⇒ By substituting such result into Euler's equation.

$$\{ \varphi = \varphi_0 + \dot\varphi_0 t \;;\; \theta = \theta_0 \;;\; \psi = \psi_0 + \dot\psi_0 t \}$$

+

$$\{ \omega_x = \dot\varphi \sin\theta \sin\psi \;;\; \omega_y = \dot\varphi \sin\theta \cos\psi \;;\; \omega_z = \dot\varphi \cos\theta + \dot\psi \}$$

$$\Downarrow$$

$$\{ \omega_x = \dot\varphi_0 \sin\theta_0 \sin\psi \;;\; \omega_y = \dot\varphi_0 \sin\theta_0 \cos\psi \;;\; \omega_z = \dot\varphi_0 \cos\theta_0 + \dot\psi_0 \}$$

+

Euler's eqn:
$$\begin{cases} \dot\omega_x + \lambda \omega_y = 0 \\ \dot\omega_y - \lambda \omega_x = 0 \\ \dot\omega_z = 0 \end{cases} \Rightarrow \begin{cases} \dot\varphi_0 \sin\theta_0 \cos\psi \cdot \dot\psi_0 + \lambda \dot\varphi_0 \sin\theta_0 \cos\psi = 0 \\ -\dot\varphi_0 \sin\theta_0 \sin\psi \, \dot\psi_0 - \lambda \dot\varphi_0 \sin\theta_0 \sin\psi = 0 \\ \omega_z = \bar\omega_z = \dot\varphi_0 \cos\theta_0 + \dot\psi_0 \end{cases}$$

Everything reduces to 3 algebraic conditions:

$$\begin{cases} [\dot\varphi_0 \dot\psi_0 + \lambda \dot\varphi_0] \cancel{\sin\theta_0 \cos\psi} = 0 \\ [\dot\varphi_0 \dot\psi_0 + \lambda \dot\varphi_0] \cancel{\sin\theta_0} \sin\psi = 0 \\ \dot\psi_0 + \dot\varphi_0 \cos\theta_0 = \bar\omega_z \end{cases} \Rightarrow \quad \dot\theta_0 = 0 \;;\; \dot\psi_0 = -\lambda \;;\; \dot\varphi_0 = \frac{\bar\omega_z - \dot\psi_0}{\cos\theta_0} = \frac{\bar\omega_z - \lambda}{\cos\theta_0}$$

313 Sequence Time (constant) rotation

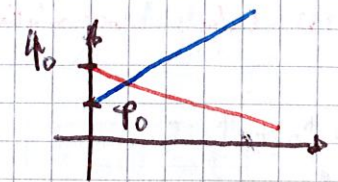
$$\dot\theta_0 = 0$$

⇒ $$\dot\psi_0 = \frac{I_x - I_z}{I_x} \cdot \bar\omega_z$$

$$\dot\varphi_0 = \frac{1}{\cos\theta_0} \left[1 - \frac{I_x - I_z}{I_x} \right] \bar\omega_z = \frac{I_z}{I_x} \frac{\bar\omega_z}{\cos\theta_0}$$

⊕ $$\theta = \theta_0 \;;\; \psi = \psi_0 + \dot\psi_0 \cdot t \;;\; \varphi = \varphi_0 + \dot\varphi_0 \cdot t$$

(a) $\lambda > 0 \Rightarrow \dot\psi_0 = \dot\psi < 0$ $(I_z > I_x)$; $\dot\psi = -\lambda$



(b) $\lambda < 0 \Rightarrow \dot\psi_0 = \dot\psi > 0$ $(I_z < I_x)$

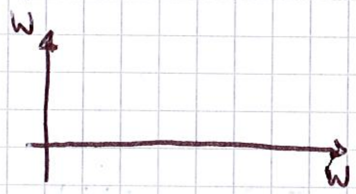
Easy to be checked numerically

"Phase-plane representation of Euler's equation"

Still considering a torque-free motion:

$$\begin{cases} I_x \dot\omega_x + (I_z - I_y) \omega_y \omega_z = 0 \\ I_y \dot\omega_y + (I_x - I_z) \omega_x \omega_y = 0 \\ I_z \dot\omega_z + (I_y - I_z) \omega_x \omega_y = 0 \end{cases} \oplus \quad 2T^{rot} = const \;;\; h^2 = const.$$

