→ " Attitude determination "

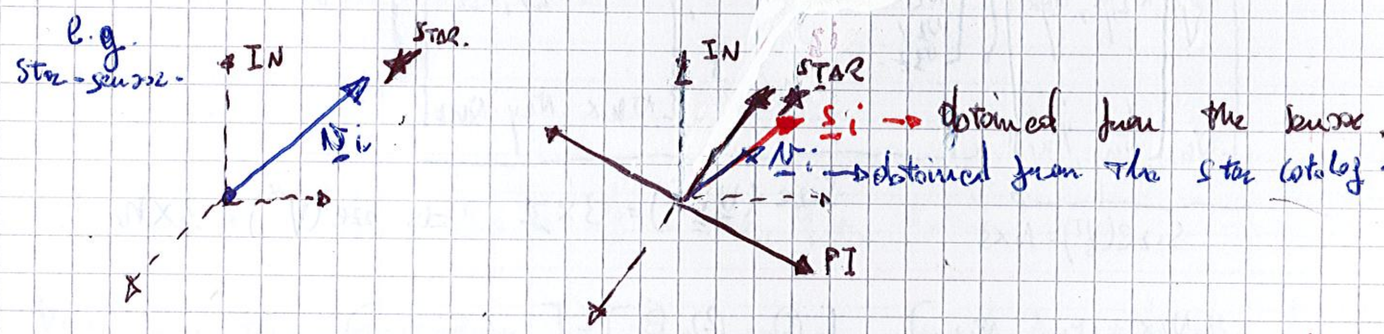
$\boxed{\text{ALGEBRAIC - METHOD}}$

* Having obtained N measurements of N celestial object: $\begin{Bmatrix} \underline{S}_1 \\ \underline{S}_2 \\ \vdots \\ \underline{S}_N \end{Bmatrix}$!! in the P.I. r.f. !!
* Known the position of such N objects in the I.N. r.f.: $\begin{Bmatrix} \underline{N}_1 \\ \underline{N}_2 \\ \vdots \\ \underline{N}_N \end{Bmatrix}$

is then possible to determine (fully determine) the attitude of the spacecraft.

<u>N=3 measurements</u>

$\forall i : \quad \underline{S}_i = \underline{\underline{A}} \cdot \underline{N}_i$



⇒ Having 3 measurements available is possible to build the 2 matrices:

$$\underline{\underline{S}} = [\, \underline{S}_1 \mid \underline{S}_2 \mid \underline{S}_3 \,] \quad ; \quad \underline{\underline{V}} = [\, \underline{N}_1 \mid \underline{N}_2 \mid \underline{N}_3 \,]$$

$\underline{\underline{A}} : \quad \underline{\underline{S}} = \underline{\underline{A}} \cdot \underline{\underline{V}}$ ⇒ IF $\underline{\underline{V}}$ IS NOT SINGULAR THEN $\exists\, \underline{\underline{V}}^{-1}$

$$\Rightarrow \overset{N=3}{\underline{\underline{A}} = \underline{\underline{S}} \cdot \underline{\underline{V}}^{-1}}$$

$N>3$ measurements.

The problem will be: $[\underline{s}_1, \underline{s}_2, \underline{s}_3, \underline{s}_4, \dots]$

$$\underbrace{[\underline{s}_1, \underline{s}_2, \underline{s}_3, \dots \underline{s}_n]}_{\text{Size}(\underline{\underline{S}}) = 3\times n} = \underbrace{\underline{\underline{A}}}_{\text{Size}(\underline{\underline{A}})=3\times 3} \underbrace{[\underline{v}_1, \underline{v}_2, \underline{v}_3, \dots, \underline{v}_n]}_{\text{Size}(\underline{\underline{V}}) = 3\times n}$$

!! Such problem is redundant therefore:
is NECESSARY to use V^* (V- pseudo inverse)
INSTEAD OF $\underline{\underline{V}}^{-1}$ (V- inverse) !!

→ Pseudo inverse matrix.

$$\underline{\underline{V}}^* \triangleq (\underline{\underline{V}}^T)(\underline{\underline{V}}\cdot\underline{\underline{V}}^T)^{-1}$$

$$\underbrace{\begin{bmatrix} v_{1x}, & v_{1y}, & v_{1z} \\ \vdots & \vdots & \vdots \\ v_{nx}, & v_{ny}, & v_{nz} \end{bmatrix}}_{\text{Size}(\underline{\underline{V}}^T) = N\times 3} \underbrace{\left(\begin{bmatrix} v_{1x} & \dots & v_{nx} \\ v_{1y} & \dots & v_{ny} \\ v_{1z} & \dots & v_{nz} \end{bmatrix} \cdot \begin{bmatrix} v_{1x}, & v_{1y}, & v_{1z} \\ \vdots & & \\ v_{nx}, & v_{ny}, & v_{nz} \end{bmatrix} \right)^{-1}}_{\text{Size}(\underline{\underline{V}}\,\underline{\underline{V}}^T) = 3\times 3} = V^*$$

$$\begin{bmatrix} v_{1x} & v_{1y} & v_{1z} \\ \vdots & \vdots & \vdots \\ v_{nx} & v_{ny} & v_{nz} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \end{bmatrix} =$$

$\Rightarrow \text{Size}(V^*) = 3\times n$

in fact.

$$\underline{s} = \underline{\underline{A}}\cdot\underline{v} \rightarrow \underline{\underline{S}} = \underline{\underline{A}}\cdot\underline{\underline{V}}$$

$$\underline{\underline{S}}\,\underline{\underline{V}}^T = \underline{\underline{A}}(\underline{\underline{V}}\,\underline{\underline{V}}^T)$$

$$\underline{\underline{S}}\,\underline{\underline{V}}^T(\underline{\underline{V}}\,\underline{\underline{V}}^T)^{-1} = \underline{\underline{A}}\,(\underline{\underline{V}}\,\underline{\underline{V}}^T)\,(\underline{\underline{V}}\,\underline{\underline{V}}^T)^{-1}$$

$$\underline{\underline{S}}\,\underbrace{\underline{\underline{V}}^T(\underline{\underline{V}}\,\underline{\underline{V}}^T)}_{\underline{\underline{V}}^*} = \underline{\underline{A}}$$

$\Rightarrow$ Γ $N>3$

$$\underline{\underline{A}} = \underline{\underline{S}}\cdot\underline{\underline{V}}^*$$

$$\underline{\underline{V}}^* = \underline{\underline{V}}^T(\underline{\underline{V}}\,\underline{\underline{V}}^T)^{-1}$$

$N=2$ Measurements.

The strategy is about to build a 3rd vector orthogonal to the other 2
$\underline{s}_3 \perp (\underline{s}_1, \underline{s}_2)$

$$[\underline{s}_1; \underline{s}_2] \rightarrow [\underline{v}_1, \underline{v}_2]$$

$$\Downarrow$$

$\underline{p}_s = \underline{s}_1$ $\qquad$ $\underline{p}_v = \underline{v}_1$

$\underline{q}_s = \dfrac{\underline{s}_1 \times \underline{s}_2}{\|\underline{s}_1 \times \underline{s}_2\|}$ $\qquad$ $\underline{q}_v = \dfrac{\underline{v}_1 \times \underline{v}_2}{\|\underline{v}_1 \times \underline{v}_2\|}$ (!! this is possible if and only if $\underline{s}_1$ is not parallel to $\underline{s}_2$!!)

$\underline{r}_s = \underline{p}_s \times \underline{q}_s$ $\qquad$ $\underline{r}_v = \underline{p}_v \times \underline{q}_v$

(!! this procedure works only if $\underline{s}_1$ is NOT ORTHOGONAL TO $\underline{s}_2$
in fact the 3 vectors $[\underline{p}_s, \underline{r}_s, \underline{q}_s]$ must not be parallel
to determine the attitude. !!)

$$\Downarrow$$

$$\underbrace{[\underline{p}_s, \underline{q}_s, \underline{r}_s]}_{\underline{\underline{S}}} = \underline{\underline{A}}\, \underbrace{[\underline{p}_v, \underline{q}_v, \underline{r}_v]}_{\underline{\underline{V}}}$$

Up to now the procedure is the same used for the attitude determination used with 3 measurement.

SINCE $\{\underline{p}_s, \underline{q}_s, \underline{r}_s\}$ AND $\{\underline{p}_v, \underline{q}_v, \underline{r}_v\}$ are orthogonal.

$(\underline{p}_s \perp (\underline{q}_s, \underline{r}_s)$; $\underline{q}_s \perp (\underline{r}_s, \underline{p}_s)$ ....)

THEN $\underline{\underline{S}}$ matrix and $\underline{\underline{V}}$ matrix are both orthogonal matrices

$\Rightarrow$ Γ $N=2$

$$\underline{\underline{A}} = \underline{\underline{S}}\,\underline{\underline{V}}^T$$

$$\underline{\underline{V}}^T = \underline{\underline{V}}^{-1}$$

obs: matrix $\underline{\underline{V}}$ is transposed (inverted) instead of $\underline{\underline{S}}$ because is not affected by errors.

obs: MEASUREMENTS $\{\underline{S}_1, \underline{S}_2, \dots, \underline{S}_M\}$ MAY COME FROM DIFFERENT SENSORS $\Rightarrow$ MAY BRING DIFFERENT ERRORS TO THE COMPLEX ERROR ON ATTITUDE DETERMINATION.

↳ Therefore optimizing the situation:

* $n=2$ $\Rightarrow$ $\boxed{\underline{p}_s = \underline{S}_1}$ ⟶ Choose $\underline{S}_1$ as the most precise measurement since $\underline{p}_s$ determine both $\underline{q}_s$ and $\underline{r}_s$

$$q_s = \frac{\underline{p}_s \times \underline{S}_2}{\|\underline{p}_s \times \underline{S}_2\|}$$

$$\underline{r}_s = \underline{p}_s \times \underline{q}_s$$

* $n=3$ $\Rightarrow$ $\underline{\underline{S}}\,\underline{\underline{V}}^{-1} = \underline{\underline{A}}$ $\Rightarrow$ I'm homogeneously distributing the error on matrix $\underline{\underline{A}}$ of the 3 measurements.

in fact:

$$\underline{S}_1 = \underline{\underline{A}}\cdot \underline{N}_1 + \underline{\varepsilon}_1$$

$$\underline{S}_2 = \underline{\underline{A}}\cdot \underline{N}_2 + \underline{\varepsilon}_2 \quad \rightarrow \quad \underline{\underline{A}} = \underline{\underline{S}}\cdot\underline{\underline{V}}^{-1} - \underline{\underline{\varepsilon}}\cdot\underline{\underline{V}}^{-1}$$

$$\underline{S}_3 = \underline{\underline{A}}\cdot \underline{N}_3 + \underline{\varepsilon}_2 \quad\quad \underline{\underline{A}} = [\underline{\underline{S}} - \underline{\underline{\varepsilon}}]\cdot \underline{\underline{V}}^{-1}$$

* $n > 3$ $\Rightarrow$ $\begin{cases} \underline{\underline{S}}\cdot\underline{\underline{V}}^{*} = \underline{\underline{A}} \\ \underline{\underline{V}}^{*} = \underline{\underline{V}}^{T}(\underline{\underline{V}}\cdot\underline{\underline{V}}^{T})^{-1}\end{cases}$ $\Rightarrow$ I'm making a sort of: mean square root of the error.

!! I'm loosing the most precise measurement !!

STATHISTICAL METHOD (Not obvious: $n \geq 3$)

↳ the global determination of the attitude is the result of many $\underbrace{\text{different measurements}}_{\neq \text{ sensors}}$ affected by different errors

$\Rightarrow$ IS POSSIBLE TO "WEIGHT" EACH MEASUREMENT AND FIND THE OPTIMAL ATTITUDE DETERMINATION.

$$\underline{S}_i = \underline{\underline{A}}\cdot\underline{N}_i + \underline{\varepsilon}_i \quad \rightarrow \quad \underline{\varepsilon}_i = \underline{S}_i - \underline{\underline{A}}\cdot\underline{N}_i$$

$$\Rightarrow \begin{cases} J = \frac{1}{2}\sum_{i=1}^{N} \lambda_i \, (\underline{\varepsilon}_i^T\cdot\underline{\varepsilon}_i) \\ \lambda_i : \sum_i \lambda_i = 1 \end{cases}$$

λ_i: weight founded to each measurement $\Rightarrow$ TO DETERMINE THROUGH OPTIMIZATION.

↳ OPTIMIZE FUNCTIONAL:

$$\begin{cases} J = \frac{1}{2}\sum_{i=1}^{N} \alpha_i \,\|\underline{S}_i - (\underline{\underline{A}}\,\underline{N}_i)\|^2 \\ \sum_{i=1}^{N}\alpha_i = 1\end{cases}$$

$\begin{cases}(\underline{\varepsilon}_i^T\cdot\underline{\varepsilon}_i) \\ \underline{\varepsilon}_i = \underline{S}_i - \underline{\underline{A}}\,\underline{N}_i\end{cases}$

Therefore:

$$J = \frac{1}{2}\sum_{i=1}^{N}\alpha_i\left(\underline{S}_i^T\underline{S}_i - \underline{S}_i^T(\underline{\underline{A}}\,\underline{N}_i) - \underline{N}_i^T\underline{\underline{A}}^T\underline{S}_i + \underline{N}_i^T\underline{\underline{A}}^T\underline{\underline{A}}\,\underline{N}_i\right)$$

$\underline{\underline{A}} = \underline{\underline{A}}^T \Rightarrow \underline{S}_i^T\underline{\underline{A}}\,\underline{N}_i = \underline{S}_i^T(\underline{\underline{A}}^T\underline{N}_i) \rightarrow$ scalar $\Rightarrow \underline{S}_i^T(\underline{\underline{A}}\,\underline{N}_i) = \underline{N}_i^T(\underline{\underline{A}}\,\underline{S}_i)$

$$J = \frac{1}{2}\sum_{i=1}^{N}\alpha_i\left(\underline{S}_i^T\underline{S}_i + \underline{N}_i^T\underline{\underline{A}}\,\underline{\underline{A}}^T\underline{N}_i - 2\,\underline{S}_i^T\underline{\underline{A}}\,\underline{N}_i\right)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\alpha_i\left(\underline{S}_i^T\underline{S}_i + \underline{N}_i^T\underline{N}_i - 2\,\underline{S}_i^T\underline{\underline{A}}\,\underline{N}_i\right)$$

↳ Supposing of working with NORMALIZED MEASUREMENTS / INERTIAL REFERENCES

$$\|\underline{S}_i\|^2 = \underline{S}_i^T\cdot\underline{S}_i = 1 \quad ; \quad \|\underline{N}_i\|^2 = \underline{N}_i^T\cdot\underline{N}_i = 1$$

$$\Rightarrow J = \sum_{i=1}^{N}\alpha_i\left(1 - \underline{S}_i^T\underline{\underline{A}}\,\underline{N}_i\right) = 1 - \sum_{i=1}^{N}\underline{S}_i^T\underline{\underline{A}}\,\underline{N}_i$$

↳ As commonly done: also for optimal control.

$$J = 1 - \sum_{i=1}^{N} \alpha_i (\underline{s}_i^T \underline{\underline{A}} \underline{v}_i) \rightarrow$$ since J is a scalar I can optimize the tr instead of J itself.

$$\Rightarrow \tilde{J} = \sum_{i=1}^{N} \alpha_i (\underline{s}_i^T \underline{\underline{A}} \underline{v}_i) = Tr(\underline{\underline{A}} \cdot \underline{\underline{B}})$$

$$J = 1 - \sum_{i=1}^{N} \alpha_i (\underline{s}_i^T \underline{\underline{A}} \underline{v}_i)$$

!! Minimizing J is coincident with maximizing $\tilde{J}$!!

$$\begin{cases} \tilde{J} = Tr\left(\sum_{i=1}^{N} \alpha_i (\underline{s}_i^T \underline{\underline{A}} \underline{v}_i)\right) = Tr(\underline{\underline{A}} \cdot \underline{\underline{B}}^T) \\ \underline{\underline{B}} = \sum_{i=1}^{N} \alpha_i \underline{s}_i \underline{v}_i^T \end{cases}$$

Trace property

(i) $Tr(\underline{\underline{A}}\,\underline{\underline{B}}) = Tr(\underline{\underline{B}} \cdot \underline{\underline{A}})$

$$\begin{cases} \text{(ii) } Tr(\underline{\underline{A}}^T) = Tr(\underline{\underline{A}}) \\ Tr(\underline{\underline{A}} + \underline{\underline{B}}) = Tr(\underline{\underline{A}}) + Tr(\underline{\underline{B}}) \text{ (iii)} \end{cases}$$

$$\tilde{J} = Tr\left(\sum_{i=1}^{N} \alpha_i (\underline{s}_i^T \underline{\underline{A}} \underline{v}_i)\right) \quad (i)$$

$$= Tr\left(\sum_{i=1}^{N} \alpha_i (\underline{\underline{A}} \underline{v}_i \underline{s}_i^T)\right) \quad (ii)$$

$$= Tr\left(\sum_{i=1}^{N} \alpha_i (\underline{s}_i \underline{v}_i^T \underline{\underline{A}}^T)\right) \quad (iii)$$

$$= Tr\left(\underbrace{\left[\sum_{i=1}^{N} \alpha_i \underline{s}_i \underline{v}_i^T\right]}_{\underline{\underline{B}}} \cdot \underline{\underline{A}}^T\right) \quad (ii)$$

$$= Tr(\underline{\underline{A}}\ \underline{\underline{B}}^T)$$

!! $J = J(\underline{\underline{A}})$ must find $\underline{\underline{A}}$ that minimizes J !!

" A direct solution of such optimization problem is not possible in a direct way ⇒ DECOMPOSITION PROCEDURE.

but: $\underline{\underline{B}}$ MUST NOT BE SINGULAR ⇒ AT LEAST 3 MEASUREMENTS ARE REQUIRED !!

⇒ The solution observed is the following:

$$\boxed{\underline{\underline{A}} = (\underline{\underline{B}}^T)^{-1} \cdot (\underline{\underline{B}}^T \underline{\underline{B}})^{1/2} = \underline{\underline{B}} (\underline{\underline{B}}^T \underline{\underline{B}})^{-1/2}}$$

detailed minimization of J

* COST FUNCTION: $\begin{cases} J = tr(\underline{\underline{A}} \cdot \underline{\underline{B}}^T) \\ \underline{\underline{B}} = \sum_{i=1}^{N} \alpha_i \underline{s}_i \underline{v}_i^T \end{cases}$

* COSTRAIN: $\underline{\underline{A}}^T \underline{\underline{A}} - \underline{\underline{I}} = \underline{0}$

orthogonality of the $\underline{\underline{A}}$ matrix.

TRACE PROPERTY DERIVATION

$$\frac{\partial\, tr(\underline{\underline{A}}\underline{\underline{B}}^T)}{\partial \underline{\underline{B}}} = \underline{\underline{A}}$$

$$J^* = Tr(\underline{\underline{A}} \cdot \underline{\underline{B}}^T) + \lambda\, tr(\underline{\underline{A}}\,\underline{\underline{A}}^T - \underline{\underline{I}}) \qquad (J^* = J^*(\underline{\underline{A}}, \lambda))$$

$$\hookrightarrow J^* = Tr(\underline{\underline{B}}^T \underline{\underline{A}}) + \lambda\, tr(\underline{\underline{A}}\,\underline{\underline{A}}^T - \underline{\underline{I}}) \underset{(ii)}{=} Tr(\underline{\underline{B}}\,\underline{\underline{A}}^T) - \lambda\, tr(\underline{\underline{A}}\,\underline{\underline{A}}^T - \underline{\underline{I}})$$

(i) $Tr(\underline{\underline{B}}^T \underline{\underline{A}}) = Tr([\underline{\underline{B}}^T \underline{\underline{A}}]^T) = Tr(\underline{\underline{A}}^T \underline{\underline{B}}) \underset{(i)}{=} Tr(\underline{\underline{B}}\,\underline{\underline{A}}^T)$

$$\frac{\partial J^*}{\partial \underline{\underline{A}}} = \underline{\underline{B}} + \lambda \underline{\underline{A}} = \underline{0} \quad (a)$$

$$\frac{\partial J^*}{\partial \lambda} = \underline{\underline{A}} \cdot \underline{\underline{A}}^T - \underline{\underline{I}} = \underline{0} \quad (b)$$

(a) $\underline{\underline{A}} = -\frac{1}{\lambda} \underline{\underline{B}}$

(b) $\frac{1}{\lambda^2} \underline{\underline{B}} \cdot \underline{\underline{B}}^T = \underline{\underline{I}} \Rightarrow \frac{1}{\lambda} = (\underline{\underline{B}} \cdot \underline{\underline{B}}^T)^{+1/2}$

⇒ CVD

$$\boxed{\underline{\underline{A}} = -(\underline{\underline{B}} \cdot \underline{\underline{B}}^T)^{1/2} \cdot \underline{\underline{B}}}$$

!! this implies that $\underline{\underline{B}}$ is a non singular matrix ⇒ 3 independent measurements

⇒ B

$$\boxed{\underline{\underline{A}} = \underline{\underline{U}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \det(\underline{\underline{U}}) \cdot \det(\underline{\underline{V}}) \end{bmatrix}}$$

$\underline{\underline{U}} = eig(\underline{\underline{B}}\,\underline{\underline{B}}^T)$

$\underline{\underline{V}} = eig(\underline{\underline{B}}\,\underline{\underline{B}}^T)$

q method → replacing $\underline{\underline{A}}$ expression with quaternions;

$$\tilde{q} \triangleq \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} \; , \; q_4$$

$$q \triangleq \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} \underline{\underline{A}} = (q_4^2 - \tilde{q}^T \tilde{q}) \underline{\underline{I}} + 2 \tilde{q} \cdot \tilde{q}^T - 2 q_4 [q\times] \\ [q\times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \end{cases}$$

constrain: $\tilde{q}^T \cdot \tilde{q} = 1$.

Avoiding the notable algebra required:

(B) $$J_q = q^T \underline{\underline{K}} \, q + \lambda (q^T q - 1)$$

where:

$$\underline{\underline{K}} = \begin{bmatrix} \underline{\underline{S}} - \sigma \underline{\underline{I}} & \underline{z} \\ \underline{z}^T & \sigma \end{bmatrix}$$

$$\sigma = \mathrm{Tr}(\underline{\underline{B}}) \quad \left(\underline{\underline{B}} = \sum_{i=1}^{N} k_i \, \underline{s}_i \, \underline{v}_i^T \right)$$

$$\underline{\underline{S}} = \underline{\underline{B}}^T + \underline{\underline{B}}$$

$$\underline{z} = [B_{3,2} - B_{23} \; ; \; B_{13} - B_{31} \; ; \; B_{12} - B_{21}]^T$$

the general derivation of a quadratic form

$$J = \{x_1, x_2\} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{x_1, x_2\} \begin{Bmatrix} a_1 x_1 + a_2 x_2 \\ a_3 x_1 + a_4 x_2 \end{Bmatrix}$$

$$= a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_1 x_2 + a_4 x_2^2 \quad (= \underline{x}^T \underline{\underline{A}} \, \underline{x})$$

$$\Rightarrow \frac{\partial J}{\partial x_1} = 2 a_1 x_1 + \cancel{\;\;\;} + a_2 x_2 + a_3 x_2 = 2 a_1 x_1 + (a_2 + a_3) x_2$$

$$\frac{\partial J}{\partial x_2} = 2 a_4 x_2 + (a_2 + a_3) x_1$$

$$\Rightarrow \nabla(J) = \begin{bmatrix} 2 a_1 & (a_2 + a_3) \\ (a_2 + a_3) & 2 a_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$\underline{\underline{A}}$ if it's symmetric ⇒ $a_2 = a_3$ ⇒ $\nabla(J) = \nabla(\underline{x}^T \underline{\underline{A}} \, \underline{x}) = 2 \underline{\underline{A}} \, \underline{x}$

$\underline{\underline{K}}$ is symmetric

$$\underline{\underline{B}} + \underline{\underline{B}}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 5 & 8 \end{bmatrix}$$ is for sure symmetric.

⇒ The result of such differentiation:

$$\nabla(J_q) = 2 \underline{\underline{K}} \, q + 2 \lambda q = 0 \Rightarrow \underline{\underline{K}} \, q = \lambda q$$

" this means that the statistical attitude determination problem can be reconducted to an eigen-values / eigen-vector problem.

→ Solution is not unique → THE EIGEN VECTOR WITH THE MAXIMUM ASSOCIATED EIGEN VALUES λ_{max} WILL BE THE FINAL QUATERNION q^{sol} //

↳ reWriting the eigen-values / eigen-vector problem in the partitioned form:

$$\underline{\underline{K}} \, q = \lambda_{max} \cdot q \Rightarrow \begin{cases} [\underline{\underline{S}} - \sigma \underline{\underline{I}}] \tilde{q} + \underline{z} \, q_4 = \lambda_{max} \tilde{q} & q_1 \cdots q_3 \\ \underline{z}^T \cdot \tilde{q} + \sigma q_4 = \lambda_{max} q_4 & q_4 \end{cases}$$

Approximation

↳ Assuming $\lambda_{max} = 1$ (NORMALIZED EIGEN-VALUES)

$$(\underline{\underline{S}} - \sigma \underline{\underline{I}}) \tilde{q} + \underline{z} \, q_4 = \tilde{q} \rightarrow (\underline{\underline{S}} - \sigma \underline{\underline{I}}) \tilde{q}/q_4 + \underline{z} = \tilde{q}/q_4$$

$$(\underline{\underline{S}} - \sigma \underline{\underline{I}} - \underline{\underline{I}}) \tilde{q}/q_4 + \underline{z} = 0$$

→ Gibbs Vector.

$$\underline{g} = \begin{Bmatrix} e_1 \tan \varphi/2 \\ e_2 \tan \varphi/2 \\ e_3 \tan \varphi/2 \end{Bmatrix} \equiv \tilde{q}/q_4 = \begin{Bmatrix} e_1 \sin \varphi/2 \\ e_2 \sin \varphi/2 \\ e_3 \sin \varphi/2 \end{Bmatrix} \frac{1}{\cos \varphi/2}$$

$$\Rightarrow [\underline{\underline{S}} - (\sigma - 1) \underline{\underline{I}}] \, \underline{g} = -\underline{z}$$

Such system is a 3×3 linear system.

