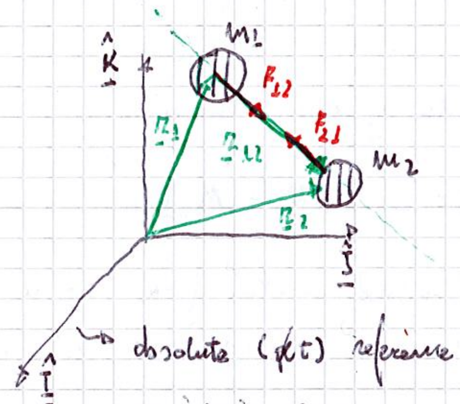
1 → " Restricted Two-body problem (R2BP) "

Considering 2 bodies, and the fact that, due to gravity, each other attracts the other body.

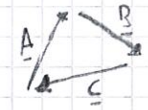


↳ absolute (fixed) reference frame

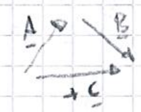
⇒ inertial frame.

$\Rightarrow \quad \underline{r}_1 + \underline{r}_{12} - \underline{r}_2 = \underline{0} \quad \rightarrow \quad \underline{r}_{12} = \underline{r}_2 - \underline{r}_1$

$\underline{r}_{12}$ → directed from 1 to 2

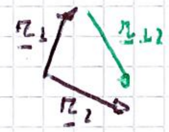


$\rightarrow \underline{A} + \underline{B} + \underline{C} = 0$



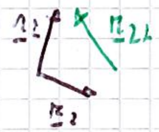
$\rightarrow \underline{A} + (-\underline{C}) + \underline{B} = 0$

$\Rightarrow \underline{B} = -\underline{A} + \underline{C} = \underline{C} - \underline{A}$



$\Rightarrow \quad \underline{r}_{12} = \underline{r}_2 - \underline{r}_1$

$\underline{r}_{12}$ → directed from 1 → 2



$\Rightarrow \quad \underline{r}_{21} = \underline{r}_1 - \underline{r}_2$

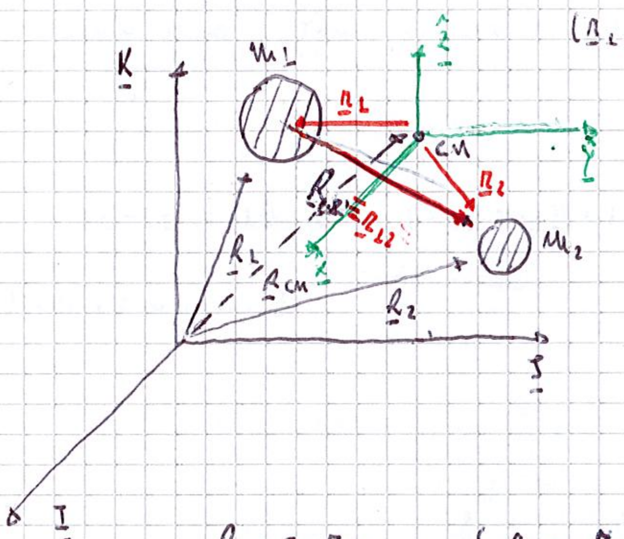
⇒ From the IIIrd dynamic principle: $\quad \|\underline{F}_{12}\| = \|\underline{F}_{21}\| \quad , \quad \underline{F}_{12} = \underline{F}_{21}$

$$\underline{F}_{12} = G \frac{m_1 m_2}{\|\underline{r}_{12}\|^3} \cdot \underline{r}_{12} \quad , \quad \underline{F}_{21} = -G \frac{m_1 m_2}{\|\underline{r}_{12}\|^3} \underline{r}_{12} = G \frac{m_1 m_2}{\|\underline{r}_{21}\|^3} \underline{r}_{21}$$

Ist → cardinal equation (momentum)

$$\begin{cases} m_1 \ddot{\underline{r}}_1 = G \dfrac{m_1 m_2}{\|\underline{r}_{12}\|^3} \cdot \underline{r}_{12} \\ \\ m_2 \ddot{\underline{r}}_2 = -G \dfrac{m_1 m_2}{\|\underline{r}_{12}\|^3} \underline{r}_{12}. \end{cases}$$

I^{st} equation must be expressed locating the origin in the center of mass of the system $\Rightarrow$ NEED TO CHANGE THE REFERENCE SYSTEM.



$(\underline{r}_1, \underline{r}_2) \rightleftarrows (\underline{R}_1, \underline{R}_2)$

$\hookrightarrow$ position in the fixed reference frame.

Center of mass theorem:

$\underline{Q} = \sum_i m_i \underline{v}_i = M \cdot \underline{V}_{CM}$.

$\underline{Q} \to$ total momentum of the system.

$M = \sum_i m_i$ (total mass of the system)

- $\underline{R}_{12} \equiv \underline{r}_{12}$, $(\underline{r}_{12} = \underline{r}_2 - \underline{r}_1)$

$\Rightarrow$ • $\underline{R}_{CM} = \dfrac{m_1}{(m_1+m_2)} \cdot \underline{R}_1 + \dfrac{m_2}{(m_1+m_2)} \underline{R}_2$

- $\begin{cases} \underline{r}_1 = \underline{R}_1 - \underline{R}_{CM} \\ \underline{r}_2 = \underline{R}_2 - \underline{R}_{CM} \end{cases}$, $\begin{cases} m_1 \ddot{\underline{R}}_1 = G \dfrac{m_1 m_2}{\|\underline{R}_{12}\|^3} \underline{R}_{12} \\ m_2 \ddot{\underline{R}}_2 = -G \dfrac{m_1 m_2}{\|\underline{R}_{12}\|^3} \underline{R}_{12} \end{cases}$

H_p^1 • Because externally of the system doesn't act any force:

$$\frac{d\underline{Q}}{dt} = m_1 \ddot{\underline{R}}_1 + m_2 \ddot{\underline{R}}_2 = (m_1+m_2)\ddot{\underline{R}}_{CM} = \underline{0} \rightarrow \ddot{\underline{R}}_{CM} = \underline{0}$$

OSS: even the 2^{nd} reference system is inertial (cause $\ddot{\underline{R}}_{CM} = \underline{0}$)

$\Rightarrow$ The expression of the I^{st} cardinal equation in the new reference frame becomes:

$$I^{st} \begin{cases} \cancel{m_1}(\ddot{\underline{r}}_1 + \overset{=0}{\cancel{\ddot{\underline{R}}_{CM}}}) = G \dfrac{\cancel{m_1} m_2}{\|\underline{r}_{12}\|^3} \underline{r}_{12} \\ \cancel{m_2}(\ddot{\underline{r}}_2 + \overset{=0}{\cancel{\ddot{\underline{R}}_{CM}}}) = -G \dfrac{m_1 \cancel{m_2}}{\|\underline{r}_{12}\|} \cdot \underline{r}_{12} \end{cases} \qquad \underline{R}_{12} = \underline{r}_2 - \underline{r}_1$$

$\Rightarrow$ Summing the 2 equations:

$$\begin{cases} \ddot{\underline{r}}_1 = G \dfrac{m_2}{\|\underline{r}_{12}\|^3} \underline{r}_{12} \\ \ddot{\underline{r}}_2 = -G \dfrac{m_1}{\|\underline{r}_{12}\|^3} \underline{r}_{12} \end{cases} \rightarrow \begin{cases} \ddot{\underline{r}}_1 = G \dfrac{m_2}{\|\underline{r}_{12}\|^3} \underline{r}_{12} \\ (+) \\ -\ddot{\underline{r}}_2 = G \dfrac{m_1}{\|\underline{r}_{12}\|^3} \underline{r}_{12} \end{cases}$$

$$\ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = G \frac{(m_1+m_2)}{\|\underline{r}_{12}\|^3} \cdot \underline{r}_{12}$$

$$\ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = -G \frac{(m_1+m_2)}{\|\underline{r}_{12}\|^3} \cdot \underline{r}_{12}$$

$\underline{r} \triangleq \underline{r}_{12}$

$\underline{r} = \underline{r}_2 - \underline{r}_1 \Rightarrow \ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1$

"Proceeding like this we find the equation of motion in function of the distance of m_1, m_2"

$$\underline{r} = \underline{r}_{12} = \underline{r}_2 - \underline{r}_1 \Rightarrow \frac{d^2\underline{r}}{dt} = -G \frac{(m_1+m_2)}{\|\underline{r}\|^3} \underline{r}$$

Γ^{R2BP}

$\underline{r} = \underline{r}_2 - \underline{r}_1 = \underline{R}_2 - \underline{R}_1 \to$ distance between (m_1, m_2)

$$\frac{d^2\underline{r}}{dt^2} = -G\frac{(m_1+m_2)}{\|\underline{r}\|^3} \underline{r}$$

H_p^2. $m_2 \ll m_1 \Rightarrow M \triangleq m_1 + m_2 \simeq m_1$

$$\Rightarrow \frac{d^2\underline{r}}{dt^2} \simeq -G \frac{m_1}{\|\underline{r}\|^3} \cdot \underline{r}, \qquad \mu \triangleq G m_1$$

$\mu \to$ reduced mass

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3} \underline{r}$$

Equation of motion for the R2BP

II^{nd} cardinal equation. (angular momentum)

$$\underline{\Gamma} = \underline{r} \times \underline{q} = \underline{r} \times \left(m \frac{d\underline{r}}{dt}\right)$$

$\underline{\Gamma} \to$ angular momentum.

II^{nd} cardinal equation: $\begin{cases} \dfrac{d\underline{\Gamma}}{dt} = \sum_i \underline{M}_i \\ \underline{\Gamma} = \sum_i \underline{r}_i \times \underline{q}_i = \sum_i \underline{r}_i \times \left(m_i \dfrac{d\underline{r}_i}{dt}\right) \end{cases}$

since . considering the R2BP we can assume that m_1 is not moving

Hp^1 . There are no external forces on the system.

$$\frac{d\underline{\Gamma}}{dt} = \underline{0} \quad , \quad \underline{\Gamma} = \underline{R}_2 \times \left(m_2 \frac{d\underline{R}_2}{dt}\right) + \underline{R}_1 \times \left(m_1 \overbrace{\frac{d\underline{R}_1}{dt}}^{=0}\right)$$

Hp^3 . Assuming that m_1 is not moving:

- $m_1 >> m_2 \Rightarrow$ the c.m. is located in m_1
- no external forces.

$$(m_1>>m_2) \overset{2^{Hp}}{\Rightarrow} \begin{cases} \frac{d\underline{Q}}{dt} = \frac{d(m_1 \dot{\underline{R}}_{cm})}{dt} = \underline{0} \overset{2^{Hp}}{\Rightarrow} \ddot{\underline{R}}_{cm} = \underline{0} \\ \underline{R}_{cm} \cong \underline{R}_1 \qquad \boxed{\ddot{\underline{R}}_1 = \underline{0}} \Leftarrow 3^{Hp} \end{cases}$$

!! $(Hp^1, Hp^2) \Rightarrow Hp^3 \Rightarrow$ m_1 is not moving

$\Rightarrow$ the c.m. of mass (and even the center of the new reference system) is located in m_1 position !!

$$\frac{d}{dt}\left[\underline{r} \times \left(m_2 \frac{d\underline{r}}{dt}\right)\right] = \frac{d\underline{\Gamma}_2}{dt}$$

$$\frac{d\underline{\Gamma}_2}{dt} = \frac{d\underline{r}}{dt} \times \cancel{\left(m_2 \frac{d\underline{r}}{dt}\right)}^{=0} + \underline{r} \times \left(m_2 \frac{d^2\underline{r}}{dt^2}\right)$$

From the Ist vectorial equation was obtained:

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3}\underline{r}$$

$$\Rightarrow \frac{d\underline{\Gamma}_2}{dt} = \underline{r} \times \left(-\frac{m_2\mu}{\|\underline{r}\|^3}\underline{r}\right) = \underline{0} \qquad \text{c.v.d.}$$

$$\underline{h} = \underline{r} \times \underline{v} = \underline{r} \times [\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}]$$

$$= \cancel{\underline{r} \times \dot{r}\hat{r}}^{=0} + r^2\dot{\theta}\,\hat{r} \times \hat{\theta}$$

$$\Downarrow$$

$$\underline{h} = r^2\dot{\theta}\,(\hat{r} \times \hat{\theta}) = r^2\dot{\theta}\,\hat{h}$$

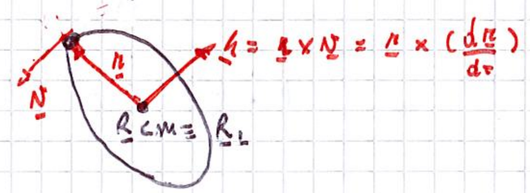
Ist integral of motion ①

$$\underline{h} \triangleq \frac{\underline{\Gamma}}{m_2} \quad , \quad \underline{h} = \underline{r} \times \frac{d\underline{r}}{dt} \Rightarrow \frac{d\underline{h}}{dt} = \underline{0}$$

$$\Rightarrow h = r^2\dot{\theta}$$

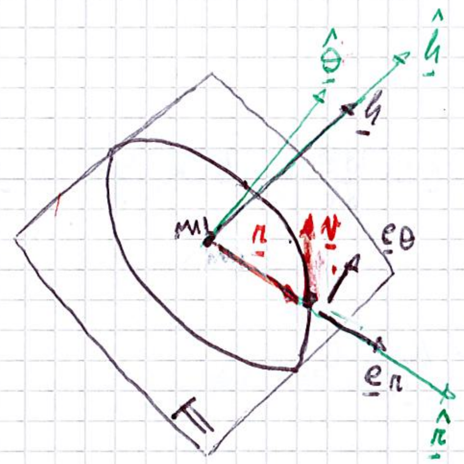
" it's proved that if m_1 is not moving $\Rightarrow$ the center of mass of the whole system is located in $m_1 \Rightarrow \underline{h}$ is a constant for the motion. "

!! The consequence of $\underline{h}$ = const is that at each time position ($\underline{r}$) and speed ($\underline{v} = \frac{d\underline{r}}{dt}$) will always be contained in the same plane normal to $\underline{h}$



!!

↳ Introducing a new reference system:

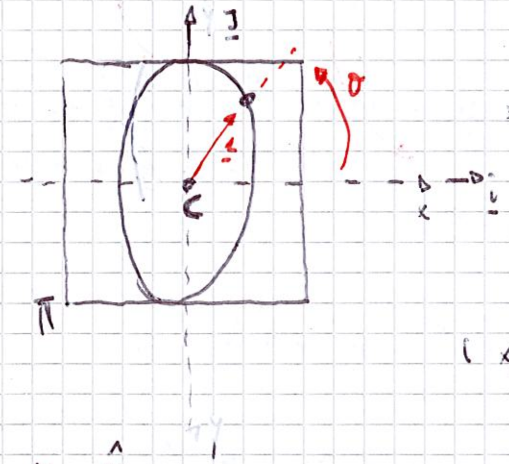


(As sketched: $\underline{v} \to$ arbitrary (tangent only if circular) $e_\theta \to$ built ortogonal to e_r.)

OSS: The systems: $\{\hat{I}, \hat{J}, \hat{K}\}$, $\{\hat{x}, \hat{y}, \hat{z}\}$ are inertial.
(located in CM)

$\{\hat{\theta}, \hat{r}, \hat{h}\}$ is not inertial.

$\Rightarrow$ considering what happens in Π-plane. (normal to $\underline{h}$)



$\Rightarrow \underline{r} = \|\underline{r}\| \cdot (\cos\theta\,\hat{I} + \sin\theta\,\hat{J})$

$= r \cdot \hat{r}$

(x, y) $\to$ arbitrary axis (at this stage)

$\underline{r} = r \cdot \hat{r}$

$$\Rightarrow \frac{d\underline{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \qquad (\hat{\theta} = -\sin\theta\,\hat{I} + \cos\theta\,\hat{J})$$

1. Calculating the quantity: $h \times v$, and deriving it later:

$$h = r \times v = r \times \frac{dr}{dt}$$

$$v = \frac{dr}{dt} \quad \Rightarrow \quad h \times v = \left[r \times \frac{dr}{dt} \right] \times \frac{dr}{dt}$$

$$a \times b \times c = (a \cdot c)\, b - (a \cdot b)\, c \quad \text{— prop}^1$$

$$\frac{d}{dt}[h \times v] = \underbrace{\frac{dh}{dt}}_{=0} \times v + h \times \frac{dv}{dt} = h \times \frac{d^2 r}{dt^2}$$

Always known that: $\frac{d^2 r}{dt^2} = -\frac{\mu}{\|r\|^3}\, r$

$$\frac{d}{dt}[h \times v] = -h \times \frac{\mu}{\|r\|^3}\, r \quad (\text{cause } h = r \times v \Rightarrow h \perp r)$$

$$= -r \times v \times \frac{\mu}{\|r\|^3}\, r \quad \text{prop}^1$$

$$= \left[-\frac{\mu}{\|r\|^3}\|r\|^2 \right] v + \left[\frac{\mu}{\|r\|^3}\, r \cdot v \right] r$$

$$r = r \cdot \hat{r}, \qquad v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\frac{d}{dt}[h \times v] = -\frac{\mu}{\|r\|}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) + \frac{\mu}{\|r\|^3}\left[r \cdot \dot{r}\hat{r} + \underbrace{r \cdot r\dot{\theta}\hat{\theta}}_{=0} \right] \cdot r\hat{r}$$

$$= -\frac{\mu}{\|r\|}\left[\cancel{\dot{r}\hat{r}} + r\dot{\theta}\hat{\theta} - \cancel{\dot{r}\hat{r}} \right]$$

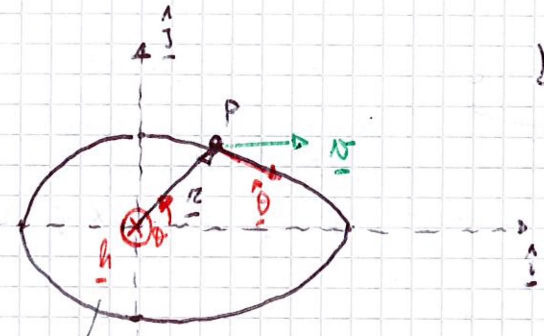
$$= -\frac{\mu}{\|r\|}\, r\dot{\theta}\hat{\theta} = -\mu\dot{\theta}\hat{\theta}$$

$$\Rightarrow \frac{d}{dt}[h \times v] = -\mu\dot{\theta}\hat{\theta}$$

But: $\hat{r} = \cos\theta\, \hat{i} + \sin\theta\, \hat{j} \rightarrow \frac{d\hat{r}}{dt} = [-\sin\theta\, \hat{i} + \cos\theta\, \hat{j}]\,\dot{\theta} = \dot{\theta}\hat{\theta}$

Follows that: $\frac{d}{dt}[h \times v] = -\mu \frac{d\hat{r}}{dt}$

Now:



For construction θ is growing apart from the motion of P:

$h \otimes \theta$, $h \odot \theta$ (sketched)

h entering the plane of the orbit: $h = r \times v$ → RIGHT HAND RULE

$$\Rightarrow \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta} \text{ if } \theta \text{ follows the motion } \xrightarrow[\text{CONSTRUCTION}]{\text{GEOM.}} \frac{d\hat{r}}{dt} = -\dot{\theta}\hat{\theta}$$

So: $\frac{d}{dt}[h \times v] = +\mu \frac{d\hat{r}}{dt} \Rightarrow \frac{d}{dt}\left[h \times v - \mu \frac{d\hat{r}}{dt}\right] = 0$

1st integral of motion (2)

$$\mu e \triangleq h \times v - \mu \hat{r}, \qquad \frac{d}{dt}[h \times v - \mu \hat{r}] = 0$$

$$\Downarrow$$

$$e = \text{const}\,|_t$$

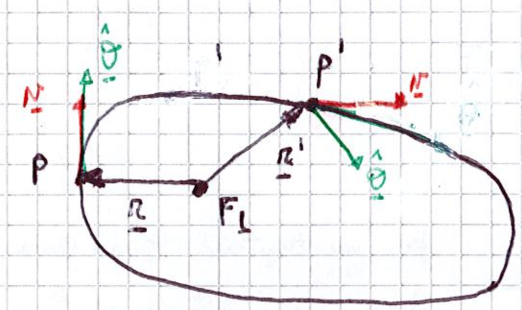
... About the meaning of vector e

$$\mu e = h \times v - \mu \hat{r}$$

- μe → constant vector in time for
  - MODULUS
  - DIRECTION.

$\Rightarrow \quad \underline{h} \times \underline{v} = r^2\dot{\theta} [\, \dot{r}\ \hat{\underline{h}} \times \hat{\underline{r}} + r\dot{\theta}\ \hat{\underline{h}} \times \hat{\underline{\theta}} \,]$

!! $\underline{h} \times \underline{v}$ → contained in the plan of the orbit BUT NOT directed as $\hat{\underline{r}}$. !!



$\hat{\underline{\theta}} \perp \hat{\underline{r}} \Rightarrow$ Tangent to the orbit

$\Rightarrow$ at the pericenter.

(IT) !! AL PERICENTRO $\underline{v}_p // \hat{\underline{\theta}} \Rightarrow (\underline{h}_p \times \underline{v}_p) // \hat{\underline{r}}$

$\Rightarrow$ POICHÉ $\mu\underline{e} = \underline{h} \times \underline{v} - \mu\hat{\underline{r}}$ È COSTANTE IN MODULO E DIREZIONE $\Rightarrow \underline{e}$ è sempre diretto dal fuoco al pericentro.

$\Rightarrow$ At the PERICENTRE is the only point $\underline{v} // \hat{\underline{\theta}} \Rightarrow \underline{h} \times \underline{v} // \underline{h} \times \hat{\underline{\theta}}$

$\Downarrow$

$\underline{h} \times \underline{v}$ directed as $\hat{\underline{r}}$

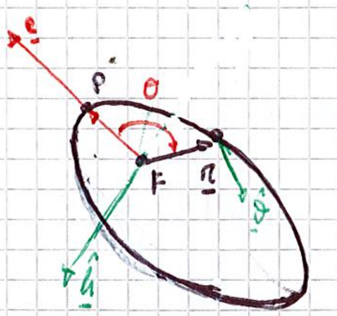
$\Rightarrow \quad \hat{\underline{e}}$ constant vector with:
- MODULUS: $\mu e = [\, h v_p - \mu \,]$
- DIRECTION AND VERSE: FROM THE FOCUS TO THE PERICENTER

↳ Pre-multiplying $\underline{e}$ vector with $\underline{r}$: (Will lead to relation $r = r(\theta)$)

$$\mu\underline{e} = \underline{h} \times \underline{v} - \mu\hat{\underline{r}}$$

$$\underline{r} \cdot \mu\underline{e} = \underline{r} \cdot [\underline{h} \times \underline{v}] - \mu\, \underline{r} \cdot \hat{\underline{r}}$$

$$\mu e \cos\theta = \underline{r} \cdot [\underline{h} \times \underline{v}] - \mu r \quad (i)$$



$\Rightarrow$ Now θ is defined as the angle between the pericenter and the focus.

$\Rightarrow \mu\underline{e} \cdot \underline{r} = \mu e \cos\theta.$

↳ $\underline{h} \times \underline{v} = h\, [\, \dot{r}\, (\hat{\underline{h}} \times \hat{\underline{r}}) + r\dot{\theta}\, (\hat{\underline{h}} \times \hat{\underline{\theta}}) \,]$

$\Downarrow \qquad \|\underline{h}\| = r^2\dot{\theta}$

$$\underline{h} \times \underline{v} = r^2\dot{\theta}\, [\, + \dot{r}\, \hat{\underline{\theta}} + r\dot{\theta}\, \hat{\underline{r}} \,]$$

Same as before $\Rightarrow \underline{h}$ entering the orbit

$\Rightarrow \hat{\underline{h}} \times \hat{\underline{\theta}} = -\underline{r}$

(i) $\quad \mu e \cos\theta = \underline{r} \cdot [\underline{h} \times \underline{v}] - \mu\hat{\underline{r}}$

$$\mu e \cos\theta = \underline{r} \cdot \underbrace{r^2\dot{\theta}\, \dot{r}\, \hat{\underline{\theta}}}_{=0} - \underline{r}\, \mu\hat{\underline{r}} + \underline{r}\, r^3\dot{\theta}^2\, \hat{\underline{r}}$$

$$\mu e \cos\theta = r^4\dot{\theta}^2 - \mu r$$

$$\mu e \cos\theta = h^2 - \mu r \quad \longrightarrow \quad r(1 + e\cos\theta) = h^2/\mu.$$

↳ Law of motion $r = r(\theta)$

$$r(\theta) = \frac{h^2/\mu}{1 + e\cos\theta} \triangleq \frac{p}{1 + e\cos\theta}$$

$p \to$ distance (r) when $\theta = \frac{\pi}{2}$

$$r\left(\tfrac{\pi}{2}\right) = \frac{h^2/\mu}{1 + e\cos\left(\frac{\pi}{2}\right)} = p$$

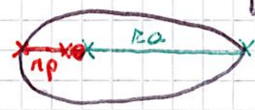
!! Problem of motion is far to be fully solved, since we know the relation $r = r(\theta)$ but we are not able to solve problem $\theta = \theta(t)$

$\Rightarrow$ ($\underline{v} = \underline{v}(t)$, $\underline{r} = \underline{r}(t)$) are not solved !!

↳ LAW OF MOTION $\Rightarrow$ MEANING OF $\|\underline{e}\|$ $\Rightarrow$ SHAPE OF THE ORBIT.

- $e = 0 \quad \Rightarrow \quad r(\theta) = \dfrac{h^2/\mu}{1+0} = \text{const}\big|_\theta \quad \Rightarrow$ CIRCONFERENCE.

- $0 < e < 1 \quad \Rightarrow \quad r_{min} = r(\theta = 0) = \dfrac{h^2/\mu}{1+e} = r_p \quad \Rightarrow$ ELLIPSE

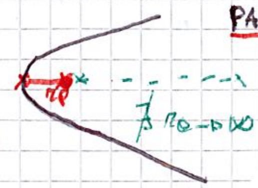


$r_{MAX} = r(\theta = \pi) = \dfrac{h^2/\mu}{1-e} = r_a, \quad 0 < r_a < \infty$!! FINITE !!

- $e = 1 \quad \Rightarrow \quad r_{min} = r(\theta = 0) = h^2/2\mu = r_p$

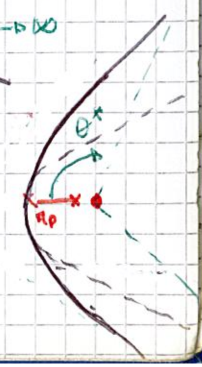
$r_{MAX} = r(\theta = \pi) = \dfrac{h^2/\mu}{0} \to r_{MAX} \to \infty$

PARABOLA



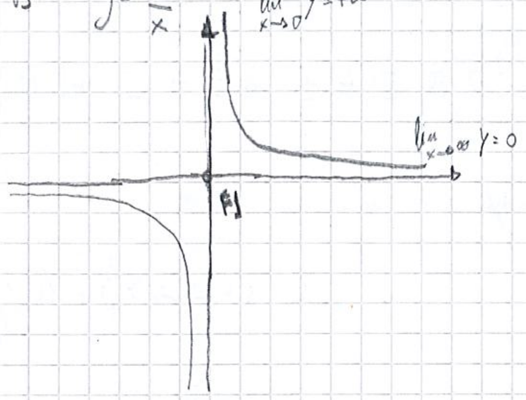
- $e > 1 \quad \Rightarrow \quad r_{min} = r(\theta = 0) = \dfrac{h^2/\mu}{1+e} = r_p$

HYPERBOLA



$\exists\ r_\infty$ for θ^*: $1 + e\cos(\theta^*) = 0, \quad \frac{\pi}{2} < \theta^* < \frac{3}{2}\pi$

Equation of Hyperbola is $y = \frac{a}{x}$ $\quad \lim_{x\to 0} y = +\infty$



III^{rd} → Energy conservation.

$T + V = \text{const} \Rightarrow \frac{d(T+V)}{dt} = \text{const}$,
$\frac{dT}{dt} + \frac{dV}{dt} = \text{const}$.

$\begin{cases} T = \frac{1}{2}\|v\|^2 & \text{(KINETIC)} \\ V = -\frac{\mu}{\|r\|} & \text{(POTENTIAL)} \end{cases}$

↳ $\frac{d\|v\|^2}{dt} = \frac{d}{dt}[v \cdot v] = v \cdot \frac{dv}{dt} + \frac{dv}{dt}\cdot v = 2\, v \cdot \frac{dv}{dt} = 2 v \frac{d^2 r}{dt^2}$

For the R2BP: $\frac{d^2 r}{dt^2} = -\mu \frac{r}{\|r\|^3}$

$\Rightarrow \frac{d\|v\|^2}{dt} = -2\mu \frac{r}{\|r\|^3}\cdot v \Rightarrow \frac{1}{2}\frac{d\|v\|^2}{dt} + \mu\frac{r\cdot v}{\|r\|^3} = 0$ (*)

↳ $\frac{d}{dt}\left(\frac{\mu}{\|r\|}\right) = \frac{d}{dt}\left[\mu\,(r\cdot r)^{-\frac{1}{2}}\right] = -\frac{1}{2}(r\cdot r)^{-\frac{3}{2}}\left[r\cdot\frac{dr}{dt} + \frac{dr}{dt}\cdot r\right]\mu$.

