dt} = \left(\frac{dh}{dt}\right)\hat{\underline{h}} + h\cdot\frac{d\hat{\underline{h}}}{dt} = \dot h\,\hat{\underline{h}} + \underline{W}\times\underline{h}$$

⊛

$$\frac{d\underline{h}}{dt} = \dot h\,\hat{\underline{h}} + \left(\frac{h}{r^2}\hat{\underline{h}} + \dot\Upsilon\hat{\underline{r}}\right)\times\underline{h} = \dot h\,\hat{\underline{h}} + \dot\Upsilon\,(\hat{\underline{r}}\times\underline{h})$$
$$= \dot h\,\hat{\underline{h}} + \dot\Upsilon h\cdot\hat{\underline{\theta}}$$

$\hat{\underline{r}}\times\underline{h} = h(\hat{\underline{r}}\times\hat{\underline{h}})$

Through comparison between (i) and (iii):

(ii) $\frac{d\underline{h}}{dt} = -r f_h\,\hat{\underline{\theta}} + r f_\theta\,\hat{\underline{h}}$

(iii) $\frac{d\underline{h}}{dt} = \dot h\,\hat{\underline{h}} + \dot\Upsilon h\,\hat{\underline{\theta}}$

$$\longrightarrow \begin{cases} \dot h = -r f_\theta \\ \dot\Upsilon h = + r f_h \end{cases}$$

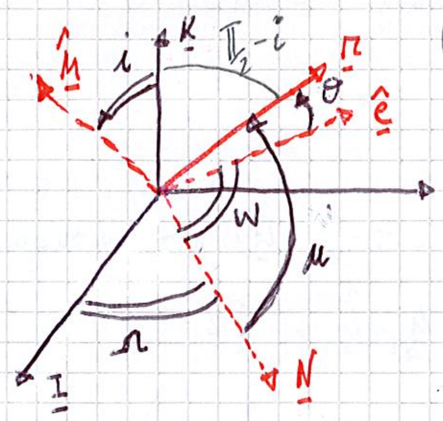
$$\Rightarrow \begin{cases} \dot h = -r f_\theta \\ \dot\Upsilon = + \dfrac{r f_h}{h} \end{cases}$$

⊛ The possibility of estimating $h\cdot\frac{d\hat{\underline{h}}}{dt} = \underline{W}\times\underline{h}$ is due to the fact: that perturbative effects don't act along $\hat{\underline{r}}$ direction.

Such effects are due to the component of $\underline{h}$ not orthogonal to $\hat{\underline{r}}$ into the perturbed model: $\underline{W} = \frac{h}{r^2}\hat{\underline{h}} + \dot\Upsilon\hat{\underline{r}}$

($\underline{W}$ in a perturbed model is no more directed as $\hat{\underline{h}}$.)

$\Rightarrow$ Expressing $\underline{W}$ in $\{\hat{K}, \hat{N}, \hat{h}\}$ reference frame:



$(\hat{r} \perp \hat{h})$

$$u \triangleq W + \theta$$
$$u \triangleq \hat{N} \wedge \hat{r}$$

$$\Rightarrow \hat{r} = \cos u \, \underline{N} - \sin i \, \underline{K} + 0 \cdot \hat{\underline{h}}$$

The transformation for a generic vector:

from $[\hat{r}, \hat{t}, \hat{h}] \longrightarrow$ to $[\hat{I}, \hat{J}, \hat{K}]$

$$\underline{a}[\hat{r}, \hat{t}, \hat{h}] = \underline{\underline{R}} \, \underline{a}[I, J, K]$$

$$\underline{a}[I, J, K] = \underline{\underline{R}}^T \, \underline{a}[\hat{r}, \hat{t}, \hat{h}]$$

orthogonal matrix:
$$\underline{\underline{A}}^{-1} = \underline{\underline{A}}^T \Rightarrow \underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$$

where:

$$\underline{\underline{R}}^T = \begin{bmatrix} \cos u \cos\Omega - \cos i \sin\Omega \sin u ; & \cos u \sin\Omega + \cos i \cos\Omega \sin u ; & \sin i \sin u \\ -\sin u \cos\Omega - \cos i \sin\Omega \cos u ; & -\sin u \sin\Omega + \cos i \cos\Omega \cos u ; & \sin i \cos u \\ \sin i \sin\Omega ; & -\sin i \cos\Omega ; & \cos i \end{bmatrix}$$