→ " Control problem → INTERNAL TORQUES "

⇒ Considering both environment and disturbance affecting the dynamic of the system:

$$\dot{\underline{h}} + \underline{w} \times \underline{h} = \underline{M}_d + \underline{M}_c$$

disturbances

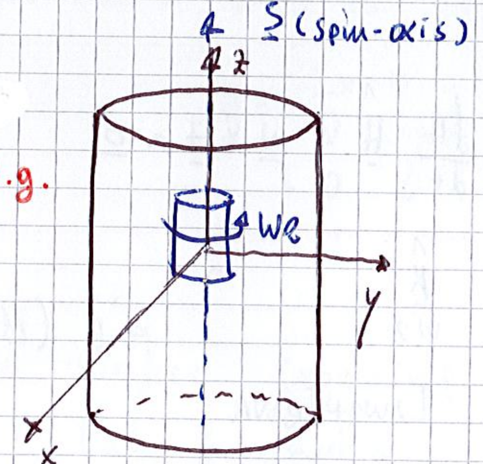
↳ EXCHANGED WITH ENVIRONMENT:

- thrusters
- Magnetic actuators.

GYROSCOPIC ACTUATOR.

⇒ require that Euler's equation is modified for the contribution of the R.W. $\underline{h} \rightarrow \underline{H} = \underline{h}_{sys} + \underline{h}_{R.W.}$

⇒ e.g.



$$\underline{H} = \underline{\underline{I}}\,\underline{w} + \underline{\underline{A}}\,\underline{h}_r$$

↳ define the orientation of the actuator in the PI reference frame (≡ spin axis direction

In this condition .. $\hat{\underline{s}} \equiv \hat{\underline{z}}$ ⇒ $\begin{cases} h_r = I_r \cdot w_r \Rightarrow \underline{\underline{A}} \cdot h_r = \begin{vmatrix} 0 \\ 0 \\ I_r w_r \end{vmatrix} \\ \underline{\underline{A}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{cases}$

In general: $\underbrace{\begin{bmatrix} 0 & 1 & 0 & \sqrt{2}/2 & \cdots \\ 0 & 0 & 1 & \sqrt{2}/2 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \end{bmatrix}}_{}$ $\cdot$ $\left\{\begin{matrix} \text{actuator } 1 \\ \text{actuator } 2 \\ \vdots \end{matrix}\right\}$ ↳ N = n° of actuators

↳ N = actuators with a different spin axis of rotation.

$$\begin{bmatrix} 0 & 1 & 0 & \sqrt{2}/2 & \cdots \\ 0 & 0 & 1 & \sqrt{2}/2 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \end{bmatrix} \begin{Bmatrix} I_{r_1} w_{r_1} \\ I_{r_2} w_{r_2} \\ \vdots \end{Bmatrix} =$$

[Therefore the correct Euler's equation should be written considering both the actuators and the system itself]

$$\dot{\underline{H}} + \underline{W} \times \underline{H} = \underline{M}_d$$

Also set = 0

$$\left.\frac{d\underline{H}}{dt}\right|_{IN} = \left.\underline{M}_{ext}\right|_{IN} \rightarrow \left.\frac{d\underline{H}}{dt}\right|_{PI} + \left.\underline{W}\right|_{PI} \times \left.\underline{H}\right|_{PI} = \left.\underline{M}_d\right|_{PI}$$

$\Rightarrow$ Terms $\begin{cases} \underline{W} \\ \underline{\underline{I}} \end{cases}$ are relative to the whole system.

Review:

↳ For the generic dual spin satellite:

$$\underline{H} = \underline{h}_B + \underline{h}_R = \begin{Bmatrix} I_x W_x \\ I_y W_y \\ I_z W_z + I_R W_R \end{Bmatrix}$$

$$\frac{d\underline{H}}{dt} = 0 \rightarrow \frac{dH}{dt} \hat{\underline{H}} + H \cdot \frac{d\hat{\underline{H}}}{dt} = \frac{dH}{dt} \hat{\underline{H}} + \underline{W} \times \underline{H} = \underline{0}$$

$$\begin{Bmatrix} I_x \dot{W}_x \\ I_y \dot{W}_y \\ I_z \dot{W}_z + I_R \dot{W}_R \end{Bmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ W_x & W_y & W_z \\ I_x W_x & I_y W_y & I_z W_z + I_R W_R \end{vmatrix} = \underline{0} \ (\underline{M}_d)$$

$$\begin{cases} I_x \dot{W}_x + I_z W_z W_y + I_R W_R W_y - I_y W_z W_y = 0 \\ I_y \dot{W}_y - I_z W_z W_x - I_R W_R W_x + I_x W_x W_z = 0 \\ I_z \dot{W}_z + I_R \dot{W}_R + I_y W_y W_x - I_x W_x W_y = 0 \end{cases}$$

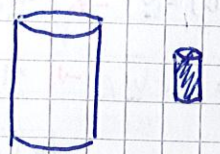
+ $I_R \dot{W}_R = M_R \approx$ electric engine (given M_R)

(IV^{th} equation is needed since the W_R degrees of freedom has been added)

$$\rightarrow \begin{cases} I_x \dot{W}_x + (I_z - I_y) W_z W_y + I_R W_R W_y = 0 \\ I_y \dot{W}_y + (I_x - I_z) W_x W_z - I_R W_R W_x \\ I_z \dot{W}_z + (I_y - I_x) W_x W_y + I_R \dot{W}_R = 0 \end{cases}$$

!! EVEN IF $W_R = 0$ THE INERTIA MOMENTS OF THE SPACECRAFT MUST BE COMPUTED BY CONSIDERING THE INERTIA OF THE ROTOR.

I_{xB}, I_{yB}, I_{zB}



I_{Rx}, I_{Ry}, I_{Rz}

$$\Rightarrow \begin{bmatrix} I_{xB} + I_{Rx} \triangleq I_x^* & & \\ & I_{yB} + I_{Ry} \triangleq I_y^* & \\ & & I_z + \underbrace{I_{Rz}}_{I_{Rz} \triangleq I_R} \triangleq I_z^* \end{bmatrix} = \underline{\underline{I}}$$

$$\Rightarrow \underline{H} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z^* \end{bmatrix} \underline{W} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \cdot I_R W_R.$$

↳ More than 1 body in relative motion: (1 FOR EACH AXIS)

$$\underline{H} = \underbrace{\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \underline{W}}_{\underline{\underline{I}} \cdot \underline{W}} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq \underline{\underline{A}}} \cdot \underbrace{\begin{Bmatrix} I_{Rx} W_x \\ I_{Ry} W_y \\ I_{Rz} W_z \end{Bmatrix}}_{\underline{h}_R}.$$

$\Rightarrow$ therefore (agreeding with this notation) i can write Euler's equation:

$$\begin{cases} \underline{H} = \underline{\underline{I}}\ \underline{W} + \underline{\underline{A}} \cdot \underline{h}_R \\ \frac{d\underline{H}}{dt} + \underline{W} \times \underline{H} = \underline{M}_d. \end{cases} \rightarrow \frac{d}{dt}[\underline{\underline{I}}\ \underline{W} + \underline{\underline{A}}\ \underline{h}_R] + \underline{W} \times [\underline{\underline{I}}\ \underline{W} + \underline{\underline{A}}\ \underline{h}_R] = \underline{M}_d$$

Proceeding with the computation:

$$\underline{\underline{I}} \frac{d\underline{W}}{dt} + \frac{d\underline{\underline{I}}}{dt} \underline{W} + \frac{d\underline{\underline{A}}}{dt} \underline{h}_R + \underline{\underline{A}} \frac{d\underline{h}_R}{dt} + \underline{W} \times (\underline{\underline{I}}\ \underline{W}) + \frac{d\underline{\underline{A}}}{dt} \underline{h}_R = 0$$

!! Since $\frac{d\underline{\underline{A}}}{dt} \neq \underline{0}$ $\Rightarrow$ also the (position) orientation of the rotors may change MODIFING THE THE INERTIA. $\Rightarrow \frac{d\underline{\underline{I}}}{dt} \neq 0$. !!

(i) $\underline{\underline{I}}\,\dot{\underline{w}} + \underline{w}\cdot\dot{\underline{\underline{I}}} + \underline{w}\times(\underline{\underline{I}}\,\underline{w}) + \dot{\underline{\underline{A}}}\,\underline{h}_r + \underline{\underline{A}}\cdot\dot{\underline{h}}_r + \underline{w}\times(\underline{\underline{A}}\,\underline{h}_r) = \underline{M}_d$

The only gyroscopic actuator that works modifying the position of the wheel is the

CMG → CONTROL MOMENT GYRO

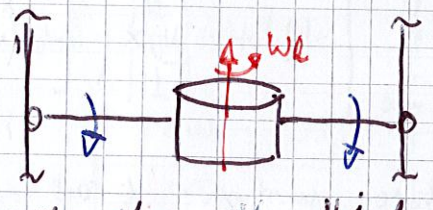
while:

RW → REACTION WHEELS → $w_R(t=0)=\underline{0}$ → h_r ↑↓

IW → INERTIA WHEELS → $w_R(t=0)=\bar{w}_R$ → h_r ↑↓

by increasing the $|w_r|$ / w_{rh} terms

CMG:

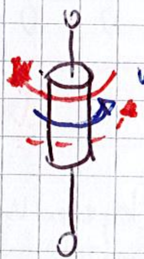


(mechanism that can be tilted)

!! anyway the term $\|\dot{\underline{\underline{A}}}\,\underline{h}_r\| >> \|\dot{\underline{\underline{I}}}\,\underline{w}\|$ since the inertia of the rotor is small if compared to the spacecraft !!

RW

IW



Working on equation (i) is then possible to put in evidence the contribution due to control

(i) $\underline{\underline{I}}\,\dot{\underline{w}} + \underline{w}\times(\underline{\underline{I}}\cdot\underline{w}) = \underline{M}_d \underbrace{- \dot{\underline{\underline{I}}}\,\underline{w} - \dot{\underline{\underline{A}}}\,\underline{h}_r - \underline{\underline{A}}\,\dot{\underline{h}}_r - \underline{w}\times(\underline{\underline{A}}\,\underline{h}_r)}_{M_c}$

Thrusters (1)

1. REACTION WHEELS

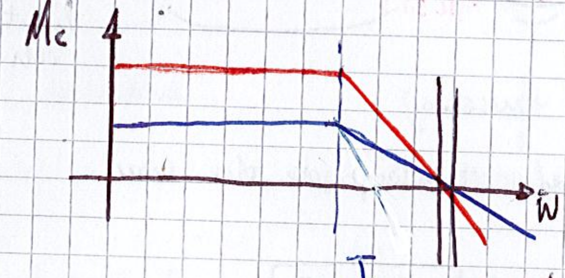
Work only modifying the term h_r ⇒ $\dot{\underline{\underline{A}}}=\underline{0}$; $\dot{\underline{\underline{I}}}=\underline{0}$

therefore equation (i) becomes:

(i) $\underline{\underline{I}}\,\dot{\underline{w}} + \underline{w}\times(\underline{\underline{I}}\,\underline{w}) = \underline{M}_d - \underline{\underline{A}}\,\dot{\underline{h}}_r - \underline{w}\times(\underline{\underline{A}}\,\underline{h}_r)$

... In order to increase the speed of reaction wheels (and so having $\dot{h}_r \neq 0$) electric (BRUSHLESS) engines are required:

⇒ The typical operational curve for a DC motor would be the following:

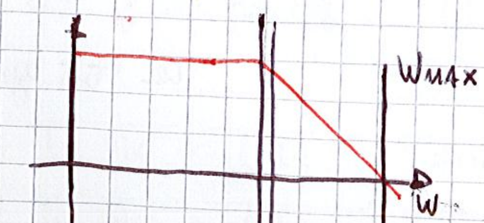


$M_c = K_t \cdot I$

Power Balance:

$\begin{cases} M_e = \dot{h}_R \\ M_e = K_t \cdot I_R \end{cases}$ ⇒ $P_{mecc} = P_{el}$ ⇒ $\begin{cases} \Delta V\, I_R = M_e \cdot w_R \\ \Delta V \simeq \text{const.} \end{cases}$

⇒ $\Delta V\, \cancel{I_R} = K_t\, \cancel{I_R}\, w_R$ → $w_R\big|_{max} = \dfrac{\Delta V}{K_t}$



Ws → saturation velocity

Assumption 1: $I_{Ri} << I_{i,s}$

it means that we can neglect into (i) equation the "gyroscopic term" of the R.W.

(i): $\underline{\underline{I}}\,\underline{\dot{W}} + \underline{W} \times (\underline{\underline{I}}\,\underline{W}) = -\underline{\underline{A}}\,\underline{\dot{h}}_R - \cancel{\underline{W} \times \underline{h}_R}$

in this example $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So equations will become

$$\begin{cases} I_x \dot{w}_x + (I_z - I_y)\,\bar{w}_z w_y + I_R \dot{w}_{Rx} = 0 \\ I_y \dot{w}_y + (I_x - I_z)\,\bar{w}_z w_x + I_R \dot{w}_{Ry} = 0 \\ I_z \dot{w}_z + (I_y - I_x)\,w_x w_y + I_R \dot{w}_{Rz} = 0 \end{cases}$$

$M_{cx} = -I_R \dot{w}_{Rx}$
$M_{cy} = -I_R \dot{w}_{Ry}$
$M_{cz} = -I_R \dot{w}_{Rz}$.

$\to$ provided by the electric engines.

Assumption 2: No torque transmitted between the rotor and the spacecraft (IDEAL TRANSMISSION)

$\Rightarrow$ Putting in evidence the control action:

$$\begin{cases} I_x \dot{w}_x + (I_z - I_y)\,\bar{w}_z w_y = -I_R \dot{w}_{Rx} = M_{cx} \\ I_y \dot{w}_y + (I_x - I_z)\,\bar{w}_z w_x = -I_R \dot{w}_{Ry} = M_{cy} \\ I_z \dot{w}_z + (I_y - I_x)\,w_x w_y = -I_R \dot{w}_{Rz} = M_{cz}. \end{cases}$$

$\to$ In designing the control torque value many strategy can be used:

e.g. $\to$ FULL-STATE FEEDBACK: (Proportional)

$$M_{cx} = K_{xx} w_x + K_{yx} w_y + K_{zx} (w_z - \bar{w}_z)$$

or PD

$$M_{cx} = K_x w_x + K_{\dot{x}} \dot{w}_x$$

$\Rightarrow$ Considering also the gyroscopic effect of the reaction wheels (relaxing Assumption 1)

$$\underline{\underline{I}}\,\underline{\dot{W}} + \underline{W} \times (\underline{\underline{I}}\,\underline{W}) = -\underline{\underline{A}}\,\underline{\dot{h}}_R - \underline{W} \times (\underline{\underline{A}}\,\underline{h}_R) = \underline{M}_c$$

$\Rightarrow$ $\begin{cases} \underline{M}_c = -\underline{\underline{A}}\,\underline{\dot{h}}_R - \underline{W} \times (\underline{\underline{A}}\,\underline{h}_R) \\ \underline{M}_c = f(w_x, w_y, w_z, \dots) \to \text{PID, LQR, } \dots \end{cases}$ + $I_R \dot{w}_R = M_R$

PROCEDURE (general)

(1) Compute $\underline{M}_c$ thanks to the control law:

(*) $\underline{M}_c = \underline{\underline{K}}_D (\dot{\underline{\alpha}} - \dot{\underline{\alpha}}_{sp}) + \underline{\underline{K}}_P (\underline{\alpha} - \underline{\alpha}_{sp})$ e.g. PD; $\underline{\alpha} \to$ attitude parameter.

(2) Determine $\underline{W}$ (by solving Euler's equation)

$$\underline{\underline{I}}\,\underline{\dot{W}} + \underline{W} \times (\underline{\underline{I}}\,\underline{W}) = \underline{M}_c \to \underline{W}$$

(3) Determine $\underline{\dot{h}}_R$

$$\underline{\dot{h}}_R = -\underline{\underline{A}}^{-1}\,\underline{M}_c - \underline{\underline{A}}^{-1}\,\underline{W} \times (\underline{\underline{A}}\,\underline{h}_R)$$

(4) Determine the power required to increase the angular momentum of the reaction wheels:

$$M_{R_i} = I_R \dot{w}_R$$

$\Rightarrow$ $M_R = K_T \cdot i_i$

i_i: current in the i-th RW feeding the DC motor

(*) in the control equation [→ CONTROL-LAW]

$$\underline{M}_c = K_D \cdot (\dot{\underline{\alpha}} - \dot{\underline{\alpha}}_{SP}) - K_P (\underline{\alpha} - \underline{\alpha}_{SP})$$

generic PD → $\underline{M}_c = K_D \, \dot{e}(t) + K_P \, e(t)$

the attitude parameter is expressed generically while: (for small deviations from target conditions) is possible to express it in a very definite form:

$\underline{\underline{A}}_T$ → "Target-direction-cosine matrix"

$\underline{\underline{A}}_m$ → "Direction cosine matrix as measured by the sensors"

⇒ $\underline{\underline{A}}_e = \underline{\underline{A}}_T - \underline{\underline{A}}_m$

BUT since both $\underline{\underline{A}}_T$ and $\underline{\underline{A}}_m$ are supposed to be orthogonal and equals.

THEN $\underline{\underline{A}}_e$ (representing the non orthogonality) can be computed as product:

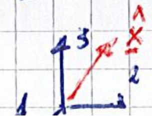
$$\boxed{\underline{\underline{A}}_e = \underline{\underline{A}}_m \cdot \underline{\underline{A}}_T^T}$$

in fact:

$$\underline{\underline{A}}_m = \begin{bmatrix} a_{11m} & a_{12m} & a_{13m} \\ a_{21m} & a_{22m} & a_{23m} \\ a_{31m} & a_{32m} & a_{33m} \end{bmatrix} = \begin{bmatrix} \underline{X}_m \\ \underline{Y}_m \\ \underline{Z}_m \end{bmatrix}$$

where: $a_{11} = \hat{\underline{x}} \cdot \hat{\underline{1}}$ → inertial r.f. ; $a_{12} = \hat{\underline{x}} \cdot \hat{\underline{2}}$ . . . . .
($\hat{\underline{x}}$: PI r.f.)

⇒ the first row of the direction cosine matrix represent the projection of the "$\hat{\underline{x}}$" axis in P.I. on the IN. reference frame



completely analogous:

$$\underline{\underline{A}}_T = \begin{bmatrix} a_{11T} & a_{12T} & a_{13T} \\ a_{21T} & a_{22T} & a_{23T} \\ a_{31T} & a_{32T} & a_{33T} \end{bmatrix} = \begin{bmatrix} \underline{X}_T \\ \underline{Y}_T \\ \underline{Z}_T \end{bmatrix}$$

Therefore:

$$\underline{\underline{A}}_e = \underline{\underline{A}}_m \cdot \underline{\underline{A}}_T^T = \begin{bmatrix} \underline{X}_m \\ \underline{Y}_m \\ \underline{Z}_m \end{bmatrix} \cdot \left(\begin{bmatrix} \underline{X}_T \\ \underline{Y}_T \\ \underline{Z}_T \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} \underline{X}_m \\ \underline{Y}_m \\ \underline{Z}_m \end{bmatrix} [\underline{X}_T^T, \underline{Y}_T^T, \underline{Z}_T^T] = \begin{bmatrix} \underline{X}_m \cdot \underline{X}_T^T ; & \underline{X}_m \underline{Y}_T^T ; & \underline{X}_m \underline{Z}_T^T \\ \underline{Y}_m \cdot \underline{X}_T^T ; & \underline{Y}_m \underline{Y}_T^T ; & \underline{Y}_m \underline{Z}_T^T \\ \underline{Z}_m \cdot \underline{X}_T^T ; & \underline{Z}_m \underline{Y}_T^T ; & \underline{Z}_m \underline{Z}_T^T \end{bmatrix}$$

⇒ each element of matrix $\underline{\underline{A}}_e$ represent the scalar product between 2 axis done in the IN. reference frame.

e.g. $\hat{\underline{i}}_m = [\hat{\underline{i}}_m \cdot \hat{\underline{1}} ; \hat{\underline{i}}_m \cdot \hat{\underline{2}} ; \hat{\underline{i}}_m \cdot \hat{\underline{3}}]$ $\quad \hat{\underline{i}}_m \rightarrow \underline{X}_m$

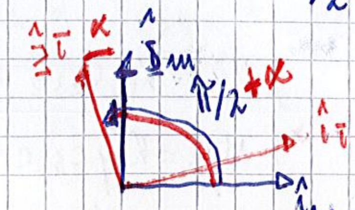
$\hat{\underline{i}}_T = [\hat{\underline{i}}_T \cdot \hat{\underline{1}} ; \hat{\underline{i}}_T \cdot \hat{\underline{2}} ; \hat{\underline{i}}_T \cdot \hat{\underline{3}}]$ $\quad \hat{\underline{i}}_T \rightarrow \underline{X}_T$

$\hat{\underline{j}}_T = [\hat{\underline{j}}_T \cdot \hat{\underline{1}} ; \hat{\underline{j}}_T \cdot \hat{\underline{2}} ; \hat{\underline{j}}_T \cdot \hat{\underline{3}}]$

for small deviation from the target condition:

$\arccos(\hat{\underline{i}}_m \cdot \hat{\underline{i}}_T) \approx 0 \, [\text{rad}]$ ⇒ $\hat{\underline{i}}_m \cdot \hat{\underline{i}}_T \approx 1$

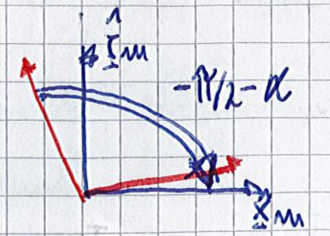
$\arccos(\hat{\underline{i}}_m \cdot \hat{\underline{j}}_T) \approx \pi/2 \, [\text{rad}]$ ⇒ $\hat{\underline{i}}_m \cdot \hat{\underline{j}}_T \approx$



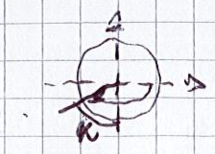
↳ in truth is equal to $\pi/2 + \alpha$ a that is a small deviation angle

⇒ $\cos(\pi/2 + \alpha) = \sin(\alpha) \approx \alpha$ (considering small angles) since this angle "α" represent the rotation around $z_{m/T}$ axis is called α_z.

WHILE considering the opposite scalar-product.



$$\Rightarrow \cos(-\pi/2-\alpha) = \sin(-\alpha) \approx -\alpha$$



$\Rightarrow$ Considering small rotation angles between the 3 PI. inertia axis

$$\underline{\underline{A}}_e = \begin{bmatrix} \underline{X}_m \underline{X}_T^T & ; & \underline{X}_m \underline{Y}_T^T & ; & \underline{X}_m \underline{Z}_T^T \\ \underline{Y}_m \underline{X}_T^T & ; & \underline{Y}_m \cdot \underline{Y}_T^T & ; & \underline{Y}_m \cdot \underline{Z}_T^T \\ \underline{Z}_m \cdot \underline{X}_T^T & ; & \underline{Z}_m \underline{Y}_T^T & ; & \underline{Z}_m \cdot \underline{Z}_T^T \end{bmatrix} \approx \begin{bmatrix} 1, & \alpha_z, & -\alpha_y \\ -\alpha_z, & 1, & \alpha_x \\ +\alpha_y, & -\alpha_x, & 1 \end{bmatrix}$$

!! This is a good approximation of the error (small variations from nominal / set-point condition) !!

As saw before.

