4 → "Orbital's parameter"

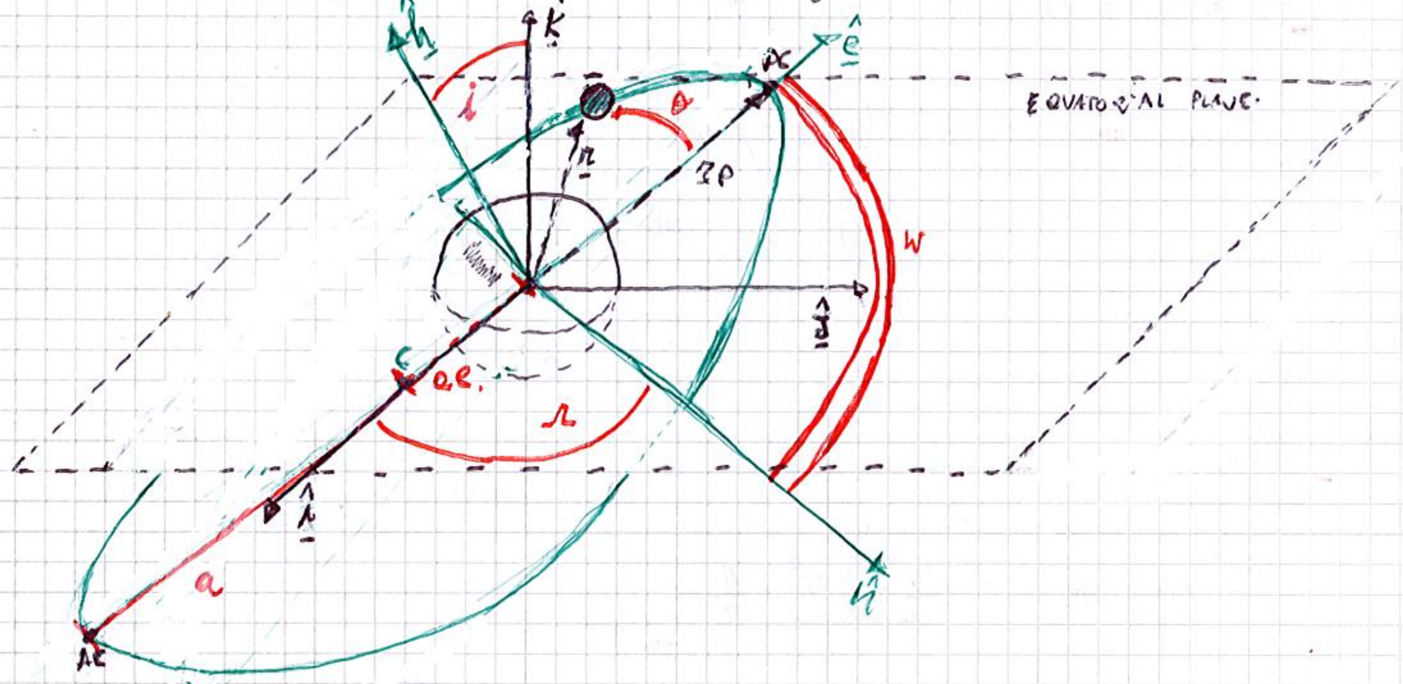
↳ From R2BP is known the orbit (trajectory) of a body under the action of a main attractor is fully contained in a plane π

⇒ is possible to individuate 2 reference systems:

1. $\{\hat{\underline{I}};\ \hat{\underline{J}};\ \hat{\underline{K}}\}$ → GEOCENTRIC (EQUATORIAL) REFERENCE SYSTEM

2. $\{\hat{\underline{e}};\ \hat{\underline{h}};\ \hat{\underline{p}}\}$ → PERICENTRIC REFERENCE SYSTEM.

!! Orbital parameters allow us to pass from one to the other. !!



• $\hat{n}$ → ascendent nodal-line: INTERCEPTION BETWEEN THE ORBITAL PLANE AND EQUATORIAL PLANE.
(linea dei nodi orbita) DEFINES A LINE CALLED NODAL-LINE

↓

Anyway Being Known that $\begin{cases} \hat{\underline{n}} \perp \hat{K} \Rightarrow \hat{\underline{n}} \text{ CONTAINED IN EQUATORIAL PLANE.} \\ \hat{n} \perp \hat{\underline{h}} \Rightarrow \hat{n} \text{ CONTAINED IN ORBITAL PLANE.} \end{cases}$

⇒ $\hat{n} = \hat{\underline{K}} \times \hat{\underline{h}}$ ⇝ ascendent node

$\hat{n}' = \hat{\underline{h}} \times \hat{\underline{K}}$ ⇝ discendent node.

⇒ $\hat{\underline{n}} = \dfrac{\hat{\underline{K}} \times \underline{h}}{\| \underline{h} \|}$ is a vector included both in orbital plane and in equatorial plane.

!! $\hat{\underline{n}}$ is orthogonal to $\underline{h}$ but not to $\hat{\underline{e}}$!!

⇒ The 6 orbital parameters are O.P. $\{a, e, i, \omega, \Omega, \theta\}$ and are enough to define the position of the satellite in both the reference systems

RV2PorOrb ⇒ $(\underline{r};\underline{v})$ KNOWN INTO GEOCENTRIC R.S. $\longrightarrow \{a, e, i, \Omega, \omega, \theta\}$

1) i → INCLINATION.

$$\begin{cases} \underline{h} = \underline{r} \times \underline{v}^{*} \\ \hat{\underline{k}} \perp \text{ to EQUATORIAL PLANE.} \end{cases} \Rightarrow \quad \cos i = \hat{\underline{h}} \cdot \hat{\underline{k}} = \frac{[\underline{r} \times \underline{v}] \cdot \hat{\underline{k}}}{\|\underline{h}\|}$$ ①

* ⇒ $\underline{r}, \underline{v}$ known in geocentric r.s. then $\underline{h}$ will be again expressed in geocentric reference system being $\underline{h} = \underline{r} \times \underline{v}$.

2) e → ECCENTRICITY

IInd INTEGRAL OF MOTION ⇒ $\mu \cdot \underline{e} = \underline{h} \times \underline{v} - \mu \hat{\underline{r}}$

$$\Rightarrow \quad \underline{e} = \frac{\underline{h} \times \underline{v}}{\mu} - \frac{\underline{r}}{\|\underline{r}\|}$$ ②

$$e = \|\underline{e}\|$$

3) a → MAJOR SEMIAXIS

IIIrd INTEGRAL OF MOTION ⇒ $\varepsilon = -\frac{\mu}{2a} = \frac{1}{2} v^2 - \frac{\mu}{r}$

$$\Rightarrow \quad a = \frac{\mu}{\frac{2\mu}{\sqrt{\underline{r}\cdot\underline{r}}} - \underline{v}\cdot\underline{v}}$$ ③

$$-\frac{\mu}{2a} = \frac{1}{2} v^2 - \frac{\mu}{r}$$

$$+\frac{1}{2a} = \frac{1}{r} - \frac{v^2}{2\mu} = \frac{2\mu - v^2}{2 r \mu} \Rightarrow \frac{2\mu - v^2}{r\mu} = \frac{1}{a}$$

$$a = \frac{r\mu}{2\mu - v^2} = \frac{\mu}{\frac{2\mu}{r} - v^2}$$

4) Ω → (IT): ASCENSIONE RETTA DEL NODO ASCENDENTE.

(EN): VERTICAL ASCENSION.

$$\begin{cases} \hat{\underline{n}} = \frac{\hat{\underline{k}} \times \underline{h}}{\|\underline{h}\|} \\ \Omega \text{ included between } \hat{\underline{i}} \text{ and } \hat{\underline{n}} \end{cases} \Rightarrow \quad \cos \Omega = \hat{\underline{n}} \cdot \hat{\underline{i}}$$ ④

$$\Downarrow$$

$$\hat{\underline{n}} = \cos\Omega\, \hat{\underline{i}} + \sin\Omega\, \hat{\underline{j}}$$

5) ω → (IT): ANOMALIA DEL PERICENTRO

(EN): PERICENTRIC ANOMALY → represent the angle (rotation) of the pericenter from the nodal-line.

$$\begin{cases} \mu \underline{e} = \underline{h} \times \underline{v} - \mu \hat{\underline{r}} \\ \hat{\underline{n}} = \frac{\hat{\underline{k}} \times \hat{\underline{h}}}{\|\underline{h}\|} \end{cases} \Rightarrow \quad \cos \omega = \frac{\underline{e} \cdot \hat{\underline{n}}}{\|\underline{e}\|}$$ ⑤

6) θ → TRUE ANOMALY

$$r = \frac{p}{1 + e\cos\theta} = \frac{a(1-e^2)}{1 + e\cos\theta} = \|\underline{r}\|$$

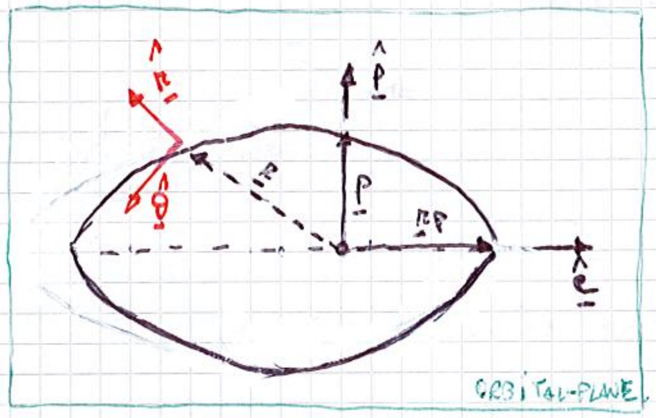
$$\Rightarrow \quad \cos\theta = \frac{\underline{e} \cdot \underline{r}}{\|\underline{e}\| \cdot \|\underline{r}\|}$$ ⑥

obs: To fully define the state $(\underline{r}, \underline{v})$ of a satellite we have a double choice. but unknown are always 6.

$$\begin{matrix} \underline{r} = \{r_x; r_y; r_z\} \\ \underline{v} = \{v_x; v_y; v_z\} \end{matrix} \longrightarrow \{a; e, i, \Omega, \omega, \theta\}$$

For Orb2rv $\Rightarrow \{a, e, i, \Omega, \omega, \theta\} \rightarrow (\underline{r} ; \underline{v})$ IN GEOCENTRIC R.S.

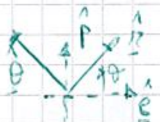
Inside the orbital plane $\underline{r}$ and $\underline{v}$ can be expressed in function of 2 reference system:



$\{\hat{r}; \hat{\theta}; (\hat{h})\}$ → LOCAL REFERENCE SYSTEM. $\propto t$ ⇒ NOT INERTIAL (acceleration of the r.f. different from zero)

$\{\hat{p}; \hat{e}; (\hat{h})\}$ → PERICENTRIC REFERENCE SYSTEM $\not\propto t$ ⇒ INERTIAL.

$$\begin{cases} \underline{r} = r \cdot \hat{r} \\ \underline{v} = \sqrt{\frac{\mu}{p}}\,[\sin\theta\,\hat{r} + (1+e\cos\theta)\,\hat{\theta}] \end{cases} ; \qquad \begin{cases} \underline{r} = r\cos\theta\,\hat{e} + r\sin\theta\,\hat{p} \\ \underline{v} = \sqrt{\frac{\mu}{p}}\,[-\sin\theta\,\hat{e} + (e+\cos\theta)\,\hat{p}] \end{cases}$$



$\Downarrow$

$$\hat{r} = \cos\theta\,\hat{e} + \sin\theta\,\hat{p} \Rightarrow \hat{\theta} = -\sin\theta\,\hat{e} + \cos\theta\,\hat{p}$$

$$\begin{Bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{e} \\ \hat{p} \\ \hat{h} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{Bmatrix} \triangleq \underline{\underline{R}}_\theta \begin{Bmatrix} \hat{e} \\ \hat{p} \\ \hat{h} \end{Bmatrix}$$

1) $(\underline{r}; \underline{v})$ are immediately known in pericentric reference system:

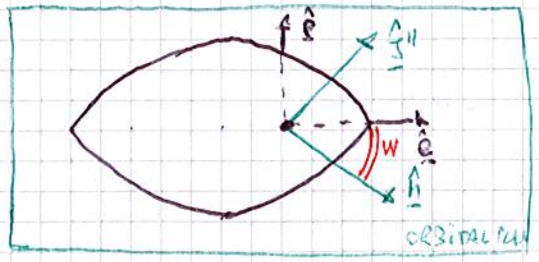
$$r = \frac{a(1-e^2)}{1+e\cos\theta} \Rightarrow \underline{r} = r\cos\theta\,\hat{e} + r\sin\theta\,\hat{p}$$

$$p = a(1-e^2) \Rightarrow \underline{v} = \sqrt{\frac{\mu}{p}}\,[-\sin\theta\,\hat{e} + (e+\cos\theta)\,\hat{p}]$$

otherwise: $\begin{Bmatrix} \hat{e} \\ \hat{p} \\ \hat{h} \end{Bmatrix} = \underline{\underline{R}}_\theta^{-1} \begin{Bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{h} \end{Bmatrix}$!! $\underline{\underline{R}}_\theta = \underline{\underline{R}}_\theta(t)$ because $\{\hat{r}, \hat{\theta}, \hat{h}\}$ is not an inertial reference system. !!

once $(\underline{r}, \underline{v})$ are known is enough to express them in r.s. $\{\hat{i}; \hat{j}; \hat{k}\}$. ⇒ FULL INTER-CHANGE THROUGH ALL THE REFERENCE SYSTEMS

2) $\{\hat{e}, \hat{p}, \hat{h}\} \rightarrow \{\hat{n}, \hat{j}'', \hat{h}\}$



$$\Rightarrow \hat{n} = \cos\omega\,\hat{e} + \sin\omega\,\hat{p} + 0\,\hat{h}$$
$$\hat{j}'' = -\sin\omega\,\hat{e} + \cos\omega\,\hat{p} + 0\,\hat{h}$$
$$\hat{h} = \hat{h}$$

$$\Rightarrow \begin{Bmatrix} \hat{n} \\ \hat{j}'' \\ \hat{h} \end{Bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{p} \\ \hat{e} \\ \hat{h} \end{Bmatrix}$$

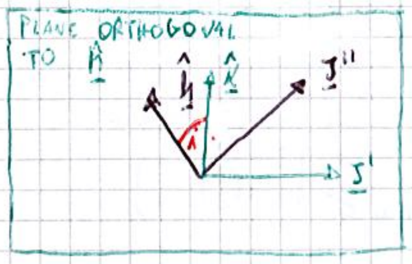
$$\begin{Bmatrix} \hat{n} \\ \hat{j}'' \\ \hat{h} \end{Bmatrix} \triangleq \underline{\underline{R}}_\omega \begin{Bmatrix} \hat{p} \\ \hat{e} \\ \hat{h} \end{Bmatrix}$$

3) $\{\hat{n}, \hat{j}'', \hat{h}\} \rightarrow \{\hat{n}, \hat{j}', \hat{k}\}$.

$$\Rightarrow \hat{n} = \hat{n}$$
$$\hat{h} = \cos i\,\hat{k} - \sin i\,\hat{j}' + 0\,\hat{n}$$
$$\hat{j}'' = +\sin i\,\hat{k} + \cos i\,\hat{j}' + 0\,\hat{n}$$

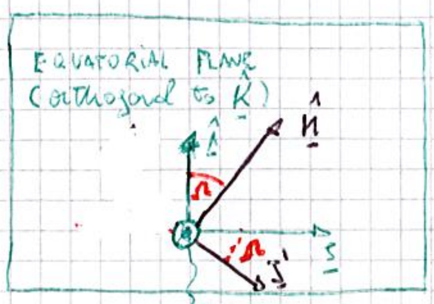
$$\Rightarrow \begin{Bmatrix} \hat{n} \\ \hat{j}'' \\ \hat{h} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & +\sin i & \cos i \end{bmatrix} \begin{Bmatrix} \hat{n} \\ \hat{j}' \\ \hat{k} \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{n} \\ \hat{j}'' \\ \hat{h} \end{Bmatrix} \triangleq \underline{\underline{R}}_i \begin{Bmatrix} \hat{n} \\ \hat{j}' \\ \hat{k} \end{Bmatrix}$$



4) $\{\hat{n}, \hat{j}', \hat{k}\} \longrightarrow \{\hat{i}, \hat{j}, \hat{k}\}$

$$\begin{Bmatrix} \hat{n} \\ \hat{j}' \\ \hat{k} \end{Bmatrix} = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$



$\hat{K}$ exiting the sheet.

$$\hat{k} = \hat{K}$$
$$\hat{n} = \cos\Omega\,\hat{i} + \sin\Omega\,\hat{j} \Rightarrow$$
$$\hat{j}' = -\sin\Omega\,\hat{i} + \cos\Omega\,\hat{j}$$

$$\begin{Bmatrix} \hat{n} \\ \hat{j}' \\ \hat{k} \end{Bmatrix} \triangleq \underline{\underline{R}}_\Omega \begin{Bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{Bmatrix}$$

⇒ the global transfer.

TOTAL) $\{\hat{i};\ \hat{j};\ \hat{k}\} \to \{\hat{r};\ \hat{\theta},\ \hat{h}\}$.

$$\begin{Bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{h}\end{Bmatrix} = \underline{\underline{R}}_\theta\, \underline{\underline{R}}_\omega\, \underline{\underline{R}}_i\, \underline{\underline{R}}_\Omega \begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix}$$

$$\begin{Bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{h}\end{Bmatrix} = \underline{\underline{R}}_\theta\, \underline{\underline{R}}_\omega\, \underline{\underline{R}}_i\, \underline{\underline{R}}_\Omega \begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix}$$

$\{\hat{r}, \hat{\theta}, \hat{h}\}$ → local reference system (NOT INERTIAL) ; $\{\hat{i};\ \hat{j},\ \hat{k}\}$ → EQUATORIAL reference system (INERTIAL)

① $$\begin{Bmatrix}\hat{e}\\ \hat{p}\\ \hat{h}\end{Bmatrix} = \underline{\underline{R}}_\theta^{-1} \begin{Bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{h}\end{Bmatrix} \qquad \underline{\underline{R}}_\theta = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

!! NOT INERTIAL TRANSFORMATION !!
($\theta = \theta(t)$)

② $$\begin{Bmatrix}\hat{e}\\ \hat{p}\\ \hat{h}\end{Bmatrix} = \underline{\underline{R}}_\omega\, \underline{\underline{R}}_i\, \underline{\underline{R}}_\Omega \begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix} \triangleq \underline{\underline{T}}_{i\omega\Omega} \begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix}$$

$$\underline{\underline{T}} = \begin{bmatrix} c\Omega\, c\omega - s\Omega\, s\omega\, ci\ ; & s\Omega\, c\omega + c\Omega\, s\omega\, si\ ; & s\omega\, si\\ -c\Omega\, s\omega - s\Omega\, c\omega\, ci\ ; & -s\Omega\, s\omega + c\Omega\, c\omega\, ci\ ; & c\omega\, si\\ s\Omega\, si\ ; & -c\Omega\, si\ ; & ci \end{bmatrix}$$

($c \triangleq \cos(\cdot)$; $s \triangleq \sin(\cdot)$)

obviously we have also: $$\begin{Bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{Bmatrix} = \underline{\underline{T}}^{-1}_{i\omega\Omega} \begin{Bmatrix}\hat{e}\\ \hat{p}\\ \hat{h}\end{Bmatrix}$$

!! BOTH DIRECT AND INVERSE TRANSFORMATION ARE INERTIAL. (INDEPENDENT IN TIME) !!

5 → "impulsive (COPLANAR) manoeuvring"

6 Under the hypothesis of a thrust acting on the satellite the equation of motion for ($m_1 \ll m_2$) becomes:

$$\begin{cases} \dfrac{d^2\underline{r}}{dt^2} = -\dfrac{\mu}{\|\underline{r}\|^3}\underline{r} + \underline{a}_c \\ \underline{a}_c = \dfrac{\underline{F}_c}{m_1} \end{cases} \Rightarrow$$

a) $a_c \gg$ (high thrust) → Thermochemical

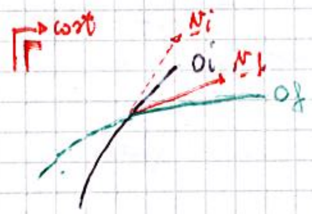
b) $a_c \ll$ (perturbation / solar sail) → other.

The high thrust is the one required to obtain a change in the orbit.

⇒ each force generates an impulse:

$\underline{F}_c \Rightarrow \underline{I} = m_1 \cdot \Delta\underline{v} \rightarrow \begin{cases} \dfrac{\underline{I}}{m_1} = \Delta\underline{v}. \\ \underline{I} \to \text{impulse} ;\ \Delta\underline{v} \to \text{required to change the orbit.} \end{cases}$

cost



$\Rightarrow \Delta\underline{v} = \underline{v}_f - \underline{v}_i \Rightarrow \begin{cases} \underline{I} = m_1 \cdot \Delta\underline{v} \\ \|\underline{I}\| = Isp_i\, g_0\, \dfrac{m_{ds,i} + m_{f,i}}{m_{ds,i}} \end{cases}$ (Tsiolkovsky)

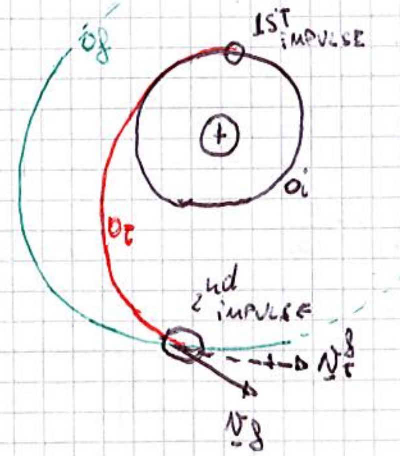
m_{ds} → mass "dry" of the satellite

m_f → mass of fuel.

!!

⇒ OBVIOUSLY: TO PASS FROM AN ORBIT TO AN OTHER ARE REQUIRED {1; 2; 3} IMPULSES. ⇒

- 1 IMPULSE → IMPULSIVE MANOEUVRE.
- 2 IMPULSES → DOUBLE IMPULSIVE MANOEUVRE ⇒ 10c
- 3 IMPULSES → E.g. BIELLITTICAL MAN. ⇒ 20c



"probably $\|\underline{v}_t^f\| > \|\underline{v}^f\|$ where $\underline{v}^f$ represent the velocity on the target orbit anyway it represent a cost.

$$\|\Delta\underline{v}\| = \|\underline{v}^f - \underline{v}_t^f\| = \frac{\|\underline{I}\|}{m_f}$$

. That's obvious because i need to reduce speed using an "OPPOSITE" THRUST.

. Otherwise i have to change the direction ⇒ i need fuel.

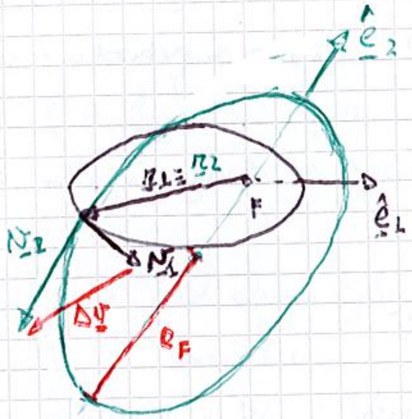
1) → IMPULSIVE MANOUVRE (IM)

IF $\exists\ O_i \cap O_f \Rightarrow \exists$ IMPULSIVE MANOUVRE TO PASS FROM $O_i \to O_f$

DOING AN IM → IS "SIMPLY" CHANGING $\boxed{\text{ECCENTRICITY}}$* AND SEMIAXIS (a)

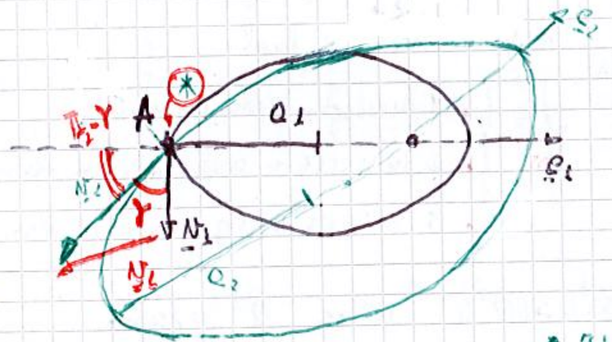
BETWEEN 2 COPLANAR ORBITS

!! $\boxed{\text{ECCENTRICITY VECTOR}}$* ⇒ CHANGING $\|\underline{e}\|$; $\hat{\underline{e}}$!!



OSS (IMPORTANT): transfer optimal (with minimum) cost must be done at the pericenter of the initial orbit where the velocity of the satellite is maximum.

... for semplicity let's consider a manouvre done at the apocenter of the initial orbit.



$\{e_1;\ e_1\} \xrightarrow[r_1=r_0]{IM_0} \{a_2;\ e_2\}$. ⇒ $\Delta v = ??$

$r_1 = r_2$

II^{nd} INVAR. OF MOTION →
$$\begin{cases} \dfrac{v_1^2}{2} - \dfrac{\mu}{r_1} = -\dfrac{\mu}{2a_1} \\ r_{A1} = a_1(1+e_1) = r_1 \end{cases};\qquad \begin{cases} \dfrac{v_2^2}{2} - \dfrac{\mu}{r_2} = -\dfrac{\mu}{2a_2} \\ r_2 = r_{A1} = a_1(1+e_1) \end{cases}$$

$r = \dfrac{a(1-e^2)}{1+e\cos\theta}$ ↓ $r_P = a(1-e)$ $r(\theta=0)$; $r_A = a(1+e)$ $r(\theta=\pi)$

⇓

$$v_2^2 = \frac{2\mu}{a_1(1+e_1)} - \frac{\mu}{a_2}$$

$$v_2^2 = 2\mu\left[\frac{1}{a_1(1+e_1)} - \frac{1}{2a_2}\right]$$

CARNOT'S THEOREM:

$\underline{c} = \underline{b} - \underline{a} \Rightarrow \|\underline{c}\| = a^2 + b^2 - 2ab\cos\gamma$

NOW: $\|\underline{h}_2\| = \|\underline{r}_2 \times \underline{v}_2\| = r_2 v_2 \sin(\frac{\pi}{2} - \gamma) = r_2 v_2 \cos\gamma$

* As sketched.

$$\begin{cases} h_2 = r_2 v_2 \cos\gamma \\ h_2 = \sqrt{p_2 \mu} \end{cases} \Rightarrow r_2^2 v_2^2 \cos^2\gamma = \mu p_2 \Rightarrow r_2^2 v_2^2 \cos^2\gamma = a_2\mu(1-e_2^2)$$

$p = h^2/\mu$ $\qquad$ $p = a(1-e^2)$

→ impulsive apocentric transfer's cost

(i) $\Delta v^2 = v_2^2 + v_1^2 - 2 v_1 v_2 \cos\gamma \Rightarrow \left(v_1 = v_{A,O1} = \sqrt{\frac{\mu}{p_1}}(1-e_1)\right)$.

(ii) $\cos^2\gamma = \dfrac{a_2\mu(1-e_2)^2}{r_2^2 v_2^2} = \dfrac{\mu a_2(1-e_2)^2}{a_1^2(1+e_1)^2 v_2^2}$

(iii) $\begin{cases} v_2^2 = 2\mu\left[\dfrac{1}{a_1(1+e_1)} - \dfrac{1}{2a_2}\right] \\ r_2^2 = a_1(1+e_1)^2 \end{cases}$

!! relations (i); (ii); (iii) permits to calculate the cost (Δv) only as a function of $\{e_1, a_1, a_2, e_2\}$!!

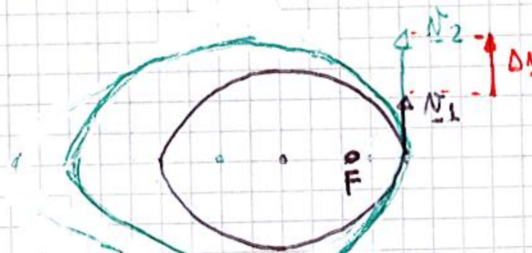
2) → PERICENTRIC / APOCENTRIC MANOUVRE (PARTICULAR I.M.)

(IT). cambiamento della quota di apogeo / perigeo.

IF the final orbit has the same w (same inclination in orbital plane)

⇒ $\hat{e}_1 = \hat{e}_f$

THEN the apocentric transfer is the LESS-COST transfer.



⇒ $\underline{r}_{P1} = \underline{r}_{P2}$.

$$-\frac{\mu}{2a_1} = \frac{1}{2}v_{P1}^2 - \frac{\mu}{r_{P1}} \to -\frac{\mu}{2a_1} = \left[\frac{1}{2}\sqrt{\frac{\mu}{p_1}}(1-e_1)\right]^2 - \frac{\mu}{a_1(1-e_1)} = \frac{1}{2}v_1^2 - \frac{\mu}{a_1(1-e_1)}$$

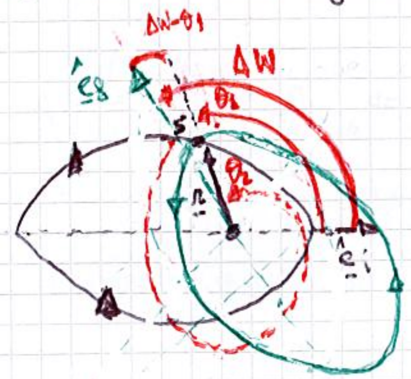
$$-\frac{\mu}{2a_2} = \frac{1}{2}v_{P2}^2 - \frac{\mu}{r_{P_2}} \to -\frac{\mu}{2a_2} = \frac{1}{2}v_2^2 - \frac{\mu}{a_1(1-e_1)}$$

→ Pericentric IM.

$$\Rightarrow \Delta v = v_{P2} - v_{P1} = \sqrt{2\left(\frac{\mu}{r_P} - \frac{\mu}{r_P + r_{A2}}\right)} - \sqrt{2\left(\frac{\mu}{r_P} - \frac{\mu}{r_P + r_{A1}}\right)}$$

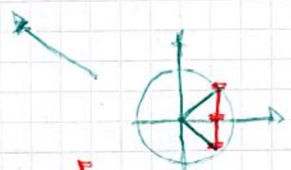
3) ⇒ CHANGE IN PERICENTRIC ANOMALY (IN) ($\Delta\omega \Rightarrow \hat{e}_1 \to \hat{e}_2$; $e_1 = e_2$; $a_1 = a_2$)

(IT): cambiato dell' argomento del perigeo (ω) ma (a, e) invariati.