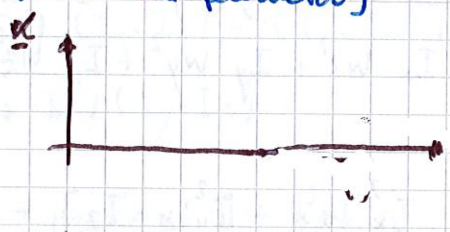
phase-plane



(NOT GOOD FOR CONTROL)

generates closed conic ⇒ cycles.

→ σ (altitude parameter)



(ALLOWS ME TO UNDERSTAND THE BANDWITH I NEED TO CONTROL)

H_p*: $I_z > I_y > I_x \Rightarrow \lambda > 0$

$$\Rightarrow I_z > \frac{h^2}{2T} \;,\; I_x < \frac{h^2}{2T} \quad \text{or} \quad I_z > \frac{h^2}{2T} \;,\; I_y < \frac{h^2}{2T}.$$

procedure*

(1) derive 1 of equation (ω_x eqn)

(2) evaluate (ω_x, ω_y) from the other 2 equations

(3) Substitute ((2) → (1))

(4) eliminate dependence on (ω_y, ω_x) thanks to
$$\begin{cases} 2T = I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \\ h^2 = I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2. \end{cases}$$

(5) Integrate in time to obtain form: $\dot\omega_i = f(\omega_i)$

(1) $I_x \ddot{w}_x + (I_z - I_y)[\dot{w}_y w_z + w_y \dot{w}_z] = 0$

(2) $$\begin{cases} \dot{w}_y = \dfrac{I_z - I_x}{I_y} w_x w_z \\ \dot{w}_z = \dfrac{I_x - I_y}{I_z} w_y w_x \end{cases}$$

(3) $I_x \ddot{w}_x + (I_z - I_y)\left[\dfrac{I_z - I_x}{I_y} w_x w_z^2 + \dfrac{I_x - I_y}{I_z} w_y^2 w_x\right] = 0$ (i)

(4) $$\begin{cases} h^2 = I_x^2 w_x^2 + I_y^2 w_y^2 + I_z^2 w_z^2 \\ 2T = I_x w_x^2 + I_y w_y^2 + I_z w_z^2 \end{cases}$$

$h^2 - 2TI_z = I_x w_x^2 [I_x - I_z] + I_y w_y^2 [I_y - I_z]$ (ii)

$h^2 - 2TI_x = I_y w_y^2 [I_y - I_x] + I_z w_z^2 [I_z - I_x]$

$h^2 - 2TI_y = I_x w_x^2 [I_x - I_y] + I_z w_z^2 [I_z - I_y]$ (iii)

(i) $I_x \ddot{w}_x + (I_z - I_y) w_x \left[\dfrac{I_z - I_x}{I_y} w_z^2 + \dfrac{I_x - I_y}{I_z} w_y^2\right] = 0$

(ii) $h^2 - 2TI_z = I_x w_x^2 [I_x - I_z] + I_y w_y^2 [I_y - I_z]$

$\Rightarrow w_y^2 (I_y - I_z) = \dfrac{h^2 - 2TI_z - I_x w_x^2 [I_x - I_z]}{I_y}$

(iii) $h^2 - 2TI_y = I_x w_x^2 [I_x - I_y] + I_z w_z^2 [I_z - I_y]$

$\Rightarrow w_z^2 (I_z - I_y) = \dfrac{h^2 - 2TI_y - I_x [I_x - I_y] w_x^2}{I_z}$

(i) to be seen in form:

$$I_x \ddot{w}_x + \left[\frac{I_z - I_x}{I_y}\right] \cdot \underbrace{(I_z - I_y) w_z^2}_{(iii)} w_x + \left[\frac{I_x - I_y}{I_z}\right] \underbrace{(I_z - I_y) w_y^2}_{(ii)} w_x$$

$$\Rightarrow I_x \ddot{w}_x + \left[\frac{I_z - I_x}{I_y}\right] \frac{h^2 - 2TI_y w_x - I_x [I_x - I_y] w_x^3}{I_z} + \left[\frac{I_x - I_y}{I_z}\right] \cdot \frac{h^2 - 2TI_z w_x - I_x [I_x - I_z] w_x^3}{I_y} = 0.$$