$= -\frac{r\cdot v}{(\|r\|^2)^{3/2}}\mu = \mu\frac{r}{\|r\|^3}\cdot v$ (*)

(*) $\Rightarrow \frac{1}{2}\frac{d\|v\|^2}{dt} + \mu\frac{r\cdot v}{\|r\|^3} = 0 \longrightarrow \frac{d}{dt}\left[\frac{1}{2}\|v\|^2 + \mu\frac{r}{\|r\|^3}\right] = 0$

I^{st} integral of motion (3)

$\frac{d\varepsilon}{dt} = 0, \quad \varepsilon \triangleq \frac{1}{2}\|v\|^2 - \frac{\mu}{\|r\|^2}, \quad \varepsilon = T + V = -\frac{\mu}{2a}$ *

Since $\varepsilon = \frac{1}{2}\|v\|^2 - \frac{\mu}{\|r\|}$

important pericenter-semiaxis relation.

$r = \frac{p}{1 + e\cos\theta}$ $\quad \begin{cases} r_p \triangleq \frac{p}{1+e} = r(\theta = 0) \\ r_a \triangleq \frac{p}{1-e} = r(\theta = \frac{\pi}{2}) \end{cases}$ → (1) $r_a = p/1-e$, $r_p = p/1+e$

$a \triangleq \frac{r_p + r_a}{2}$ (ellips. $a \to$ semiaxis. Hyp: $a < 0$; circ.: $a \to$ constant; Par: $a \to \infty$)

$\Rightarrow a = \frac{p}{2}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{p}{2}\left(\frac{2+e-e}{(1-e^2)}\right)$ → (2) $p = a(1-e^2)$

$(p = h^2/\mu)$

$a = \frac{p}{1-e^2} = \frac{h^2/\mu}{1-e^2}$

CALCULATING ε AT THE PERICENTER:

$r_p = \frac{p}{1+e} = \frac{a(1-e^2)}{1+e} = a(1-e)$ → (3) $r_p = a(1-e)$, $r_a = a(1+e)$

at the pericenter $v \parallel \hat{\theta}$ (2)

* $v_p^2 = \frac{h^2}{r_p^2} = \frac{\mu\, a(1-e^2)}{r_p^2} \Rightarrow v_p^2 = \mu\frac{a(1-e^2)}{a^2(1-e)^2} = \mu\frac{(1+e)\cancel{(1-e)}}{a(1-e)\cancel{(1-e)}}$

→ (4) $v_p^2 = \frac{\mu}{a}\frac{(1+e)}{(1-e)}$

$\Rightarrow \varepsilon = \varepsilon_p = \text{const}$.

$\varepsilon_p = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} = \frac{1}{2}\frac{\mu}{a}\frac{(1+e)}{(1-e)} - \frac{\mu}{a(1-e)}$

$= \frac{\mu}{a(1-e)}\left[\frac{1}{2}(1+e) - 1\right] = \frac{\mu}{a(1-e)}\left[\frac{1}{2} - 1 + \frac{e}{2}\right]$

$= -\frac{\mu}{\cancel{a(1-e)}}\frac{1}{2}\cancel{(1-e)} = -\frac{\mu}{2a}$ *

⇒ Circ, Ellipse: $e < 1 \to a = \frac{p}{1-e^2} > 0 \Rightarrow \varepsilon < 0$ (ATTRACTIVE!)

Par: $e = 1 \to a = \frac{p}{1-e^2} = \infty \Rightarrow \varepsilon = 0$

Hyp: $e > 1 \to a = \frac{p}{1-e^2} < 0 \Rightarrow \varepsilon > 0$ (REPULSIVE)

Syntesis of first integrals of motion

I) $\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3}\underline{r}$ ⇒ 1^{st} cardinal law

($\frac{d\underline{Q}}{dt} = \underline{0}$ → no external forces out of the system (m_1, m_2))

II) $\frac{d\underline{\Gamma}_2}{dt} = \underline{r} \times (m_2 \frac{d\underline{r}}{dt})$ ⇒ 2^{nd} cardinal law

($\frac{d\underline{\Gamma}}{dt} = \frac{d(\underline{\Gamma}_1 + \underline{\Gamma}_2)}{dt} = \underline{0}$ → no external moments acting on the system)

III) $T_2 + V_2 = \text{const}$. ⇒ Energy conservation.

↓

constants of motion: (1^{st} integrals of motion)

1) $\underline{h} \triangleq \frac{\underline{\Gamma}_2}{m_2}$ → $\underline{h} = \underline{r} \times \frac{d\underline{r}}{dt}$ ⇒ $\frac{d\underline{h}}{dt} = \underline{0}$

2) $\mu\underline{e} \triangleq \underline{h} \times \underline{v} - \mu\hat{\underline{r}}$ ⇒ $\frac{d\underline{e}}{dt} = \underline{0}$

$\underline{e}$ → directed from the focus to the pericenter. (constant × modulus, direction)

3) $\mathcal{E} = -\frac{\mu}{2a}$ ⇒ $\frac{d\mathcal{E}}{dt} = 0$.

ORBIT	ECCENTRICITY	ENERGY	SEMIAXIS
CIRCULAR	$e = 0$	$\mathcal{E} < 0$ (ATT)	$a = \frac{p}{2}$ $(a > 0)$
ELLIPS	$0 < e < 1$	$\mathcal{E} < 0$ (ATT)	$a > 0$
PARABOLA	$e = 1$	$\mathcal{E} = 0$	$a \to \infty$
HYPERBOLA	$e > 1$	$\mathcal{E} > 0$ (REP)	$a < 0$

- $\mathcal{E} = -\frac{\mu}{2a}$
- $r = \frac{p}{1 + e\cos\theta}$
- $a = \frac{p}{1 - e^2}$

!? $v = v(\theta)$?? → calculation of velocity only as function of θ. ⇒ REALLY IMPORTANT.

$$\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} \triangleq v_r \cdot \hat{\underline{r}} + v_\theta \cdot \hat{\underline{\theta}}.$$

$r = \frac{p}{1 + e\cos\theta}$ ⇒ $\frac{dr}{dt} = p\,\frac{0 - e(-\sin\theta)\dot{\theta}}{(1 + e\cos\theta)^2} = +\frac{p e \sin\theta\,\dot{\theta}}{(1 + e\cos\theta)^2}$

|| $r = \frac{p}{1 + e\cos\theta}$

$v_r = \frac{dr}{dt} = \frac{r e \sin\theta}{(1 + e\cos\theta)}\dot{\theta} = \frac{e\sin\theta}{(1 + e\cos\theta)} v_\theta$.

|| $v_\theta = r\dot{\theta}$

At the pericenter, $v_r = 0$ in fact $\sin(\theta = 0) = 0$

anyway:

$h = \text{const}$ ⇒ $h = r^2\dot{\theta}$ → $\dot{\theta} = \frac{h}{r^2}$

|| $r^2 = \frac{p^2}{(1 + e\cos\theta)^2}$

⇒ • $v_r = \frac{dr}{dt} = \frac{p e \sin\theta\,\dot{\theta}}{(1 + e\cos\theta)^2} = \frac{1}{p}\frac{p^2}{(1 + e\cos\theta)^2} e\sin\theta \cdot \frac{h}{r^2}$

$= \frac{h}{p} e \sin\theta$

• v_θ: $\dot{\theta} = \frac{h}{r^2}$ → $r\dot{\theta} = \frac{h}{r}$; $v_\theta = r\dot{\theta}$ ⇒ $v_\theta = \frac{h}{r} = \frac{h}{p}(1 + e\cos\theta)$

Vector velocity

$\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$ → $\underline{v} = \frac{h}{p}[e\sin\theta\,\hat{\underline{r}} + (1 + e\cos\theta)\hat{\underline{\theta}}]$

so:

$\underline{v} = \sqrt{\frac{\mu}{p}}[e\sin\theta\,\hat{\underline{r}} + (1 + e\cos\theta)\hat{\underline{\theta}}]$

($p = \frac{h^2}{\mu}$ → $h = \sqrt{\mu p}$ → $\frac{h}{p} = \sqrt{\frac{\mu}{p}}$)

... solving integral in time ⇒ finding $\theta = \theta(t)$

Procedure.

$$\begin{cases} \|h\| = r^2(\theta)\,\dot\theta = \text{const}|_t \\ r = \dfrac{p}{1+e\cos\theta} = \dfrac{h^2/\mu}{1+e\cos\theta} \end{cases} \Rightarrow \cancel{h} = \frac{h^4/\mu^2}{(1+e\cos\theta)^2}\frac{d\theta}{dt}$$

$$\Rightarrow h^3 \frac{d\theta}{dt} = \mu^2 (1+e\cos\theta)^2 \Rightarrow \int_{t_0}^{t^*} \frac{\mu^2}{h^3}\,dt = \int_{\theta_0}^{\theta^*} \frac{1}{(1+e\cos\theta)^2}\,d\theta.$$

!! the integral: $\int_{\theta_0}^{\theta^*} \frac{1}{(1+e\cos\theta)^2}\,d\theta$ is solvable in closed form only for $e = (0;1)$!!

a) circular orbit. $(e=0)$

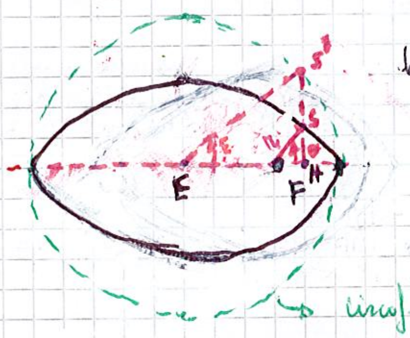
$$\int_{t_0}^{t^*} \frac{\mu^2}{h^3}\,dt = \int_{\theta_0}^{\theta^*} 1\,d\theta \Rightarrow \Delta\theta = \frac{\mu^2}{h^3}\Delta t. \Rightarrow \boxed{\theta(t) = \frac{\mu^2}{h^3}t}$$

⇒ From the relations:

- $\theta(t) = \frac{\mu^2}{h^3}t$
- $r(\theta) = \frac{p}{1+e\cos\theta} = \frac{h^2/\mu}{1+e\cos\theta}$, $r = \text{const}|_\theta$
- $\underline{v} = [\,e\sin\theta\,\hat{\underline{\theta}} + (1+e\cos\theta)\,\hat{\underline{r}}\,]$

⇒ Everything $\begin{cases} v = v(t) \\ r = r(t) \end{cases}$ is known for this system. TIME-LAW IS KNOWN.

b) Elliptic orbit $(0 < e < 1)$



obtained as the projection of s on the circumference of radius a.

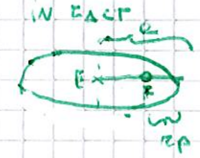
vertex of the ellipse.

⇒ circumference of radius (a) centered in the center of the ellipse.

⇒ $\overline{EH} = a\cos E$ + $\overline{FE} = e\,a.$

$\overline{FH} = r\cos\theta$

IN FACT



$\Rightarrow FE = a - r_p$

But

$r_p = r(\theta=0) = \frac{p}{1+e} = \frac{a(1-e^2)}{1+e} = a(1-e)$

$\Rightarrow \overline{FE} = a - (a - ae) = \cancel{a} - \cancel{a} + ae$

↳ (i) $\overline{EH} = \overline{EF} + \overline{FH} \Rightarrow a\cos E = r\cos\theta + ea$

$a\cos E = r\cos\theta + ea$

$a\cos E = \frac{p\cos\theta}{1+e\cos\theta} + ea \qquad \| \; p = a(1-e^2)$

$\cos E = \frac{(1-e^2)\cos\theta}{1+e\cos\theta} + e \rightarrow \cos E = \frac{\cos\theta - \cancel{e^2\cos\theta} + e + \cancel{e^2\cos\theta}}{1+e\cos\theta} = \frac{e+\cos\theta}{1+e\cos\theta}.$

↳ (ii) $\cos E = \frac{e+\cos\theta}{1+e\cos\theta}. \Rightarrow \sin E = \sqrt{1-\cos^2 E}$

$$\sin E = \sqrt{\frac{1+e^2\cos^2\theta + \cancel{2e\cos\theta} - e^2 - \cos^2\theta - \cancel{2e\cos\theta}}{(1+e\cos\theta)^2}} = \sqrt{\frac{1-e^2+\cos^2\theta(e^2-1)}{(1+e\cos\theta)^2}}$$

$$= \sqrt{\frac{(1-e^2)[1-\cos^2\theta]}{(1+e\cos\theta)^2}} = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}$$

↳ (iii) $\sin E = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}. \Rightarrow \tan E = \frac{\sin E}{\cos E}$!!Nooo!! $\tan\frac{E}{2} = \frac{\sin E}{1+\cos E}$

$\cancel{\tan\frac{E}{2} = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \cdot \frac{1+e\cos\theta}{e+\cos\theta} = \frac{\sqrt{1-e^2}\sin\theta}{e+\cos\theta}}$

$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$!

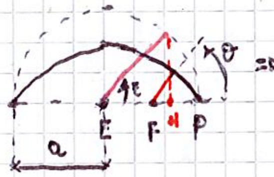
$\tan\frac{E}{2} = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} \Big/ \left(1 + \frac{e+\cos\theta}{1+e\cos\theta}\right) = \frac{\sqrt{1-e^2}\sin\theta}{\cancel{1+e\cos\theta}} \cdot \frac{\cancel{1+e\cos\theta}}{1+e(1+\cos\theta)+\cos\theta}$

$$\Rightarrow \tan\frac{E}{2} = \frac{\sqrt{1-e^2}\,\sin\theta}{1+e\cos\theta+\cos\theta+e} = \frac{\sqrt{1-e^2}\,\sin\theta}{(1+e)\,(1+\cos\theta)} = \frac{\sqrt{1-e^2}}{(1+e)}\tan\left(\frac{\theta}{2}\right)$$

$$= \sqrt{\frac{1-e^2}{(1+e)^2}}\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{(1-e)\cancel{(1+e)}}{\cancel{(1+e)}(1+e)}}\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)$$

relations $\theta \leftrightarrow E$

$$\sin E = \frac{\sqrt{1-e^2}\,\sin\theta}{1+e\cos\theta}, \quad \cos E = \frac{e+\cos\theta}{1+e\cos\theta}, \quad \tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}$$



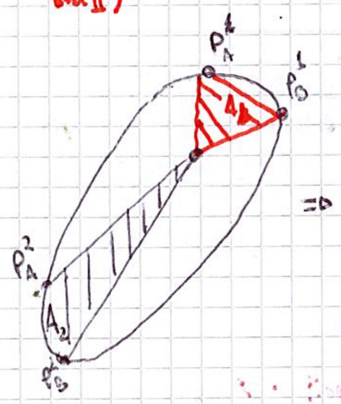
$$\Rightarrow \begin{cases} \overline{EH} = a\cos E \\ \overline{EF} = ea; \quad \overline{FH} = r\cos\theta \end{cases} \Rightarrow \overline{EH} = \overline{EF} + \overline{FH}$$

Kepler's laws

I) Planet's orbits are ellips with sun in one of the 2 fouses.

II) the radius intercepting the planet and the sun covers equal Areas in equal time gaps

dim II)



$\Rightarrow$ IF $A_1 = A_2$ THEN $\Delta t_{P_A^1 - P_B^1} = \Delta t_{P_A^2 - P_B^2}$

circular sector

$$A = \pi r^2 \frac{\theta}{2\pi} \rightarrow dA = \frac{r^2}{2}d\theta$$

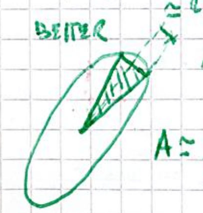
ellips sector $\Rightarrow$ $r = r(\theta)$

$$A \simeq \cancel{\pi} r^2 \frac{\theta}{2\cancel{\pi}} = \frac{1}{2}r^2\theta \rightarrow dA = \frac{2A}{2\theta}\,d\theta$$

$$\rightarrow dA = \frac{1}{\cancel{2}}\cancel{2}r\frac{dr}{d\theta}\,\theta\,d\theta + \frac{1}{2}r^2 d\theta = r\theta\,dr + \frac{1}{2}r^2 d\theta.$$

RIGHT ONE

BETTER



$\simeq r\sin\theta$

$A \simeq$ triangular

$A \simeq \frac{1}{2}\,r\cdot r\sin\theta$ $(\sin\theta \simeq \theta)$

$dA \simeq \frac{1}{2}r^2 d\theta.$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \rightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot\theta$$

BUT $\underline{h} = \underline{r}\times\underline{v} = \underline{r}\times(\dot r\,\hat{\underline{r}} + r\dot\theta\,\hat{\underline{\theta}}) = r^2\dot\theta\,\hat{\underline{h}}$

SINCE $\begin{cases} \frac{dA}{dt} = \frac{1}{2}h \\ h = \text{const}|_t \end{cases}$ THEN $\Delta A = \left(\frac{1}{2}h\right)\cdot\Delta t.$

III) The square of the period of revolution is proportional to the cube of major semiaxis:

dim III)

$$\frac{dA}{dt} = \frac{1}{2}h$$

Ellips:

$A = a\cdot b\cdot\pi$

$b = a\sqrt{1-e^2}$

$$\begin{cases} A = \pi\,2ab \\ \Delta A = \frac{1}{2}h\cdot\Delta t \end{cases} \Rightarrow \frac{\pi\,2ab}{T} = h \Rightarrow \frac{\pi\,2\cdot ab}{T} = \sqrt{p\,\mu}$$

$r = \frac{h^2/\mu}{1+e\cos\theta} \rightarrow h = \sqrt{p\,\mu}$

$p = h^2/\mu = \frac{h^2/\mu}{1+e\cos(\theta=\pi/2)}$

$$\Rightarrow \frac{2\pi ab}{T} = \sqrt{a(1-e^2)}\cdot\sqrt{\mu}$$

$$\frac{2\pi ab}{T} = \sqrt{\mu}\,\frac{a}{\sqrt a}\cdot\sqrt{1-e^2} = \sqrt{\mu}\cdot\frac{b}{\sqrt a} \Rightarrow \frac{2\pi a\cancel{b}}{T} = \sqrt{\frac{\mu}{a}}\cdot\cancel{b}$$

$b = a\sqrt{1-e^2}$

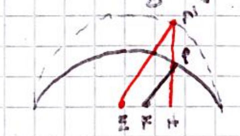
$$\Rightarrow \frac{4\pi^2}{T^2} = \frac{\mu}{a^3} \Rightarrow$$

IIIrd Kepler's law

$$T^2 = \frac{4\pi^2}{\mu}\cdot a^3 \rightarrow T^2 \propto a^3$$

a) $r = r(E)$

Starting from the relation:



$$\rightarrow \overline{EF} + \overline{FH} = \overline{EH}, \quad \begin{cases} \overline{EH} = a\cos E \\ \overline{EF} = ea \\ \overline{FH} = r\cos\theta \end{cases}$$

$$a\cos E = ea + r\cos\theta$$

BUT:

$$r = \frac{p}{1+e\cos\theta} \rightarrow p = r + er\cos\theta \rightarrow r\cos\theta = \frac{p-r}{e}$$

$\Rightarrow \quad ea + r\cos\theta = a\cos E \quad (i)$

BUT: $\begin{cases} r = \dfrac{p}{1+e\cos\theta} \\ p = a(1-e^2) \end{cases} \Rightarrow r = \dfrac{a(1-e^2)}{1+e\cos\theta}$

$\leftarrow \; a = \dfrac{r_p + r_a}{2} = \dfrac{p}{2}\left(\dfrac{1}{1+e} + \dfrac{1}{1-e}\right) = \ldots$

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \rightarrow r + re\cos\theta = a(1-e^2)$$

$$\downarrow$$

$$re\cos\theta = a(1-e^2) - r$$

$$\downarrow$$

$$r\cos\theta = \frac{a}{e}(1-e^2) - \frac{r}{e} = \frac{p-r}{e}$$

$$ea + \frac{p-r}{e} = a\cos E$$

$$-r = -e^2 a + ae\cos E + p \qquad \| \; p = a(1-e^2)$$

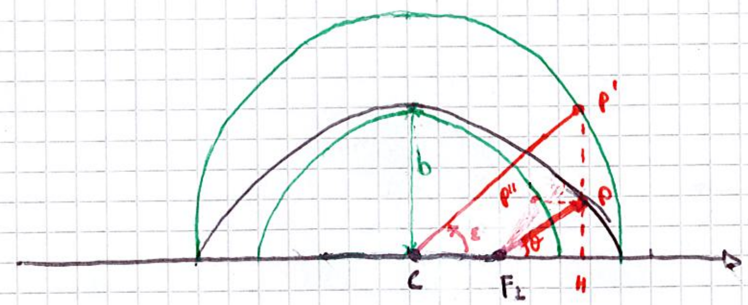
$$r = \cancel{e^2 a} - ae\cos E + a - \cancel{ae^2}$$

$$r = -ae\cos E + a.$$

$\to$ distance, as function of E.

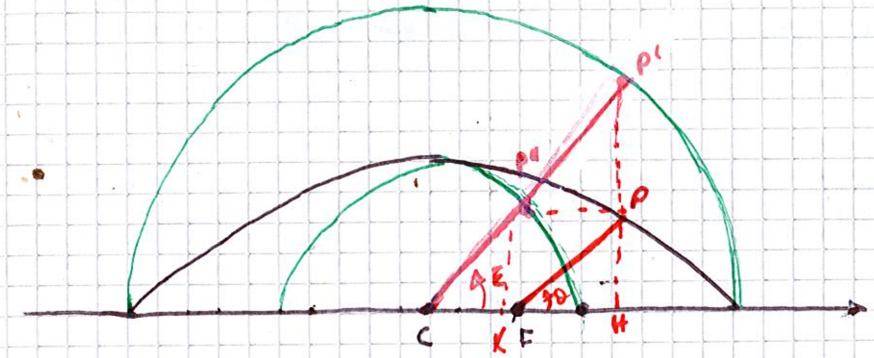
$$r = a(1-e\cos E)$$

b) $\dot{E} = \dot{E}(E)$



$\Rightarrow$ Building another circumference centered in C of radius b, then equal to the semiaxis of the ellips.

(I NEED ADDITIVE EQUATION ... TO COMPUTE VELOCITY AS FUNCTION OF E) *



$$\overline{P''K} = \overline{PH} \Rightarrow b\sin E = r\sin\theta. \rightarrow \sin\theta = \frac{b}{r}\sin E. \; * \; (ii)$$

SINCE

$\begin{cases} \underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}} \\ \underline{v} = \sqrt{\dfrac{\mu}{p}}\,[e\sin\theta\,\hat{\underline{r}} + (1+e\cos\theta)\,\hat{\underline{\theta}}] \end{cases}$ $\rightarrow$ is the velocity expressed as a function of θ.

THEN

$$r = a(1-e\cos E) \rightarrow \dot{r} = ae\sin E\,\dot{E}$$

imposing equivalence: $\quad a\cancel{e}\dot{E}\sin E = \sqrt{\dfrac{\mu}{p}}\,\cancel{e}\sin\theta$

$$\| \; (ii) \; \sin\theta = \frac{b}{r}\sin E$$

$$a\dot{E}\,\cancel{\sin E} = \sqrt{\frac{\mu}{p}}\,\frac{b}{r}\,\cancel{\sin E}$$

$$a\dot{E} = \sqrt{\frac{\mu}{p}}\,\frac{b}{r} \qquad \left\| \begin{array}{l} r = a(1-e\cos E) \\ b = a\sqrt{1-e^2} \\ p = a(1-e^2) \end{array}\right.$$

$$\cancel{a}\dot{E} = \frac{\sqrt{\mu}}{\sqrt{a}\,\cancel{\sqrt{1-e^2}}} \cdot \frac{\cancel{a}\,\cancel{\sqrt{1-e^2}}}{a(1-e\cos E)}$$

$$\dot{E} = \sqrt{\frac{\mu}{a^3}}\,\frac{1}{e(1-e\cos E)} \rightarrow$$

$\to \dot{E}(E)$

$$\dot{E} = \sqrt{\frac{\mu}{a^3}}\,\frac{1}{1-e\cos E}$$

obs

Since E is the angle between $\hat{\underline{e}}$ and the projection of P on the circumference of radius a $\Rightarrow$

- AUXILIARY CIRCONFERENCE IS A CIRCONFERENCE WITH THE SAME PERIOD OF THE ORBIT SUCH THAT: !! DIFFERENT FOCUS !!
- $\dot{E}$ IS NOT CONSTANT IN TIME !! AUX. CIRC. IS NOT A REAL ORBIT !!

⇒ Proceeding with the integration:

$$\dot E = \frac{dE}{dt} = \sqrt{\frac{\mu}{a^3}}\ \frac{1}{1-e\cos E}$$

↓

$$\frac{dE}{dt} = \sqrt{\frac{\mu}{a^3}}\ \frac{1}{1-e\cos E}$$

↓

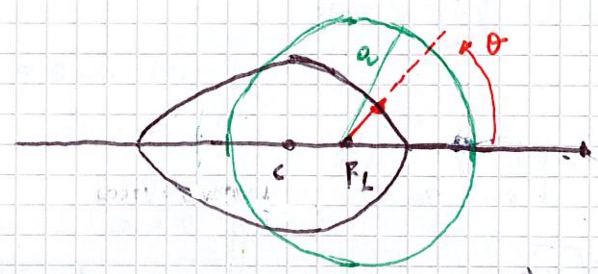
$$\int_{t_1}^{t_2}\sqrt{\frac{\mu}{a^3}}\,dt = \int_{E_1}^{E_2} 1 - e\cos E\ dE \ \rightarrow\ \sqrt{\frac{\mu}{a^3}}\,\Delta t = \left[E - e\sin E\right]_{E_1}^{E_2}$$

↓

↳ Kepler's equation

$$\sqrt{\frac{\mu}{a^3}}\,\Delta t = \left[E - e\sin E\right]_{E_1}^{E_2}$$

(IT) CONSIDERANDO UNA CIRCONFERENZA CENTRATA IN F_1 DI RAGGIO a

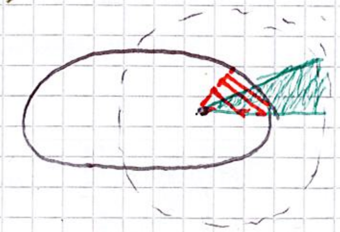


- TALE CIRCONFERENZA È • "EQUIENERGETICA" CON L'ELLISSE ($-\frac{\mu}{a} = \varepsilon$)
- HA LO STESSO PERIODO DI RIVOLUZIONE.

\+ DAL MOMENTO CHE TALE ORBITA È CIRCOLARE $\dot\theta = \sqrt{\frac{\mu}{a^3}} = \text{cost}\big|_\tau$ (for Kepler's 2nd law)

$$\left[E - e\sin E\right]_{E_1}^{E_2} \triangleq M$$

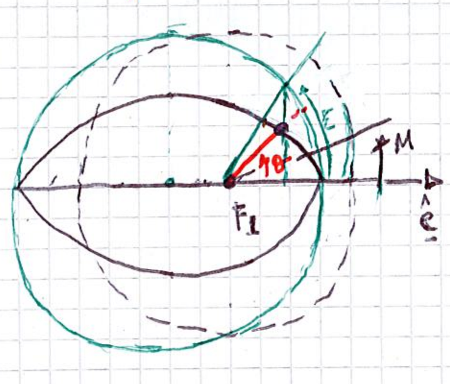
$M = \sqrt{\frac{\mu}{a^3}}\,\Delta t$ → angle covered by the planet in the same Δt on a circular centered in F_1 in the same Δt



M = θ

$A^{ellps} = A^{circ}$

The 3 Anomaly



θ → True anomaly

E → eccentric anomaly

$\begin{cases} M = E - e\sin E \rightarrow \text{Mean anomaly} \\ M = \sqrt{\frac{\mu}{a^3}}\,\Delta t \end{cases}$

FOR THE ELLIPTIC ORBIT THE PROBLEM OF TIME INTEGRATION IS SOLVED BY USING 2 COUPLED RELATION:

1) $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$ → relation { true anomaly, eccentric anomaly.