(a)

$$\frac{\mu}{2a^2} \frac{da}{dt} = \underline{f} \cdot \underline{v} \rightarrow \frac{\mu}{2a^2} \frac{da}{dt} = [e \sin\theta \cdot f_r + (1 + e\cos\theta) f_\theta] \sqrt{\frac{\mu}{a(1-e^2)}}$$

$$\rightarrow \frac{da}{dt} = \frac{2a^2}{\mu} \cdot \sqrt{\frac{\mu}{a}} \cdot \frac{1}{\sqrt{1-e^2}} [\cdot] \rightarrow \frac{da}{dt} = \frac{2\sqrt{a^3}}{\sqrt{\mu(1-e^2)}} [\cdots]$$

$$\rightarrow \frac{da}{dt} = \frac{2\sqrt{a^3}}{\sqrt{\mu(1-e^2)}} [e \sin\theta f_r + (1+e\cos\theta) f_\theta]$$

(e)

$\rightarrow$ express e^2 as a function of h and ε in such way to use (i) and (ii) eqns.

$$h = \sqrt{\mu p} \rightarrow h^2 = \mu a(1-e^2) \rightarrow \mu a e^2 = \mu a - h^2 \rightarrow e^2 = 1 - \frac{h^2}{\mu a}$$

$$\rightarrow e^2 = 1 - \frac{h^2}{\mu a} \underbrace{\left(-\frac{\mu}{2}\right)}_{\varepsilon} \left(-\frac{2}{\mu^2}\right) = 1 + 2\frac{h^2}{\mu^2} \frac{\mu}{2a}$$

$$\Rightarrow e^2 = 1 + 2\frac{h^2}{\mu^2} \varepsilon$$

$e = e(h, \varepsilon)$

$$e^2 = 1 + 2\varepsilon \frac{h^2}{\mu^2}$$

Performing derivation:

$$2e \frac{de}{dt} = 2\dot{\varepsilon} \frac{h^2}{\mu^2} + 4\frac{\varepsilon h}{\mu^2} \dot{h} \Rightarrow \frac{de}{dt} = \frac{1}{2e} \left[2\dot{\varepsilon} \frac{h^2}{\mu^2} \frac{\varepsilon}{\varepsilon} + 4\frac{\varepsilon h}{\mu^2} \dot{h} \frac{h}{h}\right]$$

$$\dot{e} = \frac{1}{2e} \underbrace{\frac{2h^2\varepsilon}{\mu^2}}_{(e^2-1)} \left[\frac{\dot{\varepsilon}}{\varepsilon} + 2\frac{\dot{h}}{h}\right]$$

$$\frac{2h^2\varepsilon}{\mu^2} = 2\mu p \frac{\varepsilon}{\mu^2} \rightarrow 2\cancel{\mu}(a)(1-e^2)\frac{\varepsilon}{\cancel{\mu^2}} = 2\cancel{a}(1-e^2)\left(-\frac{\cancel{\mu}}{\cancel{2a}}\right)\frac{1}{\cancel{\mu}} = (1-e^2)$$

$$\Rightarrow \dot{e} = +\frac{1}{2e} \cdot (e^2-1) \left[\frac{\dot{\varepsilon}}{\varepsilon} + 2\frac{\dot{h}}{h}\right]$$

- $\dot{h} \simeq \dot{h}_h = r f_\theta$. (ii^{MOD})
- $\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{d}{dt}\left(-\frac{\mu}{2a}\right) = +\frac{1}{2} \frac{\mu}{a^2} \frac{da}{dt}$.