$\Rightarrow 2\pi = \theta_2 + (\Delta\omega - \theta_1) \Rightarrow 2\pi = \theta_2 - \theta_1 + \Delta\omega$ (i)

$\Rightarrow r = \frac{p}{1 + e\cos\theta_1} = \frac{p}{1 + e\cos\theta_2} \Rightarrow \cos\theta_1 = \cos\theta_2 \Rightarrow \theta_2 = 2\pi - \theta_1$ (ii)



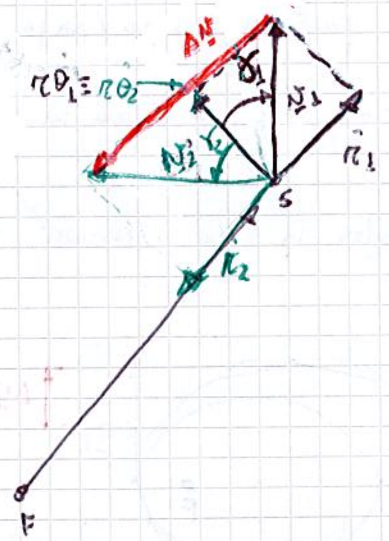
double relation:

$$\begin{cases} \theta_2 = 2\pi - \theta_1 & \text{(i)} \\ \theta_2 - \theta_1 + \Delta\omega = 2\pi & \text{(ii)} \end{cases} \Rightarrow 2\pi - \theta_1 - \theta_1 = 2\pi - \Delta\omega \Rightarrow \boxed{\theta_1 = \frac{\Delta\omega}{2}} \quad (i^*)$$

⇒ !! POINT OF MANOUVERING IS ALWAYS AT $\theta_1 = \frac{\Delta\omega}{2}$!!

using a local reference system ⇒ $\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}} = \sqrt{\frac{\mu}{p}}\,[e\sin\theta\,\hat{\underline{r}} + (1 + e\cos\theta)\,\hat{\underline{\theta}}]$

↳ At the point S, where the transfer happens, we have that $v_{\theta 1} = v_{\theta 2}$



SINCE $\begin{cases} r_1 = r_2 \\ p_1 = p_2 \quad (a_1 = a_2 ;\ e_1 = e_2) \end{cases}$

THEN $\cos\theta_1 = \cos\theta_2$

⇓

$\theta_2 = 2\pi - \theta_1$

⇓

SINCE $\underline{v} = \sqrt{\frac{\mu}{p}}\,[e\sin\theta\,\hat{\underline{r}} + (1 + e\cos\theta)\,\hat{\underline{\theta}}]$

$= \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$

THEN $\begin{cases} v_{\theta i} = v_{\theta f} \\ v_{ri} = -v_{rf} \end{cases}$ *

* ⇒ $\Delta v = \| \underline{v}_f - \underline{v}_i \|$

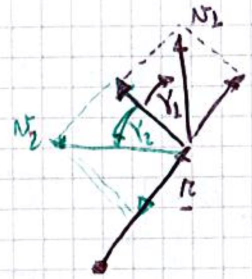
$= v_{rf} - v_{ri} = v_{rf} - (-v_{rf}) = 2 v_{rf}$.

$= 2\sqrt{\frac{\mu}{p}}\,e\sin\theta_1 \underset{(i^*)}{=} 2\sqrt{\frac{\mu}{p}}\,e\sin\left(\frac{\Delta\omega}{2}\right) \Rightarrow \boxed{\Delta v = 2\sqrt{\frac{\mu}{p}}\,e\sin\left(\frac{\Delta\omega}{2}\right)}$

↳ One other important consideration comes from energy:

$$\begin{cases} \varepsilon_1 = -\frac{\mu}{2a_1} = +\frac{1}{2}v_1^2 - \frac{\mu}{r_1} \\ \varepsilon_2 = -\frac{\mu}{2a} = \frac{1}{2}v_2^2 - \frac{\mu}{r_2} \end{cases} \quad \text{in } S \quad r_1 = r_2 \ (a_1 = a_2) \Rightarrow \|\underline{v}_1\| = \|\underline{v}_2\| \quad \text{(iii)}$$

!! since the 2 orbit have the same $\begin{cases} a \\ e \end{cases}$ ⇒ have the same $\begin{cases} h \\ \varepsilon \end{cases}$!!

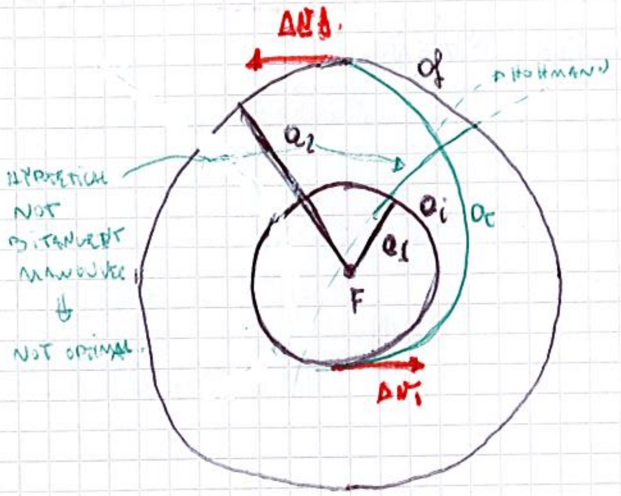


SINCE $\begin{cases} \|\underline{v}_1\| = \|\underline{v}_2\| \\ v_{r1} = -v_{r2} ;\ v_{\theta 1} = v_{\theta 2} \end{cases}$ THEN $\gamma_1 = \gamma_2 \triangleq \gamma$

⇒ $\Delta v = 2v\sin\gamma$.

4) → HOHMANN TRANSFER (2IM) [MONOELLIPTICAL - BITANGENT MANOUVRE.]

" If $(O_i ; O_f)$ can be considered circular orbits ⇒ THE HOHMANN TRANSFER IS THE OPTIMAL. in fact: such manouvre consists in a transfer orbit which is an elliptical transfer orbit where the pericenter is tangent to the O_i and the apocenter is tangent to the O_f. "



CIRC: $e=0$

$\underline{v} = \sqrt{\frac{\mu}{p}}\,[\,e \sin\theta\ \hat{r} + (1+e\cos\theta)\,\hat{\theta}\,]$

$p = a(1-e^2) = a$

$$\Rightarrow \underline{v}_{circ} = \sqrt{\frac{\mu}{p}}\ \hat{\theta} \Rightarrow \begin{cases} v_i = \sqrt{\frac{\mu}{a_i}} = \sqrt{\frac{\mu}{r_i}} \\ v_f = \sqrt{\frac{\mu}{a_f}} = \sqrt{\frac{\mu}{r_f}} \end{cases}$$

$$\Rightarrow \underline{v}_{0t}(\theta=0) = \underline{v}_p^{\,t} = (1+e_t)\sqrt{\frac{\mu}{p_t}}$$

$$\underline{v}_{0t}(\theta=\pi) = \underline{v}_a^{\,t} = (1-e_t)\sqrt{\frac{\mu}{p_t}}$$

$$\Rightarrow \Delta v = \Delta v_i + \Delta v_f = [\,(1+e_t)\sqrt{\frac{\mu}{a_t(1-e_t^2)}} - \sqrt{\frac{\mu}{a_i}}\,] + [\,-\sqrt{\frac{\mu}{a_f}} + (1-e_t)\sqrt{\frac{\mu}{a_t(1-e_t^2)}}\,]$$

$$\Rightarrow \Delta v = [\,\sqrt{\frac{\mu}{a_t(1-e_t^2)}(1+e_t)^2} - \sqrt{\frac{\mu}{a_i}}\,] + [\,\sqrt{\frac{\mu}{a(1-e_t^2)}(1-e_t)^2} - \sqrt{\frac{\mu}{a_f}}\,]$$

$$\Delta v = [\,\sqrt{\frac{\mu(1+e_t)}{a_t(1-e_t)}} - \sqrt{\frac{\mu}{a_i}}\,] + [\,\sqrt{\frac{\mu(1-e_t)}{a_t(1+e_t)}} - \sqrt{\frac{\mu}{a_f}}\,]$$

↳ The choice of $(a_t ; e)$ is constrained by the geometry of the problem.

$$\begin{cases} (1+e_t)\,a_t = a_f & (r_a^t = a_f) \\ (1-e_t)\,a_t = a_i & (r_p^t = a_i) \end{cases} \rightarrow \begin{cases} e_t = \frac{a_f}{a_t} - 1 \\ e_t = -\frac{a_i}{a_t} + 1 \end{cases}$$

$$\Rightarrow \frac{a_f}{a_t} - 1 = -\frac{a_i}{a_t} + 1 \rightarrow a_f - a_t = -a_i + a_t \Rightarrow a_t = \frac{a_f + a_i}{2}$$

$$e_t = \frac{a_f - a_i}{a_f + a_i}$$

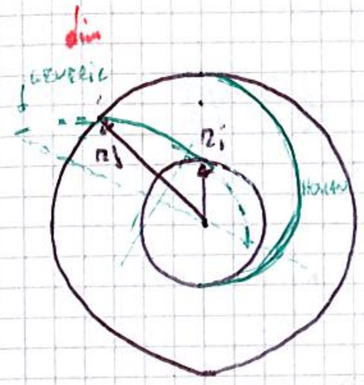
$e = 1 - \frac{r_p}{a}$ $\qquad e_t = \frac{a_f}{a_f+a_i}\cdot 2 - 1 = \frac{2a_f - a_f - a_i}{a_f + a_i} = \frac{a_f - a_i}{a_f + a_i}$

$$\Delta v^{OH} = [\,\sqrt{\frac{\mu(1+e_t)}{a_t(1-e_t)}} - \sqrt{\frac{\mu}{a_i}}\,] + [\,\sqrt{\frac{\mu(1-e_t)}{a_t(1+e_t)}} - \sqrt{\frac{\mu}{a_f}}\,]$$

$e_t = \frac{a_f - a_i}{a_f + a_i} = \frac{r_f - r_i}{r_f + r_i}$; $\quad a_t = \frac{a_f + a_i}{2} = \frac{r_f + r_i}{2}$; $\quad (a_f \equiv r_f ;\ a_i \equiv r_i)$

doing substitution is possible to obtain ⊛ →

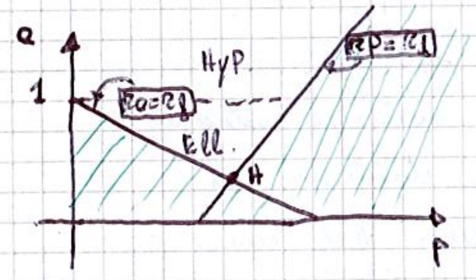
↳ " Between all the possible manoeuvres with 2 impulses between 2 circular coplanar orbits Hohmann transfer is the optimal (with minimum Δv^{TOT}) "



$$\Rightarrow \text{for the generic orbit: } \begin{cases} r_i \geq r_p \\ r_f \leq r_a \end{cases} \Rightarrow \begin{cases} \frac{p}{1+e} \leq r_i \\ \frac{p}{1-e} \geq r_f \end{cases}$$

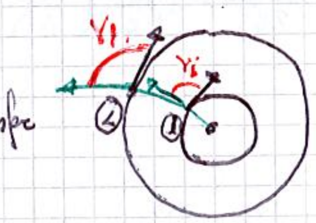
$$\Rightarrow \begin{cases} p \leq r_i(1+e) \\ p \geq r_f(1-e) \end{cases}$$

Building a plane (p, e) is possible to determine the region where is possible to determine a transfer orbit.



$\begin{cases} r_p = p(1-e^2) \\ r_a = p(1+e^2) \end{cases}$ imposing maximum value for $(r_1 ; r_2)$

⇒ for the generic transfer

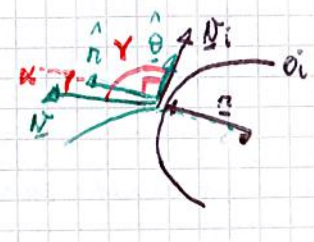


$$\begin{cases} (\Delta v_1)^2 = v_1^2 + v_{c1}^2 - 2\,v_1 v_{c1}\cos\gamma_1 \\ (\Delta v_2)^2 = v_2^2 + v_{c2}^2 - 2\,v_2 v_{c2}\cos\gamma_2 \end{cases}$$

$\varepsilon_1 = \varepsilon_2 = \frac{1}{2}v_1^2 - \frac{\mu}{r_1} = \frac{1}{2}v_f^2 - \frac{\mu}{r_2} = -\frac{\mu}{2a}$

$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) = \mu\left(\frac{2}{r} - \frac{e^2-1}{p}\right)$

$$\begin{cases} v_1^2 = \mu\left(\frac{2}{r_1} - \frac{e^2-1}{p}\right) \\ v_2^2 = \mu\left(\frac{2}{r_2} - \frac{e^2-1}{p}\right) \end{cases}$$



⇒ considering the angular momentum of the transfer orbit:

$$\underline{h} = \underline{r} \times \underline{v} \Rightarrow \begin{cases} \|\underline{h}\| = r\,v\sin\alpha \\ \gamma = \frac{\pi}{2} + \alpha \rightarrow \alpha = \gamma - \frac{\pi}{2} \end{cases} \Rightarrow h = r\,v\sin\left(\gamma - \frac{\pi}{2}\right) = r\,v\cos\gamma$$

$\cdot\ r v \cos\gamma = h \Rightarrow v \cos\gamma = \frac{h}{r} = \sqrt{\frac{\mu p}{r}}$ $\quad \| \frac{h^2}{\mu} = p \rightarrow h = \sqrt{\mu p}$

$$\cdot \begin{cases} \Delta v_1^2 = \mu\left(\frac{2}{r_1} - \frac{e^2-1}{p}\right) + v_{c1}^2 - 2 v_{c1}\sqrt{\frac{\mu p}{r_1}} \\ \Delta v_2^2 = \mu\left(\frac{2}{r_2} - \frac{e^2-1}{p}\right) + v_{c2}^2 - 2 v_{c2}\sqrt{\frac{\mu p}{r_2}} \end{cases}$$

Deriving in e: (once e is known $\Rightarrow p$ is auto-determined by the geometry of $r_1; r_2$)

$$\begin{cases} 2\Delta v_1 \cdot \frac{\partial \Delta v_1}{\partial e} = \frac{\partial}{\partial e}\left(v_1^2 + v_{c1}^2 - 2 v_{c1}\sqrt{\frac{\mu p}{r_1}}\right) = 2 v_1 \frac{\partial v_1}{\partial e} \\ 2\Delta v_2 \cdot \frac{\partial \Delta v_2}{\partial e} = \dots\dots = 2 v_2 \frac{\partial v_2}{\partial e} \end{cases}$$

$$v_1^2 = \mu\left(\frac{2}{r_1} - \frac{e^2-1}{p}\right) \Rightarrow \begin{cases} \frac{\partial(v_1^2)}{\partial e} = \frac{2\mu e}{p} \\ \frac{\partial(v_2^2)}{\partial e} = \frac{2\mu e}{p} \end{cases} \Rightarrow 2 v_1 \frac{\partial v_1}{\partial e} = 2 v_2 \frac{\partial v_2}{\partial e} = \frac{2\mu e}{p}$$

considering the total cost derived in e:

$$1)\ \begin{cases} 2\Delta v_1 \cdot \frac{\partial \Delta v_1}{\partial e} = 2 v_1 \frac{\partial v_1}{\partial e} = 2\frac{\mu e}{p} \\ 2\cdot\Delta v_2 \frac{\partial \Delta v_2}{\partial e} = 2 v_2 \frac{\partial v_2}{\partial e} = 2\frac{\mu e}{p} \end{cases} \Rightarrow \begin{cases} \frac{\partial \Delta v_1}{\partial e} = \frac{\mu e}{p}\frac{1}{\Delta v_1} \\ \frac{\partial \Delta v_2}{\partial e} = \frac{\mu e}{p}\frac{1}{\Delta v_2} \end{cases}$$

$$2)\ \frac{\partial \Delta v_{TOT}}{\partial e} = \frac{\partial \Delta v_1}{\partial e} + \frac{\partial v_2}{\partial e}$$

$$\Rightarrow \frac{\partial \Delta v_{TOT}}{\partial e} = \frac{\mu e}{p}\left(\frac{1}{\Delta v_1} + \frac{1}{\Delta v_2}\right) \checkmark$$

The lowest admissible value of e allows the minimum cost
$\Rightarrow$ the ellipse with minimum e is the Hohmann one (SEEING THE GRAPH)

$\Rightarrow$ considering the maximum p obtainable: we are considering again the minimum $\frac{\Delta v_{TOT}}{\partial e}$ with the value of e already established.

$$\frac{\partial \Delta v_{TOT}}{\partial e} = \frac{\mu e}{p}\left(\frac{1}{\Delta v_1} + \frac{1}{\Delta v_2}\right) \Rightarrow \Delta v_{TOT}\downarrow \text{ for } \begin{cases} e\downarrow \\ p\uparrow \end{cases}$$

The maximum p obtainable is for $r_f = r_a$. $\Rightarrow p|_{MAX} = r_f(1-e) \equiv r_a(1-e)$

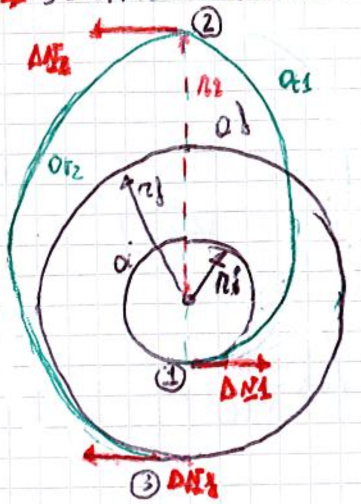
$\Rightarrow$ the best transfer is the one that has
- $e = e_{HOHMANN}$
- $r_f = r_a = a_2$

$\Rightarrow$ such orbit must be Hohmann's transfer orbit.

$\circledast$ Δv_i; Δv_f only as function of r_i; r_f

$$\Delta v_i = \sqrt{\frac{\mu}{r_i}}\left[\sqrt{\frac{2 r_f}{r_i + r_f}} - 1\right] \qquad \Delta v_f = \sqrt{\frac{\mu}{r_f}}\left[1 - \sqrt{\frac{2 r_i}{r_i + r_f}}\right], \quad \Delta v^{TOT} = \Delta v_i + \Delta v_f$$

5) BIELLIPTICAL TRANSFER. (3 IM) [BIELLIPTIC BITANGENT MANOUVER]



OT_1: $r_{p_1} = r_i$ ($\equiv a_i$)

OT_2: $\begin{cases} r_{p_2} = r_f (\equiv a_f) \\ r_{a_2} = r_{a1} \end{cases} \Rightarrow$!! $a_{OT_1} \neq a_{OT_2}$!!

choice of the designer of the orbit: THAT'S A FREE VARIABLE OF THE PROBLEM

($r_{a2} = r_{a1} \rightarrow r_2$)

$\Rightarrow$ Imposing: $\frac{r_f}{r_i} \triangleq \beta$; $\frac{r_2}{r_i} \triangleq \alpha$ and computing the total cost is possible to determine the optimal α (that minimize the cost)
$\Rightarrow$ the optimal semiaxes for the transfer orbits (OT_1; OT_2)

$$\rightarrow \begin{cases} v_i = \sqrt{\frac{\mu}{p_i}} = \sqrt{\frac{\mu}{r_i}} \\ v_f = \sqrt{\frac{\mu}{r_f}} \end{cases} ; \quad \begin{matrix} r_p = a(1-e) \\ e_0 = 0 \end{matrix}$$

$\| -\frac{\mu}{2a} = \frac{1}{2}v^2 - \frac{\mu}{r}$ $\quad v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$!! In computing transfer this form of velocity is really useful!!

$$\rightarrow OT_1: \begin{cases} v_1 = \sqrt{\frac{2\mu}{r_i} - \frac{\mu}{a_1}} \\ v_2 = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a_1}} \end{cases} \quad a_1\text{: semiaxis of } OT_1$$

$$OT_2: \begin{cases} v_2' = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a_2}} \\ v_3' = \sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_2}} \end{cases} \quad a_2\text{: semiaxis of } OT_2$$

BUT: having defined $\beta \triangleq \frac{r_f}{r_i}$; $\alpha \triangleq \frac{r_2}{r_i}$ $\Rightarrow$ $a_{OT_1} = a_1 = \frac{r_i + \alpha r_i}{2} = \frac{1+\alpha}{2} r_i$

$a_{OT_2} = a_2 = \frac{\beta r_i + \alpha r_i}{2} = \frac{\alpha+\beta}{2} r_i$

($\beta \rightarrow$ FIXED
$\alpha \rightarrow$ PROJECT PARAMETER)

$$\text{1}^{st}\text{ IM}) \ \Delta v_1 = v_1 - v_i = \sqrt{\frac{2\mu}{r_i} - \frac{2\mu}{r_i(1+\alpha)}} - \sqrt{\frac{\mu}{r_i}} = \sqrt{\frac{2\mu}{r_i}\left[1 - \frac{1}{1+\alpha}\right]} - \sqrt{\frac{\mu}{r_i}}$$

$$\text{2}^{nd}\text{ IM}) \ \Delta v_2 = v_2 - v_2' = \sqrt{\frac{2\mu}{\alpha r_i} - \frac{2\mu}{(\alpha+\beta) r_i}} - \sqrt{\frac{2\mu}{\alpha r_i} - \frac{2\mu}{(1+\alpha) r_i}} = \sqrt{\frac{2\mu}{r_i}\left[\frac{1}{\alpha} - \frac{1}{\alpha+\beta}\right]} - \sqrt{\frac{2\mu}{r_i}\left[\frac{1}{\alpha} - \frac{1}{1+\alpha}\right]}$$

$$= \sqrt{\frac{2\mu}{r_i}\left[\frac{1}{\alpha} - \frac{1}{\alpha+\beta}\right]} - \sqrt{\frac{2\mu}{r_i}\left[\frac{1}{\alpha} - \frac{1}{1+\alpha}\right]}$$

$$\text{3}^{rd}\text{ IM}) \ \Delta v_3 = v_3' - v_f = +\sqrt{\frac{2\mu}{\beta r_i} - \frac{2\mu}{(\alpha+\beta) r_i}} + \sqrt{\frac{\mu}{\beta r_i}} = -\sqrt{\frac{\mu}{\beta r_i}} + \sqrt{\frac{2\mu}{r_i}\left[\frac{1}{\beta} - \frac{1}{\alpha+\beta}\right]}$$

!! $|v_f| < |v_3'|$

⇒ $\Delta v_1 = \underbrace{\sqrt{\frac{\mu}{r_i}}}_{v_{ci}}\left[\sqrt{2\left[1-\frac{1}{1+\alpha}\right]}-1\right] = v_{ci}\left[\sqrt{\frac{2\alpha}{1+\alpha}}-1\right]$

velocity in 1st orbit r_i

$\Delta v_2 = \sqrt{\frac{\mu}{r_i}}\left[\sqrt{2\frac{\alpha+\beta-\alpha}{\alpha(\alpha+\beta)}} - \sqrt{\frac{2(1+\alpha-\alpha)}{\alpha(1+\alpha)}}\right] = v_{ci}\left[\sqrt{\frac{2\beta}{\alpha(\alpha+\beta)}}-\sqrt{\frac{2}{\alpha(1+\alpha)}}\right]$

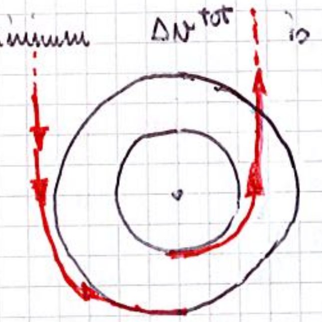
$\Delta v_3 = \sqrt{\frac{\mu}{r_i}}\left[-\sqrt{\frac{1}{\beta}}+\sqrt{\frac{2}{\beta}-\frac{2}{\alpha+\beta}}\right] = v_{ci}\left[-\sqrt{\frac{1}{\beta}}+\sqrt{\frac{2\alpha}{(\alpha+\beta)\beta}}\right]$

SINCE

⇒ $\Delta v^{TOT} = \Delta v_1+\Delta v_2+\Delta v_3$; α is the only design parameter.

THEN

minimum Δv^{TOT} is obtained for $\alpha\to\infty$ ⇒ FOR A PARABOLIC ORBIT BEING $\alpha = \frac{r_2}{r_i}$



† means imposing the limit for $\alpha\to\infty$

$$\lim_{\alpha\to\infty}\begin{cases}\Delta v_1 = v_{ci}\left[\sqrt{2}-1\right]\\ \Delta v_2 = v_{ci}\cdot 0 = 0\\ \Delta v_3 = \left[-\sqrt{\frac{1}{\beta}}+\sqrt{2}\right]\cdot v_{ci}\end{cases}$$ (*)

$\Delta v^{BITANGENT}_{PARABOLIC} = \lim_{\alpha\to\infty}\Delta v^{BITANGENT}_{ELLIPTIC}$

REALLY IMPORTANT QUESTION: FOR WICH VALUE OF $\beta = \frac{r_j}{r_i}$ IS CONVENIENT (IN TERMS OF Δv^{TOT}) CHOOSING A BITANGENT MANOUVER INSTEAD OF AN HOHMANN MANOUVER (WICH IS THE MORE CONVENIENT BETWEEN ALL THE 2 IMPULSES MANOUVER.) ??

⇒ By imposing the equivalence between:

TOTAL COST HOHMANN MANOUVRE = TOTAL COST BITANGENT MANOUVER WITH 2 PARABOLIC TRANSFER ORBITS

THE BEST POSSIBLE BETWEEN ALL THE BITANGENT MANOUVER. (Theoretical limit)

$\Delta v_1^H = \sqrt{\frac{2r_j}{r_i+r_j}}-1 \cdot \sqrt{\frac{\mu}{r_i}}$,

$\Delta v_2^H = \sqrt{\frac{\mu}{r_j}}\left[1-\sqrt{\frac{2r_i}{r_i+r_j}}\right]$

$\Delta v^H_{TOT} = \Delta v_1^H+\Delta v_2^H$.

⇒ $\Delta v^H_{TOT} = \lim_{\alpha\to\infty}\left(\Delta v^B_{TOT}\right)$

⇓

find β^*: $\Delta v^H_{TOT} = \Delta v^B_{PAR} = \Delta v^B_{\alpha\to\infty}$

(*) Then: $\lim_{\alpha\to\infty}\Delta v^B_{TOT} = \sqrt{\frac{\mu}{r_i}}\left[2\sqrt{2}-1-\sqrt{\frac{r_i}{r_j}}\right] = \Delta v^B_{PAR}$.

$\Delta v^H_{TOT} = \sqrt{\frac{2r_j}{r_i+r_j}}-1 \cdot \sqrt{\frac{\mu}{r_i}} + \sqrt{\frac{\mu}{r_j}}\left[1-\sqrt{\frac{2r_i}{r_i+r_j}}\right]$

$= \sqrt{\frac{r_j-r_i}{r_j+r_i}}\,\sqrt{\frac{\mu}{r_i}}+\sqrt{\frac{\mu}{r_j}}\left[1-\sqrt{\frac{2r_j/r_i}{1+r_j/r_i}}\right]$

$= \sqrt{\frac{r_j/r_i-1}{r_j/r_i+1}}\,\sqrt{\frac{\mu}{r_i}}+\sqrt{\frac{r_i}{r_j}}\,\sqrt{\frac{\mu}{r_i}}\left[1-\sqrt{\frac{2r_j/r_i}{1+r_j/r_i}}\right]$

$= \sqrt{\frac{\beta-1}{\beta+1}}\,\sqrt{\frac{\mu}{r_i}}+\sqrt{\frac{\mu}{r_i}}\sqrt{\frac{1}{\beta}}\left[1-\sqrt{\frac{2\beta}{1+\beta}}\right]$

⇒ $\frac{\Delta v^H_{TOT}}{v_{ci}} = \frac{\Delta v^B_{TOT,PAR}}{v_{ci}}$ ⇒ $2\sqrt{2}-1-\sqrt{\frac{1}{\beta}} = \sqrt{\frac{\beta-1}{\beta+1}}+\sqrt{\frac{1}{\beta}}-\sqrt{\frac{2}{1+\beta}}$

$2\sqrt{2}-1 = 2\sqrt{\frac{1}{\beta}}+\sqrt{\frac{\beta-1}{\beta+1}}-\sqrt{\frac{2}{1+\beta}}$ → to solve by try. double square elevation.

~B. SOLVING: $\beta = 11.94 = \frac{r_j}{r_i}$

⇒

* FOR $\beta < 11.94$ ∄ A BITANGENT TRANSFER THAT COSTS LESS THAN HOHMANN TRANSFER

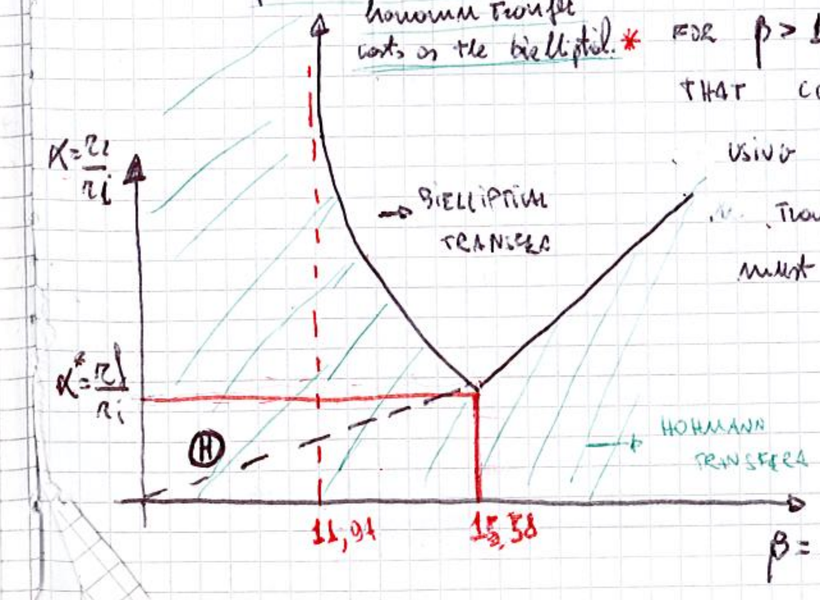
⇒ I'LL USE A HOHMANN MANOUVER.

* FOR $\beta = 11.94$ ∃ A BITANGENT MANOUVER THAT USES 2 PARABOLAS AS TRANSFER ORBITS

!! THIS SOLUTION IS POSSIBLE ONLY THEORETICALLY !!

⇒ → this distance is ∞

for each β this line represents α such that hohmann transfer costs as the bielliptical.

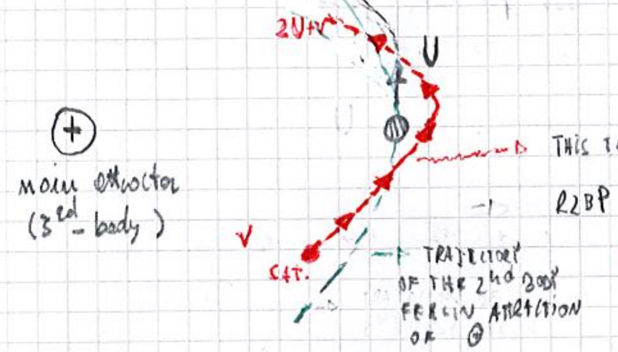


* FOR $\beta > 11.94$ ∃ A BITANGENT MANOUVER THAT COSTS LESS THAN HOHMANN TRANSFER USING 2 ELLIPS. **BUT** the semiaxis of the transfer orbit (IN EXACT DISTANCE r_2) must be taken big enough.

6. → " In-Plane gravity-assist "

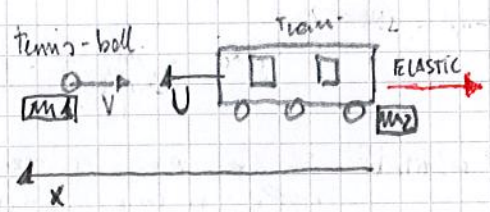
a) intuitive explanation:

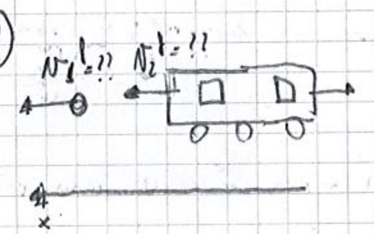
gravity assist can be seen as the exchange of momentum between 2 bodies under the influence of a 3^{rd} body wich is the main attractor.



(I'm ignoring the 3^{rd} body (main ATTRACTOR) in evaluating this phenomena.)

