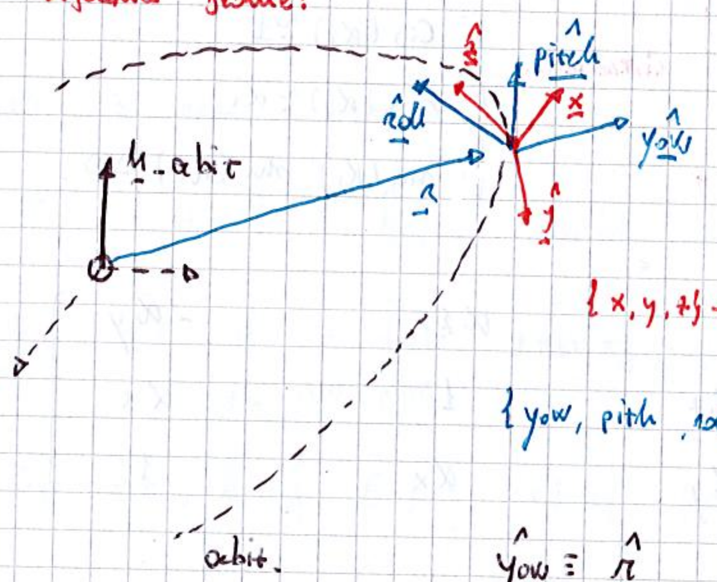
"Evaluating stability through attitude parameters ($\underline{\alpha}$) will allow the REAL-STABILITY analysis in order to see how small perturbation will increase/decrease angles instead of $\underline{w}$ vector"

→ LHLV reference frame:



$\{x, y, z\}$ → PRINCIPAL INERTIA R.F.

$\{yaw, pitch, roll\}$ → LOCAL HORIZONTAL-LOCAL VERTICAL REFERENCE FRAME.

$$\hat{yaw} \equiv \hat{r}$$

$$\hat{pitch} \equiv \hat{h}$$

$$\hat{roll} = \hat{pitch} \times \hat{yaw} \quad (= \hat{h} \times \hat{r})$$

If the orbit is circular and equatorial: $\hat{roll} \equiv \hat{v}$

$$\hat{h}_{orbit} = \underline{r} \times \underline{v} \Rightarrow \underline{h} \times \underline{r} = (\underline{r} \times \underline{v}) \times \underline{r} = -\underline{r} \times (\underline{r} \times \underline{v}) = \underline{v} \cdot (\underline{r} \cdot \underline{r}) - \underline{r}\,\underbrace{(\underline{r} \cdot \underline{v})}_{\Rightarrow 0}$$

perpendicular orbit

⇒ Defining the relative rotation between the principal inertia reference frame and local-vertical local-horizontal reference frame. I might choose a rotation-sequence:

SEQUENCE: $x \to y \to z$

α_x → rotation around x axis
↳ in the P.I. inertia reference-frame.

α_y → rotation around y' axis

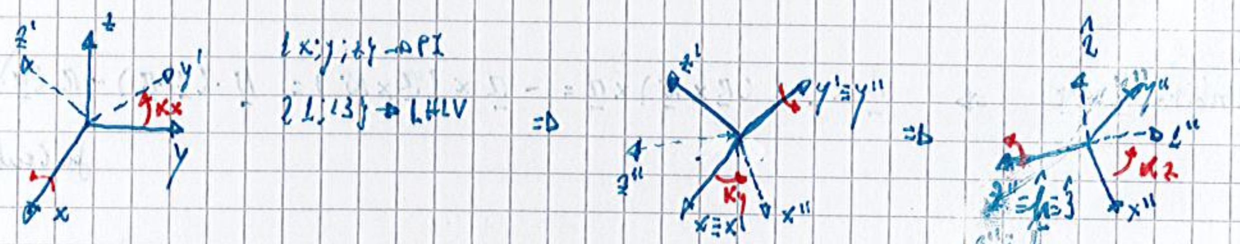
α_z → rotation around $z'' \equiv \hat{h} \equiv \hat{pitch}$ axis.

$$\Rightarrow \underline{\underline{A}}_{PI2LHLV}{}^{*} = \begin{bmatrix} c\alpha_z\, c\alpha_y \;; & c\alpha_z\, s\alpha_y\, s\alpha_x + s\alpha_z\, c\alpha_x \;; & -c\alpha_z\, s\alpha_y\, c\alpha_x + s\alpha_z\, s\alpha_x \\ -s\alpha_z\, c\alpha_y \;, & -s\alpha_z\, s\alpha_y\, s\alpha_x + c\alpha_z\, c\alpha_x \;; & s\alpha_z\, s\alpha_y\, s\alpha_x + c\alpha_z\, s\alpha_x \\ s\alpha_y \;, & -c\alpha_y\, s\alpha_x \;; & c\alpha_y\, c\alpha_x \end{bmatrix}$$

LINEARIZATION:
$$\begin{cases} \cos(\alpha_i) \simeq 1 \\ \sin(\alpha_i) \simeq \alpha_i \\ \sin(\alpha_i)\,\sin(\alpha_i) \simeq 0 \end{cases}$$

$$\underline{\underline{A}}_{PI2LHLV}\Big|_{lin} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & \alpha_x & 1 \end{bmatrix}$$

* As observed for cosine-matrix obtained from Euler's angles successive rotations starting from the original reference frame will give the cosine matrix such that:



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_x & s\alpha_x \\ 0 & -s\alpha_x & c\alpha_x \end{bmatrix} \;;\quad A' = \begin{bmatrix} c\alpha_y & 0 & s\alpha_y \\ 0 & 1 & 0 \\ -s\alpha_y & 0 & +c\alpha_y \end{bmatrix} \;;\quad A'' = \begin{bmatrix} c\alpha_z & s\alpha_z & 0 \\ -s\alpha_z & c\alpha_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{A}}'' \cdot \underline{\underline{A}}' \underline{\underline{A}} \Rightarrow \qquad \underline{a}_{LHLV} = \underline{\underline{A}} \cdot \underline{a}_{PI}$$

$$\underline{\underline{A}}_{PI2LHLV} = A''(\alpha_z)\, A'(\alpha_y)\, A(\alpha_x)$$
$$= \begin{bmatrix} c\alpha_z & s\alpha_z & 0 \\ -s\alpha_z & c\alpha_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\alpha_y \;; & +s\alpha_x\, s\alpha_y \;; & s\alpha_y \\ s\alpha_x s\alpha_y \;; & c\alpha_x \;; & s\alpha_x\, c\alpha_y \\ -c\alpha_x s\alpha_y & -s\alpha_x \;; & c\alpha_x\, c\alpha_y \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

↳ LHLV reference frame is not inertial, its orientation variates in time.

[WE HAVE TO DEAL WITH THE PROBLEM OF EXPRESSING THE ANGULAR VELOCITY OF THE BODY ($\underline{w}_B$) IN A ROTATING FRAME WITH HIS PROPER $\{\underline{w}_o\}$]

Dealing with 3 terms:

- $\tau_I : \{O;\ \hat{\underline{I}},\ \hat{\underline{J}},\ \hat{\underline{K}}\}$ → inertial, centered on earth.
- $\tau_o : \{C;\ \hat{\underline{r}},\ \hat{\underline{\theta}},\ \hat{\underline{h}}_o\}$ → LHLV ref, not inertial, centered on the satellite.
  yaw roll. pitch
- $\tau_B : \{C,\ \hat{\underline{i}},\ \hat{\underline{j}},\ \hat{\underline{k}}\}$ → PI ref, not inertial, centered on the satellite.

Dealing with ! 2 ROTATING REFERENCE WITH THEIR PROPER ANGULAR VELOCITY !

