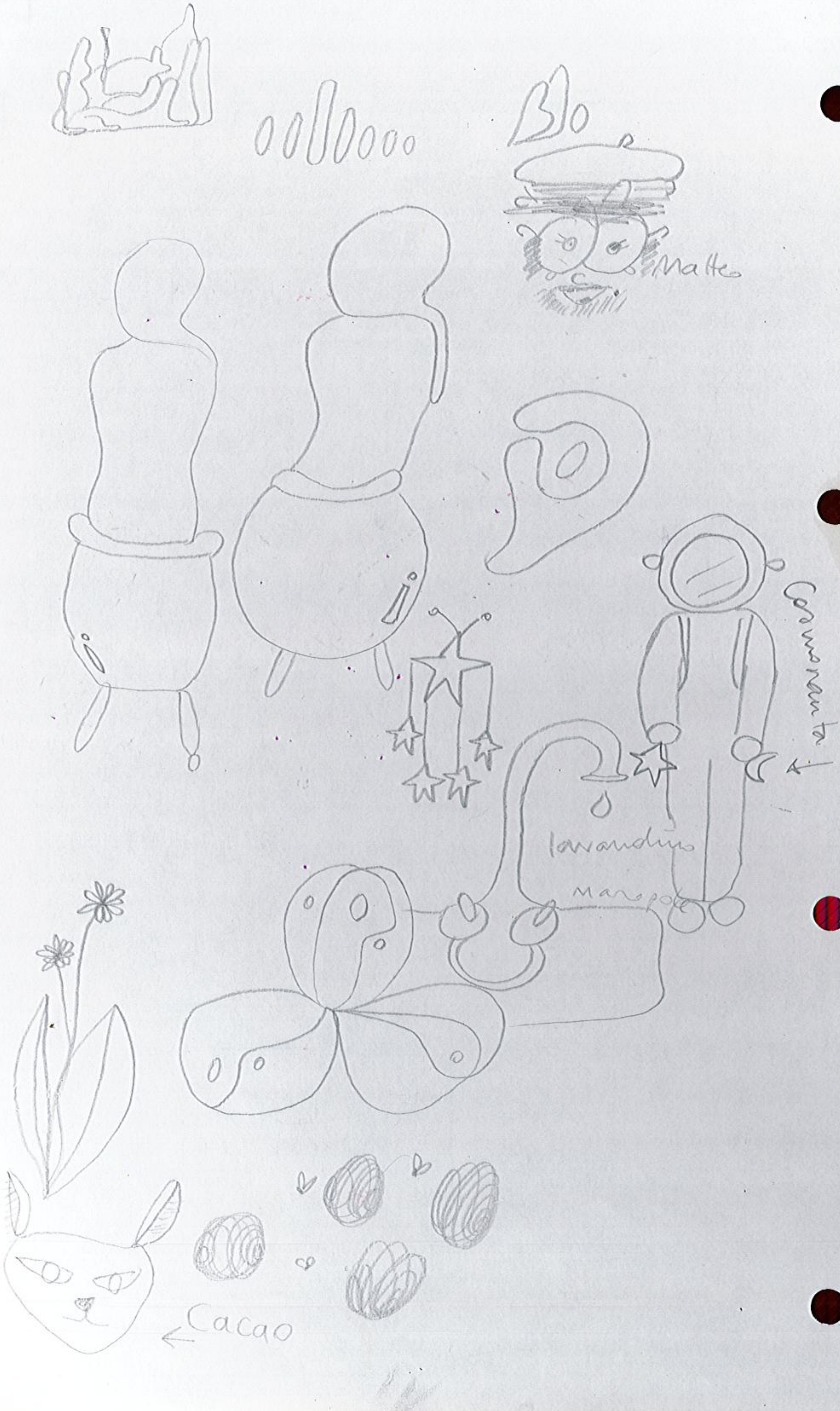


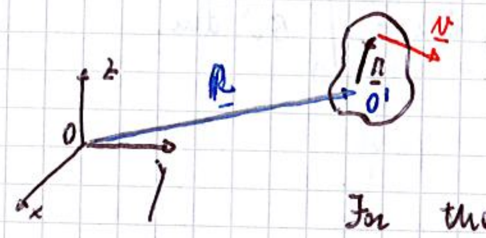
1) "Fundamental quantities"

The physical quantities fundamental to define the rigid-body motion of a spacecraft are:

1 -> Angular momentum

2 -> Kinetic energy

1 -> ANGULAR MOMENTUM



$\Rightarrow d\underline{h}_O = \underline{r} \times \underline{v}\, dm.$

For the whole rigid Body (B)

$$d\underline{h}_O = \underline{r} \times \underline{v}\, dm \Rightarrow \underline{h}_O = \int_B \underline{r} \times \underline{v}\, dm.$$

In a more general way the velocity of a body can be written as:

$$\underline{v} = \underline{v}_{O'} + \underline{w}_{O'} \times \underline{r}' + \underline{v}_{iB}$$

$\underline{v}_{O'}$ -> IS THE VELOCITY OF THE REFERENCE FRAME SET ON THE BODY

$\underline{w}_{O'}$ -> IS THE ANGULAR VELOCITY OF THE BODY
(ROTATION WITH RESPECT TO O')

$\underline{v}_{iB}$ -> IS THE RELATIVE VELOCITY (ONLY TRASLATION) BETWEEN ANY POINT OF THE BODY AND THE REFERENCE O'

$\Rightarrow$ FOR A RIGID-BODY $\underline{v}_{iB} = \underline{0}$

↳ Combining such expressions:

(i) $\underline{v} = \underline{v}_{O'} + \underline{w}_{O'} \times \underline{r}_{O'} + \underline{v}_{r,b}$ $\to = \underline{0}$;

(ii) $\underline{h}_O = \int_B \underline{r} \times \underline{v} \, dm$.

$\Rightarrow$ $\underline{h}_O = \int_B \underline{r} \times (\underline{v}_{O'} + \underline{w}_{O'} \times \underline{r}_{O'}) \, dm$

$\underline{v}_{O'} \Rightarrow$ independent from the body: is only the velocity of a generic point of the body.

$= -\underline{v}_{O'} \times \int_B \underline{r} \, dm + \int_B \underline{r} \times (\underline{w}_{O'} \times \underline{r}_{O'}) \, dm$.

obs: $\underline{r} = \underline{R} + \underline{r}_{O'} \Rightarrow \int_B \underline{r} \, dm = \int_B \underline{r}_{O'} + \underline{R} \, dm = \int_B \underline{r}_{O'} \, dm$.

$\Rightarrow$!! new notation!! $\begin{cases} \underline{r}_{O'} = \underline{r} \\ \underline{w}_{O'} = \underline{w} \end{cases}$

$\Rightarrow$ $\underline{h}_{O'} = -\underline{v}_{O'} \times \int_B \underline{r} \, dm + \int_B \underline{r} \times (\underline{w} \times \underline{r}) \, dm$.

Evaluating such integral:

$\underline{w} = \begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix}$; $\underline{r} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$.

$\Rightarrow$ $\int \underline{r} \, dm = \begin{Bmatrix} \int x \, dm \\ \int y \, dm \\ \int z \, dm \end{Bmatrix} = \begin{Bmatrix} S_x \\ S_y \\ S_z \end{Bmatrix} \triangleq \underline{S}_{O'}$

$\Rightarrow$ $\int \underline{r} \times (\underline{w} \times \underline{r}) \, dm = \underline{w} \cdot \int_B \|\underline{r}\|^2 \, dm - \int_B \underline{r} \cdot (\underline{w} \cdot \underline{r}) \, dm$.

$\ulcorner$ $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$ $\lrcorner$

Anyway:

$\underline{w} \times \underline{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix} = \hat{i}(w_y z - w_z y) + \hat{j}(w_z x - w_x \cdot z) + \hat{k}(w_x y - w_y x)$

$\underline{r} \times (\underline{w} \times \underline{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ w_y z - w_z \cdot y & w_z x - w_x \cdot z & w_x y - w_y x \end{vmatrix} =$

$= \begin{vmatrix} w_x y^2 - w_y xy - w_z xz + w_x \cdot z^2 \\ w_y x^2 - w_x yx + w_y z^2 - w_z zy \\ -w_y zy + w_z y^2 + w_z x^2 - w_x xz \end{vmatrix}$

That can be re-written as:

$$\underline{\underline{I}} \triangleq \begin{bmatrix} \int_B y^2 + z^2 \, dm ; & -\int_B xy \, dm & ; -\int_B xz \, dm \\ -\int_B yx \, dm ; & \int_B x^2 + z^2 \, dm ; & -\int_B yz \, dm \\ -\int_B xz \, dm ; & -\int_B zy \, dm & ; \int_B x^2 + y^2 \, dm \end{bmatrix}$$

$[\underline{\underline{I}}]_{ij} = -\int_B x_i \cdot x_j + [x_m^2 + x_k^2] \, \delta_{ij} \, dm$.

$\underline{\underline{I}}$: $\underline{\underline{I}}^T = \underline{\underline{I}}$

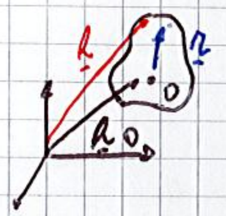
So:

$\underline{h}_{O'} = -\underline{v}_O \times \int \underline{r} \, dm + \int_B \underline{r} \times (\underline{w} \times \underline{r}) \, dm$

$= -\underline{v}_O \times \underline{S}_O + \underline{\underline{I}} \, \underline{w}$.

$\forall\ O \in B \quad ; \quad \underline{M}_O = \int_B \underline{R} \times \underline{N}\, dm = \int_B \underline{r} \times \underline{N}_O\, dm.$

$\underline{r} = \underline{R}(x,y,z) - \underline{R}_O$



$\Downarrow$

$$\underline{h}_O = -\underline{N}_O \times \underline{S}_O + \underline{\underline{I}}\, \underline{W}$$

Where:

$$\underline{S}_O = \begin{vmatrix} \int_B x\, dm \\ \int_B y\, dm \\ \int_B z\, dm \end{vmatrix} = \begin{vmatrix} \int_B (x-x_0)\, dm \\ \int_B (y-y_0)\, dm \\ \int_B (z-z_0)\, dm \end{vmatrix}$$

$$\underline{\underline{I}} = \begin{bmatrix} \int_B z^2+y^2\, dm & ; -\int_B xy\, dm & ; -\int_B xz\, dm \\ -\int_B xy\, dm & ; \int_B x^2+z^2\, dm & ; -\int yz\, dm \\ -\int_B xz\, dm & ; -\int_B yz\, dm & ; \int_B y^2+z^2\, dm \end{bmatrix}$$

The properties of inertia matrix are particularly relevant.

↳ inequalities of inertia matrix:

(i) $I_{xx} + I_{yy} = \int_B z^2+y^2+z^2+x^2\, dm = \int_B x^2+y^2+2z^2$

$I_{zz} = \int_B x^2+y^2\, dm$ $\Rightarrow$ (i) $I_{zz} \le I_{xx} + I_{yy}$.

(ii) $I_{xx} - I_{yy} = \int_B x^2 + \cancel{z^2} - y^2 - \cancel{z^2}\, dm = \int_B x^2 - y^2\, dm.$

$\Rightarrow$ (ii) $I_{zz} \ge I_{xx} - I_{yy}$

" This represent a set of 6 inequations:

$I_{kk} \le I_{ii} + I_{jj}$ (3 inequations)

$I_{kk} \ge I_{jj} - I_{ii}$ (3 inequations)

$\Rightarrow$ 6 triangular inequalities. "

(iii) $I_{xx} = \int_B (z^2+y^2)\, dm = \int_B (z+y)^2 - 2zy\, dm \triangleq A + 2\underbrace{\int_B (-zy)\, dm}_{I_{zy}}$

I_{zy} can be positive or negative.

But: $A \triangleq \int_B (x+y)^2\, dm \ge 0$

$I_{xx} \ge 0$

$A\,(\ge 0) = I_{xx} - 2 I_{zy}$ $\Rightarrow$ (iii) $I_{xx} \ge 2\int_B xy\, dm.$

↳ Triangular inequalities of inertia matrix.

(i) $I_{xx} \le I_{yy} + I_{zz}$ (3 ineq ; $I_{ii} = \int_B x_i^2\, dm$)

(ii) $I_{xx} \ge I_{yy} - I_{zz}$ (3 ineq) 6

(iii) $I_{xx} \ge 2 I_{zy}$ (3 ineq ; $I_{ij} = -\int x_i x_j\, dm$.)

2-> KINETIC ENERGY

$$dT = \frac{1}{2} v^2 \, dm = \frac{1}{2} (dm) \, \underline{v} \cdot \underline{v}.$$

for R.B. motion
$\underline{v} = \underline{v}_0 + \underline{w} \times \underline{r}$

$$\Rightarrow 2T = \int_B \underline{v} \cdot \underline{v} \, dm = \int_B [\underline{v}_0 + \underline{w} \times \underline{r}] \cdot [\underline{v}_0 + \underline{w} \times \underline{r}] \, dm$$

$$= \int_B v_0^2 + 2 \underline{v}_0 \cdot [\underline{w} \times \underline{r}] + [\underline{w} \times \underline{r}] \cdot [\underline{w} \times \underline{r}] \, dm$$

$$= M_B \cdot v_0^2 + 2 \underline{v}_0 \int_B [\underline{w} \times \underline{r}] \, dm + \int_B \| \underline{w} \times \underline{r} \|^2 \, dm.$$

Some terms (contribution to T) are easily distinguishable:

$$T_{trasl} = \frac{1}{2} M_B \| \underline{v}_0 \|^2$$

$$T_{ROT} = \underline{v}_0 \int_B [\underline{w} \times \underline{r}] \, dm + \frac{1}{2} \int_B \| \underline{w} \times \underline{r} \|^2 \, dm.$$

$$= \underline{v}_0 \cdot [\underline{w} \times \underbrace{\int_B \underline{r} \, dm}_{S_0}] + \underbrace{\frac{1}{2} \int_B \| \underline{w} \times \underline{r} \|^2 \, dm.}_{\text{2nd contribution to rotational kinetic energy}}$$

$$\Rightarrow \int_B \| \underline{w} \times \underline{r} \|^2 \, dm = \int_B (\underline{w} \times \underline{r}) \cdot (\underline{w} \times \underline{r}) \, dm.$$

$$= \int_B \begin{vmatrix} w_y z - w_z y \\ w_z x - w_x z \\ w_x y - w_y x \end{vmatrix}^T \begin{vmatrix} w_y z - w_z y \\ w_z x - w_x z \\ w_x y - w_y x \end{vmatrix} dm.$$

$$\int_B \| \underline{w} \times \underline{r} \|^2 dm = \int_B (w_y z - w_z y)^2 + (w_z x - w_y z)^2 + (w_x y - w_y x)^2 \, dm.$$

$$= \int_B [w_y^2 z^2 + w_z^2 y^2 + w_z^2 x^2 + w_y^2 z^2 + w_x^2 y^2 + w_y^2 x^2 -$$

$$- 2 w_z w_y zy - 2 w_z w_y xz - 2 w_x w_y xy] \, dm.$$

$$\Rightarrow \int_B \| \underline{w} \times \underline{r} \|^2 dm = \int_B [w_x^2 \overbrace{(y^2+z^2)}^{I_{xx}} + w_z^2 \overset{I_{zz}}{(x^2+y^2)} + w_y^2 (z^2+x^2) +$$

$$+ 2 w_z w_y \underbrace{(-z y)}_{I_{zy}} + 2 w_z w_y \underset{I_{zx}}{(-xz)} + 2 w_x w_y \underset{I_{xy}}{(-xy)}] \, dm$$

That can be re-written in form:

$$\int_B \| \underline{w} \times \underline{r} \|^2 dm = \int_B [I_{xx} w_x^2 + I_{yy} w_y^2 + I_{zz} w_z^2 + 2 w_z w_y I_{zy} +$$

$$+ 2 w_x w_y I_{xy} + 2 w_z w_x I_{zx}] \, dm.$$

$$= \underline{w}^T \cdot (\underline{\underline{I}} \, \underline{w})$$

in fact:

$$\left[\underline{\underline{I}} \right] \cdot \underline{w} = \begin{vmatrix} I_{xx} w_x + I_{xy} w_y + I_{xz} w_z \\ I_{yy} w_y + I_{xy} w_x + I_{yz} w_z \\ I_{zx} w_x + I_{zy} w_y + I_{zz} w_z \end{vmatrix}$$

$$\Rightarrow \underline{w}^T \cdot (\underline{\underline{I}} \cdot \underline{w}) = I_{xx} w_x^2 + I_{xy} w_y w_x + I_{xz} w_z w_x + I_{yy} w_y^2 + I_{xy} w_x w_y +$$

$$+ I_{yz} w_y w_z + I_{zx} w_x w_z + I_{zy} w_y w_z + I_{zz} w_z^2.$$

$$\Rightarrow \quad T = \frac{1}{2}\left[\int_B \underline{v}_O\cdot\underline{v}_O\, dm + 2\int_B \underline{v}_O\cdot[\underline{w}\times\underline{r}]\,dm + \int_B \|\underline{w}\times\underline{r}\|^2\, dm\right].$$

$$\begin{cases} T = \frac{1}{2}\int_B \underline{v}\cdot\underline{v}\,dm \\ \underline{v} = \underline{v}_O + \underline{w}\times\underline{r}\end{cases}$$

$$\int_B \underline{v}_O\cdot\underline{v}_O\, dm = M_B\|\underline{v}_O\|^2$$

$$\int_B \underline{v}_O\cdot[\underline{w}\times\underline{r}]\,dm = \underline{v}_O\cdot[\underline{w}\times \overbrace{\int_B \underline{r}\, dm}^{S_O}]$$

$$\int_B \|\underline{w}\times\underline{r}\|^2 dm = \int_B (\underline{w}\times\underline{r})\cdot(\underline{w}\times\underline{r})\, dm.$$

$$T = \frac{1}{2} M_B \|\underline{v}_O\|^2 + \underline{v}_O\cdot[\underline{w}\times\underline{S}_O] + \frac{1}{2}\underline{w}^T\cdot \underline{\underline{I}}_O\,\underline{w}$$

Where $\underline{S}_O = \int_B \begin{Bmatrix}x\\y\\z\end{Bmatrix} dm$ and $\underline{\underline{I}} = \begin{bmatrix}\int_B z^2+y^2 & -\int xy & -\int xz\\ sym & \int z^2+x^2 & -\int yz\\ & & \int x^2+y^2\end{bmatrix}$

→ Setting the reference frame (body-reference frame) in the center of mass of the body: $\Rightarrow \boxed{O' \equiv G}$

So: $$\underline{S}_O = \begin{Bmatrix}\int_B x\, dm\\ \int_B y\, dm\\ \int_B z\, dm\end{Bmatrix} = \underline{0}$$

(Such condition is not guaranteed for the extra-diagonal terms of $\underline{\underline{I}}_{O\equiv G}$ matrix)

Under such choice of the reference frame:

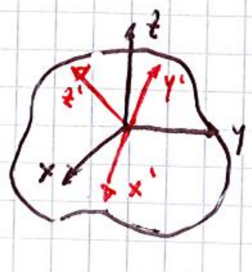
$$\bullet \begin{cases} T = T^{tral} + T^{rot} = \frac{1}{2} M_B\|\underline{v}_G\|^2 \\ T^{rot} = \underline{v}_O\cdot(\overbrace{\underline{w}\times\underline{S}_O}^{=0}) + \frac{1}{2}\underline{w}^T\underline{\underline{I}}_G\,\underline{w} \end{cases} \Rightarrow \quad T^{rot} = \frac{1}{2}\underline{w}^T\underline{\underline{I}}\,\underline{w}$$

$$\bullet \quad \underline{h}_{O\equiv G} = -\underline{v}_G\times \overbrace{\underline{S}_{O\equiv G}}^{=0} + \underline{\underline{I}}_G\cdot\underline{w} \Rightarrow \quad \underline{h} = \underline{\underline{I}}\,\underline{w}$$

(If is not specified all quantities are defined with respect to the center of the mass of the rigid-body.)

Fundamental inertia frame.

$\exists!\ \{\hat{i}', \hat{j}', \hat{k}'\}$: $\underline{\underline{I}}_{\{i', j', k'\}}$ is a diagonal matrix.



To evaluate the rotation of the fundamental inertia frame with respect to the original reference rotation cosine matrix must be evaluated.

$$\underline{\underline{A}} = \begin{bmatrix} \cos\theta\cos\psi & \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi \\ -\cos\theta\sin\psi & \cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi & \sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ \sin\theta & -\sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

Rotation cosine matrix is the matrix that allows to transform a vector (3D) from a reference to an other rotated by in the space (rotation is defined by the 3 angles (ϑ, ϕ, ψ))

$$\hat{k}_1 \quad \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_1} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix} \rightarrow \text{rotation around } \hat{k}$$

$$\begin{Bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{Bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{Bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{Bmatrix} \rightarrow \text{rotation around } \hat{j}_1$$

$$\begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = \begin{Bmatrix} \hat{i}_3 \\ \hat{j}_3 \\ \hat{k}_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \begin{Bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{Bmatrix} \rightarrow \text{rotation around } \hat{i}_2$$

$$\Rightarrow \begin{Bmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{Bmatrix} = \underline{\underline{A_3}} \cdot \underline{\underline{A_2}} \cdot \underline{\underline{A_1}} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

Rotation cosine matrix: $\underline{\underline{A}} = \begin{bmatrix} \cos\vartheta\cos\phi\,; & \cos\psi\sin\vartheta + \sin\psi\cos\vartheta\sin\phi \dots \\ \dots & \\ & \end{bmatrix}$

$\underline{\underline{A}}: \quad \underline{\underline{A}}^{-1} = \underline{\underline{A}}^T$ (orthogonal matrix)

→ useful property to write cross-product

$$\underline{W} \times \underline{r} = [\underline{W}\times]\,\underline{r} = \begin{bmatrix} 0 & -W_z & -W_y \\ -W_z & 0 & -W_x \\ -W_y & -W_x & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

So, if i want to express $\underline{W}$ in this new reference frame:

$$\forall\, \underline{a}^{W'} \Rightarrow \quad \underline{a} = \underline{\underline{A}} \cdot \underline{a}'$$

old reference frame — fundamental reference frame.

$\Rightarrow \underline{W} = \underline{\underline{A}} \cdot \underline{W}'$

Where $\underline{\underline{A}}$ is such that $\forall\, \underline{a} \rightarrow \underline{a} = \underline{\underline{A}}\,\underline{a}'$ $\quad \underline{a} \rightarrow$ old R.F.

$\underline{a}' \rightarrow$ fundamental R.F.

Going into discovering what happens to kinetic energy:

$$\begin{cases} \underline{W} = \underline{\underline{A}} \cdot \underline{W}' \\ 2T^{rot} = \underline{W}^T \underline{\underline{I}}\, \underline{W} = \underline{W}'^T \underline{\underline{I}}'\, \underline{W}' \end{cases}$$

"Since energy is a scalar must be the same if expressed in the original R.F. or in the fundamental R.F."

(supposed to be diagonal.)

$$\Rightarrow \quad 2T^{rot} = (\underline{\underline{A}}\,\underline{W})^T \underline{\underline{I}}\, \underline{W} = \underline{W}'^T \underline{\underline{I}}'\, \underline{W}'$$

$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$

$$\underline{W}^T \underline{\underline{A}}^T \underline{\underline{I}}\, \underline{\underline{A}}\, \underline{W} = \underline{W}'^T \underline{\underline{I}}'\, \underline{W}'$$

⇒ thanks to orthogonality property of rotation cosine matrix:

$$\underline{\underline{A}}^T \underline{\underline{I}}\, \underline{\underline{A}} = \underline{\underline{I}}' \Rightarrow \underbrace{\underline{\underline{A}}\, \underline{\underline{A}}^T}_{= \underline{\underline{Id}}} \underline{\underline{I}}\, \underline{\underline{A}} = \underline{\underline{A}}\, \underline{\underline{I}}'$$

$$\underline{\underline{I}}\, \underline{\underline{A}} = \underline{\underline{A}}\, \underline{\underline{I}}'$$

Now the problem is about how to find the elements of matrix $\underline{\underline{A}}$ such that $\underline{\underline{I}}'$ is a diagonal matrix knowing the relation between the 2 inertial moment tensors $\underline{\underline{I}}$ and $\underline{\underline{I}}'$ ($\underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}} \cdot \underline{\underline{I}}' \Rightarrow$?? $\underline{\underline{A}}$?? : $\underline{\underline{I}}'$ is DIAGONAL)

So: $\underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{I}}'$

$$\underline{\underline{A}} = [\,\underline{a}_1 \,;\, \underline{a}_2 \,;\, \underline{a}_3\,]$$

⇒ for $i=1,2,3$: $\underline{\underline{I}} \cdot \underline{a}_i = \lambda_i \, \underline{a}_i$.