Recovering together terms in w_x and w_x^3.

$$\ddot{w}_x + \frac{1}{I_x I_y I_z} \cdot \left[(I_z - I_x)(h^2 - 2TI_y) + (I_x - I_y)(h^2 - 2TI_z)\right] w_x - \frac{I_x}{I_x I_y I_z}\left[(I_z - I_x)(I_x - I_y) + (I_y - I_x)(I_x - I_z)\right] w_x^3 = 0$$

$(I_z - I_x)(I_x - I_y) + (I_x - I_x)(I_x - I_y) =$
$2(I_z - I_x)(I_y - I_x)$

$$\Rightarrow \ddot{w}_x + \left[\frac{(I_y - I_x)(h^2 - 2TI_z) + (I_z - I_x)(h^2 - 2TI_y)}{I_x I_y I_z}\right] w_x + \left[\frac{2(I_z - I_x)(I_y - I_x) I_x}{I_x I_y I_z}\right] w_x^3 = 0.$$

[the same procedure (procedure*) might be repeated for each axis of the principal inertia reference frame $\Leftrightarrow$ for each initial Euler's equation)]

... Avoiding heavy algebra:

$$0 = \ddot{w}_x + \left[\frac{(I_y - I_x)(h^2 - 2TI_z) + (I_z - I_x)(h^2 - 2TI_y)}{I_x I_y I_z}\right] w_x + \left[\frac{2(I_z - I_x)(I_y - I_x) I_x}{I_x I_y I_z}\right] w_x^3$$

$$0 = \ddot{w}_y + \left[\frac{(I_z - I_y)(h^2 - 2TI_x) + (I_x - I_y)(h^2 - 2TI_z)}{I_x I_y I_z}\right] w_y + \left[\frac{2(I_z - I_y)(I_y - I_z) I_y}{I_x I_y I_z}\right] w_y^3$$

$$0 = \ddot{w}_z + \left[\frac{(I_x - I_z)(h^2 - 2TI_y) + (I_y - I_z)(h^2 - 2TI_x)}{I_x I_y I_z}\right] w_z + \left[\frac{2(I_x - I_z)(I_y - I_z)}{I_x I_y I_z}\right] w_z^3$$

↳ Each equation might be written in a compact form such as:

$$\ddot{W} + P\,W + Q\,W^3 = 0$$

(5) Integrating each equation: !! Integration is performed in W !!

$$\ddot{w} = \frac{d\dot{w}}{dt} = \frac{d\dot{w}}{dW}\,\frac{dW}{dt} = \dot{w}\,\frac{d\dot{w}}{dW}$$

← I'm guaranteed by Euler's equation of $\begin{cases} \dot{w} = f(w) \\ w = w(t) \end{cases}$

$$\dot{w}\,\frac{d\dot{w}}{dw} = -pW - qw^3$$

$$\int \dot{w}\, d\dot{w} = \int -pW - qw^3\, dw$$

$$\frac{\dot{w}^2}{2} + p\,\frac{w^2}{2} + q\,\frac{w^4}{4} = K_0 \rightarrow \dot{w}^2 + pw^2 + \frac{q}{2}w^4 = 2K_0 \triangleq K$$

Therefore the 3 equations achieved are:

$$\dot{w}_x^2 + P_x w_x^2 + \frac{Q_x}{2} w_x^4 = K_x$$
$$\dot{w}_y^2 + P_y w_y^2 + \frac{Q_y}{2} w_y^4 = K_y$$
$$\dot{w}_z^2 + P_z w_z^2 + \frac{Q_z}{2} w_z^4 = K_z.$$

" Each equation must be evaluated as a conic section in $(\dot{W}, W)$ space. ⇒ $\frac{\dot{w}_x^2}{K_x} + \frac{[P_x + \frac{Q_x}{2} W_x^2]\,W_x^2}{K_x} = 1$ "