- $\underline{w}_B$ → angular velocity of τ_B with respect to τ_I
- $\underline{w}_o$ → angular velocity of τ_o with respect to τ_I
- $\underline{w}_{Bo}$ → angular velocity of τ_B with respect to τ_o

⇒ CAUSE OF COMPOSITION OF VELOCITIES:

$$\underline{w}_B = \underline{w}_o + \underline{w}_{Bo} \quad (i)$$

→ this sum MUST be done in a single R.F.
⇒ is DONE IN THE PI ref

(1) → PROJECTION OF $\underline{w}_o$ ON τ_B

for any Keplerian orbit: $\{\underline{w}_o\}_{\tau_o} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\theta} \end{Bmatrix} = \dot{\theta}\, \hat{\underline{h}}_o$

→ rotation
$\dot{\theta} \to n$

$$\Rightarrow \{\underline{w}_o\}_{\tau_o} = n \cdot \hat{\underline{h}}_o .$$

$\Rightarrow \{\underline{W}_0\}_{\underline{\tau}_B} = \underline{\underline{A}}_{LHLV2PI} \{\underline{W}_0\}_{\underline{\tau}_0} \quad (\underline{\underline{A}}_{LHLV2PI} = \underline{\underline{A}}^T_{PI2LHLV})$

$$= \begin{bmatrix} c\alpha_z c\alpha_y & c\alpha_z s\alpha_y s\alpha_x + s\alpha_z c\alpha_x & -c\alpha_z s\alpha_y c\alpha_x + s\alpha_z s\alpha_x \\ -s\alpha_z c\alpha_y & -s\alpha_z s\alpha_y s\alpha_x + c\alpha_z c\alpha_x & s\alpha_z s\alpha_y c\alpha_x + c\alpha_z s\alpha_x \\ s\alpha_y & -c\alpha_y s\alpha_x & c\alpha_y c\alpha_x \end{bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ h \end{Bmatrix}$$

$\{\alpha_x; \alpha_y; \alpha_z\} \to$ rotations around PI axis to overlap $\underline{\tau}_B$ on $\underline{\tau}_0$

this matrix is $\underline{\underline{A}}_{PI2LHLV}$

(2) $\to$ PROJECTION OF $\underline{W}_{BO}$ ON $\underline{\tau}_B$

$$\{\underline{W}_{BO}\}_{\underline{\tau}_B} = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}$$

$\to$ I what we are looking for: THE PROJECTION OF $\underline{W}_{BO} (= \underline{W}_B - \underline{W}_0)$ ON THE PI reference frame.

Since we decided to overlap the LHLV r.f. $\underline{\tau}_0$ on the PI r.f. $\underline{\tau}_B$ with a rotation sequence $\{\alpha_x, \alpha_y, \alpha_z\}$

THEN

$$\{\underline{W}_{BO}\}_{\underline{\tau}_B} = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \dot{\alpha}_x \{\hat{\underline{i}}\}_{\underline{\tau}_B} + \dot{\alpha}_y \{\hat{\underline{j}}'\}_{\underline{\tau}_B} + \dot{\alpha}_z \{\hat{\underline{h}}_0\}_{\underline{\tau}_B}$$

last rotation occurs around the LHLV axis.

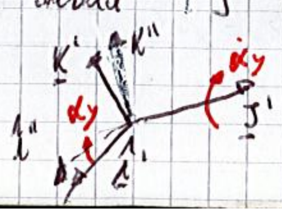
1st rotation happens around one of the inertia axis

Therefore having followed the 123 sequence.

* first rotation occurs around $\hat{\underline{i}}$ $\Rightarrow \{\hat{\underline{i}}\}_{\underline{\tau}_B} \equiv \hat{\underline{i}}$

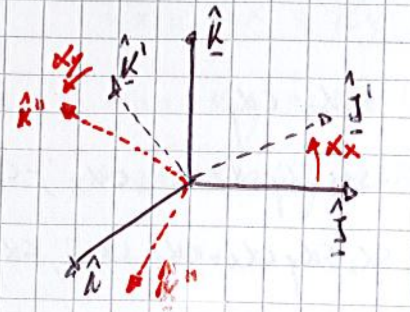
$\{\hat{\underline{i}}, \hat{\underline{j}}', \hat{\underline{k}}'\} \xrightarrow{\underline{\underline{A}}(\alpha_x)} \to PI$

* second rotation occurs around $\hat{\underline{j}}'$ $\Rightarrow \{\hat{\underline{j}}'\}_{\underline{\tau}_B} = \underline{\underline{A}}(\alpha_x)\, \hat{\underline{j}}$



* Third rotation occurs around $\hat{\underline{k}}''$ $\Rightarrow \{\hat{\underline{k}}''\}_{\underline{\tau}_B} = \underline{\underline{A}}(\alpha_y)\, \underline{\underline{A}}(\alpha_x)\, \hat{\underline{k}}$

$(\hat{\underline{k}}'' \equiv \hat{\underline{h}}_0)$



$$\Rightarrow \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} \dot{\alpha}_x \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_x & s\alpha_x \\ 0 & -s\alpha_x & c\alpha_x \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\alpha}_y \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_x & s\alpha_x \\ 0 & -s\alpha_x & c\alpha_x \end{bmatrix} \cdot \begin{bmatrix} c\alpha_y & 0 & s\alpha_y \\ 0 & 1 & 0 \\ -s\alpha_y & 0 & c\alpha_y \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\alpha}_z \end{Bmatrix}$$

therefore:

$$\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} \dot{\alpha}_x \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ c\alpha_x \dot{\alpha}_y \\ -s\alpha_x \dot{\alpha}_y \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_x & s\alpha_x \\ 0 & -s\alpha_x & c\alpha_x \end{bmatrix} \begin{Bmatrix} s\alpha_y \dot{\alpha}_z \\ 0 \\ c\alpha_y \dot{\alpha}_z \end{Bmatrix}$$

$$= \begin{Bmatrix} \dot{\alpha}_x \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ c\alpha_x \dot{\alpha}_y \\ -s\alpha_x \dot{\alpha}_y \end{Bmatrix} + \begin{Bmatrix} s\alpha_y \dot{\alpha}_z \\ s\alpha_x c\alpha_y \dot{\alpha}_z \\ c\alpha_x c\alpha_y \dot{\alpha}_z \end{Bmatrix}$$

$$\begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{Bmatrix} \dot{\alpha}_x + s\alpha_y \dot{\alpha}_z \\ c\alpha_x \dot{\alpha}_y + s\alpha_x c\alpha_y \dot{\alpha}_z \\ -s\alpha_x \dot{\alpha}_y + c\alpha_x c\alpha_y \dot{\alpha}_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & s\alpha_y \\ 0 & c\alpha_x & s\alpha_x c\alpha_y \\ 0 & -s\alpha_x & c\alpha_x c\alpha_y \end{bmatrix} \begin{Bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z \end{Bmatrix}$$

What we unconsciously obtain is the kinematic equation for 123 sequence. (BUT IN TURN THAT IS NOT P.I. ONE)

(3) → APPLICATION OF COMPOSITION OF ANGULAR VELOCITIES.

(i) $\underline{\omega}_B = \{\underline{\omega}_0\}_{\tau_B} + \{\underline{\omega}_{B0}\}_{\tau_B}$

• $$\{\underline{\omega}_0\}_{\tau_B} = \underline{\underline{A}}_{LHLV2PI} \cdot \begin{Bmatrix}0\\0\\n\end{Bmatrix} = \begin{bmatrix} c\alpha_z c\alpha_y & -s\alpha_z c\alpha_y & s\alpha_y \\ c\alpha_z s\alpha_y s\alpha_x + s\alpha_z c\alpha_x & -s\alpha_z s\alpha_y s\alpha_x + c\alpha_z c\alpha_x & -c\alpha_y s\alpha_x \\ -c\alpha_z s\alpha_y c\alpha_x + s\alpha_z s\alpha_x & s\alpha_z s\alpha_y c\alpha_x + c\alpha_z s\alpha_x & c\alpha_y c\alpha_x \end{bmatrix} \cdot \begin{Bmatrix}0\\0\\n\end{Bmatrix}$$

$$\{\underline{\omega}_0\}_{\tau_B} = \begin{Bmatrix} n\, s\alpha_y \\ -n\, c\alpha_y s\alpha_x \\ n\, c\alpha_y c\alpha_x \end{Bmatrix}$$

• $$\{\underline{\omega}_{B0}\}_{\tau_B} = \begin{Bmatrix}p\\q\\r\end{Bmatrix} = \begin{bmatrix} 1 & 0 & s\alpha_y \\ 0 & c\alpha_x & s\alpha_x c\alpha_y \\ 0 & -s\alpha_x & c\alpha_y c\alpha_x \end{bmatrix} \begin{Bmatrix}\dot\alpha_x\\ \dot\alpha_y\\ \dot\alpha_z\end{Bmatrix}$$