2) $E - e\sin E\Big|_{E_1}^{E_2} = \sqrt{\frac{\mu}{a^3}}\,\Delta t$ → Kepler's law

↳ Considering the relation between E and M:

$$E - e\sin E = M.$$

but obviously: $M = \sqrt{\frac{\mu}{a^3}}\,\Delta t \rightarrow t - \overset{=0}{t_0} = \frac{M}{2\pi}T$

important

↳ This equation (Kepler's equation) is not solvable directly ⇒ NEED TO USE NEWTON'S METHOD

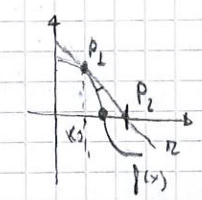
$f(E) = 0 \Rightarrow f(E) \triangleq E - e\sin E - M$

$f'(E) = 1 - e\cos E$

⇓

$$E^{i+1} = E^i - \frac{E^i - e\sin E^i - M}{1 - e\cos E^i}$$

NEWTON'S METHOD:



$m: P_1(x_0, f(x_0))$; $m = f'(x_0)$

$(x_1, 0)\ P_2$

⇓ $x_1 = ??$

$m = \frac{0 - y_0}{x_1 - x_0} \rightarrow x_1 m - x_0 m = -y_0$

$\Rightarrow x_1 = x_0 - \frac{y_0}{m}$

$\Rightarrow x^{(i)} = x^{(i-1)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$

Procedure

Given t (Known a, μ, e) ⇒ $\begin{cases} t = \frac{M}{2\pi}T \rightarrow M = \frac{2\pi t}{T} \\ T = \frac{1}{w} = \frac{1}{\dot\theta_{circ}} = \sqrt{\frac{a^3}{\mu}} \end{cases}$ ⇒ SUBSTITUTE INTO $E - e\sin E - M = f(E)$ AND SOLVE USING NEWTON ⇒ FOUND E THAN $\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$

c) Parabolic orbit ($e=1$)

For a parabolic orbit integral is soluble directly:

$$\int_{t_0}^{t^*}\sqrt{\frac{\mu}{a^3}}\,dt = \int_{\theta_0}^{\theta^*}\frac{p}{(1+\cos\theta)^2}$$

$$\cancel{\int\frac{1}{(1+\cos\alpha)^2}d\alpha = \int\frac{1}{\sin\alpha}\cdot\frac{\sin\alpha}{(1+\cos\alpha)^2}d\alpha}$$

$\mathrm{Tan}\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha}$

$$\cancel{\frac{1}{1+\cos\alpha}\cdot\frac{1}{\sin\alpha} - \int\frac{1}{1+\cos\alpha}\left(-\frac{1}{\sin^2\alpha}\cos\alpha\right)d\alpha}$$

$$\cancel{= \frac{1}{1+\cos\alpha}\cdot\frac{1}{\sin\alpha} + \int\frac{1}{1+\cos\alpha}\cdot\frac{\cos\alpha}{1-\cos^2\alpha}d\alpha}$$

$$\cancel{= \int\frac{1}{\sin\alpha}\,\mathrm{Tan}^2\left(\frac{\alpha}{2}\right)d\alpha}$$

Smart substitutions:

$$z = \mathrm{Tan}\,\frac{\theta}{2} \rightarrow dz = \left[1+\mathrm{Tan}^2\left(\frac{\theta}{2}\right)\right]\frac{1}{2}\,d\theta \rightarrow 2\,dz = \left[1+\tan^2\left(\frac{\theta}{2}\right)\right]d\theta \rightarrow d\theta = \frac{2}{[1+z^2]}\,dz \quad (i)$$

$$\mathrm{Tan}\,\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \rightarrow 1+\cos\theta = \frac{\sin\theta}{\mathrm{Tan}\frac{\theta}{2}}$$

$\sin(2x) = 2\sin x\cos x$

$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$$\rightarrow 1+\cos\theta = \frac{2\sin\left(\frac{\theta}{2}\right)\cancel{\cos\left(\frac{\theta}{2}\right)}}{\mathrm{Tan}^2\left(\frac{\theta}{2}\right)}\cdot\frac{\sin\left(\frac{\theta}{2}\right)}{\cancel{\cos\left(\frac{\theta}{2}\right)}}$$

$$\rightarrow \cos\theta = -1 + 2\sin^2\left(\frac{\theta}{2}\right)\cdot\frac{1}{\mathrm{Tan}^2\left(\frac{\theta}{2}\right)} \rightarrow \cos\theta = \frac{-\mathrm{Tan}^2\left(\frac{\theta}{2}\right) + 2\sin^2\left(\frac{\theta}{2}\right)}{\mathrm{Tan}^2\left(\frac{\theta}{2}\right)}$$

$$\rightarrow \cos\theta = \frac{-\mathrm{Tan}^2\left(\frac{\theta}{2}\right) + 1 - \left[1 - 2\sin^2\left(\frac{\theta}{2}\right)\right]}{\mathrm{Tan}^2\left(\frac{\theta}{2}\right)} = \frac{1-\mathrm{Tan}^2\left(\frac{\theta}{2}\right)}{\mathrm{Tan}^2\left(\frac{\theta}{2}\right)} - \frac{\cos\theta}{\mathrm{Tan}^2\left(\frac{\theta}{2}\right)}$$

$$\rightarrow \cos\theta\left(1+\frac{1}{z^2}\right) = \frac{1-z^2}{z^2} \rightarrow \cos\theta = \frac{1-z^2}{1+z^2} \quad (ii)$$

$$\frac{h}{p^2}\Delta t = \int_{z_0}^{z^*}\frac{1}{\left[1+\left(\frac{1-z^2}{1+z^2}\right)\right]^2}\cdot\frac{2}{(1+z^2)}\,dz$$

$$\frac{h}{p^2} = \sqrt{\frac{\mu}{a^3}}$$

$p = \frac{h^2}{\mu}$; $h = r_p v_p = \sqrt{\frac{\mu}{a}}\,(1+e)\;a(1-e) = \sqrt{\frac{\mu}{a(1-e^2)}}\cdot a(1-e^2)$

$\rightarrow p\;\frac{h}{p} = \frac{\mu}{h}$

$$\rightarrow \frac{h}{p^2}\Delta t = \int_{z_0}^{z^*}\frac{1}{\left[\frac{1+\cancel{z^2}+1-\cancel{z^2}}{1+z^2}\right]^2}\cdot\frac{2}{1+z^2}\,dz$$

$$\frac{h}{p^2}\Delta t = \int_{z_0}^{z^*}\frac{(1+z^2)^{\cancel{2}}}{\cancel{4}^2}\cdot\frac{\cancel{2}}{\cancel{1+z^2}}\,dz$$

$$2\,\frac{h}{p^2}\Delta t = \int_{z_0}^{z^*}1+z^2\,dz$$

$$2\,\frac{h}{p^2}\Delta t = z + \frac{1}{3}z^3 \qquad \Big\| \; z = \mathrm{Tan}\left(\frac{\theta}{2}\right)$$

$$\frac{h}{p^2}\Delta t = \frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \frac{1}{6}\mathrm{Tan}^3\left(\frac{\theta}{2}\right)$$

$$\boxed{\frac{h}{p^2}\Delta t = \left[\frac{1}{2}\mathrm{Tan}\left(\frac{\theta}{2}\right) + \frac{1}{6}\mathrm{Tan}^3\left(\frac{\theta}{2}\right)\right]_{\theta_0}^{\theta^*}}$$

(SOLVE USING NEWTON'S METHOD)

To determine modulus of position, procedure is the following:

$$r = \frac{p}{1+e\cos\theta} \quad \Big\downarrow e=1$$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$r = \frac{p}{1+\cos\theta} = \frac{p}{\cos^2\alpha + \cancel{\sin^2\alpha} + \cos^2\alpha - \cancel{\sin^2\alpha}} = \frac{p}{\cos^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}$$

$$r = p\left[\frac{1}{\cos^2\frac{\theta}{2}\cdot 2}\right] = \frac{p}{2}\left[\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right] = \frac{p}{2}\left(1+\mathrm{Tan}^2\frac{\theta}{2}\right)$$

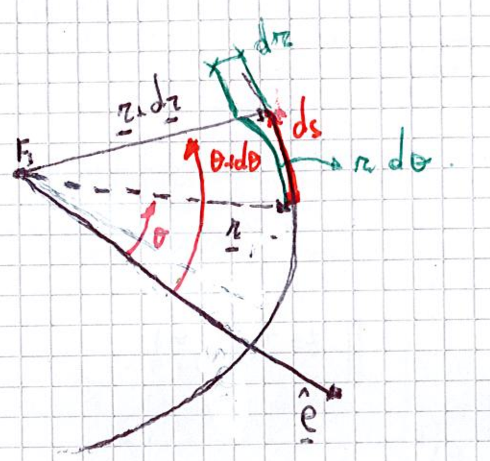
$$\boxed{r = \frac{p}{2}\left(1+\mathrm{Tan}^2\frac{\theta}{2}\right)}$$

!! this relation is dual with the one for the ellips.

$r = a(1-e\cos E)$!!

d) Hyperbolic orbit ($e>1$)

--- observation on time law. (TRAJECTORY PROBLEM)



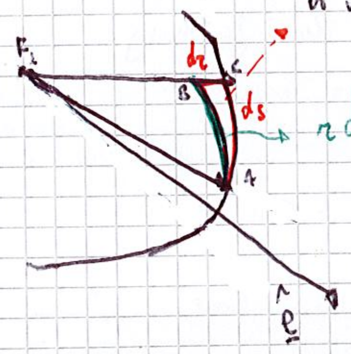
As usual, our problem is to find $\theta = \theta(t)$, in fact we can write:

$r = r(\theta(t))$

↳ a different approach can be followed calculating: $\begin{cases} s = s(t) \\ s \to \text{trajectory} \end{cases}$

$$\Rightarrow \begin{cases} s = \int ds \Rightarrow s(t) = \int \frac{ds}{dt}\, dt \\ s = s(t) \end{cases}$$

at infinitesimal quantities can be retained a triangle.* $ABC \simeq \overset{\Delta}{ABC}$ (rectum triangle)



$r\,d\theta$ because is the length of arch of a circumference ($r = \text{const}$)

in fact: $l = \underbrace{2\pi r}_{P} \cdot \frac{\theta^{rad}}{2\pi} = r\,\theta \Rightarrow dl = r\,d\theta$

A triangle *

$$\Rightarrow ds = \sqrt{(r\,d\theta)^2 + dr^2} = d\theta \cdot \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$r = \frac{p}{1+e\cos\theta} \qquad \frac{dr}{d\theta} = \frac{e p \sin\theta}{(1+e\cos\theta)^2}$$

$$\frac{ds}{dt} = \frac{d\theta}{dt}\sqrt{\frac{p^2}{(1+e\cos\theta)^2} + \frac{e p^2 \sin^2\theta}{(1+e\cos\theta)^4}}$$

$$= \frac{d\theta}{dt}\sqrt{\frac{p^2[(1+e\cos\theta)^2 + e\sin^2\theta]}{(1+e\cos\theta)^4}} = \frac{d\theta}{dt}\sqrt{\frac{p^2[1+2e\cos\theta+e^2\cos^2\theta+e\sin^2\theta]}{(1+e\cos\theta)^4}}$$

$$\frac{ds}{dt} = \dot\theta \cdot \sqrt{r^2 + \left[\frac{d}{d\theta}\left(\frac{p}{1+e\cos\theta}\right)\right]^2} \left(= \dot\theta \sqrt{\frac{p^2[1+2e\cos\theta+e^2\cos^2\theta+e\sin^2\theta]}{(1+e\cos\theta)^4}}\right)$$

⇒ IF TIME LAW IS KNOWN ($\theta = \theta(t) \to \dot\theta = \dot\theta(t)$) THEN ALSO THE LENGHT COVERED BY THE PLANET $s = \int_0^{t^*} \frac{ds}{dt}\, dt$ IS KNOWN (for a given Δt)

→ Proceeding as follow: !! remember: $a|_{hyp} < 0$ POSITIVE / REPULSIVE ENERGY !!

$$\varepsilon = -\frac{\mu}{2a}$$

$$e^2 = 1 + \frac{2p\varepsilon}{\mu} \leftarrow p = a(1-e^2) \qquad a e^2 = a - p$$

$$e^2 = 1 - \frac{p}{a} \;//\; a = -\frac{\mu}{2\varepsilon} \to \frac{1}{a} = -\frac{2\varepsilon}{\mu}$$

$$e^2 = 1 + \frac{2p\varepsilon}{\mu}$$

Doing a "strange game":

$z = a + jb \Rightarrow \bar z = a - jb$ (complex conjugate)

$$\Rightarrow a = -a' \leftarrow a + 0j = -(-a) + 0j = -(-a+0j) \triangleq -a'$$

$$b = a\sqrt{1-e^2} = -b'j \leftarrow b = a\sqrt{\underset{(>1)}{1-e^2}} = a\sqrt{e^2-1}\cdot j$$

$$\hat E = E'j \quad (E = E'j) \leftarrow E \in \mathbb{R} \Rightarrow E' = -Ej$$

$$\hat M = -M'j \quad (M = M'j) \leftarrow M = \sqrt{\frac{\mu}{a^3}}\,\Delta t \Rightarrow \dot M = \sqrt{\frac{\mu}{a^3}} \xrightarrow{a<0} M = \underbrace{\sqrt{\frac{\mu}{|a|^3}}}_{M'} \cdot \frac{1}{j}$$

But $\frac{1}{j} = \frac{1}{j}\frac{j}{j} = \frac{1}{(-1)}\cdot j = -j \Rightarrow \hat M = -M'j$

Substituting this into Kepler's equation:

$$E'j - e\sin(E'j) = -M'j \qquad (E', M') \in \mathbb{R}$$

$(E', M') \to$ needs to be defined to be used into Kepler eqn.

where: $\begin{cases} M' = \sqrt{\frac{\mu}{a'^3}} = \sqrt{\frac{\mu}{|a|^3}} \\ a = -a' \quad (a' = -a) \end{cases}$

$$\begin{cases} e^{j\alpha} = \cos\alpha + j\sin\alpha \quad (a) \\ e^{-j\alpha} = \cos\alpha - j\sin\alpha \end{cases}$$

$$\Downarrow \quad \sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \to \sin\alpha \in \mathbb{C}, \; \alpha \in \mathbb{C}$$

$$\sinh = \frac{e^{A} - e^{-A}}{2} \quad (A \in \mathbb{R}) \quad (b)$$

$$(a) \Rightarrow \sin(E'j) = \frac{e^{(E'j)j} - e^{-(E'j)j}}{2j} = \frac{e^{-E'} - e^{E'}}{2j}$$

$$(b) \Rightarrow \sin(E'j) = -\frac{\sinh(E')}{j}$$

$$\Rightarrow E'j + \frac{e\sinh(E')}{j} = -M'j$$

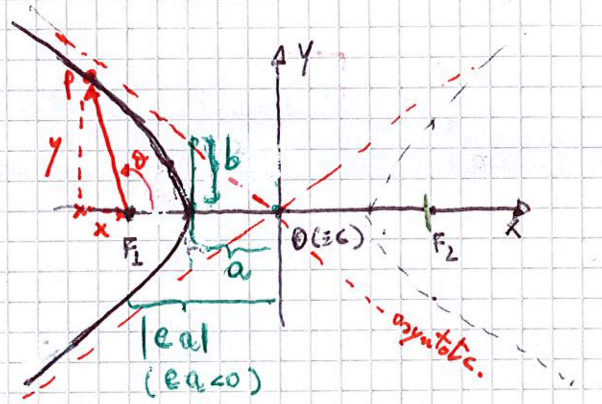
$$\Rightarrow E'j\cdot j + e\sinh(E') = -M'j\cdot j$$

$$-E' + e\sinh(E') = +M'$$

Hyperbola's relation between eccentric anomaly and mean anomaly:

$$\sqrt{\frac{\mu}{a^3}}\ \Delta t = \left[e \sinh E' - E' \right]\Big|_{E_0'}^{E'}$$

Hyperbolic geometry



The meaning of Hyperbolic eccentric anomaly (F) is : $\begin{cases} \sinh(F) = \dfrac{y}{b} \\ \cosh(F) = \dfrac{x}{a} \end{cases}$ $\quad (b = |a\sqrt{1-e^2}|)$ $(a<1,\ e>1)$ $b = -a\sqrt{e^2-1}$

Then:

$$\begin{cases} x = \cosh(F)\cdot a \\ y = \sinh(F)(-a)\sqrt{e^2-1} \end{cases} ; \quad \begin{cases} y = r\sin\theta = \dfrac{p}{1+e\cos\theta} = \dfrac{a(1-e^2)}{1+e\cos\theta} \\ x = -r\cos\theta \end{cases} \quad ; \ p = a(1-e^2)$$

$$a(1-e^2) = (-a)\cdot(e^2-1)$$

$$\Rightarrow \begin{cases} \sinh(F)\,\cancel{(-a)}\sqrt{e^2-1} = \dfrac{\cancel{a}\,(1-e^2)\sin\theta}{1+e\cos\theta} \\ \cosh(F)\,a = \dfrac{\cancel{a}\,(1-e^2)\cos\theta}{1+e\cos\theta} \end{cases} \quad \leftarrow \text{equivalence on } y$$

$$\begin{cases} \sinh(F) = \sqrt{e^2-1}\ \dfrac{\sin\theta}{1+e\cos\theta} \\ \cosh(F) = -(1-e^2)\ \dfrac{\cos\theta}{1+e\cos\theta} \end{cases} \Rightarrow \tanh\left(\frac{F}{2}\right) = \frac{\sinh F}{1+\cosh F}$$

$$\cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2} \ ; \quad \sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2} \ + \quad \tanh\left(\frac{\alpha}{2}\right) = \frac{\frac{e^{\alpha}-e^{-\alpha}}{2}}{\frac{e^{\alpha}+e^{-\alpha}}{2}+1} = \frac{e^{\alpha}-e^{-\alpha}}{2+e^{\alpha}+e^{-\alpha}}$$

$$\tanh\left(\frac{\alpha}{2}\right) = \frac{e^{2\alpha}-1}{2e^{\alpha}+e^{2\alpha}+1} = \frac{e^{2\alpha}-1}{(e^{\alpha}+1)^2} = \frac{(e^{\alpha}-1)\cancel{(e^{\alpha}+1)}}{(e^{\alpha}+1)^{\cancel{2}}} = \frac{e^{\alpha}-1}{e^{\alpha}+1} = \cancel{\frac{e^{\frac{\alpha}{2}}e^{\frac{\alpha}{2}} - e^{\frac{\alpha}{2}}e^{-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}e^{\frac{\alpha}{2}} + e^{\frac{\alpha}{2}}e^{-\frac{\alpha}{2}}}}$$

$\tanh\left(\frac{\alpha}{2}\right) = e^{\alpha/2} - e^{-\alpha/2} / e^{\alpha/2} + e^{-\alpha/2}$ → relation is demonstrated.

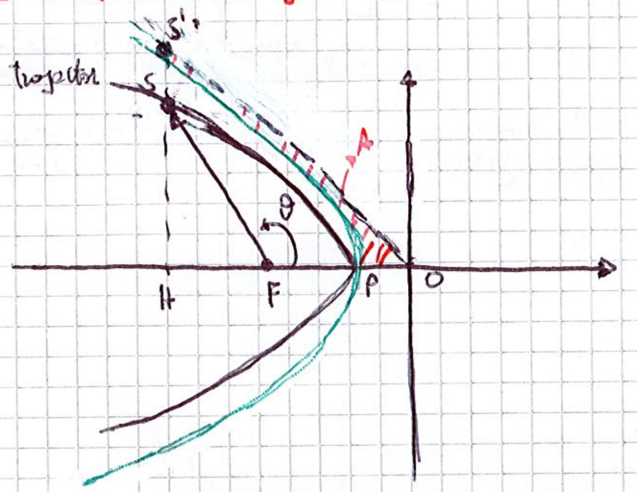
$$\Rightarrow \cancel{\tanh\left(\frac{F}{2}\right) = \frac{\sinh F}{1+\cosh F} = \frac{\sqrt{1-e^2}\sin\theta / (1+e\cos\theta)}{\frac{(1-e^2)\cos\theta}{1+e\cos\theta}+1}}$$

$$\cancel{\tanh\left(\frac{F}{2}\right) = \frac{\sqrt{1-e^2}\sin\theta}{(1-e^2)\cos\theta+1+e\cos\theta} = \frac{\sqrt{1-e^2}\sin\theta}{\cos\theta-e^2\cos\theta+1+e\cos\theta}}$$

$$\Rightarrow \begin{cases} \sinh F = \sqrt{e^2-1}\ \dfrac{\sin\theta}{1+e\cos\theta} \\ \cosh F = -(1-e^2)\ \dfrac{\cos\theta}{1+e\cos\theta} \\ \quad = (e^2-1)\ \dfrac{\cos\theta}{1+e\cos\theta} \end{cases}$$

$$\Rightarrow \tanh\left(\frac{F}{2}\right) =$$

--- interpretazione geometrica di F



$A = A_{S'HO} - A_{S'PO}$

$= \frac{1}{2} b h$

equilateral - Hyperbola: $y = \frac{K}{x^2}$

Traslated center in P: $y = \frac{ax+b}{cx+d}$ $\quad P(-\frac{d}{c}; \frac{a}{c})$

asymptotes: $\lim_{x \to +\infty} \frac{a + b/x}{c + d/x} \to y = \frac{a}{c}$ $\Rightarrow$ AUXILIARY EQUILATERAL HYPERBOLA

$O(-\frac{d}{c}; 0)$

$\lim_{y \to +\infty}$ $\quad cxy + dy = ax + b$

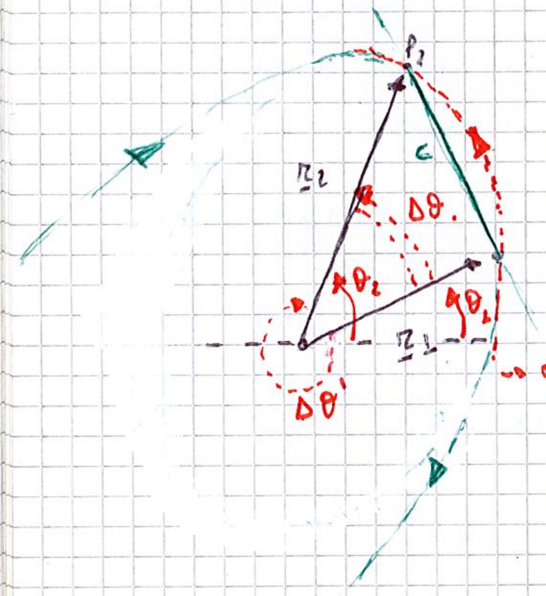
$(cy - a)x = b - dy$ $\qquad y = \frac{1}{cx+d}$

$x = \frac{b - dy}{cy - a} = \frac{-d + b/y}{c - a/y}$

$-\frac{d}{c} = a$

2 → "Lambert's problem (GEOMETRICAL INTERPRETATION)"

"LAMBERT'S PROBLEM CONSISTS INTO THE COMPUTATION OF AN ORBIT ABLE TO TRANSFER A SATELLITE FROM A POSITION P_1 TO A POSITION P_2 IN A GIVEN TIME OF FLIGHT $TOF = T_1 - T_2$ WHERE (T_1, T_2) ARE SUCH THAT $\begin{cases} P_1 = \underline{r}(T_1) \\ P_2 = \underline{r}(T_2) \end{cases}$ "



$\Delta\theta = \theta_2 - \theta_1 \Rightarrow \Delta\theta < \pi$ ANTIORARIO

$\Rightarrow \Delta\theta' = 2\pi - \Delta\theta \Rightarrow$ ORARIO

→ orbit resulting from the solution of Lambert's problem.

→ Lambert's theorem.

IL TEMPO DI VOLO (TOF) DA P_1 A P_2 È FUNZIONE DI SOLI TRE PARAMETRI

1. IL SEMIASSE MAGGIORE (a)
2. LA CORDA $(c = \|\underline{r}_2 - \underline{r}_1\|)$
3. LA SOMMA DELLE DISTANZE INIZIALE E FINALE RISPETTO AL FUOCO $(r_1 + r_2)$

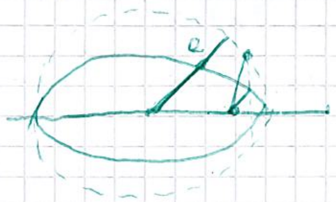
$$TOF = T_2 - T_1 = f(a, c, \|\underline{r}_1\| + \|\underline{r}_2\|) \quad \forall\, TOF;\ (\underline{r}_1, \underline{r}_2)$$

!! IL TEMPO DI VOLO NON DIPENDE DALL'ECCENTRICITÀ DELL'ORBITA (e) !!

Proof (valid for elliptical transfer)

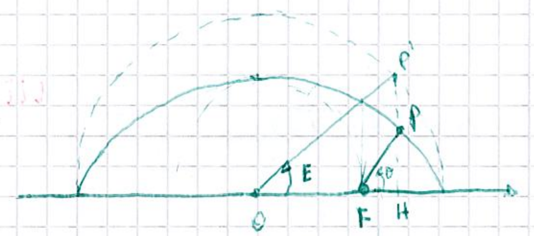
From Kepler's equation: $\sqrt{\frac{\mu}{a^3}}(T_2 - T_1) = (E_2 - e \sin E_2) - (E_1 - e \sin E_1)$

Considering the transfer orbit:



$$\Rightarrow \sqrt{\frac{\mu}{a^3}}\, T_1 = E_1 - e \sin E_1$$
$$\sqrt{\frac{\mu}{a^3}}\, T_2 = E_2 - e \sin E_2$$

Equation in polar-form of this ellipses is: $r = a(1 - e\cos E)$



$\overline{OH} = a \cos E$

$\overline{FH} = r \cos\theta \quad \Rightarrow \quad a\cos E = r\cos\theta + ae \quad (i)$

$\overline{OF} = ae$

$$a \cos E = \frac{a(1-e^2)\cos\theta}{1 + e\cos\theta} + ae \quad \ldots etc$$

→ anyway (i) $\quad a\cos E = r\cos\theta + ae$

$$r\cos\theta = a(\cos E - e) \qquad \text{But: } r = \frac{p}{1 + e\cos\theta}$$

$$\Rightarrow r + re\cos\theta = p \rightarrow re\cos\theta = p - r \rightarrow r\cos\theta = \frac{p - r}{e}$$

$$\Rightarrow \frac{p - r}{e} = a(\cos E - e) \rightarrow -r = ae(\cos E - e) + -a(1 - e^2)$$

$$r = a(1 - e^2 + e^2 - e\cos E) \rightarrow r = a(1 - e\cos E)$$

Then $\quad r_1 + r_2 = a(1 - e\cos E_1) + a(1 - e\cos E_2) = a[2 - e(\cos E_1 + \cos E_2)]$

Defining: $\quad E_P \triangleq \frac{E_2 + E_1}{2} \quad ; \quad E_M \triangleq \frac{E_2 - E_1}{2}$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\Rightarrow \begin{cases} \cos(E_P) = \cos\left(\frac{E_2}{2}\right)\cos\left(\frac{E_1}{2}\right) - \sin\left(\frac{E_2}{2}\right)\sin\left(\frac{E_1}{2}\right) \\ \cos(E_M) = \cos\left(\frac{E_2}{2}\right)\cos\left(\frac{E_1}{2}\right) + \sin\left(\frac{E_2}{2}\right)\sin\left(\frac{E_1}{2}\right) \end{cases}$$

$$\cos(E_P + E_M) = \cos\left(\frac{E_2 + E_1}{2} + \frac{E_2 - E_1}{2}\right) = \cos E_2$$

$$\cos(E_P - E_M) = \cos\left(\frac{E_2 + E_1}{2} - \frac{E_2 - E_1}{2}\right) = \cos E_1$$

$$\Rightarrow r_1 + r_2 = a\{2 - e[\cos(E_P + E_M) + \cos(E_P - E_M)]\}$$

⇒ By property of cosine:

$$\cos(E_p+E_m) + \cos(E_p-E_m) = \cos(E_p)\cos(E_m) - \cancel{\sin(E_p)\sin(E_m)} + \cos(E_p)\cos(E_m) + \cancel{\sin(E_p)\sin(E_m)}$$

$$= 2\cos(E_p)\cos(E_m)$$

Now we have 2 relations:

$$\sqrt{\frac{\mu}{a^3}}\,(t_2-t_1) = (E_2 - e\sin E_2) - (E_1 - e\sin E_1) \quad \text{(i)}$$

$$r_1 + r_2 = 2a\left[1 - e\cos(E_p)\cos(E_m)\right] \;,\quad E_p = \frac{E_1+E_2}{2} \;;\quad E_m = \frac{E_2-E_1}{2} \quad \text{(ii)}$$

↳ Calculating the chord (c):

$$c^2 = \|\underline{c}\|^2 = \|\underline{r}_2 - \underline{r}_1\|^2 \Rightarrow c^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

$$\begin{cases} x = a\cos E \\ y = b\sin E \\ (b = a\sqrt{1-e^2}) \end{cases} \Rightarrow c^2 = a^2\left[(\cos E_2 - \cos E_1)^2 + (1-e^2)(\sin E_2 - \sin E_1)^2\right]$$

$$\begin{cases} E_p = \frac{E_2+E_1}{2} \\ E_m = \frac{E_2-E_1}{2} \end{cases} \begin{cases} E_2 = E_p + E_m \\ E_1 = E_p - E_m \end{cases} \Rightarrow c^2 = a^2\left[(\cos(E_p+E_m) - \cos(E_p-E_m))^2 + (1-e^2)(\sin(E_p+E_m) - \sin(E_p-E_m))^2\right]$$

$$c^2 = a^2\left[(\cancel{\cos E_p \cos E_m} - \sin E_p \sin E_m - \cancel{\cos E_p\cos E_m} - \sin E_p \sin E_m)^2 + (1-e^2)(\cancel{\sin E_p \cos E_m} + \sin E_m \cos E_p - \cancel{\sin E_p \cos E_m} + \sin E_m \cos E_p)^2\right]$$

$$c^2 = a^2\left[(-2\sin E_p \sin E_m)^2 + (1-e^2)(2\sin E_m \cos E_p)^2\right]$$

$$c^2 = 4a^2\left[\sin^2 E_p \sin^2 E_m + (1-e^2)\sin^2 E_m \cos^2 E_p\right]$$

$$= 4a^2 \sin^2 E_m\left[\sin^2 E_p + (1-e^2)\cos^2 E_p\right]$$

$$= 4a^2 \sin^2 E_m\left[1 - \cancel{\cos^2 E_p} + \cancel{\cos^2 E_p} - e^2\cos^2 E_p\right]$$

$$c^2 = 4a^2 \sin^2 E_m\left[1 - e^2 \cos E_p\right] \quad \text{(iii)}$$

↳ Since the orbit is elliptical ⇒ $e<1$

then is possible to impose:

$$\exists\, \xi : \quad \cos\xi = e\cos E_p$$

$$\Rightarrow c^2 = 4a^2 \sin^2 E_m\left[1 - \cos^2\xi\right]$$

$$c^2 = 4a^2 \sin^2 E_m \sin^2\xi \quad \text{(iii)}$$

$$\downarrow$$

$$c = 2a\sin E_m \sin\xi$$

$$E_m = \frac{E_2-E_1}{2} \;;\quad E_p = \frac{E_2+E_1}{2} \;;\quad \xi: e\cos E_p = \cos\xi$$