$$\dot{\varepsilon} = \frac{1}{2} \frac{\mu}{a^2} \frac{da}{dt} \rightarrow$$

$$\dot{\varepsilon} = \varepsilon \frac{da}{dt} \frac{1}{a} \rightarrow \frac{\dot{\varepsilon}}{\varepsilon} = -\frac{\dot{a}}{a}$$

$$\dot{e} = \frac{1}{2e} (e^2-1) \left[2 \cdot \frac{1}{2} \frac{\mu}{a^2} \dot{a} \left(-\frac{2a}{\mu}\right) + 4\frac{r f_\theta}{h}\right]$$

$$\dot{e} = \frac{1}{2e} (e^2-1) \left[-\frac{\dot{a}}{a} + 2\frac{r f_\theta}{h}\right]$$

$$\dot e = \frac{1}{2e}(e^2-1)\left[-\frac{\dot a}{a} + 2\,\frac{r f_\theta}{h}\right] \qquad \Big\| \quad \dot a = \frac{2\sqrt{a^3}}{\sqrt{\mu(1-e^2)}}\left[e\sin\theta f_r + (1+e\cos\theta) f_\theta\right]$$

$$\dot e = -\frac{1}{2e}(1-e^2)\left[-\frac{2\sqrt{a^3}}{a}\,\frac{1}{\sqrt{\mu(1-e^2)}}\left(e\sin\theta f_r + (1+e\cos\theta)f_\theta\right) + 2\,\frac{r f_\theta}{h}\right]$$

$$\Big\| \quad r = a(1-e\cos E)\ ^{*}$$

$$h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$$

$$\Rightarrow \frac{r}{h} = \frac{a(1-e\cos E)}{\sqrt{\mu a(1-e^2)}}$$

$$\dot e = -\frac{1}{2e}(1-e^2)\left[-2\,\frac{\sqrt{a^3}}{a}\cdot\frac{1}{\sqrt{\mu(1-e^2)}}\left(e\sin\theta f_r + (1+e\cos\theta)f_\theta\right) + \frac{2a(1-e\cos E)}{\sqrt a\sqrt{\mu(1-e^2)}}\right]$$

$$\dot e = -\frac{1}{2e}\,\frac{(1-e^2)}{\sqrt{\mu(1-e^2)}}\left[-2\sqrt a\, e\sin\theta f_r - 2\sqrt a(1+e\cos\theta)f_\theta + \sqrt a(1-e\cos E)\right]$$

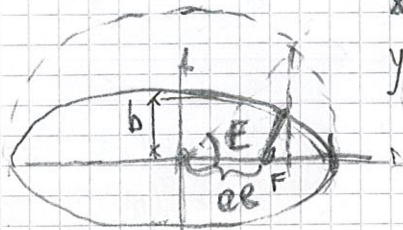
$$\dot e = -\frac{1}{2e}\sqrt{\frac{a(1-e^2)}{\mu}}\left[-2e\sin\theta f_r + 2e(\cancel{1} - e\cos E - \cancel{1} - e\cos\theta)\right]$$

$$\dot e = -\frac{1}{2e}\sqrt{\frac{a(1-e^2)}{\mu}}\left[-2e\sin\theta f_r - 2e(\cos E + \cos\theta)f_\theta\right]$$

$$\dot e = -\sqrt{\frac{a(1-e^2)}{\mu}}\left[-\sin\theta f_r - (\cos\theta + \cos E)f_\theta\right]$$

$$\rightarrow \boxed{\frac{de}{dt} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[\sin\theta f_r + (\cos\theta + \cos E)f_\theta\right]}$$