• $$\underline{\omega}_B = \{\underline{\omega}_0\}_{\tau_B} = \begin{Bmatrix} \dot\alpha_x + n\, s\alpha_y + \dot\alpha_z s\alpha_y \\ -n\, c\alpha_y s\alpha_x + c\alpha_x \dot\alpha_y - s\alpha_x c\alpha_y \dot\alpha_z \\ n\, c\alpha_y c\alpha_x - s\alpha_x \dot\alpha_y + c\alpha_y c\alpha_x \dot\alpha_z \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & s\alpha_y \\ 0 & c\alpha_x & -s\alpha_x c\alpha_y \\ 0 & -s\alpha_x & c\alpha_y c\alpha_x \end{bmatrix} \begin{Bmatrix}\dot\alpha_x\\ \dot\alpha_y\\ \dot\alpha_z + n\end{Bmatrix}.$$

(4) → LINEARIZING SUCH EQUATIONS ⇒ $\begin{cases} c(\alpha_i) \simeq 1 \\ s(\alpha_i) \simeq \alpha_i \end{cases}$

$$\begin{cases} \omega_x = \dot\alpha_x + \alpha_y \cdot (\dot\alpha_z + n) \\ \omega_y = \dot\alpha_y - \alpha_x (\dot\alpha_z + n) \\ \omega_z = -\alpha_x \dot\alpha_y + \dot\alpha_z + n. \end{cases}$$

(5) → NEGLECTING II^{nd} ORDER TERMS: $\dot\alpha\alpha \simeq 0$

$$\begin{cases} \omega_x = \dot\alpha_x + \alpha_y \cdot n \\ \omega_y = \dot\alpha_y - \alpha_x \cdot n \\ \omega_z = \dot\alpha_z + n \end{cases} \Leftrightarrow \begin{cases} \omega_x = \dot\alpha_x - \alpha_y n \\ \omega_y = \dot\alpha_y + \alpha_x \cdot n \\ \omega_z = \dot\alpha_z + n \end{cases}$$

Simply reflect a different choice in axis direction.

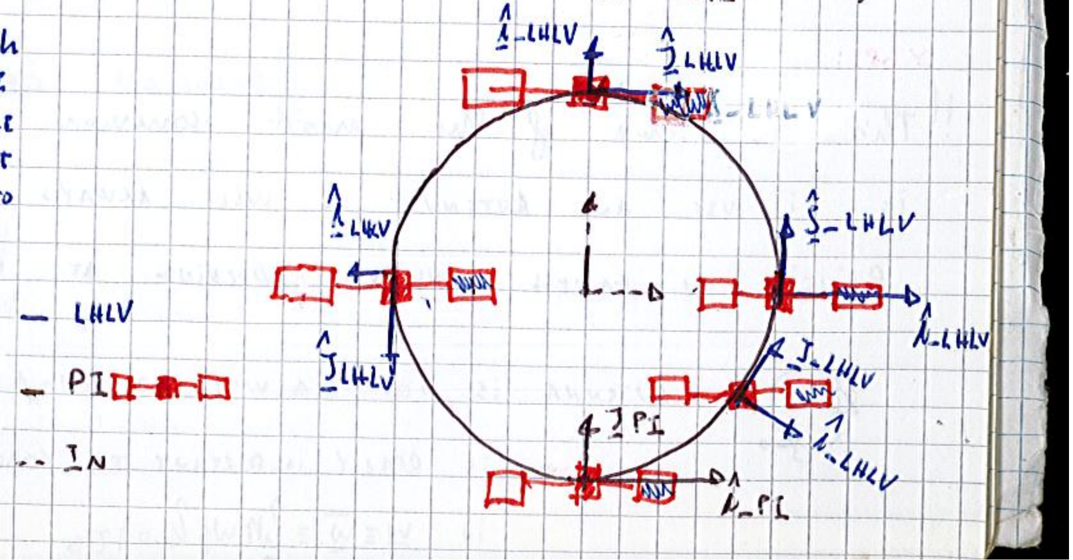
AIM: OUR AIM IS TO EVALUATE THE STABILITY CONDITION AT WICH THE PRINCIPAL INERTIAL FRAME (PI) AND THE LOCAL-VERTICAL LOCAL-HORIZONTAL FRAME (LHLV) ARE COINCIDENT AT ANY TIME.

⇒ THIS IS VERIFIED IF AND ONLY IF THE SATELLITE IS ROTATING AT ANGULAR VELOCITY n WITH RESPECT TO THE INERTIAL REFERENCE FRAME.

In a 2D Example: (⇒ looking only at the last equation $\forall t \;\; \underline{n}_0 \equiv \hat{k}_{PI}$.)

a) $\omega_z = 0 \Rightarrow \dot\alpha_z = -n$ ≃ telescope.

LHLV IS ROTATING WITH $\dot\alpha_z = n$ WITH RESPECT TO THE PI REFERENCE FRAME WHILE THE SATELLITE IS NOT MOVING WITH RESPECT TO THE INERTIAL R.F.

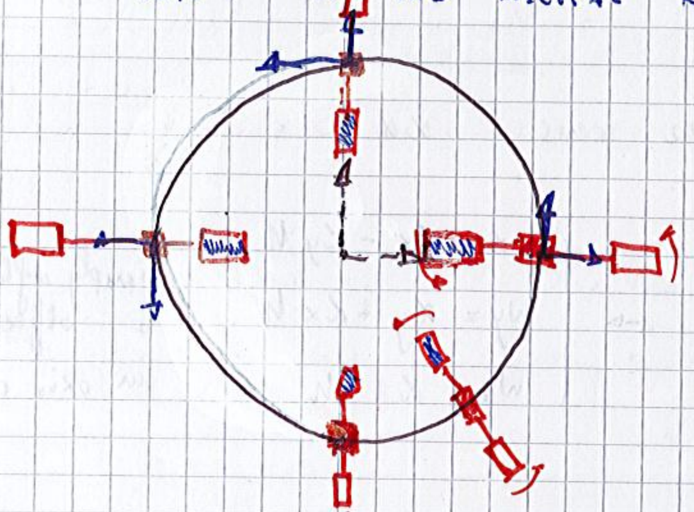


while:

b) $\dot{\alpha}_z = 0 \Rightarrow W_z = n$ (Supposing $\alpha_x = \alpha_y = 0$) ≃ GPS.

($\underline{W}_{BO}$ → no relative motion between the LHLV r.f and the PI r.f)

!! THIS OCCURS IF AND ONLY IF THE SATELLITE IS ROTATING AT THE SAME ANGULAR VELOCITY OF THE LHLV R.F WITH RESPECT TO THE INERTIAL R.F. !!



Therefore our stability condition will be: $\begin{vmatrix} \bar{W}_x \\ \bar{W}_y \\ \bar{W}_z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ n \end{vmatrix}$

that essentially mean: $\begin{vmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z \end{vmatrix} = \underline{0}$!! no relative motion between LHLV ⟷ PI !!

" Therefore analizing this special condition of a SINGLE SPIN SATELLITE with an angular velocity

$\{\underline{W}_B\}_{\tau_B} = \{\underline{W}_O\}_{\tau_B}$ ⇒ Eq: $\underline{W} = \begin{Bmatrix} -\alpha_y \cdot n \\ \alpha_x \cdot n \\ n \end{Bmatrix}$

≃ GPS:

!! This is one of the most common application !!

IF I USE AN ANTENNA I WILL ALWAYS HAVE IT POINTING ON EARTH WHILE WORKING AT CONDITION (a)

ANTENNA IS NOT ALWAYS POINTING AT EARTH.

⇒ is REALLY IMPORTANT TO EVALUATE STABILITY IN CONDITION IN $\underline{W} \equiv \overline{\underline{W}} = \{n \cdot \{\hat{h}_O\}_{\tau_B}$

- - → about n-term

* → CIRCULAR ORBIT: $\underline{R} = R\,\hat{\underline{R}}$; $\underline{V} = n \cdot R\,\hat{\underline{\theta}}$

$\underline{h}_O = \underline{R} \times \underline{V} = nR^2\,\hat{\underline{h}}_O \Rightarrow n = \frac{\|\underline{h}_O\|}{R^2} = \text{const}|_t$.