Expressing also equation (ii) as a function of (ξ, E_m)

$$r_1 + r_2 = 2a\left[1 - \cos\xi\cos E_m\right] \quad \text{(ii)}$$

And calculating the perimeter of $S = \triangle P_1 F P_2$ we obtain: (ii)+(iii)

$$r_1 + r_2 + c = 2a\left[1 - \cos\xi\cos E_m\right] + 2a\sin E_m \sin\xi$$

$$= 2a\left[1 - \cos\xi\cos E_m + \sin E_m \sin\xi\right]$$

↳ Defining $\begin{cases} \beta = \xi - E_m \\ \alpha = \xi + E_m \end{cases}$

Same property as before $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ *

⇒ • $r_1 + r_2 + c = 2a\left[1 - (\cos\xi\cos E_m - \sin E_m \sin\xi)\right]$

$$= 2a\left[1 - \cos\alpha\right]$$

$$= 2a\sin^2\left(\frac{\alpha}{2}\right)$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

• $r_1 + r_2 - c = 2a\left[1 - \cos\xi\cos E_m - \sin E_m \sin\xi\right]$

$$= 2a\left[1 - (\cos\xi\cos E_m + \sin E_m\sin\xi)\right] \; *$$

$$= 2a\left[1 - \cos\beta\right] = 2a\sin^2\left(\frac{\beta}{2}\right)$$

↳ Going back to Kepler's equation:

(i): $$\sqrt{\frac{\mu}{a^3}}\,(t_2-t_1) = [\underbrace{E_2-E_1}_{2E_M} \cancel{+}\, e\,(\underbrace{\sin E_1}_{E_1=E_p-E_M} - \underbrace{\sin E_2}_{E_2=E_p+E_M})]$$

$$\sqrt{\frac{\mu}{a^3}}\,\Delta t = [2E_M + e\,(\cancel{\sin E_p \cos E_M} - \cos E_p \sin E_M - \cancel{\sin E_p \cos E_M} - \cos E_p \sin E_M)]$$

$$\sqrt{\frac{\mu}{a^3}}\,\Delta t = [2E_M - 2e\cos E_p \sin E_M]$$

$\cos\xi = e\cos E_p$

$$\sqrt{\frac{\mu}{a^3}}\,\Delta t = 2\,[E_M - \cos\xi \sin E_M]$$

- Since $\begin{cases}\alpha = \xi + E_M \\ \beta = \xi - E_M\end{cases} \Rightarrow \alpha-\beta = \xi + E_M - (\xi - E_M) = +2E_M$

- $-2\cos\xi \sin E_M = \sin(\xi+E_M) - \sin(\xi - E_M)$

$\sin\xi\cos E_M + \cos\xi\sin E_M - (\sin\xi\cos E_M - \sin E_M\cos\xi) = 2\cos\xi\sin E_M$

$2\cos\xi\sin E_M = \sin\alpha - \sin\beta$

⇒ (i) $$\sqrt{\frac{\mu}{a^3}}\,(t_2-t_1) = \alpha - \beta - (\sin\alpha - \sin\beta)$$

NOW:

(i): $$\sqrt{\mu}\,\Delta t = \sqrt{a^3}\,[\alpha-\beta-(\sin\alpha-\sin\beta)]$$

(ii) + (iii): $$\begin{cases} r_1+r_2-c = 2a\sin^2(\frac{\beta}{2}) \\ r_1+r_2+c = 2a\sin^2(\frac{\alpha}{2})\end{cases} \rightarrow \begin{cases}\sin\frac{\beta}{2} = \sqrt{\frac{r_1+r_2-c}{4a}} \\ \sin\frac{\alpha}{2} = \sqrt{\frac{r_1+r_2+c}{4a}}\end{cases}$$

↳ $s \triangleq \frac{r_1+r_2+c}{2}$

(semiperimeter of triangle $\overset{\triangle}{P_1FP_2}$)

→ Elliptic

⇒ $$\Delta t = TOF = \sqrt{\frac{a^3}{\mu}}\,[\alpha-\beta-(\sin\alpha-\sin\beta)]$$

and $$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}\ ;\quad \sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}\ ,\quad s = \frac{1}{2}(\overset{\triangle}{P_1FP_2})$$

Then Lambert's theorem is demonstrated.

Synthesis of the proof

(i): $\sqrt{\frac{\mu}{a^3}}\,\Delta t = E_2 - e\sin E_2 - (E_1 - e\sin E_1)$ → Kepler's law SATELLITE ROTATING IN ANTICLOCK SENSE SUCH THAT $E_2>E_1$, $\theta_2>\theta_1$

(ii): $\begin{cases} r_1 = a\,(1-e\cos E_1) \\ r_2 = a\,(1-e\cos E_2)\end{cases}$ → Polar law for elliptic coordinates

⇒ calculate $r_1 + r_2$

(iii): $\begin{cases} x = a\cos E \\ y = a\sqrt{1-e^2}\,\sin E\end{cases}$ → $c^2 = (x_2-x_1)^2 - (y_2-y_1)^2$

THEN

$\begin{cases} E_p \triangleq \frac{E_2+E_1}{2} \\ E_M \triangleq \frac{E_2-E_1}{2}\end{cases}$ ⇒ EXPRESS (ii) and (iii) as function of $(E_p\,;E_M)$

⇒ EXPRESS (iii) as function of ξ: $\cos\xi \triangleq e\cos E_p$

$\begin{cases}\alpha = E_M + \xi \\ \beta = \xi - E_M\end{cases}$ ⇒ EXPRESS (ii) + (iii) $\begin{cases} r_1+r_2-c \\ r_1+r_2+c\end{cases}$ as function of (α,β) ⇒ $\sin(\frac{\alpha}{2}) = \sqrt{\frac{s}{2a}}$; $\sin(\frac{\beta}{2}) = \sqrt{\frac{s-c}{2a}}$

EXPRESS (i) as function of (α,β) ⇒ $TOF = \sqrt{\frac{a^3}{\mu}}\,[\alpha-\beta-(\sin\alpha-\sin\beta)]$

→ Hyperbolic transfer.

RESULT FOR HYPERBOLA IS PRETTY MUCH THE SAME:

$$\Delta T = TOF = \sqrt{\frac{a^3}{\mu}}\,[\sinh\alpha - \sinh\beta - (\alpha-\beta)]$$

$$\sinh\frac{\alpha}{2} = \sqrt{-\frac{r_1+r_2+c}{4a}} = \sqrt{-\frac{s}{2a}}$$

$$\sinh\frac{\beta}{2} = \sqrt{-\frac{r_1+r_2-c}{4a}} = \sqrt{-\frac{s-c}{2a}}$$

OSS:

Considering Lambert's theorem result:

$$TOF = \sqrt{\frac{a^3}{\mu}}\,[\alpha - \beta - (\sin\alpha - \sin\beta)]$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{r_1 + r_2 + c}{4a}} \quad ; \quad \sin\frac{\beta}{2} = \sqrt{\frac{r_1 + r_2 - c}{4a}}$$

⇒ FOR A GIVEN a THE TIME OF FLIGHT IS AUTOMATICALLY DETERMINED (obviously having already set $\underline{r}_1$ and $\underline{r}_2$) INDIPENDENTLY FROM THE ECCENTRICITY

⇒ SUPPOSING OF HAVING SOLVED THE PROBLEM (INVERSE)

$$(\underline{r}_1, \underline{r}_2, tof) \to a$$

analogous: $(c = \|\underline{c}\| = \|\underline{r}_2 - \underline{r}_1\| ;\ r_1, r_2, tof) \to a$

$$\bar{r}_1 = \frac{p}{1 + e\cos\theta_1} = \frac{a(1-e^2)}{1+e\cos\theta_1} \quad ; \quad \bar{r}_2 = \frac{p}{1+e\cos\theta_2} = \frac{a(1-e^2)}{1+e\cos\theta_2}$$

$(\bar{r}_1 ; \bar{r}_2)$ → DATAS OF THE PROBLEM. (→ prescribed)

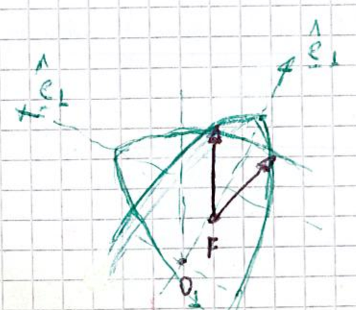
a → KNOWN FROM LAMBERT'S SOLUTION (given TOF)

$$\Rightarrow \begin{cases} \bar{r}_1 = a(1 - e\cos E_1) \\ \bar{r}_2 = a(1 - e\cos E_2) \end{cases} \Rightarrow \text{The unknown are } \begin{Bmatrix} E_1 \\ E_2 \\ e \end{Bmatrix}$$

solving by e and imposing equivalence: $\dfrac{-\bar{r}_1 + a}{\cos E_1} = \dfrac{a - \bar{r}_2}{\cos E_2}$

⇒ This equivalence is solved by an infinity of couple of values $(E_1 ; E_2)$

⇒



⇒ ellips 1 and ellips 2 may have same a but different eccentricity but may have different plane orientation.

2a ↳ GEOMETRICAL INTERPRETATION OF THIS AMBIGUITY

GIVEN $(\underline{r}_1 ; \underline{r}_2)$; $\bar{a}$

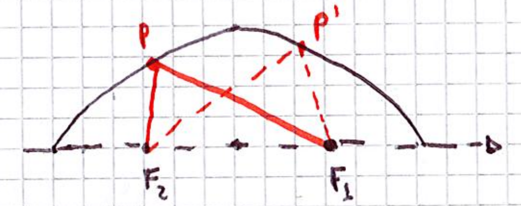
Maybe: coming from solution of Lambert's eqn.

⇒ ?? HOW MANY ORBITS (ELLIPTICAL) WITH F AS FOCUS ARE SUCH THAT, PASSING THROUGH (P_1, P_2), HAVE A SEMIAXIS a' : $a' \geq \bar{a}$??

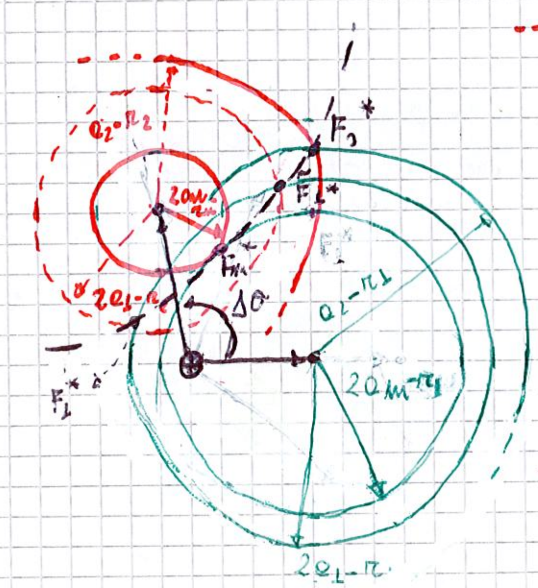
or (SAME GEOMETRICAL PROBLEM ⇒ DIFFERENT INTERPRETATION)

?? HOW MANY VACANT FOCUSES F^* (FUOCHI VACANTI) IS POSSIBLE TO DETERMINE HAVING FIXED $\begin{cases} P_1 \\ P_2 \\ a \geq \bar{a} \end{cases}$??

→ important ellips' property



$\forall P \in$ ellips ⇒ $\overline{PF_2} + \overline{PF_1} = 2a$



⇒

⇒ $\begin{cases} \overline{P_1F} + \overline{P_1F^*} = 2a \\ \overline{P_2F} + \overline{P_2F^*} = 2a \end{cases}$ (Because P_1 and P_2 belong to the same ellips)

$$\begin{cases} \overline{P_1F} + \overline{P_1F^*} = 2a \\ \overline{P_2F} + \overline{P_2F^*} = 2a \end{cases} \longrightarrow \begin{cases} \overline{P_1F^*} = 2a - r_1 \\ \overline{P_2F^*} = 2a - r_2 \end{cases}$$

⇒ $\overline{P_1F} = r_1$; $\overline{FP_2} = r_2$

⇒ As sketched the intercaption between the 2 circumferences of radius

$C_1: R_{C_1} = 2a - r_1$

$C_2: R_{C_2} = 2a - r_2$ ⇒ $C_1 \cap C_2$ in F^* or $\tilde{F}^*$

(F^* symmetrical to $\tilde{F}^*$ to the chord c)

• TUTTI I PUNTI TALI CHE $\overline{P_1F} + \overline{P_1F^*} = 2a$ (cond 1) STANNO SU UNA CIRCONFERENZA DI RAGGIO $2a - r_1$ CENTRATA IN P_1 ⇒ $\overline{P_1F^*} = 2a - r_1$

• " " " " " $\overline{P_2F} + \overline{P_2F^*} = 2a$ (cond 2) " " " " DI RAGGIO $2a - r_2$ CENTRATA IN P_2 ⇒ $\overline{P_2F^*} = 2a - r_2$

AFFINCHÉ SIA VERIFICATA SIA LA PRIMA CONDIZIONE CHE LA SECONDA IL PUNTO F^* DEVE APPARTENERE SIA ALLA 1ª CIRCONFERENZA CHE ALLA SECONDA E QUINDI DEVE TROVARSI ALL'INTERSEZIONE DELLE 2.

• The 2 ellipses satisfying the conditions of the Lambert's problem are $\begin{cases} e_2: F, \tilde{F}^*, \tilde{e} \\ e_1: F, F^*, e \end{cases}$

⇒ they have same semiaxis but different eccentricity

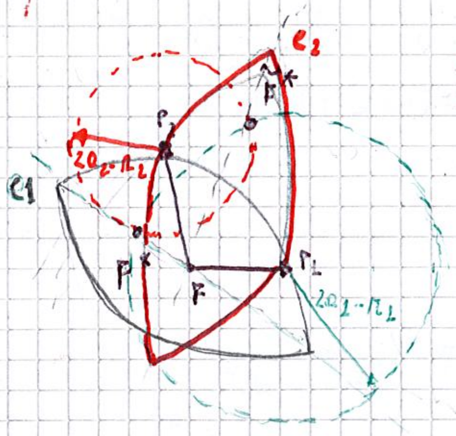
(ii)

IL FATTO CHE IL TEOREMA DI LAMBERT AFFERMI CHE IL TEMPO DI VOLO NON DIPENDA DALL'ECCENTRICITÀ SIGNIFICA CHE QUALSIASI ELLISSE DI SEMIASSE a (DUNQUE CON QUALSIASI ECCENTRICITÀ) RAGGIUNGERÀ IL PUNTO P_2 A PARTIRE DA P_1 IN $\Delta t = TOF$ TALE DA SODDISFARE LE CONDIZIONI

⇒ $\begin{cases} e_1: F; F^*, e, a \\ e_2: F; \tilde{F}^*, \tilde{e}, a \end{cases}$ ⇒ $\begin{cases} \overline{FF^*} = e a \\ \overline{F\tilde{F}^*} = \tilde{e} a \end{cases}$

SINCE $\overline{FF^*} < \overline{F\tilde{F}^*}$; $\dfrac{\overline{FF^*}}{e} = a = \dfrac{\overline{F\tilde{F}^*}}{\tilde{e}}$ ⇒ $\dfrac{\overline{FF^*}}{e} = \dfrac{\overline{F\tilde{F}^*}}{\tilde{e}}$

THEN $\tilde{e} > e$



e_1, e_2 stesso a (semiasse) ma diversa eccentricità.

↳ Starting again from relation

$$\begin{cases} \overline{P_1F^*} + \overline{P_1F} = 2a \\ \overline{P_2F^*} + \overline{P_2F} = 2a \end{cases} \longrightarrow \begin{cases} \overline{P_1F^*} = 2a - r_1 \\ \overline{P_2F^*} = 2a - r_2 \end{cases} \Rightarrow \overline{P_1F^*} - \overline{P_2F^*} = \cancel{2a} - r_1 - \cancel{2a} + r_2 = r_2 - r_1$$

⇒ The distance $\overline{P_1F^*}$ subtracted to the distance $\overline{P_2F^*}$ has a constant value equal to $r_2 - r_1$

↳ COMPUTATION OF THE ELLIPS WITH MINIMUM ENERGY.

Since energy of an orbit is computed by using $\varepsilon = -\dfrac{\mu}{2a}$ the ellips with minimum energy is obtained with $a = a_m$ ⇒ is obtained when the 2 circumferences becomes tangent.

ELLIPS MINIMUM ENERGY:

• $\varepsilon = -\dfrac{\mu}{2a_m}$

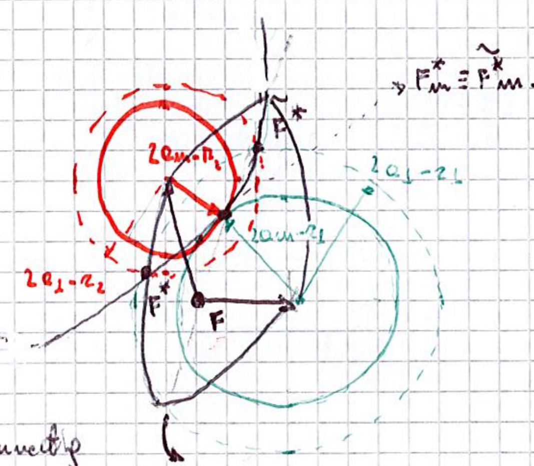
• $F^* \equiv \tilde{F}^* \triangleq F^*_m$.

⇒ $(2a_m - r_2) + (2a_m - r_1) = c$

$4a_m = c + r_2 + r_1$ } semiperimetro $s = \frac{1}{2}(c + \overline{P_1F} + \overline{FP_2})$

$4a_m = 2s$

→ $a_m = \dfrac{s}{2}$



Hyperbola obtained connecting points of interception for each possible value of a

Ellips with minimum energy.

!! The point F^*_m coincides with the vertex of the hyperbola described by the infinite set of points obtained by the interception of $\begin{cases} C_1: P_1|c \quad r = 2a - r_1 \\ \cap \\ C_2: P_2|c \quad r = 2a - r_2 \end{cases}$

$a = a_m \ldots\ldots \infty$

→ In synthesis

$$\forall a > a_m \ \exists \begin{cases} e_1 \\ e_2 \end{cases} \Rightarrow \begin{cases} e_1 : F;\ F^*;\ \tilde{e};\ a \\ e_2 : F,\ \tilde{F}^*;\ e;\ a \end{cases}$$

- PER OGNI a ESISTONO 2 POSSIBILI ELLISSI DI DIVERSA ORIENTAZIONE ($\hat{E}_1 F \hat{E}_2$) ED ECCENTRICITÀ
- OVVIAMENTE a È DETERMINATO UNA VOLTA FORNITO $\begin{cases} TOF \\ r_1;\ r_2;\ c \end{cases}$

FOR EACH ELLIPS WE HAVE 2 ARCHS CONNECTIN $P_1 \to P_2$.
$$\begin{cases} e_1 \begin{cases} \Delta\theta > \pi \to \Delta\theta' = 2\pi - \Delta\theta \\ \Delta\theta < \pi \end{cases} \\ e_2 \begin{cases} \Delta\theta > \pi \to \Delta\theta' = 2\pi - \Delta\theta \\ \Delta\theta < \pi \end{cases} \end{cases}$$

2b ↳ SOLUTION OF THE LAMBERT'S PROBLEM.

a) Elliptical transfer.

$$TOF = [\alpha - \beta - (\sin\alpha - \sin\beta)]$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}} \xrightarrow{\text{GIVEN } a} \alpha = \begin{cases} \alpha_0 (<\pi) \\ 2\pi - \alpha_0 \end{cases} \leftarrow \frac{\alpha}{2} = \sin^{-1}\left(\sqrt{\frac{s}{2a}}\right) \begin{cases} \frac{\alpha}{2} \left(\le \frac{\pi}{2}\right) \\ \left(\frac{\alpha}{2}\right)' = \pi - \frac{\alpha}{2} \end{cases}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}} \xrightarrow{\text{GIVEN } a} \beta = \begin{cases} \beta_0 \\ 2\pi - \beta_0 \ (= -\beta_0) \end{cases}$$

$\beta = \xi - E_m$

$E_m = \frac{E_2 - E_1}{2}$; $\xi_p = \frac{E_2 + E_1}{2} \Rightarrow E_m = \frac{\Delta\theta}{2} \Rightarrow \beta = \xi - \frac{\Delta\theta}{2} \Rightarrow \begin{cases} \beta < \pi \text{ if } \Delta\theta < \pi \\ \beta > \pi \text{ if } \Delta\theta > \pi \ * \end{cases}$

ξ: $e\cos(E_p) = \cos\xi \Rightarrow$ if $-\frac{\pi}{2} < \frac{E_1 + E_2}{2} < \frac{\pi}{2}$ then $\cos\xi > 0 \Leftrightarrow -\frac{\pi}{2} < \xi < \frac{\pi}{2}$

$0 < e < 1$

* in fact for $\Delta\theta > \pi \Rightarrow \xi > \frac{\pi}{2}$ because $\xi \simeq E_p$ $\Rightarrow \beta > \pi$

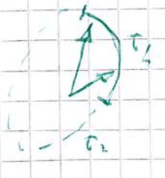
⇓

$$\begin{cases} \beta = \beta_0 & \text{for } 0 \le \Delta\theta < \pi \\ \beta = -\beta_0 & \text{for } \pi \le \Delta\theta < 2\pi \end{cases}$$

$$\begin{cases} \alpha = \alpha_0 & \text{for } \Delta t \le t_m \\ \alpha = 2\pi - \alpha_0 & \text{for } \Delta t > t_m \end{cases}$$; t_m = OBTAINED USING MINIMUM ENERGY ORBIT

⇑

Into domanda, minimum transfer is done for $0 \le \Delta\theta < \pi$
Because the point of [illegible] (tangent) is between (for $0 < \Delta\theta \le \pi$)



$t_1 < t_2$

Obs: !! t_m is not the minimum time to reach r_2; it's the time of flight associated to the path defined by the ellips of minimum energy !!

→ t_m (TOF for ellips of minimum energy)

$$E_{min} = -\frac{\mu}{2a_m} \Rightarrow a_m = \frac{s}{2} \Rightarrow s = 2a_m$$

$(a_m \equiv a|_{min\,E} \equiv a_{minimum})$

$$\Rightarrow \sin\frac{\alpha}{2}\Big|_m = \sqrt{\frac{2a_m}{2a_m}} = 1 \Leftrightarrow \alpha_0|_m = \pi \to \alpha_m = \pi$$

$\left(\sin\frac{\alpha}{2} = \left(\sqrt{\frac{s}{2a}}\right)^*\right)$

$$\Rightarrow \sin\frac{\beta}{2}\Big|_m = \sqrt{\frac{s-c}{2a_m}} = \sqrt{1 - \frac{c}{2a_m}} \to \beta_0|_m = 2\sin^{-1}\left(\sqrt{1 - \frac{c}{2a_m}}\right)$$

$$\Rightarrow \begin{cases} t_m = TOF_{on\ ell|_{E_{min}}} = [\pi - \beta_m + \sin\beta_m]\sqrt{\frac{s^3}{8\mu}} \\ \beta_m = 2\sin^{-1}\left(\sqrt{1 - \frac{c}{2a_m}}\right) \end{cases}$$

$$TOF = \sqrt{\frac{a_m^3}{\mu}}[\alpha - \beta - (\sin\alpha - \sin\beta)]$$
$\alpha_m = \pi$, $\beta_m = \beta$, $a_m = \frac{s}{2}$

→ Given a → TOF

1) GIVEN (r_1, r_2, a) → COMPUTE $\begin{cases} s = r_1 + r_2 + c \\ c = \| r_2 - r_1 \| \end{cases}$

2) COMPUTE: $\begin{cases} a_m = \frac{s}{2} \\ t_m = \sqrt{\frac{s^3}{8\mu}} \cdot [\pi - \beta_m + \sin\beta_m] \end{cases}$; $\sin\left(\frac{\beta_m}{2}\right) = \sqrt{1 - \frac{c}{2a_m}}$

3) $\begin{cases} \sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}} \\ \sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}} \end{cases} \xrightarrow[(\alpha,\,\beta)]{\text{COMPUTE}}$ DOUBLE RESULT: $\begin{cases} \alpha = \begin{cases} \alpha_0 (<\pi) \\ 2\pi - \alpha_0 \end{cases} \\ \beta = \begin{cases} \beta_0 \\ 2\pi - \beta_0 \end{cases} \end{cases}$

4) SELECT EACH POSSIBLE VALUE OF $(\alpha; \beta)$ OBTAINING 4 POSSIBLE TOF VALUES.*

$$\begin{cases} \alpha = \alpha_0 & TOF \le t_m \\ \alpha = 2\pi - \alpha_0 & TOF > t_m \end{cases} ; \quad \begin{cases} \beta = \beta_0 & 0 \le \Delta\theta \le \pi \\ \beta = -\beta_0 & \pi \le \Delta\theta \le 2\pi \end{cases}$$

* 4 VALORI CORRISPONDENTI A 2 ELLISSI CIASCUNA PERCORRIBILE IN SENSO ORARIO O IN SENSO ANTIORARIO.
ell_1: $F - F^*$ → ORARIO TOF_1, ANTIORARIO TOF_2
ell_2: $F - F^*$ → TOF_3, TOF_4

b) Parabolic transfer.

TOF → T_P ⇒ Lambert's theorem result becomes:

$$t_P = \sqrt{\frac{a^3}{\mu}}\,[\alpha - \beta - (\sin\alpha - \sin\beta)]$$

$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}} \quad , \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}}.$$

(i)
↳ Parabolic orbit: $a \to \infty$ ⇒ • $\lim_{a\to\infty} \sin\left(\frac{\alpha}{2}\right) = \lim_{a\to\infty} \sqrt{\frac{s}{2a}}$ ⇒ $\sin\frac{\alpha}{2} \simeq \frac{\alpha}{2} = \sqrt{\frac{s}{2a}}.$

• $\lim_{a\to\infty} \sin\left(\frac{\beta}{2}\right) = \lim_{a\to} \sqrt{\frac{s-c}{2a}}$ ⇒ $\sin\frac{\beta}{2} \simeq \frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}.$

obtaining then: $\begin{cases} \alpha = 2\sqrt{\frac{s}{2a}} \\ \beta = 2\sqrt{\frac{s-c}{2a}} \end{cases}$

!! REALLY SMART !! sviluppo fino al termine cubico $\begin{cases} \alpha - \sin\alpha \\ -(\beta - \sin\beta) \end{cases}$ in modo da elidere $\sqrt{\frac{a^3}{\mu}}$ ⇒ a dalle equazione

(ii)
↳ since $(\alpha, \beta) \to 0$ it's possible to make Taylor's series centered in 0

$$\alpha - \sin\alpha \simeq \alpha - \left(\alpha - \frac{\alpha^3}{3!}\right) = \frac{\alpha^3}{3!} \xrightarrow{(i)} \frac{1}{3!}\left(2\sqrt{\frac{s}{2a}}\right)^3$$

$$\sin\alpha \simeq \sin\alpha\Big|_{\alpha=0} + \cos\alpha\Big|_{\alpha=0}\,\alpha - \sin\alpha\Big|_{\alpha=0}\frac{\alpha^2}{2!} - \cos\alpha\Big|_{\alpha=0}\frac{\alpha^3}{3!} + \dots$$

$$\beta - \sin\beta \simeq \dots\dots = \frac{\beta^3}{3!} \xrightarrow{(i)} \frac{1}{3!}\left(2\sqrt{\frac{s-c}{2a}}\right)^3$$

$$\begin{cases} \alpha - \sin\alpha \simeq \dfrac{8}{6}\,\dfrac{\sqrt{s^3}}{2\sqrt{2a^3}} \\[2ex] \beta - \sin\beta \simeq \dfrac{8}{6}\,\dfrac{\sqrt{(s-c)^3}}{2\sqrt{2a^3}} \end{cases}$$

⇒ $$t_P = \frac{\sqrt{a^3}}{\sqrt{\mu}}\cdot\left[\frac{8}{6}\,\frac{\sqrt{s^3}}{2\sqrt{2a^3}} - \frac{8}{6}\,\frac{\sqrt{(s-c)^3}}{2\sqrt{2a^3}}\right]$$

$$t_P = \frac{1}{\sqrt{\mu}}\cdot\left[\frac{\sqrt{2}}{3}\sqrt{s^3} - \frac{\sqrt{2}}{3}\sqrt{(s-c)^3}\right]$$ ✱

⇒ $$\boxed{\begin{aligned} t_P &= \frac{\sqrt{2}}{3}\left[\sqrt{s^3} - \sqrt{(s-c)^3}\right] && \text{for } 0 \le \Delta\theta < \pi \\ t_P &= \frac{\sqrt{2}}{3}\left[\sqrt{s^3} + \sqrt{(s-c)^3}\right] && \text{for } \pi \le \Delta\theta < 2\pi \end{aligned}}$$

c) Hyperbolic transfer.