In fact I'm supposing that $\underline{\underline{I}}'$ is a diagonal matrix ⇒

$$\begin{bmatrix} \underline{I}_1 \\ \underline{I}_2 \\ \underline{I}_3 \end{bmatrix} \cdot \left[\, \underline{a}_1 \mid \underline{a}_2 \mid \underline{a}_3 \,\right] = \begin{bmatrix} a_{11} & \dots & a_{13} \\ & & \\ a_{31} & \dots & a_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\left[\, \underline{\underline{I}}\,\underline{a}_1 \mid \underline{\underline{I}}\,\underline{a}_2 \mid \underline{\underline{I}}\,\underline{a}_3 \,\right] = \begin{bmatrix} a_{11}\lambda_1 & a_{12}\lambda_2 & \vdots \\ a_{21}\lambda_1 & a_{22}\lambda_2 & \vdots \\ a_{31}\lambda_1 & a_{32}\lambda_2 & \vdots \end{bmatrix}$$

[FOR EACH COLUMN OF MATRIX $\underline{\underline{A}}$ WE HAVE TO SOLVE AN EIGEN-VALUES / EIGEN VALUES PROBLEM WHERE EACH OF THREE EIGEIN-VALUES FOUND WILL BE ONE OF THE ELEMENTS OF $\underline{\underline{I}}'$, AND EACH EIGEN-VECTOR FOUND WILL BE A COLOUMN OF MATRIX $\underline{\underline{A}}$]

$i = 1, 2, 3$ $\quad \underline{\underline{I}} \cdot \underline{a}_i = \lambda_i \, \underline{a}_i$

$$\Rightarrow \underline{\underline{I}}' = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} = \begin{bmatrix} I'_{x'} & & \\ & I'_{y'} & \\ & & I'_{z'} \end{bmatrix}$$

$$\underline{\underline{A}} = [\,\underline{a}_1 \mid \underline{a}_2 \mid \underline{a}_3\,]$$

Where $\underline{\underline{A}}$:

$\hat{i}' = \underline{\underline{A}}\,\hat{i}$; $\{\hat{i}', \hat{j}', \hat{k}'\}$ → foundamental inertial axis.

$\hat{j}' = \underline{\underline{A}}\,\hat{j}$ (expressed as functions of the original RF)

$\hat{k}' = \underline{\underline{A}}\,\hat{k}$

obs → the foundamental inertial frame (⇒ $\underline{\underline{A}}$) are only a "function" of → geometry of the body
→ mass distribution inside the body

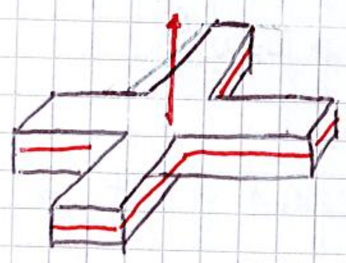
→ principal inertial reference frame:

a) IF a body has a symmetry axis AND it's homogeneous. THEN one of the principal inertia axis will lay on such axis.

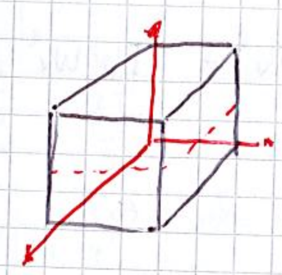
b) IF a body has a symmetry plane AND it's homogeneous

THEN 2 of the principal inertia axis will be contained in such plane.

AND the third will be orthogonal to such plane passing through the center of mass.



≃ airplane.



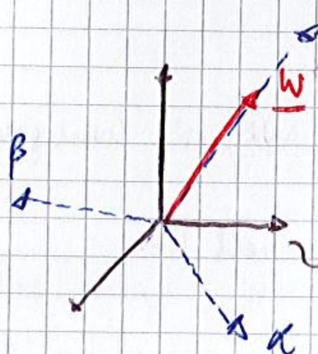
≃ satellite.

⇒ load distribution inside a satellite must be closer as possible to maintain the center of mass in the center of the satellite.

→ new notation: $\begin{cases}\{x';\ y';\ z'\}\\ \{I'_{x'};\ I'_{y'};\ I'_{z'}\}\end{cases} \rightarrow \begin{cases}\{x,\ y,\ z\}\\ \{I_x;\ I_y;\ I_z\}\end{cases}$

!! STARTING FROM NOW: I suppose of being always in the Principal inertia reference frame. !!

↳ Anyway since kinetic energy is a scalar I can imagine of choosing a R.F. such that one of the axis is perfectly coincident with the angular velocity of the body:



$\Rightarrow \{\alpha;\ \beta,\ \gamma\}:\ \begin{cases} W_\gamma = \|\underline{W}\| \\ W_\alpha = 0 \\ W_\beta = 0 \end{cases}$.

$$\Rightarrow\ 2T = I_{\gamma\gamma} W^2 = I_x W_x^2 + I_y W_y^2 + I_z W_z^2.$$

I have to consider that $\{\gamma, \beta, \alpha\}$ is not a fundamental inertial frame as $I_{\gamma,\beta} = \int_B x_\beta\, y_\beta\, dm \neq 0$

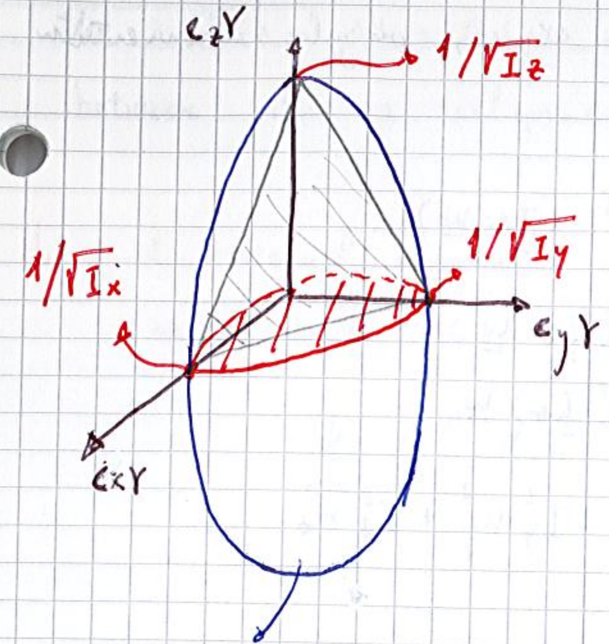
but The only contribution given to T comes from $I_{\gamma\gamma}$

$c_x \triangleq \dfrac{W_x}{W}$

$$I_x W_x^2 + I_y W_y^2 + I_z W_z^2 = I_{\gamma\gamma} W^2$$

$$\frac{I_x}{I_{\gamma\gamma}} c_x^2 + \frac{I_y}{I_{\gamma\gamma}} c_y^2 + \frac{I_z}{I_{\gamma\gamma}} c_z^2 = 1.$$

This is an ellipsoid (since all coefficients are positive) in $\{c_x\ c_y\ c_z\}$ space



(Section of an ellipsoid is an ellips, not a circumference)

Kinetic energy ellipsoid.

$$\rightarrow\ \frac{c_{x\gamma}^2}{\frac{I_{\gamma\gamma}}{I_x}} + \frac{c_{y\gamma}^2}{\frac{I_{\gamma\gamma}}{I_y}} + \frac{c_{z\gamma}^2}{\frac{I_{\gamma\gamma}}{I_z}} = 1$$

the surface described by kinetic energy ellipsoid represent all the possible (acceptable values) of $\{c_x;\ c_y;\ c_z\}$ describing kinetic energy conservation.

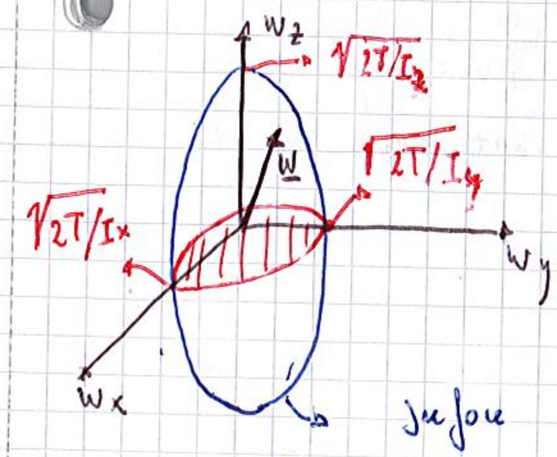
Energy conservation might be even more explicit if explicited as:

↳ (direct from w-dependence)

$$2T^{cot} = I_x w_x^2 + I_y w_y^2 + I_z w_z^2$$

$$\rightarrow\ \frac{w_x^2}{\frac{2T}{I_x}} + \frac{w_y^2}{\frac{2T}{I_y}} + \frac{w_z^2}{\frac{2T}{I_z}} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



surface representing all the possible values of $\{w_x;\ w_y;\ w_z\}$ able to express energy conservation.

In the same way i might express angular momentum conservation (as square modulus):

$$\underline{h}_G = \underline{\underline{I}}_G \underline{w} \rightarrow \| \underline{h}_G \|^2 = (\underline{\underline{I}}_G \underline{w})^T \cdot (\underline{\underline{I}}_G \underline{w})$$

diagonal! acting in principal inertia R.F.

$$= \underline{w}^T \underline{\underline{I}}_G^T \underline{\underline{I}}_G \underline{w}$$

$$= \underline{w}^T (\underline{\underline{I}}_G^T \underline{\underline{I}}_G) \underline{w}$$

$$= I_x^2 w_x^2 + I_y^2 w_y^2 + I_z^2 w_z^2$$

The ellipsoids i obtain is:

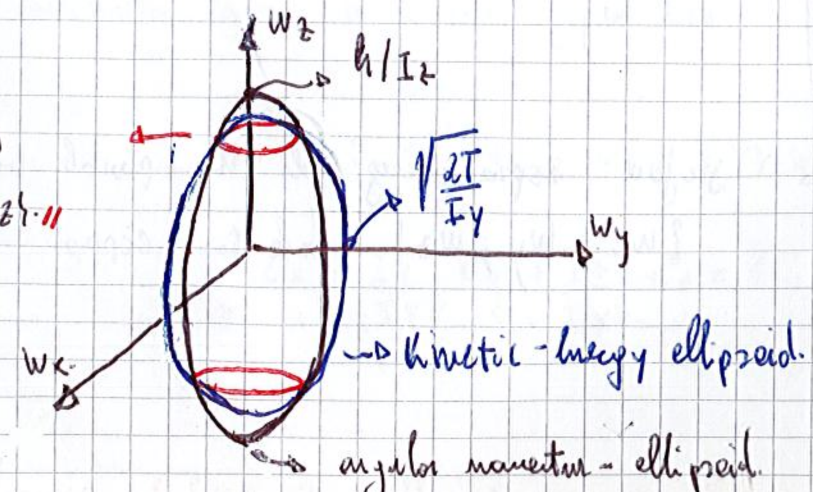
$$* \rightarrow \frac{w_x^2}{\frac{h^2}{I_x^2}} + \frac{w_y^2}{\frac{h^2}{I_y^2}} + \frac{w_z^2}{\frac{h^2}{I_z^2}} = 1 \quad \text{(ANGULAR MOMENTUM ELLIPSOID)}$$

$$* \rightarrow \frac{w_x^2}{\frac{2T}{I_x}} + \frac{w_y^2}{\frac{2T}{I_y}} + \frac{w_z^2}{\frac{2T}{I_z}} = 1 \quad \text{(KINETIC ENERGY ELLIPSOID)}$$

SINCE the first ellipsoid express $\{w_x; w_y; w_z\}$ able to conserve angular momentum

AND SINCE the 2^{nd} ellipsoid express $\{w_x; w_y; w_z\}$ able to conserve kinetic energy.

THEN the only acceptable values for $\{w_x; w_y; w_z\}$ will lays on the interception between the 2 ellipsoids



The (curve) line $w_z = \Omega(w_y, w_x)$ representing the interception between the 2 ellipsoids is called: $\boxed{\text{Polhode}}$.

obs. → An external moment acting on the R.B will make rotate the $\underline{w}$ axis according with:

$$\frac{d\underline{h}_O}{dt} = \sum_i \underline{M}_{i,O}^{ext}$$

ok modifica sia T che $\|\underline{h}_O\|$ però muterà anche la direzione di $\underline{w}$ che si sposterà nel "nuovo" poloide muovendo rispetto alla precedente inclinazione.

obs. → Initial conditions are always retrieved after a certain period.

⇒ Computing the polhode:

$$\underbrace{\frac{w_x^2}{\frac{h^2}{I_x^2}} + \frac{w_y^2}{\frac{h^2}{I_y^2}} + \frac{w_z^2}{\frac{h^2}{I_z^2}}}_{\text{inertia moment ellipsoid}} = \underbrace{\frac{w_x^2}{\frac{2T}{I_x}} + \frac{w_y^2}{\frac{2T}{I_y}} + \frac{w_z^2}{\frac{2T}{I_z}}}_{\text{kinetic energy ellipsoid}}$$

$$I_x \frac{w_x^2}{\frac{h^2}{I_x}} + \frac{w_y^2}{\frac{h^2}{I_y}} \cdot I_y + I_z \frac{w_z^2}{\frac{h^2}{I_z}} = \frac{w_x^2}{\frac{2T}{I_x}} + \frac{w_y^2}{\frac{2T}{I_y}} + \frac{w_z^2}{\frac{2T}{I_z}}$$

So: $I_x w_x^2 \cdot \left(\frac{I_x}{h^2} - \frac{1}{2T} \right) + I_y w_y^2 \left(\frac{I_y}{h^2} - \frac{1}{2T} \right) + I_z w_z^2 \left(\frac{I_z}{h^2} - \frac{1}{2T} \right) = 0.$

↳ We have three equations describing the shape of the polhode:

(i) $I_x w_x^2 \cdot \left(\frac{I_x}{h^2} - \frac{1}{2T} \right) + I_y w_y^2 \left(\frac{I_y}{h^2} - \frac{1}{2T} \right) + I_z w_z^2 \cdot \left(\frac{I_z}{h^2} - \frac{1}{2T} \right) = 0$ → interception.

(ii) $I_x w_x^2 + I_y w_y^2 + I_z w_z^2 = 2T^{tot}$ → T conservation

(iii) $I_x^2 w_x^2 + I_y^2 w_y^2 + I_z^2 w_z^2 = h_O^2$ → h conservation

(i) $W_x^2 \cdot [I_x (\frac{I_x}{h^2} - \frac{1}{2T})] + W_y^2 [I_y (\frac{I_y}{h^2} - \frac{1}{2T})] + W_z^2 [I_z (\frac{I_z}{h^2} - \frac{1}{2T})] = 0.$

!! In order to satisfy this equation:

$$\begin{cases} \left\{ (\frac{I_x}{h^2} - \frac{1}{2T}); (\frac{I_y}{h^2} - \frac{1}{2T}) \right\} > 0 \\ \left\{ (\frac{I_z}{h^2} - \frac{1}{2T}) \right\} < 0 \end{cases}$$

(one term must have different sign from the other 2)

multiply $\cdot h^2$

$$\begin{cases} I_{i,j} - \frac{h^2}{2T} > 0 \\ I_k - \frac{h^2}{2T} < 0 \end{cases} \quad \text{or} \quad \begin{cases} I_i - \frac{h^2}{2T} > 0 \\ I_{j,k} - \frac{h^2}{2T} < 0 \end{cases} \quad !!$$

[2 TERMS MUST HAVE DIFFERENT SIGN FROM THE THIRD]

ASSUMING: $I_x > I_y > I_z$

IN ORDER TO SATISFY (i) MUST BE THAT:

$$I_x - \frac{h^2}{2T} > 0 \quad \& \quad I_{y,z} - \frac{h^2}{2T} < 0 \Rightarrow \begin{cases} I_z < h^2/2T \\ I_y < h^2/2T \end{cases}$$

(i) $[I_x (I_x - \frac{h^2}{2T})] W_x^2 + [I_y (I_y - \frac{h^2}{2T})] W_y^2 + [I_z (I_z - \frac{h^2}{2T})] W_z^2 = 0$

FINAL CONDITION IS: $I_x > \frac{h^2}{2T} > I_{y,z}$.

EXTRAPOLATING W_z^2 FROM ENERGY CONSERVATION:

$$2T = I_x W_x^2 + W_y^2 I_y + W_z^2 I_z \Rightarrow I_z W_z^2 = 2T - I_x W_x^2 - I_y W_y^2.$$

SUBSTITUTING INTO POLHODE EQUATION (equ (i)) I CAN OBTAIN THE PROJECTION ON $(W_x; W_y)$ PLANE.

$$\begin{cases} I_z W_z^2 = 2T - I_x W_x^2 - I_y W_y^2 \\ -I_z W_z^2 [\frac{I_z}{h^2} - \frac{1}{2T}] = [\frac{I_x}{h^2} - \frac{1}{2T}] I_x W_x^2 + [\frac{I_y}{h^2} - \frac{1}{2T}] I_y W_y^2 \end{cases}$$

$$\Rightarrow -2T [\frac{I_z}{h^2} - \frac{1}{2T}] + I_x W_x^2 [\frac{I_z}{h^2} - \frac{1}{2T}] + I_y W_y^2 [\frac{I_z}{h^2} - \frac{1}{2T}] = $$

$$= [\frac{I_x}{h^2} - \frac{1}{2T}] I_x W_x^2 + [\frac{I_y}{h^2} - \frac{1}{2T}] I_y W_y^2.$$

$$-2T [\frac{I_z}{h^2} - \frac{1}{2T}] = [\frac{I_x}{h^2} - \cancel{\frac{1}{2T}} - \frac{I_z}{h^2} + \cancel{\frac{1}{2T}}] I_x W_x^2 + [\frac{I_y}{h^2} - \cancel{\frac{1}{2T}} - \frac{I_z}{h^2} + \cancel{\frac{1}{2T}}] I_y W_y^2$$

$$(\frac{I_x - I_z}{h^2}) I_x W_x^2 + (\frac{I_y - I_z}{h^2}) I_y W_y^2 + 2T (\frac{I_z}{h^2} - \frac{1}{2T}) = 0$$

$$2T \frac{I_z}{h^2} - 1 = \frac{2T I_z - h^2}{h^2}$$

$$(I_x - I_z) I_x W_x^2 + (I_y - I_z) I_y W_y^2 + (2T I_z - h^2) = 0$$

$$2T I_z - h^2 = -(h^2 - 2T I_z)$$

$$\left[\frac{I_x - I_z}{h^2 - 2T I_z}\right] I_x W_x^2 + \left[\frac{I_y - I_z}{h^2 - 2T I_z}\right] I_y W_y^2 = 1$$

All the curves are in form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$(a^2 = b^2) > 0 \rightarrow$ CIRCUMFERENCE

$(a \neq b) > 0 \rightarrow$ ELLIPS.

$a > 0; b < 0 \rightarrow$ HYPERBOLA.

$$\frac{W_x^2}{\underbrace{I_x \left[\frac{I_x - I_z}{h^2 - 2T I_z}\right]}_{a}} + \frac{W_y^2}{\underbrace{I_y \left[\frac{I_y - I_z}{h^2 - 2T I_z}\right]}_{b}} = 1$$

To determine the wich class of conics the projection of the polhode, we need to work on equation:

$$\left[\frac{I_x - I_z}{h^2 - 2TI_z}\right] I_x W_x^2 + \left[\frac{I_y - I_z}{h^2 - 2TI_z}\right] I_y W_y^2 = 1$$

Known that:

1 → $\{I_x, I_y, I_z\} > 0$; $I_x > I_y > I_z$ (for construction)

2 → $I_x > \frac{h^2}{2T} > I_{z,y}$ → coming from the condition on the equation obtained through the ∩ between the 2 ellipsoids.

Thanks to: 2) ⇒ $DEN = (h^2 - 2TI_z) > 0$

$I_x > \frac{h^2}{2T} > I_z$, ⇒ $h^2 - 2TI_z > 0$

1) ⇒ $I_x - I_z > 0$ ⇒ FIRST TERM > 0 ($a > 0$)

$I_y - I_z > 0$ ⇒ SECOND TERM > 0 ($b > 0$)

PROJECTION OF THE POLHOLDE (IF $I_x > I_y > I_z$) ON $\{W_x; W_y\}$ PLANE IS AN ELLIPSE ⇒ A CLOSED CONIC SUCH THAT AFTER A CERTAIN PERIOD $\underline{W}$ COMES BACK IN THE ORIGINAL POSITION.

I COULD PROCEED IN A DIFFERENT WAY.

- EXTRACT $I_x W_x^2$ and $I_y W_y^2$ from kinetic energy conservation:

$$I_x W_x^2 + I_y W_y^2 + I_z W_z^2 = 2T$$

$$\longrightarrow \begin{cases} I_x W_x^2 = 2T - I_z W_z^2 - I_y W_y^2 \\ I_y W_y^2 = 2T - I_x W_x^2 - I_z W_z^2 \end{cases}$$

- Substitute it into parametric polhode (3D) equation:

$$I_x W_x^2 \left[\frac{I_x}{h^2} - \frac{1}{2T}\right] + I_y W_y^2 \left[\frac{I_y}{h^2} - \frac{1}{2T}\right] + I_z W_z^2 \left[\frac{I_z}{h^2} - \frac{1}{2T}\right] = 0$$

- obtaining the projection of the polhode in $\left\{\begin{matrix}(x,z)\\(y,z)\end{matrix}\right\}$ planes.

$$\begin{cases} I_x W_x^2 \left[\frac{I_x}{h^2} - \frac{1}{2T}\right] + \left[\frac{I_y}{h^2} - \frac{1}{2T}\right](2T - I_z W_z^2 - I_x W_x^2) + I_z W_z^2 \left[\frac{I_z}{h^2} - \frac{1}{2T}\right] = 0 \\ I_y W_y^2 \left[\frac{I_y}{h^2} - \frac{1}{2T}\right] + \left[\frac{I_x}{h^2} - \frac{1}{2T}\right](2T - I_z W_z^2 - I_y W_y^2) + I_z W_z^2 \left[\frac{I_z}{h^2} - \frac{1}{2T}\right] = 0 \end{cases}$$

↓

$$\begin{cases} I_x W_x^2 \left[\frac{I_x}{h^2} - \cancel{\frac{1}{2T}} - \frac{I_y}{h^2} + \cancel{\frac{1}{2T}}\right] + I_z W_z^2 \left[\frac{I_z}{h^2} - \cancel{\frac{1}{2T}} - \frac{I_y}{h^2} + \cancel{\frac{1}{2T}}\right] = 1 - \frac{2T}{h^2} I_y \\ I_y W_y^2 \left[\frac{I_y}{h^2} - \cancel{\frac{1}{2T}} - \frac{I_x}{h^2} + \cancel{\frac{1}{2T}}\right] + I_z W_z^2 \left[\frac{I_z}{h^2} - \cancel{\frac{1}{2T}} - \frac{I_x}{h^2} + \cancel{\frac{1}{2T}}\right] = 1 - \frac{2T}{h^2} I_x \end{cases}$$

(x,z) plane → $\left[\frac{I_x - I_y}{h^2 - 2TI_y}\right] I_x W_x^2 + \left[\frac{I_z - I_y}{h^2 - 2TI_y}\right] I_z W_z^2 = 1$

(y,z) plane → $\left[\frac{I_y - I_x}{h^2 - 2TI_x}\right] I_y W_y^2 + \left[\frac{I_z - I_x}{h^2 - 2TI_x}\right] I_z W_z^2 = 1$

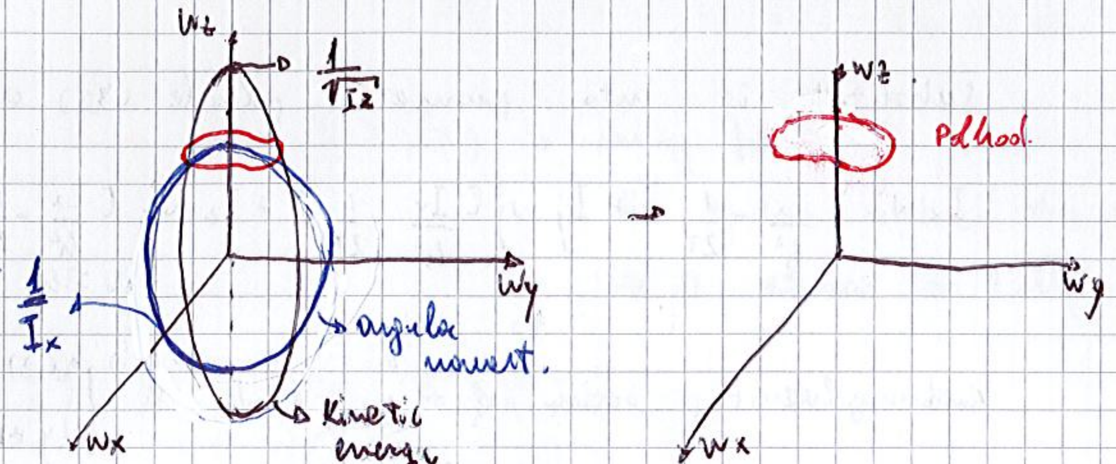
F

Foundamental inertia axis energy and moment (angular):

$$h_o^2 = I_x^2 W_x^2 + I_y^2 W_y^2 + I_z^2 W_z^2 . \qquad Hp: I_x > I_y > I_z .$$

$$2T^{rot} = I_x W_x^2 + I_y W_y^2 + I_z W_z^2$$

by interception of the 2 ellipsoids $\left\{ \begin{array}{l} \text{KINETIC ENERGY ELLIPSOID} \\ \text{ANGULAR MOMENTUM ELLIPSOID.} \end{array} \right.$



interception $\Rightarrow$ $I_x W_x^2 \left[\frac{I_x}{h^2} - \frac{1}{2T} \right] + I_y W_y^2 \left[\frac{I_y}{h^2} - \frac{1}{2T} \right] + I_z W_z^2 \left[\frac{I_z}{h^2} - \frac{1}{2T} \right] = 0 .$

$\Rightarrow$ 3D Polhode's equation.