* → GENERIC ORBIT: $\underline{R} = R(t)\,\hat{\underline{R}}$; $\underline{V} = \dot{R}\,\hat{\underline{R}} + R\dot{\theta}\,\hat{\underline{\theta}}$

$(\dot{\theta} \Leftrightarrow n(t)) \Rightarrow \underline{h}_O = R^2 \cdot n\,\hat{\underline{h}}_O \Rightarrow n(t) = \frac{h_O}{R(t)}$

Therefore in order to maintain the satellite always pointing an Earth i should follow a sort of: "ANGULAR VELOCITY TIME LAW $n(t)$" around the vector $\underline{h}_O$

(b) → STABILITY EVALUATION WITH RESPECT TO ATTITUTE PARAMETER.

$$\begin{cases} W_x = \dot{\alpha}_x - \alpha_y \cdot n \\ W_y = \dot{\alpha}_y + \alpha_x \cdot n \\ W_z = \dot{\alpha}_z + n. \end{cases}$$

* (linearized ⊕ neglecting II[nd] order term)

Eq $\begin{Bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\alpha}_z \end{Bmatrix} = \underline{0}$; $\begin{Bmatrix} \alpha_x \\ \alpha_y \end{Bmatrix} = \underline{0}$ initial condition of coincidence between LHLV and PI reference frames.

Hp: circular orbit ⇒ $n = \text{const}|_t$

$$\Rightarrow \begin{cases} \dot{W}_x = \ddot{\alpha}_x - n\,\dot{\alpha}_y \\ \dot{W}_y = \ddot{\alpha}_y + n\,\dot{\alpha}_x \\ \dot{W}_z = \ddot{\alpha}_z. \end{cases}$$

↳ Substituting such result into Euler's equation

$$\begin{cases} \dot{w}_x = \ddot{\alpha}_x - \dot{\alpha}_y n \\ \dot{w}_y = \ddot{\alpha}_y + \dot{\alpha}_x n \\ \dot{w}_z = \ddot{\alpha}_z \end{cases} ; \quad \begin{cases} I_x \dot{w}_x + (I_z - I_y) w_y w_z = 0 \\ I_y \dot{w}_y + (I_x - I_z) w_x w_z = 0 \\ I_z \dot{w}_z + (I_x - I_y) w_x w_y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} I_x(\ddot{\alpha}_x - \dot{\alpha}_y n) + (I_z - I_y)[\dot{\alpha}_y + \alpha_x n][\dot{\alpha}_z + n] = 0 \\ I_y(\ddot{\alpha}_y + \dot{\alpha}_x n) + (I_x - I_z)[\dot{\alpha}_x - \alpha_y n][\dot{\alpha}_z + n] = 0 \\ I_z \ddot{\alpha}_z + (I_x - I_y)[\dot{\alpha}_x - \alpha_y n][\dot{\alpha}_y + \alpha_x n] = 0 \end{cases}$$

$$\begin{cases} I_x \ddot{\alpha}_x - I_x \dot{\alpha}_y n + (I_z - I_y)[\underbrace{\dot{\alpha}_y \dot{\alpha}_z}_{\to 0} + n\dot{\alpha}_y + \underbrace{\alpha_x \dot{\alpha}_z n}_{\to 0} + \alpha_x n^2] \\ I_y \ddot{\alpha}_y + I_y \dot{\alpha}_x n + (I_x - I_z)[\underbrace{\dot{\alpha}_x \dot{\alpha}_z}_{\to 0} + n\dot{\alpha}_x - \underbrace{\dot{\alpha}_z \alpha_y n}_{\to 0} - \alpha_y n^2] = 0 \\ I_z \ddot{\alpha}_z + (I_x - I_y)[\underbrace{\dot{\alpha}_x \dot{\alpha}_y}_{\to 0} + \underbrace{\dot{\alpha}_x \alpha_x n}_{\to 0} - \underbrace{\alpha_y \dot{\alpha}_y n}_{\to 0} - \underbrace{\alpha_y \alpha_x n^2}_{\to 0}] = 0 \end{cases}$$

↳ Neglecting IInd order terms

* I'm linearizing around equilibrium condition $\begin{Bmatrix} \alpha_x \\ \vdots \end{Bmatrix} = \underline{0}$ therefore writing here $\alpha_x = \delta(\alpha_x)$ means to write a null perturbation

* analogous condition can be set for $\begin{Bmatrix} \dot{\alpha}_x \\ \dot{\alpha}_y \end{Bmatrix} = \underline{0}$. therefore i simply have the satellite rotating around $\hat{z} \equiv \hat{u}_0$.

$$\Downarrow$$

$$\{w\}_{\mathcal{R}_B} = n \cdot \hat{k}.$$

$$\begin{cases} I_x \ddot{\alpha}_x - I_x \dot{\alpha}_y n + (I_z - I_y)[n\dot{\alpha}_y + \alpha_x n^2] \\ I_y \ddot{\alpha}_y + I_y \dot{\alpha}_x n + (I_x - I_z)[n\dot{\alpha}_x - \alpha_y n^2] \\ I_z \ddot{\alpha}_z = 0 \end{cases}$$

$$\begin{cases} I_x \ddot{\alpha}_x + (I_z - I_y - I_x) n \dot{\alpha}_y + (I_z - I_y) n^2 \alpha_x = 0 \\ I_y \ddot{\alpha}_y + (I_y + I_x - I_z) n \dot{\alpha}_x + (I_x - I_z) n^2 \alpha_y = 0 \\ I_z \ddot{\alpha}_z = 0 \end{cases}$$

↳ Divide $\begin{cases} I^{st} \text{ equation for } I_x \\ II^{nd} \text{ equation for } I_y \end{cases} \Rightarrow \begin{cases} K_x \triangleq \dfrac{I_z - I_y}{I_x} \\ K_y \triangleq \dfrac{I_z - I_x}{I_y} \end{cases}$

in this way $(K_x, K_y) > 0$

$$\Updownarrow$$

$I_z > (I_x, I_y)$.

therefore:

$$\begin{cases} \ddot{\alpha}_x + (K_x - 1) n \dot{\alpha}_y + K_x n^2 \alpha_x = 0 \\ \ddot{\alpha}_y + (1 - K_y) n \dot{\alpha}_x + K_y n^2 \alpha_y = 0 \\ \ddot{\alpha}_z = 0 \end{cases}$$

↳ Going to Laplace domain:

$$\begin{cases} s^2 \alpha_x(s) + (K_x - 1) n s \alpha_y(s) + K_x n^2 \alpha_x(s) = 0 \\ s^2 \alpha_y(s) + (1 - K_y) n s \alpha_x(s) + K_y n^2 \alpha_y(s) = 0 \end{cases}$$

stability will only be evaluated for the first 2 equations since the third is decoupled from the other two.

$$\begin{bmatrix} s^2 + K_x n^2 ; & (K_x - 1) n s \\ (1 - K_y) n s ; & s^2 + K_y n^2 \end{bmatrix} \begin{Bmatrix} \alpha_x(s) \\ \alpha_y(s) \end{Bmatrix} = \underline{0}$$

STABILITY GUARANTEED $\Leftrightarrow$ $\det(\underline{A}(s)) = \prod_i (s - \lambda_i) \Rightarrow \forall i \; \mathrm{Re}(\lambda_i) < 0$.

$$\det(\underline{A}(s)) = \det\left(\begin{bmatrix} s^2+K_x n^2 & (K_x-1)n\,s \\ (1-K_y)n\,s & s^2+K_y n^2 \end{bmatrix}\right) = 0$$

$$(s^2+K_x n^2)\cdot(s^2+K_y n^2) - (K_x-1)(1-K_y)\,n^2 s^2 = 0$$

$$s^4 + (K_x n^2 + K_y n^2)\,s^2 + K_x K_y n^4 - (K_x - K_x K_y - 1 + K_y)\,n^2 s^2 = 0.$$

$$s^4 + [\cancel{K_x} + \cancel{K_y} - \cancel{K_x} + K_x K_y + 1 - \cancel{K_y}]\,n^2 s^2 + K_x K_y n^4 = 0.$$

$$s^4 + [1 + K_x K_y]\,n^2 s^2 + K_x K_y n^4 = 0$$

$$s^4 + b s^2 + c \;\rightarrow\; s^2_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2}$$

IF $s^2_{1,2} > 0 \rightarrow \exists\; s_1$ or $s_2 > 0 \Rightarrow$ unstable system.