Analogous to an elliptic transfer:

$$t_H = \sqrt{\frac{-a^3}{\mu}}\,[\sinh\alpha - \sinh\beta - (\alpha - \beta)]$$

$$\sinh\left(\frac{\alpha}{2}\right) = \sqrt{-\frac{r_1 + r_2 + c}{4a}} = \sqrt{-\frac{s}{2a}}.$$

$$\sinh\left(\frac{\beta}{2}\right) = \sqrt{-\frac{r_1 + r_2 - c}{4a}} = \sqrt{-\frac{s-c}{4a}}.$$

(1 possible TOF)

AT THIS STAGE: We are capable to determine the TOF for an orbit connecting $(P_1; P_2)$ located at $(\underline{r}_1; \underline{r}_2)$ for a given a (semiaxis)

?? Unique ??

MORE USEFULL: Determine (a, e) ⇒ completely define the orbit that the planet must follow to connect $(P_1; P_2)$ in a given TOF

↳ NEED TO INTRODUCE: Lagrange's invariant.

3 → " Universal variable of motion ⊕ Lagrange's invariants "

3a) → UNIVERSAL VARIABLE.

$$\ddot{\underline{r}} = -\frac{\mu}{\|\underline{r}\|^3}\,\underline{r} \longrightarrow \begin{cases} h = r^2\,\dot\theta \\ r = \dfrac{p}{1+e\cos\theta} = \dfrac{h^2/\mu}{1+e\cos\theta} \end{cases} \longrightarrow \int \frac{\mu^2}{h^3}\,dt = \int \frac{1}{(1+e\cos\theta)^2}\,d\theta$$

- $e=1$ (PAR) $\Rightarrow \theta = \theta(t)$
- $0<e<1$ (ELLIPSE) $\Rightarrow E = E(t)$ $\xrightarrow{\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\,\tan\left(\frac{\theta}{2}\right)}$ $\theta = \theta(t)$
  $\sqrt{\frac{\mu}{a^3}}\,\Delta t = [E - e\sin E]$
- $e>1$ (HYPERBOLA) $\Rightarrow E' = E'(t)$ $\xrightarrow{\tanh\left(E'/2\right) = \sqrt{\frac{1-e}{1+e}}\,\tanh\left(\frac{\theta}{2}\right)}$ $\theta = \theta(t)$
  $\sqrt{\frac{\mu}{a^3}}\,\Delta t = [E' - e\sinh(E')]$

... Instead of using 3 different angles (AUXILIARY ANGLES $\begin{cases} E \to \text{auxiliary circle} \\ F \to \text{auxiliary hyperbola equilatera} \end{cases}$) would be better if we may have a universal angle for each conic.

$$\Rightarrow \begin{cases} \varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \quad (\mathrm{III}^{2nd}) \\ \exists\ \chi : \dfrac{d\chi}{dt} = \dfrac{\sqrt{\mu}}{r} \end{cases}$$

→ Sundman's transformation.

$$\exists\ \chi : \frac{d\chi}{dt} = \frac{\sqrt{\mu}}{r} \to \sqrt{\mu}\,\frac{dt}{d\chi} = r$$

Now,

$$\begin{cases} \underline{v} = \dot r\,\hat{\underline{r}} + r\dot\theta\,\hat{\underline{\theta}} \to v^2 = \dot r^2 + r^2\dot\theta^2 \\ \underline{h} = \underline{r}\times\underline{v} = r\,\hat{\underline{r}}\times[\dot r\,\hat{\underline{r}} + r\dot\theta\,\hat{\underline{\theta}}] = r^2\dot\theta\,\hat{\underline{h}} \end{cases} \Rightarrow v^2 = \dot r^2 + \frac{h^2}{r^2}$$

$$\Rightarrow \varepsilon = \frac{\dot r^2}{2} + \frac{h^2}{2r^2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Extracting $\dot r^2$ from equation of conservation of energy:

$$\dot r^2 = -\frac{h^2}{r^2} + 2\frac{\mu}{r} - \frac{\mu}{a}$$

↳ Using the universal variable χ instead of having $r = r(\theta)$, $\theta(t)$ we have: $\begin{cases} r = r(\chi) \\ \chi = \chi(t) \end{cases}$

$$\dot r^2 = -\frac{h^2}{r^2} + 2\frac{\mu}{r} - \frac{\mu}{a} \quad (i)$$

$$\dot r = \frac{dr}{d\chi}\cdot\frac{d\chi}{dt} \quad \text{since } \chi : \frac{d\chi}{dt} = \frac{\sqrt{\mu}}{r} \to \dot r = \frac{dr}{d\chi}\,\frac{\sqrt{\mu}}{r}$$

So equation (i) in function of variable χ becomes:

$$\left[\frac{dr}{d\chi}\right]^2 \frac{\mu}{r^2} = -\frac{h^2}{r^2} + 2\frac{\mu}{r} - \frac{\mu}{a} \qquad \Big\| \quad p = h^2/\mu = a(1-e^2)$$

$$\left[\frac{dr}{d\chi}\right]^2 \frac{\mu}{r^2} = -\frac{p\mu}{r^2} + 2\frac{\mu}{r} - \frac{\mu}{a}$$

$$\left[\frac{dr}{d\chi}\right]^2 = \left(-\frac{p\cancel{\mu}}{r^2} + \frac{2\cancel{\mu}}{r} - \frac{\mu}{a}\right)\frac{r^2}{\cancel{\mu}}$$

$$\left[\frac{dr}{d\chi}\right]^2 = \left(-p + 2r - \frac{r^2}{a}\right) = \frac{-pa + 2ar - r^2}{a}$$

Adding and subtracting a^2 at numerator of previous expression:

$$\left[\frac{dr}{d\chi}\right]^2 = \frac{-pa + 2ar - r^2 + a^2 - a^2}{a}$$

$$= \frac{-(r-a)^2 + a^2 - pa}{a} = \frac{a^2\left(1-\frac{p}{a}\right) - (r-a)^2}{a}$$

$$\left[\frac{dr}{d\chi}\right]^2 = a\left(1-\frac{p}{a}\right) - \frac{a^2}{a}\left(\frac{r}{a}-1\right)^2 = a\left[\left(1-\frac{p}{a}\right) - \left(\frac{r}{a}-1\right)^2\right]$$

Only algebraic passages:

$$\frac{dr}{d\chi} = \sqrt{a}\,\sqrt{\left(1-\frac{p}{a}\right) - \left(\frac{r}{a}-1\right)^2} = \sqrt{a}\,\sqrt{1-\frac{p}{a}}\,\sqrt{1 - \left(\frac{r}{a}-1\right)^2\Big/\left(1-\frac{p}{a}\right)}$$

Preparing integration $\int dr = \int d\chi$ in order to obtain a relation $r = r(\chi)$

$$\frac{1}{\sqrt{1 - \dfrac{(r/a-1)^2}{(1-p/a)}}}\,dr = \sqrt{a\left(1-\frac{p}{a}\right)}\,d\chi$$

$$\Rightarrow \int \frac{1}{\sqrt{1-\frac{(r/a-1)^2}{(1-p/a)}}}\,dr = \int \sqrt{a}\cdot\sqrt{1-\frac{p}{a}}\,dx$$

↳ if x was 0 we should integrate $\int_{r_p}^{r} dr$ → we don't know the zero of x ⇒ let's integrate indefinite.

Must be solved by substitution

$$z = \frac{r/a-1}{\sqrt{1-p/a}} \rightarrow dz = \frac{1}{a\sqrt{1-p/a}}\,dr$$

$$\rightarrow dr = a\cdot\sqrt{1-p/a}\,dz$$

$$\int \frac{a\cancel{\sqrt{1-p/a}}}{\sqrt{1-z^2}}\,dz = \int \sqrt{a}\;\cancel{\sqrt{1-p/a}}\,dx$$

$$\int \frac{1}{\sqrt{1-y^2}}\,dy = \sin^{-1}(y)+c$$

$$\int \frac{\sqrt{a}}{\sqrt{1-z^2}}\,dz = \int dx$$

$$\sqrt{a}\,\sin^{-1}(z) = x + c_0 \rightarrow \begin{cases} \sin^{-1}(z) = \dfrac{x+c_0}{\sqrt{a}} \\ z = \dfrac{r/a-1}{\sqrt{1-p/a}} \end{cases}$$

(aim is always to obtain $r = r(x)$)

$$\rightarrow \frac{r/a-1}{\sqrt{1-p/a}} = \sin\left(\frac{x+c_0}{\sqrt{a}}\right) \rightarrow r/a = 1+\sqrt{1-p/a}\,\sin\left(\frac{x+c_0}{\sqrt{a}}\right)$$

$r = r(x)$

$$x: \quad \frac{dx}{dt} = \frac{\sqrt{\mu}}{r} \Rightarrow r = r(x) = a\left[1+\sqrt{1-p/a}\,\sin\left(\frac{x+c_0}{\sqrt{a}}\right)\right]$$

$$p = a(1-e^2) \rightarrow \sqrt{1-p/a} = \sqrt{\cancel{1}-\cancel{1}+e^2} = \sqrt{e^2}$$

$$r = r(x) = a\left[1+e\sin\left(\frac{x+c_0}{\sqrt{a}}\right)\right]$$

Once we found $r = r(x)$ relation we can easily obtain the still missing relation $x(t) = x$ but this can easily be found from the definition of x

$$\frac{dx}{dt} = \frac{\sqrt{\mu}}{r}\;; \quad \text{Known that} \quad r(x) = a\left[1+e\sin\left(\frac{x+c_0}{\sqrt{a}}\right)\right]$$

$$\rightarrow \frac{dx}{dt} = \frac{\sqrt{\mu}}{a\left[1+e\sin\left(\frac{x+c_0}{\sqrt{a}}\right)\right]}$$

$$\sqrt{\mu}\,dt = a\left[1+e\sin\left(\frac{x+c_0}{\sqrt{a}}\right)\right]dx$$

$$\int_{t_1}^{t_2}\sqrt{\mu}\,dt = \int_{x_1}^{x_2} a\left[1+e\sin\left(\frac{x+c_0}{\sqrt{a}}\right)\right]dx$$

!! It's possible to integrate from 0-value of x because the x-value is already shifted by constant c_0. !!

$$\sqrt{\mu}\,\Delta t = a\left[x - e\cos\left(\frac{x+c_0}{\sqrt{a}}\right)\Big|_0^x\right]$$

$$\sqrt{\mu}\,\Delta t = a\left[x - e\cos\left(\frac{x+c_0}{\sqrt{a}}\right) + e\cos\left(\frac{c_0}{\sqrt{a}}\right)\right]$$

$x = x(t)$

$$\sqrt{\mu}\,\Delta t = a\left[x - e\cos\left(\frac{x+c_0}{\sqrt{a}}\right) + e\cos\left(\frac{c_0}{\sqrt{a}}\right)\right]$$

(Kepler-like equation but valid for each conic)

REALLY IMPORTANT

obs: constant c_0 MUST BE USED TO ELIMINATE DEPENDANCE ON e.

$$r_0 = a\left[1+e\sin\left(\frac{\overset{=0}{x}+c_0}{\sqrt{a}}\right)\right] \Rightarrow e\sin\left(\frac{c_0}{\sqrt{a}}\right) = \frac{r_0}{a}-1$$

$$\dot{r} = a e\cos\left(\frac{x+c_0}{\sqrt{a}}\right)\frac{1}{\sqrt{a}} = e\sqrt{a}\cos\left(\frac{x+c_0}{\sqrt{a}}\right) \qquad \dot{r}_0 = e\sqrt{a}\cos\left(\frac{c_0}{\sqrt{a}}\right)$$

$$\Rightarrow e\cos\left(\frac{c_0}{\sqrt{a}}\right) = \frac{\dot{r}_0}{\sqrt{a}}$$

So when computing $r = r(x)$

$$r(x) = a + a\Big(\sin\left(\frac{x}{\sqrt{a}}\right)\underbrace{e\cos\left(\frac{c_0}{\sqrt{a}}\right)}_{=\dot{r}_0/\sqrt{a}} + \cos\left(\frac{x}{\sqrt{a}}\right)\underbrace{e\sin\left(\frac{c_0}{\sqrt{a}}\right)}_{r_0/a-1}\Big)$$

in this way dependence on e is fully eliminated.

$$\dot{r}_0 = \frac{\underline{r}_0\cdot\underline{v}_0}{r_0} = (\underline{r}_0\cdot\underline{v}_0)\,\frac{1}{a\left[1+\frac{r_0}{a}-1\right]}$$

KNOWN THE LAWS:

$r = r(\chi) \to r = a\left[1 + e\sin\left(\frac{\chi+\omega}{\sqrt a}\right)\right]$

$\chi = \chi(t) \to \sqrt{\mu}\,\Delta t = a\left[\chi - e\cos\left(\frac{\chi+\omega}{\sqrt a}\right)\right]$

χ DEFINITION $\to \frac{d\chi}{dt} = \frac{\sqrt\mu}{r} \to \dot\chi = \frac{\sqrt\mu}{r}$

⇒ TO DETERMINE ω STARTING FROM INITIAL CONDITIONS: $\begin{cases} r_0 \\ v_0 \end{cases}$ AT $\begin{cases} t=0 \\ \chi=0 \end{cases}$

1) $r_0 = a\left[1 + e\sin\left(\frac{\omega}{\sqrt a}\right)\right] \to r_0$

⇒ $e\sin\left(\frac{\omega}{\sqrt a}\right) = \frac{r_0}{a} - 1$ (i)

2) $\dot r = \frac{dr}{dt} = \frac{\partial r}{\partial \chi}\cdot\frac{d\chi}{dt}$

$= \frac{e}{\sqrt a}\cos\left(\frac{\chi+\omega}{\sqrt a}\right)\cdot\frac{\sqrt\mu}{r} = \sqrt{\frac{\mu}{a}}\,\frac{e\cos\left(\frac{\chi+\omega}{\sqrt a}\right)}{a\left[1+e\sin\left(\frac{\chi+\omega}{\sqrt a}\right)\right]}\cdot a$

SINCE $\underline{v}_0 = \dot r_0\hat{\underline r} + r_0\dot\theta\hat{\underline\theta}$ → $\dot r$ coming from derivation of $r = r(t)$

3) PASSING THROUGH SCALAR PRODUCT $\underline r\cdot\underline v$

$$\begin{cases} \underline r\cdot\underline v = \underline r \times (\dot r\hat{\underline r} + r\dot\theta\hat{\underline\theta}) = r\dot r \\ \dot r\cdot r = \sqrt{\mu a}\,e\cos\left(\frac{\chi+\omega}{\sqrt a}\right) \end{cases} \Rightarrow \underline r\cdot\underline v = \sqrt{\mu a}\,e\cos\left(\frac{\chi+\omega}{\sqrt a}\right)$$

$\dot\chi = \frac{\sqrt\mu}{r}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

$r = a\left[1 + e\sin\left(\frac{\chi+\omega}{\sqrt a}\right)\right] = a\left[1 + e\left(\sin\frac{\chi}{\sqrt a}\cos\frac{\omega}{\sqrt a} + \sin\frac{\omega}{\sqrt a}\cos\frac{\chi}{\sqrt a}\right)\right]$

→ SINCE r_0, v_0 are given at $\begin{cases} t=0 \\ \chi=0 \end{cases}$; $\underline r\cdot\underline v = \sqrt{\mu a}\,e\cos\left(\frac{\chi+\omega}{\sqrt a}\right)$

THEN $\underline r_0\cdot\underline v_0 = \sqrt{\mu a}\,e\cos\left(\frac{\omega}{\sqrt a}\right) \Rightarrow e\cos\left(\frac{\omega}{\sqrt a}\right) = \frac{\underline r_0\cdot\underline v_0}{\sqrt{\mu a}}$ (ii)

⇒ (i)+(ii) $r = a\left[1 + \frac{\underline r_0\cdot\underline v_0}{\sqrt{\mu a}}\sin\frac{\chi}{\sqrt a} + \left(\frac{r_0}{a}-1\right)\cos\frac{\chi}{\sqrt a}\right]$

"Using such considerations we had been able to eliminate dependence of ω in the formulation of universal variable. ⇒ THIS IS THE FORM TO USE."

$$r = a\left[1 + \frac{\underline r_0\cdot\underline v_0}{\sqrt{\mu a}}\sin\frac{\chi}{\sqrt a} + \left(\frac{r_0}{a}-1\right)\cos\frac{\chi}{\sqrt a}\right]$$

Given: $\begin{cases} r_0 \\ v_0 \end{cases}$ for $\begin{cases} \chi=0 \\ t=0 \end{cases}$; • $e\sin\left(\frac{\omega}{\sqrt a}\right) = \frac{r_0}{a} - 1$

• $e\cos\left(\frac{\omega}{\sqrt a}\right) = \frac{\underline r_0\cdot\underline v_0}{\sqrt{\mu a}}$ comes from scalar product $\underline r_0\cdot\underline v_0 = \dot r_0\cdot r_0$

↳ Now we have 2 trajectories equations (considering an ellipse)

$r = a(1 - e\cos E)$ — "Kepler's law"

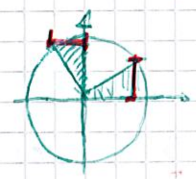
$r = a\left[1 + \frac{\underline r_0\cdot\underline v_0}{\sqrt{\mu a}}\sin\frac{\chi}{\sqrt a} + \left(\frac{r_0}{a}-1\right)\cos\frac{\chi}{\sqrt a}\right]$ — "comes from energy"

⇒ Need to be able to switch from one to the other:

$r = a(1 - e\cos E)$; $r = a\left[1 + e\sin\left(\frac{\chi+\omega}{\sqrt a}\right)\right]$

$\downarrow$

$r = a\left[1 - e\cos\left(\frac{\pi}{2} + \frac{\chi+\omega}{\sqrt a}\right)\right]$



$a(1 - e\cos E) = a\left[1 - e\cos\left(\frac{\pi}{2} + \frac{\chi+\omega}{\sqrt a}\right)\right]$

$E = \frac{\pi}{2} + \frac{\chi+\omega}{\sqrt a}$

Defining $E_0 \triangleq \frac{\pi}{2} + \frac{\omega}{\sqrt a}$

$E = E_0 + \frac{\chi}{\sqrt a}$

⇒ (i)+(ii) $\sqrt\mu\,\Delta t = a\left[\chi - e\cos\frac{\chi}{\sqrt a}\cos\frac{\omega}{\sqrt a} + \sin\frac{\chi}{\sqrt a}\sin\frac{\omega}{\sqrt a}\right]$

$= a\left[\chi - \frac{\underline r_0\cdot\underline v_0}{\sqrt{\mu a}}\cos\frac{\chi}{\sqrt a} + \left(\frac{r_0}{a}-1\right)\sin\frac{\chi}{\sqrt a}\right]$

⊢ fully solved problem. (FOR AN ELLIPS)

1) GIVEN: $\begin{cases} \underline{r}_0 \\ \underline{v}_0 \end{cases}$ AT $\begin{cases} t=0 \\ \chi=0 \end{cases}$ $\Rightarrow$ $\sqrt{\mu}\,\Delta t = a\left[\chi - \frac{\underline{r}_0\cdot\underline{v}_0}{\sqrt{a}}\cos\frac{\chi}{\sqrt{a}} + \left(\frac{r_0}{a}-1\right)\sin\frac{\chi}{\sqrt{a}}\right]$

(!! independent from eccentricity (e) !!)

2) SOLVING FOR A GIVEN t^*. $\sqrt{\mu}\,\Delta t = a\left[\chi - \frac{\underline{r}_0\cdot\underline{v}_0}{\sqrt{a}}\cos\left(\frac{\chi}{\sqrt{a}}\right) + \left(\frac{r_0}{a}-1\right)\sin\frac{\chi}{\sqrt{a}}\right]$

$\Rightarrow$ $\chi(t^*)$ is KNOWN

3) $\begin{cases} r = a\,(1-e\cos E) \\ r = a\left[1+e\sin\left(\frac{\chi+c_0}{\sqrt{a}}\right)\right] \end{cases}$

$\Rightarrow \begin{cases} e\sin\left(\frac{c_0}{\sqrt{a}}\right) = \frac{r_0}{a}-1 \\ e\cos\left(\frac{c_0}{\sqrt{a}}\right) = \frac{\underline{r}_0\cdot\underline{v}_0}{\sqrt{a}} \end{cases}$ $\Rightarrow$ COMPUTE c_0 $\Rightarrow$ $E(t^*) = \frac{\chi(t^*)}{\sqrt{a}} + \frac{c_0}{\sqrt{a}} + \frac{\pi}{2}$

$\triangleq \frac{\chi(t^*)}{\sqrt{a}} + E_0.$

4) $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}$ $\Rightarrow$ COMPUTE $\theta = \theta(t^*)$

5) $r(t^*) = \frac{p}{1+e\cos\theta(t^*)}$

⊣

3b) → EQUATIONS OF MOTION WITH UNIVERSAL VARIABLE χ

χ: $\frac{d\chi}{dt} = \frac{\sqrt{\mu}}{r}$ ⟶ $\frac{dt}{d\chi} = \frac{r}{\sqrt{\mu}}$ ⟶ $r = \sqrt{\mu}\,\frac{dt}{d\chi}$

$y = f(x)$ $\quad (f^{-1})'(y) = \frac{1}{f'(x)}$; e.g. $y = \ln x$ $\rightarrow$ $x = e^y = f^{-1}(y)$

$f'(x) = \frac{1}{x}$ $\quad (f^{-1})'(y) = e^y$ $\Rightarrow$ $e^y = \frac{1}{\frac{1}{x}}$

in this notation: $f'(t) = \frac{\sqrt{\mu}}{r}$

$e^{\ln x} = x$ $\rightarrow$ $x = x$ ok

$(f^{-1})'(\chi) = \frac{r}{\sqrt{\mu}}$; $\frac{r}{\sqrt{\mu}} = \frac{1}{f'(t)}$

(IT) Teorema della derivata inversa.

Starting from equivalence: $r^2 = \underline{r}\cdot\underline{r}$

Deriving in χ: $\cancel{2}\,r\,\frac{dr}{d\chi} = \cancel{2}\,\underline{r}\cdot\frac{d\underline{r}}{d\chi}$

$r\,\frac{dr}{d\chi} = \underline{r}\cdot\frac{d\underline{r}}{d\chi}$ (i)

up:

Since is unique and continuous the function: $t = t(\chi)$ $\Rightarrow$ $\frac{d\underline{r}}{d\chi} = \frac{d\underline{r}}{dt}\cdot\frac{dt}{d\chi}$

→ As a consequence of Sundman's transformation: $\frac{dt}{d\chi} = \frac{r}{\sqrt{\mu}}$

$\Rightarrow$ $\frac{d\underline{r}}{d\chi} = \frac{r}{\sqrt{\mu}}\cdot\underline{v}$ ← $\left(\underline{v} = \frac{d\underline{r}}{dt}\right)$ (ii)

r → scalar.

(ii) → (i)

$\cancel{r}\,\frac{dr}{d\chi} = \underline{r}\cdot\left(\frac{\cancel{r}}{\sqrt{\mu}}\cdot\underline{v}\right)$

$\frac{dr}{d\chi} = \frac{\underline{r}\cdot\underline{v}}{\sqrt{\mu}}$ $\Rightarrow$ $\boxed{\sigma \triangleq \frac{dr}{d\chi} = \frac{\underline{r}\cdot\underline{v}}{\sqrt{\mu}}}$

Doing a second differentiation in χ:

$\frac{d^2r}{d\chi^2} = \frac{d}{d\chi}\left(\frac{\underline{r}\cdot\underline{v}}{\sqrt{\mu}}\right)$

$= \frac{d}{dt}\left(\frac{\underline{r}\cdot\underline{v}}{\sqrt{\mu}}\right)\cdot\frac{dt}{d\chi} = \frac{1}{\sqrt{\mu}}\left(v^2 + \underline{r}\cdot\frac{d\underline{v}}{dt}\right)\cdot\frac{r}{\sqrt{\mu}}$

$= \frac{1}{\mu}\left(v^2 + \underline{r}\cdot\frac{d\underline{v}}{dt}\right)r$ (iii)

↳ cause of energy conservation: $\varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}.$

→ $v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$ (iv)

$$\frac{d^2r}{dx^2} = \frac{1}{\mu}\left(v^2 + \underline{r}\cdot\frac{d\underline{v}}{dt}\right) r \quad (iii) \quad ; \quad v^2 = \frac{2\mu}{r} - \frac{\mu}{a} \quad (iv)$$

(iv) $\to$ (iii)

$$\frac{d^2r}{dx^2} = \frac{1}{\mu}\left(\frac{2\mu}{r} - \frac{\mu}{a} + \underline{r}\cdot\frac{d\underline{v}}{dt}\right)\cdot r$$

$$\frac{d^2r}{dx^2} = \frac{r}{\mu}\cdot\left(\frac{2\mu}{r} - \frac{\mu}{a} + \underline{r}\cdot\frac{d\underline{v}}{dt}\right)$$

$\hookrightarrow$ Cause of equation of motion we have $\frac{d\underline{v}}{dt} = \frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{\|\underline{r}\|^3}\underline{r}$

$$\Rightarrow \frac{d^2r}{dx^2} = \frac{r}{\mu}\left(\frac{2\mu}{r} - \frac{\mu}{a} + \underline{r}\cdot\left(-\frac{\mu}{\|\underline{r}\|^3}\underline{r}\right)\right)$$

$$\frac{d^2r}{dx^2} = \frac{r}{\mu}\left(\frac{2\mu}{r} - \frac{\mu}{a} + r^2\left(-\frac{\mu}{r^3}\right)\right)$$

$$\frac{d^2r}{dx^2} = \frac{r}{\mu}\left(\frac{2\mu}{r} - \frac{\mu}{a} - \frac{\mu}{r}\right) = 1 - \frac{r}{a}.$$

$$\Rightarrow \frac{d^2r}{dx^2} = 1 - \alpha r \quad ; \quad \alpha \triangleq \frac{1}{a}$$

$\hookrightarrow$ Deriving one time more in x variable we obtain:

$$\frac{d^3r}{dx^3} + \alpha\frac{dr}{dx} = 0$$

Equation of motion (VALID FOR EACH CONIC)

$$\frac{d^3r}{dx^3} + \alpha\frac{dr}{dx} = 0 \quad \xrightarrow[\sigma = \frac{\underline{r}\cdot\underline{v}}{\sqrt{\mu}}]{\sigma \triangleq \frac{dr}{dx}} \quad \frac{d^2\sigma}{dx^2} + \alpha\sigma = 0$$

$(\alpha = \frac{1}{a})$

$$\xrightarrow{r = \sqrt{\mu}\frac{dt}{dx}} \quad \frac{d^4t}{dx^4} + \alpha\frac{dt^2}{dx^2} = 0$$

$\Rightarrow$ SOLVING THIS EQUATION OF MOTION (MORE CONFORTABLE IN σ)

$$\Rightarrow \frac{d^2\sigma}{dx^2} + \alpha\sigma = 0 \quad (i)$$

Such equation should be solved with finding a solution in form: $\sigma = A\cos(\alpha x) + B\sin(\alpha x)$

otherwise is possible to search a solution in form:

$$\sigma = \sum_{k=0}^{\infty} a_k x^k.$$

I'm using the set $P(x) = \{0, 1, x, \dots, x^n\}$ to approximate each $\mathbb{R}$ function. (*)

$$\frac{d\sigma}{dx} = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$\frac{d^2\sigma}{dx^2} = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Posing $k = n-2 \to n = k+2 \quad \Rightarrow \quad \frac{d^2\sigma}{dx^2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k$

$\Rightarrow$ Substituting into (i)

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k + \alpha\cdot\left(\sum_{k=0}^{\infty} a_k x^k\right) = 0$$

$$\Rightarrow \forall k: \quad \alpha a_k = -a_{k+2}\cdot(k+2)(k+1) \quad \Rightarrow \quad a_{k+2} = -\frac{\alpha a_k}{(k+2)(k+1)}$$

In example:

$$a_2 = -\frac{\alpha}{1\cdot 2}a_0 \;;\; a_3 = -\frac{\alpha a_1}{2\times 3} \;;\; a_4 = \frac{-\alpha a_2}{4\cdot 3} = \frac{-\alpha}{4\cdot 3}\cdot\frac{\alpha}{1\cdot 2}$$

$\Rightarrow$ In general:

$k \to$ "PARI" $\quad a_{k+2} = \frac{\alpha^k}{(k+2)!}\cdot a_0.$

$k \to$ "DISPARI" $\quad a_{k+2} = \frac{\alpha^{k-1}}{(k+2)!}a_1 \quad$ in fact $\; a_5 = -\frac{\alpha a_3}{5\cdot 4} = -\frac{\alpha^2}{5\cdot 4\cdot 3\cdot 2}a_1$

By Defining:

$$U_0 = 1 - \frac{\alpha x^2}{2!} + \frac{(\alpha x^2)^2}{4!} \dots$$

$$U_1 = x\left[1 - \frac{\alpha x^2}{3!} + \frac{(\alpha x^2)^2}{5!} - \dots\right]$$

(*) it's easy to demonstrate that: $U_0 \equiv 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = \cos(0) - \frac{\overset{=0}{\sin(0)}}{1!}x - \frac{\cos(0)}{2!}x^2 \simeq \cos x$

$$U_1 \equiv x - \frac{x^3}{3!} + \dots = \sin(0) + \frac{\cos(0)}{1!}x - \frac{\sin(0)}{2!}x^2 \simeq \sin x.$$

As obvious we obtained a linear combination of $\sin x$ and $\cos x$

$$\Rightarrow \quad \sigma = a_0 \underbrace{U_0(x, \alpha)}_{\sim \cos(\alpha x)} + a_1 \underbrace{U_1(x, \alpha)}_{\sim \sin(\alpha x)}$$

it's easy to observe also that

$$\int_0^x U_0(\tilde{x})\, d\tilde{x} = U_1(x)$$

$$\int \cos x \, dx = \sin x$$

Proceeding in the integration: Clearly we want a solution also for $\begin{cases} \dfrac{d^3 r}{dx^3} + \alpha \dfrac{dr}{dx} = 0 \\ \dfrac{d^4 t}{dx^4} + \alpha \dfrac{dt^2}{dx^2} = 0. \end{cases}$

$$U_2 = \int_0^x U_1(\tilde{x})\, d\tilde{x} \quad ; \quad U_3 = \int_0^x U_2(\tilde{x})\, d\tilde{x}$$

$$\Rightarrow U_0 = 1 - \frac{\alpha x^2}{2!} + \frac{(\alpha x^2)^2}{4!} - \dots \qquad \Rightarrow \qquad \begin{cases} U_0 + \alpha U_2 = 1 \\ U_1 + \alpha U_3 = x \end{cases}$$

$$U_1 = x \left[1 - \frac{\alpha x^2}{3!} + \frac{(\alpha x^2)^2}{5!} - \dots \right.$$

$$U_2 = x^2 \left[\frac{1}{2!} - \frac{\alpha x^2}{4!} + \frac{(\alpha x^2)^2}{6!} - \dots \right.$$

$$U_3 = x^3 \left[\frac{1}{3!} - \frac{\alpha x^2}{5!} + \frac{(\alpha x^2)^2}{7!} - \dots \right.$$

SINCE

$$\begin{cases} U_0 + \alpha U_2 = 1 & \rightarrow \quad \dfrac{dU_0}{dx} + \alpha \dfrac{dU_2}{dx} = 0 \quad \rightarrow \quad \dfrac{d^2 U_0}{dx^2} + \alpha \dfrac{d^2 U_2}{dx^2} = 0 \\ U_1 + \alpha U_3 = x & \rightarrow \quad \dfrac{dU_1}{dx} + \alpha \dfrac{dU_3}{dx} = 1 \quad \rightarrow \quad \dfrac{d^2 U_1}{dx} + \alpha \dfrac{d^2 U_3}{dx} = 0 \end{cases}$$

AND SINCE

$$\frac{d^2 U_2}{dx^2} = U_0 \quad \text{(By definition)}$$

$$\frac{d^2 U_3}{dx^2} = U_1$$

THEN

U_0 is particular solution of : $\dfrac{d^3 r}{dx^3} + \alpha \dfrac{dr}{dx} = 0$ and $\dfrac{d^4 t}{dx^4} + \alpha \dfrac{d^2 t}{dx^2} = 0$

U_1 " " " " " " " " "

1 ⇒

$(P_1) \quad \dfrac{d^2 \sigma}{dx^2} + \alpha \sigma = 0 \qquad$ sol: $\sigma(x) = a_0 U_0(x, \alpha) + a_1 U_1(x, \alpha)$

$(P_2) \quad \dfrac{d^3 r}{dx^3} + \alpha \dfrac{dr}{dx} = 0 \qquad$ sol: $r(x) = a_0 U_0(x, \alpha) + a_1 U_1(x, \alpha) + a_2 U_2(x, \alpha)$

$(P_3) \quad \dfrac{d^4 t}{dx^4} + \alpha \dfrac{d^2 t}{dx^2} = 0 \qquad$ sol: $t(x) = a_0 U_0(x, \alpha) + a_1 U_1(x, \alpha) + a_2 U_2(x, \alpha) + a_3 U_3(x, \alpha)$.