$a^2 = K_x$; $b^2 = \frac{K_x}{P_x + \frac{Q_x}{2}W_x^2}$ ⇒ $\frac{\dot{X}^2}{a^2} + \frac{X^2}{b^2} = 1$

!! In evaluating the sign of $\begin{cases} a^2 \\ b^2 \end{cases}$ must be kept into account that $b^2 = b^2(w_x) = \frac{K_x}{P_x + \frac{Q_x}{2}W_x^2}$ ⇒ $\exists\, \tilde{w}_x$: $b^2(\tilde{w}_x) = 0$. !!

Hp: $I_z > I_y > I_x$ ⇒ $I_x < \frac{h^2}{2T} < I_z$

I_y ?? ← In order to satisfy intersection between: $2T$ ellipsoid ∩ h^2 ellipsoid.

$Q_x = 2(I_z - I_x)(I_y - I_x)\,I_x$ $\boxed{>0}$; $Q_y = 2(I_z - I_y)(I_y - I_x)$ $\boxed{<0}$; $Q_z = 2(I_x - I_z)(I_y - I_z)$ $\boxed{>0}$

$P_x = (I_y - I_x)(h^2 - 2TI_z) + (I_z - I_x)(h^2 - 2TI_y)\cdot\frac{2}{I_xI_yI_z}$ ⇒ $\boxed{??}$

>0 · <0 + >0 ??

$P_y = (I_z - I_y)\underbrace{(h^2 - 2TI_x)}_{=2T(h^2/2T - I_x) > 0} + \cancel{(\dots)} + (I_x - I_y)\underbrace{(h^2 - 2TI_z)}_{=2T(h^2/2T - I_z) < 0}$ ⇒ $\boxed{>0}$

>0 · + <0 ·

$P_z = (I_x - I_z)\underbrace{(h^2 - 2TI_y)}_{2T(h^2/2T - I_y)\ ??} + (I_y - I_z)\cdot\underbrace{(h^2 - 2TI_x)}_{= h^2/2T - I_x > 0}$ ⇒ $\boxed{??}$

<0 · ?? + <0 ·

$Q_x > 0$

P_x ?? → $P_x > 0$ ⟺ I_y: $\begin{cases} I_y < h^2/2T \quad \text{SUFFICIENT CONDITION.} \\ (I_y - I_x)\cdot(\frac{h^2}{2T} - I_z) + (I_z - I_x)(\frac{h^2}{2T} - I_y) > 0 \\ \text{modulus check ⇒ comparison between ⊕ quantities.} \\ \underbrace{(I_z - I_x)}_{>0}\cdot\underbrace{(h^2/2T - I_y)}_{>0} > \underbrace{(I_x - I_y)}_{<0}\underbrace{(h^2/2T - I_z)}_{<0} \end{cases}$

→ $P_x < 0$ IN ANY OTHER CONDITION

Working on this: $-(I_z - I_x)I_y + I_y(h^2/2T - I_z) > h^2/2T(I_y - I_z) + I_x\frac{h^2}{2T} - I_xI_z$.

$I_y\cdot[h^2/2T - I_z + I_z + I_x] > 2h^2/2T\, I_x - h^2/2T\, I_z - I_xI_z$.

$I_y[h^2/2T + I_x - 2I_z] > h^2/2T\,[2I_x - I_z] - I_xI_z$

$$I_y > \frac{h^2/2T\,[2I_x - I_z] - I_xI_z}{[h^2/2T + I_x - 2I_z]}$$

Such condition is already included in the I^{ST} one.