$[\underline{\underline{A}}e]_{1,2} = a_{1,2} = \alpha_z \xrightarrow[\text{around}]{\text{drives rotation}}$ "z_axis"

$[\underline{\underline{A}}e]_{1,3} = a_{13} = -\alpha_y \xrightarrow[\text{around}]{\text{drives rotation}}$ "y_axis"

$[\underline{\underline{A}}e]_{23} = a_{23} = \alpha_x \xrightarrow[\text{around}]{\text{drives rotation}}$ "x_axis"

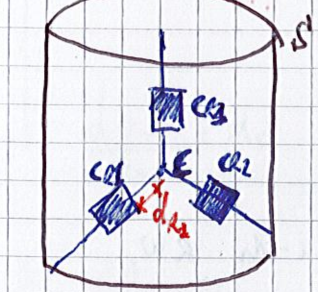
$a_{12} = \underline{X}_m \cdot \underline{Y}_T^T$; $a_{13} = \underline{X}_m \cdot \underline{Z}_T^T$; $a_{23} = \underline{Y}_m \underline{X}_T^T$

PD, control law [SMALL ROTATION]

$$\Rightarrow \begin{cases} M_x = K_{px}\alpha_x + K_{dx}\cdot\dot{\alpha}_x \\ M_y = K_{py}\alpha_y + K_{dy}\dot{\alpha}_y \\ M_z = K_{pz}\alpha_z + K_{dz}\dot{\alpha}_z \end{cases} \longrightarrow \begin{cases} M_{x,s} = K_{p_x} a_{23e} + K_{dx} w_x \\ M_{ys} = K_{py} a_{31e} + K_{dy} w_y \\ M_{z,s} = -K_{pz} a_{21e} + K_{dz} w_z \end{cases}$$

MENGALI: Control system with 3 RW aligned with the 3 inertia axis.

equilibrium condition: $\begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}$ $\rightarrow$ 1 rotation per orbit (circular) $n = \frac{2\pi}{T_{orbit}}$



C $\rightarrow$ overall center of mass.

$$\underline{H}_c = \underline{h}_{s,c} + \underline{h}_{c,R1} + \underline{h}_{c,R2} + \underline{h}_{c,R3}$$

Defining inertia moments matrices:

$$\underline{\underline{I}}_{s,c} = \begin{bmatrix} A_s & 0 & 0 \\ 0 & B_s & 0 \\ 0 & 0 & C_s \end{bmatrix}$$

$$\underline{\underline{I}}_{R1,CR1} = \begin{bmatrix} I_{R1} & 0 & 0 \\ 0 & B_{R1} & 0 \\ 0 & 0 & C_{R1} \end{bmatrix} ; \quad \underline{\underline{I}}_{R2,CR2} = \begin{bmatrix} A_{R..} & 0 & 0 \\ 0 & I_{R2} & 0 \\ 0 & 0 & C_{R2} \end{bmatrix}$$

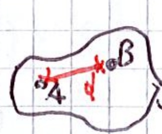
momento di inerzia rotore 1 rispetto a CR1 (centro del proprio rotore)

$$\underline{\underline{I}}_{R3,CR3} = \begin{bmatrix} A_R & 0 & 0 \\ 0 & B_R & 0 \\ 0 & 0 & I_R \end{bmatrix}$$

!! già nella terna fondamentale di inerzia !!

$\rightarrow$ Huygen's theorem

$$I_A = I_B + m\,\|(B-A)\|^2$$



(I_A, I_B inertia moment along the same axis $\Rightarrow$ (A,B) contained in the same plane) !

$$\underline{\underline{I}}_c^{TOT} = \underline{\underline{I}}_s + \underline{\underline{I}}_{R1,c} + \underline{\underline{I}}_{R2,c} + \underline{\underline{I}}_{R3,c}$$

$$= \begin{bmatrix} A_s & 0 & 0 \\ 0 & B_s & 0 \\ 0 & 0 & C_s \end{bmatrix} + \begin{bmatrix} I_{R1} & 0 & 0 \\ 0 & B_{R1}+m_{R1}d_{R1}^2 & 0 \\ 0 & 0 & C_{R1}+m_{R1}d_{R1}^2 \end{bmatrix} + \dots$$

therefore the 3 inertia moments (WITH RESPECT TO THE CENTER OF MASS OF THE OVERALL SYSTEM ($S + R_1 + R_2 + R_3$)) are the following:

$$I_x = A_s + I_{R1} + A_{R2} + m_{R2} d_{R2}^2 + A_{R3} + m_{R3} d_{R3}^2$$
$$I_y = B_s + I_{R2} + B_{R1} + m_{R1} d_{R1}^2 + B_{R3} + m_{R3} d_{R3}^2 \qquad (i)$$
$$I_z = C_s + I_{R3} + C_{R1} + m_{R1} d_{R1}^2 + C_{R2} + m_{R2} d_{R2}^2 .$$

Assumption 1

$$\{\underline{W}_{R1}\}_{\mathcal{F}B} = \begin{Bmatrix} W_x \\ W_y \\ W_z \end{Bmatrix} + \begin{Bmatrix} \Omega_1 \\ 0 \\ 0 \end{Bmatrix} \approx \begin{Bmatrix} \Omega_1 \\ W_y \\ W_z \end{Bmatrix}$$

around spin axis of the i-th RW.

body centered frame.

$$\Rightarrow \underline{W}_{R2} \approx \begin{Bmatrix} W_x \\ \Omega_2 \\ W_z \end{Bmatrix} ; \quad \underline{W}_{R3} \approx \begin{Bmatrix} W_x \\ W_y \\ \Omega_3 \end{Bmatrix} . \qquad (ii)$$

(i) + (ii) $\underline{\underline{I}}\,\underline{W}$ +

$$\underline{H}_c^{TOT} = \begin{bmatrix} A_s & 0 & 0 \\ 0 & B_s & 0 \\ 0 & 0 & C_s \end{bmatrix} \cdot \begin{Bmatrix} W_x \\ W_y \\ W_z \end{Bmatrix} + \begin{bmatrix} I_{R1} & 0 & 0 \\ 0 & B_{R1}+md^2 & 0 \\ 0 & 0 & C_{R1}+md^2 \end{bmatrix} \begin{Bmatrix} \Omega_1 \\ W_y \\ W_z \end{Bmatrix} +$$

$$+ \begin{bmatrix} A_{R2}+md^2 & 0 & 0 \\ 0 & I_{R2} & 0 \\ 0 & 0 & C_{R2}+md^2 \end{bmatrix} \begin{Bmatrix} W_x \\ \Omega_2 \\ W_z \end{Bmatrix}$$

$$+ \begin{bmatrix} A_{R3}+md^2 & 0 & 0 \\ 0 & B_{R3}+md^2 & 0 \\ 0 & 0 & I_{R3} \end{bmatrix} \begin{Bmatrix} W_x \\ W_y \\ \Omega_3 \end{Bmatrix}$$

$\Rightarrow$ Enclosing terms in $\{W_x, W_y, W_z\}$ (EVEN NEGLECTING Assumption 1) separately from the ones in $\{\Omega_1, \Omega_2, \Omega_3\}$.

$$\begin{cases} \underline{H}_c^{TOT} = \begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix} \cdot \begin{Bmatrix} W_x \\ W_y \\ W_z \end{Bmatrix} + \begin{Bmatrix} I_{R1}\Omega_1 \\ I_{R2}\Omega_2 \\ I_{R3}\Omega_3 \end{Bmatrix} \\ \dfrac{d\underline{H}_c^{TOT}}{dt} = \underline{M}_d \end{cases} ; \{ I_{R1}\Omega_1 \triangleq h_1 ; \ I_{R2}\Omega_2 = h_2 ; \ I_{R3}\Omega_3 = h_3 \}.$$

$\Rightarrow$ By defining:

$$\underline{H}_{c,s} \triangleq \begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix} \cdot \begin{Bmatrix} W_x \\ W_y \\ W_z \end{Bmatrix} .$$

capital "H" since includes also inertia contribution due to the reaction wheels.

$$\underline{H}_{cR} = \begin{Bmatrix} I_{R1}\Omega_1 \\ I_{R2}\Omega_2 \\ I_{R3}\Omega_3 \end{Bmatrix} = \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \end{Bmatrix} .$$

$$\underline{H}_c^{TOT} = \underline{H}_{c,s} + \underline{H}_{c,R}$$

$\Rightarrow$ Euler's equation will be in such condition:

$$\frac{d\underline{H}_{cs}}{dt} + \frac{d\underline{H}_{c,R}}{dt} + \underline{W} \times (\underline{H}_{c,s} + \underline{H}_{c,R}) = \underline{M}_d .$$

$\Rightarrow$ Putting again in evidence the contribution due to control:

$$\dot{\underline{H}}_{c,s} + \underline{W} \times (\underline{H}_{c,s} + \underline{H}_{c,R}) = \underline{M}_d \underbrace{- \dot{\underline{H}}_{cR}}$$

$\hookrightarrow$ $\underline{M}_c \triangleq -\dot{\underline{H}}_{c,R}$

" As observed previously the torque provided by the gravity gradient can give a sort of stabilization (or it's anyway exploitable) to open otherwise new stability regions "

* Equilibrium condition: $\begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}_{\tau_B}$

* $\{\phi, \theta, \psi\} \rightarrow$ Euler's equation determining the orientation with respect to the LVLH reference frame of the P.I. reference frame.

sequence: $x \rightarrow y \rightarrow z \Rightarrow \underline{\underline{A}}_{PI2LHLV} = \begin{bmatrix} c\alpha_z c\alpha_y ; & c\alpha_z s\alpha_y s\alpha_x + \dots \\ \dots & \dots \end{bmatrix}$

IF $Eq = \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}$; $n = \frac{2\pi}{T_{orb}}$ ($\alpha_x = \phi$; $\alpha_y = \theta$; $\alpha_z = \psi$)

!! DIFFERENTLY FROM BEFORE !!

THEN $\forall t$ $PI \equiv LHLV \Rightarrow$ I can linearize $\underline{\underline{A}}_{PI2LHLV}$ around equilibrium condition $\begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} = 0$.

$$\underline{\underline{A}}_{PI2LHLV} = \begin{bmatrix} c\alpha_z c\alpha_y ; & c\alpha_z s\alpha_y s\alpha_x + s\alpha_z c\alpha_x ; & -c\alpha_z s\alpha_y c\alpha_x + s\alpha_z s\alpha_x \\ -s\alpha_z c\alpha_y ; & -s\alpha_z s\alpha_y s\alpha_x + c\alpha_z c\alpha_x ; & s\alpha_z s\alpha_y s\alpha_x + c\alpha_z s\alpha_x \\ s\alpha_y ; & -c\alpha_y s\alpha_x ; & c\alpha_y c\alpha_x \end{bmatrix}$$

$$\begin{cases} c(\alpha_i) \approx 1 \\ s(\alpha_i) \approx \alpha_i \\ s(\alpha_i)\cdot s(\alpha_j) \approx 0 \end{cases}$$

$$\underline{\underline{A}}^{lin}_{PI2LHLV} = \begin{bmatrix} 1 ; & \alpha_z ; & -\alpha_y \\ -\alpha_z ; & 1 ; & \alpha_x \\ \alpha_y ; & \alpha_x ; & 1 \end{bmatrix}$$

* $\{\underline{W}\}_{B} = \underline{W}_0 + \underline{W}_{B0}$

$$\begin{cases} W_x = \dot{\alpha}_x + \alpha_y n \\ W_y = \dot{\alpha}_y - \alpha_x n \\ W_z = \dot{\alpha}_z + n \end{cases}$$

see previous computations.

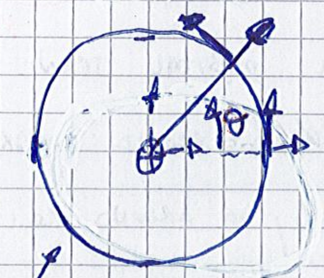
[previous exercise book]

therefore: $\underline{W} = \begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix}$

[HAVING CONSIDERED SMALL ROTATIONS, THE ORDER IN WHICH THEY ARE OBTAINED IS NOT RELEVANT.]

$\{\underline{W}\}_{\tau_B} = \{\underline{W}_{B0}\}_{\tau_B} + \{\underline{W}_0\}_{\tau_B}$

* $\{\underline{W}_0\}_{\tau_B} = \underline{\underline{A}}_{LHLV2PI} \cdot \underline{\underline{A}}_{IN2LHLV} \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}$ ($n = \sqrt{\frac{\mu_\oplus}{r^3}} \rightarrow$ circular orbit)



equatorial + circular orbit.

$\hat{n}_0 \equiv \hat{k} \Rightarrow \{\underline{W}_0\}_{\tau_I} = \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}$

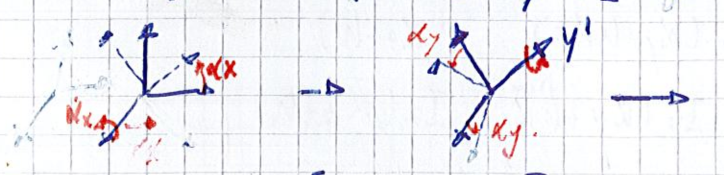
$$\underline{\underline{A}}_{IN2LHLV} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\{\underline{W}_0\}_{\tau_0} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}$$

* $\{\underline{W}_{B0}\}_{\tau_B} = \begin{bmatrix} 1 ; & \alpha_z ; & -\alpha_y \\ -\alpha_z ; & 1 ; & \alpha_x \\ \alpha_y ; & \alpha_x ; & 1 \end{bmatrix} \begin{Bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z \end{Bmatrix}$

Case of independence of rotations. $\alpha_x \rightarrow \alpha_y \rightarrow \alpha_z$

$$\underline{W}_{B0} = \underline{\underline{T}}(\alpha_x)\,\dot{\alpha}_x + \underline{\underline{T}}(\alpha_y)\left(\underline{\underline{T}}(\alpha_x)\,\dot{\alpha}_y\right) + \underline{\underline{T}}(\alpha_z)\,\underline{\underline{T}}(\alpha_y)\left(\underline{\underline{T}}(\alpha_z)\,\dot{\alpha}_z\right)$$



$\underline{\underline{T}}(\alpha_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_x & s\alpha_x \\ 0 & -s\alpha_x & c\alpha_x \end{bmatrix}$ $\quad \underline{\underline{T}}(\alpha_y) = \begin{bmatrix} c\alpha_y & 0 & s\alpha_y \\ 0 & 1 & 0 \\ -s\alpha_y & 0 & c\alpha_y \end{bmatrix}$ ..

⇒ for small rotations (as previously demonstrated) is possible to obtain $\underline{\underline{A}}$ such that:

$$A_{LVLH2PI} = \begin{bmatrix} \cdots \end{bmatrix} \cdot \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix}$$

* $$\{\underline{\omega}_B\}_{\tau_B} = \{\underline{\omega}_{BO}\}_{\tau_B} + \{\underline{\omega}_O\}_{\tau_B} = \begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix}$$

→ EULER'S EQUATION: $\dot{\underline{H}}_{c,s} + \underline{\omega} \times (\underline{H}_{c,s} + \underline{H}_{c,R}) = \underline{M}_d - \underbrace{\dot{\underline{H}}_{c,R}}_{M_c}$

→ $\underline{\omega}$ AS FUNCTION OF SMALL ROTATIONS FROM ORBITAL TERN:

$$\underline{\omega} = \begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix}$$

α_x → AROUND YAW AXIS

α_y → AROUND ROLL AXIS

α_z → AROUND PITCH AXIS

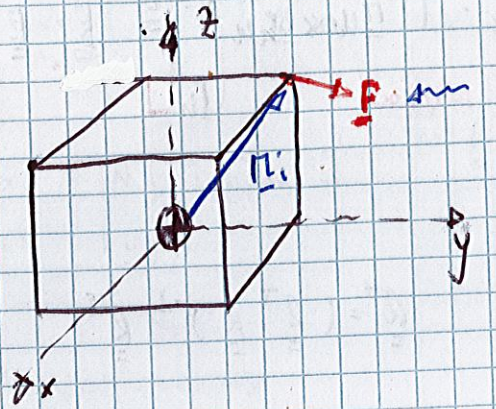
⇒ $$\underline{\underline{I}}_s \frac{d}{dt}\left(\begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix}\right) + \begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix} \times \left(\underline{\underline{I}}_s \begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix}\right) +$$

$$+ \begin{Bmatrix} \dot{\alpha}_x - n\alpha_y \\ \dot{\alpha}_y + n\alpha_x \\ \dot{\alpha}_z + n \end{Bmatrix} \times \begin{Bmatrix} I_{R1}\Omega_1 \\ I_{R2}\Omega_2 \\ I_{R3}\Omega_3 \end{Bmatrix} = \underline{M}_d + \underline{M}_c$$

→ computing separately the 2 cross products

1) $$\underline{\omega} \times \underline{H}_{c,s} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (\dot{\alpha}_x - n\alpha_y) & (\dot{\alpha}_y + n\alpha_x) & (\dot{\alpha}_z + n) \\ I_x(\dot{\alpha}_x - n\alpha_y) & I_y(\dot{\alpha}_y + n\alpha_x) & I_z(\dot{\alpha}_z + n) \end{vmatrix}$$

Thrusters: constant action ⇒ NON LINEAR CONTROL.



→ thruster are put in the "corner" of the body in order to maximize the Torque.

!! Usually thrusters have a fixed direction

⇒ $\underline{M}_i = \underline{r}_i \times \underline{F}_i = \underline{r}_i \times \underline{c}_i f_i$

$\underline{c}_i = \begin{Bmatrix} c_{x,i} \\ c_{y,i} \\ c_{z,i} \end{Bmatrix}$ projection of each i-thruster along the p.r. axis. !!

→ Considering many actuators

$$\underline{M}_i = \underline{r}_i \times \underline{F}_i \Rightarrow \underline{M} = \sum_i \underline{r}_i \times \underline{F}_i = \sum_i \underline{r}_i \times \underline{c}_i f_i$$

That in matrix form becomes:

$$\underline{M} = [\, \underline{r}_1 \times \underline{c}_1 \mid \cdots \mid \underline{r}_i \times \underline{c}_i \mid \cdots \; \underline{r}_n \times \underline{c}_n \,] \begin{Bmatrix} f_1 \\ \vdots \\ f_n \end{Bmatrix}.$$

2-Thrusters

$$\underline{M}_1 = \underline{r}_1 \times \underline{c}_1 \, f_1 = \left[\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_{1x} & r_{1y} & r_{1z} \\ c_{1x} & c_{1y} & c_{1z} \end{vmatrix}\right] f_1 = \begin{Bmatrix} r_{1y}c_{1z} - c_{1y}r_{1z} \\ r_{1z}c_{1x} - r_{1x}c_{1z} \\ r_{1x}c_{1y} - c_{1x}r_{1y} \end{Bmatrix} f_1$$

$$\underline{M}_2 = \underline{r}_2 \times \underline{c}_2 \, f_2 = \begin{Bmatrix} r_{2y}c_{2z} - c_{2y}r_{2z} \\ r_{2z}c_{2x} - r_{2x}c_{2z} \\ r_{2x}c_{2y} - c_{2x}r_{2y} \end{Bmatrix} f_2$$

$$\underline{M} = [\, \underline{r}_1 \times \underline{c}_1 \mid \underline{r}_2 \times \underline{c}_2 \,] \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} =$$

⇒ for small rotations (as previously demonstrated) is possible to obtain $\underline{\underline{A}}$ such that:

$$A_{LHLV2PI} = \begin{bmatrix} \cdots \end{bmatrix} \cdot \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix}$$

* $$\{\underline{w}_B\}_{\tau_B} = \{\underline{w}_{BO}\}_{\tau_B} + \{\underline{w}_O\}_{\tau_B} = \begin{Bmatrix} \dot\alpha_x - n\alpha_y \\ \dot\alpha_y + n\alpha_x \\ \dot\alpha_z + n \end{Bmatrix}$$

→ EULER'S EQUATION: $$\dot{\underline{H}}_{c,s} + \underline{w} \times (\underline{H}_{c,s} + \underline{H}_{c,R}) = \underline{M}_d \underbrace{- \dot{\underline{H}}_{c,R}}_{\underline{M}_c}$$

→ $\underline{w}$ AS FUNCTION OF SMALL ROTATIONS FROM ORBITAL TERN:

$$\underline{w} = \begin{Bmatrix} \dot\alpha_x - n\alpha_y \\ \dot\alpha_y + n\alpha_x \\ \dot\alpha_z + n \end{Bmatrix}$$

α_x → AROUND YAW AXIS

α_y → AROUND ROLL AXIS

α_z → AROUND PITCH AXIS

$$\underline{\underline{I}}_s \frac{d}{dt}\left(\begin{Bmatrix} \dot\alpha_x - n\alpha_y \\ \dot\alpha_y + n\alpha_x \\ \dot\alpha_z + n \end{Bmatrix}\right) + \begin{Bmatrix} \dot\alpha_x - n\alpha_y \\ \dot\alpha_y + n\alpha_x \\ \dot\alpha_z + n \end{Bmatrix} \times \left(\underline{\underline{I}}_s \begin{Bmatrix} \dot\alpha_x - n\alpha_y \\ \dot\alpha_y + n\alpha_x \\ \dot\alpha_z + n \end{Bmatrix}\right) +$$

$$+ \begin{Bmatrix} \dot\alpha_x - n\alpha_y \\ \dot\alpha_y + n\alpha_x \\ \dot\alpha_z + n \end{Bmatrix} \times \begin{Bmatrix} I_{R1}\Omega_1 \\ I_{R2}\Omega_2 \\ I_{R3}\Omega_3 \end{Bmatrix} = \underline{M}_d + \underline{M}_c$$

→ computing separately the 2 cross products

1) $$\underline{w} \times \underline{H}_{c,s} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (\dot\alpha_x - n\alpha_y) & (\dot\alpha_y + n\alpha_x) & (\dot\alpha_z + n) \\ I_x(\dot\alpha_x - n\alpha_y) & I_y(\dot\alpha_y + n\alpha_x) & I_z(\dot\alpha_z + n) \end{vmatrix}$$

$$\underline{w} \times \underline{H}_{c,s} = \hat{i}\cdot[\, I_z(\underbrace{\dot\alpha_z \dot\alpha_y}_{\to 0} + n\dot\alpha_y + \underbrace{n\alpha_x \dot\alpha_z}_{\to 0} + n^2\alpha_x) - I_y(n\dot\alpha_y + n^2\alpha_x)\,]$$
$$+ \hat{j}\,[\, I_x(n\dot\alpha_x - n^2\alpha_y) - I_z(n\dot\alpha_x - n^2\alpha_y)\,]$$
$$+ \hat{k}\,[\, I_y(\underbrace{\dot\alpha_x \dot\alpha_y}_{\to 0} + \underbrace{n\dot\alpha_x \alpha_x}_{\to 0} - \underbrace{n\alpha_y \dot\alpha_y}_{\to 0} - \underbrace{n^2\alpha_y\alpha_x}_{\to 0}) - I_x(0)\,]$$

$$\underline{w} \times \underline{H}_{c,s} = \begin{bmatrix} (I_z - I_y)\,(n\dot\alpha_y + n^2\alpha_x) \\ (I_x - I_z)\,(n\dot\alpha_x - n^2\alpha_y) \\ 0 \end{bmatrix}$$

2) $$\underline{w} \times \underline{H}_{c,R} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (\dot\alpha_x - n\alpha_y) & (\dot\alpha_y + n\alpha_x) & (\dot\alpha_z + n) \\ h_{Rx} & h_{Ry} & h_{Rz} \end{vmatrix}$$

$(h_{R1} = I_R\,\Omega_1)$ $(h_{R2} = I_R\,\Omega_2)$ $(h_{R3} = I_R\,\Omega_3)$

$$\underline{w} \times \underline{H}_{cR} = \begin{Bmatrix} h_{Rz}(\dot\alpha_y + n\alpha_x) - h_{Ry}(\dot\alpha_z + n) \\ h_{Rx}(\dot\alpha_z + n) - h_{Rz}(\dot\alpha_x - n\alpha_y) \\ h_{Ry}(\dot\alpha_x - n\alpha_y) - h_{Rx}(\dot\alpha_y + n\alpha_x) \end{Bmatrix}.$$

→ Euler's equation then becomes:

Euler's equation (w/ 3 rotors) linearized around $\{0\}$ condition.

$$\begin{cases} I_x(\ddot\alpha_x - n\dot\alpha_y) + (I_z - I_y)(n\dot\alpha_y + n^2\alpha_x) + h_{Rz}(\dot\alpha_y + n\alpha_x) - h_{Ry}(\dot\alpha_z + n) = M_x \\ I_y(\ddot\alpha_y + n\dot\alpha_x) + (I_x - I_z)(n\dot\alpha_x - n^2\alpha_y) + h_{Rx}(\dot\alpha_z + n) - h_{Rz}(\dot\alpha_x - n\alpha_y) = M_y \\ I_z\,\ddot\alpha_z + h_{Ry}(\dot\alpha_x - n\alpha_y) - h_{Rx}(\dot\alpha_y + n\alpha_x) = M_z. \end{cases}$$

$$\underline{M} = \underline{M}_d - \begin{Bmatrix} I_{R1}\dot\Omega_1 \\ I_{R2}\dot\Omega_2 \\ I_{R3}\dot\Omega_3 \end{Bmatrix} \quad \Rightarrow \quad \underline{M}_c = \begin{Bmatrix} I_{R1}\dot\Omega_1 \\ I_{R1}\dot\Omega_2 \\ I_{R3}\dot\Omega_3 \end{Bmatrix}$$

$$I_x(\ddot{\alpha}_x - n\dot{\alpha}_y) + (I_z - I_y)(n\dot{\alpha}_y + n^2\alpha_x) + h_{rz}(\dot{\alpha}_y + n\alpha_x) - h_{ry}(\dot{\alpha}_z + n) = M_x$$

$$I_y(\ddot{\alpha}_y + n\dot{\alpha}_x) + (I_x - I_z)(n\dot{\alpha}_x - n^2\alpha_y) + h_{rx}(\dot{\alpha}_z + n) - h_{rz}(\dot{\alpha}_x - n\alpha_y) = M_y$$

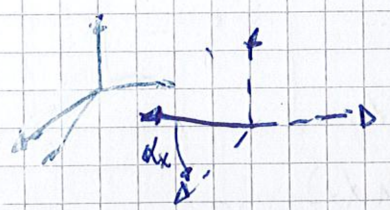
$$I_z\ddot{\alpha}_z + h_{ry}(\dot{\alpha}_x - n\alpha_y) - h_{rx}(\dot{\alpha}_y + n\alpha_x) = M_z$$

↳ $\underline{M} = \underline{M}_d + \underline{M}_c$; where (for now) the gravity gradient torque is included inside the disturbances and contributes to the dynamic of the system (even stabilizing it)

--> $\underline{M}_{gg}$

$$\underline{M}_g = 3\frac{\mu}{R^3}\begin{vmatrix}(I_z - I_y)c_2c_3 \\ (I_x - I_y)c_1c_3 \\ (I_y - I_x)c_1c_2\end{vmatrix} \qquad \begin{aligned}c_1 &= \hat{R}\cdot\hat{i} = \cos(\alpha_x) \simeq 1 \\ c_2 &= \hat{R}\cdot\hat{J} \simeq \sin(\alpha_y) \simeq \alpha_y \\ c_3 &= \hat{R}\cdot\hat{k} \simeq \alpha_z.\end{aligned}$$

$+ \quad n = \sqrt{\frac{\mu}{R^3}} \Rightarrow \frac{\mu}{R^3} = n^2$



$\sqrt{\frac{\mu}{a^3}}\,\Delta t = E - e\sin E.$

Pitch control (z-axis) → set-point: $\hat{z}$ aligned with y_0.