↳ in a duality with reality IT can be seen as an ELASTIC COLLISION BETWEEN 2 BODIES WHERE 1 HAS A MASS REALLY HIGHER THEN THE OTHER





Hp: $m_2 >> m_1$

elastic collision $\Leftrightarrow \begin{cases} E_{Ki} = E_{Kf} \\ P_i = P_f \end{cases} \Rightarrow \begin{cases} \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 U^2 = \frac{1}{2} m_1 (v_1^f)^2 + \frac{1}{2} m_2 (v_2^f)^2 \\ m_2 U \pm m_1 V = m_1 v_1^f + m_2 v_2^f \end{cases}$

verse of velocity is also an unknown for such problem

$$\Rightarrow (-) \begin{cases} m_1 [V^2 - (v_1^f)^2] = m_2 [(v_2^f)^2 - U^2] \\ m_2 [U - v_2^f] = m_1 [v_1^f + V] \end{cases}$$

$$\Rightarrow \begin{cases} m_1 [V - v_1^f][V + v_1^f] = m_2 [v_2^f - U][v_2^f + U] \quad (ii) \\ m_2 [U - v_2^f] = m_1 [V + v_1^f] \quad (i) \rightarrow m_2 [v_2^f - U] = -m_1 [V + v_1^f] \end{cases}$$

cause of (i) ⇒ (ii) becomes

$[U - v_2^f] = [V + v_1^f] \rightarrow v_1^f = -v_2^f + U - V$

$\Rightarrow m_2 v_2^f - m_2 U = -m_1 V + m_1 v_2^f - m_1 U + m_1 V$

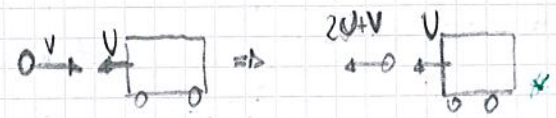
$\Rightarrow (m_2 - m_1) v_2^f = (m_2 - m_1) U + 2 m_1 V$

$$\Rightarrow \begin{cases} v_1^f = \frac{2 m_2}{(m_1 + m_2)} U + \frac{m_2 - m_1}{(m_1 + m_2)} V \\ v_2^f = \frac{m_2 - m_1}{(m_1 + m_2)} U - \frac{2 m_1}{(m_1 + m_2)} V \end{cases}$$

imposing the limit for $m_2 >> m_1 \equiv \lim_{m_2 \to \infty}$

$$\lim_{m_2 \to \infty} \begin{cases} v_1^f = 2U + V \\ v_2^f = U \end{cases}$$

REFERRING TO



$$(+) \begin{cases} m_1 [V - v_1^f][V + v_1^f] = m_2 [v_2^f - U][v_2^f + U] \\ m_2 U + m_1 V = m_1 v_1^f + m_2 v_2^f \rightarrow m_2 [U - v_2^f] = m_1 [v_1^f - V] \end{cases}$$

$m_2 [v_2^f - U] = m_1 [V - v_1^f]$

$$\Rightarrow \begin{cases} V + v_1^f = v_2^f + U \rightarrow v_2^f = V + v_1^f - U \\ m_2 [V + v_1^f - U - U] = m_1 V - m_1 v_1^f \end{cases}$$

$-2 m_2 U + m_2 V + m_2 v_1^f = m_1 V - m_1 v_1^f$

$(m_2 + m_1) v_1^f = (m_1 - m_2) V + 2 m_2 U \rightarrow v_1^f = \frac{m_1 - m_2}{m_1 + m_2} V + \frac{2 m_2}{m_1 + m_2} U$

$v_2^f = [m_1 V + \cancel{m_2 V} - m_1 U - m_2 U + m_1 V - \cancel{m_2 V} + 2 m_2 U] \frac{1}{m_1 + m_2} = \frac{2 m_1}{m_1 + m_2} V + \frac{m_2 - m_1}{m_2 + m_1} U$

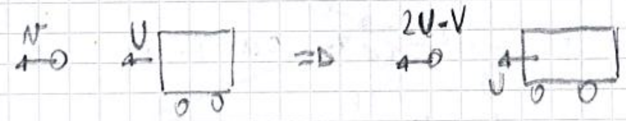
$$\Rightarrow \begin{cases} v_1^f = \frac{m_1 - m_2}{m_1 + m_2} V + \frac{2 m_2}{m_1 + m_2} U \\ v_2^f = \frac{2 m_1}{m_1 + m_2} V + \frac{(m_2 - m_1)}{(m_1 + m_2)} U \end{cases}$$

$v_1^i = -V$: $v_1^f = \frac{m_1 - m_2}{(m_1 + m_2)} (-V) + \frac{2 m_2}{m_1 + m_2} U$

$v_1^i = V$: $v_1^f = \frac{2 m_2}{m_1 + m_2} U + \frac{(m_2 - m_1)}{(m_1 + m_2)} V$

$$\lim_{m_2 \to \infty} \begin{cases} v_1^f = -(2U + V) \\ v_2^f = U \end{cases}$$

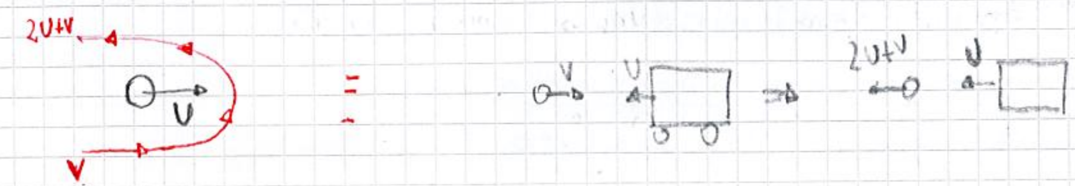
REFERRING



≈ obvious

To have a collision must be the $U > V$.

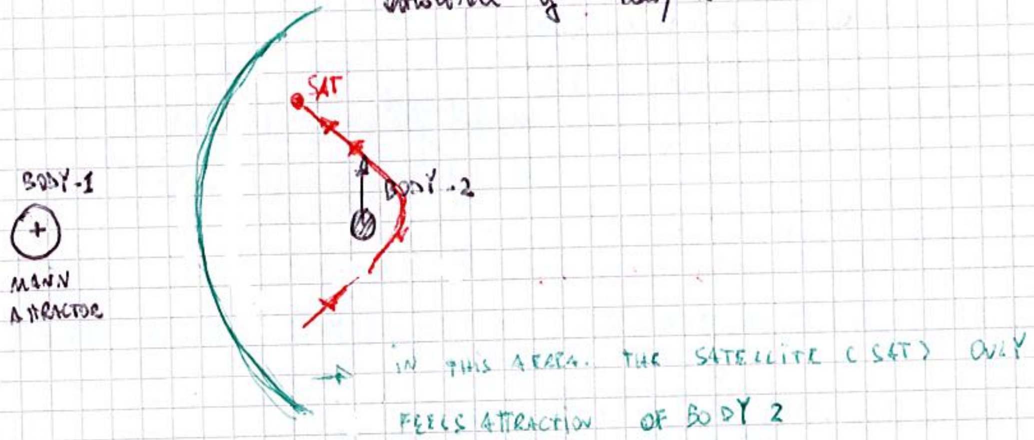
* ⇒ the train will see the ball escaping at speed V but an observer will see the ball coming to him at speed 2U+V



b) Patched / linked conic approach

!! In this case gravity assist should be considered a R3BP (i.e. restricted 3 bodies problem) because obviously we have 3 body involved in mutual attraction BUT

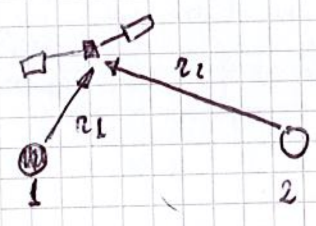
- $m_{SAT} << m_2 << m_1$
- Using patch conic method we can consider that exist a region where the sat is only under the attraction of body 2 !!



→ Sphere of influence (SOI)

SPHERE OF INFLUENCE → distance of a satellite from planet ② such that gravity effect of planet ① can be retained a disturb.

⇓



$\underline{a}$ → force per unit mass. d → disturb ; g → gravity

$$(r_1 ; r_2) : \quad \frac{|\underline{a}d_{21}|}{|\underline{a}g_{1}|} = \frac{|\underline{a}d_{21}|}{|\underline{a}g_{21}|}$$

$$r_{soi} = f_{min}(r_1 ; r_2)$$

...

→ LINKED CONIC APPROACH/APPROXIMATION

- $r_{SOI} \to \infty$
- $\hat{\underline{V}}_2 \to$ UNKNOWN
- $\Delta t \to 0$ (ISTANTANEOUS)

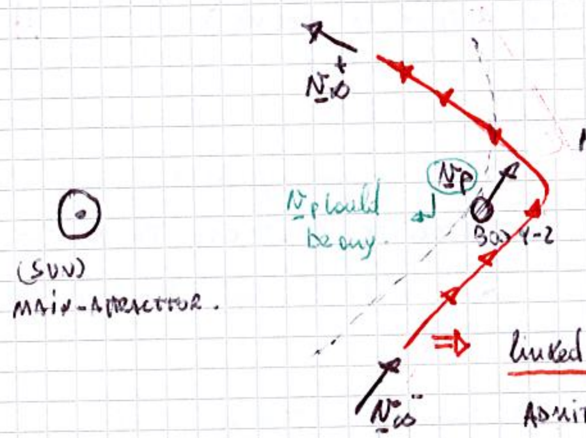
($\Delta t \to$ DURATION OF THE GRAVITY ASSIST)

" $\begin{cases} r_{SOI} \to \infty \\ \Delta t \to 0 \end{cases}$ → IT'S A BIG APPROXIMATION: MEANS THAT IN THE WHOLE FIELD, FOR THE INFINITESIMAL DURATION OF THE G.A., THE SATELLITE ONLY FEELS GRAVITY ACTION DUE TO BODY-2 "

... IN REALITY WE HAVE: → PATCHED CONIC

- $r_{SOI} \to$ DEFINED
- $\hat{\underline{V}}_2 \to$ UNKNOWN
- $\Delta t \to$ FINITE.

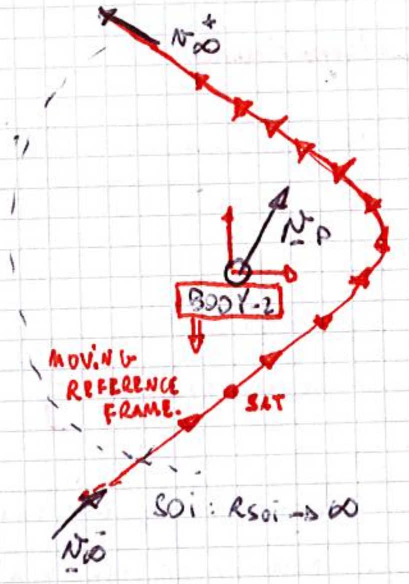
⇒ When the satellite enters the sphere of influence of body-2 will feel the influence of body-2 only for the time interval in wich the satellite is transiting the sphere of influence. Once the satellite will exit the SOI will only feel attraction due to the main attractor (TIPICALLY: SUN)



⇒ Linked-conic Approach: WE ARE ALLOWED TO ADMIT THAT, EVEN IF THE DISTANCE IS INFINITE, THE VELOCITIES OF THE SATELLITE $(\underline{V}_\infty^+ ; \underline{V}_\infty^-)$ ARE THE ONES RESULTING FROM KEPLERIAN (R2BP) INTERACTION BETWEEN BODY-2 AND THE SATELLITE ⇒ EXCLUDING THE MAIN ATTRACTOR.

PROCEEDING IN
LINKED-CONIC APPROACH

MAIN-ATTRACTOR
⇓
ABSOLUTE REFERENCE FRAME.



SINCE

INSIDE THE SOI OF THE BODY-2 THE SATELLITE IS ONLY "FEELING" BODY-2

THEN

$v_{3C} \to$ 3rd cosmic velocity

$$\varepsilon = \frac{1}{2} v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \to v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} \Rightarrow v_\infty^+ = v_\infty^- = \lim_{r \to \infty} \sqrt{-\frac{\mu}{a}}$$

$v_{3C} \to$ VELOCITY AT INFINITE DISTANCE FOR A PARABOLIC ORBIT

$$v_{3C} = v_\infty^- = v_\infty^+ = \lim_{r \to \infty} \sqrt{\mu\left(\frac{1}{r} - \frac{1}{a}\right)} \Rightarrow v_{3C} = \sqrt{-\frac{\mu}{a}} \quad (\text{HYP}\Rightarrow a<0)$$

!! THIS VELOCITY IS THE RELATIVE VELOCITY OF THE SATELLITE RESPECT TO BODY-2 (IT'S EXPRESSED IN A MOVING REFERENCE FRAME) !!

⇓

SINCE

$(v_\infty^-; v_\infty^+)$ REPRESENT THE VELOCITY OF THE SATELLITE RESPECT TO BODY-2

THEN

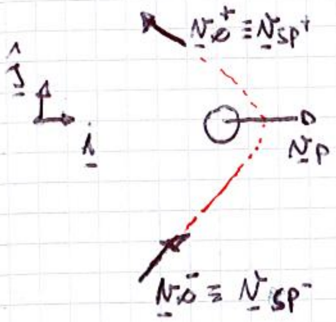
THE VELOCITY OF THE SATELLITE RESPECT TO THE ABSOLUTE REFERENCE FRAME (WICH IS NOT MOVING) MUST BE CALCUTED BY COMPOSITION OF VELOCITIES.

$v_s \to$ velocity of the satellite in the absolute r.f. ; $v_{sp} \to$ velocity of the satellite respect to the plane (relative velocity) $\equiv (v_\infty^+; v_\infty^-)$; $v_p \to$ velocity of the planet (in absolute r.f.)

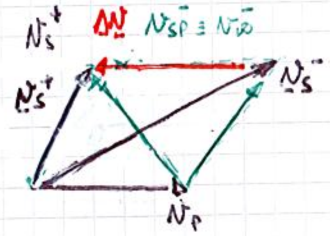
$\Rightarrow v_s = v_{sp} + v_p$

Eg:

1) Passing "IN-FRONT" of the planet.



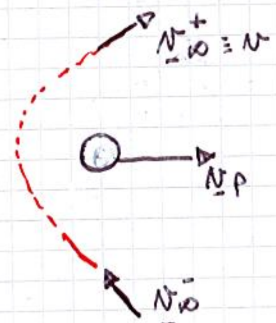
⇒ → in relative R.F. ; → in absolute R.F.



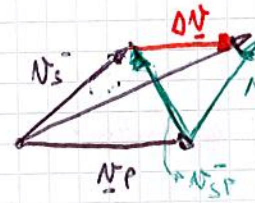
⇓

$$\begin{cases} v_s^+ < v_s^- \\ \Delta v = (-\Delta v)\,\hat{i} \end{cases}$$

2) Passing "BEHIND" the planet.



⇒ Again:

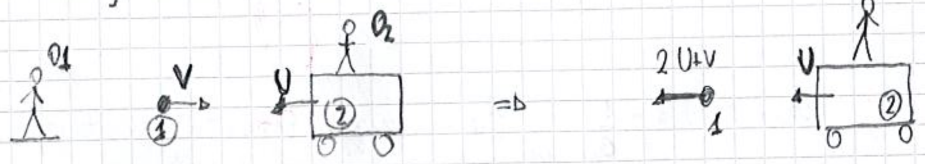


⇓

$$\begin{cases} v_s^+ > v_s^- \\ \Delta v = (+\Delta v)\,\hat{i} \end{cases}$$

obs: Body-2 will see (at infinite distances) the satellite coming to him and escaping from him at the same speed; while an observer put on the main attractor will see the satellite reaching the planet at v_s^- but escaping from it with a higher speed v_s^+

The duality with the train is then obvious

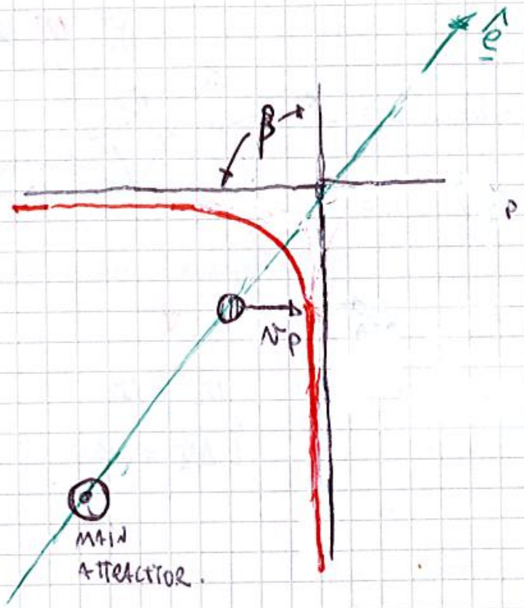


O_1 sees: $v_1 = V$; $v_2 = -U$	O_1 sees: $v_1 = -2U+V$, $v_2 = -U$
O_2 sees: $v_1 = U+V$ $v_2 = 0$	O_2 sees: $v_1 = U+V$; $v_2 = 0$

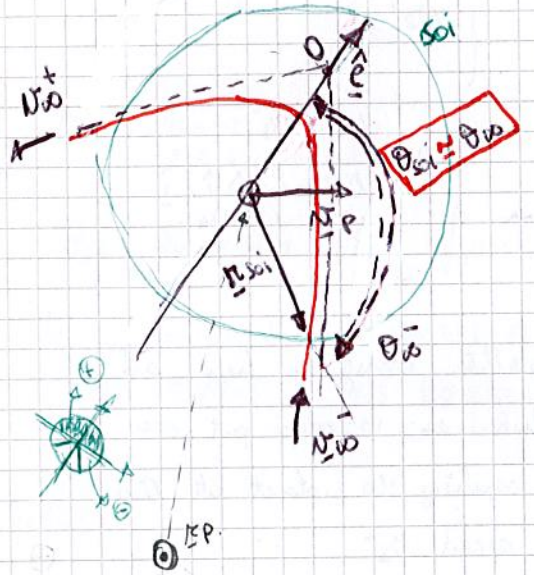
?? ⇒ the maximum Δv obtainable (FOR A FULL EXCHANGE OF MOMENTUM) by gravity assist is

$\Delta v = 2 v_p$ $(=2U)$?? obtainable for parabolic orbit ??

c) Performance of the gravity assist.



about linked conics approximation:



SINCE

for a linked conic-approach $r_{soi} \to \infty$

$$r_{soi} = \frac{p}{1 + e\cos\theta_{soi}} \rightarrow r_{soi} + e\cos\theta_{soi}\, r_{soi} = p$$

$$\cos\theta_{soi} = \frac{1}{e\, r_{soi}}(p - r_{soi})$$

THEN

$$\cos\theta_{soi} = \frac{1}{e}\left(\frac{p}{r_{soi}} - 1\right)$$

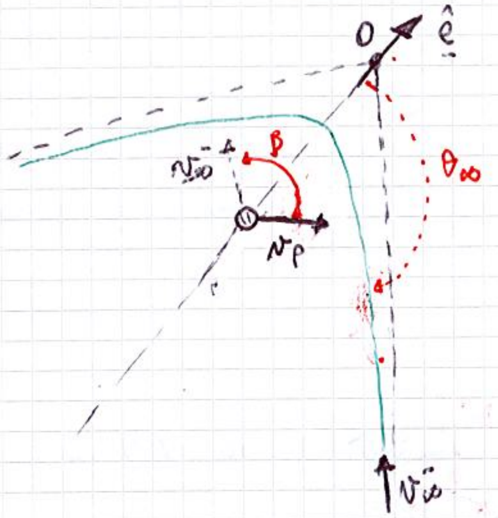
$$\cos(\theta_{soi}) \simeq \cos(\theta_\infty) = \lim_{r \to \infty} \frac{1}{e}\left(\frac{p}{r_{soi}} - 1\right)$$

$$\Rightarrow \cos(\theta_\infty) = -\frac{1}{e}$$

→ full condition entrance/exit.

$$v_\infty^+ = v_\infty^- = \sqrt{-\frac{\mu}{a}}$$ (IN PLANET R.F.) ; $\cos(\theta_\infty^-) = -\frac{1}{e}$, $\cos(\theta_\infty^+) = -\frac{1}{e}$

θ_∞ → angle between eccentricity vector and $v_\infty^\pm$;

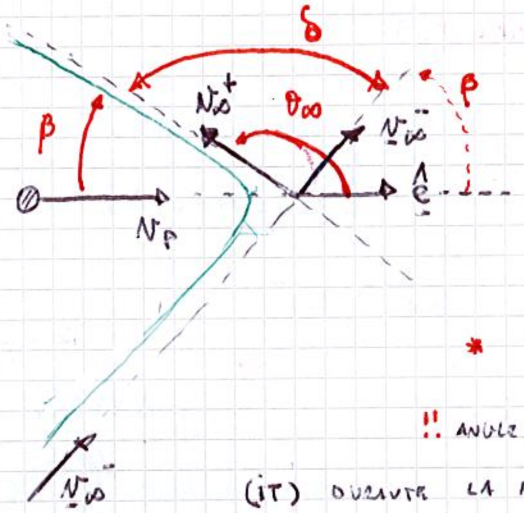


* θ_∞ → angle between ê and v_∞^- or v_∞^+

* β → angle between v_p and v_∞

O → center of the conic (hyperbola).

↳ Supposing the velocity of the planet aligned with the eccentricity of the hyperbola.



$$\Rightarrow {}^{*}\begin{cases} \beta + \theta_\infty = \pi \Rightarrow \cos\beta = +\frac{1}{e} \\ \cos\theta_\infty = -\frac{1}{e} \end{cases}$$

* δ → ANGOLO DI SVOLTA
"TURN ANGLE" ⇒ $2\beta + \delta = \pi$ **
!! ANGLE BETWEEN v_∞^- AND v_∞^+ !!

(IT) DURANTE LA FIONDA GRAVITAZIONALE. IL CAMPO GRAVITAZIONALE DEL PIANETA È IN GRADO DI RUOTARE IL SATELLITE DI UN ANGOLO PARI A δ

$$(\text{if } \hat{e} \parallel v_p) \Rightarrow \begin{cases} \theta_\infty + \beta = \pi \\ 2\beta + \delta = \pi \end{cases} \Rightarrow \begin{cases} \cos\beta = \frac{1}{e} \; * \\ \beta = \frac{\pi - \delta}{2} \; ** \end{cases}$$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

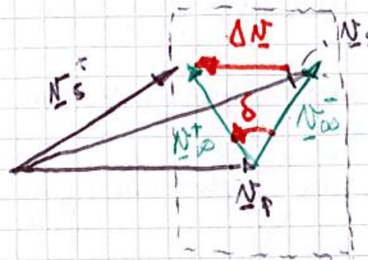
Then we find that:

$$\frac{1}{e} = \cos\left(\frac{\pi - \delta}{2}\right) = \overbrace{\cos\left(\frac{\pi}{2}\right)}^{=0}\cos\left(\frac{\delta}{2}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\delta}{2}\right)$$

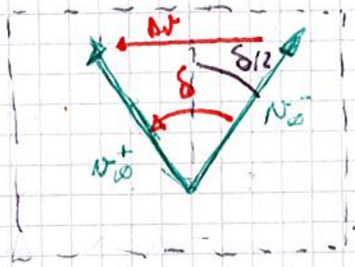
$$\Downarrow$$

$$\boxed{\frac{1}{e} = \sin\left(\frac{\delta}{2}\right)}$$

We are now ready to compute the Δv (modulus)



(velocity goes down with such configuration)



$\Rightarrow \Delta v = 2\, v_\infty \sin\frac{\delta}{2}$

$\Rightarrow$ ⌈ G.A. (with $\hat{e}$ aligned with v_p)

INTO THE PLANET (MOVING) REFERENCE SYSTEM.

$$\Delta v = \| v_s^+ - v_s^- \| \quad ; \quad \begin{cases} v_s^+ = v_p + v_\infty^+ \\ v_s^- = v_p + v_\infty^- \end{cases} \quad ; \quad \|v_\infty^+\| = \|v_\infty^-\| = -\sqrt{\frac{\mu_P}{a}}$$

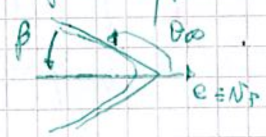
δ: angle between v_∞^+ v_∞^-

$$\Rightarrow \begin{cases} \sin\frac{\delta}{2} = \frac{1}{e} \\ \Delta v = 2\, v_\infty \sin\frac{\delta}{2} \end{cases}$$

(i) $\|v_\infty^-\| = \|v_\infty^+\| = -\sqrt{\frac{\mu}{a}}$ ← comes from the limit $\lim_{r\to\infty} \varepsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$

(ii) $\Delta v = 2\, v_\infty \sin\frac{\delta}{2}$ ← comes from composition of velocity ; $\begin{cases} \Delta v = v_s^+ - v_s^- \\ v_s^+ = v_\infty^+ + v_p \end{cases}$

(iii) $\frac{1}{e} = \sin\left(\frac{\delta}{2}\right)$ ← comes from hyperbolic geometry

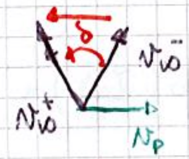


$\lim_{r\to\infty} \cos\theta_\infty = f\left(\frac{1}{r}\right) = -\frac{1}{e}$

$\theta_\infty + \beta = \pi$

⇓

$$\|\Delta v^{rel}\| = \|\Delta v^{abs}\| = \frac{2\, v_\infty}{e} = \pm\frac{2}{e}\sqrt{\frac{\mu_P}{a}}$$



$\Rightarrow \|\Delta v\| < 0$

$\Rightarrow \|\Delta v\| > 0$

⌋

obs 1: SINCE

The augment obtained with a gravity assist, in modulus, is:

$$\|\Delta v\| = \pm\frac{2}{e}\sqrt{\frac{\mu_P}{a}}$$ (Both for relative and absolute reference frame)

AND SINCE

The only two conics usable to escape from (open conics) a planet are ↗ PARABOLA → $e=1$ ↘ HYPERBOLA → $e>1$

THEN

Using a PARABOLA we could obtain the maximum ($\|\Delta v\|$) increment in speed modulus.

$$\|\Delta v\|\Big|_{MAX} \equiv \|\Delta v\|\Big|_{PAR} = 2\sqrt{\frac{\mu_P}{a}} = 2\, v_\infty$$

obs 2: • THE VELOCITY OF THE PLANET DOESN'T AFFECT THE Δv

in fact: $\begin{cases} v_s^+ = v_p + v_\infty^+ \\ v_s^- = v_p + v_\infty^- \end{cases} \Rightarrow \Delta v = v_s^+ - v_s^- = \cancel{v_p} + v_\infty^+ - \cancel{v_p} - v_\infty^- = v_\infty^+ - v_\infty^-$

• WHILE THE MASS OF THE PLANET IS THE REAL FACTOR ABLE TO MODIFY THE Δv

in fact: $\|\Delta v\| = \pm\frac{2}{e}\sqrt{\frac{\mu}{a}}$

• ALSO THE SEMIAXIS OF THE ORBIT OF THE SATELLITE AROUND THE PLANET (OBVS?) AFFECTS $\|\Delta v\|$

Consider this: $\begin{cases} v_\infty = \sqrt{-\frac{\mu_P}{a}} \rightarrow a = -\frac{\mu_P}{v_\infty^2} \\ r_P = \frac{p}{1+e\cos(\theta_0)} = \frac{a(1-e^2)}{1+e} = a(1-e) \rightarrow a = \frac{r_P}{1-e} \end{cases}$

By the equivalence: $-\frac{\mu_P}{v_\infty^2} = \frac{r_P}{1-e} \rightarrow e-1 = \frac{r_P v_\infty^2}{\mu_P}$

$\Rightarrow$ ⌈

$$e = 1 + \frac{r_P v_\infty^2}{\mu_P}$$

⌋

... going on with obs 2:

$$e = 1 + \frac{r_p v_\infty^2}{\mu_{pl}}$$

since

$$\begin{cases} \Delta v = \dfrac{2 v_\infty}{e} \\ r_p \geq r_{pl} \end{cases} \Rightarrow \quad r_p\big|_{min} = r_{pl}$$

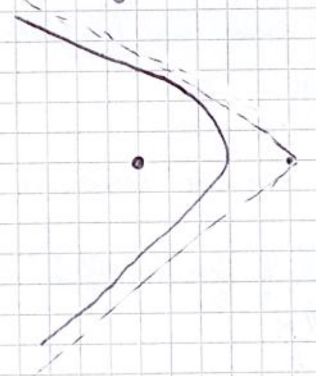
(radius of the planet itself)

$$\Rightarrow \begin{cases} \|\Delta v\|_{MAX} = \dfrac{1}{e_{MIN}} \sqrt{-\dfrac{\mu}{a}} \\ e_{MIN} = 1 + \dfrac{r_{pl} v_\infty^2}{\mu_{pl}} \end{cases}$$ (THEORITICAL VALUES)

obs 3: In planet with an atmosphere if i pass really close to them ⇒ having $r_p \simeq r_{pl}$ the $\|\Delta v\|$ obtained by gravity assist will be lost by the drag generated by the atmosphere on the satellite.

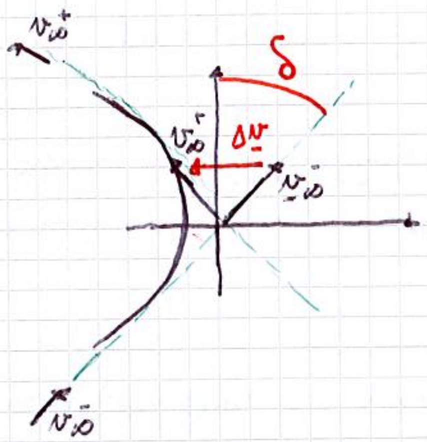
$$\varepsilon_{\text{DISSIPATED BY FRICTION}} \overset{??}{\gtrless} \varepsilon_{\text{GAINED BY G.A.}}$$

obs 4: On hyperbolic geometry:



d) Energy variation in gravity assist.

ALREADY KNOWN FOR GRAVITY ASSIST: $\|\Delta v^{rel}\| = \|\Delta v^{obs}\| = 2 v_\infty \sin\delta = \frac{2 v_\infty}{e}$



↓

MOMENTUM VARIATION MUST BE THE SAME FOR BOTH THE REFERENCE FRAME.

- $\left\| \frac{\Delta Q}{m} \right\|^{obs} = \left\| \frac{\Delta Q}{m} \right\|^{rel}$

(obviously for what happen inside the soi)

- $\Delta \varepsilon^{rel} \neq \Delta \varepsilon^{obs}$

$\Delta\varepsilon$, ENERGY VARIATION ⇒ a) PLANETARY REFERENCE FRAME

$$\begin{cases} \Delta \varepsilon = \varepsilon^+ - \varepsilon^- = 0 \\ \varepsilon = \text{const} = -\dfrac{\mu_{pl}}{2 a_{Hyp}} = \dfrac{1}{2} v^2_{rel} - \dfrac{\mu_{pl}}{r} = \dfrac{1}{2} v^2_{rel} \end{cases}$$

$$\left(= \lim_{r \to \infty} v^2_{rel} - \frac{\mu_{pl}}{r} \right)$$

But as we know: $\|v_\infty^+\| = \|v_\infty^-\|$

$$\Rightarrow \Delta\varepsilon = \frac{1}{2}(v_\infty^+)^2 - \frac{1}{2}(v_\infty^-)^2 = -\frac{1}{2}\frac{\mu}{a} + \frac{1}{2}\frac{\mu}{a} = 0$$

$\left(v_\infty = \sqrt{-\frac{\mu}{a}}\right)$

b) ABSOLUTE REFERENCE FRAME.