To have a solution of this equation must be that:

$$\frac{I_x}{h^2} - \frac{1}{2T} > 0 \quad \& \quad \left(\frac{I_y}{h^2} - \frac{1}{2T} \right) ; \left(\frac{I_z}{h^2} - \frac{1}{2T} \right) < 0 .$$

(polhode equation must have 2 terms positive and 1 negative or 2 negative and 1 positive, but having $I_x > I_y > I_z$ the only possible solution is to have the I^{st} term positive and the other 2 negative)

$$\Rightarrow I_x > \frac{h^2}{2T} \; ; \quad \left\{ \begin{array}{l} I_y < \frac{h^2}{2T} \\ I_z < \frac{h^2}{2T} . \end{array} \right.$$

the projection of the polhode on the 3 planes demonstrate the shape of the polhode.

(x, y) plane. $\Rightarrow \left[\frac{I_x - I_z}{h^2 - 2T I_z} \right] I_x W_x^2 + \left[\frac{I_y - I_z}{h^2 - 2T I_z} \right] I_y W_y^2 = 1$

DEN: $I_z < \frac{h^2}{2T}$ $\Rightarrow$ $2T I_z < h^2$ $\Rightarrow$ $h^2 - 2T I_z > 0 .$

NUM: $I_x - I_z > 0$; $I_y - I_z > 0$ $\Rightarrow$ ELLIPS.

(y, z) plane. $\Rightarrow \left[\frac{I_y - I_x}{h^2 - 2T I_x} \right] I_y W_y^2 + \left[\frac{I_z - I_x}{h^2 - 2T I_x} \right] I_z W_z^2 = 1$

DEN: $I_x > \frac{h^2}{2T}$ $\Rightarrow$ $2T I_x > h^2$ $\Rightarrow$ $h^2 - 2T I_x < 0$ (DEN<0)

NUM: $I_y - I_x < 0$; $I_z - I_x < 0 .$

Comparison with conic general shape: $\frac{x^2}{a} + \frac{y^2}{b} = 1$ $\left| \begin{array}{l} a > 0 \\ b > 0 \end{array} \right.$

$\Rightarrow$ ELLIPSOLA.

(x, z) plane. $\Rightarrow \left[\frac{I_x - I_y}{h^2 - 2T I_y} \right] I_x W_x^2 + \left[\frac{I_z - I_y}{h^2 - 2T I_y} \right] I_y W_y^2 = 1$

DEN: $I_y < \frac{h^2}{2T}$ $\Rightarrow$ $2T I_y < h^2$ $\Rightarrow$ $h^2 - 2T I_y > 0 .$

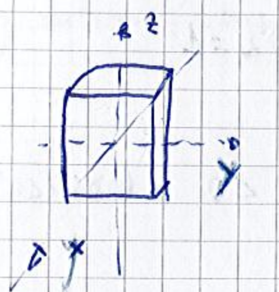
NUM: $I_x - I_y > 0$; $I_z - I_y < 0$ $\Rightarrow \left\{ \begin{array}{l} a > 0 \\ b < 0 \end{array} \right.$

$\Rightarrow$ HYPERBOLA.

(Projection of the polhoid is achieved extracting $I_h W_h^2$ from kinet. energy conservation and substituting it into 3D polhode's equation.)

L_o INSTABLE.

" We can consider the hyperbola as a sort of instability in fact if $\underline{W}$ is slightly shifted from the x-axis or from the z-axis will remain confined in a region closed to the axis, while a slight shift from y axis would cause a dramatic departure. "



Una piccola perturbazione attorno y provoca instabilità.

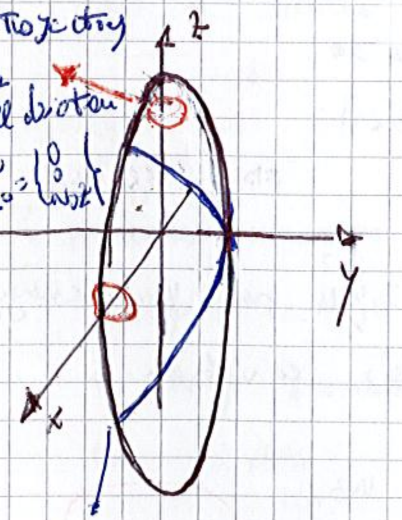
$\xi = \{\hat{i}, \hat{j}, \hat{k}\}$ one at each.

→ STARTING FROM THE $\underline{W} = W\hat{\xi}$ INITIAL CONDITION AND PERTURBING THE $\underline{W}$ VECTOR WITH A SMALL DEVIATION FROM SUCH INITIAL CONDITION (a small shift from one of the principal inertial axis)

$$2T = I_x W_x^2 + I_y W_y^2 + I_z W_z^2.$$
$$h^2 = I_x^2 W_x^2 + I_y^2 W_y^2 + I_z^2 W_z^2$$

$$\to \quad 2T = I_\xi W_\xi^2, \qquad h^2 = I_\xi^2 W_\xi^2$$

$\underline{W}$ trajectory for a small deviation from $W_0 = \{0, 0, W_{0z}\}$



$\underline{W}$ trajectory for a small deviation from $\underline{W}_0 = \{0, W_{0y}, 0\}$

→ dando W_ξ tale da mantenere invariato l'ellissoide dell'energia cinetica, a seconda dell'asse attorno a cui avviene la rotazione, la forma dell'ellissoide del momento angolare risulta invece differente da caso a caso

xò L'INSTABILITÀ SARÀ SEMPRE INTORNO A Y

(W_{0y}, W_{0z}): $I_z W_{0z}^2 = I_y W_{0y}^2$ → ci stiamo muovendo sul medesimo ellissoide dell' en...

mentre si ottengono di

3 → "Euler's equations"

In saying that $\left\{\begin{array}{l}\text{Kinetic energy}\\ \text{angular moment}\end{array}\right|$ are constant implications are:

$2T^{rot} = \text{const} \Rightarrow \frac{d}{dt}(2T^{rot}) = 0$;

$\underline{h} = \underline{\text{const}} \Rightarrow \frac{d\underline{h}}{dt} = \underline{0}$

Writing $2T^{rot}$ as a function of $\underline{h}$:

$$2T^{rot} = I_x w_x^2 + I_y w_y^2 + I_z w_z^2 = \underline{w} \cdot \underline{h}$$

$$\Rightarrow \frac{d}{dt}(2T^{rot}) = \frac{d\underline{w}}{dt} \cdot \underline{h} + \underline{w} \cdot \cancel{\frac{d\underline{h}}{dt}}^{=0} = \underline{h} \cdot \frac{d\underline{w}}{dt}$$

[IF KINETIC ENERGY AND ANGULAR MOMENT ARE BOTH CONSERVED THIS IMPLIES THAT: or $\frac{d\underline{w}}{dt} = \underline{0} \Rightarrow \underline{w} = \underline{\text{const}}$ (modulus and direction)

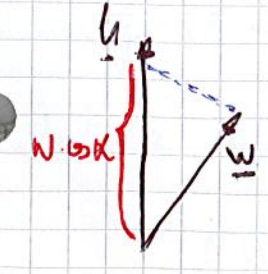
or $\underline{h} \cdot \frac{d\underline{w}}{dt} = 0 \Rightarrow \underline{h} \perp \frac{d\underline{w}}{dt}$.]

$\underline{h} \cdot \frac{d\underline{w}}{dt} = 0 \Rightarrow$ the projection of $\underline{w}$ on $\hat{\underline{h}}$ direction must be always the same in time

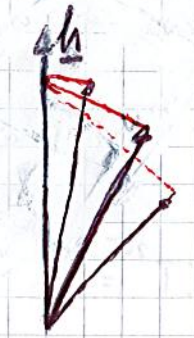
in fact:

$$(\underline{h} = \underline{\text{const}}) \Rightarrow \frac{d}{dt}(\underline{h} \cdot \underline{w}) = \cancel{\frac{d\underline{h}}{dt}}^{=0} \cdot \underline{w} + \underline{h} \cdot \frac{d\underline{w}}{dt} = 0$$

this is $2T^{rot} = \underline{h} \cdot \underline{w}$



⇒ Projection of $\underline{w}$ must be always the same in time this individuate a plane considering all the possible values of $\underline{w}$.

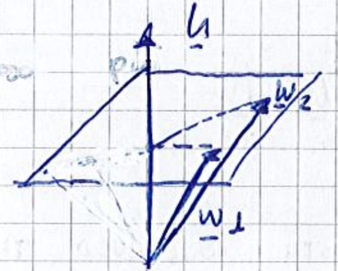


Il piano rappresenta il luogo di punti per i quali

$$\begin{cases} h \cdot w \cos\alpha = \text{cost.} \\ \alpha = \hat{h}\,\hat{w} \end{cases}$$

b:

$$w_1 < w_2 \Rightarrow \begin{cases} \hat{\underline{h}} \cdot \underline{w}_1 = h\, w_1 \cos\alpha_1 \\ \hat{\underline{h}} \cdot \underline{w}_2 = h\, w_2 \cos\alpha_2 \end{cases} \Rightarrow \cos\alpha_1 > \cos\alpha_2 \Rightarrow \alpha_2 < \alpha_1$$



→ Anyway the equation that have to be solved (integrated) in order to compute the state of our spacecraft is the II^{nd} cardinal equation:

$$\frac{d\underline{h}}{dt} = \underline{M}$$

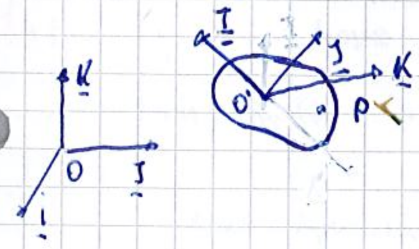
!! It's important to observe that $\underline{h}$ is expressed in the principal inertial reference frame that is solidale with the body ⇒ is rotating at $\underline{w}$ speed. !!

$$\frac{d\underline{h}}{dt} = \frac{d}{dt}(\|\underline{h}\| \cdot \hat{\underline{h}}) = \dot{h}\,\hat{\underline{h}} + \frac{d\hat{\underline{h}}}{dt}\|\underline{h}\|$$

only due to rotation of the reference frame.

$$\Rightarrow \boxed{\frac{d\underline{h}}{dt} = \dot{\hat{\underline{h}}} + \underline{w} \times \hat{\underline{h}}}$$

This is due to Poisson's formula:



$$\Rightarrow (P-O) = (P-O') + (O'-O)$$
$$= x_{O'}\,\hat{\underline{i}} + y_{O'}\,\hat{\underline{j}} + z_{O'}\,\hat{\underline{k}} + x'\,\hat{\underline{I}} + y'\,\hat{\underline{J}} + z'\,\hat{\underline{K}}$$

Under the hypotesis of rigid body: $\{x_{O'}; y_{O'}; z_{O'}\}$; $\{\underline{I}, \underline{J}, \underline{K}\}$ $\propto t$

$\{x', y', z'\}$; $\{\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}\}$ $\not\propto t$

Computing velocity of P with respect to O:

$$\frac{d}{dt}(P-O) = \dot{x}_{O'}\,\hat{\underline{i}} + \dot{y}_{O'}\,\hat{\underline{j}} + \dot{z}_{O'}\,\hat{\underline{k}} + x'\,\dot{\hat{\underline{I}}} + y'\,\dot{\hat{\underline{J}}} + z'\,\dot{\hat{\underline{K}}}$$

For a rigid body motion:

$$\underline{v}(P) = \underline{v}(O') + \underline{w} \times (P - O') \quad \text{when } O' \in R.B.$$

thanks to the comparison between the 2 expression of $\underline{v}(P)$

$$\underline{v}(o') = \dot{x}_{o'}\hat{\imath} + \dot{y}_{o'}\hat{\jmath} + \dot{z}_{o'}\hat{k}$$

$$\Rightarrow \quad \underline{w} \times (P - O') = x'\dot{\hat{I}} + y'\dot{\hat{J}} + z'\dot{\hat{K}}$$

$$\underline{w} \times (x'\hat{I} + y'\hat{J} + z'\hat{K}) = x'\dot{\hat{I}} + y'\dot{\hat{J}} + z'\dot{\hat{K}}$$

developing this vectorial products:

$$\begin{cases} \dot{\hat{I}} = \underline{w} \times \hat{I} \\ \dot{\hat{J}} = \underline{w} \times \hat{J} \\ \dot{\hat{K}} = \underline{w} \times \hat{K} \end{cases}$$

$\Rightarrow$ the second cardinal equation can be written as:

$$\frac{d\underline{h}}{dt} + \underline{w} \times \underline{h} = \underline{M}$$

And Euler's equations are simply the envelope of this:

$$\begin{cases} \dfrac{d\underline{h}}{dt} + \underline{w} \times \underline{h} = \underline{M} \\ \underline{h} = \underline{\underline{I}} \cdot \underline{w} = \begin{vmatrix} I_x w_x \\ I_y w_y \\ I_z w_z \end{vmatrix} \end{cases}$$

└ diagonal matrix in fundamental inertia axis.

$$\Rightarrow \quad \frac{d\underline{h}}{dt} + \begin{vmatrix} \hat{I} & \hat{J} & \hat{K} \\ w_x & w_y & w_z \\ I_x w_x & I_y w_y & I_z w_z \end{vmatrix} = \underline{M}.$$

// Euler's equations are simply the IInd cardinal equation expressed in the principal inertia reference frame. //

$$\frac{d\underline{h}}{dt} + \begin{vmatrix} I_z w_y w_z - I_y w_y w_z \\ I_x w_x w_z - I_z w_x w_z \\ I_y w_y w_x - I_x w_x w_y \end{vmatrix} = \underline{M}$$

Euler's equations.

$$\dot{\underline{h}} + \underline{w} \times \underline{h} = \underline{M} \qquad (\underline{h} = \underline{\underline{I}} \cdot \underline{w})$$

$$\underline{\underline{I}}\,\dot{\underline{w}} + \underline{w} \times (\underline{\underline{I}}\,\underline{w}) = \underline{M}$$

$\rightarrow$ In the principal inertia reference frame:

$$\begin{cases} I_x \dot{w}_x + (I_z - I_y) w_y w_z = M_x \\ I_y \dot{w}_y + (I_x - I_z) w_x w_z = M_y \\ I_z \dot{w}_z + (I_y - I_x) w_x w_y = M_z. \end{cases}$$

obs 1: If i have $\{M_x = M_x^c \; ; \; M_y = M_y^c\}$ settled as control torch with a total of:

euler equations (3 equ)
 control laws (2 equ) total variables $\begin{vmatrix} w_x \\ w_y \\ w_z \\ M_x^c \\ M_y^c \end{vmatrix}$

$\Rightarrow$ I will have 2 extra equations to compute $(M_x^c \; ; \; M_y^c)$ (if assume $\emptyset$ disturb torque)

Obs 2: Euler's equations present a closed solution only if 2 inertia moments (in PIF) are equals

EULER'S CLOSED SOLUTION $\Leftrightarrow$ $I_i = I_j$

LOOKING FOR EQUILIBRIUM CONDITIONS (STATE) OF EULER'S EQUATIONS:

$$Eq \Rightarrow \underline{\dot{\omega}} = \underline{0}$$

$$\begin{cases} (I_z - I_y)\,\bar{\omega}_y \bar{\omega}_z = 0 \\ (I_x - I_z)\,\bar{\omega}_x \bar{\omega}_z = 0 \\ (I_y - I_x)\,\bar{\omega}_x \bar{\omega}_y = 0. \end{cases}$$

the solution of this system is 2 ω-components equal to zero:

$$\begin{vmatrix} \omega_{ij} = 0 \\ \omega_k \neq 0 \end{vmatrix}$$

this is an ∞^2 equilibrium conditions that can be contained in one single statement:

[EQUILIBRIUM CONDITION IS GUARANTEED IF ROTATION OCCURS AROUND ONLY ONE OF THE 3 PRINCIPAL INERTIA AXIS.]

in other words:

SE UN CORPO È IN ROTAZIONE ATTORNO A UN ASSE PRINCIPALE DI INERZIA ALLORA (IN UN MOTO PER INERZIA) SI MANTIENE INDEFINITAMENTE IN COSTANTE ROTAZIONE ATTORNO A TALE ASSE.

"A linearization of Euler equations is necessary in order to evaluate the stability of this equilibrium condition"

EVALUATING THE STABILITY OF EQUILIBRIUM CONDITION $\begin{vmatrix} \omega_z \neq 0 \\ \omega_x = \omega_y = 0 \end{vmatrix}$. LINEARIZING THE EQUATIONS

$$\Rightarrow Eq: \begin{cases} \bar{\omega}_x = 0 \\ \bar{\omega}_y = 0 \\ \omega_z = \bar{\omega}_z \end{cases} \qquad \text{perturbation:} \begin{cases} \omega_x = \bar{\omega}_x + \Delta\omega_x = \Delta\omega_x \\ \omega_y = \bar{\omega}_y + \Delta\omega_y = \Delta\omega_y \\ \omega_z = \bar{\omega}_z + \Delta\omega_z \end{cases}$$

Hp: no perturbation around $\bar{\omega}_z$ ($\Delta\omega_z = 0$)

$$\begin{cases} I_x \Delta\dot{\omega}_x + (I_z - I_y)\,\omega_z\,\Delta\omega_y = 0 \\ I_y \Delta\dot{\omega}_y + (I_x - I_y)\,\omega_z \cdot \Delta\omega_x = 0 \\ I_z \dot{\omega}_z = 0 \Rightarrow \omega_z = \bar{\omega}_z \end{cases}$$

linearization around $\begin{Bmatrix} 0 \\ 0 \\ \bar{\omega}_z \end{Bmatrix}$ $\Rightarrow \begin{cases} \Delta\omega_x = \omega_x \\ \Delta\omega_y = \omega_y. \end{cases}$

$$\downarrow$$

$$\begin{cases} I_x \dot{\omega}_x + (I_z - I_y)\,\bar{\omega}_z \omega_y = 0 \\ I_y \dot{\omega}_y + (I_x - I_y)\,\bar{\omega}_z \omega_x = 0 \\ I_z \cdot \dot{\omega}_z = 0. \end{cases}$$

OBTAINING SPACE-STATE RAPPRESENTATION AND COMPUTING EIGEN-VALUES OF THE SYSTEM:

$$\begin{Bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & \bar{\omega}_z(I_z - I_y)/I_x & 0 \\ \bar{\omega}_z(I_x - I_y)/I_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{= \underline{\underline{A}}} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Eigen values can be computed as:

$$\det(s\underline{\underline{I}} - \underline{\underline{A}}) = 0$$

$$\rightarrow \det\left(\begin{bmatrix} s & -a & 0 \\ -b & s & 0 \\ 0 & 0 & s \end{bmatrix}\right) = 0.$$

$$\det(s\underline{I} - \underline{A}) = 0$$

$$s \cdot (s^2 - a \cdot b) = 0$$

$$a = \bar{w}_z \cdot \frac{(I_z - I_y)}{I_x}$$

$$b = \bar{w}_z \, \frac{(I_x - I_y)}{I_y}$$

$$s_1 = 0$$

$$s_{2,3} = \pm \sqrt{\frac{(I_z - I_y)(I_x - I_y)}{I_x \cdot I_y}} \; \bar{w}_z$$

n order to evaluate stability some coistics must be taken into account:

a) $I_x < I_y < I_z \Rightarrow$ 2 immaginary roots (with $Re = 0$) $\Rightarrow$ STABLE

b) $I_z < I_y < I_x \Rightarrow$ 2 immaginary roots $\Rightarrow$ STABLE

c) $I_y < I_z < I_x \Rightarrow$ 1 real positive root.
1 real negative root $\Rightarrow$ UN-STABLE

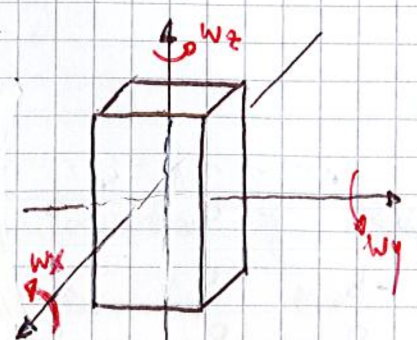
STABILITY IS GUARANTUCED IF ROTATION OCCOURS AROUND:

THE INERTIA AXIS RELATIVE TO THE MINIMUM INERTIA MOMENT.

OR

THE INERTIA AXIS RELATIVE TO THE MAXIMUM INERTIA MOMENT

$$I_x > I_y > I_z$$



$\underline{W} = \{w_x, 0, 0\} \rightarrow$ STABLE

$\underline{W} = \{0, 0, W_z\} \rightarrow$ STABLE

$\underline{W} = \{0, w_y, 0\} \rightarrow$ UNSTABLE.

APPROFONDIMENTO: moto alla poinsot

I^a card) $\quad m\,\underline{a}(G) = \underline{R}^e$

II^a card) $\quad \dfrac{d\underline{h}(G)}{dt} = \underline{M}^e(G) \; ; \quad \underline{h}(G) = \underline{\underline{I}} \cdot \underline{W}$

Sono interessati al moto di tale sistema (intorno) rispetto al centro di massa G $\Rightarrow$ come se fosse fisso.

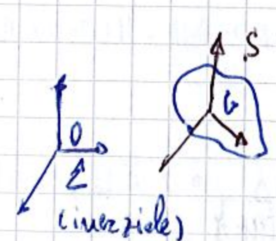
in particolare.