Assumption: Rotors' angular momenta are hereby neglected. (SUPPOSED TO BE REALLY LESS MASSIVE THAN THE SPACECRAFT)

$$h_{ry} \simeq h_{rx} \simeq 0.$$

obs: Pitch Euler's equation is decoupled from the other 2. (under Assumption 2 validity)

Model to design control can be slightly different from reality

BUT control must be enough robust to this uncertainty.

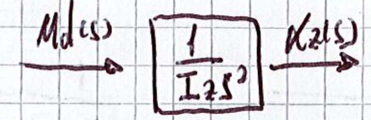
$$I_z\ddot{\alpha}_z = (3n^2(I_y - I_x))\alpha_z \xrightarrow{\mathcal{L}} I_z s^2 \alpha_z(s) = 3n^2(I_y - I_x)\alpha_z(s)$$

$$I_z s^2\alpha_z(s) + 3n^2(I_x - I_y)\alpha_z(s) = 0$$

↑ System has 2 complex-conjugated poles stable if and only if $I_x > I_y$

↳ Neglecting at this stage the gravity-gradient. (still considering it a generical disturb)

$$G(s) = \frac{\alpha_z(s)}{M_d(s)} = \frac{1}{I_z s^2}$$



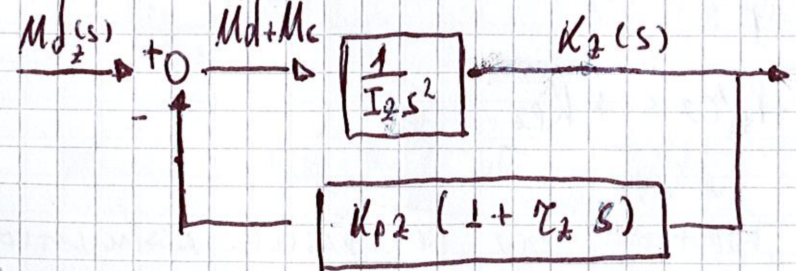
↳ Choosing of implement a PD controller

$$M_c = -\dot{h}_z = K_{pz}\alpha_z + K_{dz}\dot{\alpha}_z$$

$$\to \dot{h}_z = -K_{pz}\alpha_z - K_{dz}\dot{\alpha}_z.$$

By exploiting derivative time: $\to \dot{h}_z = -K_{pz}(\alpha_z + \tau_z\dot{\alpha}_z)$, $\tau_z \triangleq \frac{K_{dz}}{K_{pz}}$.

↳ feed-back (CL system) is then obtained.



↑ feedback transfer function would be:

$$h_z = -K_{pz}(\alpha_z + \tau_z\dot{\alpha}_z)$$

$$\downarrow \mathcal{L}$$

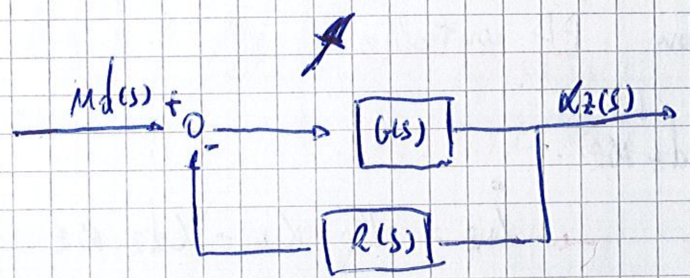
$$h_z(s) = K_{pz}(1 + \tau_z s)\,\alpha_z(s)$$

obs: SINCE the SET-POINT condition are $\begin{cases} \alpha_x = 0 \\ \alpha_y = 0 \\ \alpha_z = 0 \end{cases}$,

THEN implementing the controller for α_z instead than for the error, the effect is the same.

closed loop function will then be:

$$\frac{\alpha_z(s)}{M_d(s)} = \frac{1/(I_z s^2)}{1 + \left(\frac{1}{I_z s^2}\right)\left[K_{pz}(1 + \tau_z s)\right]}$$



$$\alpha_z(s) = G(s) \cdot \left(M_d(s) - R(s) \cdot \alpha_z(s) \right)$$

$$\alpha_z \left(1 + R(s) \cdot G(s) \right) = G(s)\, M_d(s) \Rightarrow \frac{\alpha_z(s)}{M_d(s)} = \frac{G(s)}{1 + R(s)\, G(s)}$$

$$\frac{\alpha_z(s)}{M_d(s)} = \frac{1}{I_z s^2 + \underbrace{K_{pz}\tau_z}_{2\xi w_n} s + K_{pz}} \quad , \; w_n = \sqrt{\frac{K_{pz}}{I_z}}$$

[THIS TRANSFER FUNCTION CAN BE EASILY ASSIMILATED TO A SPRING-MASS-DAMPER SYSTEM:

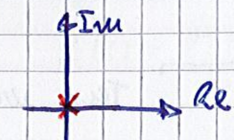
$$I_z s^2 + \tau_z s + K_{pz} \longleftrightarrow m\ddot{x} + c\dot{x} + Kx = f(t)$$

$$\Big\Downarrow \mathcal{L}$$

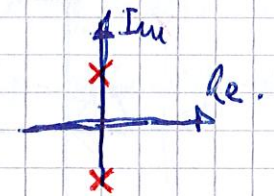
$$\left(\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + K} \right) \quad]$$

[THE POLES OF THIS CLOSED LOOP SYSTEM ARE ASINTOTICALLY STABLE]

× OL: $\frac{\alpha_z(s)}{M_d(s)} = \frac{1}{I_z s^2} \rightarrow p_{1,2} = 0$



× OL: (WITH M_{gg}) $\frac{\alpha_z(s)}{M_d(s)} = \frac{1}{I_z s^2 + 3n^2 (I_x - I_y)} \rightarrow p_{1,2} = \pm j \sqrt{\frac{3n^2 (I_x - I_y)}{I_z}}$



!! gravity gradient torque add some stiffness to the system. !!

($\Leftrightarrow$ $I_x > I_y$ // $I_{yaw} > I_{roll}$)

$$I_z s^2 + 3n^2 (I_x - I_y) = 0 \rightarrow s^2 = -\frac{3n^2 (I_x - I_y)}{I_z} \rightarrow s_{1,2} = \pm j \sqrt{\frac{3n^2 (I_x - I_y)}{I_z}}$$

× CL: $\frac{\alpha_z(s)}{M_d(s)} = \frac{1}{I_z s^2 + K_{pz}\tau_z s + K_{pz}} = \frac{1/I_z}{s^2 + \frac{K_{pz}\tau_z}{I_z} s + \frac{K_{pz}}{I_z}}$

[SUCH TRANSFER FUNCTION IS COMPLETELY EQUIVALENT TO A SPRING-MASS-DAMPER'S TRANSFER FUNCTION:

$$s^2 + \frac{K_{pz} \cdot \tau_z}{I_z} s + \frac{K_{pz}}{I_z} \longleftrightarrow s^2 + 2\xi w_n + w_n^2$$

where: $w_n = \sqrt{\frac{K_{pz}}{I_z}}$; $\xi = \frac{1}{2} \tau_z \sqrt{\frac{K_{pz}}{I_z}}$

$$p_{1,2} = -\xi w_n \pm j \sqrt{1 - \xi^2}\, w_n \qquad \left(\xi = \frac{1}{2} \frac{K_{dz}}{\sqrt{K_{pz} I_z}} \right) \quad]$$

in fact:

$$2\xi w_n = \frac{K_{pz}\tau_z}{I_z} \rightarrow \sqrt{\frac{K_{pz}}{I_z}} \cdot \sqrt{\frac{K_{pz}}{I_z}} \cdot \tau_z = w_n \tau_z \cdot \sqrt{\frac{K_{pz}}{I_z}} = 2\xi w_n$$

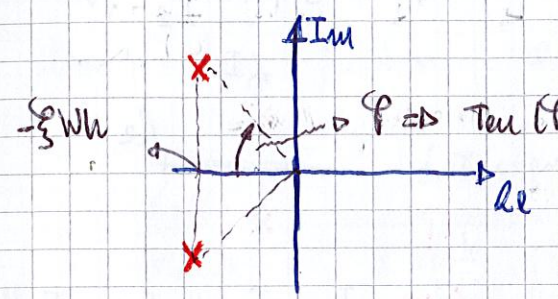
$$\Rightarrow \begin{cases} 2\xi = \tau_z \sqrt{\frac{K_{pz}}{I_z}} \\ \tau_z = \frac{K_{dz}}{K_{pz}} \end{cases} \rightarrow \begin{cases} \xi = \frac{1}{2} \tau_z \sqrt{\frac{K_{pz}}{I_z}} \\ \xi = \frac{1}{2} \frac{K_{dz}}{\sqrt{K_{pz} I_z}} \end{cases}$$

Poles in single condition will be:

$$s^2 + 2\xi\omega_n s + \omega_n^2 \to -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

($\xi < 1 \Rightarrow$ the $\sqrt{\cdots}$ term will be imaginary)

$$s^2 + 2\xi\omega_n s + \omega_n^2 \to p_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$



$\varphi \Rightarrow \tan(\varphi) = \dfrac{\omega_n\sqrt{1-\xi^2}}{\omega_n} = \sqrt{1-\xi^2}$

obs1: SINCE in the situation of INERTIAL POINTING the linearized equations are the following:

$$\begin{cases} I_x \ddot{\alpha}_x = Md_x + M_{cx} \\ I_y \ddot{\alpha}_y = Md_y + M_{cy} \\ I_z \ddot{\alpha}_z = Md_z + M_{cz} \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} I_x s^2 \alpha_x(s) = Md(s) \\ I_y s^2 \alpha_y(s) = Md(s) \\ I_z s^2 \alpha_z(s) = Md(s) \end{cases}$$

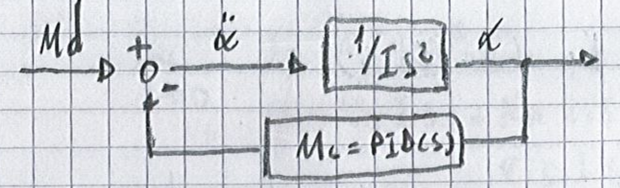
THEN (Only evaluating linearized conditions) the 3 equations are fully decoupled.

$\Rightarrow$ are coincident with the one saw just now for the control on the pitch axis: A DOUBLE INTEGRATOR TO BE CONTROLLED WITH A PID CONTROL LAW:

$i = x, y, z$

$$M_{ci} = K_{pi} \overset{A_e(ij)}{e(t)} + K_{di} \overset{(\omega_i - 0)}{\dot{e}(t)} + K_{Ii} \int_0^t e(\tau)\,d\tau$$

obs2: Only about how to design the loop (closed loop)



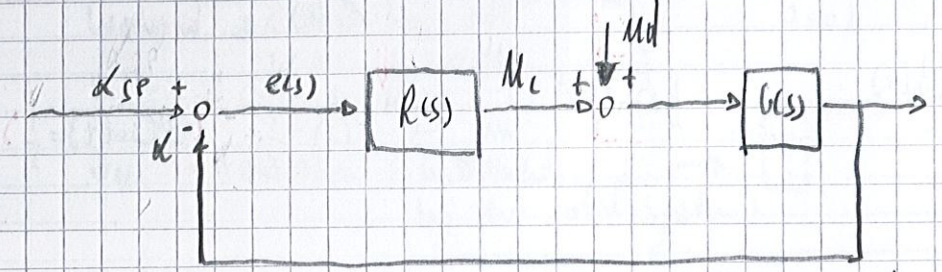
$G(s) = 1/Is^2$

$R(s) = PID(s)$

$$[Md - R(s)\cdot\alpha(s)]\cdot G(s) = \alpha(s)$$

$$G(s)\,Md = [1 + R(s)\cdot G(s)]\,\alpha(s) \Rightarrow \frac{\alpha(s)}{Md(s)} = \frac{G(s)}{[1 + R(s)G(s)]}$$

Result is the same even considering the following representation capable also of show deviation from a SP



$$(M_c + Md)\,G(s) = \alpha(s) \to [R(s)(\alpha_{sp} - \alpha) + Md]\cdot G(s) = \alpha(s)$$

$(\alpha_{sp}(s) = 0)$

$$\Rightarrow Md\,G(s) = -R(s)(\alpha_{sp} - \alpha) + \alpha \to \frac{\alpha(s)}{Md} = \frac{G(s)}{1 + R(s)G(s)}$$

The transfer function $Md \to \alpha$ // $Md \to \alpha - \alpha_{sp}$ is the same for both the configurations

!! THIS NEW DRAW IS ALSO CAPABLE OF SHOWING THE TRANSFER FUNCTION: $\alpha_{sp} \to \alpha$ (VARIATION IN THE REFERENCE) !!

$$(\alpha_{sp} - \alpha)\,R(s)\cdot G(s) = \alpha(s) \to R(s)\cdot G(s)\cdot\alpha_{sp} = \alpha(s)(1 + R(s)G(s))$$

$$\Rightarrow \frac{\alpha(s)}{\alpha_{sp}} = \frac{R(s)G(s)}{1 + R(s)G(s)}$$

!! DISTURBANCE SENSITIVITY FUNCTION $\to \dfrac{\alpha(s)}{Md(s)} = \dfrac{G(s)}{1 + R(s)G(s)}$

TRACKING SENSITIVITY FUNCTION $\to \dfrac{\alpha(s)}{\alpha_{sp}(s)} = \dfrac{R(s)G(s)}{1 + R(s)G(s)}$

Poles ...

$$\frac{R(s)}{e(s)} = K_p + K_d\, s = K_p\,(1 + \tau_d\, s)$$

PID: $$\frac{R(s)}{e(s)} = K_p + K_d\, s + \frac{K_I}{s} \Rightarrow \frac{R(s)}{e(s)} = K_p\left(1 + \tau_d s + \frac{1}{\tau_i s}\right)$$

$$\tau_d \triangleq \frac{K_d}{K_p} \quad \& \quad \tau_i \triangleq \frac{K_p}{K_i} \Rightarrow R(s) = K_p\left(\frac{\tau_i s + \tau_i \tau_d s^2 + 1}{\tau_i s}\right)$$

* final value theorem

$$\begin{cases} \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\cdot F(s) \\ F(s) = \mathcal{L}(f(t)) \end{cases}$$

to be used with the $sca(t)$ as input

$$\lim_{s\to 0} s\cdot F(s)\cdot\frac{1}{s} \qquad \left(\mathcal{L}(scalt) = \frac{1}{s}\right)$$

* REFERENCE TRACKING

PD: $$\lim_{s\to 0} s\,\frac{K_p(1+\tau_d s)\cdot\frac{1}{Is^2}}{1+\frac{K_p}{I}\frac{(1+\tau_d s)}{s^2}} = \lim_{s\to 0} \cancel{s}\cdot\frac{K_p + K_d s}{K_p + K_d s + I s^2}\cdot\frac{1}{\cancel{s}} = 1 \quad \text{(OK)}$$

PID: $$\lim_{s\to 0} s\cdot\frac{K_p\left(\frac{\tau_i s + \tau_i\tau_d s^2 + 1}{\tau_i s}\right)\cdot\frac{1}{Is^2}}{1+\frac{K_p}{I}\frac{1}{s^2}\left(\frac{\tau_i s + \tau_i\tau_d s^2 + 1}{\tau_i s}\right)}\cdot\frac{1}{s}$$

$$= \lim_{s\to 0} \cancel{s}\cdot\frac{K_p\tau_i s + K_p\tau_i\tau_d s^2 + K_p}{\tau_i I s^3 + K_p\tau_i s + K_p\tau_i\tau_d s^3 + K_p}\cdot\frac{1}{\cancel{s}} = 1 \quad \text{(OK)}$$

* DISTURBANCE REJECTION

PD: $$\lim_{s\to 0} s\,\frac{1/Is^2}{1+K_p(1+\tau_d s)\frac{1}{Is^2}} = \cancel{s}\cdot\frac{\frac{1}{Is^2}}{\frac{Is^2 + K_p + K_p\tau_d s}{Is^2}}\cdot\frac{1}{\cancel{s}}$$

$$= \lim_{s\to 0}\frac{1}{Is^2 + K_p + K_p\tau_d s} = \frac{1}{K_p} \quad \text{(NOT OK)}$$

[At steady state condition the PD CONTROLLER HAVE AN ERROR (FOR A STEP INPUT) EQUAL TO: $1/K_p$]

PID: $$\lim_{s\to 0} \cancel{s}\cdot\frac{1/Is^2}{1+\frac{K_p}{I}\frac{1}{s^2}\left(\frac{\tau_i s + \tau_i\tau_d s^2 + 1}{\tau_i s}\right)}\cdot\frac{1}{\cancel{s}}$$

$$\lim_{s\to 0}\frac{1/\cancel{I}s^2}{\frac{\tau_i I s^3 + K_p\tau_i s + \tau_i\tau_d s^2 + 1}{\tau_i s\,\cancel{I}s^2}} = \frac{\tau_i s}{\tau_i I s^3 + K_p\tau_i s + \tau_i\tau_d s^2 + 1} = 0 \quad \text{(OK)}$$

[THEREFORE IN ORDER TO HAVE $\emptyset$ ERROR FOR A STEP DISTURBANCE A PID CONTROLLER MUST BE IMPROVED]

(To tune a PID requirements of the control system must be RECASTED* into $\begin{Bmatrix}\phi_m\\ W_c\end{Bmatrix}$ specifications for the controller.)

*

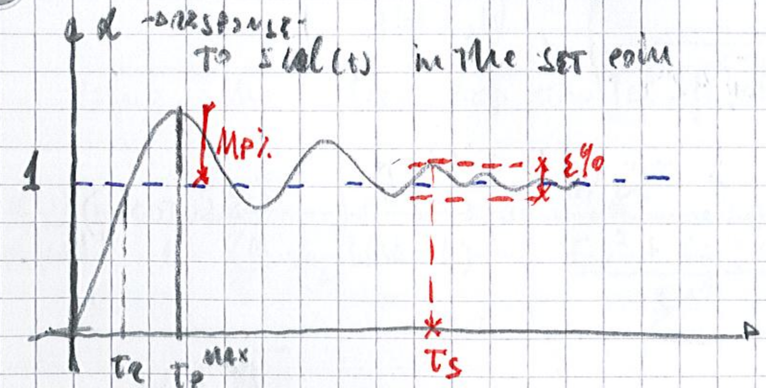
$W_c \leftrightarrow$ RISING TIME $\to$ $\boxed{T_R}$
SETTLING TIME $\to$ $\boxed{T_P}$

$\phi_m \leftrightarrow$ DAMPING (intended by the control system) $\to$ $\boxed{\xi}$ $\to$ LINKED ALSO TO THE MAXIMUM % OVERSHOOT ADMISSIBLE. *

Always looking at the response of the system for a step disturbance.

$$\phi_m \triangleq 180^0 + \measuredangle L(jW_c) = \tan^{-1}\left(\frac{2\xi}{\sqrt{\sqrt{4\xi^4+1} - 2\xi^2}}\right)$$

$$W_n \geq \frac{\pi}{T_p^{MAX}\sqrt{1-\xi^2}}, \qquad W_n \geq \frac{-\ln\left(\varepsilon\%\sqrt{1-\xi^2}\right)}{t_s^{MAX}\cdot\xi}$$



$$M_p = \theta^{MAX} \Rightarrow$$

admissible deviation from the wind condition

* $$M_p\% = e^{-\frac{\pi\xi}{\sqrt{1-\xi}}} \times 100\%$$

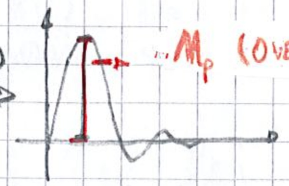
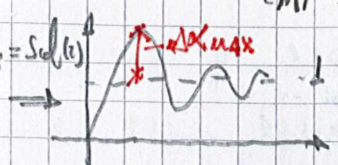
THIS CONDITION DETERMINE THE DAMPING OF THE SYSTEM.

$$\frac{\Delta\theta \text{ rad}}{1 \text{ rad}} \Rightarrow \xi \simeq 0.9\ldots$$

- procedure - MY.

(1) determine Md_{max} (perturbation analysis along each axis)

(2) Choose the maximum deviation from the $\begin{cases} \alpha\text{-st} \Rightarrow \Delta\alpha^{MAX} \\ M_P \Rightarrow \text{MAX OVERSHOOT} \end{cases}$

$Md = \text{sud}(t) \Rightarrow$  M_p (OVERSHOOT) - $\alpha_{st} = \text{sud}(t) \Rightarrow$ 

$\Rightarrow \dfrac{M_P}{M_{dmax}}$ // $\dfrac{\Delta\alpha_{max}\,\text{rad}}{1\ \text{rad}}$ → choose the most strict assignment.

(3) Determine the damping

$$\frac{M_P}{M_{dmax}} \;//\; \frac{\Delta\alpha_{max}\,\text{rad}}{1\ \text{rad}} = e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \longrightarrow \boxed{\xi}$$

(4) Choose the values for $\begin{cases} T_e^{MAX}(t_P) \\ T_S^{MAX} \end{cases}$ and value for ω_n

$$\omega_n = \frac{\pi}{t_p^{MAX}\sqrt{1-\xi^2}} \;//\; \omega_n = \frac{-\ln\left(\varepsilon\%\sqrt{1-\xi^2}\right)}{t_S^{MAX}\,\xi} \rightarrow \boxed{\omega_n}$$

(5) Recasting of the previous solutions:

$$\phi_m = \tan^{-1}\left(\frac{2\xi}{\sqrt{\sqrt{4\xi^4+1}-2\xi^2}}\right), \quad \omega_c = \omega_n$$

(6) Design the PID controller to respect $\begin{cases} \phi_m \\ \omega_c \end{cases}$ (SISOTOOL)

MENGALI: Reaction wheels sizing (and selection between manufactured model)

Control relation between the actuators and the nominal torque is the following: $\begin{cases} M_c = -\dot{h}_R \\ h = I_R\,\dot{\Omega} \end{cases}$

eg. (PD control-law)

$$\Rightarrow I_R\,\dot{\Omega} = -K_p\,(\alpha + \tau\dot{\alpha})$$

↳ Going into Laplace domain is possible to obtain a relation linking $\Omega(s)$ to the disturbance torque.

$$\mathcal{L}\left[\dot{f}(t)\right] = s\cdot F(s) - f(0)$$

$$\Rightarrow I_R\,[s\,\Omega(s) - \Omega(0)] = -K_p\,(\tau s + 1)\,\alpha(s)$$

$$I_R\,\dot{\Omega} = -K_P\,(\alpha + \tau\dot{\alpha}) \xrightarrow{\mathcal{L}} I_R\,(\Omega(s) - \Omega(0)) = -K_p\,(\alpha(s) + \tau s\,\alpha(s))$$

therefore:

$$s\,\Omega(s) = \Omega(0) - \frac{K_P(\tau s + 1)}{I_R}\,\alpha(s)$$

$$\Omega(s) = \frac{\Omega(0)}{s} - \frac{K_P(\tau s + 1)}{I_R}\cdot\frac{\alpha(s)}{s}$$

↳ Using the closed loop transfer-function: $\dfrac{\alpha(s)}{M_d(s)} = \dfrac{G(s)}{1 + R(s)\,G(s)}$

$$\frac{\alpha(s)}{M_d(s)} = \frac{1/Is^2}{1 + \frac{K_P}{I}\frac{1}{s^2}(\tau s + 1)} = \frac{1/\cancel{Is^2}}{\frac{Is^2 + K_P\tau s + K_P}{\cancel{Is^2}}} = \frac{1/I}{s^2 + K_P\frac{\tau}{I}s + \frac{K_P}{I}}$$

$$\omega \triangleq \sqrt{\frac{K_P}{I}}\,,$$

$$\Rightarrow \frac{\alpha(s)}{M_d(s)} = \frac{1/I}{s^2 + \omega^2\tau s + \omega^2}$$

Assumption (DESIGN PD): In order to have UNITARY DAMPING it's necessary to select $\tau \triangleq \dfrac{K_D}{K_P} = 2\sqrt{\dfrac{I}{K_P}}$

in fact :

$$\frac{\alpha(s)}{Md(s)} = \frac{1/I}{s^2 + w^2 \tau s + w^2} \longleftrightarrow \frac{1/I}{s^2 + 2\xi w s + w^2}$$

$$\Rightarrow \cancel{\xi = 1 = \frac{\tau}{w}} \quad \cancel{\Rightarrow \tau = w}$$

$$\Rightarrow \quad 2\xi w = w^2 \tau \xrightarrow{\xi = 1} \tau w^{\cancel{2}} = 2\cancel{w} \rightarrow \tau = 2/w = 2\sqrt{\frac{I}{Kp}}$$

therefore:

$$\frac{\alpha(s)}{Md(s)} = \frac{1/I}{s^2 + Kp\frac{\tau}{I} + \frac{Kp}{I}} = \frac{1/I}{s^2 + \tau w^2 s + w^2} = \frac{1/I}{s^2 + \frac{2}{w} w^2 s + w}$$

$$\Rightarrow \frac{\alpha(s)}{Md(s)} = \frac{1/I}{s^2 + 2ws + w} = \frac{1/I}{(s+w)^2}$$

obs2 - COMPLEMENT

By selecting in this way the PD controller (SUCH THAT $\xi = 1 \rightarrow \frac{Kd}{Kp} = 2\sqrt{\frac{I}{Kp}}$)

Using the final value theorem for watch in this condition the answer to a scal(t) disturbance

$$\lim_{s\to 0} \cancel{s} \cdot \frac{1/I}{(s+w)^2} \cdot \frac{1}{\cancel{s}} = \frac{1/I}{w^2}$$

but if w is high enough is possible to reduce the error approximately to zero.