In the absolute reference frame we have to consider the main attractor as the "potential-energy-generator"

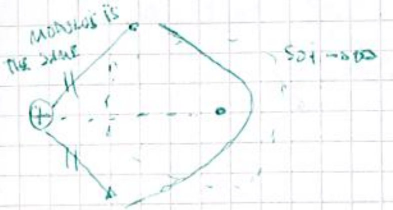
$$\varepsilon^{obs} = \frac{1}{2} v^2_{obs} - \frac{\mu_{obs}}{r_{obs}} \Rightarrow \begin{cases} \varepsilon^-_{obs} = \dfrac{1}{2}(v^-_{obs})^2 - \dfrac{\mu_{obs}}{r^-_{obs}} \\ \varepsilon^+_{obs} = \dfrac{1}{2}(v^+_{obs})^2 - \dfrac{\mu_{obs}}{r^+_{obs}} \end{cases}$$

(μ_{obs} → REDUCED MASS OF THE MAIN ATTRACTOR)

BUT WE ARE USING A LINKED CONIC APPROACH

$$\Rightarrow \text{IN OUR MODEL: } \begin{cases} \Delta t \to 0 \\ r^-_{soi} = -\infty;\ r^+_{soi} = +\infty \end{cases} \Rightarrow \underline{r}_{obs} = \underline{r}_p + \underset{\underset{r_{soi} \to \infty}{\parallel}}{\underline{r}_{rel}} \Rightarrow \underline{r}_{obs} \simeq \underline{r}_{soi}$$

⇒ $\Delta\varepsilon = \varepsilon^+ - \varepsilon^- = \frac{1}{2}(v^+_{abs})^2 - \frac{\mu}{r^+_{abs}} - \frac{1}{2}(v_{abs})^2 - \frac{1}{2}(v^-_{abs})^2 + \frac{\mu}{r_{abs}}$



⇒ $\Delta\varepsilon = \frac{1}{2}(v^+_{abs})^2 - \frac{1}{2}(v^-_{abs})^2 = \Delta T$

!! variation of potential energy from an absolute point of view is only a consequence of a variation in kinetic energy !!

an observer (as sketched), moving with the planet, will see the satellite coming to him at a velocity equal to v^-_∞ and escaping from him at a velocity equal to v^+_∞

⇓

WHAT OBSERVER SEES ARE RELATIVE VELOCITIES (v_∞)
$$\begin{cases} v^-_\infty = v^-_{abs} + v_p \\ v^+_\infty = v^+_{abs} - v_p \end{cases}$$

⇓

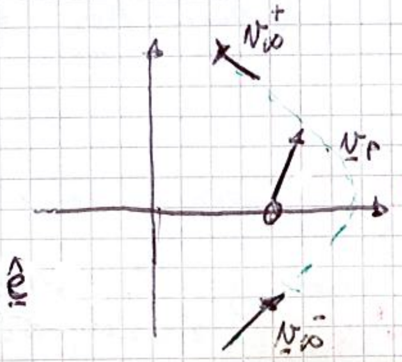
$$\begin{cases} v^-_{abs} = v^-_\infty - v_p \\ v^+_{abs} = v^+_\infty + v_p \end{cases} \qquad (\|v_\infty\| = \|v^+_\infty\|)$$

Then what we see is from an energetic point of view is:

$$\Delta T = \frac{1}{2}(v^+_{abs})^2 - \frac{1}{2}(v^-_{abs})^2 \Rightarrow 2\Delta T = (\|v^+_\infty\|^2 + \|v_p\|^2 + 2 v^+_\infty \cdot v_p) - (\|v_\infty\|^2 + \|v_p\|^2 - 2 v^-_\infty \cdot v_p)$$

$$2\Delta T = 2 v^+_\infty \cdot v_p + 2 v^-_\infty \cdot v_p$$

⇒ evaluating this 2 scalar products:



σ → angle between v_p and eccentricity vector $\hat{e}$

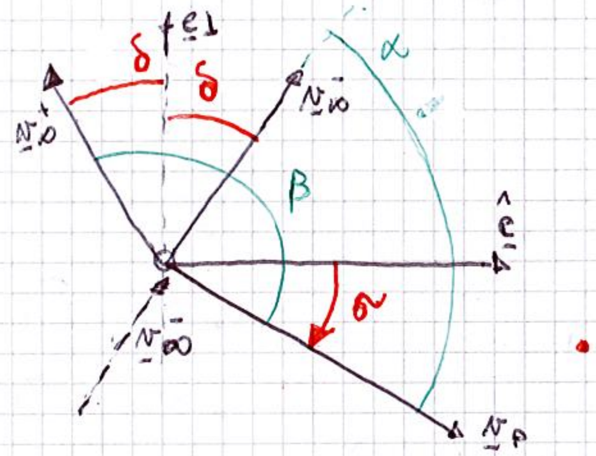
⇒ $\Delta\varepsilon = \Delta T = v^+_\infty v_p + v^-_\infty v_p$

Remembering that: $\|v^+_\infty\| = \|v^-_\infty\|$

- σ → angle between eccentricity $\hat{e}$ and velocity of the planet v_p

⇒ $\sigma : \widehat{v_p\, \hat{e}}$

- $\delta : \widehat{v^+_\infty\, \hat{e}_\perp}$; $\sin\delta = \frac{1}{e}$



- $\begin{cases} \beta = \frac{\pi}{2} + \sigma + \delta \\ \alpha = \frac{\pi}{2} - \delta + \sigma \end{cases} \Rightarrow \begin{cases} \beta : \widehat{v_p v^+_\infty} \\ \alpha : \widehat{v_p v^-_\infty} \end{cases}$

- $\|v^-_\infty\| = \|v^+_\infty\| = v_\infty \quad (= \sqrt{-\frac{\mu}{a}}$

$$\Delta\varepsilon = v_\infty v_p [\cos(\beta) + \cos(\alpha)]$$
$$= v_\infty v_p [\cos(\tfrac{\pi}{2} + \sigma + \delta) + \cos(\tfrac{\pi}{2} - \delta + \sigma)]$$
$$= v_\infty v_p [\underbrace{\cancel{\cos\tfrac{\pi}{2}}}_{=0} \cos(\sigma+\delta) - \sin\tfrac{\pi}{2} \sin(\sigma+\delta) + \underbrace{\cos\tfrac{\pi}{2}}_{=0} \cos(\sigma-\delta) - \sin\tfrac{\pi}{2} \sin(\sigma-\delta)]$$

$$\Delta\varepsilon = v_\infty v_p [-\sin(\delta+\sigma) + \sin(\sigma-\delta)]$$
$$= v_\infty v_p [-\sin\delta\cos\sigma - \cancel{\cos\delta\sin\sigma} + \cancel{\sin\sigma\cos\delta} - \sin\delta\cos\sigma]$$

$$\Delta\varepsilon = -2 v_\infty v_p \sin\delta\cos\sigma$$

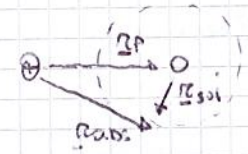
- $\sigma : \widehat{\hat{e}\, v_p}$
- $\delta : \widehat{\hat{e}_\perp\, v^+_\infty}$

$\boxed{\Delta\varepsilon = \Delta T}$ ⇒ $\Delta\varepsilon = -2 v_\infty \sin\delta \cos\sigma$ → $-\frac{\pi}{2} < \sigma < \frac{\pi}{2}$ → DECREASING SPEED

→ $\frac{\pi}{2} < \sigma < \frac{3}{2}\pi$ → INCREASING SPEED.

SINCE

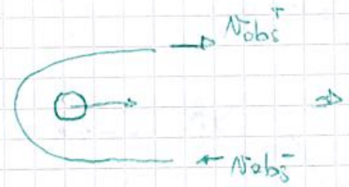
$$\text{L.C.}\begin{cases} r^{+}_{soi} \to \infty \\ \underline{r}_{obs} = \underline{R}_p + \underline{r}_{r} \end{cases}$$



$\Rightarrow r_{soi} = r_r \Rightarrow r^{+}_{obs} = r^{-}_{obs}$
(modulus)

$$\Rightarrow \quad \varepsilon = \frac{1}{2}\|v^{+}_{obs}\|^2 - \cancel{\frac{\mu_{obs}}{r^{+}_{obs}}} - \frac{1}{2}\|\underline{v}^{-}_{obs}\|^2 + \cancel{\frac{\mu_{obs}}{r^{-}_{obs}}} \equiv \Delta T$$

$$\begin{cases} v^{+}_{\infty} = \underline{v}_{abs} - \underline{v}_p \\ \underline{v}^{-}_{\infty} = \underline{v}_{obs} + \underline{v}_p \end{cases}$$



THEN

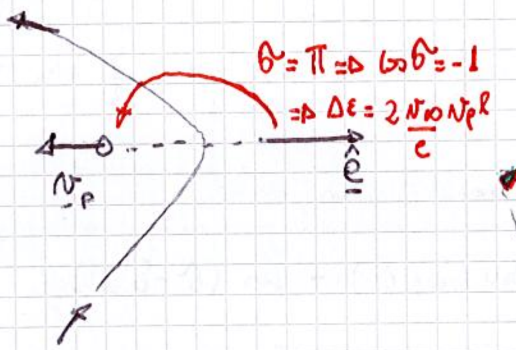
$$\begin{cases} \Delta\varepsilon = \Delta T = \frac{1}{2}\left(\|v^{+}_{obs}\|^2 - \|v^{-}_{obs}\|^2\right) \\ \Delta\varepsilon = -2\, v_\infty\, v_{pl} \sin\delta \cos\sigma = -2\,\frac{v_\infty}{e}\, v_{pl} \cos\sigma \end{cases}$$

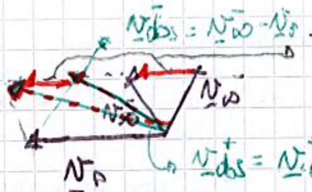
- σ : $\widehat{v_p\, \hat{e}}$
- δ : $\frac{\widehat{v^{-}_{\infty}\, v^{+}_{\infty}}}{2}$

- $\frac{\pi}{2} < \sigma < \frac{3}{2}\pi \Rightarrow \cos\sigma < 0 \Rightarrow \Delta\varepsilon > 0 \Rightarrow \Delta T > 0 \Rightarrow \|v^{+}_{obs}\| > \|v^{-}_{obs}\|$

$\Downarrow$

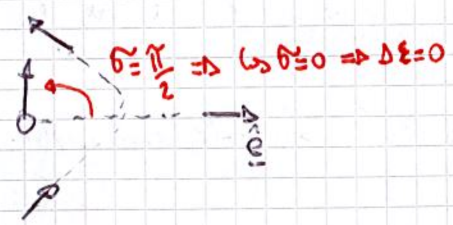
INCREASING VELOCITY

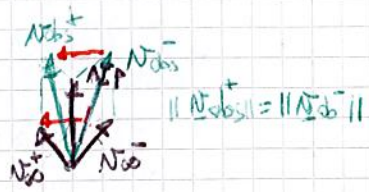




The 2 Δv are the same.

- $\sigma = \frac{\pi}{2} \Rightarrow \cos\sigma = 0 \Rightarrow \Delta\varepsilon = 0 \Rightarrow \Delta T = 0 \Rightarrow \|\underline{v}^{-}_{obs}\| = \|v^{+}_{obs}\|$

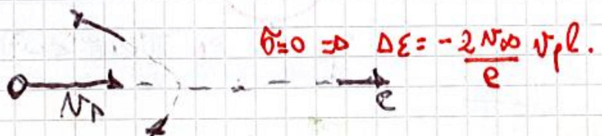




- $-\frac{\pi}{2} < \sigma < \frac{\pi}{2} \Rightarrow \cos\sigma > 0 \Rightarrow \Delta\varepsilon < 0 \Rightarrow \Delta T < 0 \Rightarrow \|v^{+}_{obs}\| < \|v^{-}_{obs}\|$

$\Downarrow$

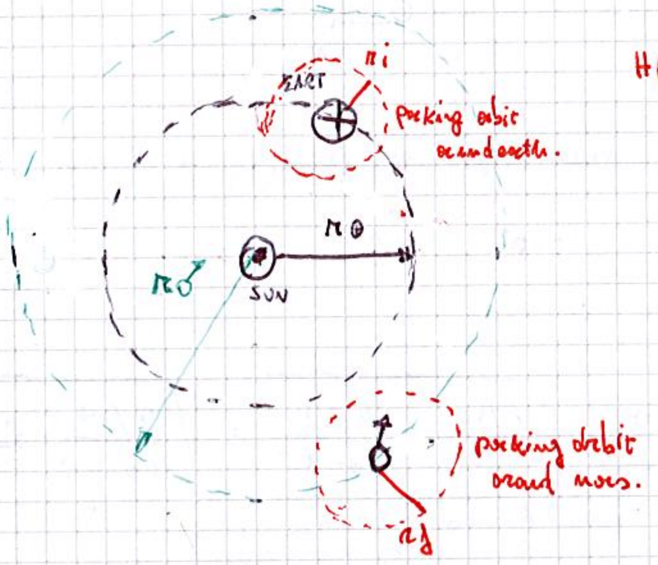
DECREASING VELOCITY



e) Analysis of a interplanetary mission. (NOT REFERRED TO G.A ⇒ BETTER UNDERSTAND LINKED CONIC APPROACH)

" Linked / Patched conic approach allows us TO APPROXIMATE the motion of a satellite using a serie of Keplerian motion "

Eg: MISSION



Hp:

(i) All orbits are coplanar.

(ii) All planetary orbits are circular

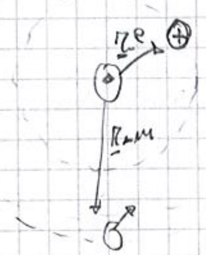
↳ In such mission can be distinguished 3 phases

1. PLANETOCENTRIC (EARTH) ESCAPE FROM CIRCULAR PARKING ORBIT.
2. HELIOCENTRIC TRANSFER. (ELLIPS e.g. HOHMANN TRANSFER)
3. PLANETOCENTRIC (MARS) CAPTURE AROUND MARS ON THE FINAL PARKING ORBIT.

2 → HELIOCENTRIC TRANSFER.

" Heliocentric phase is obviously the 2^{nd} in order of time, but must be selected as first, BECAUSE WILL GIVE US THE VELOCITY RESPECT TO THE SUN AT THE EXIT FROM THE SPHERE OF INFLUENCE OF EARTH AND AT THE ENTERING OF THE SPHERE OF INFLUENCE OF MARS "

From a computational POV mean to solve an optimized Lambert problem

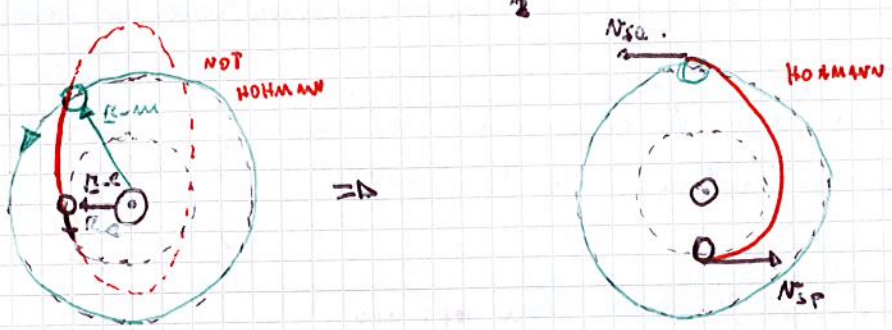


$\forall\ (\underline{r}_e\ ;\ \underline{R}_m) \Rightarrow$ Solve Lambert's problem $\forall$ TOF

FORK-CHOP DIAGRAM.

$\boxed{\text{result MUST be}}$ Hohmann Transfer ⇒ done at the instants of time when planets are aligned ⇒ when the distance between them is equal to $r_♂ - r_\oplus$

⇒ Hohmann transfer: $a = \dfrac{r_\oplus + r_♂}{2}$ (By definition of hohmann transfer)



HOHMANN TRANSFER CAN START ONLY WHEN EARTH AND MARS WILL BE ALIGNED

analitically: $t = t^*$: $\| \underline{r}_m(t^*) - \underline{r}_e(t^*) \| = r_♂ + r_\oplus$

velocity at the pericenter of the transfer orbit ⇒ $v_{SP} = \sqrt{\dfrac{2\mu_\odot}{r_\oplus} - \dfrac{\mu_\odot 2}{r_\oplus + r_♂}} = \sqrt{\dfrac{2\mu_\odot}{r_\oplus + r_♂}\left(\dfrac{r_♂}{r_\oplus}\right)}$

$\mathcal{E} = \dfrac{1}{2} v^2 - \dfrac{\mu}{r} \rightarrow \dfrac{1}{2} v^2 = \dfrac{\mu}{r} - \dfrac{\mu}{2a} \rightarrow v = \sqrt{\dfrac{2\mu}{r} - \dfrac{\mu}{a}}$

velocity at apocenter of the transfer orbit ⇒ $v_{sa} = \sqrt{\dfrac{2\mu_\odot}{r_♂} - \dfrac{2\mu_\odot}{r_\oplus + r_♂}} = \sqrt{\dfrac{2\mu_\odot}{r_\oplus + r_♂}\left(\dfrac{r_\oplus}{r_♂}\right)}$

1 → ESCAPE PHASE

(i) • SATELLITE MUST GO ON AN OPEN ORBIT (hyperbola) TO ESCAPE FROM THE SPHERE OF INFLUENCE OF EARTH.

(ii) • FOR CONTINUITY AT EDGE OF EARTH'S SOI (which is infinite) THE VELOCITY OF THE SATELLITE MUST BE THE SAME OF THE TRANSFER ORBIT !! MUST BE THE SAME IN THE ABSOLUTE REFERENCE SYSTEM !!

⇓

$v_{SP} = v_\infty^{\oplus} + v_\oplus$ (v_{SP} tangent to the circular orbit of earth around the sun)

$v_\infty^{\oplus}$ → velocity of the hyperbola in the relative reference system.

$v_\infty^{\oplus}$ → This indicates the reference frame in wich the velocity is indicate. If not indicated it means it's expressed in absolute reference frame.

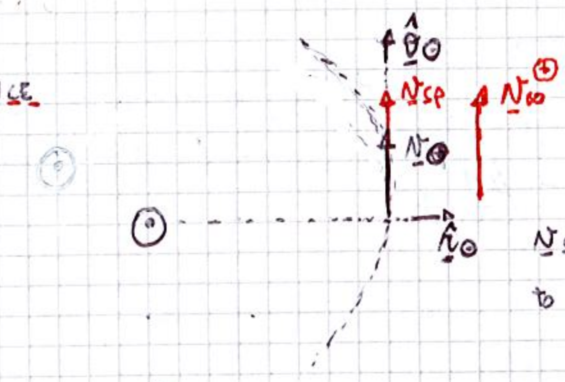
☞ Continuity : linked-conic Approach.

$$\underline{v}_{ot}\Big|_{\substack{SOI\\pl.\\(\to\infty)}} = \underline{v}_\infty^{pl} + \underline{v}_{pl}$$

- $\underline{v}_{ot}$ → VELOCITY OF THE SATELLITE OBTAINED FROM THE CHOOSEN TRANSFER ORBIT (EXPRESSED IN THE ABSOLUTE REFERENCE FRAME)
- $\underline{v}_\infty^{pl}$ → VELOCITY OF THE OPEN CONIC AT "INFINITUM" FROM THE PLANET (EXPRESSED IN PLANETARY REFERENCE FRAME)

SINCE $\underline{v}_{SP} = \underline{v}_\infty^{\oplus} + \underline{v}_\oplus$ (! in absolute reference frame!)

AND SINCE



$(\underline{v}_{SP}, \underline{v}_\oplus) \parallel \hat{\theta}_\oplus$

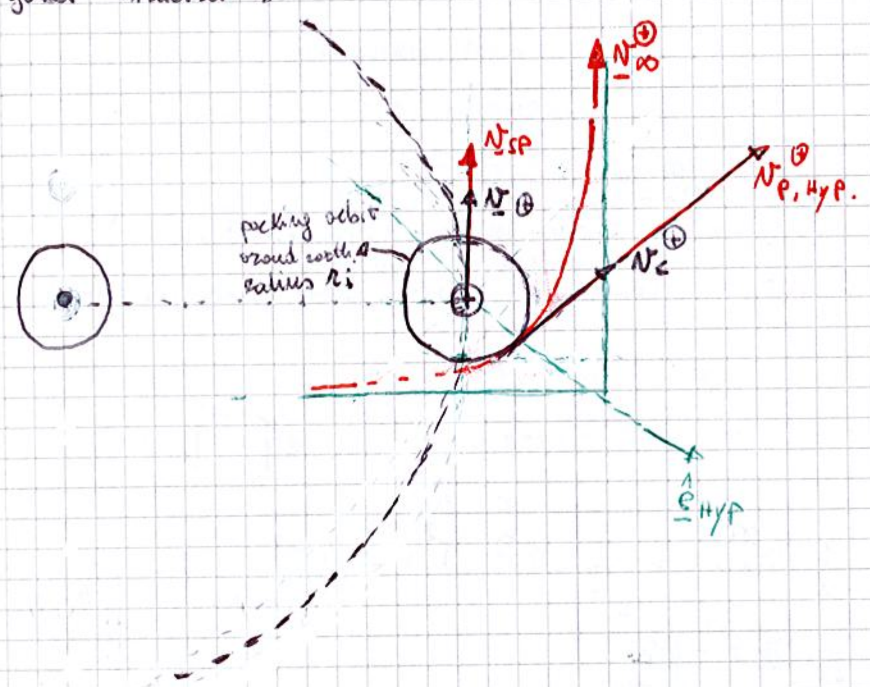
v_{SP} and $v_\oplus$ are both parallel to $\hat{\theta}$, orthogonal to the position vector of earth.

THEN $\underline{v}_\infty^{\oplus} = \underline{v}_{SP} - \underline{v}_\oplus$ ⇒ $\underline{v}_\infty^{\oplus} \parallel \hat{\theta}_\oplus$

and continuity can be expressed as an equivalence between modules

$$\Rightarrow \begin{cases} v_\infty^{\oplus} = \|\underline{v}_{SP}\| - \|\underline{v}_\oplus\| = \sqrt{\dfrac{2\mu_\odot}{r_\oplus + r_♂}\left(\dfrac{r_♂}{r_\oplus}\right)} - \sqrt{\dfrac{\mu_\odot}{r_\oplus}} \\ \underline{v}_\infty^{\oplus} = v_\infty^{\oplus} \cdot \hat{\theta}_\oplus \end{cases}$$

the global situation is:



obviously we will have: $\|\underline{v}_c^{\oplus}\| = \sqrt{\dfrac{\mu_\oplus}{r_i}}$

Since: WE WANT TO MINIMIZE THE COST THE MANEUVER TO GET ON A HYPERBOLIC ORBIT

⇒ THE PERICENTER OF THE ORBIT MUST COINCIDE WITH THE CIRCULAR PARKING ORBIT

(wanna minimize cost ⇒ manouver at the pericenter WHERE THERE'S NO CHANGE IN DIRECTION)

Now:

- $v_c^{\oplus} = \sqrt{\frac{\mu_\oplus}{r_i}}$

- $v_{p,hyp}^{\oplus} = \sqrt{\frac{2\mu_\oplus}{r_i} - \frac{\mu_\oplus}{a_{hyp}}}$ (velocity of hyperbolic trajectory evaluated at his pericenter)

$$\left[\varepsilon\big|_p = -\frac{\mu_\oplus}{2a_{hyp}} = \frac{1}{2} v_{p,hyp}^2 - \frac{\mu_\oplus}{r_i} \right]$$

$$v_{p,hyp}^2 = \frac{2\mu_\oplus}{r_i} - \frac{\mu_\oplus}{a_{hyp}} \rightarrow v_{p,hyp} = \sqrt{\frac{2\mu_\oplus}{r_i} - \frac{\mu_\oplus}{a_{hyp}}}$$

- $\Delta v_i = v_{p,hyp}^{\oplus} - v_c^{\oplus}$

(cost of initial impulse to get on a hyperbolic trajectory)

$$\Rightarrow \Delta v_i = \sqrt{\frac{2\mu_\oplus}{r_i} - \frac{\mu_\oplus}{a_{hyp}}} - \sqrt{\frac{\mu_\oplus}{r_i}}$$

!! BUT a_{hyp} IS UNKNOWN ⇒ CAN BE OBTAINED BY THE EQUIVALENCE:

$$\begin{cases} v_\infty^{\oplus} = \|v_{SP}\| - \|v_\oplus\| = \sqrt{\frac{2\mu_\odot}{r_\oplus + r_{\sigma^\prime}}\left(\frac{r_{\sigma^\prime}}{r_\oplus}\right)} - \sqrt{\frac{\mu_\odot}{r_\oplus}} \\ v_\infty^{\oplus} = \sqrt{-\frac{\mu_\oplus}{a_{hyp}}} \end{cases}$$

$$\varepsilon = -\frac{\mu}{2a} = \frac{1}{2}v^2 - \frac{\mu}{r} \Rightarrow v_\infty^{\oplus} = \lim_{r \to \infty} \sqrt{\frac{2\mu_\oplus}{r} - \frac{\mu_\oplus}{a_{hyp}}} = \sqrt{-\frac{\mu_\oplus}{a_{hyp}}}$$

But THERE'S A MUCH MORE SMART WAY OF DOING !!

MUCH smarter way of doing: FROM ENERGY CONSERVATION ($\varepsilon\big|_{r\to\infty} = \varepsilon\big|_p$)

(to obtain $v_{p,hyp}^{\oplus}$)

$$\frac{1}{2}(v_\infty^{\oplus})^2 = \frac{1}{2}(v_{p,hyp}^{\oplus})^2 - \frac{\mu_\oplus}{r_{p,hyp}} \qquad \Big\| \; r_{p,hyp} \equiv r_i$$

$$\frac{1}{2}(v_\infty^{\oplus})^2 - \underbrace{\frac{\mu_\oplus}{(r\to\infty)}}_{=0}$$

$$\Downarrow$$

$$v_{p,hyp}^{\oplus} = \sqrt{(v_\infty^{\oplus})^2 + 2\frac{\mu_\oplus}{r_i}}$$

$$\Rightarrow \Delta v_i = \sqrt{(v_\infty^{\oplus})^2 + 2\frac{\mu_\oplus}{r_i}} - \sqrt{\frac{\mu_\oplus}{r_i}}$$

↳ cost computation without knowing orbital parameters for the hyperbola.

$$\Delta v_i = \sqrt{(v_\infty^{\oplus})^2 + 2\frac{\mu_\oplus}{r_i}} - \sqrt{\frac{\mu_\oplus}{r_i}} \quad (i)$$

$\Delta v_i \rightarrow$ cost necessary to escape from a circular orbit around earth. ⇒ transfer done at the pericenter of the hyperbola. (MINIMUM COST)

Where:

$$v_\infty^{\oplus} = \sqrt{\frac{2\mu_\odot}{r_\oplus + r_{\sigma^\prime}}\left(\frac{r_{\sigma^\prime}}{r_\oplus}\right)} - \sqrt{\frac{\mu_\odot}{r_\oplus}}$$

↳ generally speaking: $v_{OT} = v_\infty^{(pl)} + v_{(pl)}$

- $v_{OT} \rightarrow$ VELOCITY IN ABS R.F. AT THE POINT OF INTERCEPTION FOR THE DESIRED ORBIT.
- $v_\infty^{(pl)} \rightarrow$ IN PLANETARY R.F. IS THE VELOCITY OF SATELLITE AT THE BOUNDARY OF AN INFINITE SOI.

⇓ IF $v_{OT} \parallel v_{(pl)}$ (TANGENT MANEOUVER)

THEN $v_\infty^{(pl)} \parallel (v_{OT};\, v_{(pl)})$

⇓ CHOOSING A PERICENTRIC TANGENT MANEOUVER (THE ONE THAT MINIMIZES COST) PARAMETERS FOR THE HYPERBOLA ARE "LOCKED"

in fact

$v_{p,hyp}$ ⇒ WILL BE OBTAINED FROM ENERGY CONSERVATION ⇒ $\begin{cases} \varepsilon\big|_{r\to\infty} = \frac{1}{2}(v_\infty^{\oplus})^2 - \underbrace{\frac{\mu_\oplus}{r_{\to\infty}}}_{=0} \\ \varepsilon\big|_{r_p} = \frac{1}{2}(v_p^{\oplus})^2 - \frac{\mu_\oplus}{r_p} \end{cases}$

$\varepsilon\big|_{r\to\infty} = \varepsilon\big|_{r_p} \Rightarrow v_p^{\oplus}$ FOR THE EXIT ORBIT WILL BE KNOWN

↳ SUCH WAY OF DOING CAN BE REPEATED EVEN IF THE PARKING ORBIT IS NOT CIRCULAR e.g. manoeuvring at the apocenter of an ellipse

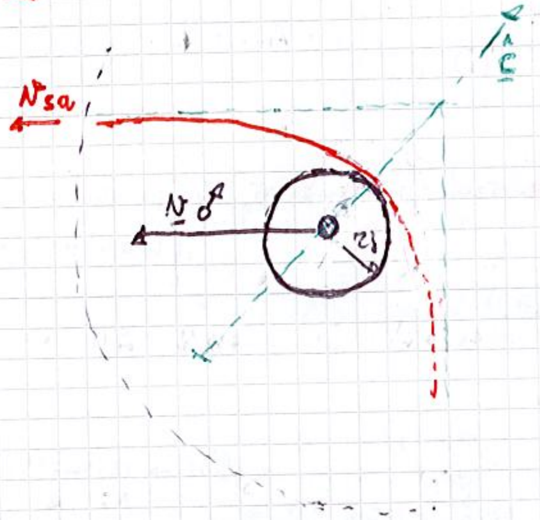
⇒ ORBITAL PARAMETERS ARE LOCKED BECAUSE:

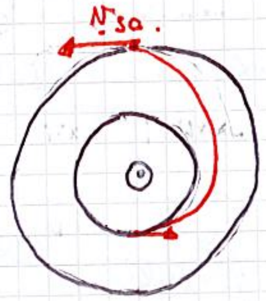
- $-\frac{\mu_\oplus}{2a_{hyp}} = \frac{1}{2}(v_\infty)^2 \Rightarrow a_{hyp} = -\frac{\mu_\oplus}{(v_\infty)^2}$ ⟹
- $r_p = r_{MANOUVRE} = a_{hyp}(1 - e_{hyp}) \Rightarrow e_{hyp} = 1 - \frac{a_{hyp}}{r_p}$

ALSO SHAPE OF THE HYPERBOLA IS FULLY DEFINED

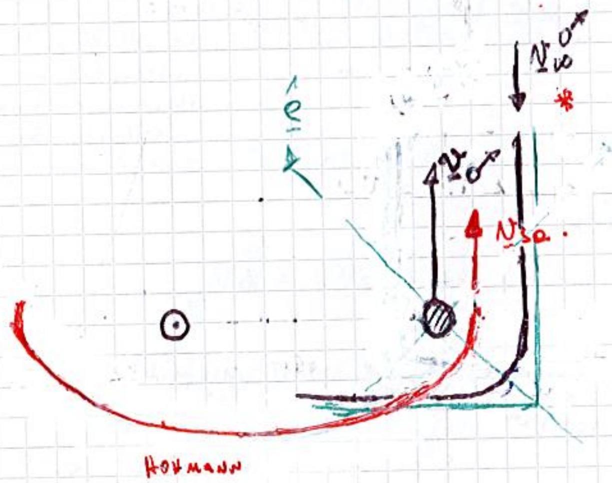
⇒ $\sin\delta = \frac{1}{e}$

3. CAPTURE PHASE





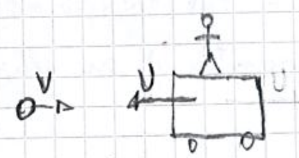
Keeping hohmann transfer in mind



↳ about velocities composition:

ALWAYS TRUE: $\underline{v}_{obs} = \underline{v}_{pl} + \underline{v}_\infty^{pl}$

a)



$\underline{v}_{obs} = V\hat{\underline{i}}$

$\underline{v}_{pl} = -U\hat{\underline{i}}$

$\Rightarrow V\cdot\hat{\underline{i}} = -U\hat{\underline{i}} + \underline{v}_\infty^{pl}$

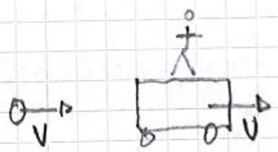
$\Downarrow$

$\underline{v}_\infty^{pl} = (V+U)\hat{\underline{i}}$

$\Downarrow$

$v_\infty^{pl} = v^{pl} + v_{obs}$

b)



$\underline{v}_{obs} = V\cdot\hat{\underline{i}}$

$\underline{v}_{pl} = U\hat{\underline{i}}$

$\Rightarrow V\hat{\underline{i}} = U\hat{\underline{i}} + \underline{v}_\infty^{pl}$

$\Downarrow$

$\underline{v}_\infty^{pl} = (V-U)\hat{\underline{i}}$

if $U > V \Rightarrow \underline{v}_\infty^{pl} = |v_\infty^{pl}|(-\hat{\underline{i}})$

if $U < V \Rightarrow \underline{v}_\infty^{pl} = |v_\infty^{pl}|\cdot\hat{\underline{i}}$

... Still on b) situation:

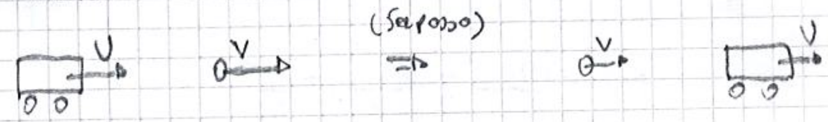
if $\begin{cases} U > V \\ v_{pl} > v_{sa} \end{cases} \Rightarrow \underline{v}_\infty^{pl} = V\hat{\underline{i}} - U\hat{\underline{i}} = (V-U)\hat{\underline{i}} \equiv (v_{sa} - v_{pl})\cdot\hat{\underline{i}}$

$\Downarrow$

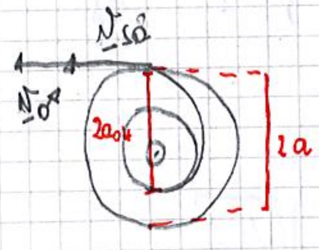
$\underline{v}_\infty^{pl} = \underline{v}_{sa} - \underline{v}_{pl} \Rightarrow \underline{v}_\infty^{pl} = (v_{sa} - v_{pl})\hat{\underline{i}} = (v_{pl} - v_{sa})(-\hat{\underline{i}})$ *

⇒ NOTHING ABSURDUS; ≃ OBVIUS

- IF THE TRAIN IS FASTER WILL SEE THE BALL GOING AWAY FROM HIM BUT THEY CAN MEET ANYWAY IN EXAMPLE WITH THIS CONFIGURATION:



THAT'S EXACTLY WHAT HAPPENS IN THE CAPTURE



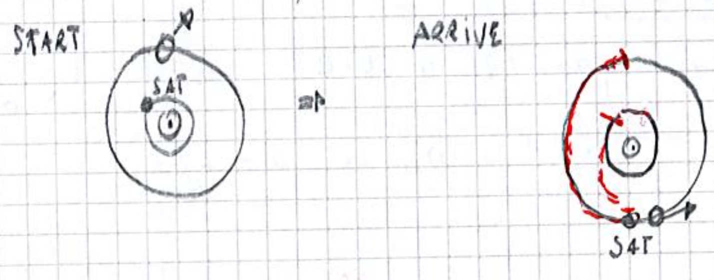
$v_\sigma > v_{sa}$ in fact $\underline{r}_{sa} = \underline{r}_\sigma = r$

$\varepsilon^{oH} = -\frac{\mu_\odot}{r} + \frac{1}{2}v_{sa}^2 = -\frac{\mu_\odot}{2a_{oH}}$

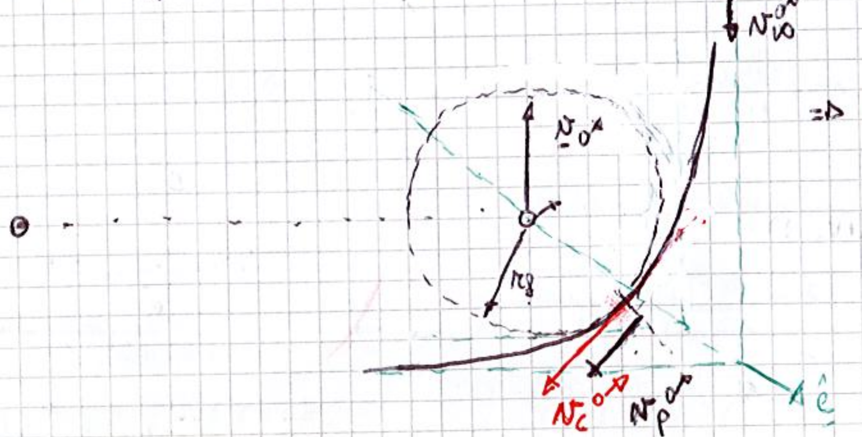
$\varepsilon^{\sigma} = -\frac{\mu_\odot}{r} + \frac{1}{2}v_\sigma^2 = -\frac{\mu}{2a}$

↳ equal ⇒ since $2a > 2a_{oH}$ THEN $v_\sigma > v_{sa}$.