" Si analizza il caso in cui $\underline{M}^e(G) = \underline{0}$ e dunque il moto del corpo avviene semplicemente per inerzia. "

$\Rightarrow$ Euler equations:
$$\begin{cases} I_x \dot{w}_x + (I_z - I_y)\, w_y w_z = 0 \\ I_y \dot{w}_y + (I_x - I_z)\, w_x w_z = 0 \\ I_z \dot{w}_z + (I_y - I_x)\, w_y w_x = 0 \end{cases}$$

In un problema di questo tipo (non forzato) gli integrali primi del moto sono l'energia cinetica e il momento angolare:

$\underline{h}$



$\Rightarrow \quad O \equiv G \Rightarrow \left[\dfrac{d\underline{h}_o}{dt}\right]_\Sigma = 0$

$$\left[\frac{d\underline{h}_o}{dt}\right]_\Sigma = \left[\frac{d\underline{h}_o}{dt}\right]_S + \underline{w} \times \underline{h}$$

T^{rot}

$$T^{rot} = \frac{1}{2}\, \underline{h}_o \cdot \underline{w}(0)$$

velocità angolare a $t = 0$ (c.i.)

"

moto di precessione $\Rightarrow \underline{a}_G = \underline{0}$ (o non è di interesse)

moto di precessione x inerzia $\Rightarrow \underline{M}^e = \underline{0}$ "

Th: " In un moto di precessione per inerzia, l'ellissoide di inerzia del sistema relativo al polo $O \equiv G$ rotola senza strisciare su un piano fisso dalla giacitura normale al vettore $\underline{h}_0$. "

ellissoide di inerzia → $2T^{tot} = I_x w_x^2 + I_y w_y^2 + I_z w_z^2$

$\exists\ \hat{\underline{h}} : \underline{w} = w\, \hat{\underline{h}}$

$\Rightarrow 2T^{tot} = I_h w^2 = I_x w_x^2 + I_y w_y^2 + I_z w_z^2$

$I_h = I_x\, w_x^2/w^2 + I_y\, w_y^2/w^2 + I_z\, w_z^2/w$

sarebbe il momento di inerzia che dovrebbe avere il corpo (calcolato intorno a $\hat{h}$) in modo da conservare l'energia cinetica. calcolato in un riferimento non principale di inerzia $\{\hat{h}, \hat{\beta}, \hat{\gamma}\}$

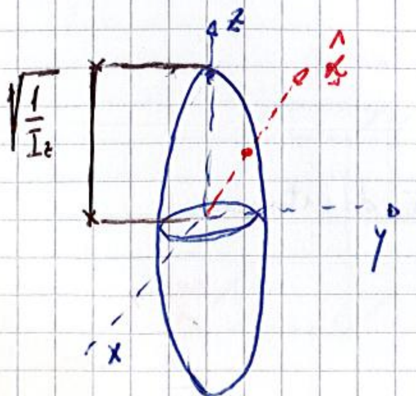
$l_{hx} \triangleq w_x/w$, $l_{hy} \triangleq w_y/w$, $l_{hz} \triangleq w_z/w$

ellissoide di inerzia: $\dfrac{l_{hx}^2/I_h}{1/I_x} + \dfrac{l_{hy}^2/I_h}{1/I_y} + \dfrac{l_{hz}^2/I_h}{1/I_z} = 1$

* Equ: $\dfrac{l_{hx}^2/I_h}{1/I_x} + \dfrac{l_{hy}^2/I_h}{1/I_y} + \dfrac{l_{hz}^2/I_h}{1/I_z}$

* point coordinates: $\{ l_{hx}/\sqrt{I_h} ; l_{hy}/\sqrt{I_h} ; l_{hz}/\sqrt{I_h} \}$

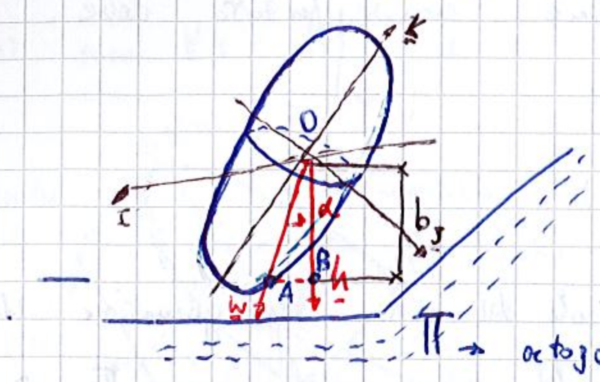
* semi-axis $\{ 1/\sqrt{I_x} ; 1/\sqrt{I_y} ; 1/\sqrt{I_z} \}$



Tracciando una generica direzione $\hat{\alpha} = \{\alpha_x ; \alpha_y ; \alpha_z\}$. L'intersezione con l'ellissoide avviene a distanza dall'origine $1/\sqrt{I_\alpha}$

$I_\alpha : \quad 2T^{tot} = I_x w_x^2(\alpha) + I_z w_y^2(\alpha) + I_y w_y^2(\alpha) = I_\alpha w^2(\alpha)$.

$\Rightarrow I_\alpha$ RAPPRESENTA IL MOMENTO DI INERZIA RISPETTO AL BARICENTRO CALCOLATO IN UN QUALSIASI (non principale di inerzia) SISTEMA DI RIFERIMENTO DOVE UN ASSE È PROPRIO $\hat{\alpha}$



$\hat{\underline{w}} \wedge \hat{\underline{h}} = \alpha$

Π → ortogonale ad $\underline{h}$ e tangente al

⇒ ESSENDO $\underline{h} = $ const$|_\Gamma$ il piano Π ha giacitura costante

MOTO ALLA POINSOT: durante il moto il punto O si mantiene alla stessa distanza dal piano tangente all'ellissoide (che si costruisce con polo in O) nel punto in cui $\underline{w}$ interseca l'ellissoide.

- $\cos\alpha = \dfrac{\underline{w}\cdot\underline{h}}{w \cdot h}$

- $(A-O) = \dfrac{\lambda}{\sqrt{I_w}}\, \hat{\underline{w}}$ (λ → parametro di omotetia dell'ellissoide

  DEF. ELLISSOIDE ⇒ $\|A-O\| = 1/\sqrt{I_w}$

  $\lambda : \quad \lambda \cdot \dfrac{L}{\sqrt{I_w}} = L$

  L → dimensione "fisica" del corpo (UNA LUNGHEZZA))

- $\|(O-B)\| = \|A-O\| \cdot \cos\alpha$.

$\|(O-B)\| \triangleq l = \dfrac{\lambda}{\sqrt{I_w}} \cdot \cos\alpha = \dfrac{\lambda}{\sqrt{I_w}}\, \dfrac{\underline{w}\cdot\underline{h}}{w \cdot h}$.

poiché: $\rightarrow \underline{w}\cdot\underline{h} = 2T^{tot} = 2T_0^{tot}$.

$\rightarrow \underline{h} = $ const $\Rightarrow \|\underline{h}\| = $ const.

$\rightarrow \sqrt{I_w}\cdot w = $ cost. ← infatti $2T^{tot} = I_w w^2 = I_x w_x^2 + I_y w_y^2 + I_z w_z^2$

$\sqrt{I_w}\cdot w = \sqrt{2T^{tot}}$.

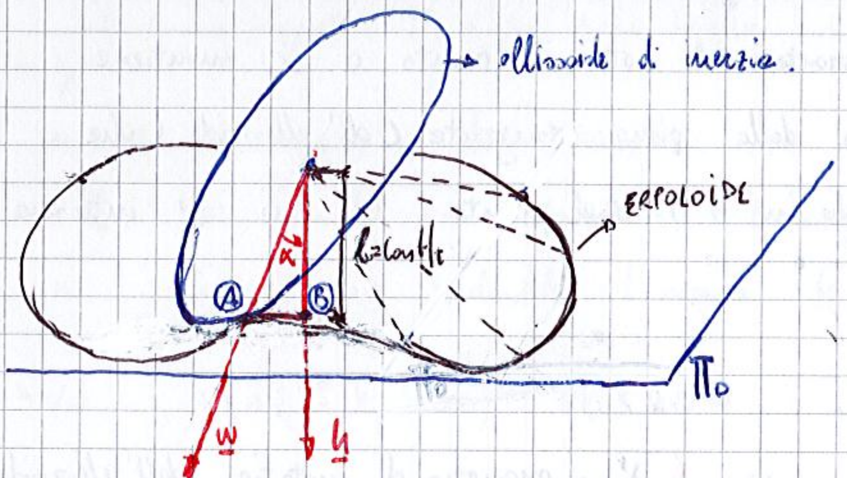
$\Rightarrow l = \dfrac{\lambda}{\sqrt{I_w}} \cdot \cos\alpha = \dfrac{\lambda}{\sqrt{I_w}}\, \dfrac{\underline{w}\cdot\underline{h}}{h \cdot w} = \dfrac{\lambda}{h}\, \dfrac{2T^{tot}}{\sqrt{2T^{tot}}}$.

$$\boxed{l = \text{const}|_t = \frac{\lambda}{h}\cdot\sqrt{2T_0}}$$

" In questo particolare problema non frazioni dove.

$$\begin{cases} 2T^{\,\text{cost}} = 2T_0^{\,\text{cost}} \\ \underline{h} = \underline{\text{cost}}|_t \quad (= \underline{h}(0)) \end{cases}$$

si ha un ulteriore integrale del moto che garantisce che la distanza del baricentro del corpo dal piano Π_0 parallelo lungo $\underline{w}$ è costante nel tempo. "

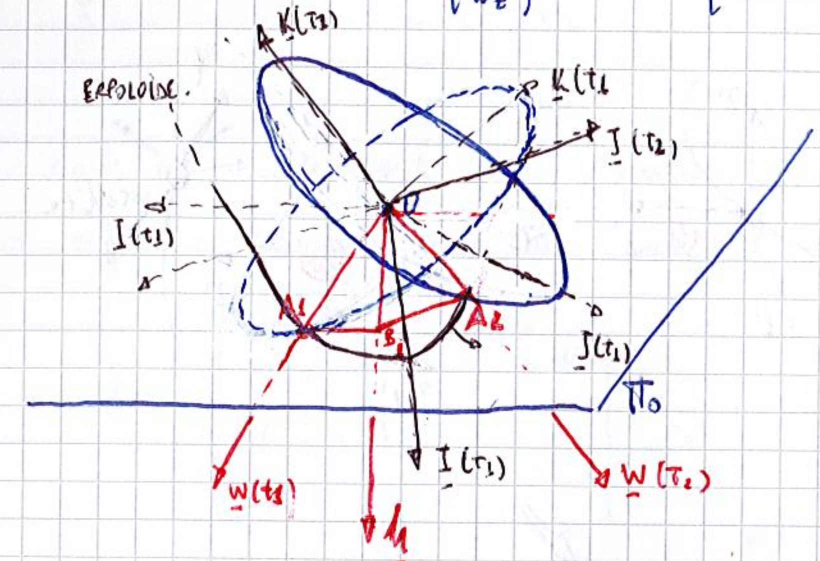


[L'ELLISSOIDE DI INERZIA ROTOLA SENZA STRISCIARE SUL PIANO Π_0 (CHE È FISSO NEL TEMPO) E IN QUESTO MOTO SI HA LA ROTAZIONE DELLA TERNA FONDAMENTALE DI INERZIA $\Rightarrow$ LA CURVA DESCRITTA SU Π_0 IN QUESTO MOTO DI PURA ROTAZIONE ($O \equiv G$) VIENE DESCRITTA UNA CURVA CHIAMATA ERPOLOIDE.]

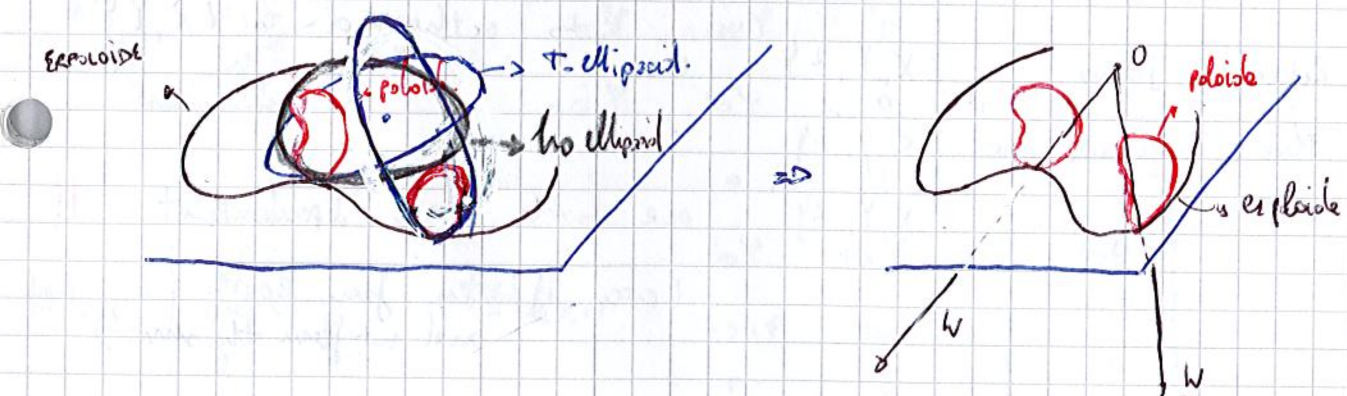
[IL PUNTO Ⓐ, CHE RAPPRESENTA LA PROIEZIONE DI $\underline{w}$ LUNGO $\underline{h}$, È L'INTERSEZIONE DI $\underline{w}$ CON IL PIANO Π_0, È ISTANTANEAMENTE FISSO e DURANTE IL MOTO ~~DL~~ L'ELLISSOIDE SI PORTA DIETRO IL CORPO (O VICEVERSA) A CAUSA DELLA ROTAZIONE IN O DELL'ASSE DI ROTAZIONE.]

Dal punto di vista degli ellissoidi di $\begin{cases}\text{energia inetica} \\ \text{momento angolare.}\end{cases}$

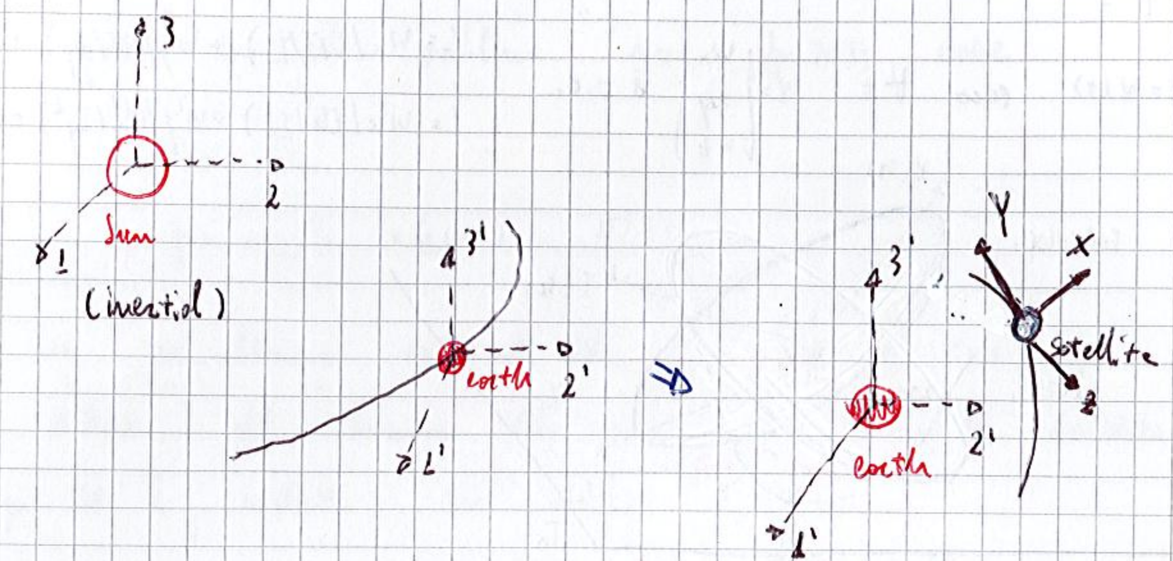
$\underline{w} = \underline{w}(t)$ però $\forall t$: $\underline{w} = \begin{Bmatrix} w_x \\ w_y \\ w_z \end{Bmatrix}$ è t.c. $\begin{cases} 1 = w_x^2/(2T/I_x) + w_y^2/(2T/I_y) + \dots \\ 1 = w_x^2/(h^2/I_x^2) + w_y^2/(h^2/I_y^2) + \dots \end{cases}$

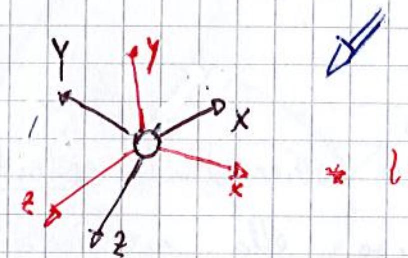


Dall'intersezione dei 2 ellissoidi si ottiene il poloide. che risulta "costretta" di conseguenza alla rotazione dell'ellissoide di inerzia (d'altronde l'ellissoide di inerzia è semplicemente scalato [e su assi diversi] rispetto all'ellissoide dell'energia cinetica)



→ " Attitude Parameters "





* $\{x, y, z\}$ → satellite's principal inertia axis

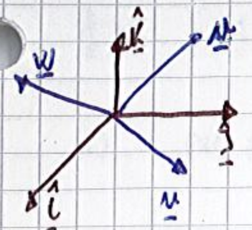
* $\begin{cases} X \to \text{satellite local vertical direction} \\ Y \to \text{satellite velocity direction} \\ Z \to \text{orthogonal to } \{X, Y\} \end{cases}$

!! The reference frames: $\{x, y, z\}$
$\{X, Y, Z\}$ are both time dependent !!
(even if seen from earth and not from the sun.)

" To define the rotation of the R.F. with respect to the inertial one at least 3 parameters are required. but more can be used to define the situation:

DIRECTION COSINES MATRIX → 9 Par.
EULER AXIS / ANGLE → 4 Par.
QUATERNIONS → 4 Par.
EULER ANGLES → 3 Par.
GIBBS VECTOR → 3 Par.

1 → DIRECTION COSINE MATRIX



$$\to \underline{\underline{A}} = \begin{bmatrix} u_1 , & u_2 , & u_3 \\ v_1 , & v_2 , & v_3 \\ w_1 , & w_2 , & w_3 \end{bmatrix}$$

where each row represent the projection of one axis of the current reference frame on the inertial reference frame:

$$\underline{\underline{A}} = \begin{bmatrix} \hat{i}\cdot\underline{u} ; & \hat{j}\,\underline{u} ; & \hat{k}\cdot\underline{u} \\ \hat{i}\cdot\underline{v} ; & \hat{j}\,\underline{v} ; & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

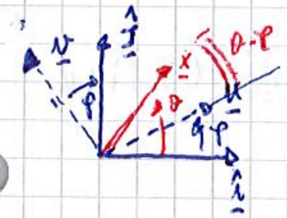
obviously: . $\forall i_j \; [A]_{ij} < 1$

. $\| (A)_i \| = 1$

norm of each row is equal to 1.

⇒ $\underline{\underline{A}}$: $\{\underline{x}\}_{u,v,w} = \underline{\underline{A}} \, \{\underline{x}\}_{i,j,k}$

es. considering an IN PLANE ROTATION:



$$\to \underline{\underline{A}} = \begin{bmatrix} \cos\varphi ; & \sin\varphi ; & 0 \\ -\sin\varphi ; & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{\underline{x}\}_{ijk} = \begin{Bmatrix} \cos\theta X \\ \sin\theta X \\ 0 \end{Bmatrix} \to \{\underline{x}\}_{u,v,w} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} \cos\theta X \\ \sin\theta X \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} \cos\varphi\cos\theta + \sin\theta\sin\varphi \\ -\sin\varphi\cos\theta + \cos\varphi\sin\theta \\ 0 \end{Bmatrix} = \begin{Bmatrix} \cos(\theta-\varphi) \\ \sin(\theta-\varphi) \\ 0 \end{Bmatrix} . \text{ cvd.}$$

2 orthogonality.

$$\underline{\underline{A}}^T \cdot \underline{\underline{A}} = \underline{\underline{I}} \to \begin{bmatrix} c\varphi & s\varphi & 0 \\ -s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c\varphi & -s\varphi & 0 \\ s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \overbrace{c^2\varphi + s^2\varphi}^{=1} ; & \overbrace{-c\varphi s\varphi + c\varphi s\varphi}^{=0} & 0 \\ 0 & 1 & \\ 0 & & 1 \end{bmatrix}$$

$\ulcorner$ I^{ST}

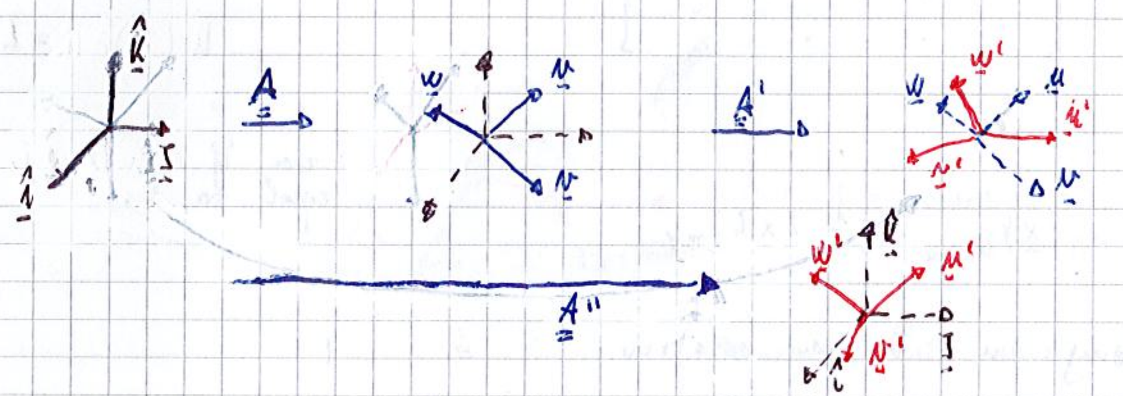
$$\underline{\underline{A}} = \begin{bmatrix} \underline{u}\cdot\hat{\underline{i}} & \underline{u}\cdot\hat{\underline{j}} & \underline{u}\cdot\hat{\underline{k}} \\ \underline{v}\cdot\hat{\underline{i}} & \dots & \\ \dots & & \end{bmatrix}$$

prop: (i) $\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$ (orthogonal matrix)

(ii) $\forall_{ij}\ [A]_{ij} \le 1$

$\|\{A\}_i\| = 1$ (norm of each row)

Consecutive rotations:



$$\begin{cases} \{\underline{a}\}_{uvw} = \underline{\underline{A}}\,\{\underline{a}\}_{ijk} \\ \{\underline{a}\}_{u'v'w'} = \underline{\underline{A}}'\,\{\underline{a}\}_{uvw} \end{cases} \Rightarrow \{\underline{a}\}_{u'v'w'} = \underbrace{\underline{\underline{A}}'\,\underline{\underline{A}}}_{\underline{\underline{A}}''}\,\{\underline{a}\}_{ijk}$$

$\Rightarrow$ (iii) $\underline{\underline{A}}'' = \underline{\underline{A}}'\,\underline{\underline{A}}$

IF $\underline{\underline{A}}$: $\underline{\underline{A}}^T\,\underline{\underline{A}} = \underline{\underline{1}}$ (orthogonal matrix)

THEN $\exists!$ $\underline{e}$: $\underline{\underline{A}}\cdot\underline{e} = \underline{e}$ (unit eigenvalue)

Proof:

$\underline{\underline{A}}$: $\underline{\underline{A}}^T\cdot\underline{\underline{A}} = \underline{\underline{I}} \Rightarrow \exists!\ \underline{e} : \underline{\underline{A}}\cdot\underline{e} = \underline{e}$

A) pre-proof: $\underline{\underline{A}}^T\cdot\underline{e} = \underline{e}$

in fact $(\underline{\underline{A}}\,\underline{e})^T = \underline{e}^T\cdot\underline{\underline{A}}^T$

$(\underline{\underline{A}}\,\underline{e})^T\cdot\underline{e} = \underline{e}^T\cdot\underline{\underline{A}}^T\cdot\underline{e}$ but: $\begin{cases} (\underline{\underline{A}}\cdot\underline{e})^T = \underline{e}^T \\ (\underline{\underline{A}}^T\cdot\underline{e}) = \underline{e} \quad \text{FOR HYPOTESIS} \end{cases}$

$\Downarrow$

$\underline{e}^T\cdot\underline{e} = \underline{e}^T\cdot\underline{e}\ (=1)$

B) $\underline{\underline{A}}\cdot\underline{e} = \underline{e} \rightarrow \underline{\underline{A}}^T\cdot\underline{\underline{A}}\cdot\underline{e} = \underline{\underline{A}}^T\underline{e} \rightarrow \underline{\underline{I}}\cdot\underline{e} = \underline{\underline{A}}^T\cdot\underline{e} \rightarrow \underline{e} = \underline{e}$.