↳ Substituting this expression into the $\Omega(s)$ transfer function,

$$\begin{cases} \Omega(s) = \frac{\Omega(0)}{s} - \frac{Kp(\tau s+1)}{IR} \cdot \frac{\alpha(s)}{s} \\ \frac{\alpha(s)}{Md(s)} = \frac{1/I}{(s+w)^2} \quad \left(w = \sqrt{\frac{Kp}{I}}\right) \end{cases}$$

$$\Rightarrow \quad \Omega(s) = \frac{\Omega(0)}{s} - \frac{Kp(\tau s+1)}{IR} \cdot \frac{1}{s} \cdot \frac{1/I}{(s+w)^2}$$

$$\boxed{\Omega(s) = \frac{\Omega(0)}{s} - \frac{2\tau s + 1}{IR\, s(\tau s+1)^2} Md(s)}$$

$$\Omega(s) = \frac{\Omega(0)}{s} - \underbrace{\frac{Kp}{I}}_{w^2} \frac{(\tau s+1)}{IR} \cdot \frac{1}{s} \cdot \frac{1}{w^2\left(\frac{1}{w}s+1\right)^2}$$

$$= \frac{\Omega(0)}{s} - \cancel{w^2} \frac{(\tau s+1)}{IR} \frac{1}{s} \frac{1}{\cancel{w^2}\left(\frac{1}{w}s+1\right)^2} \quad \Big|\Big| \quad \tau = \frac{2}{w}$$

* Response to imp(t) ($\mathcal{L}[\text{imp}(t)] = 1$)

$$\lim_{s\to 0} s \cdot \Omega(s) = \left(\frac{\Omega(0)}{\cancel{s}} - \frac{(2\tau s+1) Md}{IR\,\cancel{s}(\tau s+1)}\right)\cancel{s} = \Omega(t\to\infty)$$

$$\Omega(t\to\infty) = \Omega(0) - \frac{Md}{IR}$$

$$\boxed{\underbrace{\Omega(t\to\infty) - \Omega(0)}_{\Delta\Omega max} = \frac{Md^{max}}{IR}}$$

* Response to scal(t)

$$\lim_{s\to 0} \cancel{s} \cdot \Omega(s) \cdot \frac{1}{\cancel{s}} = \frac{\Omega(0)}{s} - \frac{2\tau s + 1 \cdot Md}{IR(\tau s+1)}$$

theoretically ∞ but practically the result is the same obtained for the impulse response.

* Periodical disturb ($\mathcal{L}[\sin(w^* t)] = \frac{w^*}{s^2 + w^{*2}}$)

$$\lim_{s\to 0} \frac{w}{s^2+w^2} \cdot \Omega(s) \cdot s = \Omega_0 \frac{w^*}{s^2 + w^{*2}} - \frac{2\tau s + 1}{IR(\tau s+1)} Md \cdot \frac{w^*}{s^2 + w^{*2}}$$

$$\boxed{\Omega(t\to\infty) - \Omega(0) = \frac{Md}{w^* IR_2}}$$

if $w^* < 1$ this disturbance can beat the impulse one.

→ Reaction wheels disposition:

ONCE $\underline{M}_c$ has been computed (thanks to → LQR, → PID, → ETC..)

THEN $\underline{M}_r$ can be computed

Euler with control: → $\dot{\underline{H}} + \underline{W} \times \underline{H} = \overbrace{-\underline{\underline{A}}\,\dot{\underline{h}}_r - \underline{W} \times (\underline{\underline{A}}\,\underline{h}_r)}^{\underline{M}_c} + \underline{M}_d$

capital $\underline{H}$ to remember that it's also including rotor's inertia.

$$\Rightarrow \underline{M}_c = -\underline{\underline{A}}\,\dot{\underline{h}}_r - \underline{W} \times (\underline{\underline{A}}\,\underline{h}_r)$$

$$\underline{\underline{A}}\,\dot{\underline{h}}_r = -\underline{M}_c - \underline{W} \times (\underline{\underline{A}}\,\underline{h}_r)$$

In order to compute the term $\dot{\underline{h}}_r$ needed to compute the torque that the electric engine have to provide:

$\boxed{\text{Sensors + error determination}} \xrightarrow[\text{LQR...}]{\text{PID...}} \boxed{\text{select the } h_r \text{ for each reaction wheel}} \longrightarrow \begin{cases} M_{r_i} = \dot{h}_{r_i} \\ M_{r_i} = K_t \cdot I_i \end{cases} \longrightarrow \boxed{I_i}$

(controlled in current electric motors)

⇒ to compute the $\dot{h}_r$ term must solve the equation: $\underline{\underline{A}}\,\dot{\underline{h}}_r = -\underline{M}_c - \underline{W} \times (\underline{\underline{A}}\,\underline{h}_r)$

BUT IN GENERAL → there are more than 3 actuators
→ actuators are not along the 3 PI axis.

THEN → $\underline{\underline{A}}$ is not a 3×3 matrix
→ $\underline{\underline{A}}$ is not a square matrix $\Rightarrow \nexists\, \underline{\underline{A}}^{-1}$

IN FACT 3 rotors are enough to solve exactly the initial problem, but in practice a 4th is added for redundancy

---

→ Pseudo-inverse matrix.

size $(\underline{\underline{A}}) = 3 \times n$ → n° of actuators.

$$\underline{\underline{A}}_{3,n} \cdot \underline{x}_{n,1} = \underline{y}_{3,1} \Rightarrow \underline{\underline{A}}^T_{3,n}\,\underline{\underline{A}}_{3,n}\,\underline{x}_{n,1} = \underline{\underline{A}}^T_{3,n} \cdot \underline{y}_{3,1}$$

$(\underline{\underline{A}}_{3\times n} \cdot \dot{\underline{h}}_r = -\underline{M}_c)$

if $\exists!$ the inverse of $(\underline{\underline{A}}^T_{3,n}\,\underline{\underline{A}}_{3,n})$

then $(\underline{\underline{A}}^T_{n,3} \cdot \underline{\underline{A}}_{3,n})^{-1} (\underline{\underline{A}}^T_{3,n} \cdot \underline{\underline{A}}_{3,n}) \cdot \underline{x}_{n,1} = (\underline{\underline{A}}^T_{n,3}\underline{\underline{A}}_{3,n})^{-1}\underline{\underline{A}}^T_{3,n}\,\underline{y}_{31}$

$$\underline{\underline{A}} \cdot \underline{\underline{A}}^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} \end{bmatrix} = \begin{bmatrix} 3\times 3 \end{bmatrix}$$

$$\underline{\underline{A}} \cdot \underline{h}_r = \begin{bmatrix} a_{11} & \cdots & a_{1,n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \end{bmatrix} \begin{bmatrix} I_{R_1} W_{R_1} \\ \vdots \\ I_{Rn} W_{R,n} \end{bmatrix} = \begin{bmatrix} 3\times 1 \end{bmatrix}$$

$$\underline{\underline{A}}^T \cdot \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n_3} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1,n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \end{bmatrix} = \begin{bmatrix} n\times n \end{bmatrix}$$

⇒ same size for $(\underline{\underline{A}}^T\underline{\underline{A}})^{-1}$

therefore:

$$\underbrace{(\underline{\underline{A}}^T_{n,3}\,\underline{\underline{A}}_{3,n})^{-1} (\underline{\underline{A}}^T_{3,n}\,\underline{\underline{A}}_{3,n})}_{\underline{\underline{I}}_{n,n}} \cdot \underline{x}_{n,1} = \underbrace{(\underline{\underline{A}}^T_{n,3}\,\underline{\underline{A}}_{3,n})^{-1}\,\underline{\underline{A}}^T_{3,n}}\,\underline{y}_{31}$$

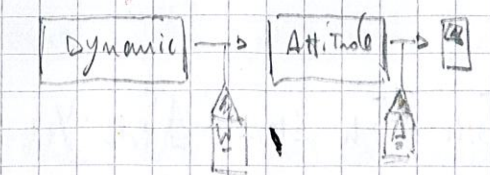
$$\begin{bmatrix} a_{11} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n_3} \end{bmatrix} = \begin{bmatrix} n\times 3 \end{bmatrix} \Rightarrow \begin{bmatrix} n\times 3 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \vdots \end{bmatrix}_{n\times 1}$$

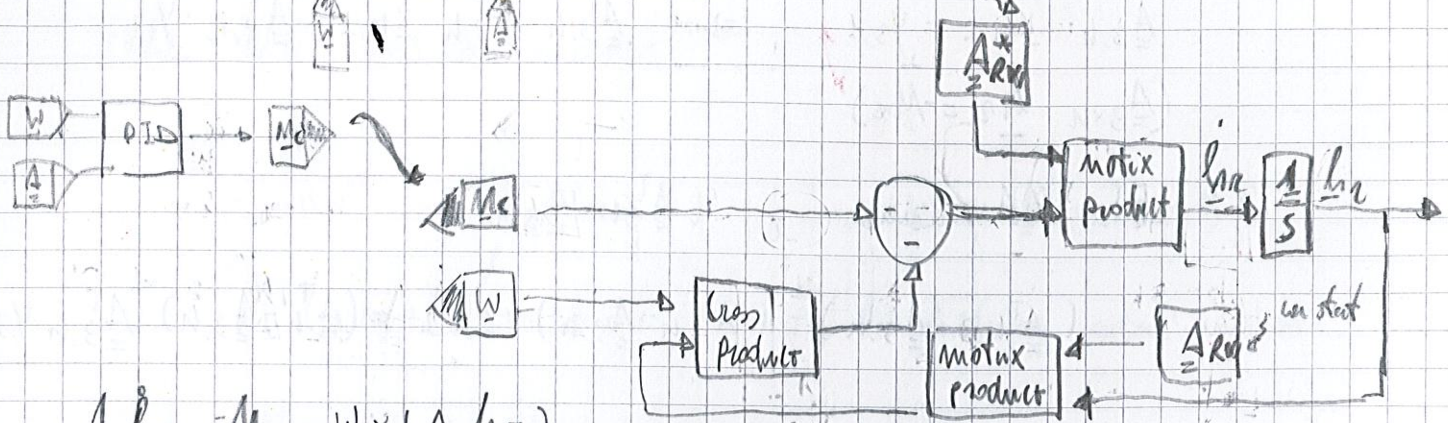
$$\Rightarrow \begin{cases} \underline{x}_{n,1} = \underline{\underline{A}}^*_{n,3}\,\underline{y}_{31} \\ \underline{\underline{A}}^* = (\underline{\underline{A}}^T_{n,3}\,\underline{\underline{A}}_{3,n})^{-1}\,\underline{\underline{A}}^T_{3,n} \end{cases}$$

(size $(\underline{\underline{A}}^*) = n\times 3$)

($\underline{x}_{n,1} \rightleftarrows \dot{\underline{h}}_r$
$\underline{y}_{31} \rightleftarrows \underline{M}_c = \underline{M}_c(\text{PID})$)

a simulink configuration for this might be the following:



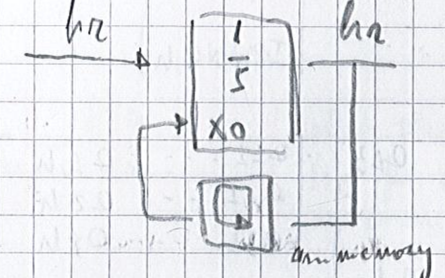


$$\underline{\underline{A}}_{RW}\dot{\underline{h}}_{R} = \underline{M}_c - \underline{W}\times(\underline{\underline{A}}_{RW}\underline{h}_R)$$

$$\rightarrow \quad \dot{\underline{h}}_{R} = -\underline{\underline{A}}^*_{RW}\underline{M}_c - \underline{\underline{A}}^*_{RW}\underline{W}\times(\underline{\underline{A}}\,\underline{h}_R)$$

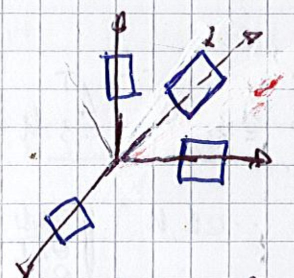
$$\underline{\underline{A}}^*_{RW} = (\underline{\underline{A}}^T_{RW}\,\underline{\underline{A}}_{RW})^{-1}\cdot\underline{\underline{A}}_{RW}$$

to avoid heavy computation => select instead



=> 4 rotors common configurations are:

a)



$$\Rightarrow \underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 & 1/\sqrt{3} \\ 0 & 1 & 0 & 1/\sqrt{3} \\ 0 & 0 & 1 & 1/\sqrt{3} \end{bmatrix}$$

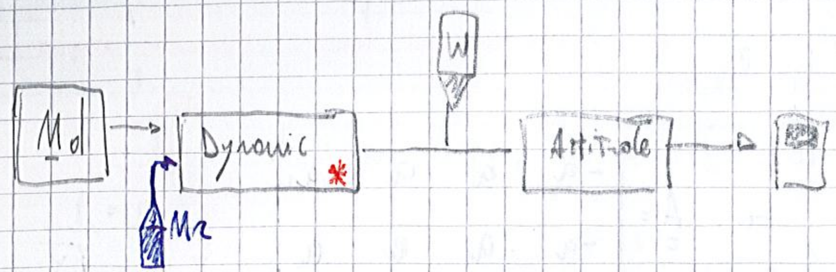
3 RW. aligned with the PI inertia axis

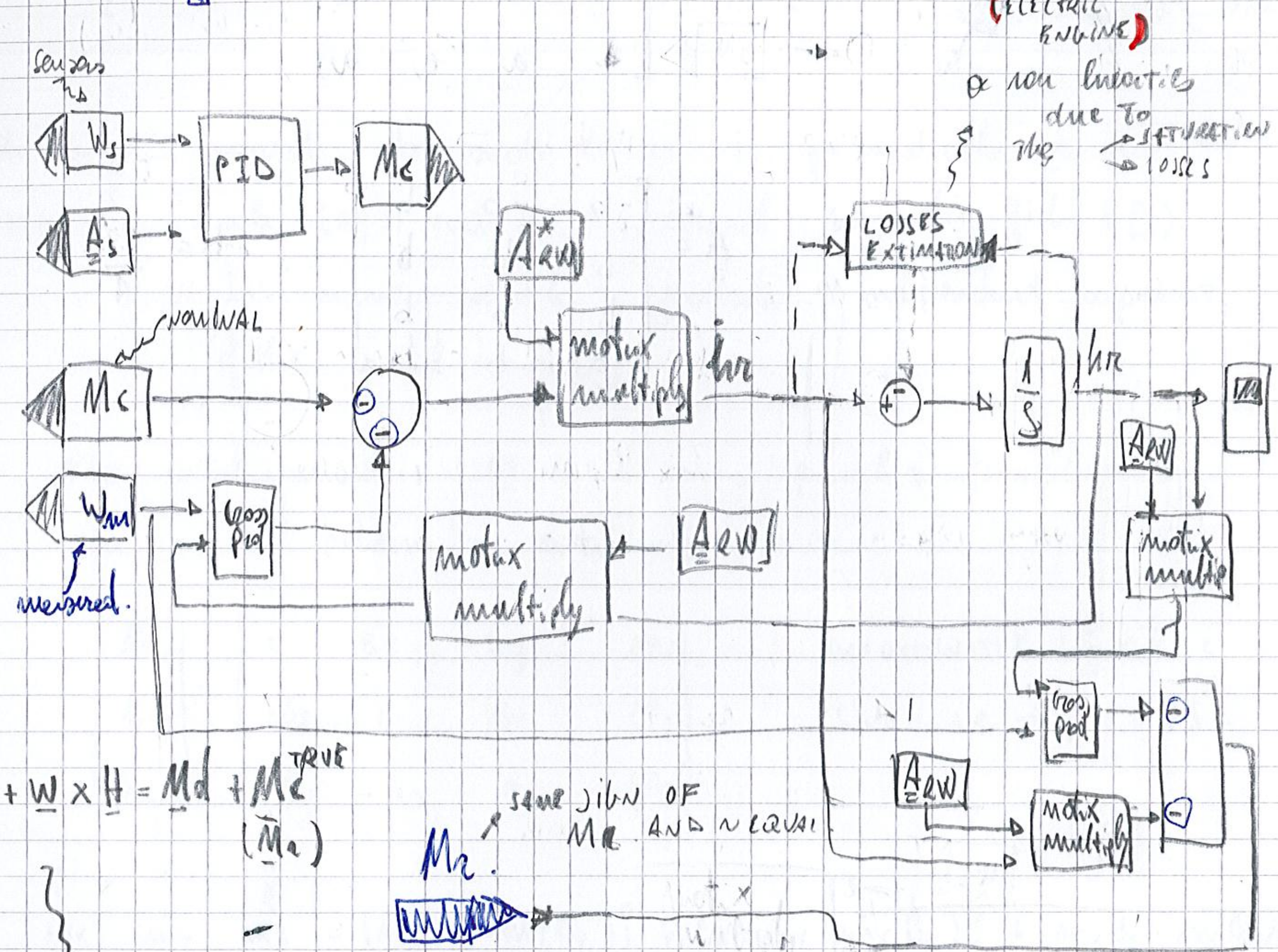
4th RW. equal component along the 3 axis

$$\sqrt{\left(\frac{1}{\sqrt3}\right)^2+\left(\frac{1}{\sqrt3}\right)^2+\left(\frac{1}{\sqrt3}\right)^2} = \left(\frac{3}{3}\right)^{1/2} = 1$$

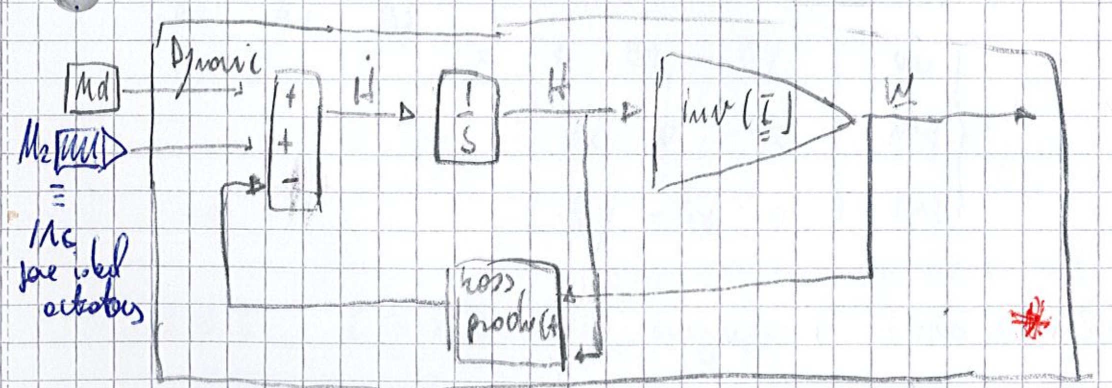
$$\Rightarrow \underline{\underline{A}}^* = \begin{bmatrix} 5/6 & -1/6 & -1/6 \\ -1/6 & 5/6 & -1/6 \\ -1/6 & -1/6 & 5/6 \\ 1/2\sqrt3 & 1/2\sqrt3 & 1/2\sqrt3 \end{bmatrix}$$

For non ideal actuators { introduce energy losses / problems of various form





$$\dot{\underline{H}} + \underline{W}\times\underline{H} = \underline{M}_d + \underline{M}_c^{TRUE} \quad (\bar{M}_c)$$

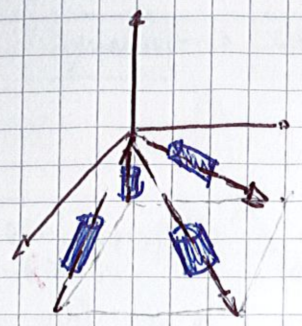


Obs: With this 4 rotors configuration a control to be applied along a single axis will generate a control action along each rotor

$$\underline{M}_c = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \underline{M}_r = \underline{\underline{A}}^*(-\underline{M}_c - \underbrace{\underline{W}\times\underline{h}_r}_{\text{supposed to be zero}}) = \begin{Bmatrix} 5/6 \\ -1/6 \\ -1/6 \\ 1/2\sqrt3 \end{Bmatrix} \Rightarrow \|\underline{M}_c\| = \sqrt{\frac{30}{36}}$$

(less than 1)

b) The other possible configuration is the quadratic-based pyramid



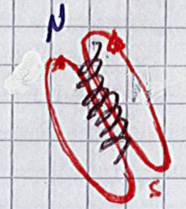
$$\Rightarrow \underline{\underline{A}} = \begin{bmatrix} -a & a & a & -a \\ -a & -a & a & a \\ a & a & a & a \end{bmatrix} \quad ; \quad a = \frac{1}{\sqrt{3}}$$

$\Downarrow$

$$\underline{\underline{A}}^* = \begin{bmatrix} -b & -b & b \\ b & -b & b \\ b & b & b \\ -b & b & b \end{bmatrix} \quad ; \quad b = \frac{\sqrt{3}}{4}$$

→ " Magnetic actuators "

→ Dipole moment induced in a coil.



$$\Rightarrow \quad \underline{m} = \mu \cdot N S \cdot \underline{i}$$

$$\Rightarrow \quad \underline{M} = \underline{m} \times \underline{B} \qquad odG: 10^{-3} \div 10^{-6} \; Nm$$

[The torque generated by such devices is perpendicular to the magnetic dipole ($\underline{m}$) and to the external magnetic field ($\underline{B}$)

⇒ It's never possible to generate 3 independent component of the control torque]

↳ The control command is $\underline{m}$ (i) ⇒ is possible to formulate the control problem in matrix form as:

$$\begin{Bmatrix} M_{cx} \\ M_{cy} \\ M_{cz} \end{Bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

!! UNFORTUNATELY $[\underline{\underline{B}}\times]$ IS SINGULAR $\Rightarrow \nexists \, [\underline{\underline{B}}\times]^{-1}$!!

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ m_x & m_y & m_z \\ B_x & B_y & B_z \end{vmatrix} = (m_y B_z - m_z B_y)\hat{i} + (m_z B_x - m_x B_z)\hat{j} + (m_x B_y - m_y B_x)\hat{k}$$

$$= \begin{bmatrix} 0 & B_z & -B_y \\ B_x & 0 & -B_z \\ B_y & -B_x & 0 \end{bmatrix} \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

SINCE $[\underline{\underline{B}} \times]$ is singular (THIS MEANS THAT IT'S IMPOSSIBLE TO GENERATE 3 INDEPENDENT MOMENT ALONG THE 3 P.I. AXIS) $\begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix} = \underline{\underline{B}}^{-1} \begin{Bmatrix} M_{cx} \\ M_{cy} \\ M_{cz} \end{Bmatrix}$

THEN A solution for coupled dynamic is required.

↳ CONSIDER A SOLUTION WHERE THE CONTROL ACTION IS REQUIRED ONLY ALONG 2 AXIS.

e.g. control action required only along (y, z) components

$$\begin{Bmatrix} 0 \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \cdot \begin{Bmatrix} m_x \\ m_y \\ m_z \end{Bmatrix}$$

↳ imposing $m_x = 0$
(∅ current circulating inside the "x" coil)

Then the problem is fully decoupled:

$$\begin{cases} M_y = B_x m_z \\ M_z = -B_x m_y \end{cases} \longrightarrow \begin{cases} m_z = \dfrac{M_y}{B_x} \\ m_y = -\dfrac{M_z}{B_x} \end{cases}$$

BUT
↳ A residual torque along "x" will be generated.

$$M_{xR} = +B_z m_y - B_y m_z \longrightarrow M_{xR} = -M_z \frac{B_z}{B_x} - M_y \frac{B_y}{B_x}$$

RESIDUAL → NOT REQUIRED BY CONTROL ACTION

THEN
↳ Another actuator is required to provide control along such axis
↳ a torque must counter-balance this residual moment.