*



$x = ae + r\cos\theta$; $x = a\cos E$

$y = r\sin\theta$

Ell: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a\cos E = ae + r\cos\theta \rightarrow r\cos\theta = (a\cos E - ae) = a(\cos E - e)$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \rightarrow r(1+e\cos\theta) = a(1-e^2)$$

$$r + re\cos\theta = a(1-e^2)$$

$$r = -re\cos\theta + a(1-e^2) = a[-e\cos E + \cancel{e^2} + 1 - \cancel{e^2}]$$

$$= a(1 - e\cos E)$$

(i)

from i definition: $\cos i = \dfrac{\underline{h}\cdot\underline{K}}{\|\underline{h}\|}$ through derivation:

$$-\sin i\,\frac{di}{dt} = \frac{\frac{d\underline{h}}{dt}\cdot\underline{K} - \frac{dh}{dt}(\underline{h}\cdot\underline{K})}{h^2}$$

(ii) $\dfrac{d\underline{h}}{dt} = \underline{r}\times\underline{a}_p \xrightarrow{\text{opp}} \dot h \simeq r f_\theta$.

$$\frac{d\underline{h}}{dt} = -r f_h\,\hat{\underline\theta} + r f_\theta\,\hat{\underline h}$$

$$\Rightarrow \frac{di}{dt} = -\frac{1}{\sin i}\,\frac{h[-r f_h\,\hat\theta\cdot\hat{\underline K} + r f_\theta\,\hat{\underline h}\cdot\hat{\underline K}] - r f_\theta\,\underline{h}\cdot\hat{\underline K}}{h^2}$$

$\Big\|$ $-r f_\theta\,\underline h\cdot\hat{\underline K} = -r f_\theta h\,\hat{\underline h}\cdot\hat{\underline K} = -r f_\theta h\cos i$

$$= -\frac{1}{\sin i}\cdot\frac{\cancel{h r f_\theta\cos i} - h r f_h\,\hat\theta\cdot\hat{\underline K}^{*} - \cancel{r f_\theta\cos i\, h}}{h^2} \Rightarrow \frac{di}{dt} = \frac{1}{\sin i}\,\frac{r f_h}{h}\,\hat\theta\cdot\hat{\underline K}$$

$$^{*}\ \hat{\underline\theta}\cdot\hat{\underline K} \quad = -\frac{1}{\sin i}\cdot\frac{h r f_h\,\hat\theta\cdot\hat K}{h^2} = -\frac{1}{\sin i}\,\frac{r f_h\,\hat\theta\cdot\hat K}{h}$$

$$\underline a_{\{\hat r,\hat\theta,\hat h\}} = \underline{\underline R}\;\underline a_{\{\hat i,\hat j,\hat K\}} \Rightarrow \hat\theta\cdot\hat K = \begin{Bmatrix}0\\1\\0\end{Bmatrix}^T\cdot\underline{\underline R}\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$ (scalar product in $\{\hat r,\hat\theta,\hat h\}$)

$$\hat\theta\cdot\hat K = \left(\underline{\underline R}^T\begin{Bmatrix}0\\1\\0\end{Bmatrix}\right)^T\cdot\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$ (scalar product in $\{\hat i,\hat j,\hat K\}$)

$$\underline{\underline{R}} = \underline{\underline{Rot}} \cdot \underline{\underline{Rot}}^T \Rightarrow \hat{\theta}\cdot\hat{K} = \begin{Bmatrix}0\\1\\0\end{Bmatrix} \underline{\underline{Rot}}\ \underline{\underline{Rot}}^T \begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$\hat{\theta}\cdot\hat{K} = \begin{Bmatrix}0\\1\\0\end{Bmatrix}^T \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\omega\, c\Omega - s\omega\, ci\, s\Omega ; & c\omega\, s\Omega + s\omega\, ci\, c\Omega ; & s\omega\, si \\ -s\omega\, c\Omega - c\omega\, ci\, s\Omega ; & -s\omega\, s\Omega + c\omega\, ci\, c\omega ; & c\omega\, si \\ si\, s\Omega ; & -si\, c\Omega ; & ci \end{bmatrix} \cdot \begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$= \begin{Bmatrix}0\\1\\0\end{Bmatrix}^T \begin{bmatrix} Rot \end{bmatrix} \begin{Bmatrix} s\omega\, si \\ c\omega\, si \\ ci \end{Bmatrix} = \begin{Bmatrix}0\\1\\0\end{Bmatrix}^T \begin{Bmatrix} c\theta\, s\omega\, si + s\theta\, c\omega\, si \\ -s\theta\, s\omega\, si + c\theta\, c\omega\, si \\ ci \end{Bmatrix}$$