2 ⇒

U_2 is solution (particular) of $\dfrac{d^2 \sigma}{dx^2} + \alpha \sigma = 0$

In facts $\begin{cases} U_0 = \dfrac{d^2 U_2}{dx^2} \\ U_0 \Rightarrow \text{solution of } \dfrac{d^2 \sigma}{dx^2} + \alpha \sigma = 0 \end{cases} \Rightarrow \dfrac{d^4 U_2}{dx^4} + \alpha \dfrac{d^2 U_2}{dx^2} = 0 \Rightarrow$ OBTAINING AGAIN EQUATION $\dfrac{d^4 t}{dx^4} + \alpha \dfrac{d^2 t}{dx^2} = 0$. THAT IS CONSEQUENCE OF EQ ON σ.

substituting in $\dfrac{d^2 \sigma}{dx^2} + \alpha \sigma$.

so solving $\dfrac{d^4 t}{dx^4} + \alpha \dfrac{d t^2}{dx^4}$ in a general solution (without imposing b.c.)

$$\sqrt{\mu}\,(t - t_0) = a_0 U_0 + a_1 U_1 + a_2 U_2 + a_3 U_3$$

since $\{a_1; a_2; a_3; a_0\}$ are coefficients determined by b.c. i write solution in such form that is more similar to the immediate Kepler's law: $\sqrt{\frac{\mu}{a^3}} = E - e \sin E$.
FORMALLY writing a formula as $t = t(x)$ has the same meaning.

a_0

b.c.$_0$ $\begin{cases} t = t_0 \\ x = 0 \end{cases}$ for $x = 0$ $\{U_1; U_2; U_3\} = 0 \Rightarrow a_0 = 0$.

a_1

SUNDMAN'S TRANSFORMATION $\quad r = \sqrt{\mu} \dfrac{dt}{dx} \rightarrow r = a_1 \dfrac{dU_1}{dx} + a_2 \dfrac{dU_2}{dx} + a_3 \dfrac{dU_3}{dx}$

$= a_1 U_0 + a_2 U_1 + a_3 U_2$

b.c.$_1$ $\begin{cases} t = t_0 \\ x = 0 ; \; r = r_0 \end{cases}$

$\Rightarrow$ for $x = 0$ $\begin{cases} \{U_1; U_2; U_3\} = 0 \\ U_0 = 1 \end{cases} \Rightarrow r_0 = a_1 \Rightarrow a_1 = r_0$

$a_2; a_3$

$$\sigma \triangleq \frac{dr}{dx} = \frac{r \cdot v}{\sqrt{\mu}} \rightarrow \sigma = r_0 \frac{dU_0}{dx} + a_2 \frac{dU_1}{dx} + a_3 \frac{dU_2}{dx}$$

$$= -\alpha r_0 U_1 + a_2 U_0 + a_3 U_1$$

$$\frac{dU_0}{dx} = -\alpha U_1$$

b.c.$_2$ $\begin{cases} t = t_0 \\ x = 0 ; \; \sigma = \sigma_0 = \dfrac{r_0 \cdot v_0}{\sqrt{\mu}}. \end{cases}$

!! There is only 1 set of b.c. i add the number: b.c. $t = t_0$ $\begin{cases} x = 0 \\ r = r_0 \end{cases}$; $v = v_0$; $\left. \dfrac{d\sigma}{dx} \right|_{t = t_0} = 1 \cdot \alpha$

$$\Rightarrow \quad \sigma = -\alpha r_0 U_1 + a_2 U_0 + a_3 U_1 \qquad bc_2 \begin{cases} \tau = \tau_0 \\ x = 0; \ \sigma = \sigma_0 = \frac{r_0 \cdot v_0}{\sqrt{\mu}} \end{cases}$$

Derivating a 2^{nd} time on σ.

$$\frac{d\sigma}{dx} = -\alpha r_0 U_0 + \alpha a_2 U_1 + a_3 U_0 \qquad bc_3 \begin{cases} \tau_0 \\ x=0; \ r = r_0. \end{cases}$$

$$\frac{d\sigma}{dx} = \frac{d^2 r}{dx^2} = 1 - \alpha r \quad \Rightarrow \begin{cases} 1 - \alpha r_0 = -\alpha r_0 U_0\Big|_{\substack{\tau=\tau_0 \\ x=0}} - \alpha a_2 \cancel{U_1}^{=0}\Big|_{\substack{\tau=\tau_0 \\ x=0}} + a_3 U_0\Big|_{\substack{\tau=\tau_0 \\ x=0}} \\ \sigma_0 = \frac{r_0 v_0}{\sqrt{\mu}} = -\alpha r_0 U_1\Big|_{\substack{\tau=\tau_0 \\ x=0}} + a_2 U_0\Big|_{\substack{\tau=\tau_0 \\ x=0}} + a_3 U_1\Big|_{\substack{\tau=\tau_0 \\ x=0}} \end{cases}$$

$$\left(U_0\Big|_{\substack{\tau=\tau_0 \\ x=0}} = 1 \ ; \quad U_1\Big|_{\substack{\tau=\tau_0 \\ x=0}} = 0. \right)$$

$$\Rightarrow \begin{cases} 1 - \cancel{\alpha r_0} = a_3 - \cancel{\alpha r_0} \rightarrow a_3 = 1 \\ \sigma_0 = a_2. \end{cases}$$

Full solution of equation of motion - $\tau = \tau(x)$

$$\frac{d^3 r}{dx^3} + \alpha \frac{dr}{dx} = 0 \quad \xrightarrow{dt/dx = \frac{r}{\sqrt{\mu}}} \quad (P) \ \frac{d^4 t}{dx^4} + \alpha \frac{d^2 t}{dx^2} = 0 \ + \ \underline{b.c.}$$

$$\left(\alpha = \frac{1}{a}\right)$$

sol: $\sqrt{\mu}(\tau - \tau_0) = a_0 U_0 + a_1 U_1 + a_2 U_2 + a_3 U_3$

$$U_0 = 1 - \frac{\alpha x^2}{2!} + \dots \ ; \quad U_1 = x\left[1 - \frac{\alpha x^2}{3!} \dots\right]$$

$$U_2 = \int U_1 \, dx = x^2 \left[\frac{1}{2!} - \frac{\alpha x^2}{4!} \dots\right]$$

$$U_3 = \int U_2 \, dx = x^3 \left[\frac{1}{3!} - \frac{\alpha x^2}{5!} \dots\right]$$

$$\underline{b.c.} \ \text{for } \tau = \tau_0 \begin{cases} x = 0 \ (1) \\ r = r_0 \\ v = v_0 \\ \sigma = \sigma_0 = \frac{r_0 \cdot v_0}{\sqrt{\mu}} \end{cases}$$

$$\frac{d\sigma}{dx}\Big|_{\tau_0} = 1 - \alpha r_0 \quad \Rightarrow \begin{cases} a_0 = 0 \\ a_1 = r_0 \\ a_2 = \sigma_0 \ ; \ a_3 = 1 \end{cases}$$

THIS bc SUBSTITUTE $\sigma_0 = \frac{r_0 \cdot v_0}{\sqrt{\mu}} = \sigma\Big|_{t=t_0}$ that is useless because r_0, v_0 are not known already knowing conditions.

FROM THE DERIV: $\quad r^2 = \underline{r} \cdot \underline{r} \quad \rightarrow \quad 2r \frac{dr}{dx} = 2\underline{r} \cdot \frac{d\underline{r}}{dx}$

$$\rightarrow \frac{d\underline{r}}{dx} = \frac{d\underline{r}}{dt}\frac{dt}{dx} = \frac{d\underline{r}}{dt}\frac{r}{\sqrt{\mu}} = \frac{r}{\sqrt{\mu}} \underline{v} \quad \Rightarrow \quad \cancel{r}\frac{dr}{dx} = \frac{\cancel{r}}{\sqrt{\mu}} \underline{v} \cdot \underline{r} \overset{\Delta}{=} \sigma$$

$$\rightarrow \sigma = \frac{dr}{dx} \quad \Rightarrow \quad \frac{d^2 r}{dx^2} = \frac{d\sigma}{dx} = \frac{d}{dt}\left(\frac{\underline{r} \cdot \underline{v}}{\sqrt{\mu}}\right)\frac{dt}{dx} = \underset{\text{ENERGY, SOLUTION, EQ. OF MOTION } \ddot{\underline{r}} = -\frac{\mu}{r^3}\underline{r}}{\dots} \quad \Rightarrow \quad \frac{d\sigma}{dx} = 1 - \alpha r$$

$$\Rightarrow \boxed{\sqrt{\mu}(t - t_0) = r_0 U_1 + \left(\frac{r_0 \cdot v_0}{\sqrt{\mu}}\right) U_2 + U_3}$$

SAME "PROCEDURE" CAN BE TO DETERMINE $r = r(x)$

!!! Big difference to determine $r = r(x)$ instead of $\underline{r} = \underline{r}(x)$!!

$$r = a_0 U_0 + a_1 U_1 + a_2 U_2$$

a_0

$$r_0 = a_0$$

a_1

$$\frac{dr}{dx} = -\alpha r_0 U_1 + a_1 U_0 + a_2 U_1 \Rightarrow \sigma_0 = a_1$$

a_2

$$\frac{d^2 r}{dx^2} = -\alpha r_0 U_0 + \alpha a_1 U_1 + a_2 U_0 \Rightarrow 1 - \cancel{\alpha r_0} = -\cancel{\alpha r_0} + a_2 \Rightarrow a_2 = 1$$

$\Rightarrow r = r(x)$

$$\boxed{r = r_0 U_0 + \left(\frac{r_0 \cdot v_0}{\sqrt{\mu}}\right) U_1 + U_2.}$$

3c) → LAGRANGE'S COEFFICIENTS

SINCE

motion is contained in a plane (fully contained) $\perp \hat{h}$, $\in F_1 F_2$

THEN

$\{\underline{r}_0; \underline{v}_0\}$ constitute a complete set of basis to describe the motion

$\Rightarrow$ $\underline{r}, \underline{v}$ can be expressed as linear combination of $\{\underline{r}_0, \underline{v}_0\}$

$\Downarrow$

$\exists\ F(t);\ g(t)\ :\ \forall t$ $\quad \underline{r}(t) = F(t)\,\underline{r}_0 + g(t)\,\underline{v}_0$

$\Downarrow$

$\underline{v}(t) = \dot{F}(t)\,\underline{r}_0 + \dot{g}(t)\,\underline{v}_0.$

Calculating the angular momentum (for unit-mass) $\underline{h}$

$$\Rightarrow \underline{h} = \underline{r} \times \underline{v} = (F\,\underline{r}_0 + g\,\underline{v}_0) \times (\dot{F}\,\underline{r}_0 + \dot{g}\,\underline{v}_0)$$

$$= \underbrace{F\dot{F}\,\underline{r}_0 \times \underline{r}_0}_{=0} + F\dot{g}\,\underline{r}_0 \times \underline{v}_0 + g\dot{F}\,\underline{v}_0 \times \underline{r}_0 + 0$$

$$\Rightarrow \underline{h} = (F\dot{g} - g\dot{F})\,\underline{h}_0$$

$$1 = F\dot{g} - g\dot{F}$$

!! Because motion is Keplerian $\underline{h}$ is a 1st integral of motion (of int)

$\Rightarrow \underline{h}_0 = \underline{r}_0 \times \underline{v}_0 = \underline{r}(t) \times \underline{v}(t) = \underline{h}(t)\ \forall t$!!

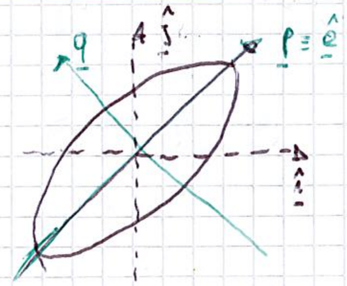
$\Rightarrow$ $F(t);\ g(t)$: $\quad \underline{r} = F\,\underline{r}_0 + g\,\underline{v}_0 \ \rightarrow\ \underline{v} = \dot{F}\,\underline{r}_0 + \dot{g}\,\underline{v}_0$

$$\Rightarrow F\dot{g} - g\dot{F} = 1$$

↳ 1st interest is to express F; g as a function of θ.

otherwise is possible again to use χ (universal variable)

Known that:



Using perifocal system of coordinates.

$\hat{i}; \hat{j}$ are such that $\hat{i} \equiv \hat{e}$

$\Rightarrow \{\underline{i}; \underline{j}\} \rightarrow \{\underline{p}; \underline{q}\}$.

$$\Rightarrow \underline{r} = r\cos\theta\,\underline{p} + r\sin\theta\,\underline{q}$$

Velocity expressed in perifocal reference system.

$$\underline{r} = r\cos\theta\,\hat{\underline{p}} + r\sin\theta\,\hat{\underline{q}} \quad ; \quad r = \frac{p}{1+e\cos\theta} \quad ; \quad \underline{h} = \underline{r} \times \underline{v}$$

$$\underline{h} = \underline{r} \times \underline{v} = r\,\hat{\underline{r}} \times (\dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}) = r^2\dot{\theta}\,\hat{\underline{h}} \ \Rightarrow\ h = r^2\dot{\theta}$$

$\Downarrow$

$$\begin{cases} \underline{v} = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\,\hat{\underline{p}} + (\dot{r}\sin\theta + \dot{\theta}\,r\cos\theta)\,\hat{\underline{q}} \\ \dot{r} = -\dfrac{p\,(-e\sin\theta)\,\dot{\theta}}{(1+e\cos\theta)^2} = \dfrac{p\,e\sin\theta\,\dot{\theta}}{(1+e\cos\theta)^2} \end{cases}$$

$$\underline{v} = \left[\frac{\dot{\theta}\,p\,e\sin\theta\cos\theta}{(1+e\cos\theta)^2} - \frac{p\sin\theta\,\dot{\theta}}{(1+e\cos\theta)}\right]\hat{\underline{p}} + \left[\frac{\dot{\theta}\,p\,e\sin^2\theta}{(1+e\cos\theta)^2} + \frac{p\cos\theta\,\dot{\theta}}{(1+e\cos\theta)}\right]\hat{\underline{q}}$$

$$= \left[\frac{\cancel{\dot{\theta}\,p\,e\sin\theta\cos\theta} - \dot{\theta}\,p\sin\theta - \cancel{p\,e\sin\theta\cos\theta}\,\dot{\theta}}{(1+e\cos\theta)^2}\right]\hat{\underline{p}} + \left[\frac{p\,e\sin^2\theta\,\dot{\theta} + p\cos\theta\,\dot{\theta} + p\,e\cos^2\theta\,\dot{\theta}}{(1+e\cos\theta)^2}\right]\hat{\underline{q}}$$

$\sin^2\theta + \cos^2\theta = 1$

$$\underline{v} = -\frac{p}{(1+e\cos\theta)^2}\,\dot{\theta}\,\sin\theta\,\hat{\underline{p}} + \left[\frac{pe + p\cos\theta}{(1+e\cos\theta)^2}\,\dot{\theta}\right]\hat{\underline{q}}$$

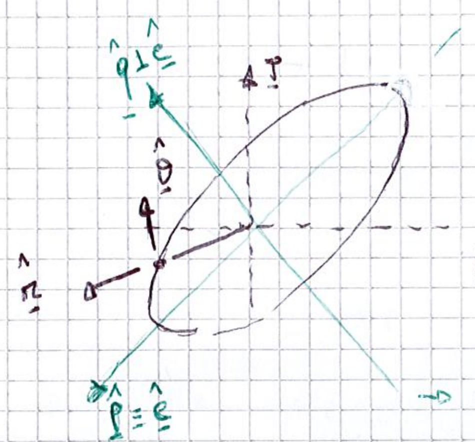
$$= -\frac{1}{p}\,\underbrace{\frac{p^2}{(1+e\cos\theta)^2}\,\dot{\theta}}_{h = r^2\dot{\theta}}\,\sin\theta\,\hat{\underline{p}} + \left[\frac{1}{p}\,\frac{p^2}{(1+e\cos\theta)^2}\,\dot{\theta}\,(e+\cos\theta)\right]\hat{\underline{q}}$$

$$= -\frac{h}{p}\sin\theta\,\hat{\underline{p}} + \frac{h}{p}(e+\cos\theta)\,\hat{\underline{q}} \quad ; \quad p \triangleq \frac{h^2}{\mu} \rightarrow h = \sqrt{\mu p}$$

$$\Rightarrow \underline{v} = \sqrt{\frac{\mu}{p}}\left[\sin\theta\,\hat{\underline{p}} + (e+\cos\theta)\,\hat{\underline{q}}\right]$$

$\underline{v}$ → Velocity in perifocal system reference.

Obs: About the reference system. !! The form of velocity hodograph is different !!



$\rightarrow \underline{v} = \sqrt{\frac{\mu}{p}} \left[\sin\theta \, \hat{r} + (1 + e\cos\theta) \hat{\theta} \right]$

$\{ \hat{r} ; \hat{\theta} \} \rightarrow$ local reference System.

$\{ \hat{r} ; \hat{\theta} \} \rightarrow d$ on T $\Rightarrow$ NOT INERTIAL.

$\rightarrow \underline{v} = \sqrt{\frac{\mu}{p}} \left[-\sin\theta \, \hat{p} + (e + \cos\theta) \hat{q} \right]$

$\{ \hat{p}, \hat{q} \} \rightarrow$ Perifocal system reference.

$\{ \hat{p}, \hat{q} \} \not{d}$ on T. $\Rightarrow$ INERTIAL.

$$\begin{Bmatrix} \hat{r} \\ \hat{\theta} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \hat{p} \\ \hat{q} \end{Bmatrix}$$

$\Rightarrow$ Also the reference system $\{ \underline{r}_0, \underline{v}_0 \}$ is time independent.

$$\Rightarrow \begin{Bmatrix} \underline{r}_0 \\ \underline{v}_0 \end{Bmatrix} = \begin{bmatrix} r_0 \cos\theta_0 & ; & r_0 \sin\theta_0 \\ -\sqrt{\frac{\mu}{p}} \sin\theta_0 & ; & \sqrt{\frac{\mu}{p}} (e + \cos\theta_0) \end{bmatrix} \begin{Bmatrix} \hat{p} \\ \hat{q} \end{Bmatrix} \rightarrow \begin{Bmatrix} \underline{r}_0 \\ \underline{v}_0 \end{Bmatrix} = \left[\underline{\underline{A}} \right] \begin{Bmatrix} \hat{p} \\ \hat{q} \end{Bmatrix}$$

!

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \underline{\underline{A}}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(\underline{\underline{A}}) = r_0 \cos\theta_0 \sqrt{\frac{\mu}{p}} \, e + r_0 \sqrt{\frac{\mu}{p}} \cos^2\theta_0 + r_0 \sqrt{\frac{\mu}{p}} \sin^2\theta_0$$

$$= r_0 \sqrt{\frac{\mu}{p}} \left(e\cos\theta_0 + 1 \right)$$

$$\underline{\underline{A}}^{-1} = \frac{1}{r_0 \sqrt{\frac{\mu}{p}} (e\cos\theta_0 + 1)} \begin{bmatrix} \sqrt{\frac{\mu}{p}} (e + \cos\theta_0) & ; & -r_0 \sin\theta_0 \\ \sqrt{\frac{\mu}{p}} \sin\theta_0 & ; & r_0 \cos\theta_0 \end{bmatrix}$$

!! don't be too stupid !!

$$r_0 = \frac{p}{1 + e\cos\theta_0} \Rightarrow \frac{p}{1 + e\cos\theta_0} \sqrt{\frac{\mu}{p}} \cancel{(1 + e\cos\theta_0)} = \sqrt{\mu p} \rightarrow \det(\underline{\underline{A}}) = \sqrt{\mu p}$$

$$\sqrt{\frac{\mu}{p}} \cdot \frac{1}{\sqrt{\mu p}} = \frac{1}{p}$$

$$\Rightarrow \underline{\underline{A}}^{-1} = \begin{bmatrix} \frac{e + \cos\theta_0}{p} & ; & -\frac{r_0 \sin\theta_0}{\sqrt{\mu p}} \\ \frac{\sin\theta_0}{p} & ; & \frac{r_0 \cos\theta_0}{\sqrt{\mu p}} \end{bmatrix} \quad ; \text{ similarly } \begin{Bmatrix} \hat{p} \\ \hat{q} \end{Bmatrix} = \left[\underline{\underline{A}}^{-1} \right] \begin{Bmatrix} \underline{r}_0 \\ \underline{v}_0 \end{Bmatrix}.$$

$$\hat{p} = \frac{e + \cos\theta_0}{p} \underline{r}_0 - \frac{r_0 \sin\theta_0}{\sqrt{\mu p}} \underline{v}_0$$

$$\hat{q} = \frac{\sin\theta_0}{p} \underline{r}_0 + \frac{r_0 \cos\theta_0}{\sqrt{\mu p}} \underline{v}_0 .$$

$\rightarrow$ to obtain $\underline{r}, \underline{v}$ is enough to substitute the $\begin{Bmatrix} \hat{p} \\ \hat{q} \end{Bmatrix}$ into $\begin{Bmatrix} \underline{r} \\ \underline{v} \end{Bmatrix}$ considering:

$$\Delta\theta \triangleq \theta(t) - \theta_0 .$$

$$\underline{r} = \cos\theta \, r \, \hat{p} + r \sin\theta \, \hat{q} = \frac{\cos\theta \, p}{1 + e\cos\theta} \hat{p} + \frac{p \sin\theta}{1 + e\cos\theta} \hat{q}$$

$$\underline{r} = \frac{\cancel{p}\cos\theta}{1 + e\cos\theta} \cdot \frac{e + \cos\theta_0}{\cancel{p}} \underline{r}_0 - \frac{p\cos\theta}{1 + e\cos\theta} \cdot \frac{r_0 \sin\theta_0}{\sqrt{\mu p}} \underline{v}_0 +$$

$$+ \frac{\cancel{p}\sin\theta}{1 + e\cos\theta} \cdot \frac{\sin\theta_0}{\cancel{p}} \underline{r}_0 + \frac{p\sin\theta}{1 + e\cos\theta} \cdot \frac{r_0 \cos\theta_0}{\sqrt{\mu p}} \underline{v}_0$$

$$\underline{r} = \left[\frac{e\cos\theta}{1 + e\cos\theta} + \frac{\cos\theta\cos\theta_0}{1 + e\cos\theta} + \frac{\sin\theta\sin\theta_0}{1 + e\cos\theta} \right] \underline{r}_0 +$$

$$+ \sqrt{\frac{p}{\mu}} \left[\frac{-\sin\theta_0\cos\theta}{1 + e\cos\theta} + \frac{\sin\theta\cos\theta_0}{1 + e\cos\theta} \right] \cdot r_0 \, \underline{v}_0$$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

& $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$\underline{r} = \left[\frac{e\cos\theta}{1 + e\cos\theta} + \frac{\cos\Delta\theta}{1 + e\cos\theta} \right] \underline{r}_0 + \sqrt{\frac{p}{\mu}} \frac{p}{1 + e\cos\theta_0} \left[\frac{\sin\Delta\theta}{1 + e\cos\theta} \right] \underline{v}_0 .$$

$$\sqrt{\frac{p}{\mu}} \frac{1}{p} \overbrace{\frac{p}{1 + e\cos\theta}}^{r} \overbrace{\frac{p}{1 + e\cos\theta_0}}^{r_0} = \frac{r \, r_0}{\sqrt{\mu p}}$$

$$! \quad r = \frac{p}{1 + e\cos\theta} \rightarrow e\cos\theta = \frac{p - r}{r} \, !$$

$$\cdot \frac{e\cos\theta}{1 + e\cos\theta} = \frac{p - r}{r p} \frac{p}{1 + e\cos\theta} = \frac{p - r}{\cancel{r} p} \cancel{r} = \frac{p - r}{p} = 1 - \frac{r}{p}$$

$$\cdot \frac{\cos\Delta\theta}{1 + e\cos\theta} = \frac{p}{1 + e\cos\theta} \cdot \frac{\cos\Delta\theta}{p} = \frac{r}{p} \cos\Delta\theta$$

$$\underline{r} = \left[1 - \frac{r}{p} \left(1 - \cos\Delta\theta \right) \right] \underline{r}_0 + \frac{r \, r_0}{\sqrt{\mu p}} \sin\Delta\theta \, \underline{v}_0 . \quad (i)$$

while:

$$\underline{v} = \sqrt{\frac{\mu}{p}}\left[-\sin\theta\,\hat{\underline{r}} + (e+\cos\theta)\,\hat{\underline{q}}\right]$$

$$= \sqrt{\frac{\mu}{p}}\left[-\sin\theta\,\frac{e+\cos\theta_0}{p}\,\underline{r}_0 - \sin\theta\,\frac{(-r_0\sin\theta_0)}{\sqrt{\mu p}}\,\underline{v}_0 + (e+\cos\theta)\,\frac{\sin\theta_0}{p}\,\underline{r}_0 + (e+\cos\theta)\,\frac{r_0\cos\theta_0}{\sqrt{\mu p}}\,\underline{v}_0\right]$$

$$\underline{v} = \sqrt{\frac{\mu}{p}}\cdot\frac{1}{p}\left[-e\sin\theta - \cos\theta_0\sin\theta + e\sin\theta_0 + \cos\theta\sin\theta_0\right]\underline{r}_0 + \sqrt{\frac{\mu}{p}}\cdot\frac{1}{\sqrt{\mu p}}\left[+r_0\sin\theta\sin\theta_0 + r_0\cos\theta\cos\theta_0 + e\,r_0\cos\theta_0\right]$$