Anyway is possible that SAT travelled a smaller distance then was



⇒ capture will occur again at the pericenter of the hyperbola with an impulse which has the opposite direction of the velocity of the planet.



$\Rightarrow \Delta v^{\sigma} = v_p^{\sigma} - v_c^{\sigma}$

⇒ $\| \Delta v^{\delta} \| = \| v_c^{0\delta} - v_p^{0\delta} \| = v_p^{0\delta} - v_c^{0\delta}$

↳ Δv is always defined as $\| \Delta v \| = \| v_f - v_i \|$
but obviously we have $v_p^{0\delta} > v_c^{0\delta}$

$$\varepsilon^{Hyp} = -\frac{\mu}{2a_{Hyp}} = \frac{1}{2} v_p^2 - \frac{\mu}{r_f} \Rightarrow v_p^2 + \frac{\mu}{a_{Hyp}} = v_c^2 + \frac{\mu}{2r_f}$$

$$\varepsilon^{circ} = -\frac{\mu}{2r_f} = \frac{1}{2} v_c^2 - \frac{\mu}{r_f}$$

$(a_{Hyp} < 0)$ ⇒

$$v_p^2 = v_c^2 + \frac{\mu}{r_f} - \frac{\mu}{a_{Hyp}}$$

- $\varepsilon = \frac{(v_\infty^{0\delta})^2}{2} = \frac{(v_p^{0\delta})^2}{2} + \frac{\mu_{0\delta}}{r_f}$

⇒ → $v_p^{0\delta} = \sqrt{(v_\infty^{\delta})^2 + 2\frac{\mu_{0\delta}}{r_f}}$

- $v_c^{0\delta} = \sqrt{\frac{\mu_{0\delta}}{r_f}}$

- $\Delta v^{\delta} = \sqrt{(v_\infty^{0\delta})^2 + 2\frac{\mu_{0\delta}}{r_f}} - \sqrt{\frac{\mu_{\delta}}{r_f}}$

!! IF I DON'T GIVE THIS LAST "BRAKE" IMPULSE I WILL ESCAPE FROM THE PLANET (MARS) ON THE OTHE ASYNTOTE OF THE HYPERBOLA. !!

- Hyperbola properties: $\begin{cases} r_p = r_f = a(1-e) \rightarrow e = 1 - \frac{r_p}{a} \\ -\frac{\mu_{0\delta}}{2a} = \frac{1}{2}(v_\infty^{0\delta})^2 \rightarrow a = -\frac{\mu_\delta}{(v_\infty^{0\delta})^2} \end{cases}$

⇒ $\sin\delta = \frac{1}{e}$

EXCERCISE: Gravity-assist on Jupiter and earth escape.

$\oplus \begin{cases} e = 0 \\ i_\oplus = i_♃ \end{cases}$ (all coplanar heliocentric orbits are circular) + $v_{\infty}^{♃}$ in shadow

$\oplus \rightarrow ♃$ ⇒ Hohmann.

$h_\oplus|_{min} = 300$ Km

$h_♃|_{min} = 1000$ Km

DATAS

$\mu_\oplus = 398600$ Km³/s² ; $\mu_\odot = 1{,}327 \times 10^{11}$ Km³/s²

$m_♃ = 18\,986 \times 10^{23}$ Kg ; $m_\oplus = 59{,}736 \times 10^{23}$ Kg

→ $\mu_♃ = \frac{m_♃}{m_\oplus}\mu_\oplus = 1{,}2670$ Km³/s²

$R_\oplus = 6371$ Km ; $r_\oplus \simeq a_\oplus = 1$ AU $= 149\,597871$ Km

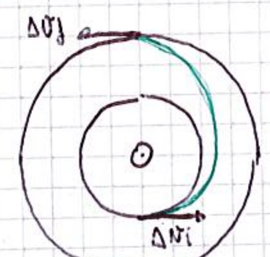
$R_♃ = 69\,911$ Km ; $r_♃ \simeq a_♃ = 5{,}2093$ AU

⇓

$r_\oplus = 149597871$ Km

$r_♃ = 7{,}79 \times 10^8$ Km

a) HOHMANN TRANSFER



$v_\oplus = \sqrt{\frac{\mu_\odot}{r_\oplus}} = 29{,}78$ Km/s

$v_♃ = \sqrt{\frac{\mu_\odot}{r_♃}} = 13{,}06$ Km/s

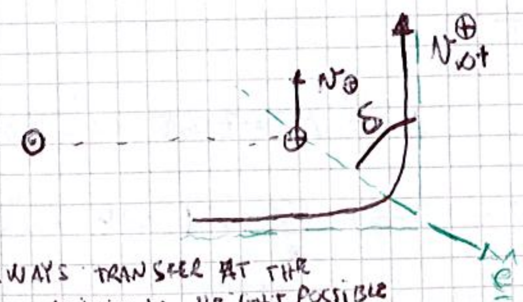
$r_{pt} = r_\oplus$, $r_{at} = r_♃$ ⇒ $a_t = \frac{r_\oplus + r_♃}{2} = 4{,}638 \times 10^{11}$ m $= 4{,}638 \times 10^8$ Km.

$e_t = \frac{r_♃ - r_\oplus}{r_♃ + r_\oplus} = 0{,}677$ ⇒ $p_t = a_t(1 - e_t^2) = 2{,}51 \times 10^8$ Km

⇒ $v_{pt} = \sqrt{\frac{\mu_\odot}{p_t}}\,(1 + e_t) = 38{,}57$ Km/s

$v_{at} = \sqrt{\frac{\mu_\odot}{p_t}}\,(1 - e_t) = 7{,}46$ Km/s

b) ESCAPE FROM EARTH



ALWAYS TRANSFER AT THE MINIMUM HEIGHT POSSIBLE

↳ $r_p = h_\oplus|_{min} + R_\oplus = 6671$ Km.

$v_{pt} = v_\oplus + v_\infty^{\oplus +}$ ⇒ $v_\infty^{\oplus +} = v_{pt} - v_\oplus$

$v_\infty^{\oplus +} = 8{,}79$ Km/s

⇒ $a_{hyp} = \frac{\mu_\oplus}{(v_\infty^{\oplus})^2} = 5159$ Km

$\varepsilon = -\frac{\mu}{2a} = \frac{1}{2}v^2 - \frac{\mu}{r_{hyp}}$ → $v_\infty = \sqrt{-\frac{\mu}{a}}$ → $a = \frac{\mu}{v_\infty^2}$

$\Rightarrow a_{Hyp} = 5139\ Km \Rightarrow e_{Hyp} = 1 - \frac{r_{p,Hyp}}{a_{Hyp}} = 1{,}7733 \Rightarrow \sin\delta = \frac{1}{e}$

$r_{p,Hyp} = 6671\ Km$

$\Downarrow$

$\delta = 34{,}32°$

$r_p = a(1-e) \Rightarrow \frac{r_p}{a} = 1-e \Rightarrow e = 1 - \frac{r_p}{a}$

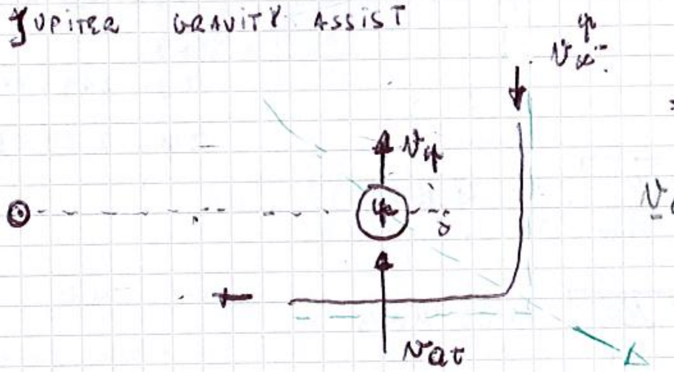
$p_{Hyp} = a_{Hyp}(1 - e^2_{Hyp}) = +1{,}106 \times 10^4\ Km.$

$v_{p,Hyp} = \sqrt{\frac{\mu_\oplus}{p_{Hyp}}}\ (1 + e_{Hyp}) = 10{,}64\ Km/s$

$\Rightarrow v_c = \sqrt{\frac{\mu_\oplus}{r_{p,Hyp}}} = 7{,}7298 \Rightarrow \Delta v = v_{p,Hyp} - v_c = 8{,}91\ Km/s$

$(r_{p,Hyp} \equiv r_{pc} \equiv r.)$

c) Jupiter gravity assist



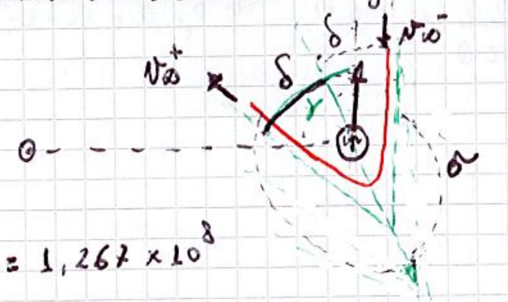
$\Rightarrow v_\infty^{4-} = v_{4} - v_{a\tau}$

$\underline{v}_{a\tau} = \underline{v}_{4} + \underline{v}_\infty^{4}$

$\Rightarrow \underline{v}_\infty^{4-} = \underline{v}_{a\tau} - \underline{v}_{4}$

$v_{4} = 13{,}06\ Km/s\ ;\ v_{a\tau} = 7{,}40\ Km/s \Rightarrow v_\infty^{4-} = 5{,}6\ Km/s = v_\infty^{4+}$

~~[illegible]~~ good approximation:



$r_{p_{Hyp}} = R_{4} + h_{4}|_{min} = 70\,911\ Km.$

$\mu_{4} = 1{,}267 \times 10^8$

$\begin{cases} e_{Hyp} = 1 - \frac{r_{pHyp}}{a_{Hyp}} = 1{,}018 \\ a_{Hyp} = -\frac{\mu_{4}}{v_\infty^2} = -3{,}924 \times 10^6\ Km. \end{cases}$

$\Rightarrow \delta = \sin^{-1}\left(\frac{1}{e}\right) = 79{,}21°$

$\boxed{\theta + \delta = \pi}$

correct:

$\gamma\ \widehat{v_\infty^+ v_{pl}} \Rightarrow 2\delta = \gamma$

energy: $\Delta\mathcal{E}^{obs} = \Delta T = -2\cos\theta\,\sin\delta\, v_\infty v_{pl}.$

$\|v_{obs}^+\| = \sqrt{v_{pl}^2 + v_\infty^2 + 2\cos(2\delta)\, v_\infty \cdot v_{pl}}$

$= 8{,}11$

$\Delta\mathcal{E}^{obs} = -2\cos(\pi - \delta) \cdot \frac{v_\infty}{e} \cdot v_{pl} = 22{,}26$

$\Delta T = \Delta\mathcal{E}^{obs} = -\frac{1}{2} v_{a\tau}^2 + \frac{1}{2} v_{obs}^{2+}$

$v_{obs}^+ = \sqrt{2\Delta T + \underbrace{v_{a\tau}^2}_{v_{obs}}} = 10{,}13\ Km/s.$

8) Connection with "intuitive" (a) gravity assist.

(showing that results obtained are fully coherent with momentum conservation and energy conservation)

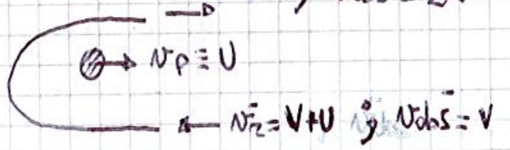
result from momentum/energy conservation.

- Elastic collision means exchange of momentum for $\Delta t \to 0$ and conservation of T (kinetic energy)

-  elastic collision having $M \gg m$. (train) >> (tennis ball)

$\Downarrow$

$v_r^- = V + U\ ;\ v_{obs}^+ = 2U + V$

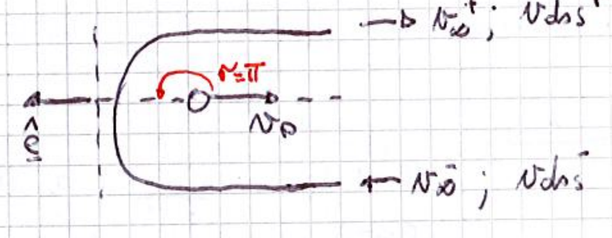


$\Rightarrow$ from obtained before: $\begin{cases} \Delta\mathcal{E} = -2\, v_\infty v_p \sin\delta\cos\theta \Rightarrow \Delta\mathcal{E}|_{max} = 2\, v_\infty v_p \text{ obtained for } \begin{cases} \theta = \pi \\ \delta = \frac{\pi}{2} \end{cases} \\ !!\ \Delta\mathcal{E} = \Delta T = \frac{1}{2}(v_{obs}^+)^2 - \frac{1}{2}(v_{obs}^-)^2\ !! \end{cases}$

in fact for a linked conic approach $\begin{cases} r \to \infty \\ r_{SOI} \to \infty \end{cases}$

$\mathcal{E} = \frac{1}{2} v^2 - \frac{\mu}{r \to \infty} = \frac{1}{2} v^2 = \Delta T$

- $\Delta\mathcal{E}|_{max} \Rightarrow \begin{cases} \theta = \pi \\ \delta = \frac{\pi}{2} \end{cases} \Rightarrow \sin\delta = \frac{1}{e} \Rightarrow e = 1$



- $\Delta\mathcal{E}|_{max} = \Delta T \Rightarrow 2\, v_\infty v_p = \frac{1}{2}(v_{obs}^+)^2 - \frac{1}{2}(v_{obs}^-)^2$

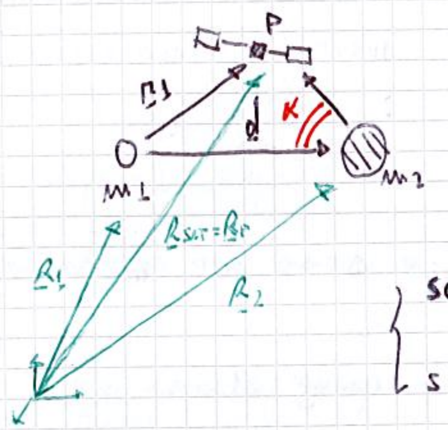
$\underline{v}_{obs}^- = \underline{v}_\infty^- + \underline{v}_p = -v_p\,\hat{i} + v_\infty\,\hat{i} \Rightarrow |v_{obs}^-| = v_\infty - v_p \Rightarrow v_\infty = v_{obs}^- + v_p$

$v_p \cdot 2(v_{obs}^- + v_p) = \frac{1}{2}(v_{obs}^+)^2 - \frac{1}{2}(v_{obs}^-)^2$

$4 v_p v_{obs}^- + 4 v_p^2 = (v_{obs}^+)^2 - (v_{obs}^-)^2$

$\Rightarrow (v_{obs}^+)^2 = 4 v_p v_{obs}^- + 4 v_p^2 + (v_{obs}^-)^2 = (v_{obs}^- + 2 v_p)^2 \Rightarrow (v_{obs}^+)_{max} = 2 v_p + v_{obs}^-$ proved.

8) Sphere of influence's computation



$(r_{soi,1} \; ; \; r_{soi,2})$

$\Downarrow$

$$\begin{cases} soi_1: & r_{soi_1}=r_1 \\ soi_2: & r_{soi_2}=r_2 \end{cases} \quad r_1; r_2 \text{ t.c. } \frac{\| a_{d_{2\to 1}} \|}{\| a_{g1} \|} = \frac{\| a_{d_{1\to 2}} \|}{\| a_{g2} \|}$$

$a_{g1} = a_{g1}(r_1)$

$a_{g2} = a_{g2}(r_2)$

$a_{d_{2\to1}} = a_{d_{2\to1}}(r_2, r_1)$

$a_{d_{1\to2}} = a_{d_{1\to2}}(r_1; r_2)$

P = SAT

$g_1 = -\frac{m_1 G}{\|r_1\|^3} r_1$

$$\frac{d^2 R_P}{dt^2} = g_1(P) + g_2(P) \quad \to \text{ SAT DYNAMIC}$$

$$\frac{d^2 R_1}{dt^2} = g_P(1) + g_2(1) \quad \to m_1 \text{ DYNAMIC}$$

$g_P = + \frac{m_{SAT} G}{\|r_1\|^3} r_1$ 	 clearly $\to 0$

$$\frac{d^2 R_2}{dt^2} = \underset{\simeq 0}{g_P(1)} + g_1(2)$$

$g_P(1)$ = P attracting 1

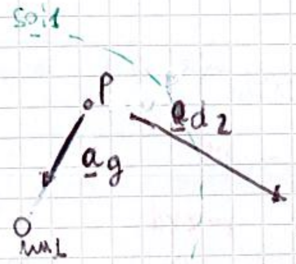
Considering the dynamic of the satellite into the 2 different soi is possible to define a_d

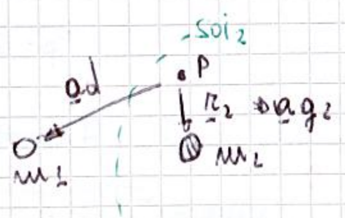
$\boxed{\text{INTO } soi_1}$ 	 P $\to$ 1 (g) 	 P $\to$ 2 (d)

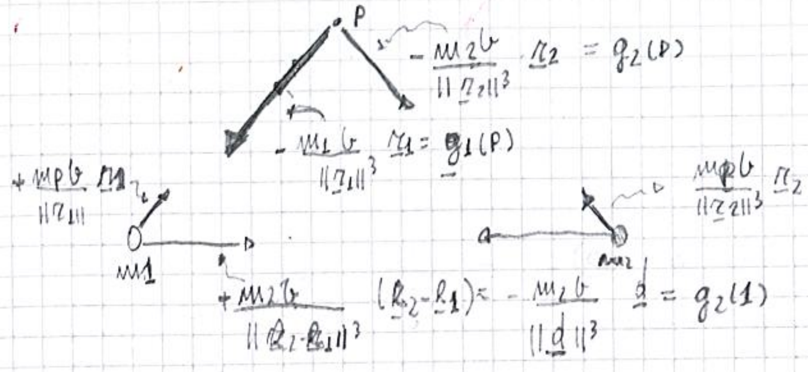
$$\frac{d^2 r_1}{dt^2} = \underbrace{g_1(P) - g_P(1)}_{a_{g1}} + \overbrace{g_2(P) - g_2(1)}^{a_{d2}}$$

$\boxed{\text{INTO } soi_2}$ 	 P $\to$ 1 (d) 	 P $\to$ 2 (g)

$$\frac{d^2 r_2}{dt^2} = \underbrace{g_2(P) - g_P(2)}_{a_{g2}} + \overbrace{g_1(P) - g_1(2)}^{a_{d1}}$$







$$\frac{d^2 R_1}{dt^2} = \frac{m_2 G}{\|R_2 - R_1\|}(R_2 - R_1); \quad r_1 = R_P - R_1$$

important

$$\frac{d^2 R_P}{dt^2} = -\frac{m_1 G}{\|r_1\|^3} r_1 - \frac{m_2 G}{\|r_2\|^3} r_2$$

$\Downarrow$

$$\frac{d^2 r_1}{dt^2} = \underbrace{-\frac{m_1 G}{\|r_1\|^3} r_1 +}_{g} \underbrace{-\frac{m_2 G}{\|r_2\|^3} r_2 - \frac{m_2 G (R_2 - R_1)}{\|R_2 - R_1\|^3}}_{d}$$

* $\varepsilon_1 \triangleq \frac{\| a_{d2} \|}{\| a_{g1} \|} \quad \to \quad \varepsilon_1 = \frac{\| g_2(P) - g_2(1) \|}{\| g_1(P) - g_P(1) \|} = \frac{(N)}{(D)}$

$\varepsilon \triangleq \frac{\text{disturb acc. due to (2) acting on P}}{\text{gravity acc. due to (1) acting on P}}$

(D): $\quad -\frac{G m_1}{\|r_1\|^3} r_1 - \left(+ \frac{G m_P}{\|r_1\|} r_1 \right) \simeq -\frac{G m_1}{\|r_1\|^3} r_1 = -\frac{G m_1}{\|r_1\|^2} \hat{r}_1 \Rightarrow + \frac{G m_1}{r_1^2}$

(N): $\quad -\frac{G m_2}{\|r_2\|^3} r_2 - \left(+ \frac{G m_2}{\|d\|^3} d \right) = -\frac{G m_2}{\|r_2\|^2} \hat{r}_2 - \frac{G m_2}{\|d\|^2} \hat{d}$

$$\Rightarrow \| g_2(P) - g_2(1) \|^2 = \left[\frac{1}{r_2^4} + \frac{1}{d^4} + 2 \frac{\hat{r}_2 \cdot \hat{d}}{r_2^2 \, d^2} \right]^{\frac{1}{2}} (G m_2)^2$$

normal scalar product $(a+b)\cdot(a+b) = a^2 + b^2 + 2 a \cdot b$

$\rho \triangleq \frac{r_2}{d}$

$Hp^* \Rightarrow \rho \ll 1 \quad \Rightarrow \quad \| g_2(P) - g_2(1) \|^2 = (G m_2)^2 \left[\frac{1}{r_2^4} + \frac{1}{d^4} + 2 \frac{\hat{r}_2 \cdot \hat{d}}{r_2^2 \, d^2} \right]$

$$= \frac{(G m_2)^2}{r_2^4} \left[1 + \rho^4 + 2 (\hat{r}_2 \cdot \hat{d}) \rho^2 \right]$$

$\Downarrow$ Hp^* $(\rho \ll 1)$

$$\| g_2(P) - g_2(1) \|^2 = \frac{(G m_2)^2}{r_2^4} \Rightarrow \| g_2(P) - g_2(1) \| = \frac{G m_2}{r_2^2}$$

obtaining:

$$\begin{cases} \varepsilon_1 = \frac{(N)}{(D)} = \frac{G m_2}{r_2^2} \cdot \frac{r_1^2}{G m_1} = \frac{m_2}{m_1} \frac{r_1^2}{r_2^2} \\ r_1^2 = \| r_2 + d \|^2 = (r_2^2 + d^2 + 2 r_2 \cdot d) = d^2 (\rho^2 + 1 - 2\rho \cos\alpha) \end{cases}$$

$\Downarrow$ Hp^*

$r_1^2 \simeq d^2 \Rightarrow r_1 \simeq d$

$$\Rightarrow \varepsilon_1 \simeq \frac{m_2}{m_1} \frac{d^2}{r_2^2}$$

* $\varepsilon_2 \triangleq \dfrac{\|g_1(P) - g_1(2)\|}{\|g_2(P) - g_P(2)\|} = \dfrac{(N)}{(D)}$

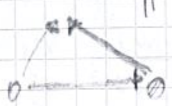
(D): $g_2(P) - g_P(2) \simeq g_2(P) \Rightarrow (D) \simeq -\dfrac{Gm_2}{\|r_2\|^2}$

(N): $-\dfrac{Gm_1}{\|r_1\|^2}\hat{r}_1 - \left(-\dfrac{Gm_1}{\|d\|^2}\hat{d}\right) = -\dfrac{Gm_1}{\|r_1\|^2}\hat{r}_1 + \dfrac{Gm_1}{\|d\|^2}\hat{d}$

$= -\dfrac{Gm_1}{\|r_1\|^3} r_1 + \dfrac{Gm_1}{\|d\|^3} d$

writing obvious relation: $r_1 = r_2 + d$

$\Uparrow$



$\rightarrow r_1 - d = r_2 \Rightarrow r_1 = r_2 + d.$

(N): $-\dfrac{Gm_1}{(r_2^2 + d^2 + r_2 d\, \hat{d}\cdot\hat{r}_2)^{\frac{3}{2}}}(r_2 + d) + \dfrac{Gm_1}{\|d\|^3} d$

taylor expansion.

$(1+x)^n \simeq 1 + nx + \dfrac{n\cdot(n-1)}{2}x^2 + O(x^3)$

$\Rightarrow (r_2^2 + d^2 + r_2 d\, \hat{d}\cdot\hat{r}_2)^{\frac{3}{2}} =$

$(r_2^2 + d^2 + r_2 d \cos\alpha)^{\frac{3}{2}} =$

$[r_2^2(1 + \dfrac{d^2}{r_2^2} + \dfrac{d}{r_2}\cos\alpha)]^{\frac{3}{2}} = r_2^3(1 + \dfrac{d^2}{r_2^2} + \dfrac{d}{r_2}\cos\alpha)^{\frac{3}{2}}$

$(r_2^2 + d^2 + r_2 d\, \hat{d}\cdot\hat{r}_2)^{-\frac{3}{2}} = [d^2(\dfrac{r_2^2}{d^2} + 1 + \dfrac{r_2}{d}\cos\alpha)]^{-\frac{3}{2}} = d^{-3}\cdot(1 + \dfrac{r_2^2}{d^2} + \dfrac{r_2}{d}\cos\alpha)^{-\frac{3}{2}}$

$\simeq d^{-3}[1 - \dfrac{3}{2}(\dfrac{r_2^2}{d^2} + \dfrac{r_2}{d}\cos\alpha) + \dfrac{3}{2}(\dfrac{-3}{2} - 1)\cdot\dfrac{1}{2}(\dfrac{r_2^2}{d^2} + \dfrac{r_2}{d}\cos\alpha)^2]$

$\simeq d^{-3}[1 + \dfrac{3}{2}\dfrac{r_2^2}{d^2} + \dfrac{3}{2}\dfrac{r_2}{d}\cos\alpha + \dfrac{9}{8}(\dfrac{r_2^4}{d^4} + \dfrac{r_2^2}{d^2}\cos^2\alpha + \dfrac{2r_2^3}{d^3}\cos\alpha)]$

$\simeq d^{-3}[1 + \dfrac{3}{2}\rho^2 + \dfrac{3}{2}\rho\cos\alpha + \dfrac{9}{8}\rho^4 + \dfrac{9}{8}\cos^2\alpha\,\rho^2 + \dfrac{9}{4}\rho^3\cos\alpha]$

$\Rightarrow$ (N): $-\dfrac{Gm_1}{d^3}[r_2^2 - 3\, r_2\cdot\hat{d}] = \dfrac{Gm_1}{d^3}[r_2^2 + 9r_2^2\cos^2\alpha - 6r_2^2\cos^2\alpha]^{1/2}$

$= \dfrac{Gm_1}{d^3}[r_2^2 + 3r_2^2\cos^2\alpha]^{1/2}$

$= -\dfrac{Gm_1}{d^3} r_2 [1 + 3\cos^2\alpha]^{1/2}$

$\Rightarrow \varepsilon_2 = \dfrac{(N)}{(D)} = -\dfrac{\cancel{G}m_1}{d^3}\cdot -\dfrac{\|r_2\|^2}{\cancel{G}m_2}\cdot r_2[1 + 3\cos^2\alpha]^{\frac{1}{2}}$

$= \dfrac{m_1}{m_2}\dfrac{r_2^3}{d^3}[1 + 3\cos^2\alpha]$

SOI computation

$\varepsilon_1 = \varepsilon_2 \Rightarrow \dfrac{m_1}{m_2}\rho^3[1 + 3\cos^2\alpha]^{\frac{1}{2}} = \dfrac{m_2}{m_1}\cdot\dfrac{1}{\rho^2}$ $(\cos\alpha = \hat{r}_2\cdot\hat{d})$

$\Big\downarrow$ solve

$\rho = \dfrac{r_2}{d} \Rightarrow \begin{cases} r_2 = r_{soi,2} = \rho\cdot d \\ d = \|R_2 - R_1\| \end{cases}$

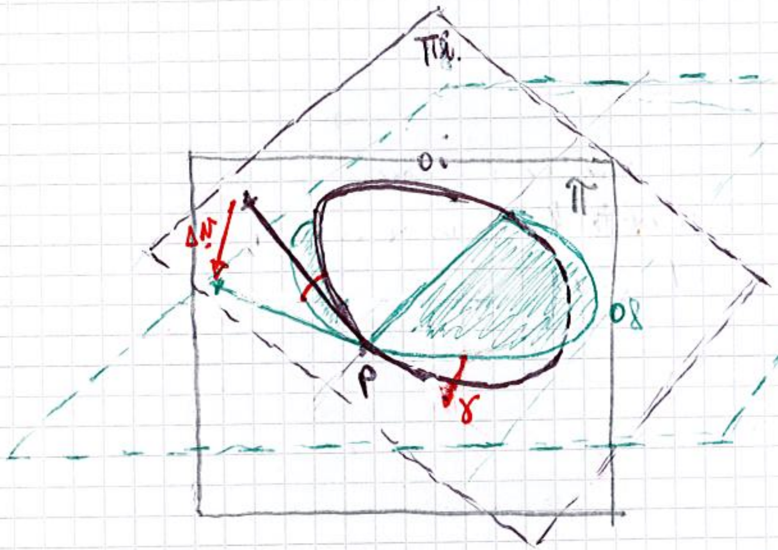
$\Big\downarrow$

$\begin{cases} r_2 = r_{soi,2} = \rho\cdot d \\ r_1 = \sqrt{r_2^2 + d^2} = r_{soi,1} \end{cases} \Rightarrow r_{soi,1} = d\cdot\left(\dfrac{m_1}{m_2}\right)^{\frac{2}{5}}\cdot\dfrac{1}{\sqrt[10]{1 + 3\cos^2\alpha}}$

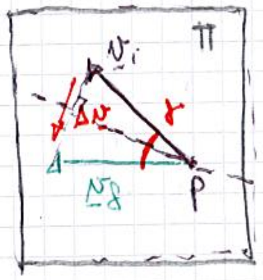
$\simeq d\left(\dfrac{m_1}{m_2}\right)^{\frac{2}{5}}$

7 → " NON-Coplanar transfer impulsive manouver."