$Q_y < 0$

$P_y < 0$

$Q_z > 0$

$P_z > 0$ ⟺ $\begin{cases} I_y > h^2/2T \\ \dots\dots \text{ analogous consideration} \end{cases}$

Sign of this quantity will establish the shape of the conic.

$$\dot{w}_i + \underbrace{\left[P_i + \frac{Q_i}{2} w_i^2 \right]}_{} w_i^2 = k_i \qquad \| \qquad I_z > I_y > I_x.$$

($i \to$ generic inertia axis)

$(i \equiv x)$ $\quad P_x \gtrless 0$ (a)/(b) $\quad ; \quad Q_x > 0$

(a) $P_x > 0 \; ; \; Q_x > 0 \Rightarrow \forall\, w_x \quad P_x + \frac{Q_x}{2} w_x^2 > 0 \Rightarrow \boxed{\text{ELLIPS}}$

$(\Rightarrow I_y < h^2/2T)$

(b) $P_x < 0 \; ; \; Q_x > 0 \Rightarrow \exists\, \tilde{w}_x : P_x + \frac{Q_x}{2} \tilde{w}_x^2 = 0 \Rightarrow$ $\boxed{\text{HYPERBOLA } w_x < \tilde{w}_x \;\; \text{ELLIPSE } w_x > \tilde{w}_x}$

$(\Rightarrow I_y > h^2/2T)$

$\Downarrow$ WILL PRODUCE A "MIXED" CONIC NOT CENTERED IN THE ORIGIN.

$(i \equiv y)$ $\quad P_y > 0 \; ; \; Q_y < 0.$

$\Rightarrow \exists\, \tilde{w}_y : \quad P_y + \frac{Q_x}{2} \tilde{w}_y^2 = 0 \Rightarrow$ $\boxed{\text{ELLIPS } w_y < \tilde{w}_y \;\; \text{HYPERBOLA } w_y > \tilde{w}_y}$ $\Rightarrow$ SUCH CONIC IS CENTERED IN THE ORIGIN.

$(i \equiv z)$ $\quad P_z \gtrless 0$ (b)/(a) $\quad ; \quad Q_z > 0$

(a) $P_z < 0 \; ; \; Q_z > 0 \Rightarrow \exists\, \tilde{w}_z : P_z + \frac{Q_z}{2} \tilde{w}_z^2 = 0 \Rightarrow$ $\boxed{\text{HYPERBOLA } w_z < \tilde{w}_z \;\; \text{ELIPSE } w_z > \tilde{w}_z}$

$(\Rightarrow I_y < h^2/2T)$

$\Downarrow$ NOT CENTERED CONIC ("MIXED" CONIC)

(b) $P_z > 0 \; ; \; Q_z > 0 \Rightarrow \forall\, w_z \quad P_z + \frac{Q_z}{2} w_z^2 > 0 \Rightarrow \boxed{\text{ELLIPS}}$

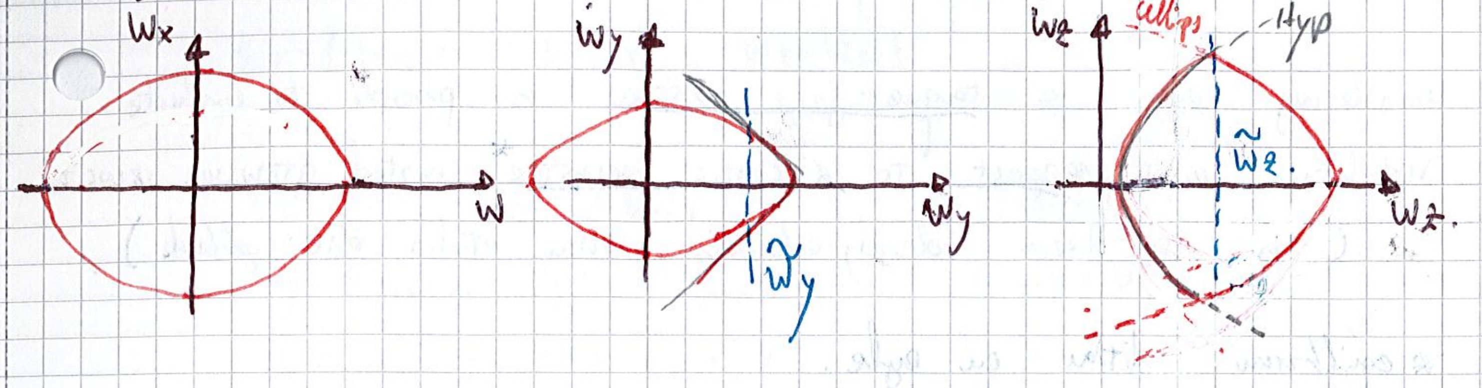
$(\Rightarrow I_y > h^2/2T)$

[IN EVERY CASE (DEPENDENTLY ON THE BODY AND ON $\{h^2 ; 2T\}$) the initial condition