$$\hat{\theta}\cdot\hat{K} = -s\theta\, s\omega\, si + c\theta\, c\omega\, si$$

$$= si\,[\,c\theta\, c\omega - s\theta\, s\omega\,] = \sin i \cos(\theta+\omega)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\Rightarrow \frac{di}{dt} = \frac{1}{\sin i}\,\frac{r f_h}{h}\,\hat{\theta}\cdot\hat{K} = \frac{1}{\sin i}\,\sin i \cos(\theta+\omega)\,\frac{r f_h}{h}$$

$$\frac{di}{dt} = \frac{p}{1+e\cos\theta}\,\frac{1}{\sqrt{\mu p}}\,f_h \cos(\theta+\omega) = \frac{\sqrt{p}\,\cos(\theta+\omega)}{\sqrt{\mu}\,(1+e\cos\theta)}\,f_h$$

$$\frac{di}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)}\cos(\theta+\omega)\cdot f_h$$

$$\boxed{\frac{di}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)}\cdot\cos(\theta+\omega)\, f_h}$$

(Ω)

Through Ω definition: $\cos\Omega = \underbrace{\hat{N}\cdot\hat{i}}_{3} = \left(\frac{\hat{K}\times h}{\|\hat{K}\times h\|}\right)\cdot\hat{i}$

Derivating: $$-\sin\Omega\,\frac{d\Omega}{dt} = \frac{\overbrace{[(\hat{K}\times\frac{dh}{dt})\cdot\hat{i}]\,\|\hat{K}\times h\|}^{} - \overbrace{\frac{d}{dt}(\|\hat{K}\times h\|)}^{1}\cdot\overbrace{[(\hat{K}\times h)\cdot\hat{i}]}^{2}}{\|\hat{K}\times h\|^2}$$ *

1 → $\|\hat{K}\times h\| = h\sin i \Rightarrow \frac{d\|\hat{K}\times h\|}{dt} = h\cos i\,\frac{di}{dt} + \frac{dh}{dt}\sin i$

As previously observed: $\bullet\ \frac{dh}{dt} = \tilde{h} \simeq r f_\theta$ (a-opp)

$\bullet\ \frac{di}{dt} = \sqrt{\frac{p}{\mu}}\,\frac{\cos(\theta+\omega)}{1+e\cos\theta}\,f_h$

$$\frac{d\|\hat{K}\times h\|}{dt} = \sqrt{\mu p}\cdot\cos i\,\sqrt{\frac{p}{\mu}}\cdot\frac{\cos(\theta+\omega)}{1+e\cos\theta}\,f_h + \frac{p}{1+e\cos\theta}\,f_\theta\sin i$$

$$= \underbrace{\frac{p}{1+e\cos\theta}}_{r}\,[\,\cos(\theta+\omega)\cdot\cos i\, f_h + \sin i\, f_\theta\,] = r\,[\cos(\theta+\omega)\ldots]$$

2 → $$\hat{K}\times h = \hat{K}\times(\underline{\underline{Rot}}^T\,\underline{\underline{Rot}}\cdot h) = \hat{K}\times\left(\underline{\underline{Rot}}^T\cdot\begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}0\\0\\h\end{Bmatrix}\right)$$

$$= \hat{K}\times\left(\underline{\underline{Rot}}^T\cdot\begin{Bmatrix}0\\0\\h\end{Bmatrix}\right)$$!! equivale a selezionare solo la 3a colonna di $\underline{\underline{Rot}}^T$!!