1) CHANGE IN ORBITAL PLANE (maintaining the shape of the orbit) ⇒ 1 I.M.



⇒



↳ Supposing $(O_i; O_f)$ CIRCULAR ORBIT ⇒ $\begin{cases} e_{0i} = 0 \\ e_{0f} = 0 \end{cases}$ ⇒ $v_{ci} = v_{cf} = \sqrt{\frac{\mu}{r}} = v_c$

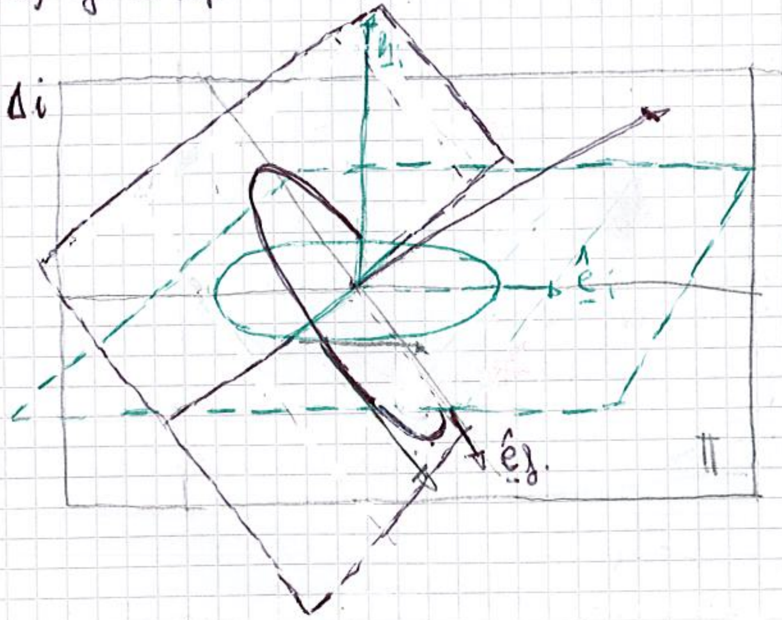
⇒ $\underline{v}_{ci}$ and $\underline{v}_{cf}$ are both contained in plane Π ⇒ $\Pi \perp (\Pi_i; \Pi_f)$

$\Delta\underline{v} = \underline{v}_i - \underline{v}_f$ ⇒

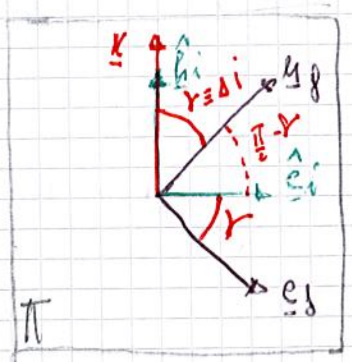
$$\|\Delta\underline{v}\| = v_c \cdot 2\sin\left(\frac{\gamma}{2}\right) = 2\sqrt{\frac{\mu}{r}}\sin\left(\frac{\gamma}{2}\right)$$

"Such manouver is really expensive!! e.g. to have $\gamma = 60^\circ$ ⇒ $\Delta v = v_c$.
⇒ velocity of the satellite must be doubled."

obs: $\gamma \equiv \Delta i$

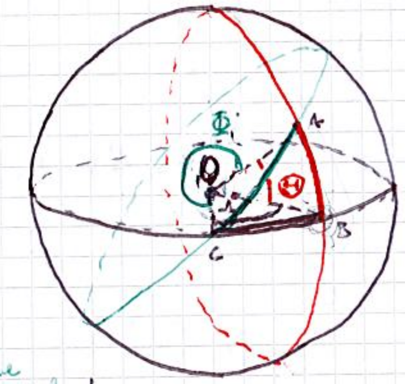


⇒



" IN GENERAL: A VARIATION IN ORBITAL PLANE ⇒ • VARIATION OF INCLINATION (Δi)
• VARIATION OF ASCENDENDENT NODE $(\Delta\Omega)$ "

--→ Spherical Triangles



• maximum circumference → $\Pi \cap sp$; $\Pi \cap O$

(IT) una circonferenza massima si ottiene intersecando la calotta sferica con un piano Π passante per il centro della sfera.

Since we have a trigonometrical sphere $(R=1)$ each arch (= lane) can be considered an angle a the center

⇓

name of the lane has the capital greek letter of the of the opposite angle

e.g. Λ is THE LANE OPPOSITE TO THE ANGLE λ

• INTERSECTION OF 3 MAXIMUM CIRCUMFERENCES GENERATES A SPHERICAL TRIANGLE.

⇕ IF AND ONLY IF

1. THE SUM OF 2 LANES (LATO) IS MAJOR OF THE 3rd

$\overline{CA} + \overline{CB} > \overline{AB}$ or $\overline{CA} + \overline{AB} > \overline{CB}$ or $\overline{AB} + \overline{CB} > \overline{CA}$

2. SUM OF THE 3 ANGLES IS MAJOR THAN 180°

$\phi + \lambda + \theta > 180$

3. EACH SPHERICAL ANGLE IS LESS THEN 180°

$(\phi; \lambda; \theta) < 180$

IF WE HAVE A SPHERICAL TRIANGLE.

THEN FOLLOWING RELATIONS ARE VALID

* SIN-LAW: $\frac{\sin\Theta}{\sin\theta} = \frac{\sin\Lambda}{\sin\lambda} = \frac{\sin\Phi}{\sin\varphi}$

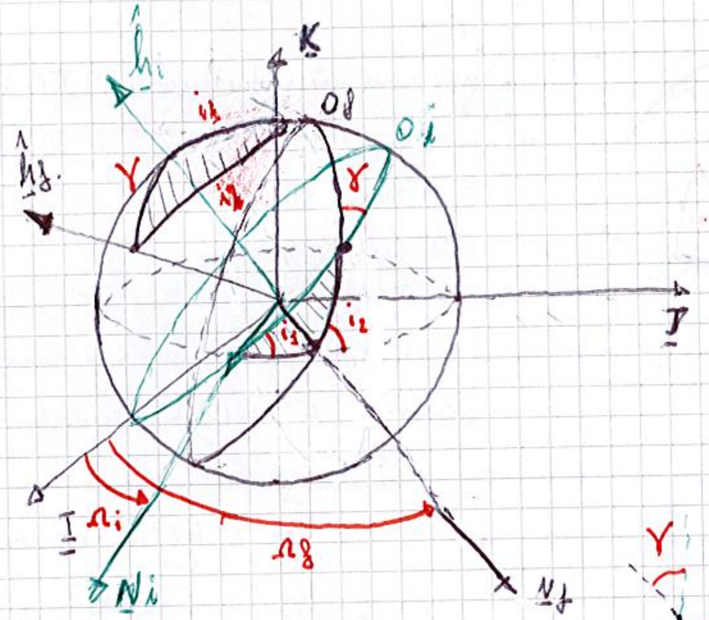
" the ratio between angle at the surface and angle at the center is equal.

* COS-LAW FOR LANES: $\begin{cases} \cos\Lambda = \cos\Theta\cos\Phi + \sin\Theta\sin\Phi\cos\lambda \\ \cos\Phi = \cos\Lambda\cos\Theta + \sin\Lambda\sin\Theta\cos\varphi \\ \cos\Theta = \cos\Phi\cos\Phi + \sin\Phi\sin\Theta\cos\theta \end{cases}$

* COS-LAW FOR ANGLES $\begin{cases} \cos\lambda = -\cos\theta\cos\varphi + \sin\theta\sin\varphi\cos\Lambda \\ \cos\varphi = -\cos\theta\cos\lambda + \sin\theta\sin\lambda\cos\Phi \\ \cos\theta = -\cos\varphi\cos\lambda + \sin\varphi\sin\lambda\cos\Theta \end{cases}$

↳ considering always 2 non coplanar circular orbits.

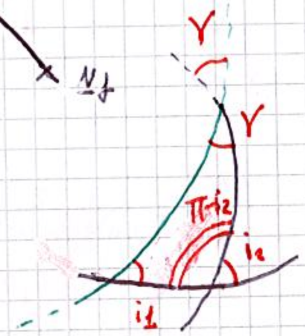
⇒ will be possible to define a sphere.



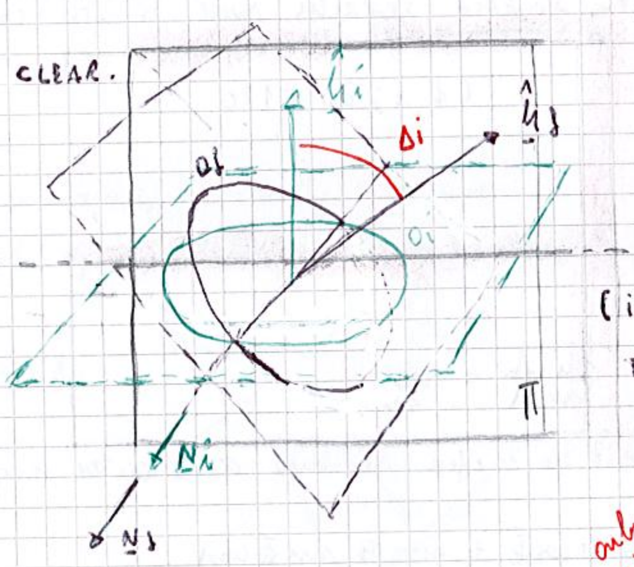
⇒ $\underline{N}_f = \underline{K} \times \underline{h}_f$

$\underline{N}_i = \underline{K} \times \underline{h}_i$

⇒



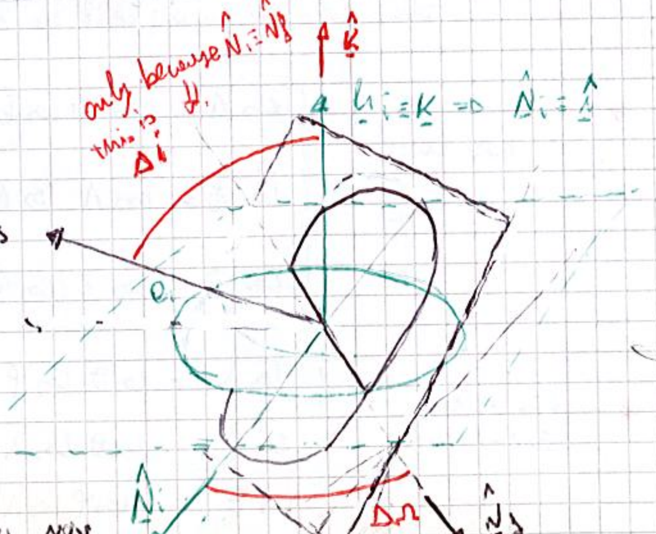
TO BE CLEAR.

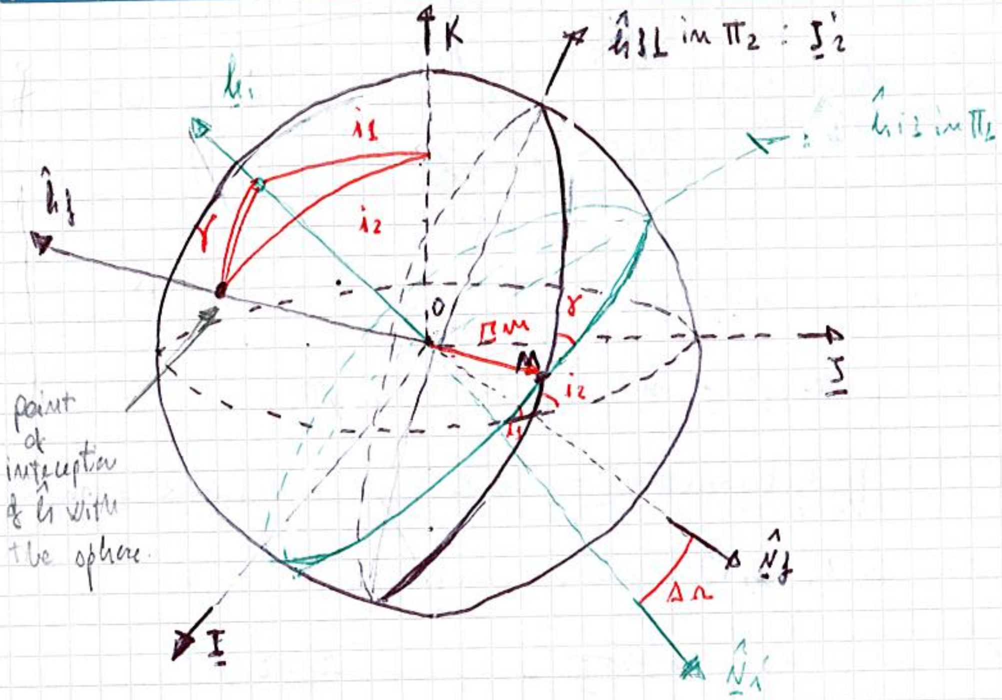


* $\Delta i \neq 0$, $\Delta\Omega = 0$ ⟺ $\hat N_i = \hat N_f$; $\gamma = \Delta i$

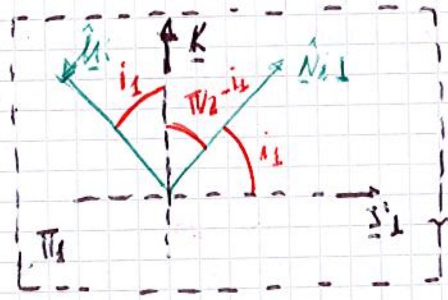
(it means that $\hat h_i$ and $\hat h_f$ belongs to the same plane π normal to $\hat N_i ; \hat N_f$)

* $\Delta i \neq 0$, $\Delta\Omega \neq 0$ ⇒ IF THE 2 ORBITS h VECTORS ARE NOT CONTAINED IN THE SAME PLANE π (WHICH HAS $\hat N_i = \hat N_f$ AS NORMAL VECTOR) A CHANGE IN INCLINATION IMPLIES ALSO A CHANGE IN ASCENDING NODE.



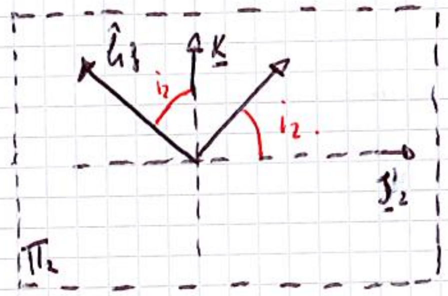


Π_1 : ⊥ $\hat N_i$ ⇒ (passing through O)



⟶ $\underline I'_1$ generated by interception of Π_1 with equatorial plane.

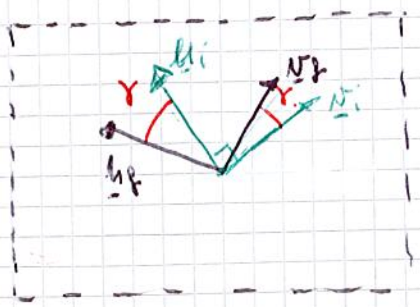
Π_2 : ⊥ $\hat N_f$ ⇒ (passing through O)



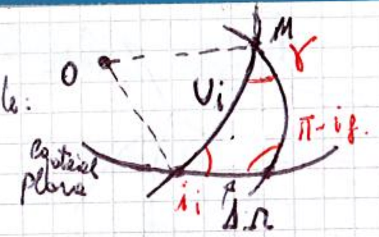
M point of interception of the 2 orbits

$\underline h_i = \underline r_M \times \underline v_M|_{O_i}$; $\underline h_f = \underline r_M \times \underline v_M|_{O_f}$ (supposing of translating $(\underline h_i ; \underline h_f)$ in M)

Π_3 : ⊥ $\underline r_M$ ⇒ (passing through M)



$\Rightarrow$ Considering the spherical triangle:



$\Rightarrow$ U_i latitude of initial manoeuvre.

$\cos(\pi - i_f) = -\cos i_i \cos\gamma + \sin i_i \sin\gamma \cos U_i$

$\cos\gamma = -\cos i_i \cos(\pi - i_f) + \sin i_i \sin(\pi - i_f) \cos \Delta\Omega$

$\Downarrow$ $(\cos(\pi - \alpha) = -\cos\alpha)$; $\sin(\pi - \alpha) = \sin\alpha)$

$$\begin{cases} \cos i_f = \cos i_i \cos\gamma + \sin i_i \sin\gamma \cos U_i \\ \cos\gamma = \cos i_i \cos i_f + \sin i_i \sin i_f \cos(\Omega_2 - \Omega_i) \end{cases}$$

obs 1: $\Delta\Omega = 0 \Leftrightarrow \begin{cases} U_i = 0 \\ i_i = 0 \end{cases}$ $\Rightarrow$ THE IMPULSIVE MANEUVER CHANGES ONLY i_f ONLY IF HAPPENS AT THE EQUATORIAL PLANE.

$\longrightarrow$

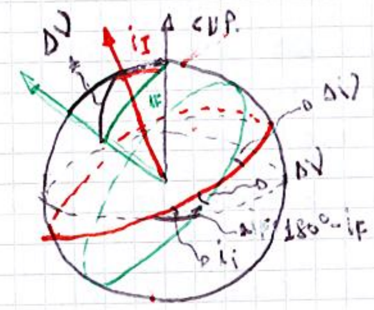
$\Rightarrow \begin{cases} \cos i_f = \cos\gamma - 0 \Rightarrow i_f = \gamma \end{cases}$

E.g.

Ⓘ $\begin{cases} e = 0 \\ i = 45° \\ r = 7000 \text{ km} = a \end{cases}$; Ⓕ $\begin{cases} e = 0 \\ r = r_i \\ \Delta\Omega = 6{,}4° ; \ i = 50° \end{cases}$ $\Rightarrow$?? $\|\Delta \vec{v}\|_{i \to f}$; $\Delta T = T_H - T_N$

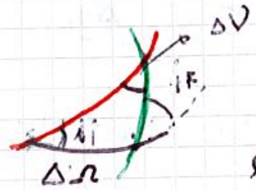
$\Rightarrow v_{ci} = v_{cf} = \sqrt{\frac{\mu_\oplus}{r}} = 7{,}54600 \text{ km/s} = v_c$

$\varepsilon = -\frac{\mu}{2a} = \frac{1}{2} v^2 - \frac{\mu}{r} \rightarrow \frac{\mu}{2r} = \frac{1}{2} v^2 \rightarrow v = \sqrt{\ }$



$\Rightarrow \Delta v = 2 v_c \sin\left(\frac{\Delta V}{2}\right)$ (isoc)

$\Rightarrow$



cosine-law:

~~[illegible]~~

~~[illegible]~~

$\Rightarrow \cos(\Delta V) = -\cos(i_i)\cos(\pi - i_f) + \sin(i_i)\sin(\pi - i_f)\cos\Delta\Omega$

$= \cos(i_i)\cos(i_f) + \sin(i_i)\sin(i_f)\cos\Delta\Omega = 0{,}9928$

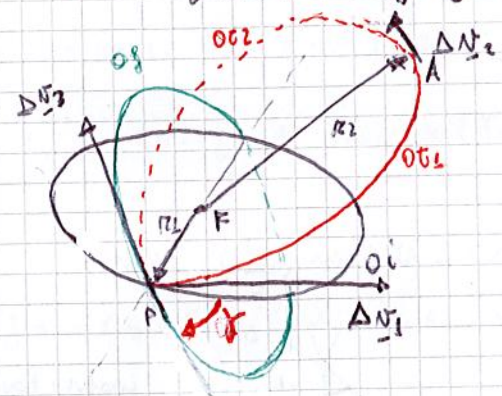
$\Rightarrow \Delta V = 6{,}87°$

SPEED ARE THE SAME TRIANGLE IS ISOSCELE.

$\Rightarrow \Delta v = 2 v_c \cdot \sin\left(\frac{\Delta V}{2}\right) = \sqrt{2 v_c^2 (1 - \cos\Delta V)} = 893{,}6 \text{ m/s}$

2) CHANGE IN ORBITAL PLANE WITH 3-IMPULSES MANEOUVER. ⇒ 3 IM.

Considering $o_i ; o_f$: $\begin{cases} e_i = 0 \\ a_i = r_i \end{cases} ; \begin{cases} e_f = 0 \\ a_f = r_f \gtrless r_i \end{cases}$ ⇒ 2 circular orbits.



" since single impulse maneouver cost a lot
⇓
in some cases biellptic maneouver (with requires 3 impulses) is cheaper.

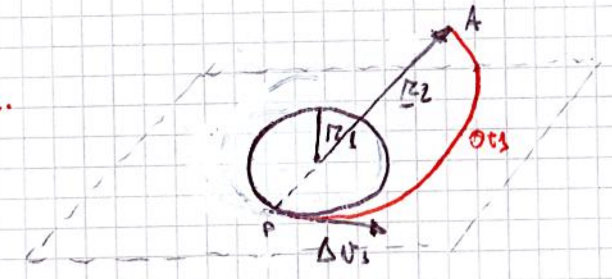
USUALLY: $\gamma \simeq 40^0$ is better a bitangent maneouver. "

$|\overline{FP}| = r_1$; $\overline{FA} = r_2$

$\overline{FP} = r_1$ → 1st impulse tangent to oi ⇒ $ot_1 \in \Pi_1$ (Π_1: orbital plane of oi)

$\overline{FA} = r_2$ → 3rd impulse tangent to of ⇒ $ot_2 \in \Pi_2$ (Π_2: orbital plane of of)

1st impulse.

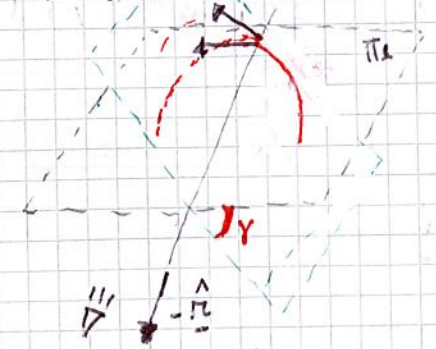


⇒ ot_1: $a_{ot_1} = \frac{r_1 + r_2}{2}$

$\Delta v_1 = v_p^{ot_1} - v_{c,i}$

2nd impulse

ONCE THE SATELLITE REACHES A (THE APOCENTER OF OT1) THE 2nd IMPULSE DOESN'T MODIFIES THE MODULUS OF V BUT ONLY CHANGES THE DIRECTION OF THE VELOCITY VECTOR.



⇒

Carnot's Theorem

$a^2 + b^2 - 2ab\cos\alpha$

$2a^2 - 2a^2\cos\alpha$

$\Delta v_2 = 2 (v_a^{ot_1})^2 (1 - \cos\gamma)$

3rd impulse

is "ONLY" NEEDED TO RE-CIRCULARIZE THE ORBIT

≃ Because the 2 transfer orbits (ot_1; ot_2) if watched in each orbital plane are equal ⇒ $v_p^{ot_1} = v_p^{ot_2}$

(!! as modulus, orbital planes are different!!)

$\Delta \underline{v}_3 = \underline{v}_{c,f} - \underline{v}_p^{ot_2}$ ⇒ $\|\Delta \underline{v}_3\| = v_p^{ot_2} - v_{c2}$

equal to v_{c1}

But : $\begin{cases} \|\Delta v_3\| = v_p^{ot_2} - v_{cf} \\ a_{ot_1} = a_{ot_2} ; \quad v_p^{ot_1} = v_p^{ot_2} ; \quad v_{ci} = v_{cf} \end{cases}$ ⇒ $\Delta v_3 = \Delta v_1$

⇒ It's possible to write the total cost of the maneouver such that

- $\Delta v_{TOT} = 2\Delta v_1 + \Delta v_2$.

- $\Delta v_1 = v_p^{ot_1} - v_{ci}$

$\varepsilon = -\frac{\mu}{2a} = -\frac{\mu}{r} + \frac{1}{2}v^2$ ⇒ $-\frac{\mu}{2 \cdot \frac{r_1+r_2}{2}} = -\frac{\mu}{r_1} + \frac{1}{2}v_p^2$ ⇒ $v_p^2 = 2\mu\left(\frac{1}{r_1} - \frac{1}{r_1+r_2}\right)$

$v_p^2 = \frac{2\mu}{r_1}\left(1 - \frac{1}{1+\frac{r_2}{r_1}}\right)$ ↳ Defining quantity $\rho \triangleq \frac{r_2}{r_1}$

$v_p = \sqrt{\frac{\mu}{r_1}} \cdot \sqrt{2\left(1 - \frac{1}{1+\rho}\right)} = \sqrt{2\frac{\rho}{1+\rho}} \cdot \sqrt{\frac{\mu}{r_1}} = v_{ci}\sqrt{\frac{2\rho}{1+\rho}}$

⇒ $\frac{\Delta v_1}{v_{ci}} = \sqrt{\frac{2\rho}{1+\rho}} - 1$

- $\Delta v_2 = 2 v_a^{ot_1} \sin\left(\frac{\gamma}{2}\right)$

$\varepsilon = -\frac{\mu}{2\frac{r_1+r_2}{2}} = -\frac{\mu}{r_2} + \frac{1}{2}v_a^2$ ⇒ $v_a^2 = 2\mu\left(\frac{1}{r_2} - \frac{1}{r_1+r_2}\right) = \frac{2\mu}{r_1}\left(\frac{1}{\frac{r_2}{r_1}} - \frac{1}{1+\frac{r_2}{r_1}}\right)$

$v_a = \sqrt{\frac{\mu}{r_1}} \cdot \sqrt{2\left(\frac{1}{\rho} - \frac{1}{\rho+1}\right)} = \sqrt{\frac{\mu}{r_1}} \cdot \sqrt{2\frac{\rho+1-\rho}{\rho(\rho+1)}} = \sqrt{\frac{\mu}{r_1}}\sqrt{\frac{2}{\rho(\rho+1)}}$

⇒ $\frac{\Delta v_2}{v_{ci}} = 2\sqrt{\frac{2}{\rho(1+\rho)}} \cdot \sin\left(\frac{\gamma}{2}\right)$

↳ Now will be easy to find the ratio $\rho = \frac{r_2}{r_1}$ such that the cost will be minimized. for a given angle of change of orbital plane (γ)

$$\frac{\Delta v_{tot}}{v_{ci}} = 2\sqrt{\frac{2}{\rho(1+\rho)}}\sin\left(\frac{\gamma}{2}\right) + 2\sqrt{\frac{2\rho}{1+\rho}} - 2 = 2\left[\sqrt{\frac{2}{\rho(1+\rho)}} + \sqrt{\frac{2\rho}{1+\rho}} - 1\right]$$

$$\Rightarrow \rho|_{opt} : \quad \frac{\partial \frac{\Delta v_{tot}}{v_{ci}}}{\partial \rho} = 0 \quad \text{or} \quad f(\rho) \triangleq \frac{\Delta v_{tot}}{v_{ci}} \Rightarrow \frac{\partial f(\rho)}{\partial \rho}\Big|_{\rho_{opt}} = 0$$

$$\frac{\partial f(\rho)}{\partial \rho} = \cancel{2}\left(\frac{2}{\rho(1+\rho)}\right)^{-\frac{1}{2}}\frac{1}{\cancel{2}}\cdot\sin\left(\frac{\gamma}{2}\right)\cdot\frac{(-2)[1+\rho+\rho]}{\rho^2(1+\rho)^2} + \cancel{2}\left(\frac{2\rho}{(1+\rho)}\right)^{-\frac{1}{2}}\frac{1}{\cancel{2}}\frac{2\rho+2-2\rho}{(1+\rho)^2}$$

$$= \left(\frac{2}{\rho(1+\rho)}\right)^{-\frac{1}{2}}\sin\left(\frac{\gamma}{2}\right)\frac{-2-4\rho}{\rho^2(1+\rho)^2} + \left(\frac{2\rho}{1+\rho}\right)^{-\frac{1}{2}}\frac{2}{(1+\rho^2)} = 0$$

$$\Rightarrow f(\rho) = 2\left[\sqrt{\frac{2\rho}{1+\rho}}\left(-\frac{\sin\gamma/2}{\rho^2}\right) + \left(1+\frac{\sin\gamma/2}{\rho}\right)\cdot\frac{\sqrt{1+\rho}}{2\sqrt{2\rho}}\cdot\frac{2(1+\rho)-2\rho}{(1+\rho)^2} = 0\right.$$

$$-\frac{\sin\gamma/2}{\rho^2}\cdot\sqrt{\frac{2\rho}{1+\rho}} + \frac{\rho+\sin\gamma/2}{\rho\sqrt{2\rho}(1+\rho)\sqrt{1+\rho}} = 0$$

$$\Rightarrow \quad -2\rho(1+\rho)\sin(\gamma/2) + \rho^2 + \rho\sin(\gamma/2) = 0$$

$$-2\rho\sin\frac{\gamma}{2} + -2\rho^2\sin\gamma/2 + \rho^2 + \rho\sin\gamma/2 = 0$$

$$\rho^2(+1-2\sin\gamma/2) + \rho\cdot\sin\gamma/2\,(1-2) = 0$$

$\rho|_{opt} = 0 \rightarrow$ impossible

$\rho|_{opt} = \frac{+\sin\gamma/2}{+1-2\sin\gamma/2}$

↳ such writing is valid for $\rho|_{opt} \geq 1$ ($\rho_{opt} = \frac{r_2}{r_1} = 1 \Rightarrow$ A SINGLE IMPULSE MANEOUVER.)

⇒ at the boundary: $\rho_{opt} = 1 \Rightarrow 1 = \frac{+\sin\gamma/2}{+1-2\sin\gamma/2}$.