$s^2_{12} \in \mathbb{C} \rightarrow$ e.g. $s^2_{12} = a + jb \Rightarrow s_1 = a_1,\; s_2 = a_2 + jb_2$ $\begin{cases} a = a_1 a_2 \\ b = a_1 b_2 \end{cases}$

$\Rightarrow a_1$ or $a_2 > 0$ TO HAVE $Re<0$ (STABLE ...)

ELSE IF $s^2_{12} \leq 0 \rightarrow (s_1, s_2) \in Im \Rightarrow$ system is (SIMPLY) stable.

This is verified if and only if: $\begin{cases} \sqrt{b^2-4ac} > 0 & (\text{IF NOT } s^2_{12} \in \mathbb{C}) \\ b > \sqrt{b^2-4ac} \end{cases}$

$\Downarrow$

STABILITY $\Leftrightarrow$ $\begin{cases} [(1+K_xK_y)^2 - 4K_xK_y]\,n^4 > 0 \\ -(1+K_xK_y)\,n^2 + \sqrt{[(1+K_xK_y)^2 - 4K_xK_y]\,n^4} \leq 0. \end{cases}$

other condition (WITH THE OPPOSITE SIGN) is: $-(1+K_xK_y) - \sqrt{\cdots}$ is satisfied $\forall (K_x K_y)$

$$\begin{cases} (1+K_xK_y)^2 - 4K_xK_y > 0 \\ (1+K_xK_y) \geq \sqrt{(1+K_xK_y)^2 - 4K_xK_y} \end{cases}$$

$$\begin{cases} 1 + 2K_xK_y + (K_xK_y)^2 - 4K_xK_y = (K_xK_y)^2 - 2K_xK_y + 1 > 0 \\ (1+K_xK_y) > \sqrt{(K_xK_y)^2 - 2K_xK_y + 1} \end{cases}$$

$$(K_xK_y)^2 - 2(K_xK_y) + 1 = (K_xK_y - 1)^2$$

$$\Rightarrow \sqrt{(K_xK_y)^2 - 2K_xK_y + 1} = \pm(K_xK_y - 1)$$

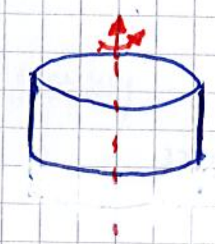
$\begin{cases} (K_xK_y - 1)^2 > 0 \rightsquigarrow \text{satisfied } \forall (K_xK_y) \\ 1 + K_xK_y > K_xK_y - 1 \rightsquigarrow \text{satisfied } \forall (K_xK_y) \\ \cancel{1} + K_xK_y > \cancel{1} - K_xK_y \end{cases}$

$\Rightarrow$ STABILITY CONDITION.

$$K_xK_y > 0 \;;\; \begin{cases} K_x = \dfrac{I_z - I_y}{I_x} \\ K_y = \dfrac{I_z - I_x}{I_y} \end{cases}$$

$\Downarrow$

a) $\begin{cases} I_z > I_y \\ I_z > I_x \end{cases}$



b) $\begin{cases} I_z < I_y \\ I_z < I_x \end{cases}$

[THE RESULT JUST ACHIEVED SEEMS TO BE THE SAME OBTAINED SIMPLY WORKING ON EULER'S EQUATIONS: (Eq: $\dot{\underline{W}} = 0 \rightarrow$ LIN $\rightarrow \frac{d}{dt} \rightarrow 2 \rightarrow$ ROOTS) BUT NOW WE ARE DEALING WITH THE STABILITY OF A SPACECRAFT WITH $\underline{W} = n\{\underline{u}_0\}_{\tau_B}$ AND WITH SMALL PERTURBATIONS AROUND SUCH EQUILIBRIUM CONDITION.

(P.I. reference frame $\equiv$ LHLV reference frame.)]

[ONLY FOR THIS REALLY SPECIFIC CASE ($\underline{w}_z = \{\underline{h}_0\} r_B$)

THE 2 CONDITIONS $\begin{cases} I_z > (I_x, I_y) \\ I_z < (I_x, I_y) \end{cases}$ GUARANTEE STABILITY WITH RESPECT TO ATTITUDE PARAMETERS

IN OTHER WORDS : IF $I_z > (I_x, I_y)$ OR $I_z < (I_x, I_y)$

AND IF $\hat{k} = \{\hat{h}\}$

$\hat{z}$ axis is aligned with h_0

Spacecraft is rotating with the axis of $\begin{cases} \text{MINIMUM} \\ \text{MAXIMUM} \end{cases}$ inertia and a direction parallel to the angular momentum of the orbit itself.

THEN THE SYSTEM IS STABLE.]

[SINCE $s^2_{12} = n^2 \dfrac{-(1+K_xK_y) \pm \sqrt{(1-K_xK_y)^2}}{2} = -\dfrac{n^2}{2}[K_x K_y]$

only $\ominus$ condition is meaning-full.

THEN stability is independent from SINGLE SPIN ANGULAR VELOCITY ($n = \overline{w}_z$)

BUT IF $\|\underline{w}\| = n$

THEN AT ANY TIME (WITH A SMALL DEVIATION FROM NOMINAL CONDITION) THE LHLV r.f. IS COINCIDENT WITH THE PI r.f.

$\Rightarrow K_x = K_{yaw}$

$K_y = K_{roll}$ *]

* $K_x = \dfrac{I_z - I_y}{I_x}$; $K_y = \dfrac{I_z - I_x}{I_y}$ → this will be relevant once the perturbations effect will depend on the relative position between the PI and the {pitch, yaw, roll}

$K_r \hat{=} \dfrac{I_h - I_{yaw}}{I_{roll}}$; $K_{yaw} = \dfrac{I_h - I_z}{}$

in this configuration, $\underline{w} = w\{\hat{h}_0\} r_B$

$\left\{\begin{matrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{matrix}\right\} = 0$

$\Rightarrow K_x \equiv K_{yaw}$

$K_y \equiv K_r$

obs. If the nominal angular velocity (n) is small.

$\Rightarrow$ POLES $= \pm j\sqrt{n^2/2\,[K_x K_y]}$

this means that the oscillations around equilibrium condition will occur at low frequency

→ " Stability of simple spin satellite under kinetic energy losses "

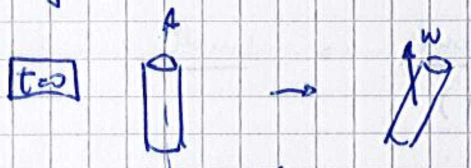
Hp: $(2\dot{T}^{rot}) < 0$

$\|\underline{h}\| = const|_t$

This is possible if $\{w_x, w_y, w_z\}$ satisfy: $I_x^2 w_x^2 + I_y^2 w_y^2 + I_z^2 w_z^2 = h^2$

but not $2T = I_x w_x^2 + I_y w_y^2 + I_z w_z^2$.

therefore the satellite that at t=0 have



w_z will be distributed in a more conservative way for kinetic energy.

⇒ We evaluate the 2 stable conditions for the single spin satellite:

$2T_{I_z max} = I_{z max} w_{z max}^2$ ← $I_z = I_{max}$

$2T_{I_z min} = I_{z min} w_{z min}^2$ ← $I_z = I_{min}$

!! We are dealing with 2 spacecrafts single spin stabilized with different kinetic energy but with the same angular momentum. * !!

a)

$2T_{I_z max} = I_{z max} w_{z max}^2$

b)

$2T_{I_z min} = I_{z min} w_{z min}^2$

* that would be the same of analyzing a spacecraft rotating around 2 of his inertia axis: THE MINIMUM AND THE MAXIMUM ONES.