$\rfloor \rightarrow$ EULER AXIS/ANGLE.

$\ulcorner$ II^{nd}

$\cdot\underline{e}$: $\underline{\underline{A}}\cdot\underline{e} = \underline{e}$; $\underline{e} \rightarrow$ Euler axis

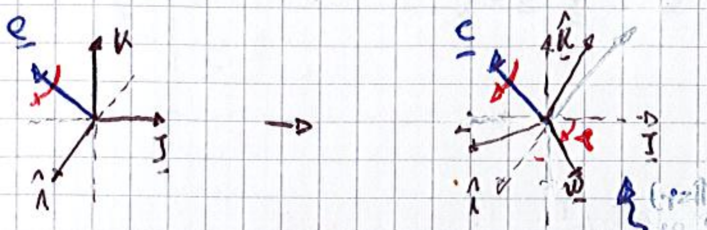
$\varphi \rightarrow$ amplitude (angle) of rotation (Euler angle) $\rfloor$

[SINCE $\exists\ \underline{e}$: $\underline{\underline{A}}\cdot\underline{e} = \underline{e}$ (! $\underline{\underline{A}}$ represents the direction cosine matrix !)

THEN this means that $\underline{e}$ is invariant to $\underline{\underline{A}}$ transformation.

$\Rightarrow$ $\underline{e}$ DON'T CHANGE DURING ROTATION (REPRESENTED BY MATRIX $\underline{\underline{A}}$). THIS IS POSSIBLE IF AND ONLY IF ROTATION OCCURS AROUND $\underline{e}$]

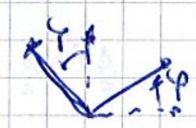
This is evident if:



the projection of $\underline{e}$ on "rigidly" rotated axes respect the same angles.

180° rotation because is hard to draw so projection of $\hat{e}$ on $\{\underline{u},\underline{w},\underline{v}\}$ is the same that the one on $\{\hat{\imath},\hat{\jmath},\hat{k}\}$.

this is even more evident if $\underline{e}$ coincides with one of the inertial axes $\{\hat{\imath},\hat{\jmath},\hat{k}\}$



$\underline{e}\equiv\underline{k}\equiv\underline{w}$ $\quad \underline{e}=\begin{Bmatrix}0\\0\\1\end{Bmatrix} \rightarrow \underline{\underline{A}}_3=\begin{bmatrix}c\varphi & s\varphi & 0\\ -s\varphi & c\varphi & 0\\ 0 & 0 & 1\end{bmatrix}$

$\underline{e}\equiv\underline{\jmath}\equiv\underline{v}$ $\quad \underline{e}=\begin{Bmatrix}0\\1\\0\end{Bmatrix} \rightarrow \underline{\underline{A}}_2=\begin{bmatrix}c\varphi & 0 & s\varphi\\ 0 & 1 & 0\\ -s\varphi & 0 & c\varphi\end{bmatrix}$

$\underline{e}\equiv\hat{\imath}\equiv\underline{u}$ $\quad \underline{e}=\begin{Bmatrix}1\\0\\0\end{Bmatrix} \rightarrow \underline{\underline{A}}_1=\begin{bmatrix}1 & 0 & 0\\ 0 & c\varphi & s\varphi\\ 0 & -s\varphi & c\varphi\end{bmatrix}$

$\Rightarrow$ In each case: $\quad Tr(\underline{\underline{A}}_1)=Tr(\underline{\underline{A}}_2)=Tr(\underline{\underline{A}}_3)=Tr(\underline{\underline{A}})$

$$\boxed{Tr(\underline{\underline{A}})=1+2\cos\varphi \rightarrow \cos\varphi=\frac{1}{2}\left[Tr(\underline{\underline{A}})-1\right]}$$

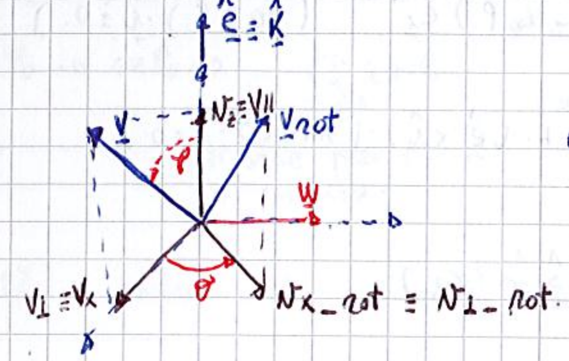
$(\hat{\underline{e}},\varphi)\rightarrow\underline{\underline{A}}$ (ALWAYS POSSIBLE)

$$\underline{\underline{A}}=\begin{bmatrix}\cos\varphi+e_1^2(1-\cos\varphi) & e_1e_2(1-\cos\varphi)+e_3\sin\varphi & e_1e_3(1-\cos\varphi)-e_2\sin\varphi\\ e_1e_2(1-\cos\varphi)-e_3\sin\varphi & \cos\varphi+e_2^2(1-\cos\varphi) & e_2e_3(1-\cos\varphi)+e_1\sin\varphi\\ e_1e_3(1-\cos\varphi)+e_2\sin\varphi & e_2e_3(1-\cos\varphi)-e_1\sin\varphi & \cos\varphi+e_3^2(1-\cos\varphi)\end{bmatrix}$$

$$\underline{\underline{A}}=\underline{\underline{I}}\cos\varphi+(1-\cos\varphi)\,\underline{e}\,\underline{e}^T-\sin\varphi\,[\underline{e}\wedge]$$

$[\underline{e}\times]$ is the cross product: $[\underline{e}\times]=\begin{bmatrix}0 & -e_3 & e_2\\ e_3 & 0 & -e_1\\ -e_2 & e_1 & 0\end{bmatrix}$

This is nothing more than Rodrigues rotation formula.



rotation around $\hat{\underline{k}}$

$\underline{v}_{\parallel}=(\hat{\underline{k}}\cdot\underline{v})\,\hat{\underline{k}}$; $\quad \underline{v}_{\perp}=\underline{v}-\underline{v}_{\parallel}=\underline{v}-(\hat{\underline{k}}\cdot\underline{v})\,\hat{\underline{k}} \Rightarrow \underline{v}=\underline{v}_{\perp}+\underline{v}_{\parallel}$

(parallel to $\hat{\underline{k}}$) $\quad$ ($\perp$ to $\hat{\underline{k}}$)

$\underline{w}\triangleq\hat{\underline{k}}\times\underline{v} \quad\Rightarrow\quad \underline{w}=\hat{\underline{k}}\times\underline{v}=\underbrace{\hat{\underline{k}}\times\underline{v}_{\parallel}}_{=0}+\hat{\underline{k}}\times\underline{v}_{\perp}=\hat{\underline{k}}\times\underline{v}_{\perp}$.

$\|\underline{v}_{\perp}\|=\|\cancel{v\cos\varphi\,\hat{\underline{k}}}+v\sin\varphi\,\hat{\underline{k}}-\cancel{(v\cos\varphi)\,\hat{\underline{k}}}\|=v\sin\varphi$

$\|\underline{w}\|=v\sin\varphi \Rightarrow \begin{cases}\|\underline{v}_{\perp}\|=\|\underline{w}\|\\ \underline{v}_{\perp}\text{ ROTATED BY }90^\circ\text{ WITH RESPECT TO }\underline{w}\end{cases}$

$\begin{cases}\underline{v}_{\perp}^{rot}=\underline{v}_{\perp}\cos\theta+\underline{w}\sin\theta\\ \underline{v}_{\parallel}=\underline{v}_{\parallel}^{rot.}\end{cases}$ (obvious) $\rightsquigarrow \underline{v}_{\parallel}^{rot}=(\hat{\underline{k}}\cdot\underline{v})\cdot\hat{\underline{k}}$

$\Rightarrow \underline{v}^{rot}=\underline{v}_{\perp}^{rot}+\underline{v}_{\parallel}^{rot}=\underline{v}_{\perp}\cos\theta+\underline{w}\sin\theta+(\hat{\underline{k}}\cdot\underline{v})\cdot\hat{\underline{k}}$

$$\underline{v}^{rot}=\left[\underline{v}-(\hat{\underline{k}}\cdot\underline{v})\,\hat{\underline{k}}\right]\cos\theta+(\hat{\underline{k}}\times\underline{v})\sin\theta+(\hat{\underline{k}}\cdot\underline{v})\,\hat{\underline{k}}$$

$\rightarrow$ Rodrigues rotation formula

$$\underline{v}^{rot}=\cos\varphi\,\underline{v}+(1-\cos\varphi)(\hat{\underline{e}}\cdot\underline{v})\cdot\hat{\underline{e}}+(\hat{\underline{e}}\times\underline{v})\sin\varphi$$

$\varphi\rightarrow$ Euler angle

$\hat{e}\rightarrow$ Euler axis

↳ Applying Rodrigues formula to $\{\hat{i}, \hat{j}, \hat{k}\}$

$$\underline{u} = \mathrm{Rf}(\hat{i}) = \hat{i} \cdot \cos\varphi + (1-\cos\varphi)\,(\hat{\underline{e}} \cdot \hat{i}) \cdot \hat{\underline{e}} + (\hat{\underline{e}} \times \hat{i}) \sin\varphi$$

$$= \hat{i} \cos\varphi + (1-\cos\varphi)\, e_1\, \hat{\underline{e}} + (\hat{\underline{e}} \times \hat{i}) \sin\varphi$$

$$\Rightarrow \quad [\underline{\underline{A}}]_{1,1} = \underline{u} \cdot \hat{i} = \cos\varphi + (1-\cos\varphi)\, e_1^2 \qquad ((\underline{e} \times \hat{i}) \cdot \hat{i} = 0.)$$

$$[\underline{\underline{A}}]_{1,2} = \underline{u} \cdot \hat{j} = (1-\cos\varphi) e_1 e_2 + (\hat{\underline{e}} \times \hat{i}) \cdot \hat{j} \sin\varphi \quad (\hat{j} \cdot \hat{i} = 0)$$

$$\hat{\underline{e}} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e_1 & e_2 & e_3 \\ 1 & 0 & 0 \end{vmatrix} = \hat{j}\,(-e_3) + \hat{k} \cdot (e_2)$$

$$\Rightarrow [\underline{\underline{A}}]_{1,2} = (1-\cos\varphi)\, e_1 e_2 - e_3 \sin\varphi$$

$\vdots$

and so on Matrix $\underline{\underline{A}}$ is obtained in the previous form.

Rf $\rightarrow$ $$\underline{N}^{rot} = \cos\varphi \cdot \underline{N} + (1-\cos\varphi)\,(\hat{\underline{e}} \cdot \underline{N}) \cdot \hat{\underline{e}} + (\hat{\underline{e}} \times \underline{N}) \sin\varphi$$

$\downarrow$

$$\underline{\underline{A}} = \cos\varphi \cdot [\hat{i} \,\vdots\, \hat{j} \,\vdots\, \hat{k}] + (1-\cos\varphi)\,(\hat{\underline{e}}^T \cdot [\hat{i} \,\vdots\, \hat{j} \,\vdots\, \hat{k}]) \cdot \hat{\underline{e}} + (\hat{\underline{e}} \times [\hat{i} \,\vdots\, \hat{j} \,\vdots\, \hat{k}]) \sin\varphi$$

$\downarrow$

$\underline{\underline{A}}^{rot}$ $\rightarrow$ $$\underline{\underline{A}} = \underline{\underline{I}} \cos\varphi + (1-\cos\varphi) \cdot \underline{e}^T \cdot \underline{e} + [\underline{e} \times] \sin\varphi$$

$$[\underline{e} \times] = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

$(\underline{\underline{A}}) \rightarrow (\underline{e}, \varphi)$ (NOT ALWAYS POSSIBLE.) $\rightarrow$ SINGULARITY

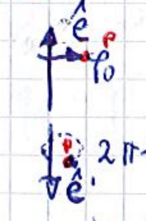
(1): $\mathrm{Tr}(\underline{\underline{A}}) = 1 + 2\cos\varphi \rightarrow \varphi = \cos^{-1}\left[\frac{1}{2}\left(\mathrm{Tr}(\underline{\underline{A}}) - 1\right)\right]$

Solve for φ

~ PROBLEM 1: FAKE — I have 2 solution for φ (φ_0; $2\pi - \varphi_0$) that will provide 2 different axis $\hat{\underline{e}}$ and $-\hat{\underline{e}}$ BUT $\pm\hat{\underline{e}}$ will be simply ROTATED OF π (we are looking at the same axis with different orientation)

$\varphi = \varphi_0 < \pi/2 \rightarrow \sin\varphi_0 > 0 \Rightarrow$

$\varphi = 2\pi - \varphi_0 \rightarrow \sin\varphi_0 < 0$



but same point P is reached.

(2) Solve for $\underline{e}$

$\varphi = \varphi_0 < \pi/2 \Rightarrow \{e_1; e_2; e_3\}$

$\varphi = 2\pi - \varphi_0 \Rightarrow \{e_1' = -e_1;\ e_2' = -e_2 \ldots\}$

$$e_1 = \frac{(A_{23} - A_{32})}{2\sin\varphi}$$

$$e_2 = \frac{(A_{31} - A_{13})}{2\sin\varphi}$$

$$e_3 = \frac{(A_{12} - A_{21})}{2\sin\varphi}$$

in fact: $A_{12} = -e_1 e_2 (\cos\varphi - 1) + e_3 \sin\varphi$

$A_{21} = e_1 e_2 (1 - \cos\varphi) + e_3 \sin\varphi$ $\Rightarrow A_{12} + A_{21} = 2 e_3 \sin\varphi$

PROBLEM 2: $\varphi = n\pi$ represent singularity case of undetermined rotation:

$e_1 = \frac{0}{0}$; $e_2 = \frac{0}{0}$ ... $\Rightarrow$ AXIS NOT UNIVOCALLY DETERMINED.

UNDETERMINED AXIS $\Rightarrow$ it means that the final configuration might be reached through (2) consecutive rotations of the same values $\varphi = n\pi$ but such (2) axis are undetermined.

obs → Euler axis/angle is useful cause of the 4 par used instead of the 9 par of direction cosine matrix (!! THIS MEANS 5-LESS EQUATION TO INTEGRATE IN TIME)

BUT PROBLEM 3: THERE'S NO RULE FOR CONSECUTIVE ROTATIONS THROUGH EULER AXIS/ANGLE.

3 → "QUATERNIONS"

IIIrd $(\underline{e}, \varphi) \to \underline{q}$

$(\underline{e}, \varphi) \to$

$q_1 = e_1 \sin(\varphi/2)$

$q_2 = e_2 \sin(\varphi/2)$

$q_3 = e_3 \sin(\varphi/2)$

$q_4 = \cos(\varphi/2)$

$\underline{q}: \quad \|q\| = q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$

(in fact: $\underline{e}: \|\underline{e}\| = 1$

$$\|q\| = \underbrace{(e_1^2 + e_2^2 + e_3^2)}_{=1} \sin^2(\varphi/2) + \cos^2 \varphi/2 =$$

$$= \sin^2 \varphi/2 + \cos^2 \varphi/2 = 1 \quad)$$

" Advantage of quaternions is in the fact that there's no singularity in the transformation:

QUATERNIONS ⟶ DIRECTION COSINE MATRIX

EULER AXIS/ANGLE $\not\longrightarrow$ DIRECTION COSINE MATRIX (singularity).

$\underline{q} \to \underline{\underline{A}}$ (DIRECT)

$$\underline{\underline{A}} = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 \,; & 2(q_1 q_2 + q_3 q_4) \,; & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) \,; & -q_1^2 + q_2^2 - q_3^2 - q_4^2 \,; & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) \,; & 2(q_2 q_3 - q_1 q_4) \,; & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

$$\Downarrow$$

$$\underline{\underline{A}} = (q_4^2 - \underline{q}^T \underline{q}) \cdot \underline{\underline{I}} + 2\, \underline{q} \cdot \underline{q}^T - 2 q_4 [\underline{q} \times]$$

where:

* VECTORIAL REPRESENTATION FOR QUATERNIONS → $\underline{q} = \begin{vmatrix} q_1 \\ q_2 \\ q_3 \end{vmatrix} \,; \; q_4$

* $[\underline{q} \times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$

This is pretty easy to verify:

* $A_{11}: A_{11}(\underline{e}, \varphi) = \cos\varphi + e_1^2 (1 - \cos\varphi)$

$\cos\varphi = 1 - 2\sin^2(\varphi/2)$

$A_{11}(\underline{q}, q_4) = q_1^2 - q_2^2 - q_3^2 - q_4^2 =$

$-\sin^2(\varphi/2) + \cos^2(\varphi/2) = \cos(\varphi)$

$\cos(\varphi/2) \cdot \sin(\varphi/2) = \sin(\varphi)$

$= e_1^2 \sin^2(\varphi/2) - e_2^2 \sin^2(\varphi/2) - e_3^2 \sin^2(\varphi/2) - e_4^2 \sin^2(\varphi/2)$

$= e_1^2 \sin^2(\varphi/2) + e_1^2 \cos^2(\varphi/2) - e_1^2 \cos^2 \varphi/2 - (e_2^2 + e_3^2 + e_4^2) \sin^2(\varphi/2)$

$= e_1^2 \sin^2(\varphi/2) + e_1^2 - e_1^2 \sin^2(\varphi/2) - e_1^2 \cos^2(\varphi/2) - (1 - e_1^2) \sin^2(\varphi/2)$

$= -e_1^2 \overbrace{(\cos^2(\varphi/2) - \sin^2(\varphi/2))}^{\cos\varphi} + e_1^2 - e_1^2 \sin^2(\varphi/2) - (1 - e_1^2) \sin^2(\varphi/2)$

$= e_1^2 (1 - \cos\varphi) + \cos^2(\varphi/2) - \sin^2(\varphi/2) = \cos\varphi + e_1^2 (1 - \cos\varphi)$

* $A_{21} = e_1 e_2 (1 - \cos\varphi) + e_3 \sin\varphi$

$= 2 (e_1 e_2 \underbrace{\sin^2(\varphi/2)}_{2\sin^2(\varphi/2) = 1 - \cos\varphi} + \overbrace{\cos(\varphi/2) \sin(\varphi/2)}^{\sin\varphi} e_3) = e_1 e_2 (1 - \cos\varphi) + e_3 \sin\varphi.$

IF $\underline{A} \to \underline{q}$ (INVERSE)

$$q_1 = \frac{1}{4q_4}(A_{23} - A_{32}) \qquad q_3 = \frac{1}{4q_4}(A_{12} - A_{21})$$

$$q_2 = \frac{1}{4q_4}(A_{31} - A_{13}) \qquad q_4 = \pm\frac{1}{2}(1 + A_{11} + A_{22} + A_{33})^{1/2}$$

$$\left(\underline{q} = \begin{vmatrix} e_1 \sin\varphi/2 \\ e_2 \sin\varphi/2 \\ e_3 \sin\varphi/2 \end{vmatrix}, \quad q_4 = \cos\varphi/2. \right)$$

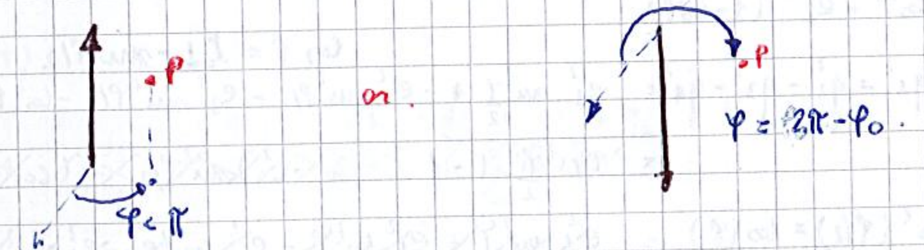
ambiguity: $q_4 = \pm\frac{1}{2}(1 + A_{11} + \dots)$

↳ solve for $\varphi = 2\cos^{-1}\left[\frac{1}{2}(1 + A_{11} + \dots)\right]$

that means $\varphi = \varphi_0 < \pi$ ($\varphi_0 < \pi$)

$\varphi = 2\pi - \varphi_0$

Same ambiguity as for Euler axis/angle:



φ Singularity: !! BOTH in INVERSE / DIRECT TRANSFORMATION !!

(inverse). $\underline{A} \to \underline{q}$ ⇒ → I^st: solving for q_4 ⇒ $q_4 = 0$ ⇒ $\varphi = (2k+1)\frac{\pi}{2}$

→ II^nd: computing $\underline{q}$: • $\sin\varphi/2 = \sin\left[(2k+1)\frac{\pi}{2}\right] = 1$

• $q_1 = \frac{1}{4q_4}(A_{23} - A_{32}) = \frac{(\cdot)}{0}$

IF $q_4 = 0$

THEN $q_1 = \frac{(\cdot)}{0}$; $q_2 = \frac{(\cdot)}{0}$; $q_3 = \frac{(\cdot)}{0}$; $q_4 = \frac{(\cdot)}{0}$

that mean to have a singularity in inverse transformation.

BUT I can always choose an alternative order of inverse transformation.

$$q_1^2 = \pm\frac{1}{2}\sqrt{1 + A_{11} - A_{22} - A_{33}}$$

✓ if $\boxed{q_1^2 \neq 0}$

$$q_2^2 = \frac{1}{4q_1^2}(A_{12} + A_{21}) \; ; \quad q_3^2 = \frac{1}{4q_1^2}(A_{13} + A_{31}) \; ; \quad q_4^2 = \frac{1}{4q_1^2}(A_{23} - A_{32})$$

OR

$$q_2^3 = \pm\frac{1}{2}\sqrt{1 - A_{11} + A_{22} - A_{33}}$$

✓ if $\boxed{q_2^3 \neq 0}$

$$q_1^3 = \frac{1}{4q_2^3}(A_{12} + A_{21}) \; ; \quad q_3^3 = \frac{1}{4q_2^3}(A_{23} + A_{32}) \; ; \quad q_4^3 = \frac{1}{4q_2^3}(A_{31} - A_{13})$$

OR

$$q_3^4 = \pm\frac{1}{2}\sqrt{1 - A_{11} - A_{22} + A_{33}}$$

✓ if $\boxed{q_3^4 \neq 0}$

$$q_1^4 = \frac{1}{4q_3^4}(A_{13} + A_{31}) \; ; \quad q_2^4 = \frac{1}{4q_3^4}(A_{23} + A_{32}) \; ; \quad q_4^4 = \frac{1}{4q_3^4}(A_{12} - A_{21})$$

Easy to verify formulas for each inverse transformation.

$$q_4 = \pm\frac{1}{2}\left(1 + \cancel{q_1^2} - \cancel{q_2^2} - \cancel{q_3^2} - \cancel{q_4^2} - \cancel{q_1^2} + \cancel{q_2^2} - q_3^2 + q_4^2 - q_1^2 - q_2^2 + \cancel{q_3^2} + \cancel{q_4^2}\right)^{1/2}$$

$$= \pm\frac{1}{2}\left(1 - \underbrace{(q_1^2 + q_2^2 + q_3^2 + q_4^2)}_{=1} + 2q_4^2\right)^{1/2} = q_4. \quad \text{cvd}$$

$$q_1 = \frac{1}{4q_4^{2/2}}\left(2\cancel{q_2 q_3} + 2q_1q_4 - 2\cancel{q_2 q_3} + 2q_1q_4\right) = q_1 \quad \text{cvd.}$$

→ consecutive rotations

$$\underline{q}'' = \begin{bmatrix} q_4' & q_3' & -q_2' & +q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \underline{q}$$

⊛→ Better explained in A.P. Kinematic.