- $e\cos\theta_0 = \frac{p-r_0}{r_0} \rightarrow e\,r_0\cos\theta_0 = p-r_0 \rightarrow e\,r_0\cos\theta_0 =$ ...
- $-\sin(\alpha-\beta) = \cos\alpha\sin\beta - \sin\alpha\cos\beta$; $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

$$\underline{v} = \frac{1}{p}\sqrt{\frac{\mu}{p}}\left[-e\sin\theta + e\sin\theta_0 - \sin\Delta\theta\right]\underline{r}_0 + \frac{1}{p}\left[r_0\cos\Delta\theta + p - r_0\right]\underline{v}_0$$

$$= \frac{\sqrt{\mu}}{p\sqrt{p}}\left[-e\sin(\theta_0+\Delta\theta) + e\sin\theta_0 - \sin\Delta\theta\right]\underline{r}_0 + \left[\frac{r_0}{p}\cos\Delta\theta + 1 - \frac{r_0}{p}\right]\underline{v}_0$$

- $\sin(\theta_0+\Delta\theta) = \sin\theta_0\cos\Delta\theta + \cos\theta_0\cdot\sin\Delta\theta$

$$\underline{v} = \frac{\sqrt{\mu}}{p\sqrt{p}}\left[-e\sin\theta_0\cos\Delta\theta - e\cos\theta_0\sin\Delta\theta + e\sin\theta_0 - \sin\Delta\theta\right]\underline{r}_0 + \left[1+\frac{r_0}{p}(\cos\Delta\theta-1)\right]$$

$$= \frac{\sqrt{\mu}}{p\sqrt{p}}\left[e\sin\theta_0(1-\cos\Delta\theta) - \sin\Delta\theta\,\underbrace{(1+e\cos\theta_0)}_{r_0=\frac{p}{1+e\cos\theta} \rightarrow 1+e\cos\theta=\frac{p}{r_0}}\right]\underline{r}_0 + \left[1+\frac{r_0}{p}(\cos\Delta\theta-1)\right]\underline{v}_0$$

$$= \underline{\frac{\sqrt{\mu}}{p\sqrt{p}}}\left[e\sin\theta_0(1-\cos\Delta\theta) - \frac{p}{r_0}\sin\Delta\theta\right]\underline{r}_0 + \left[1+\frac{r_0}{p}(\cos\Delta\theta-1)\right]\underline{v}_0.$$

But:

$$\underline{r}\cdot\underline{v} = r\,\hat{r}\cdot(\dot{r}\,\hat{\underline{r}} + r\dot\theta\,\hat{\underline{\theta}}) = r\cdot\dot r\,\hat{\underline{r}} = \frac{p}{1+e\cos\theta}\cdot\frac{+p\,e\sin\theta}{(1+e\cos\theta)^2}\,\dot\theta$$

$$h = r^2\dot\theta \;;\quad \frac{h^2}{\mu} = p \rightarrow h = \sqrt{\mu p}$$

$$\Rightarrow \frac{e\sin\theta}{1+e\cos\theta}\,h = \underline{r}\cdot\underline{v} \Rightarrow \underline{r}\cdot\underline{v} = r\,e\sin\theta\cdot\frac{h}{p} = \frac{r\sqrt{\mu}}{\sqrt{p}}\,e\sin\theta.$$

$$\Rightarrow \underline{v} = \left[\frac{1}{p\,r_0}\,\underline{r}_0\cdot\underline{v}_0 - \frac{\sqrt{\mu}}{p\sqrt{p}}\,\frac{p}{r_0}\sin\Delta\theta\right]\underline{r}_0 + \left[1+\frac{r_0}{p}(\cos\Delta\theta-1)\right]\underline{v}_0.\ \text{(ii)}$$

Lagrange invariants functions of $\Delta\theta$.

$$F, G:\quad \underline{r} = F\,\underline{r}_0 + G\,\underline{v}_0 \;;\quad \underline{v} = \dot F\,\underline{r}_0 + \dot G\,\underline{v}_0$$

$$\Rightarrow F\dot G - G\dot F = 1 \quad \text{(o)}$$

$$\underline{r} = \left[1 - \frac{r}{p}(1-\cos\Delta\theta)\right]\underline{r}_0 + \frac{r\,r_0}{\sqrt{\mu p}}\sin\Delta\theta\,\underline{v}_0$$

$$\begin{cases} \underline{v} = \frac{\sqrt{\mu}}{p\,r_0}\left[\sigma_0(1-\cos\Delta\theta) - \sqrt{p}\sin\Delta\theta\right]\underline{r}_0 + \left[1 - \frac{r_0}{p}(1-\cos\Delta\theta)\right]\underline{v}_0 \\ \sigma_0 = \frac{\underline{r}_0\cdot\underline{v}_0}{\sqrt{\mu}} \quad \left(= \frac{r_0}{\sqrt{p}}\,e\sin\theta_0\right) \end{cases}$$

$\Downarrow$

$$F = 1 - \frac{r}{p}(1-\cos\Delta\theta) \;;\quad G = \frac{r\,r_0}{\sqrt{\mu p}}\sin\Delta\theta.$$

$\downarrow$ more convenient if got from property (o).

$$\dot G = 1 - \frac{r_0}{p}(1-\cos\Delta\theta)$$

$$\dot F = \frac{F\dot G - 1}{G} = \sqrt{\frac{\mu}{p}}\tan\frac{\Delta\theta}{2}\left[\frac{1-\cos\Delta\theta}{p} - \frac{1}{r} - \frac{1}{r_0}\right]$$

3d) ⇒ PHYSICAL INTERPRETATION OF UNIVERSAL VARIABLE χ

" Evaluating link between χ and the 3 possible eccentric anomaly $\begin{cases} E \\ F \\ D \end{cases}$ is possible to understand the physical meaning of χ "

$$\sqrt{\frac{\mu}{a^3}}\,\Delta t = E - e\sin E \quad \rightarrow \text{Kepler's law} \Rightarrow ??\ \chi = f(E,F,D)\ ??$$

!! PHYSICAL INTERPRETATION IS DIFFERENT FOR EACH CONIC. !!

a) ELLIPS ($\chi \rightarrow E$)

$\sqrt{\frac{\mu}{a^3}}\,\Delta t = E - e\sin E$ ⇒ differentiating Kepler's law, is possible to obtain:

$$\sqrt{\frac{\mu}{a^3}}\,dt = dE - e\cos E\,dE = (1-\cos E)\,dE.$$

$$\begin{cases} \sqrt{\frac{\mu}{a^3}}\,dt = (1-e\cos E)\,dE & \rightarrow \text{Kepler's law} \\ \frac{d\chi}{dt} = \frac{\sqrt{\mu}}{r} & \rightarrow \chi\text{'s definition.} \end{cases}$$

$$\begin{cases} \sqrt{\mu}\,dt = a\sqrt{a}\,(1-e\cos E)\,dE \\ \sqrt{\mu}\,dt = r\,d\chi \end{cases} + \quad r = a(1-e\cos E)$$

from calculation of distance as a function of E.

⇒ imposing the equivalence:

$$\cancel{a}\sqrt{a}\,(\cancel{1-e\cos E})\,dE = \cancel{a}\,(\cancel{1-e\cos E})\,d\chi$$

$$d\chi = \sqrt{a}\,dE \quad \rightarrow \quad \boxed{\begin{array}{l} \chi = 0 \text{ for } t = 0 \\ \chi = \sqrt{a}\,(E - E_0) \end{array}}$$

obs: χ is simply "amplified" of a factor $\sqrt{a}$

in fact "time-relations" are pretty much the same.

in E): $\sqrt{\frac{\mu}{a^3}}\,\Delta t = E - e\sin E.$

in χ): $\sqrt{\mu}\,\Delta t = a\left[\chi - \frac{r_0 \cdot v_0}{\sqrt{a}}\cos\left(\frac{\chi}{\sqrt{a}}\right) + \left(\frac{r_0}{a} - 1\right)\sin\frac{\chi}{\sqrt{a}}\right] \ldots E_0 =$

b) PARABOLA. ($\chi \rightarrow D \triangleq \tan(\frac{\theta}{2})$)

Procedure is the same as before ⇒ differentiating time-law and using variable's transformation.

$$\frac{h}{p^2}\,\Delta t = \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2} \xrightarrow{D \triangleq \tan(\frac{\theta}{2})} \frac{h}{p^2}\,\Delta t = \frac{1}{2}D + \frac{1}{6}D^3$$

should be determined imposing $\theta(t=0) = 0$.

$$\Delta t = \frac{p^2}{h}\left[\frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{6}\tan^3\frac{\theta}{2}\right]_{\theta_0}^{\theta^+}$$

$$\frac{h^2}{\mu} = p \rightarrow h = \sqrt{\mu p} \rightarrow \frac{h}{p^2} = \sqrt{\frac{\mu p}{p^4}} = \sqrt{\frac{\mu}{p^3}}$$

$$\begin{cases} \sqrt{\frac{\mu}{p^3}}\,dt = \left(\frac{1}{2} + \frac{1}{2}D^2\right)dD \\ \frac{d\chi}{dt} = \frac{\sqrt{\mu}}{r} \end{cases} \longrightarrow \begin{cases} \sqrt{\mu}\,dt = \sqrt{p^3}\cdot\frac{1}{2}\,(1+D^2)\,dD \\ \sqrt{\mu}\,dt = r\,d\chi \end{cases} + \quad r = \frac{p}{2}\cdot(1+D^2)$$

$$\Rightarrow \sqrt{p^3}\cdot\cancel{\frac{1}{2}}\,\cancel{(1+D^2)}\,dD = \sqrt{p^2}\cdot\cancel{\frac{1}{2}}\,\cancel{(1+D^2)}\,d\chi$$

$$\sqrt{p}\,dD = d\chi \quad \Rightarrow \quad \boxed{\begin{array}{l} \chi = 0 \text{ for } t = 0 \\ \chi = \sqrt{p}\,(D - D_0) \end{array}}$$

c) HYPERBOLA.

$$\sqrt{\frac{\mu}{a^3}}\,\Delta t = F - e\sinh F \quad \rightarrow$$

$$\begin{cases} \sqrt{\frac{\mu}{a^3}}\,dt = (1 - e\cosh F)\,dF \\ \frac{d\chi}{dt} = \frac{\sqrt{\mu}}{r} \end{cases} \rightarrow \begin{cases} \sqrt{\mu}\,dt = \sqrt{a^3}\,(1-e\cosh F)\,dF \\ d\chi \cdot r = \sqrt{\mu}\,dt \end{cases} + \quad r = -a\,(1-\cosh F)$$

$$-a\,(1-\cancel{\cosh F})\,d\chi = \sqrt{a^3}\,(1-\cancel{e\cosh F})\,dF$$

$$\Rightarrow \boxed{\begin{array}{l} \chi = 0 \text{ for } t = 0 \\ \chi = \sqrt{a}\,(F - F_0) \end{array}}$$

30) → "Lagrange's invariants as function of "x" ⊕ continuation of Lambert's problem"

a is known from the solution of the computational problem:

Lambert's theorem → $TOF = [\alpha - \beta - (\sin\alpha - \sin\beta)]$ ⇒ a KNOWN

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}} \quad ; \quad \sin\frac{\beta}{2} \sqrt{\frac{s-c}{2a}}$$

(?? e/p ??)

BUT the orbit is not fully defined because eccentricity is not known.

!! TO KNOW THE ECCENTRICITY OF THE "TRANSFER" ORBIT WE NEED TO EXPRESS LAGRANGE INVARIANTS BOTH AS FUNCTION OF $\Delta\theta$ BUT ALSO AS A FUNCTION OF x ⇒ FROM THE EQUIVALENCE WILL BE POSSIBLE TO FULLY SOLVE LAMBERT'S PROBLEM. !!

$$\frac{dx}{dt} = \frac{\sqrt{\mu}}{r} \rightarrow \begin{cases} \dfrac{dt}{dx} = \dfrac{r}{\sqrt{\mu}} \\ x = x(t) \quad \exists!\ t : t = t(x)\ (Hp) \end{cases}$$

← Sundman's transformation (AS USUAL)

I^{st} differentiation in x

$$\frac{d\underline{r}}{dx} = \frac{d\underline{r}}{dt} \cdot \frac{dt}{dx} \Rightarrow \frac{d\underline{r}}{dx} = \frac{r}{\sqrt{\mu}} \cdot \underline{v}$$

II^{nd} differentiation (of $\underline{r}$) in x

$$\frac{d^2\underline{r}}{dx^2} = \frac{1}{\sqrt{\mu}} \frac{dr}{dx} \cdot \underline{v} + \frac{r}{\sqrt{\mu}} \frac{d\underline{v}}{dx}$$

$\begin{cases} r = r(t) \\ t = t(x) \end{cases}$ e.g. $\begin{cases} \sqrt{\dfrac{\mu}{a^3}}\, \Delta t = E - e \sin E \\ x = \sqrt{a}\,(E - E_0) \end{cases}$

$$= \frac{1}{\sqrt{\mu}} \frac{dr}{dt} \cdot \frac{dt}{dx} \cdot \underline{v} + \frac{r}{\sqrt{\mu}} \cdot \frac{d\underline{v}}{dt} \cdot \frac{dt}{dx}$$

(don't differentiate 1st term)

$$\frac{d\underline{v}}{dt} = -\frac{\mu}{r^3} \underline{r}$$

$$= \frac{1}{\sqrt{\mu}} \cdot \frac{dr}{dx} \cdot \underline{v} + \frac{r}{\sqrt{\mu}} \left(-\frac{\mu}{r^3}\right) \underline{r} \cdot \frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{r}{\sqrt{\mu}}$$

$$= \frac{1}{\sqrt{\mu}} \underline{v} \cdot \frac{dr}{dx} - \frac{r}{\sqrt{\mu}} \cdot \frac{\mu}{r^3} \cdot \underline{r} \cdot \frac{r}{\sqrt{\mu}}$$

$$\Rightarrow \frac{d^2\underline{r}}{dx^2} = \frac{1}{\sqrt{\mu}} \underline{v} \cdot \frac{dr}{dx} - \frac{\cancel{\mu}}{r} \cdot \underline{r}$$

PREVIOUSLY DEFINED:

$$\sigma \triangleq \frac{\underline{r} \cdot \underline{v}}{\sqrt{\mu}} = \frac{dr}{dx} \quad ; \quad \alpha \triangleq \frac{1}{a}$$

$$\frac{d^2\underline{r}}{dx^2} = \frac{\sigma}{\sqrt{\mu}} \underline{v} - \frac{\underline{r}}{r}$$

III^{rd} differentiation in x

$$\frac{d^3\underline{r}}{dx^3} = \frac{1}{\sqrt{\mu}} \frac{d\sigma}{dx} \underline{v} + \frac{\sigma}{\sqrt{\mu}} \frac{d\underline{v}}{dx} - \frac{d}{dx}\left(\frac{1}{r}\right) \underline{r} - \frac{1}{r} \frac{d\underline{r}}{dx}$$

PREVIOUS RESULT:

$$\frac{d\sigma}{dx} = 1 - \alpha r$$

$$\Rightarrow \frac{d^3\underline{r}}{dx^3} = \frac{\underline{v}}{\sqrt{\mu}} (1 - \alpha r) + \frac{\sigma}{\sqrt{\mu}} \cdot \frac{d\underline{v}}{dx} - \frac{\underline{r}}{r^2} \frac{dr}{dx} - \frac{1}{r} \frac{d\underline{r}}{dx}$$

$$\frac{d(\cdot)}{dx} = \frac{d(\cdot)}{dt} \cdot \frac{dt}{dx} = \frac{d(\cdot)}{dx} \frac{r}{\sqrt{\mu}} \ !$$

$$\Rightarrow \frac{d^3\underline{r}}{dx^3} = \frac{\underline{v}}{\sqrt{\mu}} (1 - \alpha r) + \frac{\sigma}{\sqrt{\mu}} \cdot \frac{d\underline{v}}{dt} \cdot \frac{dt}{dx} - \frac{\sigma}{r^2} \underline{r} - \frac{1}{r} \frac{d\underline{r}}{dt} \frac{dt}{dx}$$

$$\frac{d\underline{v}}{dt} = \frac{\mu}{\|\underline{r}\|^3} \underline{r} \ ; \ \frac{d\underline{r}}{dt} = \underline{v} \ !$$

$$\Rightarrow \frac{d^3\underline{r}}{dx^3} = \frac{\underline{v}}{\sqrt{\mu}} (1 - \alpha r) + \frac{\sigma}{\sqrt{\mu}} \cdot \frac{\mu}{r^3} \underline{r} \cdot \frac{r}{\sqrt{\mu}} - \frac{\sigma}{r^2} \cdot \underline{r} - \frac{1}{r} \frac{r}{\sqrt{\mu}} \cdot \underline{v}$$

$$\frac{d^3\underline{r}}{dx^3} = \frac{\underline{v}}{\sqrt{\mu}} (1 - \alpha r) + \cancel{\frac{\sigma}{r^2} \underline{r}} - \cancel{\frac{\sigma}{r^2} \cdot \underline{r}} - \frac{\underline{v}}{\sqrt{\mu}} = \frac{\underline{v}}{\sqrt{\mu}} \cdot (\cancel{1} - \alpha r - \cancel{1})$$

$$= -\alpha r \cdot \frac{\underline{v}}{\sqrt{\mu}} = -\frac{\alpha}{\sqrt{\mu}} r \frac{d\underline{r}}{dt} = -\frac{\alpha}{\sqrt{\mu}} r \cdot \frac{d\underline{r}}{dx} \cdot \frac{dx}{dt} = -\frac{\alpha \cancel{r}}{\cancel{\sqrt{\mu}}} \frac{\cancel{\sqrt{\mu}}}{\cancel{r}} \frac{d\underline{r}}{dx} = -\alpha \frac{d\underline{r}}{dx}$$

$$\Rightarrow \frac{d^3\underline{r}}{dx^3} + \alpha \frac{d\underline{r}}{dx} = 0$$

← SIMILAR TO PREVIOUS DIFFERENTIAL EQUATION: WAS NOT SURE IT WAS VALID ALSO FOR E VECTORIAL ?

$$(P) \quad \frac{d^3\underline{r}}{dx^3} + \alpha \frac{d\underline{r}}{dx} = 0$$

⇒ Sol: $\underline{r} = \underline{a}_0 U_0 + \underline{a}_1 U_1 + \underline{a}_2 U_2$.

b.c.: for $t = t_0$ $\begin{cases} x = 0 \\ \underline{r} = \underline{r}_0 \\ \underline{v} = \underline{v}_0 \end{cases}$ $\left(\Rightarrow \sigma_0 = \dfrac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu}} \right)$

$$\underline{r} = \underline{a}_0 U_0 + \underline{a}_1 U_1 + \underline{a}_2 U_2 \qquad + \text{b.c. for } t = t_0 \begin{cases} x = 0 \rightarrow U_1 = U_2 = 0 \;;\; U_0 = 1 \\ \underline{r} = \underline{r}_0 \\ \underline{v} = \underline{v}_0 \end{cases}$$

$\underline{a}_1$ *

$$\begin{cases} \dfrac{d\underline{r}}{dx} = \dfrac{d\underline{r}}{dt} \cdot \dfrac{dt}{dx} = \underline{v} \dfrac{r}{\sqrt{\mu}} \\[2ex] \dfrac{d\underline{r}}{dx} = \underline{a}_0 \dfrac{dU_0}{dx} + \underline{a}_1 \dfrac{dU_1}{dx} + \underline{a}_2 \cdot \dfrac{dU_2}{dx} \end{cases}$$

$$\frac{dU_0}{dx} = -\alpha U_1 \;;\quad \frac{dU_1}{dx} = U_0 \;;\quad \frac{dU_2}{dx} = U_1$$

$\downarrow$

$$\begin{cases} \dfrac{d\underline{r}}{dx} = \underline{v} \cdot \dfrac{r}{\sqrt{\mu}} \\[2ex] \dfrac{d\underline{r}}{dx} = -\underline{a}_0 \alpha U_1 + \underline{a}_1 U_0 + \underline{a}_2 \cdot U_1 \end{cases} \Rightarrow \begin{cases} \left.\dfrac{d\underline{r}}{dx}\right|_{\substack{t=t_0 \\ x=0}} = \underline{v}_0 \dfrac{r_0}{\sqrt{\mu}} \\[2ex] \left.\dfrac{d\underline{r}}{dx}\right|_{\substack{t=t_0 \\ x=0}} = \underline{a}_1 \end{cases} \Rightarrow \underline{a}_1 = \frac{r_0}{\sqrt{\mu}} \underline{v}_0$$

$\underline{a}_2$ *

$$\begin{cases} \dfrac{d^2\underline{r}}{dx^2} = \dfrac{d\underline{v}}{dx} \dfrac{r}{\sqrt{\mu}} + \dfrac{1}{\sqrt{\mu}} \underbrace{\dfrac{dr}{dx}}_{\sigma} \underline{v} = \dfrac{r}{\sqrt{\mu}} \underbrace{\dfrac{d\underline{v}}{dt}}_{= -\frac{\mu}{r^3}\underline{r}} \underbrace{\dfrac{dt}{dx}}_{= r/\sqrt{\mu}} + \dfrac{1}{\sqrt{\mu}} \sigma \underline{v} = \dfrac{r}{\sqrt{\mu}} \cdot \left(-\dfrac{\mu}{r^3}\right) \dfrac{r}{\sqrt{\mu}} \underline{r} = -\dfrac{\underline{r}}{r} + \dfrac{1}{\sqrt{\mu}} \sigma \underline{v} \\[2ex] \dfrac{d^2\underline{r}}{dx^2} = -\underline{a}_0 \alpha \dfrac{dU_1}{dx} - \underline{a}_1 \dfrac{dU_0}{dx} + \underline{a}_2 \dfrac{dU_1}{dx} \end{cases}$$

$\downarrow$

$$\begin{cases} \dfrac{d^2\underline{r}}{dx^2} = -\dfrac{\underline{r}}{r} + \dfrac{1}{\sqrt{\mu}} \sigma \underline{v} \\[2ex] \dfrac{d^2\underline{r}}{dx^2} = -\underline{a}_0 \alpha U_0 - \alpha \underline{a}_1 U_1 + \underline{a}_2 U_0 \end{cases} \Rightarrow \begin{cases} \left.\dfrac{d^2\underline{r}}{dx^2}\right|_{\substack{t=t_0 \\ x=0}} = -\dfrac{\underline{r}_0}{r_0} + \dfrac{\sigma_0}{\sqrt{\mu}} \underline{v} \\[2ex] \left.\dfrac{d^2\underline{r}}{dx^2}\right|_{\substack{t=t_0 \\ x=0}} = -\alpha \underline{a}_0 + \underline{a}_2 \; * \end{cases}$$

$\underline{a}_0$ *

$$\underline{r}_0 = \underline{a}_0 \left. U_0 \right|_{\substack{t=t_0 \\ x=0}} + \underline{a}_1 \cancel{\left. U_1 \right|_{\substack{t=t_0 \\ x=0}}}^{=0} + \underline{a}_2 \cancel{\left. U_2 \right|_{\substack{t=t_0 \\ x=0}}}^{=0} \Rightarrow \underline{a}_0 = \underline{r}_0$$

$$* \Rightarrow \quad -\alpha \underline{r}_0 + \underline{a}_2 = \frac{\sigma_0 \underline{v}_0}{\sqrt{\mu}} - \frac{\underline{r}_0}{r_0} \Rightarrow \underline{a}_2 = \frac{\underline{r}_0 \cdot \underline{v}_0 \underline{v}_0}{\sqrt{\mu}} + \left(\alpha - \frac{1}{r_0}\right) \underline{r}_0 .$$

$$\Rightarrow \underline{r} = \underline{r}_0 U_0 + \frac{r_0}{\sqrt{\mu}} \underline{v}_0 U_1 + \left[\left(\frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu}}\right) \underline{v}_0 + \left(\frac{1}{a} - \frac{1}{r_0}\right) \underline{r}_0 \right] U_2 .$$

$$U_0 = 1 + \frac{\alpha x^2}{2!} + \frac{(\alpha x^2)^2}{4!} + \dots \rightarrow \cos(\alpha x) \simeq \cos(0) + [\sin(\alpha 0) \cdot \alpha] x + \cos(\alpha 0) \alpha^2 \frac{x^2}{2!} + \dots$$

$$U_1 = x \left[1 - \frac{\alpha x^2}{3!} + \frac{(\alpha x^2)^2}{5!} - \dots \right] \qquad U_2 = \int U_1 \, dx = x^2 \left[\frac{1}{2!} - \frac{\alpha x^2}{4!} + \frac{(\alpha x^2)^2}{6!} - \dots \right]$$

POSITION FROM AN ABSOLUTE REFERENCE SYSTEM AS FUNCTION OF UNIVERSAL VARIABLE x.

↳ Lagrange's invariants are such that: $\begin{cases} \underline{r} = F \underline{r}_0 + G \underline{v}_0 \\ \underline{v} = \dot{F} \underline{r}_0 + \dot{G} \underline{v}_0 \end{cases}$

PREVIOUSLY SHOWN $U_0 + \alpha U_2 = 1$

⇒ it's immediate to show that:

$$F = U_0 + \Big(\underbrace{\frac{1}{a}}_{\alpha} - \frac{1}{r_0}\Big) U_2 = \left[1 - \frac{\alpha x^2}{2!} + \frac{(\alpha x^2)^2}{4!} - \dots \right] + \overbrace{\left[\alpha \frac{x^2}{2!} - \frac{\alpha^2 x^4}{4!} + \dots \right]}^{\alpha U_2} - \frac{1}{r_0} U_2$$

$$\Rightarrow F = 1 - \frac{U_2}{r_0} .$$

$$G = \frac{r_0}{\sqrt{\mu}} U_1 + \frac{\underline{r}_0 \cdot \underline{v}_0}{\sqrt{\mu}} U_2$$

$$\Rightarrow \sqrt{\mu}\, G = r_0 U_1 + (\underline{r}_0 \cdot \underline{v}_0) U_2$$

!! r_0 is a constant !!

$$\dot{F} = \frac{dF}{dt} = -\frac{1}{r_0} \underbrace{\frac{dU_2}{dx}}_{U_1} \cdot \underbrace{\frac{dx}{dt}}_{\frac{\sqrt{\mu}}{r}} = -\frac{1}{r_0} U_1 \frac{\sqrt{\mu}}{r}$$

$$\Rightarrow \dot{F} = -\frac{\sqrt{\mu}}{r r_0} U_1$$

$$F\dot{G} - G\dot{F} = 1 \longrightarrow \dot{G} = \frac{G\dot{F} + 1}{F}$$

!! Better if I got it deriving $\underline{r}$ expression !!