$$= \hat{K}\times\left(\begin{bmatrix} \ldots, & \ldots, & si\, s\Omega \\ \ldots, & \ldots, & -si\, c\Omega \\ \ldots, & \ldots, & ci \end{bmatrix}\begin{Bmatrix}0\\0\\h\end{Bmatrix}\right)$$

$$= h\cdot\left(\hat{K}\times\begin{Bmatrix} si\, s\Omega \\ -si\, c\Omega \\ ci\end{Bmatrix}\right) = h\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{K} \\ 0 & 0 & 1 \\ si\, s\Omega ; & -si\, c\Omega ; & ci \end{vmatrix}$$

$$= h\,[\,+si\, c\Omega\,\hat{i} + si\, s\Omega\,\hat{j} + 0\,\hat{K}\,]$$

$$\Rightarrow (\hat{K}\times h)\cdot\hat{i} = h\, si\, c\Omega.$$

$$3 \rightarrow \quad \underline{\hat{K}} \times \frac{d\underline{h}}{dt} = \underline{\hat{K}} \times [\, r f_\theta \underline{\hat{h}} - r f_h \underline{\hat{\theta}} \,]$$

$$\left(\frac{d\underline{h}}{dt} = + r f_\theta \underline{\hat{h}} - r f_h \underline{\hat{\theta}} \right)$$

↳ Need to consider in final: $\left(\underline{\hat{K}} \times \frac{d\underline{h}}{dt}\right) \cdot \underline{\hat{i}}$

As before: • $\underline{\hat{K}} \times \underline{\hat{h}} = \begin{Bmatrix} s_i c_\Omega \\ + s_i s_\Omega \\ c_i \end{Bmatrix}$

$$\hat{\theta}\begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} = \underline{\underline{ROT}}^T \; \underline{\underline{ROT}}^T \cdot \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \underline{\underline{ROT}}^T \cdot \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$= \underline{\underline{ROT}}^T \begin{Bmatrix} -s_\theta \\ c_\theta \\ 0 \end{Bmatrix} = \begin{bmatrix} c_w c_\Omega - s_w c_i s_\Omega \,, & - c_\Omega s_w - s_\Omega c_w c_i \,, & \dots \\ s_\Omega c_w + c_\Omega s_w c_i \,, & - s_\Omega s_w + c_\Omega c_w c_i \,, & \dots \\ s_w s_i \,, & c_w s_i \,, & \dots \end{bmatrix}$$

anche in qst caso posso escludere la 3ª colonna

$$\Rightarrow \underline{\hat{\theta}}\begin{Bmatrix} i \\ j \\ k \end{Bmatrix} = \begin{Bmatrix} - s_\theta c_w c_\Omega + s_\theta s_w c_i s_\Omega - c_\theta c_\Omega s_w - c_\theta s_\Omega c_w c_i \\ - s_\theta s_\Omega c_w - s_\theta c_\Omega s_w c_i - c_\theta s_\Omega s_w + c_\theta c_\Omega c_w c_i \\ - s_\theta s_w s_i + c_\theta c_w s_i \end{Bmatrix}$$

$$= \begin{Bmatrix} c_i s_\Omega (- s_w s_\theta - c_\theta c_w) + c_\Omega (- s_\theta c_w - c_\theta s_w) \\ c_\Omega c_i (- s_\theta s_w + c_\theta c_w) + s_\Omega (- s_\theta c_w - c_\theta s_w) \\ s_i (- s_\theta s_w + c_\theta c_w) \end{Bmatrix}$$

$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \cos\beta$
$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$!

$$\hat{\theta}\begin{Bmatrix} i \\ j \\ k \end{Bmatrix} = \begin{vmatrix} - c_i s_\Omega \sin(\theta+w) + c_\Omega \sin(\theta+w) \\ c_\Omega c_i \cos(\theta+w) - s_\Omega \sin(\theta+w) \\ + s_i \sin(\theta - w) \end{vmatrix}$$