4 → "GIBBS VECTOR"

Γ IV th. $(\underline{q}) \to \underline{g}$

$\underline{g}$ → gibbs vector.

$$g_1 = \frac{q_1}{q_4} = e_1 \tan(\varphi/2)$$

← Not proper phisiel / geometrical meaning.

$$g_2 = \frac{q_2}{q_4} = e_2 \tan(\varphi/2)$$

$$g_3 = \frac{q_3}{q_4} = e_3 \tan(\varphi/2)$$

Gibbs vector is a minimal attitude minimization (3 par. to describe relative rotation of 3 axis from a fixed reference frame)

Γ $\underline{g} \to \underline{A}$ (DIRECT) $\varnothing$ singularity

$$\underline{\underline{A}} = \frac{1}{1+g_1^2+g_2^2+g_3^2}\begin{bmatrix} 1+g_1^2-g_2^2-g_3^2 & 2(g_1g_2+g_3) & 2(g_1g_3-g_2) \\ 2(g_1g_2-g_3) & 1-g_1^2+g_2^2-g_3^2 & 2(g_2g_3+g_1) \\ 2(g_1g_3+g_2) & 2(g_2g_3-g_1) & 1-g_1^2-g_2^2+g_3^2 \end{bmatrix}$$

$$\underline{\underline{A}} = \frac{(1-g^2)\,\underline{\underline{I}} + 2\,\underline{g}\,\underline{g}^T - 2\,[\underline{g}\times]}{(1+g^2)}$$

$$[g\times] = \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix}$$

Γ $\underline{A} \to \underline{g}$ (inverse) !! Singularity !!

$$g_1 = \frac{A_{23} - A_{32}}{1 + A_{11} + A_{22} + A_{33}}$$

→ Den: $1 + \cancel{q_1^2} - \cancel{q_2^2} - \cancel{q_3^2} - \cancel{q_4^2} - q_1^2 + \cancel{q_2^2} - q_3^2 - q_4^2 - \cancel{q_1^2} - q_2^2 + \cancel{q_3^2} + \cancel{q_4^2} = 1 + (q_1^2 - q_2^2 - q_3^2 - q_4^2) + q_4 = q_4$.

$$g_2 = \frac{A_{31} - A_{13}}{1 + A_{11} + A_{22} + A_{33}}$$

$$g_3 = \frac{A_{12} - A_{21}}{1 + A_{11} + A_{22} + A_{33}}$$

Singularity: for $\varphi = (2k+1)\pi \Rightarrow q_4 = \cos(\varphi/2) = 0$.

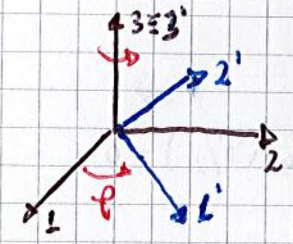
Γ consecutive rotations:

$$\underline{g}'' = \frac{\underline{g} + \underline{g}' - \underline{g}' \times \underline{g}}{1 - \underline{g} \cdot \underline{g}'}$$

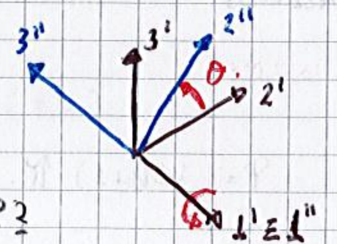
5 → " Euler angles."

[It's always possible to overlap 2 orthogonal reference frames by appropriate rotations of 1 of the 2 R.F. around its reference axis]

e.g.: $\{1,2,3\} \xrightarrow{\text{overlap}} \{u, v, w\}$.

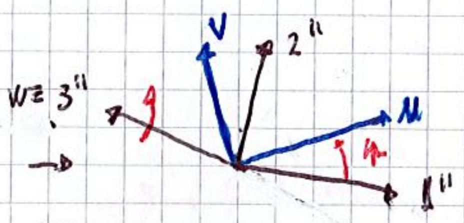


$$\begin{cases} 3' \equiv 3 \\ 1' = \cos\varphi\, 1 + \sin\varphi\, 2 \\ 2' = -\sin\varphi\, 1 + \cos\varphi\, 2 \end{cases} \rightarrow$$



→

$$\underline{\underline{A}}_3^{(1)} = \begin{bmatrix} c\varphi & s\varphi & 0 \\ -s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underline{\underline{A}}_1^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}$$



$\varphi \rightarrow I^{st}$ rotation

$\theta \rightarrow II^{nd}$ rotation

$\psi \rightarrow III^{rd}$ rotation.

$$\underline{\underline{A}}_3^{(3)} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \underline{\underline{A}}_{313} = \underline{\underline{A}}_3 \cdot \underline{\underline{A}}_1 \cdot \underline{\underline{A}}_3 . \\ \underline{\underline{A}}_{313} : \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underline{\underline{A}}_{313} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}. \end{cases}$$

$$A_{313} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\varphi & s\varphi & 0 \\ -s\varphi & c\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix} = \underline{\underline{A}}_3^{(1)} \cdot \begin{Bmatrix} 1' \\ 2' \\ 3' \end{Bmatrix} ; \quad \begin{Bmatrix} 1'' \\ 2'' \\ 3'' \end{Bmatrix} = \underline{\underline{A}}_1^{(2)} \begin{Bmatrix} 1' \\ 2' \\ 3' \end{Bmatrix} ; \quad \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underline{\underline{A}}_3^{(3)} \begin{Bmatrix} 1'' \\ 2'' \\ 3'' \end{Bmatrix}.$$

$$\Rightarrow \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underline{\underline{A}}_3^{(3)} \cdot \underline{\underline{A}}_1^{(2)} \, \underline{\underline{A}}_3^{(1)} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}.$$

$$\underline{\underline{A}}_{313} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\varphi & s\varphi & 0 \\ -c\theta s\varphi & c\theta c\varphi & s\theta \\ -s\theta s\varphi & -s\theta c\varphi & c\theta \end{bmatrix} = \begin{bmatrix} c\psi c\varphi - s\psi c\theta s\varphi ; & c\psi s\varphi + s\psi c\theta c\varphi ; & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$\Rightarrow \underline{\underline{A}}_{313} = \begin{bmatrix} c\psi c\varphi - s\psi s\varphi c\theta ; & c\psi s\varphi + s\psi c\varphi c\theta ; & s\psi s\theta \\ -s\psi c\varphi - c\psi s\varphi c\theta ; & -s\psi s\varphi + c\psi c\varphi c\theta ; & c\psi s\theta \\ s\varphi s\theta ; & -c\varphi s\theta ; & c\theta \end{bmatrix}$$

$\Gamma\; \underline{\underline{A}} \rightarrow \{\theta, \varphi, \psi\}$ (inverse)

Starting from θ $\quad \underline{\underline{A}} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \boxed{c\theta} \end{bmatrix}$

$$\theta = \cos^{-1}(A_{33}) \rightarrow \psi = \tan^{-1}\left(\frac{A_{31}}{A_{32}}\right)$$

$$\varphi = \tan^{-1}\left(-\frac{A_{13}}{A_{23}}\right)$$

Singularity:

for $\theta = k\pi$ $\Rightarrow$ $\begin{cases} A_{31} = s\varphi s\theta \\ A_{32} = c\varphi s\theta \\ A_{13} = s\theta s\psi \\ A_{23} = c\psi s\theta \end{cases} = 0$

$(\sin\theta \rightarrow 0)$

$\Rightarrow \psi = \tan^{-1}\left(\frac{0}{0}\right)$; $\varphi = \tan^{-1}\left(\frac{0}{0}\right)$ that are undetermined ⅃

I could (theoretically) 27 rotations matrices:

e.g.: starting with rotation around 1

$\cancel{1}\,1\,\cancel{1}$ → I can't overlap 2 R.F. rotating around 1 only axis (2 d...)

$\cancel{113}$, $1\,\cancel{1}\,2$ → I " " " " " 2 times consecutively around 1 axis.

$\{121, 123\}$ → OK $\quad 1\,\cancel{2}\,\cancel{2}$

$\{131, 132\}$ → OK $\quad 1\,\cancel{3}\,3$

9 → theoretical

4 → admissible.

If this analogous count is made for all the other 2 axis $\Rightarrow$ 12 = 3 · 4 admissible sequencies:

$\Rightarrow$ I can write 12 rotation cosine matrix

* $\rightarrow$ All different indices: {312, 313, 213, 231; 123; 132} $\Rightarrow$ 6

+

* $\rightarrow$ 2 equal indices: {313; 323; 212; 232; 131; 121} $\Rightarrow$ 6

=

12.

linearization: if $x = \{\theta, \varphi, \psi\} < 10-15°$

$$\Rightarrow \begin{cases} \cos x \simeq 1 \\ \sin x \simeq x \end{cases} ; \quad x_1 \cdot x_2 \simeq 0$$

THEN

$$\underline{A}_{312} = \begin{bmatrix} c\psi c\varphi - s\psi s\varphi s\theta & c\psi s\varphi + s\psi c\varphi s\theta & -s\psi c\theta \\ -s\varphi c\theta & c\varphi c\theta & s\theta \\ s\psi c\varphi + c\psi s\varphi s\theta & s\psi s\varphi - c\psi c\varphi s\theta & c\theta c\psi \end{bmatrix}$$

lin:

$$\underline{A}^{lin}_{312} = \begin{bmatrix} 1 & \varphi & -\psi \\ -\varphi & 1 & \theta \\ \psi & -\theta & \psi \end{bmatrix}$$

$$\underline{A}_{313} = \begin{bmatrix} c\psi c\varphi - s\psi s\varphi c\theta & c\psi s\varphi + s\psi c\varphi c\theta & s\psi s\theta \\ -s\psi c\varphi - c\psi s\varphi c\theta & -s\psi s\varphi + c\psi c\varphi c\theta & c\psi s\theta \\ s\varphi s\theta & -c\varphi s\theta & c\theta \end{bmatrix}$$

$$\xrightarrow{lin} \quad \underline{A}_{313} = \begin{bmatrix} 1 & \varphi+\psi & 0 \\ -\varphi-\psi & 1 & \theta \\ 0 & -\theta & 1 \end{bmatrix}$$

* $\Rightarrow$ rotation with all different indices:

$$\underline{A}^{lin}_{312}(\varphi,\theta,\psi) = \begin{bmatrix} 1 & \varphi & -\psi \\ -\varphi & 1 & \theta \\ \psi & -\theta & 1 \end{bmatrix} = \underline{A}^{lin}_{321}(\varphi,\psi,\theta) = \underline{A}^{lin}_{213}(\psi,\theta,\varphi) \ldots$$

$$\left\ulcorner \underline{A}^{lin} = \underline{I} - \left[\begin{Bmatrix} \varphi \\ \theta \\ \psi \end{Bmatrix} \times \right] \right\lrcorner$$

$\leftarrow$ generic operator:

$$[\underline{a} \times] = \begin{bmatrix} 0 & -a_1 & +a_3 \\ +a_1 & 0 & -a_2 \\ -a_3 & +a_2 & 0 \end{bmatrix}$$

If want to build

$$\underline{A}^{lin}_{1,2,3} = \underline{I} - \left[\begin{Bmatrix} \theta \\ \varphi \\ \psi \end{Bmatrix} \times \right] = \begin{bmatrix} 1 & \varphi & -\psi \\ -\varphi & 1 & +\theta \\ \psi & -\theta & 1 \end{bmatrix}$$

* $\Rightarrow$ rotation with 2 equal indices:

" the first and the last indices are coincident, due to loss of importance (course of small angles) $\Rightarrow$ the Ist and the IIIrd rotation are almost about the same axis "

$$\left\ulcorner \underline{A}^{lin} = \underline{I} - \left[\begin{Bmatrix} \varphi+\psi \\ \theta \\ 0 \end{Bmatrix} \times \right] \right\lrcorner$$

the 2 rotation axis 3; 3" collapse onto the same axis.

$$\underline{A}^{lin}_{313} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\varphi-\psi & 0 \\ \varphi+\psi & 0 & -\theta \\ \theta & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \varphi+\psi & 0 \\ -\psi-\varphi & 1 & \theta \\ -\theta & 0 & 1 \end{bmatrix}$$

end

→ " Attitude Kinematic "

In order to define how parameters variate in time we need to have a relation of the form:

(P) $$\frac{d\underline{\alpha}}{dt} = f(\underline{w}) \rightarrow \underline{\alpha} = \begin{vmatrix} \underline{\underline{A}} \\ \underline{q} \\ \{\varphi, \theta, \psi\} \end{vmatrix}$$

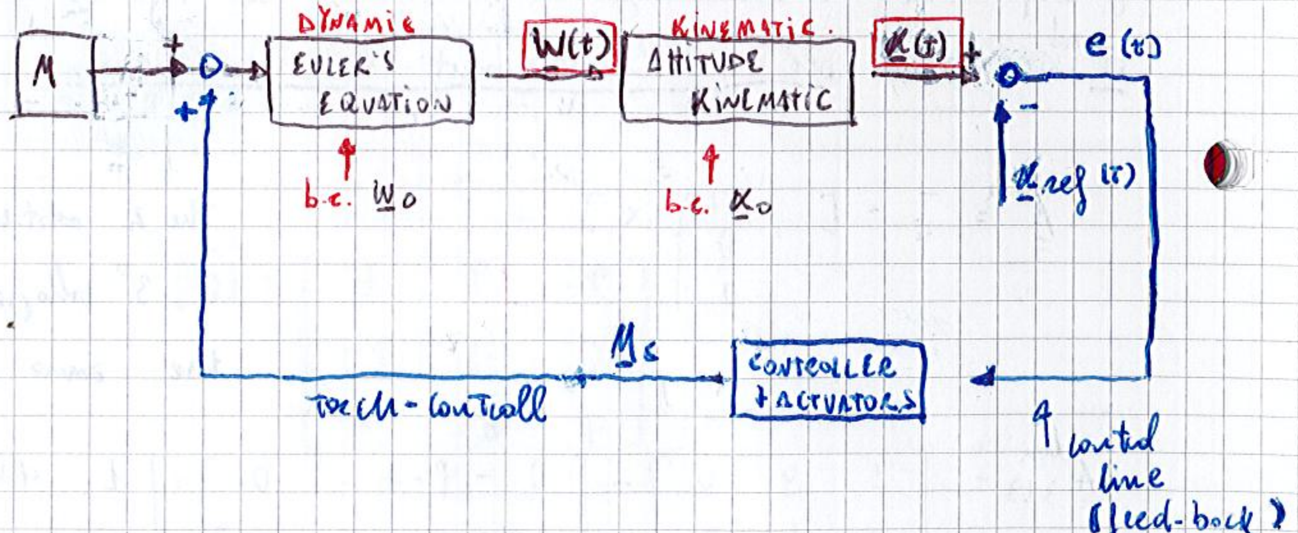
$\underline{\alpha} = A.P.$ → generic attitude parameter's set.

$\underline{w}$ → angular velocity in P.I.F. reference frame.

(not inertial → rotating R.F.)

(b.c.) $\underline{\alpha}_0 = \underline{\alpha}(t=0)$ $\quad (\underline{\alpha} = A.P.)$

therefore the overall differential problem (to be integrated in time) is the following:



4. In order to obtain the kinematic law for all the attitude parameters there's a common procedure:

→ procedure. (∀ A.P. $\underline{\alpha}$)

(1) Rule for consecutive rotation. (recall)

(2) consider rotations at times $(t, t+\Delta t)$

$$\underline{\alpha}(t) \; ; \; \underline{\alpha}(t+\Delta t) \Rightarrow \underline{\alpha}(t+\Delta t) = \underline{\alpha}'(\underline{w}) \cdot \underline{\alpha}(t).$$

CONSECUTIVE ROTATIONS AS A FUNCTION OF $\underline{w}$.

(3) Impose small rotations

* $\underline{\alpha}(t+\Delta t) \approx \underline{\alpha}(t) \quad (\varphi \to 0)$
* $\varphi \approx w \cdot \Delta t$

$\boxed{\underline{w} = w \cdot \underline{e}}$

(4) Solve incremental ratio: $\displaystyle\lim_{\Delta t \to 0} \frac{\underline{\alpha}(t+\Delta t) - \underline{\alpha}}{\Delta t}$

$I^{ST}_{A.P.}$ → DIRECTION COSINE MATRIX $(\underline{\underline{A}})$

(1) $\underline{\underline{A}}(t+\Delta t) = \underline{\underline{A}}' \, \underline{\underline{A}}(t)$

(2) $\underline{\underline{A}}' = \underline{\underline{I}} \cdot \cos\varphi + (1-\cos\varphi)\, \underline{e} \cdot \underline{e}^T - \sin\varphi \, [\underline{e} \times]$

$\underline{w} = \|\underline{w}\| \cdot \underline{e}$

($\underline{e}$ → defines the direction with of the PIF(t+Δt) is rotated with respect to PIF(t). and it's expressed properly in PIF(t).

⇒ $\underline{\underline{A}} = \underline{\underline{A}}'$)

Small angles assumption:

$\varphi \to 0 \Rightarrow \begin{cases} \cos\varphi \to 1 \\ \sin\varphi \to \varphi \end{cases} \Rightarrow \underline{\underline{A}}' = \underline{\underline{I}} + \varphi \, [\underline{e} \times]$

SINCE $\underline{e}$ represent the axis around which occurs the rotation of the PIF $\{i, j, k\}$ of magnitude φ.

AND SINCE we are dealing with an infinitesimal instant of time $(\Delta t \to 0)$

THEN $\varphi = w \cdot \Delta t$. (linearized kinematic)

(3) $\begin{cases} \underline{\underline{A}}' = \underline{\underline{I}} - \varphi \cdot [\underline{e}\times] \\ \varphi = W \cdot \Delta t \, ; \ \underline{W} = W \underline{e} \end{cases} \Rightarrow \underline{\underline{A}}' = \underline{\underline{I}} - W \cdot [\underline{e}\times]$

$\Rightarrow \underline{\underline{A}}' = \underline{\underline{I}} - W[\underline{e}\times] = \underline{\underline{I}} - [\underline{W}\times]$

$(\underline{W} = W\underline{e} \Rightarrow W[\underline{e}\times] = [(W\underline{e})\times] = [\underline{W}\times])$

Therefore the limit of incremental ratio will be

(4) $\underline{\underline{A}}(t+\Delta t) = \underline{\underline{A}}' \, \underline{\underline{A}}(t) = (\underline{\underline{I}} - [\underline{W}\times]) \cdot \underline{\underline{A}}(t)$

$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot [\underline{\underline{A}}(t+\Delta t) - \underline{\underline{A}}(t)] = \frac{1}{\Delta t} \cdot [\cancel{\underline{\underline{A}}(t)} - [\underline{W}\times]\underline{\underline{A}}(t) - \cancel{\underline{\underline{A}}(t)}]$

$\Rightarrow$

$\underline{\underline{A}}(t) = \underline{\underline{I}} \cdot \cos\varphi(t) + (1-\cos\varphi(t))\,\underline{e}^T(t) \cdot \underline{e}(t) - \sin\varphi(t)[\underline{e}\times]$

(p): $\frac{d\underline{\underline{A}}}{dt} = -[\underline{W}\times] \, \underline{\underline{A}}$

where $[\underline{W}\times] = \begin{bmatrix} 0 & -W_3 & W_2 \\ W_3 & 0 & -W_1 \\ W_2 & +W_1 & 0 \end{bmatrix}$

obs 1 → ~~A~~ closed solution for $\frac{d\underline{\underline{A}}}{dt} = -[\underline{W}\times]\underline{\underline{A}}$ therefore such integration have to be performed numerically

$\underline{\underline{A}}(t+\Delta t) = \underline{\underline{A}}(t^*) + \int_{t^*}^{+\Delta t} \dot{\underline{\underline{A}}} \, dt$

$= \underline{\underline{A}}(t^*) + \underbrace{\int_{t^*}^{t^*+\Delta t} [-\underline{W}(\tau)\times] \, \underline{\underline{A}}(\tau) \, d\tau}_{\text{NUMERICAL INTEGRATION}}$

⇒ Error of integration (NUMERICAL) can imply that matrix $\underline{\underline{A}}(t+\Delta t)$ is anymore orthogonal

$(\underline{\underline{A}}^T(t+\Delta t) \cdot \underline{\underline{A}}(t+\Delta t) \neq \underline{\underline{I}})$

→ orthogonalization procedure:

$\underline{\underline{A}}^{(0)}_{(t)}$ (not orthogonal) $\rightarrow \underline{\underline{A}}^{(1)}_{(t)} = \underline{\underline{A}}^{(0)}_{(t)} \cdot \frac{3}{2} - \underline{\underline{A}}^{(0)}_{(t)} \, \underline{\underline{A}}^{(0)T}_{(t)} \cdot \underline{\underline{A}}^{(0)}_{(t)} \cdot \frac{1}{2}$

$\vdots$

$\underline{\underline{A}}^{(K+1)} = \underline{\underline{A}}^{(K)} \cdot \frac{3}{2} - \underline{\underline{A}}^{(K)} \, \underline{\underline{A}}^{(K)T} \, \underline{\underline{A}}^{(K)} \cdot \frac{1}{2}$

WHILE $(\underline{\underline{A}}^{(K)T}_{(t)} \cdot \underline{\underline{A}}^{(K)}_{(t)}) \not\approx \underline{\underline{I}}$

THEN go to successive time instant that

II^{nd} A.P. → QUATERNIONS

(1) $\dot{\underline{q}} = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \underline{q}$

(2) → $\underline{\underline{A}}'' = \underline{\underline{A}}' \cdot \underline{\underline{A}} \ ; \ \underline{\underline{A}} = (q_4^2 - \underline{q}^T \cdot \underline{q})\underline{\underline{I}} + 2\underline{q} \cdot \underline{q}^T - 2q_4[\underline{q}\times]$

Therefore: $\underline{q}(t+\Delta t) = \left[\; q_1' \; q_2' \; q_3' \; q_4' \;\right] \underline{q}(t)$

$$\begin{cases} q_1' = e_u \sin \varphi/2 \\ q_2' = e_v \sin \varphi/2 \\ q_3' = e_w \sin \varphi/2 \\ q_4' = \cos \varphi/2 \end{cases}$$

decoupling $\sin\varphi$ part from $\cos\varphi$ part in order to impose later small rotations assumption.

$$\Rightarrow \underline{q}(t+\Delta t) = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \underline{q}(t)$$

So:

$$\underline{q}(t+\Delta t) = \left[\sin(\varphi/2) \begin{bmatrix} 0 & e_3' & -e_2' & e_1' \\ -e_3' & 0 & e_1' & e_2' \\ e_2' & -e_1' & 0 & e_3' \\ -e_1 & -e_2 & -e_3 & 0 \end{bmatrix} + \cos(\varphi/2)\, \underline{\underline{I}} \right] \underline{q}(t)$$

(2, 3) $\boxed{\Delta t \to 0} \Rightarrow \varphi \simeq w \cdot \Delta t$

$\boxed{\varphi \to 0} \Rightarrow \begin{cases} \sin(\varphi/2) \to \frac{\varphi}{2} \\ \cos(\varphi/2) \to 1 \end{cases} \Rightarrow \begin{cases} \sin\varphi/2 \simeq \frac{w \cdot \Delta t}{2} \\ \cos\varphi/2 \simeq 1 \end{cases}$

$$\underline{q}(t+\Delta t) = \left[1/2\, (w \cdot \Delta t)\, [\underline{e} \times] + \underline{\underline{I}} \right] \underline{q}(t)$$

& but size($[\underline{e}\times]$) = 4x4.