NOW
↳ supposing that the control is required along (x, y) ⇒ the "parasite" torque will be generated along "z"

$$\begin{Bmatrix} M_x \\ M_y \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \cdot \begin{Bmatrix} m_x \\ m_y \\ 0 \end{Bmatrix} \longrightarrow \begin{cases} M_x = m_y B_z \rightarrow m_y = \dfrac{M_x}{B_z} \\ M_y = -B_z m_x \rightarrow m_x = -\dfrac{M_y}{B_z} \\ M_{z,R} = -\dfrac{B_y}{B_z} M_y - \dfrac{B_x}{B_z} M_x \end{cases}$$

THEREFORE
↳ Granting stability along z axis thanks to a RW the control problem must be re-formulated including the torque provided by the RW

⇒

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{Bmatrix} m_x \\ m_y \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -h_z \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & B_z & 0 \\ -B_z & 0 & 0 \\ B_y & -B_x & 1 \end{bmatrix} \begin{Bmatrix} m_x \\ m_y \\ h_z \end{Bmatrix}$$

↳ this time such matrix is non-singular.
and the control problem can be solved directly

$$\longrightarrow \begin{Bmatrix} m_x \\ m_y \\ h_z \end{Bmatrix} = \frac{1}{B_z} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ B_x & B_y & B_z \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix}$$

obs: SUCH CONTROL PROBLEM IS SOLVABLE IF AND ONLY IF THE "z" COMPONENT OF THE MAGNETIC FIELD IS DIFFERENT FROM ZERO (in P.I reference frame)

Alternative procedure. (≠ CONTROL STRATEGY)

$$\underline{M}_c = \underline{m} \times \underline{B} \longrightarrow \underline{B} \times \underline{M}_c = \underline{B} \times (\underline{m} \times \underline{B})$$

|| $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{c} \cdot \underline{a}) - \underline{c}(\underline{a} \cdot \underline{b})$

therefore: $\underline{B} \times \underline{M}_c = \underline{m} \cdot (\underline{B} \cdot \underline{B}) - \underline{B}(\underline{m} \cdot \underline{B})$

↳ GENERATING A DIPOLE MOMENT EXACTLY PERPENDICULAR TO B || $\underline{m} \perp \underline{B} \Downarrow \underline{m} \cdot \underline{B} = 0$

$$\underline{B} \times \underline{M}_c = \|\underline{B}\|^2 \underline{m} \longrightarrow \underline{m} = \frac{\underline{B} \times \underline{M}_c}{\|\underline{B}\|^2}$$

(100% orthogonal to $\underline{B}$)

↳ THEREFORE THE EFFECTIVE CONTROL TORQUE GENERATE WILL BE.

$$\underline{M}_{eff} = \underline{m} \times \underline{B} \longrightarrow \underline{M}_{eff} = \frac{1}{B^2} (\underline{B} \times \underline{M}_c) \times \underline{B}$$

!! $\underline{M}_{eff} \equiv \underline{M}_c \iff \underline{m} \perp \underline{B}$!!

that is the desired control torque.

in fact: $\begin{cases} \underline{M}_{eff} = \underline{m} \times \underline{B} \\ \underline{m} = \dfrac{\underline{M}_c \times \underline{B}}{B^2} \end{cases} \Rightarrow \underline{M}_{eff} = \dfrac{1}{B^2} \, (\underline{B} \times \underline{M}_c) \times \underline{B}$

$\Rightarrow \underline{M}_{eff} = \dfrac{1}{B^2} \left[\underline{M}_c (\underline{B} \cdot \underline{B}) - \underline{B} \cdot (\underline{M}_c \cdot \underline{B}) \right] \quad \Big\| \quad \underline{M}_c = \underline{m} \times \underline{B}$

$= \dfrac{1}{B^2} \left[\underline{M}_c (\underline{B} \cdot \underline{B}) - \underline{B} \cdot \big(\underbrace{(\underline{m} \times \underline{B}) \cdot \underline{B}}_{=0} \big) \right]$

$= \underline{M}_c$

$\Rightarrow$ therefore $\begin{cases} \underline{M}_c = \underline{m} \times \underline{B} \\ \underline{m} \perp \underline{B} \end{cases}$ can generate a $\underline{M}_c$ torque orthogonal to $\underline{B}$

!! this control method really works when the control torque required are (FOR THEIR NATURE) almost $\perp$ to $\underline{B}$.

→ "Minimum time maneuver"

$$\min J^* = \underbrace{\int_{T_0}^{t_f} dt}_{\text{COST FUNCTION } (J)} + \lambda \cdot \underbrace{\int_{T_0}^{t_f} (\dot{\underline{x}} + \underline{\underline{A}}\, \underline{x} - \underline{\underline{B}}\, \underline{u})}_{\text{COSTRAIN: Dynamic of the system. (expressed in the } I^{st} \text{ order)}}$$

↳ Rotation along a single axis ⇒ $M_i = I_i \ddot{\alpha}_i$ represent the linearized equation of dynamic for such system. !! ROTATIONS AROUND 1 AXIS IS ONE OF THE SMALL NUMBER OF CASES WITH ANALYTICAL SOLUTION!

$$u = \ddot{\alpha} = \frac{M}{I} \xrightarrow{\int dt} \begin{cases} \dot{\alpha} = \dot{\alpha}_0 + u t \\ u = \dfrac{M}{I} \end{cases} \xrightarrow{\int dt} \begin{cases} \alpha = \alpha_0 + \dot{\alpha}_0 t + \frac{1}{2} u t^2 \\ u = \dfrac{M}{I} \end{cases}$$

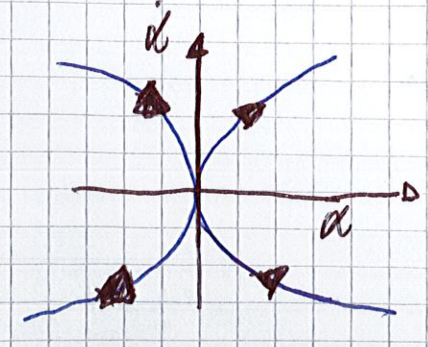
therefore 2 equations are available:

$\begin{cases} \dot{\alpha} = \dot{\alpha}_0 + u t \\ \alpha = \alpha_0 + \dot{\alpha}_0 t + \frac{1}{2} u t^2 \end{cases}$ → compute time from I^{st} equation ⇒ $t = \dfrac{\dot{\alpha} - \dot{\alpha}_0}{u}$ ← substitute

$\Rightarrow \begin{cases} \alpha = \alpha_0 + \dot{\alpha}_0 \dfrac{(\dot{\alpha} - \dot{\alpha}_0)}{u} + \dfrac{1}{2} \left(\dfrac{\dot{\alpha} - \dot{\alpha}_0}{u} \right)^2 \\ u = \dfrac{M}{I} \end{cases}$ (M: control torque along the rotation axis)

IF M = const. → $M = M^*$

THEN the PHASE-SPACE representation of such curve is a parabola whose concavity depends on the torque limit (M^*)



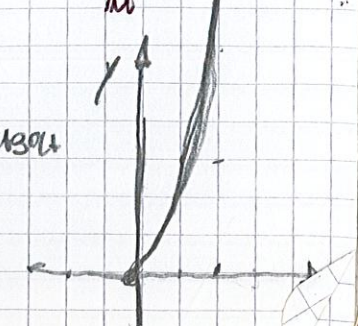
$$(\alpha - \alpha_0) = \frac{1}{2u} (\dot{\alpha} - \dot{\alpha}_0)^2 + \frac{1}{u} \dot{\alpha}_0 (\dot{\alpha} - \dot{\alpha}_0)$$

parabola ⇒ is a "RAMO" DI PARABOLA

$y = a x^2 + b x$

[a=1; b=2]

y	x	y	x
0	0	0	-2
8	2	3	1



I^st design concept: $\rightarrow$ $\mathring{\alpha}_i = 0$ & $\mathring{\alpha}_j = 0$

$\rightarrow$ BOUNDED MAXIMUM TORQUE

$\Rightarrow \begin{cases} \dot{\alpha} \text{ is maximum} \\ t \text{ is minimum} \end{cases}$

therefore having choosen $\mathring{\alpha}_i = 0$ the parabola reduces to:

$$\alpha = \alpha_0 + \frac{1}{2u} \cdot (\dot{\alpha} - \dot{\alpha}_0)^2 + \frac{\overset{=0}{\cancel{\dot{\alpha}_0}}}{u} (\dot{\alpha} - \alpha_0)$$

$\leadsto$ still constant and equal to $\frac{M^*}{I}$

therefore the dependendence of the angle from the angular velocity will be:

$$\alpha = \alpha_0 + \frac{1}{2u} \dot{\alpha}^2$$

e.g. X-axis: $\alpha_x = [\underline{\underline{A}}e]_{2,3} \rightarrow \dot{\alpha}_x = [\dot{\underline{\underline{A}}}e]_{2,3} \rightarrow$!

POSITION SET-POINT $\longrightarrow$ $[\underline{\underline{A}}e]_{2,3}(t) = [\dot{\underline{\underline{A}}}e]_{2,3}\Big|_{t=0} + \frac{1}{2(M^*/I_x)} (M^*/I_x)^2 t^2$

VELOCITY SET-POINT $\longrightarrow$ $W_x \approx [\dot{\underline{\underline{A}}}e_{,3}] = \overset{=0}{\cancel{[\dot{A}_{2,3}]}}\Big|_{t=0} + u \cdot t = \frac{M^*}{I_x} \cdot t$

$$\Rightarrow \begin{cases} [\underline{\underline{A}}e]_{2,3}(t) = \frac{1}{2}(M^*/I_x)\, t^2 \\ W_{x\,SP} = \frac{M^*}{I_x}\, t \end{cases}$$

the problem is at this point to determine when (for wich angle) the torque (M^*=Max_torque) must be changed in sign.

$\rightarrow$ according with the control logic.

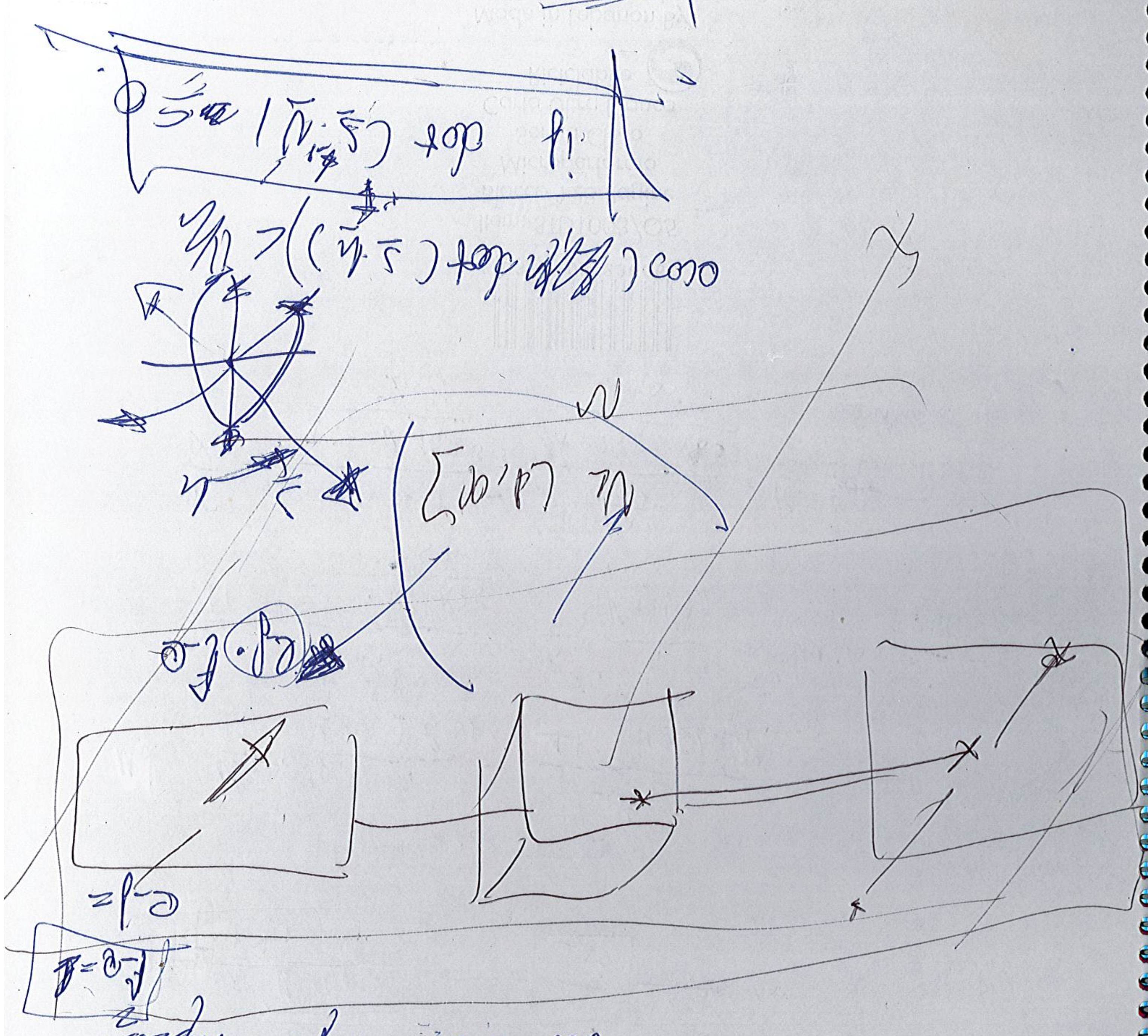
$$M = -M_{max}\, \text{sing}(\alpha)$$

$\rightarrow$ in order to stabilize the system:

$$M = -M_{max}\, \text{sgn}(\alpha + K\dot{\alpha})$$

Theta → R.A.

phi → CO_DEC.



* Statistical method for attitude determination (93)
* Instability for Euler's equation → plot on inertia ellipsoid

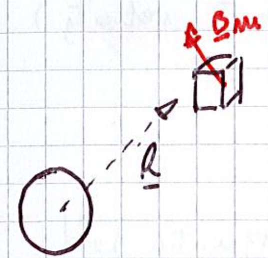
Missing before end:

[1,0,0] [0,1,0] [-1,0,0]

SENSOR 1) magnetic field sensor.

This sensor provide the measurement of one vector. (B_m)

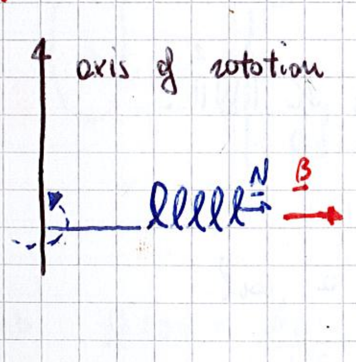
⇒ By coupling such measurement with the magnetic field's mathematical model is possible to get the attitude.



$$\Rightarrow \quad \underline{B}(\underline{R}) = \underline{\underline{A}}\,\underline{B}m$$

$$-\frac{\partial \Phi_s(\underline{B})}{\partial t} = FEM \Rightarrow i = \frac{F.E.M}{R} = -\frac{1}{R}\frac{\partial \Phi_s(\underline{B})}{\partial t}$$

e.g.



$$\begin{cases} \phi_{coil}(\underline{B}) = \int_{coil} \underline{B} \cdot d\underline{A} = B \cdot N \cdot A_{coil} \cdot \cos(wt) \\ \alpha = wt \qquad (\underline{B} \cdot \underline{N} = \cos\alpha) \end{cases}$$

Therefore for a constant magnetic field the recorded current will be a sinusoidal form:

$$i(t) = -\frac{1}{R}\frac{d\Phi(\underline{B})}{dt} = W\frac{B \cdot N\, A_{coil}}{R} \sin(wt) \rightarrow \text{measured.}$$

$$\frac{di(t)}{dt} = \frac{W \cdot B \cdot N\,A}{R} \cdot \cos(wt) \Rightarrow B\cos(wt) \propto \frac{di(t)}{dt}$$

($B \cdot \cos\alpha$) → represent the projection of the magnetic field on the coil axis.

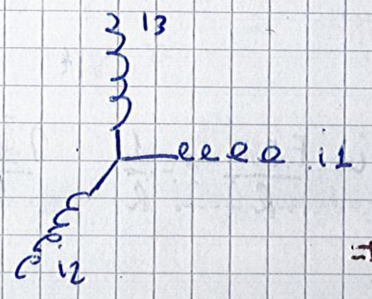
!! this means that the derivative of the signal is in fact proportional to the projection of the magnetic field on the coil's axis. !!

this means that:

IF The spacecraft is rotating (at constant velocity)

AND IF Three coils are used

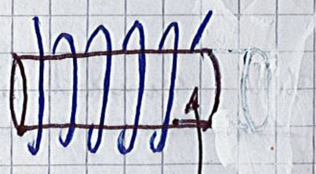
THEN is possible to determine the magnetic field by measuring 3 currents



$$i_1 \propto B_x \cos(w_x t) \cdot w_x^2$$
$$\Rightarrow \quad i_2 \propto B_y \cos(w_y t) \cdot w_y^2$$
$$i_3 \propto B_z \cos(w_z t)\, w_z^2.$$

$$I = i_1 + i_2 + i_3 \propto \|\underline{w}\|^2.$$

↳ Adding a ferromagnetic material inside a coil.



FERRO-MAGNETIC MATERIAL.

this is done in order to increase the magnetic field.

In general:

- INDUCTOR: Device capable of immagazinating electic energy into a magnetic field

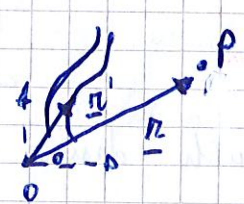
- CIRCUITS: $\xrightarrow{i}$ o—ℓℓℓℓ—o (ΔV) $\Rightarrow \quad \Delta V(t) = L \cdot \frac{di(t)}{dt}$

In order to compute L:

⌐→ Ampere's law / Laplace-formula.

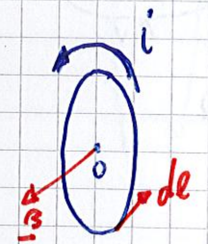
Ampere's law: $\oint_{\partial S} \underline{B} \cdot d\underline{r} = \mu_0 \cdot \sum_i^{\text{conc}} i_i$

Laplace's formula: $d\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\, i\, \frac{d\underline{l} \times (\underline{r} - \underline{r}')}{\|\underline{r} - \underline{r}'\|^3}$



$$\Rightarrow \quad \underline{B}(\underline{r}) = \frac{\mu_0 i}{4\pi} \int_\Gamma \frac{d\underline{l} \times (\underline{r} - \underline{r}')}{\|\underline{r} - \underline{r}'\|^3}$$

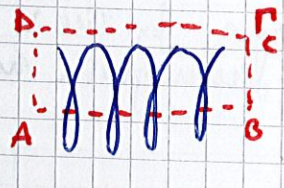
* CIRCULAR SINGLE COIL → magnetic field in the origin.



$$\begin{cases} \underline{B}(0) = \frac{\mu_0 i}{4\pi} \int_\Gamma - \frac{(d\underline{l} \times \underline{r}')}{\|-\underline{r}'\|^3} \\ d\underline{l} \perp \underline{r}' \; ; \quad dl = R \sin(d\theta) \simeq R\, d\theta. \end{cases}$$

$$\Rightarrow \quad \underline{B}(0) = \frac{\mu_0 i}{4\pi} \cdot \frac{1}{R^3} R^2 \int_0^{2\pi} d\theta = \frac{\mu_0 i}{2R}$$

* N COILS → N singular circular coils constituting a winding.



Hp^*: L solenoide → ∞ ⇒ $\underline{B}$ external to the winding is null.

$$\Rightarrow \quad \int_\Gamma \underline{B} \cdot d\underline{r} = \int_A^B \underline{B} \cdot d\underline{r} + \int_B^C \underbrace{\underline{B} \cdot d\underline{r}}_{(\underline{B} \perp d\underline{r})} + \int_C^D \underbrace{\underline{B} \cdot d\underline{r}}_{\underline{B}=0} + \int_D^A \underbrace{\underline{B} \cdot d\underline{r}}_{=0.}$$

$$\Rightarrow \quad B \cdot L = N \cdot i \cdot \mu_0 \Rightarrow \begin{cases} B = n \cdot i\, \mu_0. \\ n \triangleq N/L \end{cases}$$

concatenated currents.

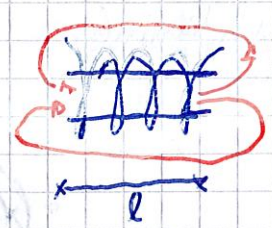
* N COILS WINDING A FERRO MAGNETIC MATERIAL → $\mu = \mu_r \cdot \mu_0$

$$B = \mu_0 \, n \cdot i$$

* CONSTITUTIVE LAW FOR INDUCTORS → circuital element.

⌐→ Faraday's law

$$\frac{d\Phi(B)}{dt} = - FEM$$



Therefore having such device acting on a $i(t)$

$$\begin{cases} B = \mu \cdot n \cdot i(t) \\ \Phi(B) = (NA) \cdot B \end{cases} \Rightarrow FEM = \Delta V(t) = \mu \cdot n \cdot N \cdot A \, \frac{di(t)}{dt}$$

$$\left(n = \frac{N}{l} \right)$$

$$\Phi_{sd}(B) = \sum_{i=1}^{N} A_i \, \Phi_{A_i}(B) = N \cdot A \cdot B$$

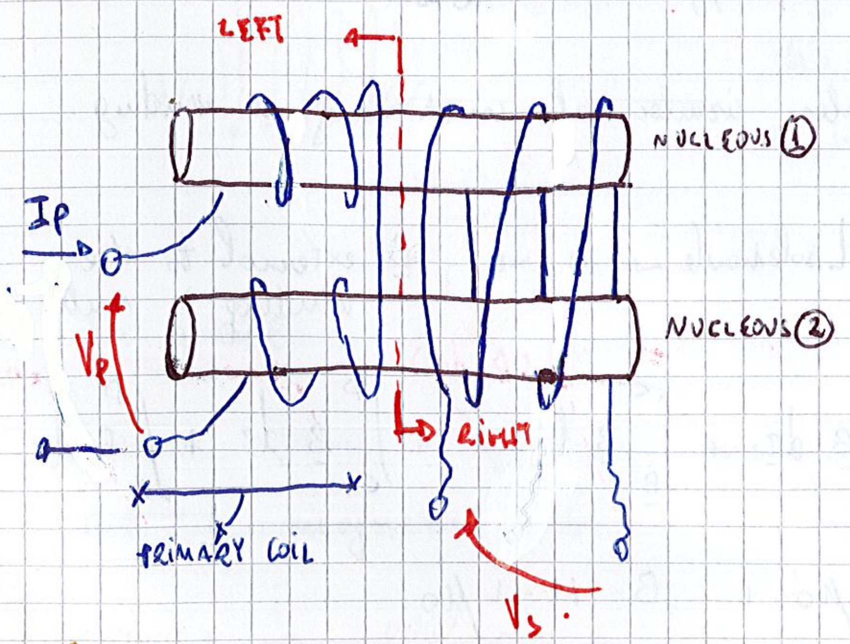
i → o—ℓℓℓℓ—o, ΔV

$$\Delta V(t) = \frac{\mu N^2}{l} A \cdot \frac{di(t)}{dt}$$

$$L \triangleq \frac{\mu N^2}{l} \cdot A$$

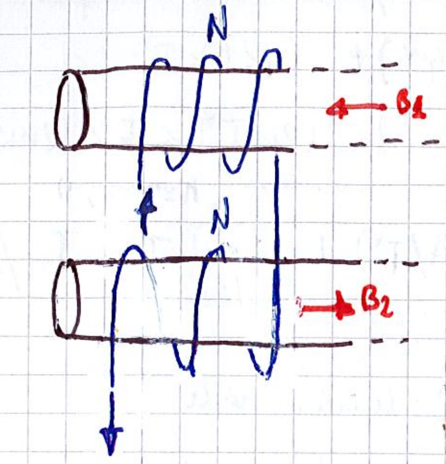
" Having a constant current ⇒ ∅ potential losses will be detected "

↳ To get an attitude determination:



S → sectional cross area. (Same for the 2 ferromagnetic nucleous)

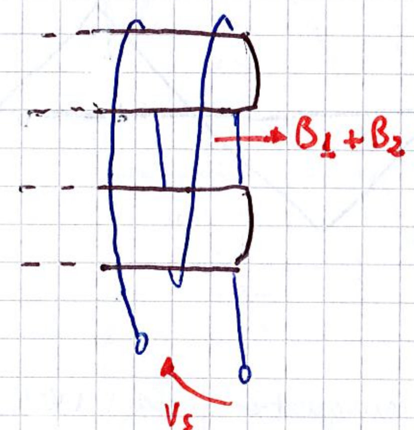
LEFT SIDE



$$B_2 = - B_1 = \mu \, n \cdot i(t)$$

This magnetic field depending by time is capable to induce a F.E.M. in the right side of the sensor.

RIGHT SIDE.



Thanks to Faraday's law:

$$V_s = - \frac{d\phi(B_1 + B_2)}{dt} = 2 \mu \frac{N^2}{l} A_{sx} \frac{di}{dt} \triangleq L \frac{di}{dt}$$

l → length of this second (RIGHT) winding around the 2 nucleis.

A → area of the winding on the right side

IF a ∅ EXTERNAL MAGNETIC FIELD is taken into account.

THEN $B_1 = -B_2$ → $\phi_{S_{RIGHT}}(B) = -\phi_{S_{RIGHT}}(B_2)$ → $V_s = \frac{d}{dt} \phi(B_1 + B_2) = 0$

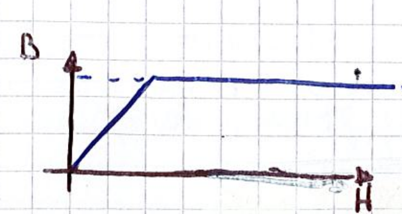
WHILE considering the WORLD MAGNETIC FIELD the true magnetic flow across the right winding will be:

$$V_s = \frac{d}{dt} \phi_{S_{RIGHT}}(B_1 + B_2) + \underbrace{\frac{d}{dt} \phi \, B_{W.F.}}_{\text{NEGLIGIBLE}}$$

⌐→ Magnetic induction.