• $\underline{\hat{K}} \times \underline{\hat{\theta}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ - c_i s_\Omega \sin(\theta+w) + c_\Omega \sin(\theta+w); & c_\Omega c_i \cos(\theta+w) - s_\Omega \sin(\theta+w); & - s_i \sin(\theta w) \end{vmatrix}$

↳ the real aim is to compute:

$$\left(\underline{\hat{K}} \times \frac{d\underline{h}}{dt}\right) \cdot \underline{\hat{i}} = r f_\theta (\underline{\hat{K}} \times \underline{h}) \cdot \underline{\hat{i}} - r f_h (\underline{K} \times \underline{\hat{\theta}}) \cdot \underline{\hat{i}}$$

$$\Rightarrow (\underline{\hat{K}} \times \underline{\hat{h}}) \cdot \underline{\hat{i}} = s_i c_\Omega$$

$$(\underline{\hat{K}} \times \underline{\hat{\theta}}) \cdot \underline{\hat{i}} = \begin{Bmatrix} - c_\Omega c_i \cos(\theta+w) + s_\Omega \sin(\theta+w) \\ \dots \\ 0 \end{Bmatrix}^T \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$= - c_\Omega c_i \cos(\theta+w) + s_\Omega \sin(\theta+w)$$

$$\Rightarrow \left(\underline{\hat{K}} \times \frac{d\underline{h}}{dt}\right) \cdot \underline{\hat{i}} = r f_\theta s_i c_\Omega - r f_h [\, - c_\Omega c_i \cos(\theta+w) + s_\Omega \sin(\theta+w) \,]$$

$$= r f_\theta s_i c_\Omega + r f_h c_\Omega c_i \cos(\theta+w) - r f_h s_\Omega \sin(\theta+w)$$

$\Rightarrow$ substituting everything into "*": $- \sin\Omega \frac{d\Omega}{dt} = \dfrac{\|\underline{\hat{K}} \times \underline{h}\| \left[\left(\underline{\hat{K}} \times \frac{d\underline{h}}{dt}\right) \cdot \underline{\hat{i}}\right] - \frac{d}{dt}\left(\|\underline{\hat{K}} \times \underline{\hat{h}}\|\right) \left[(\underline{K} \times \underline{h}) \cdot \underline{\hat{i}}\right]}{\|\underline{\hat{K}} \times \underline{h}\|^2}$

$$- \sin\Omega \frac{d\Omega}{dt} = \frac{\cancel{h \sin i} \,(r f_\theta s_i c_\Omega + r f_h c_\Omega c_i \cos(\theta+w) - r f_h s_\Omega \sin(\theta+w))}{h^{\cancel{2}} \sin^{\cancel{2}} i} \; -$$

$$- \frac{\cancel{h \sin i}\, r c_\Omega [\, \cos(\theta+w) \cos i \, f_h + \sin i \, f_\theta \,]}{h^2 \sin^2 i}$$

$$- s_\Omega \frac{d\Omega}{dt} = \{ r f_\theta [\, s_i \cancel{c_\Omega} - s_i \cancel{c_\Omega} \,] + r f_h [\, c_\Omega c_i \cos(\theta w) - s_\Omega \sin(\theta+w) - c_\Omega c_i \cos(\theta w) \,] \}$$

$$(- \cancel{s_\Omega}) \frac{d\Omega}{dt} = r f_h \cdot \frac{1}{h \sin i} \cdot (- \cancel{s_\Omega}) \sin(\theta+w)$$

$$\frac{r}{h} = \frac{p}{1+e\cos\theta} \cdot \frac{1}{\sqrt{p\mu}} = \sqrt{\frac{p}{\mu}} \frac{1}{1+e\cos\theta}$$

$$\rightarrow \boxed{\frac{d\Omega}{dt} = \frac{\sqrt{a(1-e^2)}}{\sqrt{\mu}\,(1+e\cos\theta)} \cdot \frac{\sin(\theta w)}{\sin i} f_h}$$