$\|\underline{h}\| = I_{z max} w_{z max} = I_{z min} w_{z min}$.

$$\Rightarrow w_{z max} = \frac{I_{z min}}{I_{z max}} w_{z min}$$

($w_{z max} < w_{z min}$)

therefore:

$$2T_{I_z max} = I_{z max} w_{z max}^2 = I_{z max}\left(\frac{I_{z min}}{I_{z max}}\right)^2 w_{z min}^2$$

$$2T_{I_z max} = \frac{I_{z min}}{I_{z max}} \cdot \underbrace{\left(I_{z min} w_{z min}^2\right)}_{2T_{I_z min}} = \frac{I_{z min}}{I_{z max}}\, 2T_{I_z min}$$

($2T_{I_z max} < 2T_{I_z min}$)

⇒ [IN CASE OF ENERGY LOSSES, THE SATELLITE WILL TEND TO THE MOTION CONDITION THAT HAS THE MINIMUM ENERGY

⇓

WILL TEND ROTATING AROUND THE MAXIMUM INERTIA AXIS (I_{max})

⇓

THE ONLY STABLE CONDITION FOR A SATELLITE IS TO ROTATE AROUND ONE OF THE PRINCIPAL INERTIA AXIS (this is a requirement for equilibrium) AND SUCH AXIS MUST BE THE MAXIMUM ONE (if also kinetic energy loss is taken into account)]

IF the spacecraft is symmetric ⇒ $I_x = I_y$

THEN $\forall t$ the $\underline{w}$ vector can be represented only by 2 components ⇒ $\underline{w} = w_z \hat{k} + w_n \hat{n}$ ($\hat{n} = \hat{n}(t)$)

Saw before:

$$\begin{cases} I_y \dot w_y + (I_x - I_z) w_x w_z = 0 \\ I_x \dot w_x + (I_z - I_y) w_z w_y = 0 \\ I_z \dot w_z + (I_x - I_y) w_x w_y = 0 \end{cases} \xrightarrow{I_x = I_y} \begin{cases} I_y \dot w_y + (I_x - I_z) \bar w_z w_x = 0 \\ I_x \dot w_x + (I_z - I_y) \bar w_z w_y = 0 \\ I_z \dot w_z = 0 \Rightarrow w_z = \bar w_z \end{cases}$$

$$\xrightarrow{I_x = I_y = I_n} \begin{cases} I_n \dot w_y + (I_n - I_z) \bar w_z w_x = 0 \\ I_n \dot w_x + (I_n - I_z) \bar w_z w_y = 0 \end{cases} \xrightarrow{\lambda \triangleq \frac{I_n - I_z}{I_n} \cdot \bar w_z} \begin{cases} \dot w_y - \lambda w_x = 0 \\ \dot w_x + \lambda w_y = 0 \end{cases}$$

$$\Rightarrow \dot w_y + w_x \left(\frac{\dot w_x}{w_y} \right) = 0 \rightarrow \dot w_y w_y + w_x \dot w_x = 0$$

$$\frac{d}{dt}\left(w_y^2 + w_x^2 \right) = 0 \Rightarrow \|w_h\| = \text{const}.$$

Anyway:

$$\begin{cases} \dot w_x + \lambda w_y = 0 \\ \dot w_y - \lambda w_x = 0 \end{cases} \xrightarrow{d/dt} \ddot w_x + \lambda \dot w_y = 0 \Rightarrow \ddot w_x + \lambda^2 w_x = 0$$

$$\Rightarrow w_x(t) = w_{x,0} \cos(\lambda t) + \frac{\dot w_{x,0}}{\lambda} \sin(\lambda t)$$

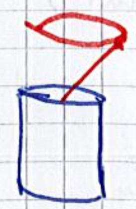
$$w_x(t) = w_{x,0} \cos(\lambda t) + \frac{\dot w_{x,0}}{\lambda} \sin(\lambda t)$$

$$w_y = -\frac{1}{\lambda} \dot w_x \Rightarrow w_y(t) = -w_{x,0} \sin(\lambda t) + w_{y,0} \cos(\lambda t)$$

$$\left(\frac{\dot w_{x,0}}{\lambda} = w_{y,0} \right)$$

$$\Rightarrow \hat h = \hat h(t) = \frac{(w_{x,0} - w_{y,0})}{\sqrt{w_{y,0}^2 + w_{x,0}^2}} \left[\cos(\lambda t) + \sin(\lambda t) \right]$$

In fact a cone is described



Analytical results are hereby presented for such situation of $I_x = I_y$.

⇒ $\forall t$:

• $h^2 = I_z^2 w_z^2 + I_n^2 w_n^2$

$$\frac{d(h^2)}{dt} = 2 I_z^2 w_z \dot w_z + 2 I_n^2 w_n \dot w_n = 0$$

$$\Rightarrow w_n \dot w_n = -\frac{I_z^2}{I_n^2} w_z \dot w_z \quad *$$

• $2\dot T < 0$

$$2T = I_z w_z^2 + I_n w_n^2$$

$$\frac{d(2T)}{dt} = 2 I_z w_z \dot w_z + 2 I_n w_n \dot w_n < 0$$

* I cannot say anymore that $\dot w_z = 0$ since we have a kinetic energy loss.

$$\Rightarrow 2\dot T = 2 I_z w_z \dot w_z + 2 I_n \left(-\frac{I_z^2}{I_n^2} w_z \dot w_z \right) < 0$$

$$w_z \dot w_z \left[I_z \left(1 - \frac{I_z}{I_n} \right) \right] < 0 \quad (i)$$

b) $I_z = I_{min} \Rightarrow I_z < I_n$

$$(i) \; w_z \dot w_z \underbrace{\left[\frac{I_z}{I_n} (I_n - I_z) \right]}_{\text{positive quantity}} < 0 \Rightarrow w_z \dot w_z < 0$$

this inequalities is satisfied for:

$$\begin{cases} w_z > 0 \; ; \; \dot w_z < 0 \\ \boxed{w_z < 0 \; ; \; \dot w_z > 0} \end{cases}$$

In both cases w_z decrease in amplitude and w_n increases

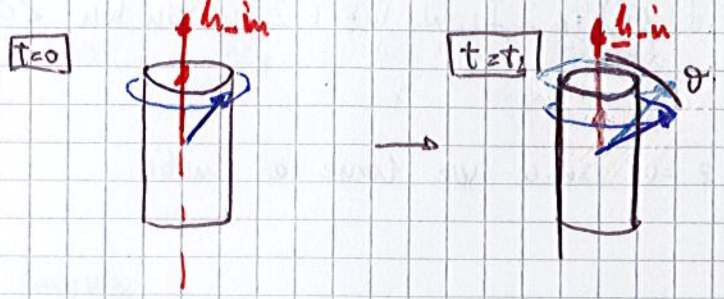
$\boxed{w_z < 0 \;\; \dot w_z > 0}$ in fact $\underbrace{w_z \dot w_z}_{(<0)} = -\frac{I_z^2}{I_n^2} \underbrace{w_n \dot w_n}_{(>0)}$

$$\Rightarrow w_n \dot w_n > 0 \Rightarrow w_n \dot w_n > 0 \Rightarrow \begin{cases} \dot w_n > 0 \; , \; w_n > 0 \\ \dot w_n < 0 \; ; \; w_n < 0 \end{cases}$$

!! in both cases w_n is increasing in his direction of rotation !!