(1)

$$\lim_{\Delta t \to 0} \frac{\Delta \underline{q}}{\Delta t} = \frac{1}{2} w\, [\underline{e} \times] = \frac{1}{2} [\underline{w} \times]$$

it's not the same $\underline{w}$ matrix as before $\Rightarrow [\underline{w}\times] \neq [\underline{\Omega}\times]$

This time is a 4x4 matrix.

$$\Rightarrow \frac{d\underline{q}}{dt} = \frac{1}{2} [\underline{\Omega} \times] \underline{q}$$

where

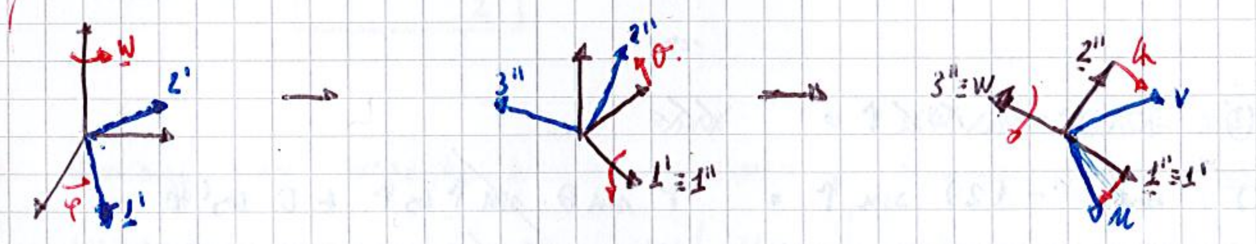
$$[\underline{\Omega} \times] = \begin{bmatrix} 0 & w_w & -w_v & w_u \\ -w_w & 0 & w_u & w_v \\ w_v & -w_u & 0 & w_w \\ -w_u & -w_v & -w_w & 0 \end{bmatrix}$$

III 2nd A.P ⇒ EULER ANGLES.

"Since there's no rule for consecutive rotations for Euler's angles ⇒ it's impossible to follow the previous procedure."

e.g. ⇒ considering again the {3, 1, 3} sequence.

3) $\underline{w} = \dot\varphi \cdot \underline{3}$ 1) $\underline{w} = \dot\theta \cdot \underline{1}'$ 3) $\underline{w} = \dot\psi\, \underline{w}$



$$\Rightarrow \underline{w} = \dot\varphi\, \underline{3} + \dot\theta\, \underline{1}' + \dot\psi\, \underline{w}$$

Projecting $\underline{w}$ vector on the final (u,v,w) reference frame what i obtain is:

$$w_u = \underline{w}\cdot\underline{u} = \dot\varphi\, \underline{3}\cdot\underline{u} + \dot\theta\, \underbrace{\underline{1}'\cdot\underline{u}}_{\cos\psi} + \dot\psi\, \underbrace{\underline{w}\cdot\underline{u}}_{=0}$$

$$w_v = \underline{w}\cdot\underline{v} = \dot\varphi\, \underline{3}\cdot\underline{v} + \dot\theta\, \underbrace{\underline{1}'\cdot\underline{v}}_{\sin\psi} + \dot\psi\, \underbrace{\underline{w}\cdot\underline{v}}_{=0}$$

$$w_w = \underline{w}\cdot\underline{w} = \dot\varphi\, \underline{3}\cdot\underline{w} + \dot\theta\, \underbrace{\underline{1}'\cdot\underline{w}}_{=0} + \dot\psi\, \underbrace{\underline{w}\cdot\underline{w}}_{=1}$$

↳ critical ⇒ need to consider Lyme matrix $\underline{A} = \underline{A}(\theta, \varphi, \psi)$

In computing $\underline{3}\cdot\underline{u}$ i might help directly with the direction cosine matrix

$$\underline{\underline{A}}_{313} = \begin{bmatrix} \cdot & ; & ; & \underline{3}\cdot\underline{u} \\ & ; & ; & \underline{3}\cdot\underline{v} \\ & ; & ; & \underline{3}\cdot\underline{w} \end{bmatrix}$$

$\underline{3}\cdot\underline{u} = \sin\psi \sin\theta$ $\quad$ $\underline{3}\cdot\underline{w} = \cos\theta.$

$\underline{3}\cdot\underline{v} = \cos\psi \sin\theta$

⇒ !! Valid only for the 313 sequence. !!

$$\begin{cases} w_u = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi & (1) \\ w_v = \dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi & (2) \\ w_w = \dot{\varphi}\cos\theta + \dot{\psi}. & (3) \end{cases}$$

⇒

$w_u\cos\psi - w_v\sin\psi =$

$$= (1)\cdot\cos\psi - (2)\cdot\sin\psi = \dot{\varphi}\sin\theta\cancel{\sin\psi\cos\psi} + \dot{\theta}\cos^2\psi - \dot{\varphi}\sin\theta\cancel{\cos\psi}\sin\psi + \dot{\theta}\sin^2\psi$$

⇒ $\dot{\theta} = w_u\cos\psi - w_v\sin\psi$

$$w_u = \dot{\varphi}\sin\theta\sin\psi + w_u\cos^2\psi - w_v\sin\psi\cos\psi$$

$$w_u\sin^{\cancel{2}}\psi = \dot{\varphi}\cancel{\sin\psi}\sin\theta + w_v\cancel{\sin\psi}\cos\psi \Rightarrow \dot{\varphi} = \frac{1}{\sin\theta}\left[w_u\sin\psi + w_v\cos\psi\right]$$

⇒ $\dot{\psi} = w_w - \tan\theta\left[w_u\sin\psi + w_v\cos\psi\right]$

⇒ 313

$$\begin{cases} \dot{\varphi} = \frac{1}{\sin\theta}\left[w_u\sin\psi + w_v\cos\psi\right] \\ \dot{\psi} = w_w - \tan\theta\left[w_u\sin\psi + w_v\cos\psi\right] \\ \dot{\theta} = w_u\cos\psi - w_v\sin\psi. \end{cases}$$

(3 fully coupled equations)

⇒ I might write other 11 equations equal to this

Singularity problems.

* $\boxed{313}$ → $\boxed{\theta = 0}$ is the discontinuity

While considering an other rotation set:

$$123 \Rightarrow \begin{cases} \dot{\varphi} = \frac{1}{\cos\theta}\left[w_u\cos\psi - w_v\sin\psi\right] \\ \dot{\theta} = w_v\cos\psi + w_u\sin\psi \\ \dot{\psi} = w_w - \left[w_u\cos\psi - w_v\sin\psi\right]\tan\theta. \end{cases}$$

* $\boxed{123}$ → $\boxed{\theta = (2k+1)\frac{\pi}{2}}$ is the discontinuity

⇒ Choosing the correct set of equations, basing on the initial conditions is possible to avoid the discontinuity.

Proof: integrability of Euler's angles Time equation.

$$\begin{Bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \underline{f}(\theta, \varphi, \psi, \underline{w}) \rightarrow \begin{cases} \varphi = \varphi_0 + \int_0^t \dot{\varphi}\, d\tau \\ \theta = \theta_0 + \int_0^t \dot{\theta}\, d\tau \\ \psi = \psi_0 + \int_0^t \dot{\psi}\, d\tau \end{cases}$$

↳ Thanks to integrability condition

$z = f(x, y) \rightarrow dz = A\cdot dx + B\, dy$ where $A \triangleq \frac{\partial f}{\partial x}$; $B \triangleq \frac{\partial f}{\partial y}$

$f \in C^2(\Psi(y)) \Leftrightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. $\Rightarrow$ Can be integrated.

(continuity of 1^{st} derivative guarantee possibility of integrate f) (Schwartz theorem)

Therefore $df = A\,dx + B\,dy \Rightarrow$ check on $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$.

e.g.

$d\varphi = \dot{\varphi}\,dt$; $d\theta = \dot{\theta}\,dt$.

$\{ W_u = \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$

$\Rightarrow \quad W_u\,dt = \sin\theta \sin\psi\, \dot{\varphi}\,dt + \cos\psi\, \dot{\theta}\,dt$

$= \underbrace{\sin\theta \sin\psi}_{A}\, d\varphi + \underbrace{\cos\psi}_{B}\, d\theta.$

* In order to have an integrable function

$\int W_u\,dt \;\exists \Leftrightarrow \frac{\partial A}{\partial \theta} \overset{??}{=} \frac{\partial B}{\partial \varphi}$ $\begin{cases} \frac{\partial(\sin\theta \sin\psi)}{\partial\theta} = \cos\theta \sin\psi \\ \frac{\partial \cos\psi}{\partial\varphi} = 0. \end{cases}$ $\Rightarrow$ NOT INTEGRABLE.

* Some condition might be imposed on:

$\dot{\varphi} = W_u \frac{\cos\psi}{\cos\theta} - W_v \frac{\sin\psi}{\cos\theta}$ (313 sequence)

$d\varphi = \dot{\varphi}\,dt = (W_u\,dt) \overbrace{\frac{\cos\psi}{\cos\theta}}^{A} - (W_v\,dt) \overbrace{\frac{\sin\psi}{\cos\theta}}^{B}$

↳ instantaneous rotation angle around u (PSE)

Keep in mind:
$\begin{cases} W_u\,dt = \sin\theta \sin\psi\, \dot{\varphi}\,dt + \cos\psi\, \dot{\theta}\,dt = \sin\theta \sin\psi\, d\varphi + \cos\psi\, d\theta \; (1) \\ W_v\,dt = \cos\psi \sin\theta\, \dot{\varphi}\,dt - \sin\psi\, \dot{\theta}\,dt = \sin\theta \cos\psi\, d\varphi - \sin\psi\, d\theta \; (2) \\ W_w\,dt = \dot{\varphi}\cos\theta + \dot{\psi} \; (3) \end{cases}$ ⊕ inverse equations.

↳ previously found from Kinematic

$\begin{cases} W_u = \dots \dot{\varphi} + \dots \\ W_v = \dots \\ W_w = \dots \end{cases}$

... solvable for $\{\dot{\varphi}\} = \{ f(W_u, \varphi, \theta, \psi) ;\; f(W, \varphi, \theta, \psi) \}$

$\Rightarrow$ * $A = \frac{\cos\psi}{\cos\theta} \rightarrow \frac{\partial A}{\partial(W_v dt)} = \frac{\partial\left(\frac{\cos\psi}{\cos\theta}\right)}{\partial(W_v dt)}$

$= \frac{1}{\cos^2\theta} \left[\sin\psi \cos\theta \frac{\partial\psi}{\partial(W_v dt)} + \cos\psi \sin\theta \frac{\partial\theta}{\partial(W_v dt)} \right]$

$\frac{\partial\psi}{\partial(W_v dt)} = \frac{\partial}{\partial W_v}\left(\frac{d\psi}{dt}\right) = \frac{\partial\dot{\psi}}{\partial W_v} = -\frac{\sin\theta \cos\psi}{\cos\theta}.$

always possible to interchange variable order.

$\dot{\psi} = W_w - \tan\theta\,[W_u \sin\psi - W_v \cos\psi]$

$\frac{\partial\theta}{\partial(W_v dt)} = \frac{\partial}{\partial W_v}\left(\frac{d\theta}{dt}\right) = \frac{\partial\dot{\theta}}{\partial W_v} = -\sin\psi.$

$\dot{\theta} = W_u \cos\psi - W_v \sin\psi$

$\Rightarrow \frac{\partial A}{\partial(W_v dt)} = \frac{1}{\cos^2\theta}\left[-\cos\psi \sin\psi \sin\theta + \cos\psi \sin\psi \sin\theta\right]$

$\Rightarrow$ * $B = -\frac{\sin\psi}{\cos\theta} \rightarrow \frac{\partial B}{\partial(W_u dt)} = -\frac{\partial\left(\frac{\sin\psi}{\cos\theta}\right)}{\partial(W_u dt)}$

$= \frac{-1}{\cos^2\theta}\left[\cos\psi \cos\theta \frac{\partial\psi}{\partial(W_u dt)} + \sin\psi \sin\theta \frac{\partial\theta}{\partial(W_u dt)}\right]$

$\frac{\partial\psi}{\partial(W_u dt)} = \frac{\partial\dot{\psi}}{\partial W_u} = +\frac{\sin\theta \sin\psi}{\cos\theta}$

$\frac{\partial\theta}{\partial(W_u dt)} = \frac{\partial\dot{\theta}}{\partial W_u} = +\cos\psi.$

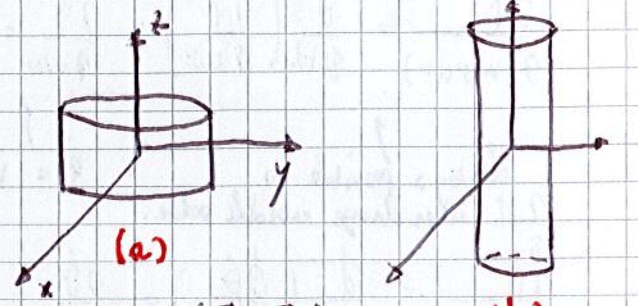
$\Rightarrow \frac{\partial B}{\partial(W_u dt)} = -\frac{1}{\cos^2\theta}\left[\sin\theta \sin\psi \cos\psi + \sin\psi \cos\psi \sin\theta\right]$

end of the proof.

→ "Euler's equation exact solution"

[ ] EXACT SOLUTION $\Leftrightarrow \exists (i,j) : I_i = I_j$]

therefore supposing: $\begin{cases} I_x = I_y \\ I_z \end{cases}$



$I_z > (I_x = I_y)$ $\qquad$ $I_z < (I_x = I_y)$

$I_x = I_y$

$\Rightarrow \int_B (x - x_G)\, dm = \int_B (y - y_G)\, dm$

means (FOR AN HOMOGENEOUS BODY) the axial symmetry.

therefore Euler equations become: (FOR INERTIA MOTION)

$$\begin{cases} I_x \dot{\omega}_x + (I_z - I_x)\,\omega_z \omega_y = 0 \\ I_y \dot{\omega}_y + (I_y - I_z)\,\omega_z \omega_x = 0 \\ I_z \dot{\omega}_z + (\cancel{I_x - I_y})\,\omega_x \omega_y = 0 \end{cases} \longrightarrow \dot{\omega}_z = \text{const} \rightarrow \omega_z = \bar{\omega}_z$$

(CONSTANT IN TIME.)

$I_x = I_y \Rightarrow \lambda \triangleq \frac{I_z - I_x}{I_x}\,\bar{\omega}_z = \frac{I_z - I_y}{I_y}\,\bar{\omega}_z$.

Therefore euler's equation become:

$$\begin{cases} \dot{\omega}_x + \lambda \omega_y = 0 \\ \dot{\omega}_y - \lambda \omega_x = 0. \end{cases}$$

& $\dot{\omega}_y + \frac{(I_y - I_z)}{I_y}\,\bar{\omega}_z\,\omega_x = \dot{\omega}_y - \overbrace{\frac{(I_z - I_x)}{I_x}\,\bar{\omega}_z}^{\lambda}\,\omega_x = 0$

(1) multiply I^{st} and II^{nd} equ. for ω_x and ω_y

$$\begin{cases} \dot{\omega}_x + \lambda \omega_y \\ \dot{\omega}_y - \lambda \omega_x \end{cases} \xrightarrow[\cdot(\omega_y)]{\cdot(\omega_x)} \begin{cases} \dot{\omega}_x \omega_x + \lambda \omega_y \omega_x = 0 \\ \dot{\omega}_y \omega_y - \lambda \omega_x \omega_y = 0 \end{cases}$$

(2) summing the two Euler's equations:

$$\dot{\omega}_x \omega_x + \dot{\omega}_y \omega_y = 0$$

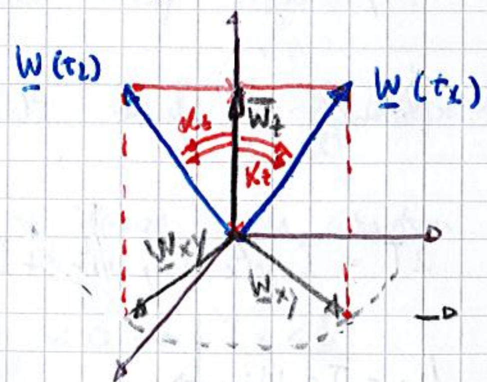
$\Rightarrow \ulcorner\ d(\omega_x^2 + \omega_y^2) = 0$

$d(\|\underline{\omega}_{xy}\|^2) = 0$

$\rightarrow \|\underline{\omega}_{xy}\| = \text{const}|_t \ \lrcorner$

" Projection of $\underline{\omega}$ vector on (x,y) plane has a constant modulus (in time) "

⇒ Representing $\underline{\omega}$ vector on the principal inertia axis:



→ Projection of $\underline{\omega}$ on z axis is constant ⇒ $\alpha_z = \text{const}|_T$.

→ motion of $\underline{\omega}_{xy}$ on xy plane represents a circumference

obs: SINCE $\omega_z = \bar{\omega}_z$

AND SINCE $\omega_x^2 + \omega_y^2 = \bar{\omega}_{xy}$

THEN $\|\underline{\omega}\|^2 = \omega_z^2 + \omega_x^2 + \omega_y^2 \rightarrow \ulcorner\ \|\underline{\omega}\| = \text{const}. \lrcorner$

↳ The same result might be obtained considering the constant kinetic energy of the system:

$$2T^{rot} = \underline{w}\cdot(\underline{\underline{I}}\cdot\underline{w}) = I_x w_x^2 + I_y w_y^2 + I_z w_z^2 .$$

$I_x = I_y \rightarrow \quad 2T^{rot} = I_x (w_x^2 + w_y^2) + I_z w_z^2$

$w_z = \bar{w}_z \rightarrow \quad w_x^2 + w_y^2 = const.$

↳ Looking at vector $\underline{h}$ (IN THE PRINCIPAL INERTIA R.F.)

Torch-free motion, $w_z = \bar{w}_z$, $I_x = I_y$

$$\Rightarrow \quad \underline{h} = \begin{vmatrix} \boxed{I_z \bar{w}_z} \\ I_x w_y \\ I_x w_x \end{vmatrix} \rightarrow \text{constant in time}^{*}$$

this means that if at $t=0$ $\underline{h} = \underline{h}_{in}$ component h_z will remain coincident to the $\underline{h}_{in}$.

$$h_z = I_z \bar{w}_z$$

$$\underline{h}_{xy} = I_x w_x \hat{i} + I_y w_y \hat{j} = I_x (w_x \hat{i} + w_y \hat{j}) = I_x \underline{w}_{xy}$$

↳ from $\|\underline{h}\|$ and $2T^{rot}$ is possible to derive that:

$$h^2 = w_x^2 I_x^2 + w_y^2 I_x^2 + w_z^2 I_z^2 \quad ; \quad 2T^{rot} = I_x w_x^2 + I_y w_y^2 + I_z w_z^2 .$$

$$\frac{h^2}{I_z^2} = \frac{2 I_x^2}{I_z^2} (w_x^2 + w_y^2) + w_z^2 \qquad\qquad \frac{2T^{rot}}{I_z} = \frac{2 I_x}{I_z} (w_x^2 + w_y^2) + w_z^2$$

$$\frac{h^2}{I_z^2} = 2\left(\frac{I_x}{I_z}\right)^2 w_{xy}^2 + w_z^2 \qquad\qquad \frac{I_x}{I_z}\cdot\frac{2T^{rot}}{I_z} = 2\left(\frac{I_x}{I_z}\right) w_{xy}^2 + w_z^2 \cdot \frac{I_x}{I_z}$$

$$2\left(\frac{I_x}{I_z}\right)^2 w_{xy}^2 = \frac{h^2}{I_z^2} - w_z^2 .$$

$$h^2 = I_x^2 w_x^2 + I_y^2 w_y^2 + I_z^2 w_z^2 = 2 I_x^2 w_{xy}^2 + I_z^2 w_z^2 .$$

$$* \quad \frac{h^2}{I_z^2} = w_z^2 + 2\frac{I_x^2}{I_z^2}\cdot w_{xy}^2$$

$$2T = 2 I_x w_{xy}^2 + I_z w_z^2 \rightarrow \frac{2T}{I_z} = 2\frac{I_x}{I_z} w_{xy}^2 + w_z^2$$

$$\rightarrow * \quad 2\left(\frac{I_x}{I_z}\right) w_{xy}^2 = \frac{2T}{I_z} - w_z^2 .$$

$$\Rightarrow \quad \frac{h^2}{I_z^2} = w_z^2 + \left(\frac{I_x}{I_z}\right)\left[\frac{2T}{I_z} - w_z^2\right]$$

$$\frac{h^2}{I_z^2} = w_z^2 + \frac{2T}{I_z^2} I_x - \frac{w_z^2}{I_z}$$

$$[h^2 - 2T I_x] = I_z^2 w_z^2 - I_z w_z^2 \rightarrow \underline{[h^2 - 2T I_x]} = h_z^2 - I_z w_z^2 .$$

from kinetic energy and moment ellipsoids intersection

$$h^2 = I_x^2 w_x^2 + I_x^2 w_y^2 + I_z^2 w_z^2 \quad ; \quad 2T = I_x w_x^2 + I_x w_y^2 + I_z w_z^2 .$$

$$\frac{w_x^2}{h^2/I_x^2} + \frac{w_y^2}{h^2/I_y^2} + \frac{w_z^2}{h^2/I_z^2} = 1 \quad ; \quad \frac{w_x^2}{2T/I_x} + \frac{w_y^2}{2T/I_y} + \frac{w_z^2}{2T/I_z} = 1$$

polhode: $I_x\left[\frac{I_x}{h^2} - \frac{1}{2T}\right] w_x^2 + I_x\left[\frac{I_x}{h^2} - \frac{1}{2T}\right] w_y^2 + I_z\left[\frac{I_z}{h^2} - \frac{1}{2T}\right] w_z^2 = 0$

In order to have such equation satisfied, must be that:

$$\begin{cases} \frac{I_x}{h^2} - \frac{1}{2T} < 0 \rightarrow I_x < \frac{h^2}{2T} \\ \frac{I_z}{h^2} - \frac{1}{2T} > 0 \rightarrow I_z > \frac{h^2}{2T} \end{cases} \quad \text{OR} \quad \begin{cases} \frac{I_x}{h^2} - \frac{1}{2T} > 0 \rightarrow I_x > \frac{h^2}{2T} \\ \frac{I_z}{h^2} - \frac{1}{2T} < 0 \rightarrow I_z < \frac{h^2}{2T} \end{cases}$$

(a) $I_z > I_x \; (= I_y)$ (b) $I_z < I_x \; (= I_y)$

$h^2 - 2T I_x < 0 \rightarrow h_z^2 - I_z w_z^2 < 0$ if $I_z > I_x$ (a) 

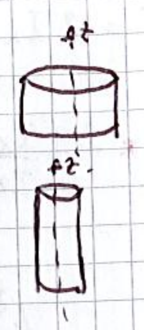
$h^2 - 2T I_x > 0 \rightarrow h_z^2 - I_z w_z^2 > 0$ if $I_z < I_x$ (b) 

$$\begin{cases} h_z^2 - I_z w_z^2 = \alpha \\ h_z = I_z w_z \end{cases} \Rightarrow \quad h_z \cdot (h_z - w_z) \gtrless 0$$

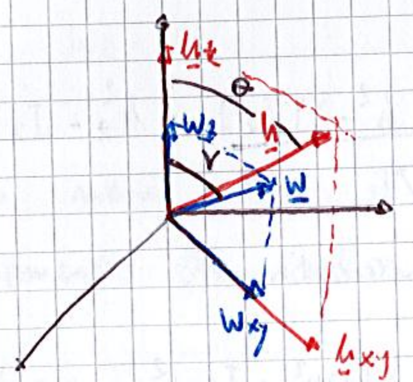
sign to be evaluated.

if $I_z > I_x \Rightarrow h_z > w_z$ (a)

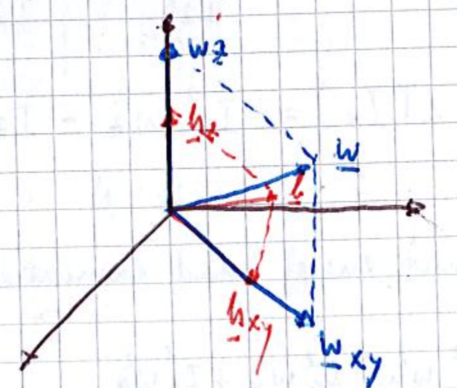
if $I_z < I_x \Rightarrow h_z < w_z$ (b)



(a)



(b)