$(\dot{w}_i \; ; \; w_i) \to$ ELLIPS $\qquad (\dot{w}_K \; ; \; w_K) \to$ HYP + ELL. NOT CENTERED

$(\dot{w}_J \; ; \; w_J) \to$ HYP + ELL CENTERED IN THE ORIGIN $\qquad$ HAVING: $I_i < I_J < I_K$.]

(a) $\to I_y < h^2/2T$. $\qquad (I_z > I_y > I_x)$



In (b) situation $(\dot{w}_x , w_x)$ is switched with $(\dot{w}_z ; w_z)$.

→ "Attitude stability: single spin satellite"

Considering again a torque-free motion is possible to evaluate stability with respect to kinematic parameter* insted of angular velocity w (that has been already achieved when studied the polhode).

* equilibrium condition on cycle.

(1) → individuate equilibrium condition

Euler eqn:
$$\begin{cases} I_x \dot w_x + (I_z - I_y)\, w_y w_z = 0 \\ I_y \dot w_y + (I_x - I_z)\, w_x w_z = 0 \\ I_z \dot w_z + (I_y - I_x)\, w_y w_x = 0 \end{cases}$$

Eq: $\dot{\underline{w}} = \underline{0}$. (system rotating around a constant axis).

$$\text{Eq} \begin{cases} w_y w_z = 0 \\ w_x w_z = 0 \\ w_x w_y = 0 \end{cases} \iff \exists\ w_i = w_j = 0 \ \cup\ w_k \neq 0$$

Equilibrium condition is verified if and only if is rotating around one of the principal inertia axis.

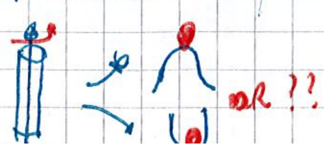
[Under inertia motion (torque free) equilibrium conditions are represented by rotations (with constant $\bar w_i$) around only one inertia axis]

Not an equilibrium condition ⟹ w is destinated to variate in time

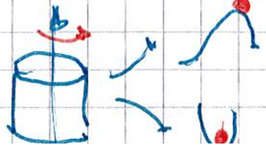
Equilibrium ⟹ if not perturbed will keep rotating around such axis indefinitely

We are eventually now asking around with principal axis a rotation is stable

Ideas



OR ??



(2) → consider linearized equation (linearized around equilibrium position)

Eq: $\{ w_x = 0 \ ;\ w_y = 0 \ ;\ w_z = \bar w_z \}$.

$$\begin{cases} w_x = w_x|_{eq} + \delta w_x = \delta w_x \\ w_y = w_y|_{eq} + \delta w_y = \delta w_y \\ w_z = w_z|_{eq} + \delta w_z = \bar w_z + \delta w_z \end{cases} \xrightarrow{\text{new notation}} \begin{cases} w'_x = w_x \\ w'_y = w_y \\ w'_z = \bar w_z + w_z \end{cases}$$

$$\Rightarrow \begin{cases} I_x \dot w_x + (I_z - I_y)\, w_y (\bar w_z + w_z) = 0 \\ I_y \dot w_y + (I_x - I_z)\, w_x (\bar w_z + w_z) \\ I_z \dot w_z + w_x w_y = 0 \end{cases}$$

Since linearization procedure consider small perturbations around equilibrium condition, then second order infinitesimal can be neglected:

$$\delta w_i \, \delta w_j = w_i w_j = 0 .$$

$$\begin{cases} I_x \dot w_x + (I_z - I_y)\, w_y \bar w_z + (I_z - I_y)\, \underbrace{w_y w_z}_{\to 0} = 0 \\ I_y \dot w_y + (I_x - I_y)\, \bar w_z w_x + (I_x - I_z)\, \underbrace{w_x w_z}_{\to 0} = 0 \\ I_z \dot w_z + (I_y - I_x)\, \underbrace{w_x w_y}_{\to 0} = 0 \end{cases}$$

→ Euler equations linearized.

$$\begin{cases} I_x \dot w_x + (I_z - I_y)\, \bar w_z w_y = 0 \\ I_y \dot w_y + (I_x - I_z)\, \bar w_z w_x = 0 \\ I_z \dot w_z = 0 \end{cases}$$

(Eq: $\{ w'_x = 0 ,\ w'_y = 0 ;\ w_z = \bar w_z \}$.)