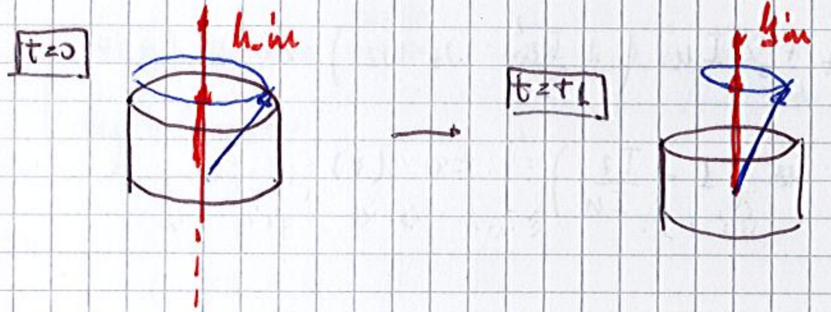
(a) $I_z \neq I_{max} \Rightarrow I_z > I_{max}$

(i) $\omega_z \dot{\omega}_z \underbrace{\left[\frac{I_z}{I_n} (I_n - I_z) \right]}_{\text{negative quantity}} < 0 \Rightarrow \omega_z \dot{\omega}_z > 0$



$\Rightarrow \tan\vartheta = \frac{I_n \omega_n}{I_z \omega_z}$

ϑ increase



ϑ decrease.

$$\omega_n \dot{\omega}_n = -\left(\frac{I_z}{I_n}\right)^2 \underbrace{\omega_z \dot{\omega}_z}_{>0} \Rightarrow \dot{\omega}_n < 0$$

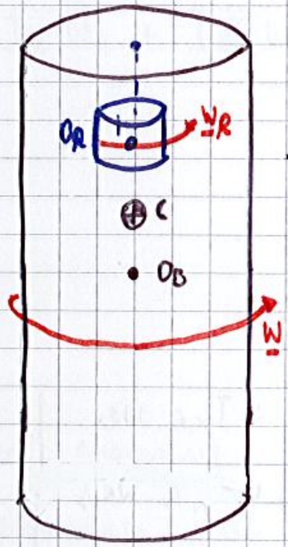
→ "Dual-spin satellite stabilization"

" Single spin satellites have the typical limitation that they cannot be used for an efficient communication earth-satellite because antenna cannot be pointed on earth. (WITH ARBITRARY SPIN VELOCITY) "

except for the situation where

$\{\omega\} = n \hat{k}$

$\hat{h} = \hat{k}$; $n \rightarrow$ orbital angular velocity



$h^B_{O_B} = \underline{\underline{I}}_B \cdot \omega_B$; $h^R_{O_R} = \underline{\underline{I}}_R \cdot \omega_R$

Angular momentum of the body with respect to O_B → CENTER OF MASS OF THE BODY

Angular momentum of the rotor with respect to O_R, O_R → CENTER OF MASS OF THE ROTOR.

$H_C = h^B_C + h^R_C$

Angular moment of the system BODY + ROTOR with respect to C → CENTER OF MASS OF THE SYSTEM BODY + ROTOR.

→ Angular momentum transport formula.

$$h_P = h_O + m\,(O - C) \times v_P$$

$$h^B_C = h^B_{O_B} + m_B\,(O_B - C) \times v_C$$

$$h^R_C = h^R_{O_C} + m_R\,(O_C - C) \times v_C$$

$\Rightarrow \begin{cases} \frac{dH_C}{dt} = M \\ H_C = h^B_C + h^R_C \end{cases}$ represents the IInd cardinal equation for such configuration.

Therefor for a <u>torque free motion</u>:

$$\frac{d\underline{H}_c}{dt}=\underline{0} \;\Rightarrow\; \frac{d}{dt}\left[\underline{h}^B_{OB} + m_B(O_B-C)\times \underline{v}_c + \underline{h}^R_{OR} + m_R(O_R-C)\times\underline{v}_c\right] = \underline{0}$$

$$\frac{d}{dt}\left(\underline{h}^B_{OO} + \underline{h}^R_{OC}\right) + \frac{d}{dt}\left[m_B(O_B-C)\times \underline{v}_c + m_R(O_R-C)\times\underline{v}_c\right]=\underline{0}$$

$$\frac{d}{dt}\left(\underline{h}^B_{OB}+\underline{h}^R_{OC}\right) + \frac{d}{dt}\left[M\, \frac{m_B(O_B-C)+m_R(O_R-C)}{M} \times \underline{v}_c\right]=0$$

Thanks to the definition of the center of mass

$$(C-O) = \frac{1}{M}\int_M (P-O)\, dm$$

$$\underline{0} = (C-C) = \frac{1}{M}\int_M (P-C)\,dm$$

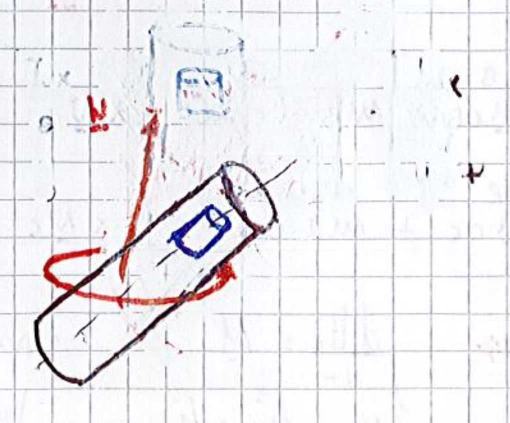
$$\Rightarrow m_B(O_B-C)+m_R(O_R-C)=\underline{0}$$

$$\frac{d}{dt}\left(\underline{h}^B_{OO}+\underline{h}^R_{OC}\right)=\underline{0} \;\Rightarrow\; \frac{d}{dt}\begin{Bmatrix} I_x w_x + I_{x,R}\, w_{Rx} \\ I_y w_y + I_{y,R}\, w_{R,y} \\ I_z w_z + I_{z,R}\, w_{R,z}\end{Bmatrix} = \underline{0}$$

$$\underline{h}_{OB} = \underline{\underline{I}}_B\, \underline{w}_B \;;\; \underline{h}_{OR} = \underline{\underline{I}}_R\, \underline{w}_R$$

Assuming (AS SKETCHED) that the rotor is rotating around z-axis of the body (z axis aligned for $\{R, B\}$)

$$\Rightarrow \underline{H} = \begin{Bmatrix} I_x w_x \\ I_y w_y \\ I_z w_z + I_R w_R \end{Bmatrix}$$



Therefore in this configuration Euler's equations become:

$$\frac{d\underline{H}}{dt} = \frac{dH}{dt}\,\hat{\underline{H}} + H\,\frac{d\hat{\underline{H}}}{dt} = \frac{dH}{dt}\,\hat{\underline{H}} + H\, \underline{w}\times\hat{\underline{H}} = \underline{M}$$

$$\begin{Bmatrix} I_x \dot{w}_x \\ I_y \dot{w}_y \\ I_z \dot{w}_z + I_R \dot{w}_R \end{Bmatrix}\hat{\underline{H}} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_x & w_y & w_z \\ I_x w_x & I_y w_y & (I_z w_z + I_R w_R)\end{vmatrix} = \underline{M}$$

$$\begin{cases} I_x \dot{w}_x + (I_z - I_y) w_z w_y + I_R w_R w_y = M_x \\ I_y \dot{w}_y + (I_x - I_z) w_x w_z - I_R w_R w_x = M_y \\ I_z \dot{w}_z + \cancel{I_R \dot{w}_R} + (I_y - I_x) w_y w_x = M_z \end{cases} \rightarrow \text{External torques}$$

$$+ I_R \dot{w}_R = \underbrace{M_R}$$

→ Internal torque.

RELATIVE TORQUE BETWEEN THE BODY AND THE ROTOR.

→ In evaluating the stability of such system the same procedure will be followed.

(1) linearization and equilibrium condition.

$$\text{Eq}: \begin{Bmatrix} \dot{w}_x \\ \dot{w}_y \\ \dot{w}_z \\ \dot{w}_R \end{Bmatrix} = \underline{0}$$

$$\Rightarrow \begin{cases} (I_z - I_y)\, w_z w_y + I_R w_R w_y = 0 \\ (I_x - I_z)\, w_x w_z - I_R w_R w_x = 0 \\ (I_y - I_x)\, w_y w_x = 0 \end{cases} \Rightarrow \underline{1 \text{ only possible solution}}$$

$$\text{Eq}\; \begin{aligned} & w_x = w_y = 0 \\ & w_z = \bar{w}_z \\ & w_R = \bar{w}_R . \end{aligned}$$

$$\Rightarrow \begin{cases} \omega_x \simeq 0 + \delta\omega_x \\ \omega_y \simeq 0 + \delta\omega_y \\ \omega_z \simeq \bar\omega_z + \delta\omega_z \\ \omega_R \simeq \bar\omega_R + \delta\omega_R \end{cases}$$

$$\Rightarrow \begin{cases} I_x\dot\omega_x + (I_z - I_y)[\bar\omega_z\delta\omega_y + \underbrace{\delta\omega_z\delta\omega_y}_{\to 0}] + I_R[\bar\omega_R\delta\omega_y + \underbrace{\delta\omega_R\delta\omega_y}_{\to 0}] = 0 \\ I_y\dot\omega_y + (I_x - I_z)[\bar\omega_z\delta\omega_x + \underbrace{\delta\omega_z\delta\omega_x}_{\to 0}] - I_R[\bar\omega_R\delta\omega_x + \underbrace{\delta\omega_R\delta\omega_x}_{\to 0}] = 0 \\ I_z\dot\omega_z + (I_y - I_z)[\underbrace{\delta\omega_x\delta\omega_y}_{\to 0}] + I_R\dot\omega_R = 0 \\ + \quad I_R\dot\omega_R = 0 \end{cases}$$