⇒ deriving G in time

$$\dot{G} = \frac{r_0}{\sqrt{\mu}} \frac{dU_1}{dx} \cdot \frac{dx}{dt} + \frac{\sigma_0}{\sqrt{\mu}} \cdot \frac{dU_2}{dx} \cdot \frac{dx}{dt} .$$

Trick: $+\frac{U_2}{r} - \frac{U_2}{r}$

$$= \frac{r_0}{\sqrt{\mu}} U_0 \cdot \frac{\sqrt{\mu}}{r} + \frac{\sigma_0}{\sqrt{\mu}} U_1 \frac{\sqrt{\mu}}{r} = \underbrace{\frac{r_0}{r} U_0 + \frac{\sigma_0}{r} U_1 + \frac{U_2}{r}}_{=1} - \frac{U_2}{r}$$

PREVIOUS RESULT

$$r = a_0 U_0 + a_1 U_1 + a_2 U_2 = r_0 U_0 + \sigma_0 U_1 + U_2$$

$$\Rightarrow \dot{G} = 1 - \frac{U_2}{r} .$$

Lagrange coefficients as function of x

$$F = 1 - \frac{U_2}{r_0} \;, \qquad G = r_0 U_1 + (\underline{r}_0 \cdot \underline{v}_0) U_2$$

$$\dot{F} = -\frac{\sqrt{\mu}}{r_0 r} U_1 \;; \qquad \dot{G} = 1 - \frac{U_2}{r}$$

where:

$$r = r_0 U_0 + (\underline{r}_0 \cdot \underline{v}_0) U_1 + U_2$$

$$U_0 = 1 - \frac{\alpha x^2}{2!} + \frac{(\alpha x^2)^2}{4!} - \dots$$

$$U_1 = x \left[1 - \frac{\alpha x^2}{3!} + \frac{(\alpha x^2)^2}{5!} - \dots \right]$$

$$U_2 = x^2 \left[\frac{1}{2!} - \frac{\alpha x^2}{4!} + \frac{(\alpha x^2)^2}{6!} - \dots \right]$$

... NOW THAT WE HAVE BEEN ABLE TO EXPRESS $\begin{cases} r = r(\chi) \\ t = t(\chi) \end{cases}$; $\begin{cases} F = F(\chi) \\ \dot{F} = \dot{F}(\chi) \end{cases}$; $g = g(\chi)$; $\dot{g} = \dot{g}(\chi)$

AND

$\chi = \chi(E, F, \Delta)$

WE NEED TO EXPRESS LAGRANGE'S COEFFICIENTS AS FUNCTION OF (E, F, Δ)

Defining: $z \triangleq \alpha \chi^2$

$$C(z) \triangleq \frac{1}{2!} - \frac{z}{4!} + \frac{z^2}{6!} - \frac{z^3}{8!} + \dots = \sum_{i=0}^{+\infty} \frac{(-z)^i}{(2i+2)!}$$

$$S(z) \triangleq \frac{1}{3!} - \frac{z}{5!} + \frac{z^2}{7!} - \dots = \sum_{i=0}^{+\infty} \frac{(-z)^i}{(2i+3)!}$$

$$\Rightarrow U_0 = 1 - \frac{\alpha\chi^2}{2!} + \frac{(\alpha\chi^2)^2}{4!} \dots = 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} = 1 - z\,C(z)$$

$$U_1 = \chi\left[1 - \frac{\alpha\chi^2}{3!} + \frac{(\alpha\chi^2)^2}{5!} - \dots\right] = \chi\left[1 - \frac{z}{3!} + \frac{z^2}{5!} \dots\right] = \chi\,[1 - z\,S(z)]$$

$$U_2 = \chi^2\left[\frac{1}{2!} - \frac{\alpha\chi^2}{4!} + \frac{(\alpha\chi^2)^2}{6!} - \dots\right] = \chi^2 C(z)$$

$$U_3 = \chi^3\left[\frac{1}{3!} - \frac{\alpha\chi^2}{5!} + \frac{(\alpha\chi^2)^2}{7!} - \dots\right] = \chi^3 S(z)$$

↳ $t = t(\chi)$

$$\sqrt{\mu}\,(t - t_0) = r_0 U_1 + \left(\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\right) U_2 + U_3$$

$$= \chi^3 S(z) + \left(\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\right)\chi^2 C(z) + r_0\,[1 - z\,S(z)]$$

→ Kepler's EQUATION FOR EACH CONIC

$$\sqrt{\mu}\,(t - t_0) = \chi^3 S(z) + \left(\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\right)\chi^2 C(z) + r_0\,[1 - z\,S(z)]$$

↳ $r = r(\chi)$

$$r = r_0 U_0 + \left(\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\right) U_1 + U_2$$

$$= r_0\,[1 - z \cdot C(z)] + \left(\frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\right)\chi\,[1 - z\,S(z)] + \chi^2 C(z)$$

$$r = \chi^2 C(z) + \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{\sqrt{\mu}}\,\chi\,[1 - z\,S(z)] + r_0\,[1 - z\,C(z)]$$

NOW:

$$C(z) = \sum_{i=0}^{+\infty} \frac{(-z)^i}{(2i+2)!} =$$

→ $z = 0$: $C(z) = \frac{1}{2!}$

→ $z > 0$: $C(z) = \frac{1}{2!} - \frac{z}{4!} + \frac{z^2}{6!} - \frac{z^3}{8!} \simeq \frac{1 - \cos\sqrt{z}}{z}$.

$$\cos(\sqrt{z}) \simeq \cos(0) + \frac{1}{2} z^{-\frac{1}{2}} \sin(\sqrt{z})\Big|_{z=0} + \frac{1}{2!}\,\frac{1}{3}\, z^{-\frac{3}{2}} \cos(\sqrt{z})\Big|_{z=0}$$

$$+ \frac{1}{2!}\,\frac{1}{2}\, z^{-\frac{1}{2}}\, z^{-\frac{1}{2}} \cdot 1$$

→ $z < 0$: $C(z) = \frac{1}{2!} + \frac{|z|}{4!} + \frac{|z|^2}{6!} + \frac{z^3}{8!} \simeq \frac{1 - \cosh\sqrt{-z}}{z}$.

$$S(z) = \sum_{i=0}^{+\infty} \frac{(-z)^i}{(2i+3)!} =$$

→ $z = 0$: $S(z) = \frac{1}{3!}$

→ $z > 0$: $S(z) = \dots \simeq \frac{\sqrt{z} - \sin\sqrt{z}}{\sqrt{z^3}}$

→ $z < 0$: $S(z) = \dots \simeq \frac{\sinh(\sqrt{-z}) - \sqrt{-z}}{\sqrt{-z^3}}$

THEN:

UNIVERSAL KEPLER'S LAW

ELLIPS e.g.

$$\sqrt{\mu}\,(t - t_0) = \chi^3 S(z) + \sigma_0\,\chi^2 C(z) + r_0\,[1 - z\,S(z)] \qquad z = \alpha\chi^2$$

$$\simeq \cancel{\chi^3} \cdot \frac{\sqrt{\alpha}\,\chi - \sin(\sqrt{\alpha}\,\chi)}{\alpha^{\frac{3}{2}} \cdot \cancel{\chi^3}} + \sigma_0\,\cancel{\chi^2}\,\frac{1 - \cos(\sqrt{\alpha}\,\chi)}{\alpha\,\cancel{\chi^2}} + r_0 - \alpha\,\cancel{\chi^2}\, r_0\,\frac{\sqrt{\alpha}\,\chi - \sin(\sqrt{\alpha}\,\chi)}{\alpha^{\frac{3}{2}}\,\chi^3}$$

... TO REALLY DETERMINE LAGRANGE'S COEFFICIENT AS FUNCTION OF ΔE:

$$U_0(x,\alpha) = 1 - \frac{\alpha x^2}{2!} + \frac{(\alpha x^2)^2}{4!} - \dots \simeq \cos(\sqrt{\alpha}\, x)$$

$$\cos(\sqrt{\alpha} x) \simeq \cos(0) - \sqrt{\alpha}\sin(\sqrt{\alpha}\,0)\, x - \frac{\sqrt{\alpha}\cdot\sqrt{\alpha}\cos(\sqrt{\alpha}\,0)}{2!} x^2 + \frac{\alpha\sqrt{\alpha}\sin(\sqrt{\alpha}\,0)}{3!} x^3 + \alpha\sqrt{\alpha}\sqrt{\alpha}\cos(\sqrt{\alpha}\,0)\frac{x^4}{4!}$$

$$\simeq 1 - \frac{\alpha x^2}{2!} + \frac{\alpha^2 x^4}{4!}$$

$$U_1(x,\alpha) = x - \frac{\alpha x^3}{3!} + \frac{\alpha^2 x^5}{5!} - \dots \simeq \frac{\sin(\sqrt{\alpha} x)}{\sqrt{\alpha}}$$

$$\frac{\sin(\sqrt{\alpha} x)}{\sqrt{\alpha}} \simeq \frac{\sin(0)}{\sqrt{\alpha}} + \frac{\sqrt{\alpha}\cos(\sqrt{\alpha}\,0)}{\sqrt{\alpha}}\, x - \frac{\sin(\sqrt{\alpha}\,0)\sqrt{\alpha}\sqrt{\alpha}}{\sqrt{\alpha}} + \frac{\cos(\sqrt{\alpha}\,0)\,\alpha\sqrt{\alpha}}{\sqrt{\alpha}}\frac{x^3}{3!} \dots$$

$$U_2(x,\alpha) = \frac{x^2}{2!} - \frac{\alpha x^4}{4!} + \frac{(\alpha x^2)^2}{6!} - \dots \simeq \frac{1-\cos(\sqrt{\alpha} x)}{\alpha}$$

$$\left[1 - \left(1 - \frac{\alpha x^2}{2!} + \frac{\alpha^2 x^4}{4!} - \dots\right)\right]\frac{1}{\alpha} = \frac{x^2}{2!} + \frac{\alpha x^4}{4!} \dots = U_2(x)$$

this is enough to determine F as function of x (THIS IS ALSO ENOUGH TO SOLVE LAMBERT'S PROBLEM FOR AN ELLIPS → OUR ORIGINAL INTENT)

$$F = 1 - \frac{U_2}{r_0} = - \frac{1-\cos(\sqrt{\alpha} x)}{\alpha r_0} + 1 \qquad \Big\| \qquad x = \sqrt{a}(E-E_0)\ ;\ \alpha = \frac{1}{a}$$

$$= 1 - \frac{1-\cos\left[\sqrt{\frac{1}{a}}\cdot\sqrt{a}(E-E_0)\right]}{\alpha r_0}$$

$$= 1 - \frac{a}{r_0}\left[1-\cos(E-E_0)\right]$$

Ellips.

$$\Rightarrow F = 1 - \frac{a}{r_0}\left[1-\cos(E-E_0)\right]$$

$$= 1 - \frac{a}{r_0}\left[1-\cos\Delta E\right]$$

WITHOUT CALCULATING IT:

Hyperbola

$$\Rightarrow F = 1 - \frac{a}{r_0}\left[1-\cosh(F-F_0)\right]$$

$$= 1 - \frac{a}{r_0}\left[1-\cosh(\Delta F)\right]$$

... TO FULLY SOLVE LAMBERT'S PROBLEM: (ELLIPS TRANSFER)

$$TOF = [\alpha - \beta - (\sin\alpha - \sin\beta)] \Rightarrow a \text{ KNOWN.}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}\ ;\ \sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$$

$\boxed{F = F(\theta)}$ → $F = 1 - \frac{r}{p}(1-\cos\Delta\theta)$

$\boxed{F = F(E)}$ → $F = 1 - \frac{a}{r_0}[1-\cos\Delta E]$

F: $\underline{r} = F\,\underline{r}_0 + G\,\underline{v}_0$

SINCE The goal of the Lambert's is to define the orbit(s) able to connect $\underline{r}_1$ and $\underline{r}_2$.

THEN $\underline{r}_2$ can be expressed as linear combination of $\underline{r}_1$; $\underline{v}_1$

$$\underline{r}_2 = F\,\underline{r}_1 + G\,\underline{v}_1.$$

WE CAN NOW EXPRESS IT BOTH AS FUNCTION OF Δθ AND ΔE. (AND LUCKILY ONE DEPEND ON r_2 AND THE OTHER DEPENDS ON r_1)

⇓

Known thanks to Lambert's theorem

$$1 - \frac{r_2}{p}(1-\cos\Delta\theta) = 1 - \frac{a}{r_1}[1-\cos\Delta E]$$

given by the problem

$\underline{r}_1\cdot\underline{r}_2 = \|\underline{r}_1\|\|\underline{r}_2\|\cos\Delta\theta$

$\alpha - \beta = 2E_m = E_2 - E_1 = \Delta E$

⇓

$$p = \frac{r_1 r_2}{a}\cdot\frac{1-\cos\Delta\theta}{1-\cos\Delta E}\ ;$$

1) $\cos\Delta\theta = \frac{\underline{r}_1\cdot\underline{r}_2}{\|\underline{r}_1\|\cdot\|\underline{r}_2\|}$

$$p = a(1-e^2) \rightarrow e = \sqrt{1-\frac{p}{a}}$$

2) $\Delta E = \alpha - \beta = 2\sin^{-1}\left(\sqrt{\frac{s}{2a}}\right) - 2\sin^{-1}\left(\sqrt{\frac{s-c}{2a}}\right)$

$\left(\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}}\ ;\ \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-c}{2a}}\right)$

HOW TO SELECT THE TRANSFER → elliptical / → hyperbolic.

Lambert:

$$tof = \sqrt{\frac{a^3}{\mu}} \cdot [\alpha - \beta - (\sin\alpha - \sin\beta)] = f(r_1, r_2, c)$$

$$s = r_1 + r_2 + c \Rightarrow \sin\frac{\alpha}{2} = \pm\sqrt{\frac{s}{2a}}$$

$$\sin\frac{\beta}{2} = \pm\sqrt{\frac{s-c}{2a}}$$

This is the most general form of Lambert's Theorem.

Problem.

$$(r_1, r_2, tof) \rightarrow (a)$$

ELLIPS MINIMUM ENERGY

FOR AN ASSIGNED a ⇒ I CAN FIND 4 POSSIBLE COMBINATIONS OF (α, β)
(instead of 8 because α is always positive)

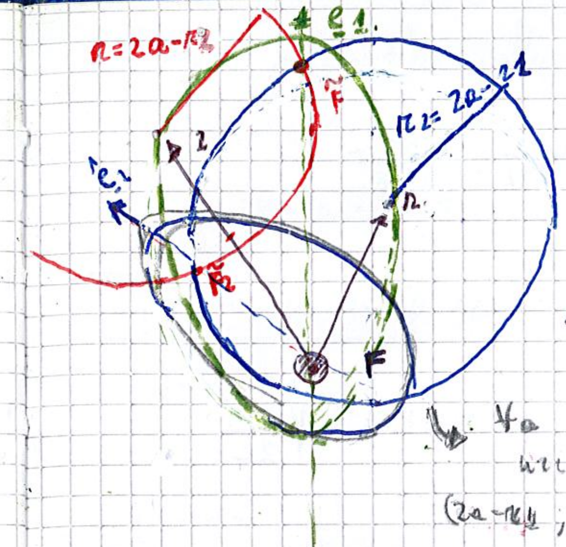
$$\begin{cases} \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} \rightarrow 1 - \cos\alpha = 2\sin^2\frac{\alpha}{2} \rightarrow \cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - \frac{s}{a} \\ \sin\frac{\alpha}{2} = \pm\sqrt{\frac{s}{2a}} \end{cases}$$

$$\rightarrow \begin{cases} \alpha \stackrel{\Delta}{=} \xi + E_M \\ \xi : e\cos E_p = \cos\xi \end{cases} ; \quad E_p = \frac{E_1 + E_2}{2} \Rightarrow \alpha \text{ IS ALWAYS POSITIVE}$$

$$\begin{cases} \sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}} \rightarrow \cos\beta = 1 - \frac{s-c}{a} \\ \sin\frac{\beta}{2} = \pm\sqrt{\frac{s-c}{2a}} \end{cases}$$

$$\alpha = \begin{cases} \alpha_0 = a\cos\left(1 - \frac{s}{a}\right) \\ \alpha = 2\pi - \alpha_0 = 2\pi - a\cos\left(1 - \frac{s}{a}\right) \end{cases} \qquad \beta = \begin{cases} a\cos\left(1 - \frac{s-c}{a}\right) \\ -a\cos\left(1 - \frac{s-c}{a}\right) \end{cases}$$

$$\rightarrow \begin{cases} \beta \stackrel{\Delta}{=} \xi - E_M \\ \xi : e\cos E_p = \cos\xi \end{cases} ; \Rightarrow \text{ALWAYS } \beta < \alpha.$$



ellips property: $\forall P: \overline{PF_1} + \overline{PF_2} = 2a$

$$\Rightarrow \exists (\tilde{F}_1; \tilde{F}_2) : \begin{cases} \overline{P_1F} + \overline{P_1\tilde{F}} = 2a \\ \overline{P_2F} + \overline{P_2\tilde{F}} = 2a \end{cases} \rightarrow \begin{cases} \overline{P_1\tilde{F}} = 2a - r_1 \\ \overline{P_2\tilde{F}} = 2a - r_2 \end{cases}$$

(vacant focuses)

↓

sistono 2 punti di intersezione tra le 2 circonferenze.

$$\begin{cases} \overline{P_1\tilde{F}} = 2a - r_1 \\ \overline{P_2\tilde{F}} = 2a - r_2 \end{cases}$$

∀a I could draw 2 circumferences of radius $(2a - r_1, 2a - r_2)$

FOR EACH a ASSIGNED EXIST 4 DIFFERENT ELLIPTICAL ARCHS CONNECTING P_1, P_2 BELONGING TO 2 DIFFERENT ELLIPSES.

ell 1: $\overline{F\tilde{F}_1} = \hat{e}_1$; $e_1 \neq e_2$

ell 2: $\overline{F\tilde{F}_2} = \hat{e}_2$

arch 1: ell 1 CLOCK WISE. → tof_1

arch 2: ell 1 ANTI-CLOCK WISE → tof_2

arch 3: ell 2 CLOCK-WISE. → tof_3

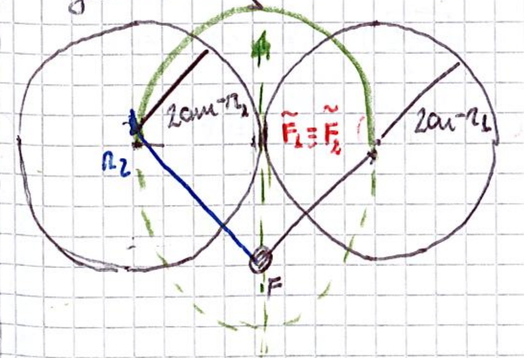
arch 4: ell 2 ANTI-CLOCK WISE. → tof_4

INTERCEPTION
⇕
$2a - r_1 + 2a - r_2 \geq c$.

$$\Rightarrow \exists\, a_m : \begin{cases} \tilde{F}_1 \equiv \tilde{F}_2 \\ 2a_m - r_1 + 2a_m - r_2 = c \end{cases} \Rightarrow 4a_m = r_1 + r_2 + c$$

$$a_m = \frac{s}{2} \Rightarrow \sin\left(\frac{\alpha}{2}\right)\Big|_{a_m} = \sqrt{\frac{s}{2a_m}} = \sqrt{\frac{s}{s}} = 1 \Rightarrow \alpha\Big|_{a_m} = \pi$$

geometrically



→ same Time of flight on both the

→ angular counting.

GIVEN (a) ; $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{s}{2a}}$; $\sin\frac{\beta}{2} = \pm\sqrt{\frac{s-c}{2a}}$

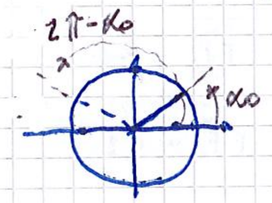
(i) $\alpha \triangleq \frac{\xi + E_M}{2} \Rightarrow \alpha > 0 \Rightarrow \sin\frac{\alpha}{2} > 0 \Rightarrow \sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$

(ii) $\beta \triangleq \frac{\xi - E_M}{2} \Rightarrow \begin{cases} \beta \geq 0 \\ \beta < \alpha \end{cases} \Rightarrow \sin\frac{\beta}{2} = \pm\sqrt{\frac{s-c}{2a}}$

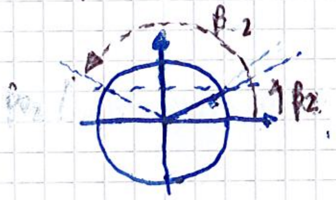
(iii) $\sin\left(\sqrt{\frac{s}{2a}}\right) > \sin\left(\sqrt{\frac{s-c}{2a}}\right)$

⇒ $\alpha = 2\,\text{asin}\left(\sqrt{\frac{s}{2a}}\right) = \begin{cases} \alpha_0 \\ 2\pi - \alpha_0 \end{cases}$

↳ $\text{asin}\left(\sqrt{\frac{s}{2a}}\right) = \begin{cases} \frac{\alpha_0}{2} \\ \pi - \frac{\alpha_0}{2} \end{cases} \rightarrow \begin{cases} \alpha_0 \\ 2\pi - \alpha_0 \end{cases}$



⇒ $\begin{cases} \beta = 2\,\text{asin}\left(\sqrt{\frac{s-c}{2a}}\right) = \begin{cases} \beta_1 = \beta_{01} \\ \beta_2 = \pi - \beta_{01} \end{cases} \\ \beta = 2\,\text{asin}\left(-\sqrt{\frac{s-c}{2a}}\right) = \begin{cases} \beta_3 = -\beta_{01} \\ \beta_4 = -(\pi - \beta_{01}) = \pi + \beta_{01} \end{cases} \end{cases}$



;

ACCEPTABLE COUPLES ⇒ 4 TRANSFER ARCHS ⇒ 4 TIMES OF FLIGHT.

1) $\begin{cases} \alpha = \alpha_0 \\ \beta = \beta_0 \end{cases}$ → ToF$_1$

$\alpha_0 \triangleq 2\,\text{asin}\left(\sqrt{\frac{s}{2a}}\right)$ ONLY ACUTE ANGLE.

$\beta_0 \triangleq 2\,\text{asin}\left(\sqrt{\frac{s-c}{2}}\right)$ ONLY ACUTE ANGLE.

$(\alpha_0 > \beta_0)$

2) $\begin{cases} \alpha = \alpha_0 \\ \beta = -\beta_0 \end{cases}$ → ToF$_2$

3) $\begin{cases} \alpha = \pi - \alpha_0 \\ \beta = \beta_0 \end{cases}$ → ToF$_3$

4) $\begin{cases} \alpha = \pi - \alpha_0 \\ \beta = -\beta_0 \end{cases}$ → ToF$_4$.

→ That's completely analogous to imposing the 4 possible value:

$\alpha = \begin{cases} \text{acos}\left(1 - \frac{s}{a}\right) \\ 2\pi - \text{acos}\left(1 - \frac{s}{a}\right) \end{cases}$; $\beta = \begin{cases} \text{acos}\left(1 - \frac{s-c}{a}\right) \\ -\text{acos}\left(1 - \frac{s-c}{a}\right) \end{cases}$

$\boxed{\text{PARABOLIC TIME OF FLIGHT}}$

Parabolic trajectory $\begin{cases} r = \frac{p}{1 + e\cos\theta} \Rightarrow r(\theta = \pi) = \infty \; ; \; r_a = a(1+e) \Rightarrow a \to \infty \\ e = 1 \end{cases}$

so $\sin\frac{\alpha}{2} = \lim_{a \to \infty} \sqrt{\frac{s}{2a}} = 0 \Rightarrow \sin\frac{\alpha}{2} \simeq \frac{\alpha}{2} = \sqrt{\frac{s}{2a}} \rightarrow \alpha = 2\sqrt{\frac{s}{2a}}$

$\sin\frac{\beta}{2} = \lim_{a \to \infty} \sqrt{\frac{s-c}{2a}} \Rightarrow \sin\frac{\beta}{2} \simeq \frac{\beta}{2} = \sqrt{\frac{s-c}{2a}} \rightarrow \beta = 2\sqrt{\frac{s-c}{2a}}$

↳ Using a Taylor serie to approximate $t_P = [\alpha - \sin\alpha - (\beta - \sin\beta)]\sqrt{\frac{a^3}{\mu}}$

$\alpha - \sin\alpha \simeq 0 + (1 - \cos\alpha)\big|_{\alpha=0} \cdot \alpha + \frac{1}{2}(0 + \sin\alpha)\big|_{\alpha=0} \cdot \alpha^2 + \frac{1}{6}\cos\alpha\big|_{\alpha=0} \cdot \alpha^3$

⇒ $\alpha - \sin\alpha \simeq \frac{8}{6} \cdot \left(\frac{s}{2a}\right)^{3/2}$; $\beta - \sin\beta \simeq \frac{8}{6}\left(\frac{s-c}{2a}\right)^{3/2}$

$t_P = \frac{1}{\sqrt{\mu}} \cdot \frac{8}{6}\left[\left(\frac{s\,\cancel{a}}{2\,\cancel{a}}\right)^{3/2} - \left(\frac{(s-c)\,\cancel{a}}{2\,\cancel{a}}\right)^{3/2}\right] = \frac{1}{\sqrt{\mu}} \cdot \frac{4}{3}\left[\left(\frac{s}{2}\right)^{3/2} - \left(\frac{s-c}{2}\right)^{3/2}\right]$

$t_P = \frac{1}{\sqrt{\mu}} \frac{\sqrt{2}}{3}\left[\sqrt{s^3 \frac{\cancel{2}}{\cancel{2}}} - \sqrt{\frac{(s-c)^3\,\cancel{2}}{\cancel{2}}}\right]$

/ $\frac{4}{3} = \frac{2\sqrt{2}\cdot\sqrt{2}}{3} = 2^{3/2} \cdot \frac{\sqrt{2}}{3}$

$t_P = \frac{1}{\sqrt{\mu}} \cdot \frac{\sqrt{2}}{3}\left[\sqrt{s^3} - \text{sign}(\Delta\theta)\sqrt{(s-c)^3}\right]$

FULL LAMBERT'S PROCEDURE.

$(\underline{r}_1, \underline{r}_2)$, ToF ⇒ GIVEN

(1) compute

$$\begin{cases} t_p = \dfrac{1}{\sqrt{\mu}} \cdot \dfrac{\sqrt{2}}{3} \cdot \left[\sqrt{s^3} - \mathrm{sgn}(\Delta\theta) \sqrt{(s-c)^3} \right] \\ \Delta\theta = \arccos\left(\dfrac{\underline{r}_1 \cdot \underline{r}_2}{\|\underline{r}_1\| \, \|\underline{r}_2\|} \right) \end{cases}$$

$\Delta\theta < \pi \Rightarrow \beta > 0$

$0 < \Delta\theta < \pi \Rightarrow \beta < 0.$

IF $ToF > t_p$ THEN ∃ elliptic transfer able to connect $\underline{r}_1$ and $\underline{r}_2$.

(3) compute t_m ⇒ TIME OF FLIGHT ASSOCIATED TO THE ELLIPS OF MINIMUM ENERGY

; $a_m = \dfrac{s}{2}$ ⇒ $\sin\dfrac{\alpha_m}{2} = \sqrt{\dfrac{s}{2\frac{s}{2}}}$ ⇒ $\alpha_m = \pi$

$\sin\dfrac{\beta}{2} = \sqrt{\dfrac{s-c}{s}} = \sqrt{1 - \dfrac{c}{s}}$

$t_m = [\alpha_m - \beta_m - (\sin\alpha_m - \sin\beta_m)]$

⇒ $\beta_m = -\arcsin\sqrt{1 - \dfrac{c}{s}}$

IF $ToF > t_m$ ⇒ $\begin{cases} \alpha = 2\pi - \alpha_0. \\ \alpha_0 = \arccos\left(1 - \dfrac{s}{a}\right) \quad (\alpha_0 < \pi) \end{cases}$

IF $ToF < t_m$ ⇒ $\alpha = \alpha_0$

(4) select the correct β : $\beta < \alpha$

$\beta = \beta_0$ IF $\Delta\theta < \pi$

$\beta = -\beta_0$ IF $\pi < \Delta\theta < 2\pi$

→ consider with selecting if the orbit is going clock-wise or anti clock-wise.

(5) solve numerically the correct Lambert problem.

a) $ToF < t_m$ ∪ $\Delta\theta < \pi$

SOLVE: $\sqrt{\dfrac{\mu}{a^3}} \cdot ToF = [\alpha - \beta - (\sin\alpha - \sin\beta)]$

$\alpha = \arccos\left(1 - \dfrac{s}{a}\right)$, $\beta = \arccos\left[1 - \dfrac{(s-c)}{a}\right]$

b) $ToF < t_m$ ∪ $\Delta\theta > \pi$

SOLVE: $\sqrt{\dfrac{\mu}{a^3}}\, ToF = [\alpha_0 + \beta_0 - \sin\alpha_0 - \sin\beta_0]$

c) ToF t_m ∪ $\Delta\theta < \pi$

SOLVE: $\sqrt{\dfrac{\mu}{a^3}} \cdot ToF = [2\pi - \alpha_0 - \beta_0 + \sin\alpha_0 + \sin\beta_0]$

d) $ToF > t_m$ ∪ $\Delta\theta > \pi$

SOLVE: $\sqrt{\dfrac{\mu}{a^3}} \cdot ToF = [2\pi - \alpha_0 + \beta_0 - \sin\alpha_0 - \sin\beta_0]$

to recall

DEFINE A SYSTEM OF FUNCTIONS (NOT LINEAR)

!! MUST BE WRITTEN IN ANOTHER FILE (SIMPLICIT FUNCTION) ⇒ MUST BE WRITTEN AS A FUNCTION !!

```
FUNCTION  Fn = MSF(t, y)      INDEPENDENT VARIABLE / DESCRIB. VAR.
[n, m] = size(y)
Fn = zeros(n, m)
Fn(1) = y(2)
Fn(2) = -k/m y(1) + sin(Ωt)
RETURN
```

⇒ [T_h, u_h] = ode45('msf', [tmin, tmax], y0)

"How to use "ODE" function in MATLAB"

ODE can solve only 1^{st} order problem.

e.g. 1

$$y'(t) = f(t,y) = \frac{y(t)}{1+t} - [y(t)]^2$$

DEFINE A FUNCTION IN MATLAB → `f = @(t,y) y./(1+t)-y.^2`
(IMPLICIT FUNCTION)

DEFINE A SUBSET IN TIME → `t_dis = linspace(Tmin, Tmax, 2000)`

ODE choose step by her own.

USE THE "ODE" → `[t23, u23] = ode23(f, [Tmin Tmax], y0)`

- t23: discretized time, as choosed by "ODE"
- u23: result.
- f: implicit function !! in form $y' = f(x,t)$!!
- [Tmin Tmax]: time integral
- y0: boundary condition, referred to Tmin (time instant 0)

PLOTTING THE RESULT → `plot(t23, u23)`

$$\frac{dy}{dt} = \frac{y}{1+t} - y^2 \rightarrow$$

eg. 2 → mass-spring system.

$$m\ddot{x} + Kx = \sin(\Omega t) \rightarrow \begin{cases} m\dot{y} = -Kx + \sin(\Omega t) \\ y = \dot{x} \end{cases}$$

$$(m\ddot{x}(t) + Kx(t) = \sin(\Omega t))$$

$$\begin{cases} \dot{y} = -\frac{K}{m}x + \sin(\Omega t) \\ y = \dot{x} \end{cases}$$

adding damping to this:

$$m\ddot{x} + c\dot{x} + Kx = \sin(\Omega t)$$

$$\begin{cases} \dot{y} = -\frac{c}{m}y - \frac{K}{m}x + \sin(\Omega t) \\ y = \dot{x} \end{cases}$$

$$\begin{cases} \dot{y} = -\frac{K}{m}x + \frac{1}{m}\sin(\Omega t) \\ \dot{x} = y \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \underline{F}(t, y) = [f_1, f_2] \\ \underline{y}(t=0) = y_0 \end{cases}$$