!! USUALLY IT'S NOT LINEAR ⇒ LINEAR WITH AN HYSTERESIS !!

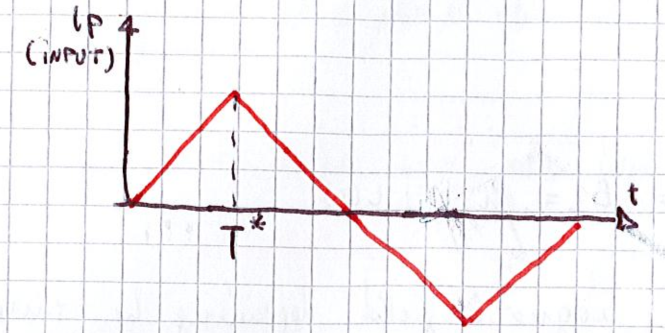
↳ there's a limit in the energy storable into ferromagnetic nucleus.



$$H = \frac{B}{\mu_0} - M$$

↖ campo magnetico aggiuntivo dovuto al materiale

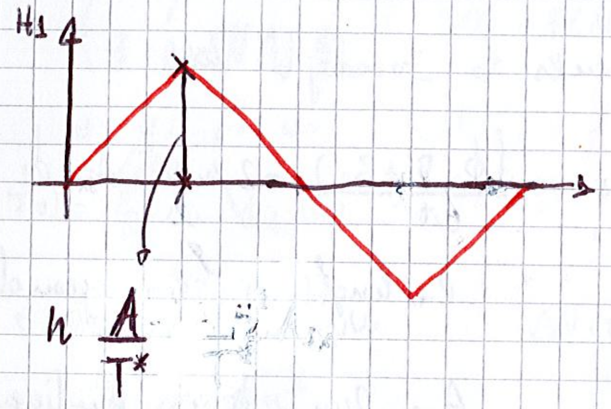
↳ Setting ip as a triangular signal:

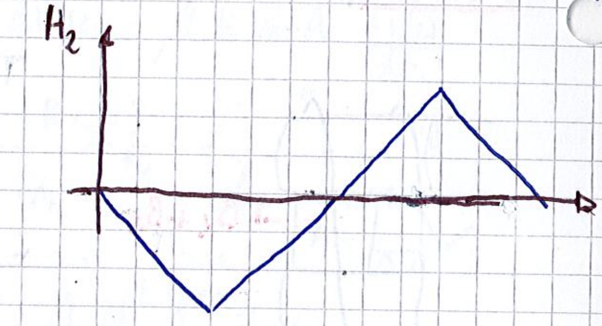


A → amplitude of the current.

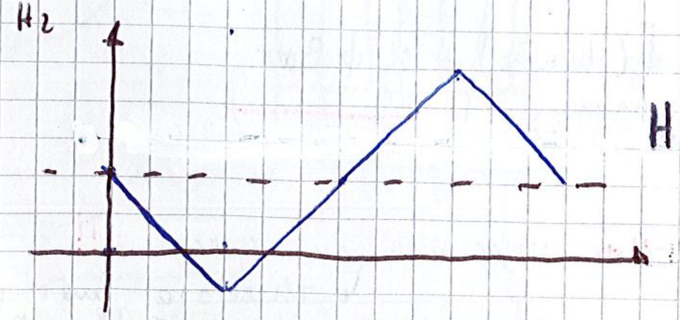
$$ip = \begin{cases} (A/T^*)\,t & \text{for } 2nT^* < t < (2n+1)T^* \quad n = 0, \dots, N \\ -(A/T^*)\,t \ ; & T^* = T_{ip}/4 \end{cases}$$

↳ The current circulating in the 2 nuclei will be opposite in "direction" ⇒ able to induce to induce opposite magnetic induction.





↳ Adding the effect of a constant magnetic field (WMF)



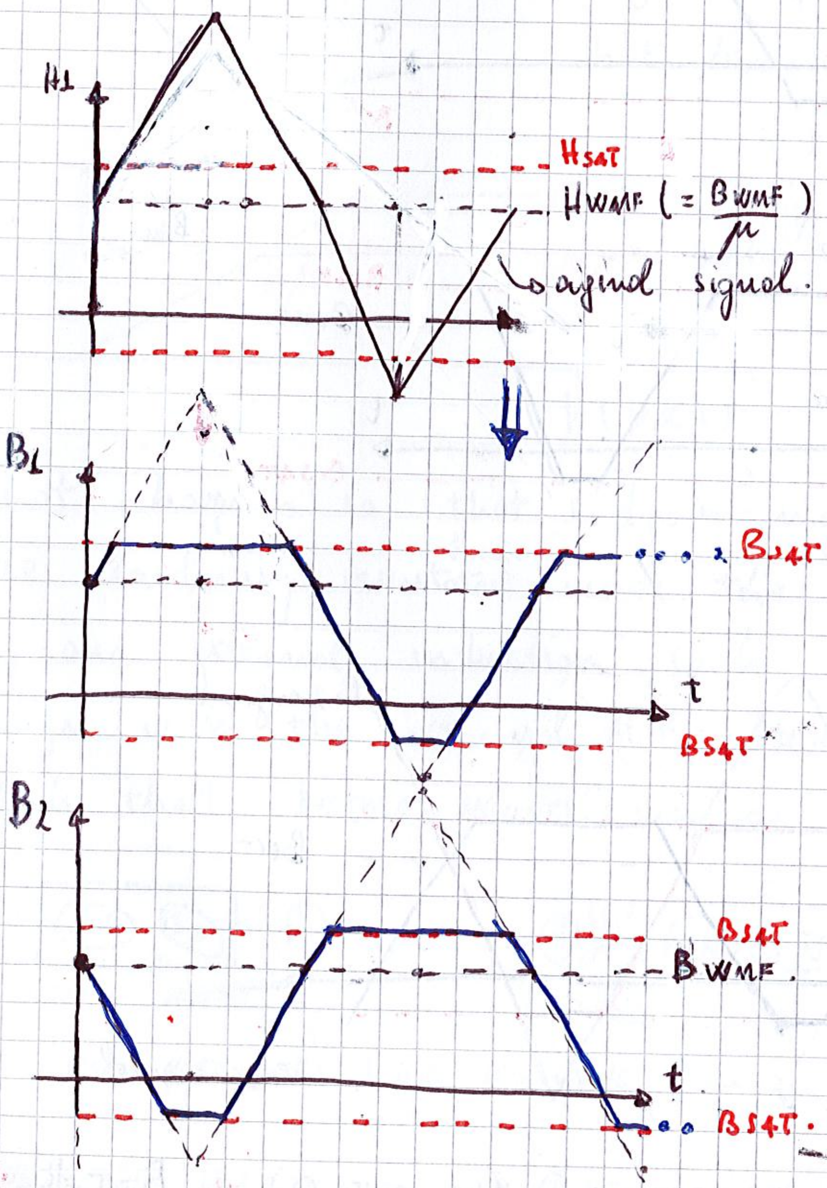
↳ Considering the non linearity of the ferromagnetic material: SATURATION

$$\begin{cases} B = \mu \cdot n \cdot i(t) \ (\to \text{solenoide}) \\ B = \mu H \end{cases} \Rightarrow H = n\, i(t)$$

!! there's a limit in the storable energy inside the ferromagnetic nucleus !!

this means that if: $|B_{1/2} + B_{ext}| > B_{SAT} \Rightarrow B_{1/2} = B_{SAT}^+$

$|B_{1/2} + B_{ext}| < B_{SAT}^- \Rightarrow B_{1/2} = B_{SAT}^-$



$$\Rightarrow H_{WMF} = \frac{B_{WMF}}{\mu} + n \cdot i(t)$$

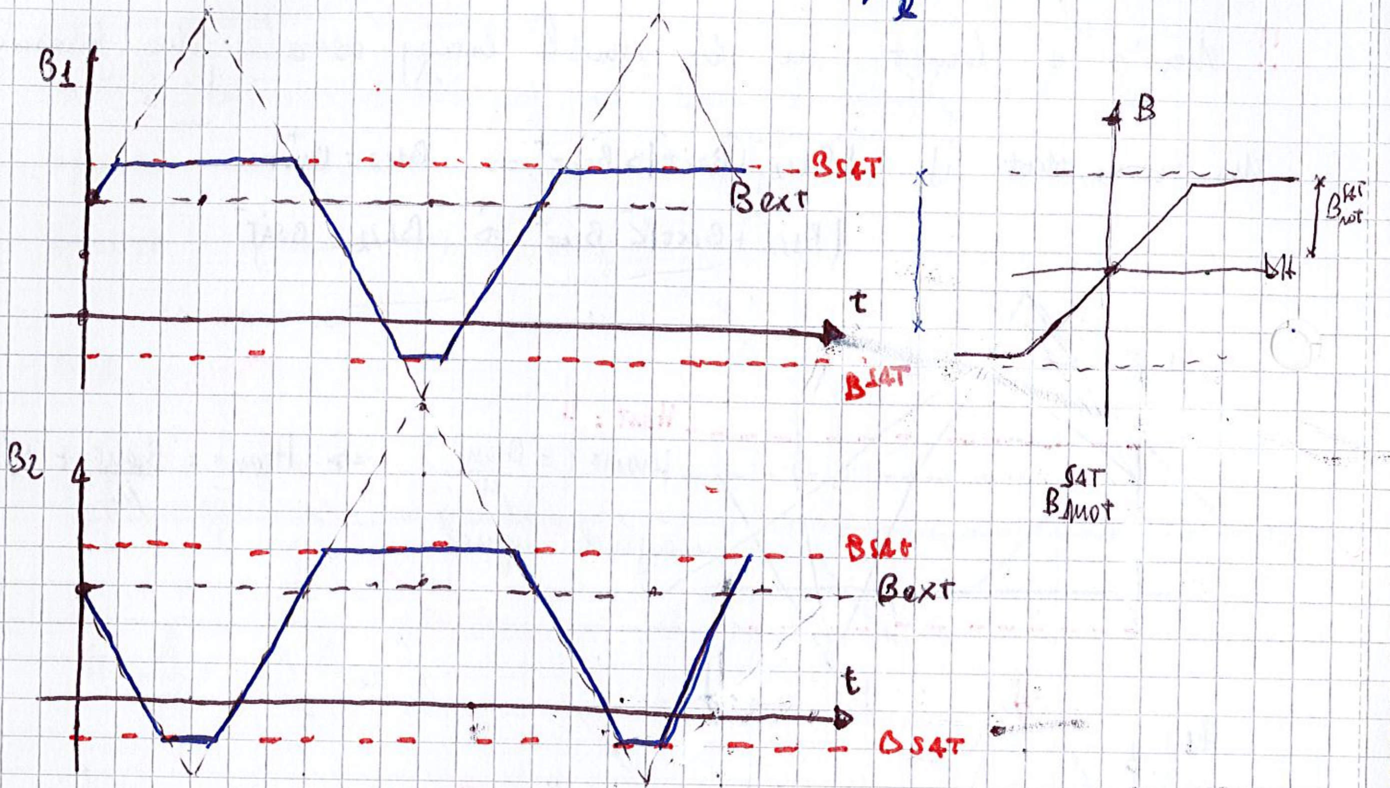
↳ To evaluate the Vs the derivative of this signal must be taken into account.

according with Faraday's law:

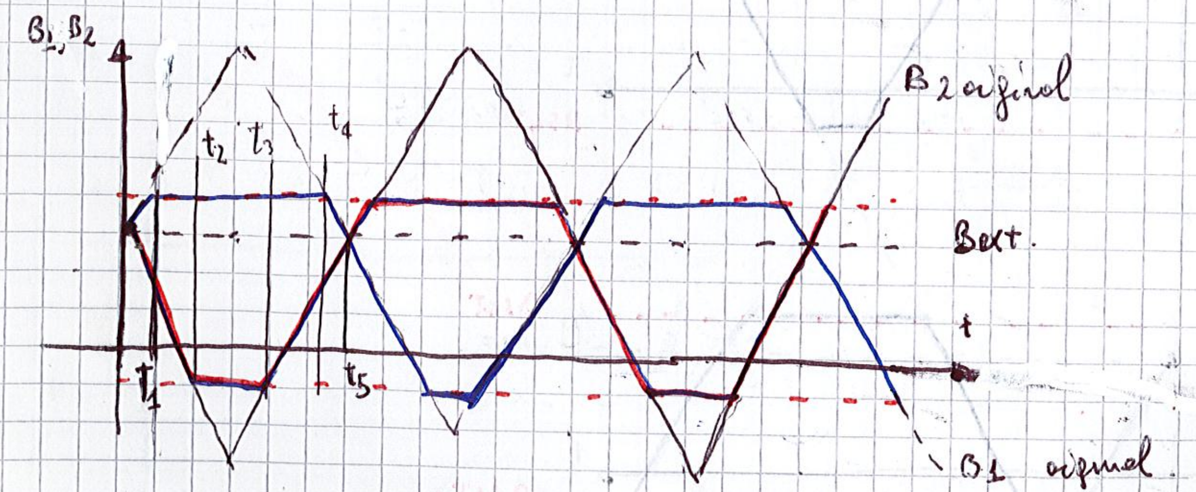
$$V_s = -\frac{d\Phi(B_1+B_2)}{dt} = \mu n N A \cdot \frac{di_p}{dt}$$

$$= \mu (N A) \frac{dH}{dt} \quad \left(= L \cdot \frac{di}{dt}\right)$$

$$L = \frac{\mu N^2}{\ell} A$$



=



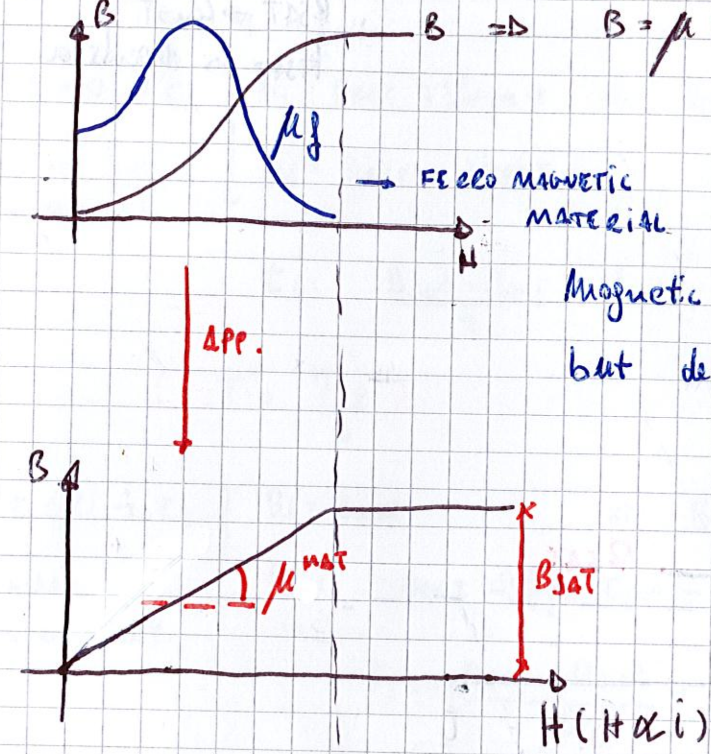
a) $B_1 \cup B_2$ SATURATED (ONE SATURATED ⊕ THE OTHER ⊖) ⇒ $B_{TOT} = B_{ext}$ ⇒ $V_s = 0$

b) $B_1 \cup B_2$ NOT SATURATED ⇒ $B_{TOT} = B_{ext} + slope \cdot T + B_{ex} - slope \cdot T$

$= 2 B_{ext}$ ⇒ $V_s = 0$

c) B_1 or B_2 NOT SATURATED { $B_1 =$

Details about saturation



B ⇒ $B = \mu H = \mu(H) \cdot H$

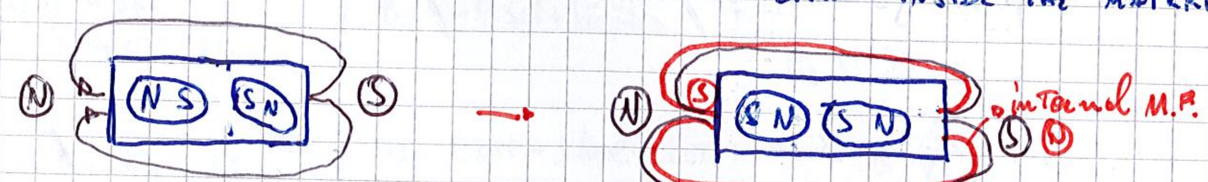
(solenoid: $H = n \cdot i$)

Magnetic permeability is not constant but depends NOT LINEARY by H.

e.g. Fe-C: $\mu_{Fe\text{-}C} = 8.75 \times 10^{-4}$

$B_{SAT} \approx 2T$.

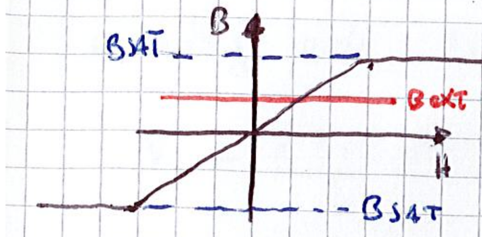
What happens is that dipoles (magnetic dip) originally are randomly orientated inside the material. Cause of an external induction such dipoles tend to align with the external M.F. creating a "contra-M.F." such that FURTHER MAGNETIC INDUCTION INSIDE THE MATERIAL IS ENABLED

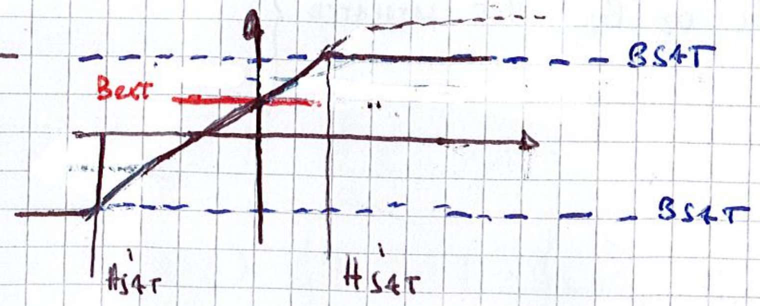


IF An external (NOT INDUCTIVE) magnetic field is present.

THEN SATURATION WILL OCCUR AT LOWER VALUES OF H IN THE DIRECTION OF THE EXTERNAL M.F. AND AT HIGHER VALUES OF H IN THE OPPOSITE DIRECTION OF THE EXTERNAL M.F.

⇒ THIS CORRESPOND TO A TRANSLATION↑ OF THE PREVIOUS GRAPH.

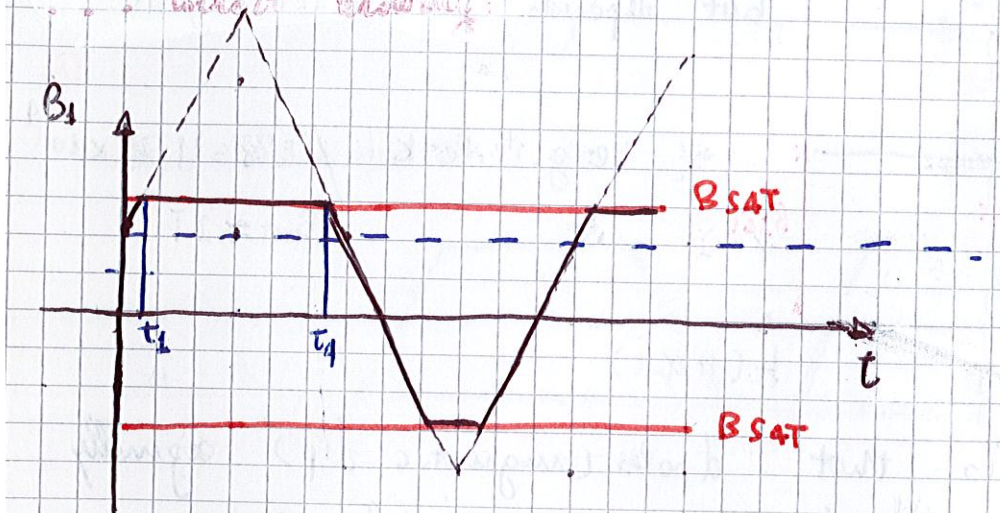




$|H'^{(-)}_{SAT}| > |H^{(-)}_{SAT}|$
opposite direction of Bext

$H'^{(+)}_{SAT} < H^{+}_{SAT}$

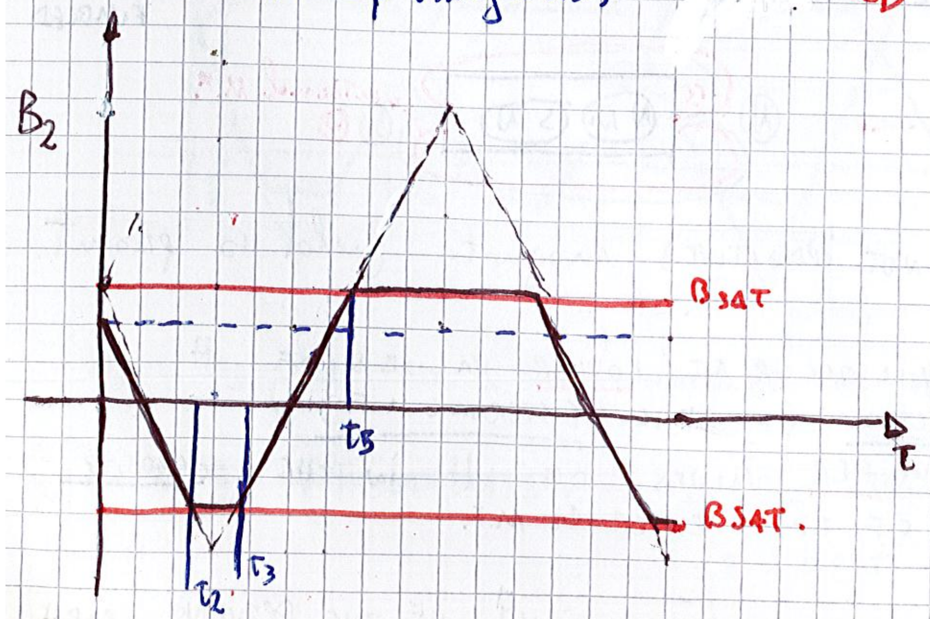
BSAT → const
HSAT → depends on Bext

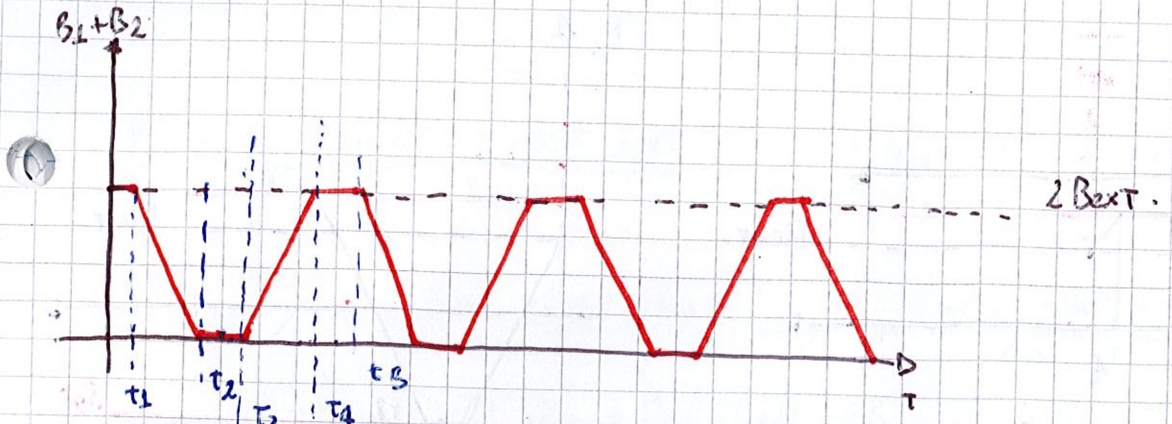


!! In order to have an interception with BSAT must be that

$$I_p \cdot \mu N > (B_{SAT} + B_{ext})$$

$(i = I_p \text{ triangle}(t))$ $\Rightarrow$ $\mu H^{MAX} > (B_{SAT}+B_{ext})\big|_{MAX}$!!





* $t = 0 \div t_1$

1 → NOT SAT.
2 → NOT SAT.

$$\begin{cases} B_1 = B_{ext} + \text{slope}\cdot t \\ B_2 = B_{ext} - \text{slope}\cdot t \end{cases} \Rightarrow B_{TOT} = 2B_{ext} \Rightarrow V_s = -\frac{d\Phi(B)}{dt} = 0$$

$$t_1: \frac{B_{sat}-B_{ext}}{t_1} = \text{slope} = \frac{B_D}{T^*} \rightarrow t_1 = T^* \frac{(B_{sat}-B_{ext})}{B_D}$$

$$\left(T^* = \frac{T_{ip}}{4} \right)$$

* $t = t_1 \div t_2$

1 → SAT
2 → NOT. SAT.

$$\begin{cases} B_1 = B_{sat} \\ B_2 : B_{ext} - \text{slope}\cdot t_2 = -B_{sot} \end{cases} \Rightarrow B_{TOT} = B_{SAT} + B_2(t_1) - \text{slope}\cdot t.$$

$$t_2 = \frac{B_{ext}+B_{sat}}{\text{slope}} = \frac{B_{ext}+B_{sat}}{B_D} T^*$$

$$V_s = -\frac{d\Phi}{dt} = \text{slope}$$

$$V_s = \frac{B_D}{T^*}$$

* $t = t_2 \div t_3$

1 → SAT.
2 → SAT.

$$\begin{cases} B_1 = B_{sat} \\ B_2 = -B_{sat} \end{cases} \Rightarrow B_{tot} = 0 \Rightarrow V_s = 0.$$

* $t = t_3 \div t_4$

1 → SAT
2 → NOT SAT.