$+1-2\sin\gamma/2 = +\sin\gamma/2 \Rightarrow \sin\frac{\gamma}{2} \geq \frac{1}{3} \Rightarrow \gamma = 38,94°$

Π

⇒ $\gamma < 38,94$ → CHOOSE A SINGLE A SINGLE IMPULSE MANEOUVER.

$38,94 \leq \gamma \leq 60°$ → ρ_{opt} is GIVEN FROM EQUATION: $\rho_{opt} = \frac{\sin\gamma/2}{1-2\sin\gamma/2}$.

in fact for $\gamma = 60°$ we have $\rho|_{opt} = \frac{1/2}{1-2\cdot\frac{1}{2}} = +\infty$

→ Technically: Selecting $r_2 \to \infty$ for $\theta > 60°$ we have the optimum condition to minimize the cost.

Π

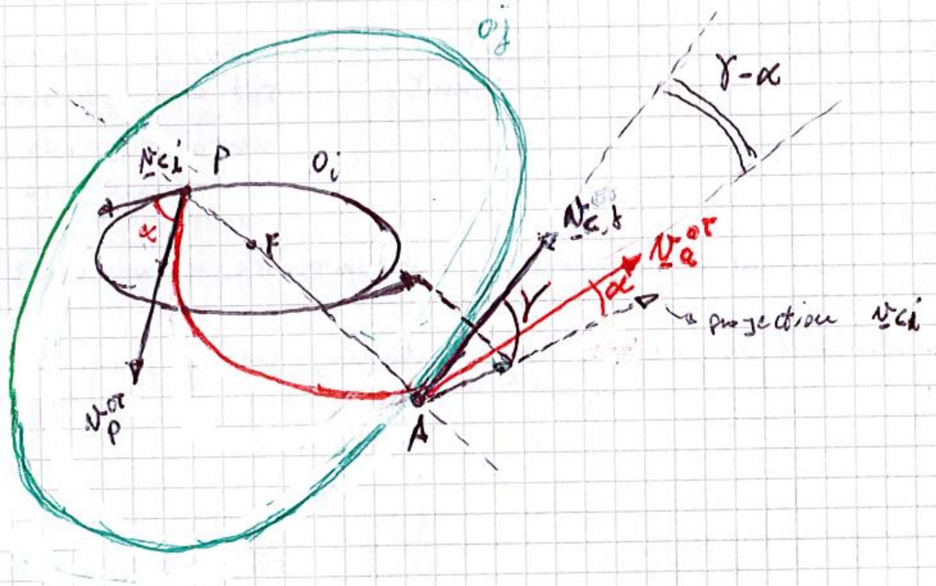
- $\gamma < 38,94°$ → ALWAYS CONVENIENT A SINGLE IMPULSE MANEOUVER.
- $38,94° < \gamma < 60°$ → $\rho|_{opt} = \frac{\sin\gamma/2}{1-2\sin\gamma/2}$ ⇒ MAKING A COMPARISON BETWEEN THE COST OF SINGLE IMPULSE MANEOUVER AND A 3 IMPULSES ONE IS POSSIBLE TO SELECT THE CORRECT MANEOUVER.
- $\gamma > 60°$ → ALWAY CONVENIENT A 3 IMPULSES MANEOUVER $\forall \rho|_{opt}$.

⇒ r_2 will be selected by constrains coming from time.

3) NON-COPLANAR HOHMANN TRANSFER → 2 i M.

(hohmann transfer with orbital plane change)

$$(o_i ; o_f) : \begin{cases} e_i = 0 \\ r_i = a_i \end{cases} ; \quad \begin{cases} e_f = 0 \\ r_f = a_f \quad (r_f > r_i) \end{cases}$$



→ $ot : \begin{cases} r_p^{OT} = r_1 \\ r_a^{OT} = r_2 \end{cases}$ " like a normal hohmann transfer but: since we have a plane change at the pericenter and at the apocenter of the transfer orbit → costs will be higher. "

$$-\frac{\mu}{r_i + r_f} = -\frac{\mu}{r_i} + \frac{1}{2} v_p^2 \rightarrow v_p^2 = \mu \cdot \left(\frac{1}{r_i} - \frac{1}{r_i + r_f} \right) = \mu \cdot \frac{r_i + r_f - r_i}{r_i (r_i + r_f)}$$

↳ Using carnot's theorem:

$$P : \begin{cases} v_{ci} = \sqrt{\frac{\mu}{r_i}} ; \quad v_p^{OT} = \sqrt{\frac{r_f}{r_i} \cdot \frac{\mu}{r_i + r_f}} \\ \Delta v_1^2 = v_{ci}^2 + v_p^{OT\,2} - 2 v_{ci} v_p^{OT} \cos\alpha \end{cases}$$

$$A : \begin{cases} v_{cf} = \sqrt{\frac{\mu}{r_f}} ; \quad v_a^{OT} = \sqrt{\frac{r_i}{r_f} \cdot \frac{\mu}{r_i + r_f}} \\ \Delta v_2^2 = v_{cf}^2 + v_a^{OT\,2} - 2 v_{cf} v_a^{OT} \cos(\gamma - \alpha) \end{cases}$$

↳ $\Delta v_{tot} = \Delta v_1 + \Delta v_2 = \sqrt{v_{ci}^2 + v_p^{OT\,2} - 2 v_{ci} v_p^{OT} \cos\alpha} + \sqrt{v_{cf}^2 + v_a^{OT\,2} - 2 v_{ci} v_p^{OT} \cos(\gamma - \alpha)}$

!! α is a degree of freedom designing the hohmann transfer !!

⇒ it's possible to compute $\alpha|_{opt}$ → the optimal inclination of ot that minimizes the total cost of the transfer.

$$v_{ci} = \sqrt{\frac{\mu}{r_i}} ; \quad v_p^{OT} = \sqrt{\frac{\mu}{r_i + r_f} \cdot \left(\frac{r_f}{r_i}\right)} \quad \Rightarrow \Delta v_1 = v_{ci}^2 + v_p^{OT\,2}$$

$$v_{cf} = \sqrt{\frac{\mu}{r_f}} ; \quad v_a^{OT} = \sqrt{\frac{\mu}{r_i + r_f} \cdot \left(\frac{r_i}{r_f}\right)}$$

$$\Delta v_{TOT} = \Delta v_1 + \Delta v_2 = \sqrt{v_{ci}^2 + v_p^{OT\,2} - 2 v_{ci} v_{opt} \cos\alpha} + \sqrt{v_{cf}^2 + v_a^{OT\,2} - 2 v_{ci} v_a^{OT\,2} \cos(\gamma - \alpha)}$$

$$\frac{\partial \Delta v_{TOT}}{\partial \alpha} = \frac{v_p^{OT} v_{ci} \sin\alpha}{\sqrt{v_{ci}^2 + \dots}} - \frac{v_{cf} v_a^{OT} \sin(\gamma - \alpha)}{\sqrt{v_{cf}^2 + \dots}}$$

~procedure:

1 → solve NUMERICALLY $\frac{\partial \Delta v_{TOT}}{\partial \alpha} = 0 \rightarrow \alpha|_{opt}$.

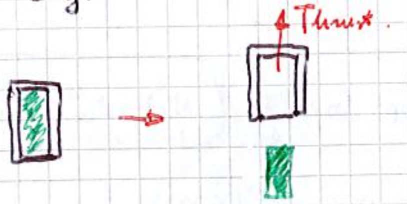
2 → COMPUTE Δv_{TOT}

can be immediately computed because v_p^{OT}, v_a^{OT} are only function of $(r_i ; r_f)$

8) → " Vertical ascent "

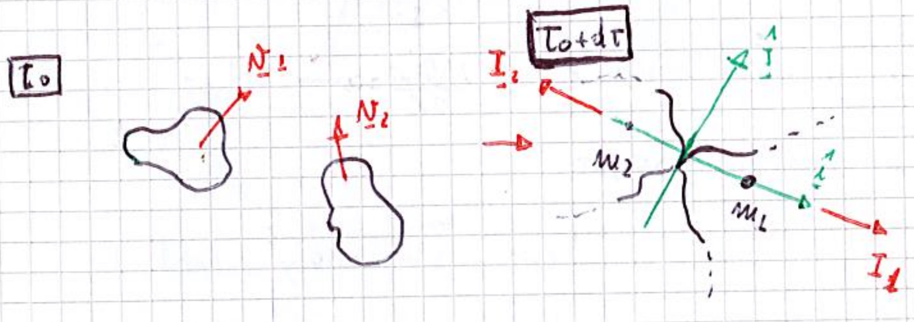
a) Impulsive dynamic.

dealing with momentum conservation it's necessary to deal with impulsive dynamic → e.g.



a1

↳ considering a 2D elastic collision:



- $\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}$; $\begin{Bmatrix} \underline{v}_1^- \\ \underline{v}_2^- \end{Bmatrix}$ → KNOWN
- $\begin{Bmatrix} \underline{v}_1^+ \\ \underline{v}_2^+ \end{Bmatrix}$; $\begin{Bmatrix} \underline{I}_1 \\ \underline{I}_2 \end{Bmatrix}$ → UNKNOWN

→ Impulse's Theorem..

$$d\underline{I} = \lim_{\Delta T \to 0} \underline{F} \cdot \Delta T \Rightarrow \underline{I} = \int_{T_0}^{T_1} \underline{F}\, dt \quad ; \quad T_1 - T_0 = \Delta T$$

impulse = variation of momentum in a finite time interval (ΔT)
(due to a force $\underline{F}$ acting on a body for the duration of ΔT)

Newton's law:
$$\begin{cases} \underline{F} = \dfrac{d\underline{q}}{dt} = \dfrac{d(m\underline{v})}{dt} \\ d\underline{I} = \underline{F}\, dt \end{cases} \Rightarrow \underline{I} = \Delta \underline{q} \; ; \; \underline{I} = \int_{T_0}^{T_1} \underline{F}\, dt .$$

→ Principle of action - reaction.

(IT). Per ogni forza che un corpo A esercita su un corpo B ne esiste ISTANTANEAMENTE un'altra uguale ma opposta in verso (stessa direzione e stesso modulo) esercitata dal corpo B che agisce sul corpo A

in terms of $\underline{I}$ ⇒ $\underline{F}_{A\to B} = -\underline{F}_{B\to A}$ ⇒ $\int_0^t \underline{F}_{A\to B}\, dt = \int_0^t -\underline{F}_{B\to A}\, dt$

⇒ $\underline{I}_A = -\underline{I}_B$

⇒ solving the problem of the elastic collision:

EQUIVALENT CONDITION: $\underline{I} = \Delta \underline{q}$

impulse's theorem $\begin{cases} \underline{I}_1 = \Delta \underline{q}_1 = m_1 (\underline{v}_1^+ - \underline{v}_1^-) \\ \underline{I}_2 = \Delta \underline{q}_2 = m_2 (\underline{v}_2^+ - \underline{v}_2^-) \end{cases}$ + action reaction $\underline{I}_1 = -\underline{I}_2$ ($\Delta \underline{Q}^{sys} = \underline{0}$; $\underline{I} = \Delta \underline{q}$

$\underline{0} = \Delta \underline{q}_1 + \Delta \underline{q}_2$
↓
$\Delta \underline{q}_1 = -\Delta \underline{q}_2$)

considering such bi-dimensional problem:

$$\begin{cases} I_{1,i} = -I_{2,i} \\ I_{1,j} = I_{2,j} = 0 \end{cases} \Rightarrow \begin{cases} m_1 v_{1,i}^+ - m_1 v_{1,i}^- = m_2 v_{2,i}^+ - m_2 v_{2,i}^- \\ \cancel{m_1} v_{1,j}^+ = \cancel{m_1} v_{1,j}^- \;;\; \cancel{m_2} v_{2,j}^+ = \cancel{m_2} v_{2,j}^- \end{cases}$$

$v_{1,j}^+ = v_{1,j}^-$ $\qquad$ $v_{2,j}^+ = v_{2,j}^-$

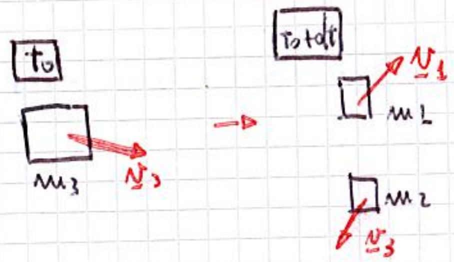
≈ Need also to impose energy conservation to fulfill the problem with:

8 unknown $\begin{Bmatrix} v_{1,i}^+ \\ v_{1,j}^+ \\ v_{2,i}^+ \\ v_{2,j}^+ \\ \underline{I}_1 \\ \underline{I}_2 \end{Bmatrix}$; 8 equations: $\begin{Bmatrix} \underline{I}_1 = \Delta \underline{q}_1 \ (2) \\ \underline{I}_2 = \Delta \underline{q}_2 \ (2) \\ \underline{I}_1 = -\underline{I}_2 \ (2) \\ \Delta T = 0 \quad (1) \\ I_{1,j} = I_{2,j} = 0 \ (1) \end{Bmatrix}$ $\underline{i}$ → collision direction, $\underline{j}$ → ⊥ $\underline{i}$

↳ it's only 1 equation because is redundant with $\underline{I}_1 = -\underline{I}_2$.

a2

↳ considering mass expulsion (or capture)



- $\{m_1\}$; $\{\underline{v}_3\}$ → KNOWN
- $\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix}$; $\begin{Bmatrix} \underline{v}_1 \\ \underline{v}_2 \end{Bmatrix}$ → UNKNOWN

!! This is an impossible problem ⇒ WE HAVE TO SUPPOSE TO KNOW $(m_1; m_2)$!!

⇒ We need 4 scalar equation (for a 2D problem)

(2) $\underline{q}|_{T_0} = \underline{q}|_{T_0+dT}$ considering only 1 direction ($\underline{i}$) (1) $m_3 v_3 = m_1 v_1 + m_2 v_2$

(1a) $\Delta T = 0$ $\qquad$ $m_3 v_3^2 = m_1 v_1^2 + m_2 v_2^2$

(1b) $\Delta T = \Delta E_{chemical}$ $\qquad$ $m_3 v_3^2 - (m_2 v_2^2 + m_1 v_1^2) = \Delta E_{chemical}$.

↳ in both cases (1a ; 1b) the problem is fully solved.

fully solving the 1-D problem

$$\begin{cases} m_3 v_3 = m_1 v_1 + m_2 v_2 \\ m_3 v_3^2 - (m_2 v_2^2 + m_1 v_1^2) = 2\Delta T \end{cases}$$

ΔT caused by an internal energy (chemical energy) variation

$$\Delta T = T_{t_0} - T_{t_0+dt}$$

$$\begin{cases} v_2 = \dfrac{m_3 v_3 - m_1 v_1}{m_2} \\ m_3 v_3^2 - \cancel{m_2} \dfrac{m_3^2 v_3^2 + m_1^2 v_1^2 - 2 m_1 m_3 v_1 v_3}{m_2^{\cancel{2}}} - m_1 v_1^2 = 2\Delta T \end{cases}$$

$\downarrow$

$m_3 = m_1 + m_2$

$$\begin{cases} v_2 = \dfrac{m_3 v_3 - m_1 v_1}{m_2} \\ (m_1 + m_2)\, m_2 v_3^2 - (m_1+m_2)^2 v_3^2 + m_1^2 v_1^2 + 2 m_1 (m_1+m_2) v_1 v_3 - m_1 m_2 v_1^2 = m_2 2\Delta T \end{cases}$$

$$\begin{cases} v_2 = \dfrac{m_3 v_3 - m_1 v_1}{m_2} \\ m_1 m_2 v_3^2 + \cancel{m_2^2 v_3^2} - m_1^2 v_3^2 - \cancel{m_2^2 v_3^2} - 2 m_1 m_2 v_3^2 + m_1^2 v_1^2 + 2 m_1^2 v_1 v_3 \end{cases}$$

$$+ 2 m_1 m_2 v_1 v_3 - m_1 m_2 v_1^2 = m_2 \Delta T$$

$$\begin{cases} v_2 = \dots \\ m_1 [\, m_2 - m_1 - 2 m_2 \,] v_3^2 + 2 m_1 [\, m_1 + m_2 \,] v_1 v_3 - m_1 [\, m_1 + m_2 \,] v_1^2 = m_2 \Delta T . \end{cases}$$

$$\begin{cases} v_2 = \dots \\ - m_1 [\, m_1 + m_2 \,] \underbrace{\{ v_3^2 - 2 v_1 v_3 + v_1^2 \}}_{= (v_3 - v_1)^2} = m_2 \Delta T \end{cases}$$

$\Downarrow$

$$-(v_3 - v_1)^2 = \Delta T \cdot \frac{m_2}{m_1 (m_1 + m_2)}$$

$\Delta T < 0$
if warm increase speed.

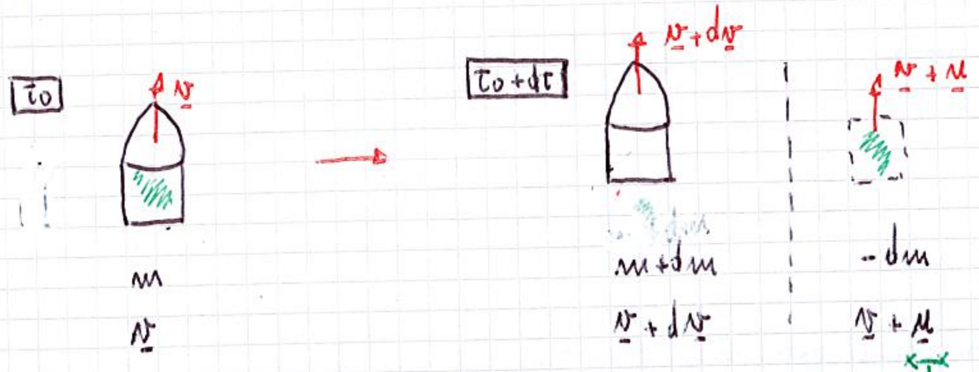
$\triangleright$ velocity
$m_1 \to$ LAUNCHER
$m_3 \to$ LAUNCHER + FUEL

important:

!! without the energy introduced by the chemistry would be impossible to have an increase of velocity

if $\Delta T|_{chem} = 0$ THEN $-(v_3 - v_1)^2 = 0 \Rightarrow v_3 = v_1$ | IF $\Delta T|_{chem} < 0$ THEN $v_1 > v_3$

b) Thrust formulation.



CHEMICAL $\to$ KINETIC conversion of the energy of the gas.

$\mapsto$ Imposing momentum conservation:

$$m \underline{v} = (m + dm)(\underline{v} + d\underline{v}) - dm (\underline{v} + \underline{u})$$

$$\cancel{m \underline{v}} = \cancel{m \underline{v}} + m\, d\underline{v} + \cancel{dm\, \underline{v}} + dm\, d\underline{v} - \cancel{dm\, \underline{v}} - dm\, \underline{u}$$

APPROXIMATION: let's suppose that 2^{nd} order infinitesimal quantity are negligible.

$$0 = m\, d\underline{v} - dm\, \underline{u}$$

$$m\, d\underline{v} = dm\, \underline{u}$$

$\mapsto$ Considering $\begin{Bmatrix} d\underline{v} \\ dm \end{Bmatrix}$ as the variation of mass and velocity (infinitesimal variations) that occurs in an infinitesimal time instant. $\Rightarrow$ dividing for such infinitesimal time instant

$$m \frac{d\underline{v}}{dt} = \underline{u} \cdot \frac{dm}{dt} \quad \Rightarrow \text{ we find back Newton's law. } (F = ma)$$

$$F^{gas} = \frac{dm}{dt} \cdot \underline{u} \overset{\Delta}{=} -\alpha \cdot \underline{u}$$

$F^{gas} \to$ force exercited by the expellant gas on the launcher.

$\begin{cases} \underline{u} \to \text{velocity of the gas exiting.} \\ \alpha \to \text{mass flow } \alpha \overset{\Delta}{=} -\dfrac{dm}{dt} \end{cases}$

$\Downarrow$

$$\begin{cases} m \dfrac{d\underline{v}}{dt} = \underline{T} \\ \underline{T} = \dfrac{dm}{dt} \cdot \underline{u} \overset{\Delta}{=} -\alpha \underline{u} \end{cases} \Rightarrow \underline{T} = F^{gas} = -\alpha \underline{u}$$

c) Phases of the Typical mission

1 → VERTICAL ASCENSION (Always present) → our interest now!!

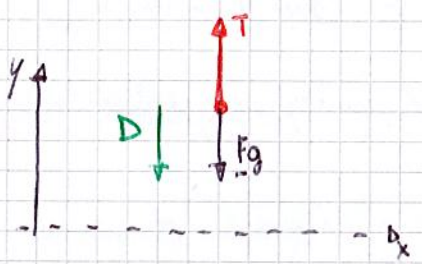
2 → TURN - OVER MANEOUVER.

3 → GRAVITY TURN MANEOUVER.

4 → BALLISTIC PHASE (optional ; $\Delta t = t_{burn} - t_{GTend}$)

5 → IMPULSIVE MANEOUVER.

1 → VERTICAL DYNAMIC



obs: we have ∅ lift in fact.

$$L = \frac{1}{2} \rho(h) (\underline{v} \cdot \underline{v}) S \; C_L(\alpha) \qquad \alpha = 0$$

rocket has ∅ lift while is going like this!!!

Hp*: (opp)

(i) D is negligible ⇒ let's suppose $\underline{D} = \underline{0}$

(ii) $\underline{g} \not\propto y$ in fact we should have

$$\underline{g} = -\frac{G m_\oplus}{(R_\oplus + y)^2} \approx -\frac{G m_\oplus}{R_\oplus} = \underline{g}_0$$

⇒ The rocket dynamic becomes:

$$\begin{cases} m \dfrac{d\underline{v}}{dt} = \underline{T} + m \underline{g}_0 \\ \underline{T} = \dfrac{dm}{dt} \underline{u} = -\alpha \underline{u} \end{cases} \quad + \quad m = m(t) = m_0 - \alpha t$$

$$\left(\alpha \triangleq \frac{dm}{dt} = \dot{m} \right)$$

$$\Rightarrow \quad (m_0 - \alpha t) \frac{d\underline{v}}{dt} = \underline{T} + (m_0 - \alpha t)\underline{g}_0$$

$$d\underline{v} = \frac{\underline{T}}{m_0 - \alpha t} dt + \underline{g}_0 \, dt$$

$$\Delta \underline{v} = \int_{t_0}^{t_f} \frac{\underline{T}}{m_0 - \alpha t} dt + \underline{g}_0 \Delta t$$

Performing this integration we will obtain Tsiolkowsky's law.

$$\int_{t_0}^{t_f} \frac{T}{m_0 - \alpha t} dt = \int_{t_0}^{t_f} \frac{T}{(m_0 - \alpha t)} \frac{-\alpha}{-\alpha} dt = -\frac{T}{\alpha} \ln(m_0 - \alpha t) \Big|_{t_0}^{t_f} \qquad \begin{cases} t_0 = 0 \\ t_f = t \text{ (generic)} \end{cases}$$

$$= -\frac{T}{\alpha} \ln(m_0 - \alpha t) \Big|_0^t$$

$$= -\frac{T}{\alpha} [-\ln(m_0) + \ln(m_0 - \alpha t)]$$

$$= -\frac{T}{\alpha} \ln\left(\frac{m_0}{m_0 - \alpha t}\right)$$

$$\underline{g}_0 = -g_0 \hat{\underline{j}} = -\frac{G m_\oplus}{R_\oplus} \hat{\underline{j}}$$

$$\underline{T} = T \cdot \hat{\underline{j}}$$

$$\Delta \underline{v} = v(t) \hat{\underline{j}} - v(0) \hat{\underline{j}}$$

usually equal to ∅

Tsiolkowsky

$$v(t) = \frac{T}{\alpha} \ln \frac{m_0}{m_0 - \alpha t} - g_0 \Delta t + v_0$$

* $I_{sp} \triangleq \dfrac{T}{\alpha g_0}$; $\alpha \triangleq \dfrac{dm}{dt}$

I_{sp} → specific impulse.

IMPULSE, given by the thrust.

PHYSICAL MEANING OF SPECIFIC IMPULSE : $I_{sp} = \dfrac{I}{W} = \dfrac{\text{Impulse}}{\text{weight}}$.

$$\Rightarrow I_{sp} = \frac{T \cdot \Delta t}{m \cdot g_0} = \frac{T}{\frac{m}{\Delta t} g_0} = \frac{T}{\alpha g_0}.$$

* $r \triangleq \dfrac{m_0}{m_0 - \alpha t_f} = \dfrac{m_0}{m_f}$ ⇒ $\Delta v = v_f - v_i = v(t_f) - v(t_0)$

$$\Rightarrow \begin{cases} \Delta v = I_{sp} g_0 \ln(r) - g_0 \Delta t. \\ \Delta v = v(t_f) - v(t=0) \end{cases}$$

* sometimes also term of $g_0 \Delta t$ is negligible e.g. once we are in open space.

obs: $t_{br} = \frac{m_0}{\alpha}$

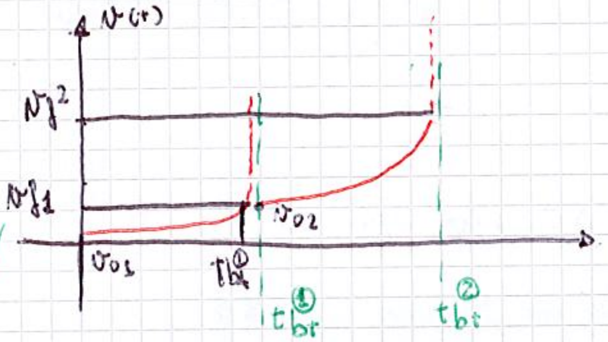
t_{br} → burning time ⇒ time required to burn all the fuel.

$\frac{m_0}{m_0 - \alpha t}$ ⇒ $m_f = 0$ ⟺ $t_{br} = \frac{m_0}{\alpha}$

→ In such condition the velocity is infinite: obviously is wrong.

IT'S IMPOSSIBLE BECAUSE AT THE $t = t_{br}$ I SHOULD HAVE NO MASS

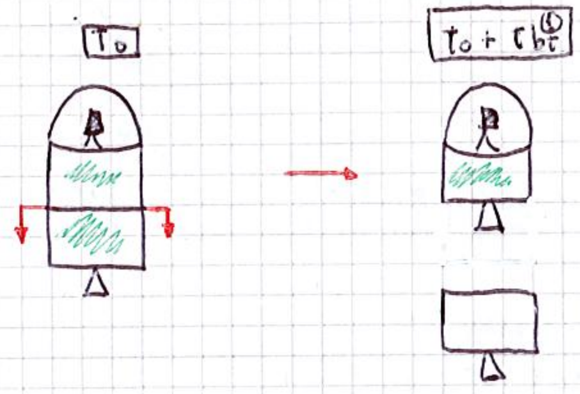
It's immediate to see that the only solution is to augment number of stages.



v_{f1}: $m_f^1 = m(t)$

↳ calling t_{bA}^1 the time required to burn the fuel (without burning the rocket.)

d) Stadiation



Using a N-staged Tsiolkowsky law what we obtain is:

$$v(t) = \sum_{i=1}^{N} \frac{T_i}{\alpha_i} \ln \frac{m_{0i}}{m_{0i} - m_{fi}} - g_0 \Delta t + v_0$$

$v_0 = v_{01}$ relative to velocity before 1st stage is fired.

$$Isp \triangleq \frac{T}{\alpha g_0} \; ; \quad r_i \triangleq \frac{m_{0i}}{m_{0i} - m_{fi}} = \frac{m_{dry} + m_{fuel}}{m_{dry}}$$

r_i → ponderal ratio.

$$v(t) = \sum_{i=1}^{N} Isp_i \ln r_i - g_0 \Delta t + v_0$$

⇒ $$v(t_f) = \sum_{i=1}^{N} Isp_i \ln r_i - g_0 \Delta t + v_0$$ $N = 1, \dots, n$ → n° of stages.

$\Delta t = (t_f - t_0)$

→ $Isp_i = \frac{T_i}{\alpha_i g_0}$

→ $r_i = \frac{m_{0i}}{m_{0i} - m_{fi}}$ where: $\begin{cases} m_{0i} = m_{dry\,i} + m_{fuel\,i} = m_{di} + m_{fi} \\ m_{0i} = m_{fi} + \underbrace{m_{si} + m_{pl\,i}}_{m_{dry}} \end{cases}$

→ $m_{dry} = m_s + m_{pl}$ = STRUCTURAL MASS + PAY-LOAD MASS.

APP: ignoring gravity losses:

↳ $$\Delta v = \sum_{i=1}^{N} Isp_i \, g_0 \ln \frac{m_{0i}}{m_{di}} = \sum_{i=1}^{N} Isp_i \, g_0 \ln r_i$$

↳ Integrating in time Tsiolkowsky law is possible to obtain the height of the rocket as function of time:

$$\Delta h = \Delta y = \int_{t_0}^{t_f} v(t)\,dt \qquad ; \qquad v(t) = I_{sp}\, g_0 \ln \frac{m_0}{m_0 - \alpha t} - g_0 (t - \underbrace{\cancel{t_0}}_{=0}) + \underbrace{\cancel{v_0}}_{=0}$$

$$\Rightarrow \Delta h = \int_0^{t_f} I_{sp}\, g_0 \ln\left(\frac{m_0}{m_0 - \alpha t}\right) - g_0 t \; dt$$

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - x = x(\ln x - 1)$$

$$\Rightarrow \Delta h = \int_0^{t_f} I_{sp}\, g_0 \ln\left(\frac{1}{1 - \frac{\alpha}{m_0} t}\right) dt - \frac{1}{2} g_0 t_f^2 \qquad \Bigg\| \; \beta \triangleq \frac{\alpha}{m_0}$$

$$= \int_0^{t_f} I_{sp}\, g_0 \ln\left(\frac{1}{1 - \beta t}\right) dt - \frac{1}{2} g_0 t_f^2 \qquad *$$

$$* \; \Bigg\| \; z \triangleq \ln(1 - \beta t) \qquad e^z = 1 - \beta t \qquad e^z\, dz = -\beta\, dt$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\Rightarrow \Delta h = \int_0^{t_f} - I_{sp}\, g_0 \ln(1 - \beta t)\, dt - \frac{1}{2} g_0 t_f^2$$

$$= + \int_0^{t_f} \frac{I_{sp}\, g_0}{\beta}\, z\, e^z\, dz - \frac{1}{2} g_0 t_f^2 = - I_{sp}\, g_0 \left[e^z (z - 1) \right]_{z_0}^{z} - \frac{1}{2} g_0 t_f^2$$

$$\int z\, e^z\, dz = z\, e^z - \int e^z\, dz = e^z (z - 1)$$

obviously we can say $t_f = t$ (general time)

$$\Delta h(t) = + \frac{I_{sp}\, g_0}{\beta} \left[(1 - \beta t) \left[\ln(1 - \beta t) - 1 \right] \right]_0^t - \frac{1}{2} g_0 t^2 + v_0 t \qquad \Rightarrow$$

$$\Rightarrow \Delta h(t) = \frac{I_{sp}\, g_0}{\beta} \left\{ (1 - \beta t) \left[\ln(1 - \beta t) - 1 \right]_0^t \right\} - \frac{1}{2} g_0 t^2$$

$$\beta \triangleq \frac{\alpha}{m_0} = \frac{\dot m}{m_0}$$

↳ In designing a lander we have many degrees of freedom.

* GLOBAL UNKNOWN (SUPPOSING $N \to$ FIXED) $\to \begin{Bmatrix} m_{0i} \\ m_{si} \\ m_{fi} \\ m_{pli} \end{Bmatrix} \Rightarrow 4N$ unknown.

BUT $\begin{cases} m_{0i} = m_{si} + m_{fi} + m_{pli} \\ m_{0\,i+1} = m_{pli} \end{cases}$

for each stage the payload represents the real payload + the mass of the following stage.