(3) → pass from 2 I^{st} order to 1 II^{nd} order eqn.

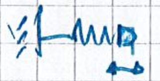
(as done for Euler's exact solution computation)

$$\begin{cases} \dot w_x + \dfrac{(I_z - I_y)}{I_x}\, \bar w_z \, w_y = 0 \\ \dot w_y + \dfrac{(I_x - I_z)}{I_y}\, \bar w_z \, w_x = 0 \end{cases}$$

$$\begin{cases} \dot{w}_x + \dfrac{(I_z - I_y)}{I_x}\,\bar{w}_z\, w_y = 0 \\ \dot{w}_y + \dfrac{(I_x - I_z)}{I_y}\,\bar{w}_z\, w_z = 0 \end{cases} \xrightarrow{d/dt} \begin{cases} \ddot{w}_x + \dfrac{(I_z - I_y)}{I_x}\,\bar{w}_z\, \dot{w}_y = 0 \\ \dot{w}_y = -\dfrac{(I_x - I_z)}{I_y}\,\bar{w}_z\, w_x \end{cases}$$

$$\Rightarrow \quad \ddot{w}_x + \overbrace{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y}\,\bar{w}_z^2}^{\text{constant value}}\, w_x = 0$$

"This is a II^{nd} order (undamped) system whose stability is guaranteed if poles are at least conjugate-complex"



$\Rightarrow \quad m\ddot{x} + Kx = 0 \quad \mathcal{L}$

$(m s^2 + K)\, X(s) = 0 \qquad s_{1,2} = \pm\sqrt{-K/m}$

$K>0 \Rightarrow s_{1,2} \in Im \Rightarrow x(t) = A \cdot e^{j\sqrt{K/m}\cdot t} + B\, e^{-j\sqrt{K/m}\cdot t}$

that means an oscillating (stable) motion.

$(A; B) \in \mathbb{C}$ and: response in real like a $\cos(\sqrt{K/m}\, t) + \sin(\sqrt{K/m}\, t)$.

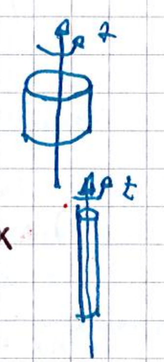
(P) $\quad \ddot{w}_x + \lambda^2 w_x = 0 \xrightarrow{\mathcal{L}} w_x(s^2 + \lambda^2) = 0$

$$\left(\lambda^2 = \frac{(I_z - I_y)(I_z - I_x)}{I_x I_y}\,\bar{w}_z^2 \right)$$

sol: $\quad w_x = a \cdot e^{\lambda t}$

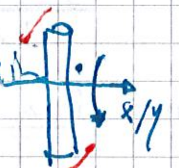
$\Rightarrow$ [SYSTEM IS STABLE IF AND ONLY IF $\lambda^2 > 0$

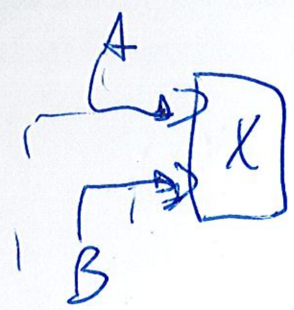
$\Rightarrow$ IF AND ONLY IF · (a) $I_z > I_y > I_x$



· (b) $I_z < I_y \cup I_z < I_x$]

The only unstable condition is verified when the rotation occurs around the inertia axis related to the intermediate inertia moment.





A · B

B..

Kubuntu: 14.10

ubuntu gnome remix: 14.10

Windows usb downloadable tool.