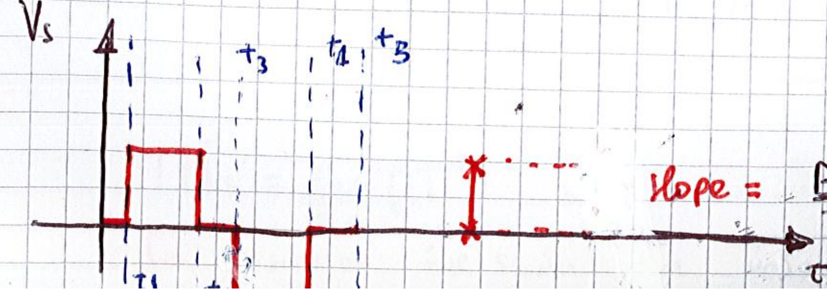
$$\begin{cases} B_1 = B_{sat} \\ B_2 = -B_{sat} + \text{slope}(t - t_3) \end{cases} \Rightarrow B_{tot} = \text{slope}\cdot t \Rightarrow V_s = -\frac{d\Phi}{dt} = -\text{slope}.$$

$B_2(t_4) =$

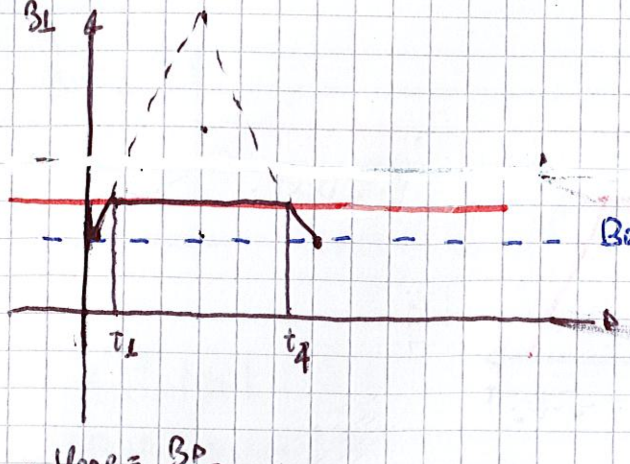
* $t = t_4 \div t_5$

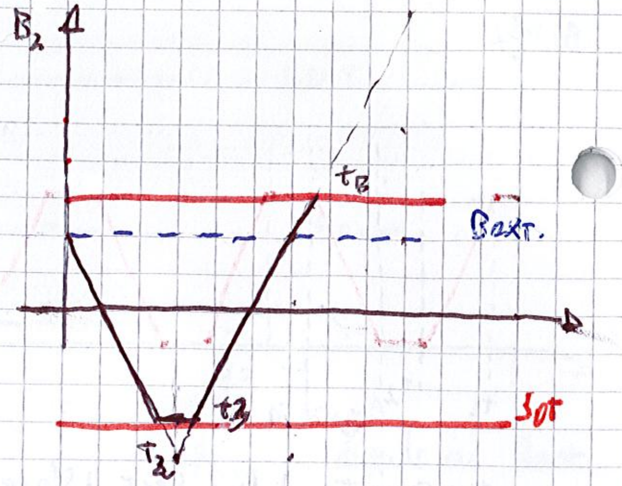
1 → NOT SAT.
2 → NOT SAT.

$$\begin{cases} B_1 = B_{sat} - \text{slope}(t - t_4) \\ B_2 = B_2(t_4) + \text{slope}(t - t_4) \end{cases} \Rightarrow V_s = 0.$$



$$\text{slope} = \frac{B_D}{T^*} = \frac{\mu N I_p}{T_{ip}/4}$$





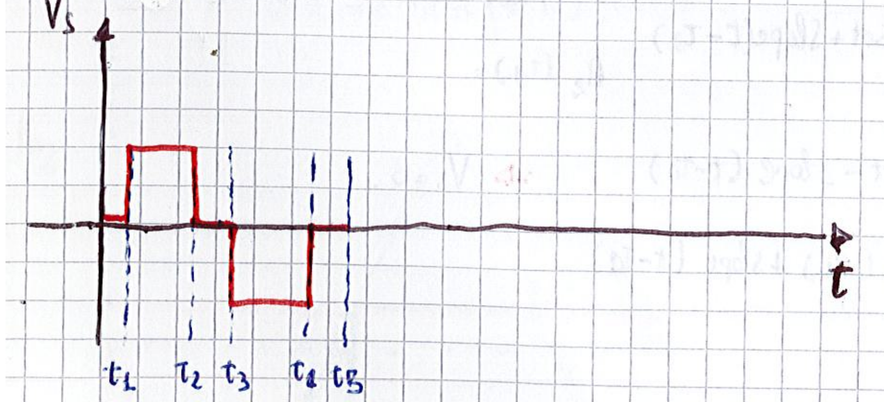
- $\begin{cases} \text{slope} = \dfrac{B_D}{T^*} \\ \dfrac{B_{sat}-B_{ext}}{t_1-0} = \text{slope} \rightarrow t_1 = \dfrac{B_{sat}-B_{ext}}{B_D}\cdot T^* \end{cases}$

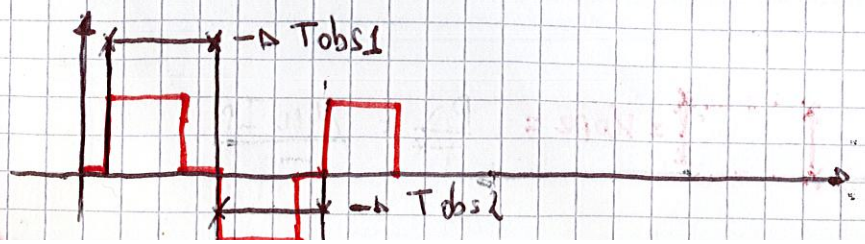
- $\dfrac{B_{ext}-(-B_{sat})}{t_2-0} = \text{slope} \rightarrow t_2 = \dfrac{B_{sat}+B_{ext}}{B_D}\cdot T^*$

- $(B_{ext}-B_D) + \text{slope}\,(t_3-T^*) = -B_{sat} \rightarrow \dfrac{-B_{sat}-B_{ext}+B_D}{B_D}\cdot T^* + T^* = t_3$

- $(B_{ext}+B_D) - \text{slope}\,(t_4-T^*) = B_{sat} \rightarrow \dfrac{-B_{sat}+B_{ext}+B_D}{B_D}\,T^* + T^* = t_4$

- $B_{ext}-B_d + \text{slope}\cdot(t_5-T^*) = B_{sat} \rightarrow \dfrac{-B_{ext}+B_D+B_{sat}}{B_D}\cdot T^* + T^* = t_5$





$$t_2-t_1 = \frac{B_{sat}+B_{ext}-B_{sat}+B_{ext}}{B_D}\,T^* = \frac{2B_{ext}}{B_D}\,T^* \qquad \left(T^* = T_{ip}/4\right)$$

$$t_4-t_3 = \cancel{T^*} + \frac{-B_{sat}+B_{ext}+B_D}{B_D}\,T^* - \cancel{T^*} - \frac{-B_{sat}-B_{ext}+B_D}{B_D}\,T^*$$

$$= \left[\frac{-B_{sat}+B_{ext}+\cancel{B_D}+B_{sat}+B_{ext}-\cancel{B_D}}{B_D}\right]T^* = \left[\frac{2B_{ext}}{B_D}\right]T^* \text{ cvd.}$$

$t_2-t_1 = t_4-t_3$

(different length.)

$$t_3-t_2 = T^* + \frac{-B_{sat}-B_{ext}+B_D}{B_D}\,T^* - \frac{B_{sat}+B_{ext}}{B_D}\,T^*$$

$$= T^* + \frac{-2B_{sat}-2B_{ext}}{B_D}\,T^* + \frac{B_D}{B_D}\,T^* = 2T^*\left(1-\frac{B_{sat}+B_{ext}}{B_D}\right)$$

$$t_5-t_4 = \cancel{T^*} + \frac{-B_{ext}+B_D+B_{sat}}{B_D}\,T^* - \cancel{T^*} - \frac{-B_{sat}+B_{ext}+B_D}{B_D}\,T^*$$

$$= \left[\frac{-B_{ext}+\cancel{B_D}+B_{sat}+B_{sat}-B_{ext}-\cancel{B_D}}{B_D}\right]T^* = \frac{2(B_{sat}-B_{ext})}{B_D}\,T^*$$

$$T_{obs1} = (t_2-t_1)+(t_3-t_2) = 2T^*\left[1-\frac{B_{sat}+B_{ext}}{B_D}+\frac{B_{ext}}{B_D}\right]$$

$$= 2T^*\left[1-\frac{B_{sat}}{B_D}\right]$$

$$T_{obs2} = (t_4-t_3)+(t_5-t_4) = \cancel{\frac{2B_{ext}}{B_D}}\,T^* + \frac{2B_{sat}}{B_D}\,T^* - \cancel{\frac{2B_{ext}}{B_D}}\,T^*$$

$$= 2T^*\left[\frac{B_{sat}}{B_D}\right]$$

→ Defining $K \triangleq \dfrac{B_{sat}}{B_D}$

* $T_{obs1} = 2T^*(1-K)$

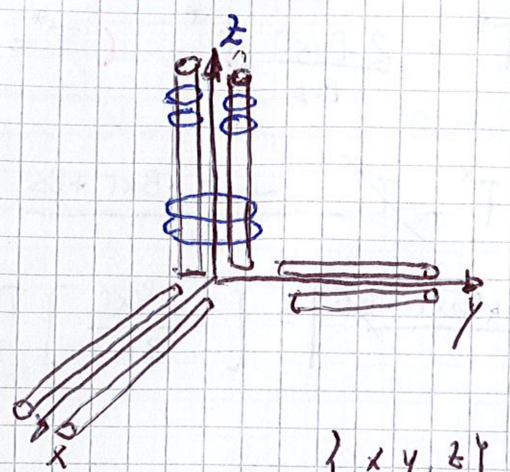
* $T_{obs2} = 2T^*K$

F

$\Delta T_{obs} = \dfrac{B_{ext}(2T^*)}{B_D} \rightarrow B_{ext} = \dfrac{B_D\,\Delta T_{obs}}{2T^*}$ → $B_D = \dfrac{\mu N^2 A}{l} I_P$; → $T^* = T_{ip}/4$

→ LENGTH OF THE SIGNAL FOR NON Ø VALUES

Using a tuple Setup : ⇒ it's possible to know the value along the three axis of Earth's magnetic field.



$\{x, y, z\}$ → body-frame

(depending on the mounting)

!! this measurement have to be referenced to the Mathematical model of World's Magnetic field but KEEPING INTO ACCOUNT THE DIFFERENT REFERENCE FRAMES !!

Sensor 2) Star Sensors (STAR TRACKER)

The sensor follows the procedure:

[DATA ACQUISITION] → [CORRECT POSITIONING IN SPACE (of the aquired data)]

↓

[INTERPRETATION OF DATA (Star identification) => ANALYSIS OF A STAR CATALOGUE]

↓

[ATTITUDE DETERMINATION]

• ECLIPSE MAY OCCUR

• CORRECT CHOICE OF THE STAR TO OBSERVE.

really heavy from computational POV.

→ " Matrix sensor (CCD)

→ Image is projected in a plane. ")

→ " In the FOV (field of view) of the sensor even if it's really narrow, the star captured might be really numerous

Sensor

θ → FOV.

Star tracker

* TILTING CAPABILITY

=> detects always the same star(s^*) in such a way that s^* is always in the middle of the FOV.

CCD Sensor. (STAR MAPPER SENSOR)

* FIXED ON THE SPACECRAFT.

⇓

RECONSTRUCT THE MAP OF THE OBSERVED SKY.

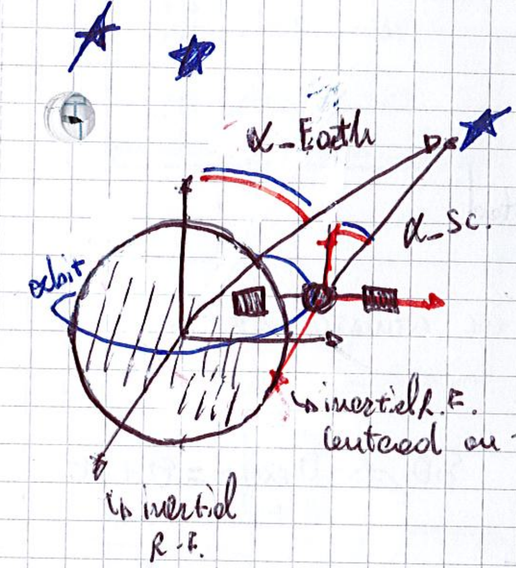
Assumption: [REASONABLE] Star sensor catalog is given in the EARTH-CENTERED-INERTIAL-REFERENCE FRAME.

But:

SINCE stars are theoretically at ∞ distance from Earth.

THEN Each observation (data acquisition) made from the spacecraft will give the same results (angles) as if it was done from Earth

!! Where the star-catalogue has been build !!



=> $\alpha\_Earth \approx \alpha\_spacecraft.$

e.g. catalogue:

NAM	SAO	RA	DEC	M
ALDEBARAN	1234	10°	20°	6

→ $\{s_1, \dots, s_N\}$ → Stars in the catalogue.

$\{\hat{o}_1, \dots, \hat{o}_n\}$ → Observation available.

$n \ll N$ → depending on the FOV of the sensor.

≃ obvious: At this stage $\underline{A}$ IN 2Pi is not available.

($\underline{A}$ → normally used)

AIM: is to associate each observation (in the sensor reference frame) with one star of the catalogue.

$\hat{o}_1 \xrightarrow{\underline{A}^T} o \longrightarrow s_1$

$\hat{o}_n \xrightarrow{\underline{A}^T} o_n \longrightarrow s_1$

STRICTLY NECESSARY.

!! THE PROCESS REQUIRES AS FIRST GUESS THE ESTIMATION OF MATRIX A !!

$\hat{O}_1 \xrightarrow{A^T} O_1 \dashrightarrow S_1$

$\hat{O}_2 \xrightarrow{A^T} O_2 \dashrightarrow S_2$

$\hat{O}_3 \xrightarrow{A^T} O_3 \dashrightarrow S_3$

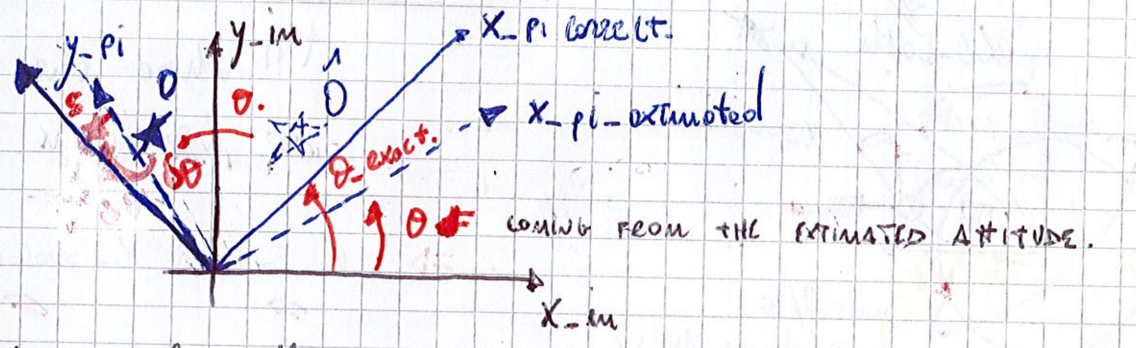
$\vdots$

(*)

$\hat{O}_4 \xrightarrow{A^T} O_4 \dashrightarrow S_4$

↳ STAR POSITION IN THE IN r.f.

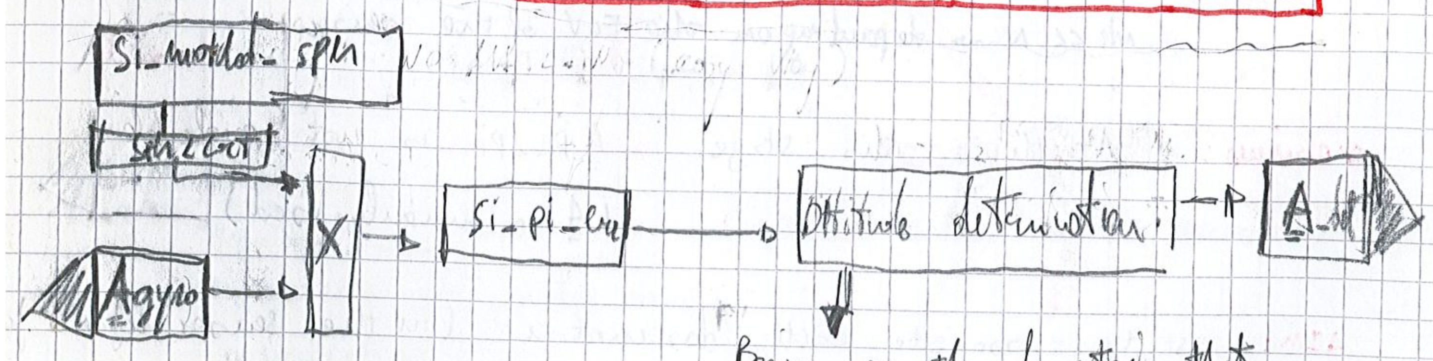
STAR POSITION IN THE PI r.f.



⇒ the angle will be corrected for $\delta\theta \Rightarrow \theta_{exact} = \theta + \delta\theta$

SUCH OPERATION IS POSSIBLE IF AND ONLY IF THE ASSOCIATION BETWEEN $\hat{O}$ AND $\hat{S}$ IS CORRECT.

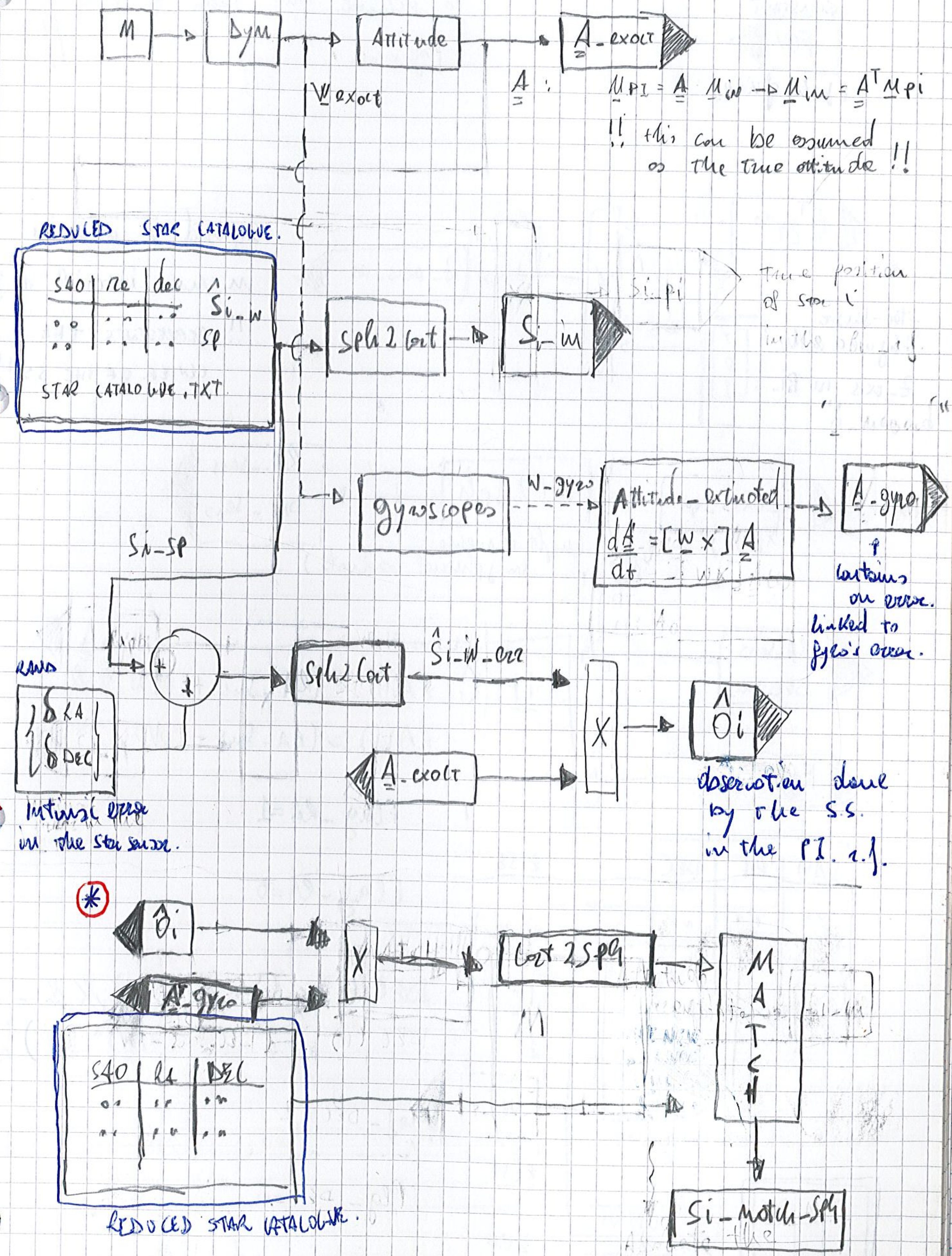
(*)



Being on the hypothesis that they many measurements NO ERROR WILL BE ALLOW

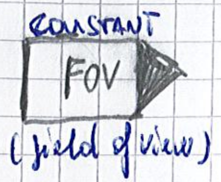
>> PROJECT_MY

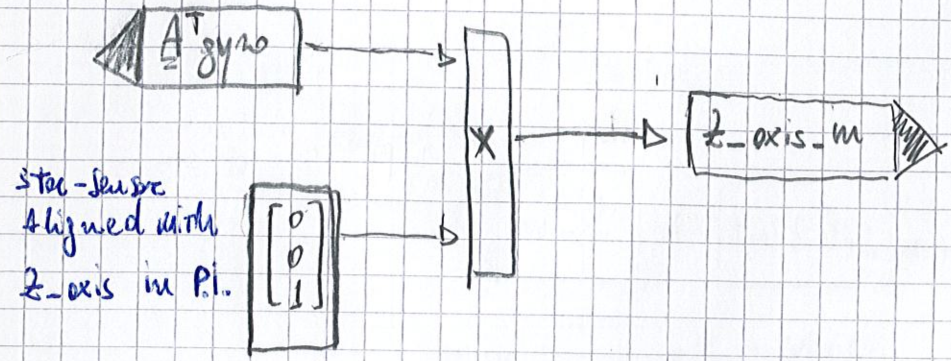
A reasonable simulink procedure is the following:



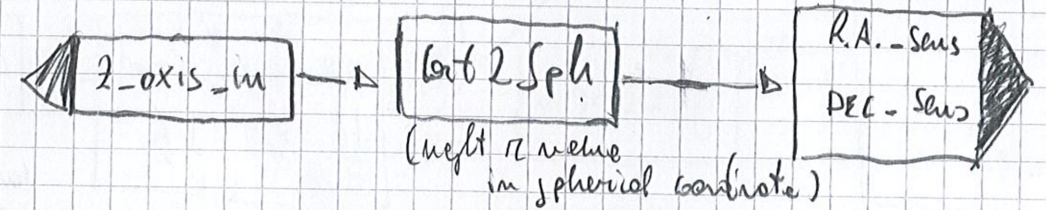
>> To select the correct star [PORTION OF CELESTIAL SPHERE]

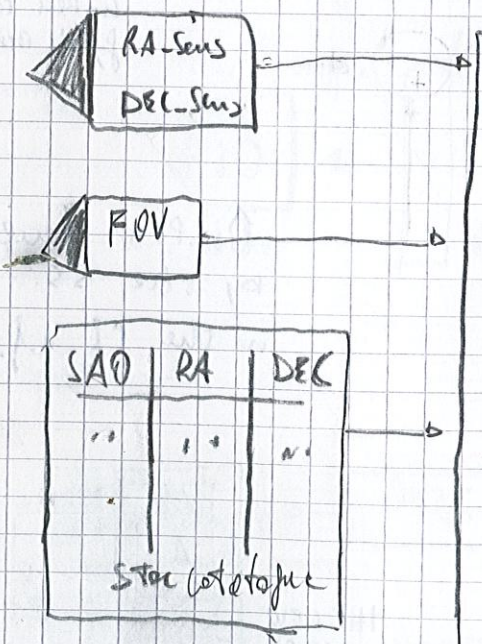
!! To increase the speed of the program !!





Supposed z_axis_in in the inertial r. f.
!! REPRESENTS THE CENTER OF THE S.S. !!






```
iF ( RA(i) < (RA_Sens +FOV) &&
     RA(i) > (RA_Sens - FOV)    )

     Flag_RA=1
ELSE
     Flag_RA=0
END

iF ( DEC(i) < (DEC_Sens +FOV) &&
     DEC(i) > (DEC_Sens - FOV)   )

     Flag_DEC=1
ELSE
     Flg_DEC=0
END

iF  (Flag_RA && flag_DEC )
     S(i) S_REDUCED (i)
END
```