⇓

(REAL) GLOBAL UNKNOWN $\to$ $2N-1$ *

* KNOWN $\to \begin{Bmatrix} m_{pl} \to \text{real payload} \\ N \end{Bmatrix}$

* Supposing known $\begin{cases} N = 2 \\ m_{pl} = m_{pl}^* \end{cases}$ $\quad DOF_g = \begin{Bmatrix} m_{01} \\ m_{s1} \\ m_{f1} \\ m_{pl1} \\ m_{02} \\ m_{s2} \\ m_{f2} \\ m_{pl}^* \end{Bmatrix} \Rightarrow$ 8 unknown $\to$ 7 unknown (known m_{pl}^*)

but: $m_{02} = m_{pl1}$ $\to$ 6 unknown ($m_{pl1} = m_{02}$)

$m_{02} = m_{s2} + m_{f2} + m_{pl}^*$ $\to$ 3 unknown $= 2N-1$

It's really smarter express the problem as a function of:

$$r_i = \frac{m_{0i}}{m_{0i} - m_{fi}} > 1 \qquad r_i \to \text{propellant ratio } (3 \div 4{,}5)$$

$$s_i = \frac{m_{si}}{m_{si} + m_{fi}} < 1 \qquad s_i \to \text{structural ratio } (0{,}12 \div 0{,}20)$$

⇒ Using Lagrange's multiplier's method is possible to generate the $2N-1$ equations that allow us to find the optimal solution for such problem.

Expressing the problem in function of $(r_i ; s_i)$

$$2N \rightarrow \text{UNKNOWN} \begin{Bmatrix} r_i \\ s_i \end{Bmatrix} \quad \forall i = 1 \dots N$$

$+$

$$1 \rightarrow \text{UNKNOWN} \begin{Bmatrix} m_{plN} \\ m_{01} \end{Bmatrix}$$

m_{plN} is real payload early called m_{pl}^*

$\Rightarrow m_{01}$ is the real unknown.

Using Lagrange's multiplier's method:

- $J = f(\underbrace{r_i ; s_i}_{N}) = \frac{m_{01}}{m_{plN}} \rightarrow$ COST FUNCTION
  "What I want to minimize"
- $\Delta V = \sum_{i=1}^{N} I_{sp}\, g_0 \ln r_i \rightarrow 1^{ST}$ CONSTRAIN

(let's use ΔV law ignoring gravity losses.)

another possible constrain is the following.

- $m_{si} = g(m_{fi}) \Leftrightarrow s_i = W(r_i) \rightarrow 2^{nd}$ POSSIBLE CONSTRAIN

From Lagrange method we have to extract $2N+1$ equation $\Rightarrow$

$\frac{\partial J}{\partial r_i}\ (i = 1 \dots N) \rightarrow N$ equations

$\frac{\partial J}{\partial s_i}\ (i = 1 \dots N) \rightarrow N$ equation.

$\frac{\partial \mathcal{L}}{\partial \lambda} \rightarrow 1$ equation

$\mathcal{L} = J + \lambda_1 v_1 +$

$$\Rightarrow \begin{cases} \mathcal{L} = J + \lambda_1 g_1 + \lambda_2 g_2 + \dots \\ g_1 = \sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i - \Delta V. \\ g_2 = W(r_i ; s_i) \end{cases}$$

(1) BUILD COST FUNCTION J

$$J = \frac{m_{01}}{m_{plN}} \rightarrow \text{must be expressed as a function of } \{r_1, s_1, \dots, r_N, s_N\}$$

SINCE

$$\begin{cases} m_{pl\,i-1} = m_{0i} \\ J = \frac{m_{01}}{m_{02}} \cdot \frac{m_{02}}{m_{03}} \cdots \frac{m_{0N}}{m_{plN}} = \frac{m_{01}}{m_{plN}} \end{cases}$$

THEN $J = \frac{m_{01}}{m_{pl1}} \cdot \frac{m_{02}}{m_{pl2}} \cdots \frac{m_{0N}}{m_{plN}}$

$$\Downarrow$$

$$J = \prod_{i=1}^{N} \frac{m_{0i}}{m_{pli}} = \frac{m_{01}}{m_{plN}}$$

Anyway J must be expressed as a function of $\{r_1, s_1, \dots, r_N, s_N\}$

$$J = \prod_{i=1}^{N} \frac{m_{0i}}{m_{pli}} = \frac{m_{01}}{m_{plN}} \rightarrow \boxed{\prod_{i=1}^{N} \frac{m_{0i}}{m_{pli}} = \frac{m_{01}}{m_{plN}}}$$

$$\forall i : \begin{cases} \frac{m_{0i}}{m_{pli}} = \frac{m_{0i}}{m_{0i} - m_{si} - m_{fi}} \\ r_i = \frac{m_{0i}}{m_{0i} - m_{fi}} ; \quad s_i = \frac{m_{si}}{m_{si} + m_{fi}} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{m_{0i}}{m_{pli}} = \frac{m_{0i}/m_{0i}}{m_{0i}/m_{0i} - m_{si}/m_{0i} - m_{fi}/m_{0i}} = \frac{1}{1 - m_{si}/m_{0i} - m_{fi}/m_{0i}} \\ r_i = \frac{m_{0i}/m_{0i}}{m_{0i}/m_{0i} - m_{fi}/m_{0i}} = \frac{1}{1 - \frac{m_{fi}}{m_{0i}}} ; \quad s_i = \frac{m_{si}/m_{si}}{m_{si}/m_{si} + m_{fi}/m_{si}} = \frac{1}{1 + \frac{m_{fi}}{m_{si}}} \end{cases}$$

$$\begin{cases} \frac{m_{0i}}{m_{pli}} = \dots \\ r_i - r_i \left(\frac{m_{fi}}{m_{0i}} \right) = 1 \rightarrow \frac{m_{fi}}{m_{0i}} = \frac{r_i - 1}{r_i} ; \end{cases}$$

$$s_i = \frac{1}{1 + \frac{m_{fi}}{m_{si}}} = \frac{1}{1 + \frac{m_{fi}}{m_{0i}} \cdot \left(\frac{m_{0i}}{m_{si}} \right)} = \frac{1}{1 + \left(\frac{r_i - 1}{r_i} \right) \left(\frac{m_{0i}}{m_{si}} \right)}$$

$$s_i + s_i \left(\frac{r_i - 1}{r_i} \right) \left(\frac{m_{0i}}{m_{si}} \right) = 1 \Rightarrow \frac{m_{0i}}{m_{si}} = \frac{1 - s_i}{s_i \left(\frac{r_i - 1}{r_i} \right)}$$

(i) $\frac{m_{oi}}{m_{pli}} = \frac{1}{1 - \frac{m_{si}}{m_{oi}} - \frac{m_{fi}}{m_{oi}}}$; $J = \prod_{i=1}^{N} \frac{m_{oj}}{m_{pli}}$

(ii) $\frac{m_{fi}}{m_{oi}} = \frac{r_i - 1}{r_i}$ (iii) $\frac{m_{oi}}{m_{si}} = \frac{1-s_i}{s_i\left(\frac{r_i-1}{r_i}\right)}$

$\Downarrow$

$\frac{m_{si}}{m_{oi}} = s_i\left(\frac{r_i-1}{r_i}\right) / (1-s_i) = \frac{s_i\left(\frac{r_i-1}{r_i}\right)}{1-s_i}$

(iii), (ii) $\rightarrow$ (i): $\frac{m_{oi}}{m_{pli}} = \frac{1}{1 - \frac{s_i\left(\frac{r_i-1}{r_i}\right)}{1-s_i} - \frac{r_i-1}{r_i}} = \frac{r_i(1-s_i)}{1-r_i s_i}$

$\Uparrow$

$\frac{1}{(1-s_i)r_i}\left[(1-s_i)r_i - s_i(r_i-1) - (1-s_i)(r_i-1)\right]$

$\cancel{r_i} - s_i r_i - \cancel{s_i r_i} + \cancel{s_i} - \cancel{r_i} + 1 + \cancel{s_i r_i} - \cancel{s_i} = 1 - s_i r_i$

$\Rightarrow$ $J(r_i; s_i) = \prod_i \frac{r_i(1-s_i)}{1-r_i s_i}$ where $r_i = \frac{m_{oi}}{m_{oi}-m_{fi}}$; $s_i = \frac{m_{si}}{m_{si}+m_{fi}}$

(2) BUILD THE LAGRANGE'S POTENTIAL FUNCTION $\mathcal{L}$

$$\mathcal{L} = \prod_{i=1}^{N} \frac{r_i(1-s_i)}{1-r_i s_i} + \lambda_1\left(\sum_i I_{spi}\, g_0 \ln r_i - V\right) + \lambda_2\, W(r_i; s_i)$$

When will proceede with the minimization we will obtain:

$\nabla \mathcal{L} = 0 \Leftrightarrow \begin{cases} \frac{\partial \mathcal{L}}{\partial r_i} = 0 \rightarrow N \text{ eq} \\ \frac{\partial \mathcal{L}}{\partial s_i} = 0 \rightarrow N \text{ eq} \\ \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \rightarrow N \text{ eq} \end{cases}$ $N-1 \begin{cases} m_{oi} = f(r_i, s_i) \\ m_{oi} = m_{pl\,i-1} \\ r_i \\ s_i \end{cases}$ $\Rightarrow$ 2N+1 DEGREES OF FREEDOM.

$\Rightarrow$ λ_i eq $i = 1 \dots N$ eqution.

SINCE J grows monothonically

THEN minimizing $\ln(J)$ we obtain the same result. (mantaing the same contraints)

$\Rightarrow$ $J' = \ln(J) = \ln\left(\prod_{i=1}^{N} \frac{r_i(1-s_i)}{1-r_i s_i}\right) = \sum_{i=1}^{N} \left[\ln r_i + \ln(1-s_i) - \ln(1-r_i s_i)\right]$

$$\mathcal{L}' = \sum_{i=1}^{N} \left[\ln r_i + \ln(1-s_i) - \ln(1-r_i s_i)\right] + \lambda_1\left[\sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i - V\right] + \lambda_2 \cdot \cancel{W(r_i; s_i)}$$

We don't know the function able to link for each stage (i) r_i with s_i or a similar function to link m_s with m_f.

!! WE ARE SUPPOSING TO KNOW FOR EACH STAGE THE COEFFICIENTS s_i FOR EACH STAGE.

$\Rightarrow$ "HOW MUCH STRUCTURE WE NEED TO CONTAIN (m_{si}) A MASS OF FUEL (m_{fi})" !!

(3) FINAL COMPUTATION

$\nexists$ $\nabla(\mathcal{L}') = 0$; $\mathcal{L}' = \mathcal{L}'(r_{i=1\dots N}; \lambda_1)$

$\nabla(\mathcal{L}') = 0$ $\begin{cases} \frac{\partial \mathcal{L}'}{\partial r_i} = 0 & i = 1, 2, \dots N \\ \frac{\partial \mathcal{L}'}{\partial \lambda_1} = 0 \end{cases}$

$\Rightarrow \begin{cases} \frac{\partial \mathcal{L}'}{\partial r_i} = \frac{\partial}{\partial r_i}\left[\ln r_i + \ln(1-s_i) - \ln(1-r_i s_i)\right] + \frac{\partial}{\partial r_i} \lambda_1 I_{spi}\, g_0 \ln r_i = 0 \\ \frac{\partial \mathcal{L}'}{\partial \lambda_1} = \sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i - V = 0 \end{cases}$

$\begin{cases} \frac{\partial \mathcal{L}'}{\partial r_i} = \frac{1}{r_i} - \left(\frac{-s_i}{1-r_i s_i}\right) + \lambda_1 I_{spi}\, g_0 \frac{1}{r_i} = 0 & i = 1, 2, \dots N \\ V = \sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i \end{cases}$

$\begin{cases} \frac{1 - \cancel{r_i s_i} + \cancel{r_i s_i}}{r_i(1-r_i s_i)} + \lambda_1 I_{spi}\, g_0 \frac{1}{r_i} = \frac{1 + \lambda_1 I_{spi}\, g_0 (1-r_i s_i)}{r_i(1-r_i s_i)} = 0 & i = 1, 2, \dots N \\ V = \sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i \end{cases}$

$\Rightarrow \begin{cases} 1 + \lambda_1 I_{spi}\, g_0 (1 - r_i s_i) = 0 \\ V = \sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i \end{cases}$ $\Rightarrow \begin{cases} r_i = \frac{\lambda_1 I_{spi}\, g_0 + 1}{\lambda_1 I_{spi}\, g_0\, s_i} \\ V = \sum_{i=1}^{N} I_{spi}\, g_0 \ln r_i \end{cases}$

general solution

$$\begin{cases} r_i = \dfrac{1+\lambda I_{spi}\, g_0}{\lambda I_{spi}\, g_0\, s_i} \\ \sum_i I_{spi}\, g_0 \ln r_i = \Delta v. \end{cases}$$

; Hp: $\forall i$ $s_i \to$ KNOWN $\Rightarrow s_i = \bar{s}_i$

- solving: $\sum_{i=1}^{N} I_{spi}\, g_0 \ln\left[\dfrac{1+\lambda I_{spi}\, g_0}{\lambda I_{spi}\, g_0\, \bar{s}_i}\right] \Rightarrow$ FIND λ
- for each stage: $r_i = \dfrac{1+\lambda I_{spi}\, g_0}{\lambda I_{spi}\, g_0\, \bar{s}_i}$ *

If we want to proceed with an analytical computation we have to make some assumptions

Hp*
(i) $\forall i$ $s_i \to$ KNOWN $\Rightarrow s_i = \bar{s}_i$ (as before)
(ii) $\forall i$ $I_{spi} = I_{sp}$ (All the stages has the same specific impulse)

$$\begin{cases} r_i = \dfrac{1+\lambda I_{sp}\, g_0}{\lambda I_{sp}\, g_0\, \bar{s}_i} \\ \sum_i I_{sp}\, g_0 \ln r_i = \Delta v \end{cases}$$

same procedure $\Rightarrow$ $\Delta v = I_{sp} g_0 \left(\sum_i \ln\left[\dfrac{1+\lambda I_{sp}\, g_0}{\lambda I_{sp}\, g_0\, \bar{s}_i}\right]\right)$

$$\Delta v = I_{sp}\, g_0 \left\{ \sum_i \ln\left[\left(\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0}\cdot\frac{1}{\bar{s}_i}\right)\right]\right\} = I_{sp} g_0 \left\{ \sum_i \left[\underbrace{\ln\left(\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0}\right)}_{\text{this is a constant.}} - \ln(\bar{s}_i)\right]\right\}$$

$$\Delta v = I_{sp}\, g_0 \cdot \left\{ N \cdot \ln\left(\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0}\right) - \sum_i \ln(\bar{s}_i)\right\}$$

$$\frac{\Delta v}{I_{sp} g_0} = N \ln\left(\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0}\right) - \sum_i \ln(\bar{s}_i)$$

$$\ln\left(\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0}\right) = \frac{1}{N}\left[\frac{\Delta v}{I_{sp} g_0} + \sum_i \ln(\bar{s}_i)\right]$$

$$\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0} = e^{\frac{1}{N}\left[\frac{\Delta v}{I_{sp} g_0} + \sum_i \ln(\bar{s}_i)\right]} \quad (**)$$

Procedure

Hp: (i) $\forall i$ $I_{spi} = I_{sp}$

(ii) $\forall i$ $s_i = \bar{s}_i$ (known $\bar{s}_i = \dfrac{m_{si}}{m_{si}+m_{pi}}$)

$\Downarrow$

$$\frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0} = e^{\frac{1}{N}\left[\frac{\Delta v}{I_{sp} g_0} + \sum_i \ln(\bar{s}_i)\right]}$$

(! $\Delta v = \Delta v$ Hp: $\Delta v_0 = 0$!)

$$\lambda I_{sp} g_0 \cdot e^{\frac{1}{N}\left[\frac{\Delta v}{I_{sp} g_0} - \sum_i \ln(\bar{s}_i)\right]} = 1 + \lambda I_{sp} g_0$$

$$\begin{cases} \lambda = \dfrac{1}{I_{sp} g_0 \left\{ e^{\frac{1}{N}\left[\frac{\Delta v}{I_{sp} g_0} + \sum_i \ln(\bar{s}_i)\right]} - 1\right\}} \\ r_i = \dfrac{1+\lambda^* I_{sp} g_0}{\lambda^* I_{sp} g_0\, \bar{s}_i} \end{cases}$$

1st SOLVE FOR λ
$\Downarrow$
$\lambda = \lambda^*$

$\Downarrow$

$$\begin{cases} \Delta v = \Delta v = \sum_i \Delta v_i \\ \Delta v_i = I_{sp}\, g_0 \ln(r_i) \end{cases}$$

obs 1:

Since for each stage we have * $r_i = \dfrac{1+\lambda I_{spi}\, g_0}{\lambda I_{spi}\, g_0\, \bar{s}_i}$ once we assume that each stage has the same specific impulse ($\forall i$ $I_{spi} = I_{sp}$)

$$\Rightarrow \forall i: \quad r_i \bar{s}_i = \frac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0}$$

!! this is a constant value for each stage "i" !!

$$\Rightarrow \quad \forall i, j: \; r_i \bar{s}_i = r_j \bar{s}_j$$

obs 2:

Since we obtained relation (**) $\Rightarrow$ $r_i \bar{s}_i = \dfrac{1+\lambda I_{sp} g_0}{\lambda I_{sp} g_0} = e^{\frac{1}{N}\left[\frac{\Delta v}{I_{sp} g_0} + \sum_i \ln(\bar{s}_i)\right]}$

We obtain a faster relation to compute r_i WITHOUT COMPUTING λ !!

Example 1: optimized launcher.

Datos: $s_1 = 0,148$ $(N=2)$?? $\frac{m_{01}}{m_{plN}}$

$s_2 = 0,125$

$v_{bo_2} = 8,3\,km/s$

$I_{spi} = 320\,s$

I_{spi} if i change the mass of payload $m_{pl}' = m_{pl} + 10\%\,m_{pl}$??

Supposing an optimal stadiation of the launcher:

$$\begin{cases} \dfrac{m_{01}}{m_{plN}} = \displaystyle\prod_{i=1}^{N} \frac{m_{0i}}{m_{pli}} \ (=J) \Rightarrow \ln\left(\frac{m_{01}}{m_{plN}}\right) = \sum_{i=1}^{N} \ln\left(\frac{m_{0i}}{m_{pli}}\right) \\ m_{pli} = m_{0i+1} \end{cases}$$

dealing with optimal stadiation

- $J = \dfrac{m_0}{m_{plN}} = \displaystyle\prod_{i=1}^{N} \frac{m_{0i}}{m_{pli}}$ (since $m_{pli} = m_{0i+1}$)
- $\forall (i;j):\ r_i \bar{s}_i = r_j \bar{s}_j = e^{\frac{1}{N}\left[\frac{v}{I_{sp} g_0} + \sum_i \ln(\bar{s}_i)\right]}$

($\Leftrightarrow \forall i\ I_{spi} = I_{sp}$)

$$\Rightarrow \begin{cases} r_1 s_1 = r_2 s_2 \\ r_1 s_1 = e^{\frac{1}{2}\left[\frac{8,3}{320 \cdot 9,81 \times 10^{-3}} + \ln(0,148) + \ln(0,125)\right]} = 0,51017 \end{cases}$$

$r_1 = \dfrac{r_1 s_1}{s_1} = 3,4471$; $r_2 = \dfrac{r_2 s_2}{s_2} = \dfrac{r_1 s_1}{s_2} = 4,081$

$$\frac{m_{01}}{m_{pl2}} = \frac{m_{01}}{m_{pl1}} \cdot \frac{m_{pl1}}{m_{pl2}} = \frac{m_{01}}{m_{02}} \cdot \frac{m_{02}}{m_{pl2}} = \prod_{i=1}^{n} \frac{m_{0i}}{m_{pli}}$$

Important relation $J = J(r_i, \bar{s}_i)$

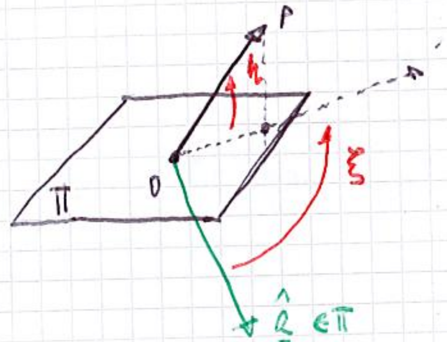
$$J = \frac{m_0}{m_{plN}} = \prod_{i=1}^{N} \frac{r_i(1-s_i)}{1 - r_i s_i}$$

$$\Rightarrow \frac{m_0}{m_{plN}} = \frac{r_1(1-s_1)}{1 - r_1 s_1} \cdot \frac{r_2(1-s_2)}{1 - r_2 s_2} = 5,995 \cdot 7,289 = 43,69.$$

9) → "Reference systems in astronomical problems"

To define the position of a satellite we have 2 solutions:

$\underline{r} = \{ r_x ; r_y ; r_z \}$ ⇄ $\underline{r} : \{ \|\underline{r}\| ; \xi ; \eta \}$



⇒ η → elevation

ξ → deviation.

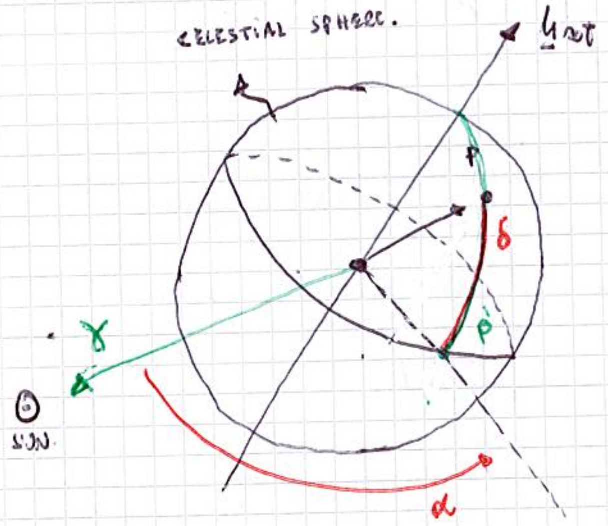
!! Would be nice if $\hat{R}$ is an inertial reference ($\hat{R} \in C$) !!

⇓

Each reference system is based on • 1 PLANE (⇒ 1 NORMAL VECTOR) • 2 ANGLES • 1 REFERENCE AXIS (IN PLANE)

a) EQUATORIAL REFERENCE SYSTEM (IJK)

(IT): SISTEMA DI RIFERIMENTO GEOCENTRICO EQUATORIALE.



⇒ $P = P(\delta, \alpha)$

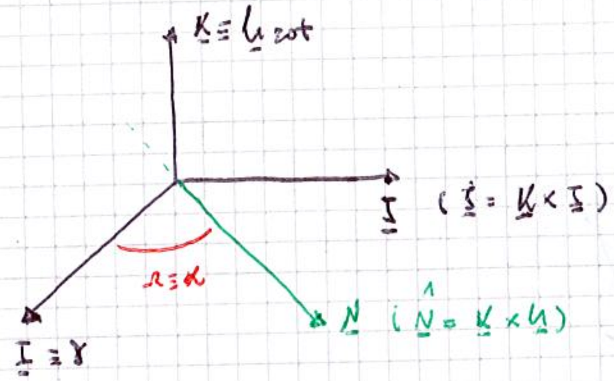
$\alpha = (0 ; 2\pi)$

$\delta = (-\frac{\pi}{2} ; +\frac{\pi}{2})$

CELESTIAL SPHERE: Spere of arbitrary radius with the center coinciding with the center of earth.

in fact: for many astronomical problem the distance of a body from earth is not relevant.

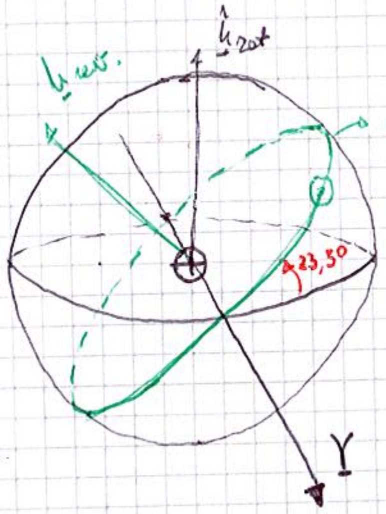
Geocentric reference system is also called $\{\underline{I}, \underline{J}, \underline{K}\}$ reference system, in fact the 3 fundamental versor are rule defined:



→ $\Omega = RAAN \equiv \alpha_{AN}$

RAAN: angle between ascendent node and γ direction.

↳ γ defined by the interception between equatorial plane and ecliptic plane.



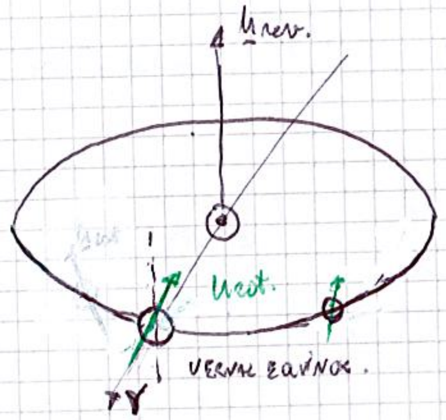
→ apparent rotation of sun around earth → ECLIPTIC

the rotation (apparent) of sun around earth is about 1°/day ⇒ the plane in witch happens this motion is called eliptic.

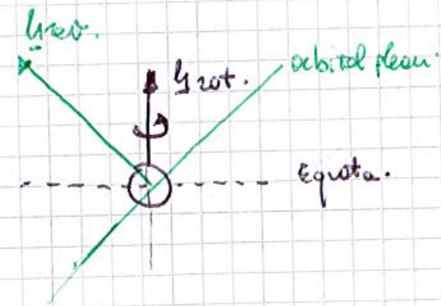
⇒ ORBITAL PLANE OF EARTH AND EQUATORIAL PLANE OF EARTH INTERCEPTS ONLY IN EQUINOXES.

⇓

the duration of the day during equinoxes is the same all over the earth.



⇒



→ Equatorial reference system.

$\underline{r} : (\alpha, \delta)$

$\alpha = [0 ; 2\pi]$

$\delta = [-\frac{\pi}{2} ; +\frac{\pi}{2}]$

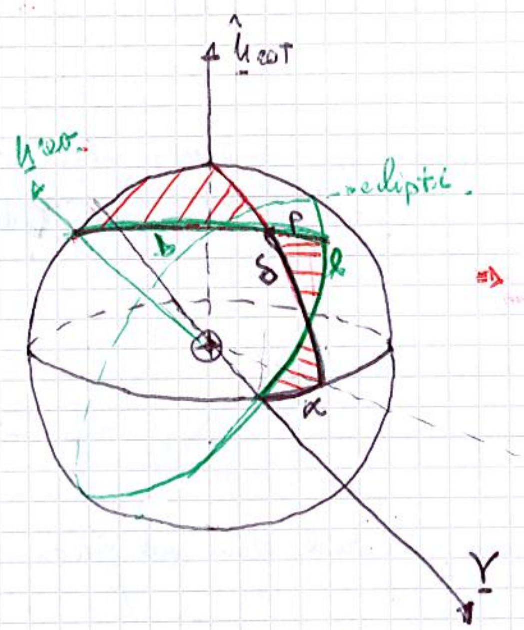
;

1) $\hat{K} \equiv \hat{h}_{rot}$

2) $\hat{I} \equiv \hat{\gamma}$ → $\hat{\gamma}$ interception between equator and ecliptic plane.

3) $\hat{J} = \hat{I} \times \hat{K}$

b) ECLIPTIC REFERENCE SYSTEM



→ Ecliptic reference system.

$\underline{r}: (l, b)$ 1) $\hat{h}_{rev\,\oplus}$

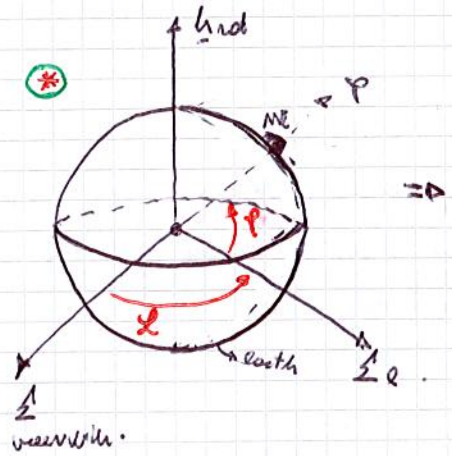
$l = [0; 2\pi]$; 2) $\oplus - \odot$

$b = [-\frac{\pi}{2}; \frac{\pi}{2}]$ 3) $♈ \equiv \hat{R}$

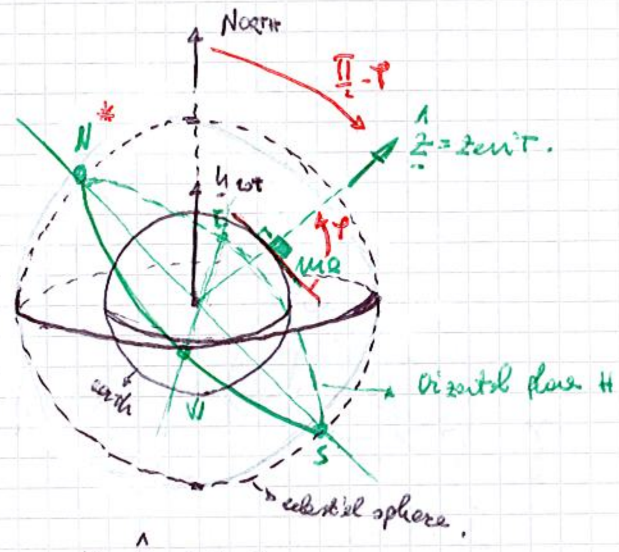
→ Using spherical triangles properties is possible to pass from ecliptic to equatorial r.s.

c) LOCAL REFERENCE SYSTEM (SEZ)

(HORIZONTAL - TOPOCENTRIC)



⇒



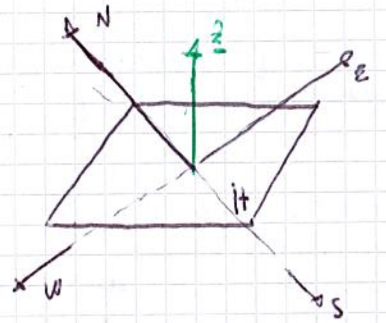
1) $\hat{z}$ → Horizon.

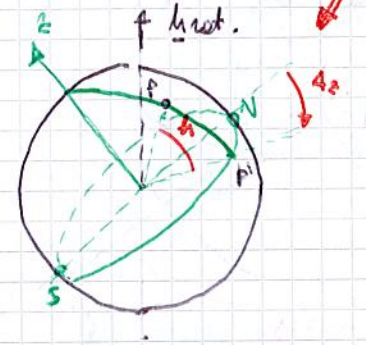
2) PLANET ($\oplus$)

3) $\hat{R} \equiv$ NORTH.

* The actual north (N) of the observer (me) will be the point nearer to the north of the celestial sphere.

⇒





→ Local reference system.

$\underline{r}: (A_z, h)$ 1) $\hat{z}$

$A_z = [0; 2\pi]$ 2) W-E

$h = [-\frac{\pi}{2}; +\frac{\pi}{2}]$ 3) $N \equiv \hat{R}$

N → local north of the observer.

!! Local reference system is NOT INERTIAL because obviously earth is rotating around herself. !!

* Position of the observer on earth is defined by another reference system. (*)

$P: (\lambda, \varphi)$ 1) $\hat{h}_{rot} \equiv$ North

$\lambda = [0; 2\pi]$ North → real north.

$\varphi = [-\frac{\pi}{2}; +\frac{\pi}{2}]$ 2) E

3) $\hat{E}_{Greenwich} \equiv \hat{R}$