θ: $\widehat{h_z\, h_{xy}} \Rightarrow \tan\theta = \frac{I_x w_{xy}}{I_z w_z}$

γ: $\widehat{w_z\, w_{xy}} \Rightarrow \tan\gamma = \frac{w_{xy}}{w_z}$

$\Rightarrow \quad \tan\theta = \frac{I_x}{I_z}\tan\gamma$

$\Rightarrow$ (a) $I_z > I_x \Rightarrow h_z > w_z \Rightarrow \theta < \gamma$

(b) $I_x > I_z \Rightarrow h_z < w_z \Rightarrow \theta > \gamma$.

going back to Euler's equations:

$$\begin{cases} \dot w_x + \lambda w_y = 0 \\ \dot w_y - \lambda w_x = 0 \\ w_z = \bar w_z \end{cases}$$

(1) deriving the 1st in time

(2) substitute $\dot w_y$

(3) substitute w_x solution into IInd and integrate.

$$\dot w_x + \lambda w_y \xrightarrow{(1)} \begin{cases} \ddot w_x + \lambda \dot w_y = 0 \\ \dot w_y - \lambda w_x = 0 \end{cases} \xrightarrow{(2)} \ddot w_x + \lambda^2 w_x = 0$$

(P) $\ddot w_x + \lambda^2 w_x = 0$

← This is a common 2nd order system [illegible]

Sol: $w_x = w_{x,0}\cos(\lambda t) + \frac{\dot w_{x,0}}{\lambda}\sin(\lambda t)$

IInd order. $w_x(t) = A\cos(\lambda t) + B\sin(\lambda t) \to \dot w_x(t) = -A\lambda\sin(\lambda t) + B\lambda\cos(\lambda t)$

$w_x(0) = A = w_0$ & $\dot w_x(0) = B\lambda = \dot w_{x,0} \to B = \frac{\dot w_{x,0}}{\lambda}$

$$\dot w_y - \lambda w_x = 0 \xrightarrow{(3)} \dot w_y = \lambda w_{x,0}\cos(\lambda t) + \dot w_{x,0}\sin(\lambda t)$$

$$w_y = +\lambda\frac{w_{x,0}}{\lambda}\sin(\lambda t) - \frac{\dot w_{x,0}}{\lambda}\cdot\lambda\cos(\lambda t)$$

$\Rightarrow$ in virtue of $\begin{Bmatrix} 2^{nd} \\ 1^{st} \end{Bmatrix}$ Euler's equation: $\begin{cases} -\dot w_{x,0}/\lambda = w_{y,0} \\ \lambda \cdot w_{x,0} = \dot w_{y,0} \end{cases}$

therefore:

$w_z = \bar w_z$

$\underline{w}_{xy} = [w_{x,0}\cos(\lambda t) + \frac{\dot w_{x,0}}{\lambda}\sin(\lambda t)]\,\hat i + [\,w_{y,0}\cos(\lambda t) + w_{x,0}\sin(\lambda t)]\,\hat j$

$w_z = \bar w_z$

$w_{x,y} = (w_{x,0}\,\hat i + w_{y,0}\,\hat j)\cdot(\cos(\lambda t)\,\hat i + \sin(\lambda t)\,\hat j)$

→ Im/Re representation

$$W_{xy} = \underbrace{(W_{0x} + j\,W_{0y})}_{\|W_{xy}\| = \|W_{xy,0}\|}\;[\cos(\lambda t) + j\sin(\lambda t)] \rightarrow$$ is the projection of $\underline{W}$ on (x,y); describe motion of a point

Re → x ; Im → y

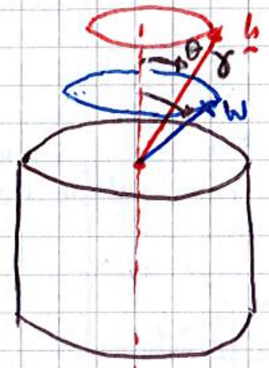
" W_{xy} is the projection of $\underline{W}$ on (x,y) $\begin{cases}\text{CONSTANT MODULUS} \\ \text{PROJECTION as } \cos/\sin\end{cases}$ ⇒ DESCRIBE A CIRCUMFERENCE. "

↳ Sitting on the PRINCIPAL INERTIA REFERENCE FRAME. i will see both W_{xy} and h_{xy} rotating around $\hat{W}_z \hat{e}_z$.

(a) $I_z > I_{y=x} \rightarrow \theta < \gamma$

$(h_z > W_z)$.

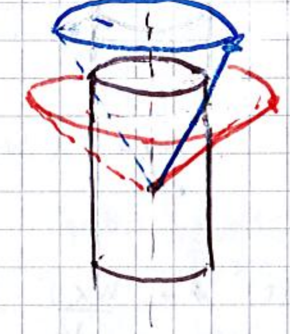
$\left(\lambda = \dfrac{I_z - I_x}{I_z} > 0\right)$



(b) $I_z < I_{x=y} \rightarrow \theta > \gamma$

$(h_z < W_z)$

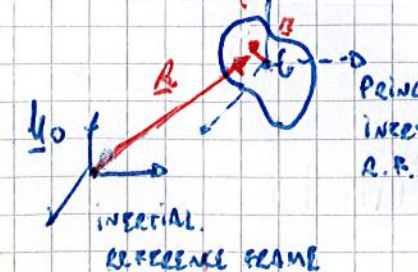
$\left(\lambda = \dfrac{I_z - I_x}{I_z} < 0\right)$



↳ Sitting on the INERTIAL REFERENCE FRAME.

$$\frac{dh}{dt} = M \xrightarrow{M=0} h = \text{const}|_T \quad \text{(in the inertial reference frame).}$$

dominantly as now before:



⇒ $$h_o = \int_B \underline{R} \times \underline{v}\, dm \qquad \| \underline{R} = \underline{R}_G + \underline{r}$$

$$= \int_B \underline{R}_G \times \underline{v}\, dm + \int_B \underline{r} \times \underline{v}.$$

for a rigid body: $\begin{cases}\underline{v} = \underline{v}_{O'} + \underline{W} \times \underline{r} \; ; \\ O' \in B\end{cases}$

$$\Rightarrow h_o = \int_B \underline{R}_G \times [\underline{v}_G + \underline{W} \times \underline{r}]\, dm + \int_B \underline{r} \times \underline{v}_G + \underline{r} \times (\underline{W} \times \underline{r})\, dm.$$

$$= M(\underline{R}_G \times \underline{v}_G) + \underline{R}_G \times \int_B \underline{W} \times \underline{r}\, dm - \underline{v}_G \times \int_B \underline{r}\, dm + \int_B \underline{r} \times (\underline{W} \times \underline{r})\, dm$$

$a \times b \times c = \underline{b}\,(\underline{c} \cdot \underline{a}) - \underline{c}\,(\underline{a} \cdot \underline{b})$

$$h_o = M \cdot (\underline{R}_G \times \underline{v}_G) + \underline{R}_G \times \underline{W} \times \underbrace{\begin{Bmatrix}\int_B x\,dm \\ \int_B y\,dm \\ \int_B z\,dm\end{Bmatrix}}_{S_G} - \underline{v}_G \times \underbrace{\begin{Bmatrix}\int_B x\,dm \\ \int_B y\,dm \\ \int_B z\,dm\end{Bmatrix}}_{S_G} + \int_B \underline{r} \times (\underline{W} \times \underline{r})\, dm$$

($\underline{S}_G = \underline{0}$ if computed in P.I.F.)

$$h_o = M \cdot (\underline{R}_G \times \underline{v}_G) + \underbrace{\int_B (\underline{r} \times \underline{W} \times \underline{r})\, dm.}_{}\;\circledast$$

→ This contribution will become $\underline{\underline{I}} \cdot \underline{W}$.

$$h_o = M(\underline{R}_G \times \underline{v}_G) + \underline{\underline{I}} \cdot \underline{W}$$

therefore by deriving in Time, must be kept into account that:

⊛ I performed such integration expressing $\underline{W}$ in the principal inertia reference frame that is rotating around $\underline{W}$.

Anyway: $$\frac{dh_o}{dt} = M(\underbrace{\underline{v}_G \times \underline{v}_G}_{=0}) + \underbrace{M\underline{R}_G \times \frac{d\underline{v}_G}{dt}}_{\sum \underline{F}_i} + \frac{d}{dt}(\underline{\underline{I}} \cdot \underline{W})$$

$\sum \underline{F}_i$ → overall forces acting on the body I^{st}: $\sum_i F_i = M \cdot \dfrac{d\underline{v}_G}{dt}$.

$$\underline{M}_o = -\left(\sum_i \underline{F}_i\right) \times \underline{R}_G + \underline{M}$$ → "pure couples"

"moment due to external forces."

↳ If no pure Torque acts ⇒ $\dfrac{d}{dt}\left(\int_B \underline{r} \times (\underline{W} \times \underline{r})\right) = \underline{0}$.

this integration in the

(a) PRINCIPAL INERTIA REFERENCE FRAME: $\{\hat{i}, \hat{j}, \hat{k}\}$ $\quad \int_B (\underline{r} \times \underline{W} \times \underline{r})\, dm = \underline{\underline{I}} \cdot \underline{W}$.

but $$\frac{d}{dt}(\underline{\underline{I}} \cdot \underline{W}) = \frac{d}{dt}(\|\underline{\underline{I}} \cdot \underline{W}\|) + \|\underline{\underline{I}} \cdot \underline{W}\| \cdot \begin{Bmatrix} d\hat{i}/dt \\ d\hat{j}/dt \\ d\hat{k}/dt \end{Bmatrix}$$ → poisson's formula.

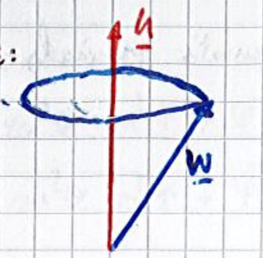
$\rightarrow \{ d\hat{i}/dt ; d\hat{j}/dt ; d\hat{k}/dt \} = \{ \underline{w} \times \hat{i} ; \underline{w} \times \hat{j} ; \underline{w} \times \hat{k} \}$

Where $\{\hat{i}, \hat{j}, \hat{k}\}$ represent the PRINCIPAL INERTIA REFERENCE FRAME.

$$\frac{d\underline{h}_o}{dt} = \underline{0} \Rightarrow \frac{d(\|\underline{\underline{I}}\cdot\underline{w}\|)}{dt} \cdot (\widehat{\underline{\underline{I}}\cdot\underline{w}}) + \underline{w} \times (\underline{\underline{I}}\cdot\underline{w}) = \underline{0}$$

$$\frac{d\underline{h}_{-PI}}{dt} + \underline{w} \times \underline{h}_{-PI} = \frac{d\underline{h}_{IN}}{dt} \quad (=\underline{0})$$

$\hookrightarrow$ INERTIAL REFERENCE FRAME:



In fact $\underline{w}$ expressed in the inertial reference frame can be expressed as:

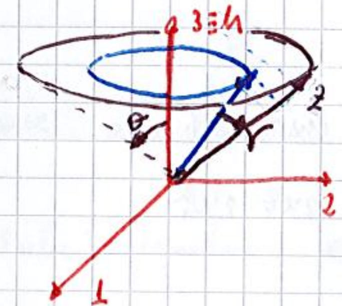
$$\underline{w}_{PI} = \bar{w}_z \hat{k} + w_{xy,0} \cdot [\cos(\lambda t)\,\hat{i} + \sin(\lambda t)\,\hat{j}]$$

principal inertia reference frame.

while:

$\underline{w}_{IN} = \underline{w}_X$

Inertial reference frame.



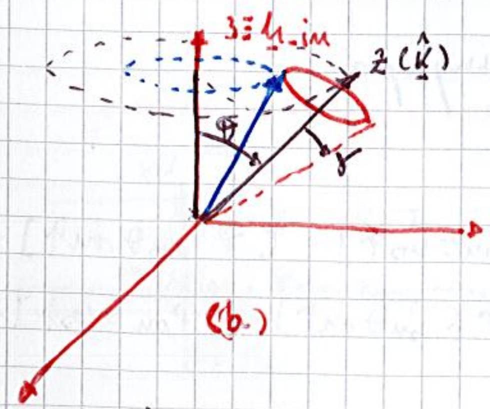
z-axis cone is due to the constant projection in time of $\underline{h}$ vector on z axis.

* $\hat{z}$ axis describes a cone with aperture θ.
* $\underline{w}$ describes a cone with aperture γ around z

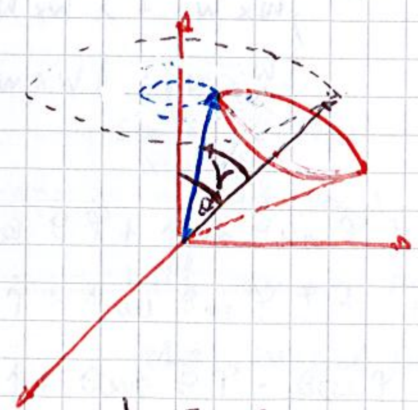
$\downarrow$

$\underline{w}$ will also describe a circular motion around z generating another cone. of aperture γ

[SINCE PROJECTIONS OF $\underline{w}$ AND $\underline{h}$ ALONG $\hat{z}_{-PI}$ ARE CONSTANTS IN TIME ($\bar{h}_z ; \bar{w}_z$)]



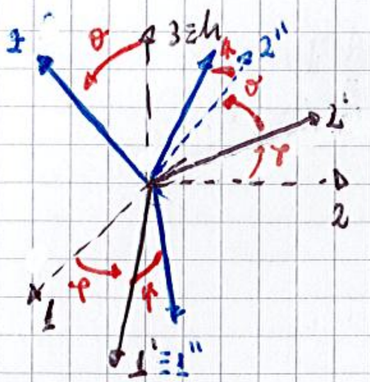
$$\lambda = \frac{I_z - I_x}{I_x} < 0. \quad (\theta > \gamma)$$



$$\lambda = \frac{I_z - I_x}{I_x} > 0 \quad (\theta > \gamma)$$

↳ Applying an attitude parameter (thanks to kinematic previously defined)

313 Euler's angles
$$\begin{cases} \omega_x = \dot\varphi \sin\theta \sin\psi \\ \omega_y = \dot\varphi \sin\theta \cos\psi \\ \omega_z = \dot\varphi \cos\theta + \dot\psi \end{cases}$$



← θ angles coincide with the aperture of body cone of $\underline{h}$ and $\hat{z}$

Taking first derivative:

$$\begin{cases} \dot\omega_x = \ddot\varphi \sin\theta \sin\psi + \dot\varphi\dot\theta \cos\theta \sin\psi + \dot\varphi\dot\psi \sin\theta \cos\psi \\ \dot\omega_y = \ddot\varphi \sin\theta \cos\psi + \dot\varphi\dot\theta \cos\theta \cos\psi - \dot\varphi\dot\psi \sin\theta \sin\psi \\ \dot\omega_z = \ddot\varphi \cos\theta - \dot\varphi\dot\theta \sin\theta + \ddot\psi \end{cases}$$

Substituting this into Euler equation (remember $I_x = I_y$)

$$\begin{cases} \dot\omega_x + \lambda \omega_y = 0 \\ \dot\omega_y - \lambda \omega_x = 0 \\ \dot\omega_z = 0 \end{cases} \longrightarrow \begin{cases} \dot\omega_z = 0 \\ \dot\omega_x \cdot \omega_x + \dot\omega_y \, \omega_y = 0 \end{cases}$$

$$\begin{cases} \dot\omega_x \omega_x + \lambda \omega_x \omega_y = 0 \\ \dot\omega_y \omega_y - \lambda \omega_x \omega_y = 0 \end{cases} \oplus \rightarrow \dot\omega_x \omega_x + \dot\omega_y \omega_y = 0$$

$$\Rightarrow \begin{cases} \overbrace{[\ddot\varphi \sin\theta \sin\psi + \dot\varphi\dot\theta \cos\theta \sin\psi + \dot\varphi\dot\psi \sin\theta \cos\psi]}^{\dot\omega_x} \cdot \overbrace{[\dot\varphi \sin\theta \sin\psi]}^{\omega_x} + \\ + \overbrace{[\ddot\varphi \sin\theta \cos\psi + \dot\varphi\dot\theta \cos\theta \cos\psi - \dot\varphi\dot\psi \sin\theta \sin\psi]}^{\dot\omega_y} \cdot \overbrace{[\dot\varphi \sin\theta \cos\psi]}^{\omega_y} = 0 \\ \ddot\varphi \cos\theta - \dot\varphi\dot\theta \sin\theta + \ddot\psi = 0 \end{cases}$$

So:

$$\ddot\varphi\dot\varphi \sin^2\theta \sin^2\psi + \dot\varphi^2\dot\theta \sin\theta \cos\theta \sin^2\psi + \dot\varphi^2\dot\psi \sin^2\theta \cos\psi \sin\psi +$$
$$+ \ddot\varphi\dot\varphi \sin^2\theta \cos^2\psi + \dot\varphi^2\dot\theta \sin\theta \cos\theta \cos^2\psi - \dot\varphi^2\dot\psi \sin^2\theta \cos\psi \sin\psi = 0.$$

$$0 = \ddot\varphi\dot\varphi \sin^2\theta \, [\underbrace{\sin^2\psi + \cos^2\psi}_{=1}] + \dot\varphi^2\dot\theta \sin\theta \cos\theta \, [\underbrace{\sin^2\psi + \cos^2\psi}_{=1}] + \dot\varphi^2\dot\psi \sin^2\theta \, [\cancel{\cos\psi \sin\psi - \cos\psi \sin\psi}^{=0}]$$

That means:

$$\begin{cases} \ddot\varphi\dot\varphi \sin^2\theta + \dot\varphi^2\dot\theta \sin\theta \cos\theta = 0 \\ \ddot\varphi \cos\theta + \ddot\psi - \dot\varphi\dot\theta \sin\theta = 0 \end{cases}$$

$\boxed{\dot\theta = 0}$ → Since θ represent the angle between $\underline{h}_{in}$ and the $\hat{z}$ axis and it's constant in time

θ: $\hat{\underline{h}} \wedge \hat{z} = \hat{h}_{in} \wedge \hat{z}$ ($\underline{h}$ describe a cone round z axis in P.I. reference).

ANGULAR MOMENTS CONE

⇒ Euler's equation as function of Euler's angles becomes:

$$\begin{cases} \ddot\varphi\dot\varphi \sin^2\theta + \dot\varphi^2 \cancel{\dot\theta}^{=0} \sin\theta \cos\theta = 0 \\ \ddot\varphi \cos\theta + \ddot\psi - \dot\varphi \cancel{\dot\theta}^{=0} \sin\theta \end{cases} \longrightarrow \begin{cases} \ddot\varphi \cdot \dot\varphi = 0 \\ \ddot\varphi \cos\theta + \ddot\psi = 0 \end{cases}$$

$$(P) \quad \begin{cases} \ddot\varphi\dot\varphi = 0 \\ \ddot\varphi \cos\theta + \ddot\psi = 0 \end{cases} \Rightarrow ;$$

Sol: $\dot\varphi$ = const. ⇒ $\dot\psi$ = const.

(φ = const is not admissible since contradicts Euler solution)

Sequence 313 Euler's exact solution.

$\psi(t) = \psi_0 + \dot\psi_0 \cdot t$

$\varphi(t) = \varphi_0 + \dot\varphi_0 \cdot t$

$\theta = \theta_0$.

← since $\begin{cases} \dot\varphi = \text{const} \\ \dot\psi = \text{const} \end{cases}$ will remain equal to the initial conditions characterizing the spacecraft.

↳ Applying an attitude pro... result into Euler's equation.

$\psi = \psi_0 + \dot{\psi}_0 t$)

313 Euler's ...

... $\omega_y = \dot\varphi \sin\theta \sin\psi$; $\omega_z = \dot\varphi \cos\theta + \dot\psi$)

... $= \dot\varphi_0 \sin\theta_0 \cos\psi$; $\omega_z = \dot\varphi_0 \cos\theta_0 + \dot\psi_0$).

Euler's equ:
$$\begin{cases} \dot\omega_x + \lambda\, \omega_y = 0 \\ \dot\omega_y - \lambda\, \omega_x = 0 \\ \dot\omega_z = 0 \end{cases} \Rightarrow \begin{cases} \dot\varphi_0 \sin\theta_0 \cos\psi \cdot \dot\psi_0 + \lambda \dot\varphi_0 \sin\theta_0 \cos\psi = 0 \\ -\dot\varphi_0 \sin\theta_0 \sin\psi\, \dot\psi_0 - \lambda \dot\varphi_0 \sin\theta_0 \sin\psi = 0 \\ \omega_z = \bar\omega_z = \dot\varphi_0 \cos\theta_0 + \dot\psi_0 \end{cases}$$

Everything reduces to 3 algebraic conditions:

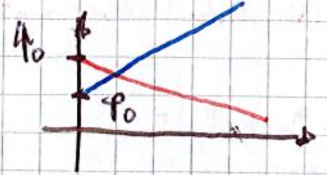
$$\begin{cases} [\dot\varphi_0 \dot\psi_0 + \lambda \dot\varphi_0] \cancel{\sin\theta_0} \cancel{\cos\psi} = 0 \\ [\dot\varphi_0 \dot\psi_0 + \lambda \dot\varphi_0] \cancel{\sin\theta_0} \cos\psi = 0 \\ \dot\psi_0 + \dot\varphi_0 \cos\theta_0 = \bar\omega_z \end{cases} \Rightarrow \quad \dot\theta_0 = 0 \quad \dot\psi_0 = -\lambda \quad \dot\varphi_0 = \frac{\bar\omega_z - \dot\psi_0}{\cos\theta_0} = \frac{\bar\omega_z - \lambda}{\cos\theta_0}.$$

313 Sequence Time (constant) rotation

$$\Rightarrow \quad \dot\theta_0 = 0 \qquad \dot\psi_0 = \frac{I_x - I_z}{I_x} \cdot \bar\omega_z \qquad \dot\varphi_0 = \frac{1}{\cos\theta_0} \cdot \left[1 - \frac{I_x - I_z}{I_x}\right] \bar\omega_z = \frac{I_z}{I_x} \frac{\bar\omega_z}{\cos\theta_0}$$

⊕ $\theta = \theta_0$; $\psi = \psi_0 + \dot\psi_0 \cdot t$; $\varphi = \varphi_0 + \dot\varphi_0 \cdot t$

(a) $\lambda > 0 \Rightarrow \dot\psi_0 = \dot\psi < 0$ ($I_z > I_x$) $\dot\psi = -\lambda$



(b) $\lambda < 0 \Rightarrow \dot\psi_0 = \dot\psi > 0$ ($I_z < I_x$)

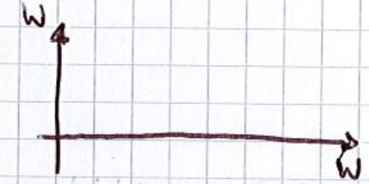
Easy to be checked numerically

→ " Phase-plane representation of Euler's equation "

Still considering a torque-free motion:

$$\begin{cases} I_x \dot\omega_x + (I_z - I_y)\, \omega_y \omega_z = 0 \\ I_y \dot\omega_y + (I_x - I_z)\, \omega_x \omega_y = 0 \\ I_z \dot\omega_z + (I_y - I_x)\, \omega_x \omega_y = 0 \end{cases} \quad ⊕ \quad 2T^{rot} = \text{const} \quad h^2 = \text{const.}$$

→ phase-plane



(NOT GOOD FOR CONTROL)

generates closed conic ⇒ circles.

→

δ (attitude parameter)

(ALLOWS ME TO UNDERSTAND THE BANDWITH I NEED TO CONTROL)

H_p^* $I_z > I_y > I_x \Rightarrow \lambda > 0$

$$\Rightarrow I_z > \frac{h^2}{2T},\ I_x < \frac{h^2}{2T} \quad \text{or} \quad I_z > \frac{h^2}{2T},\ I_y < \frac{h^2}{2T}.$$

→ procedure *

(1) derive 1 of equation (ω_x equ)

(2) evaluate (ω_x, ω_y) from the other 2 equations

(3) Substitute ((2) → (1))

(4) eliminate dependence on (ω_y, ω_x) thanks to
$$\begin{cases} 2T = I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 \\ h^2 = I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2. \end{cases}$$

(5) Integrate in time to obtain form: $\dot\omega_i = f(\omega_i)$