Therefore the linearized equations will become:

$$\begin{cases} I_x\dot\omega_x + (I_z - I_y)\bar\omega_z\omega_y + I_R\bar\omega_R\omega_y = 0 \\ I_y\dot\omega_y + (I_x - I_z)\bar\omega_z\omega_x - I_R\bar\omega_R\omega_x = 0 \\ I_z\dot\omega_z + I_R\dot\omega_R = 0 \\ I_R\dot\omega_R = 0 \end{cases}$$

(2) Obtain IInd order equation from 2 of the Ist

DEFINING:

$$\lambda_x \triangleq \frac{(I_z - I_y)\bar\omega_z + I_R\bar\omega_R}{I_x}$$

$$\lambda_y \triangleq \frac{(I_z - I_x)\bar\omega_z + I_R\bar\omega_R}{I_y}$$

$$\Rightarrow \begin{cases} \dot\omega_x + \lambda_x\omega_y = 0 \\ \dot\omega_y - \lambda_y\omega_x = 0 \\ I_z\dot\omega_z + I_R\dot\omega_R = 0 \\ I_R\dot\omega_R = 0 \end{cases}$$

$$\begin{cases} \dot\omega_x + \lambda_x\omega_y = 0 \xrightarrow{d/dt} \ddot\omega_x + \lambda_x\dot\omega_y = 0 \\ \dot\omega_y - \lambda_y\omega_x = 0 \longrightarrow \dot\omega_y = +\lambda_y\omega_x \\ I_z\dot\omega_z + I_R\dot\omega_R = 0 \\ \dot\omega_R = 0 \end{cases} \Rightarrow \ddot\omega_x + \lambda_x\lambda_y\omega_x = 0$$

(3) Laplace domain and roots computation

$$\ddot\omega_x + \lambda_x\lambda_y\omega_x = 0 \xrightarrow{\mathcal{L}} \omega_x s^2 + \lambda_x\lambda_y\omega_x = 0$$

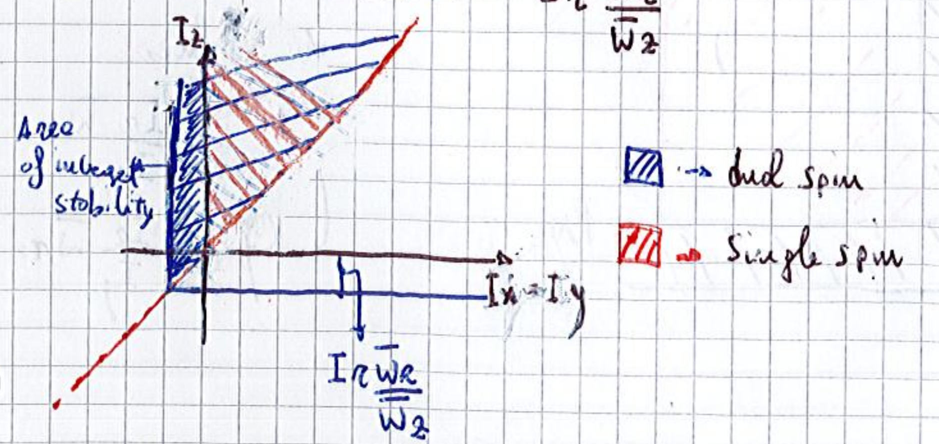
STABILITY CONDITION: $\lambda_x\lambda_y > 0$ → (a) $\begin{cases}\lambda_x > 0 \\ \lambda_y > 0\end{cases}$ → (b) $\begin{cases}\lambda_x < 0 \\ \lambda_y < 0\end{cases}$

↳ SUPPOSING $\begin{Bmatrix}\bar\omega_z \\ \bar\omega_R\end{Bmatrix}$ AS GIVEN QUANTITY ⇒ STABILITY DEPENDS ONLY ON INERTIA MOMENTS.

(a) CONSIDERING (a) SITUATION $\begin{cases}\lambda_x > 0 \\ \lambda_y > 0.\end{cases}$

$$\Rightarrow \begin{cases} (I_z - I_y)\bar\omega_z + I_R\bar\omega_R > 0 \\ (I_z - I_x)\bar\omega_z + I_R\bar\omega_R > 0 \end{cases} \Rightarrow \begin{cases} I_z > I_y - I_R\dfrac{\bar\omega_R}{\bar\omega_z} \\ I_z > I_x - I_R\dfrac{\bar\omega_R}{\bar\omega_z} \end{cases}$$

[FROM THE COMPARISION WITH THE SINGLE-SPIN STABILIZED SATELLITE CAN BE SEEN THAT THE STABILITY RANGE IS INCREASED BY A QUANTITY $I_R\dfrac{\bar\omega_R}{\bar\omega_z}$



(b) $\begin{cases} \lambda_x < 0 \\ \lambda_y < 0 \end{cases}$ considering (b) situation:

$$\begin{cases} (I_z - I_y)\overline{\omega}_z + I_r \overline{\omega}_r < 0 \\ (I_z - I_x)\overline{\omega}_z + I_r \overline{\omega}_r < 0 \end{cases} \Rightarrow \begin{cases} I_z < I_y - I_r \dfrac{\overline{\omega}_r}{\overline{\omega}_z} \\ I_z < I_x - I_r \dfrac{\overline{\omega}_r}{\overline{\omega}_z} \end{cases}$$

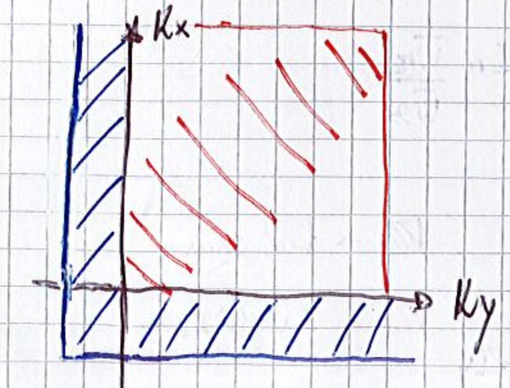
both for conditions (a) and (b) can be seen that the stability is increased by a quantity $I_r \dfrac{\overline{\omega}_r}{\overline{\omega}_z}$ in fact:

*→ SINGLE SPIN : $K_x \cdot K_y > 0$ where: $\begin{cases} K_x = \dfrac{I_z - I_y}{I_x} \overline{\omega}_z > 0 \\ K_y = \dfrac{I_z - I_x}{I_y} \overline{\omega}_z > 0 \end{cases}$

(stability condition)

*→ DUAL SPIN : $\lambda_x \cdot \lambda_y > 0$

where: $$\begin{cases} \lambda_x = \dfrac{(I_z - I_y)\overline{\omega}_z + I_r \overline{\omega}_r}{I_x} = K_x + \dfrac{I_r}{I_x}\overline{\omega}_r > 0 \\ \lambda_y = \dfrac{(I_z - I_x)\overline{\omega}_z + I_r \overline{\omega}_r}{I_y} = K_y + \dfrac{I_r}{I_y}\overline{\omega}_r > 0 \end{cases}$$



↓ are admissible all the values:

$$\begin{cases} K_x > -\dfrac{I_r}{I_x}\overline{\omega}_r \\ K_y > -\dfrac{I_r}{I_y}\overline{\omega}_r \end{cases}$$

[AVOIDING COMPUTATIONS, IN PRESENCE OF KINETIC-ENERGY LOSSES, $\dfrac{dT^{rot}}{dt} < 0$ ⇒ RESULT WILL BE THE SAME OBTAINED FOR THE SINGLE SPIN SATELLITE.]

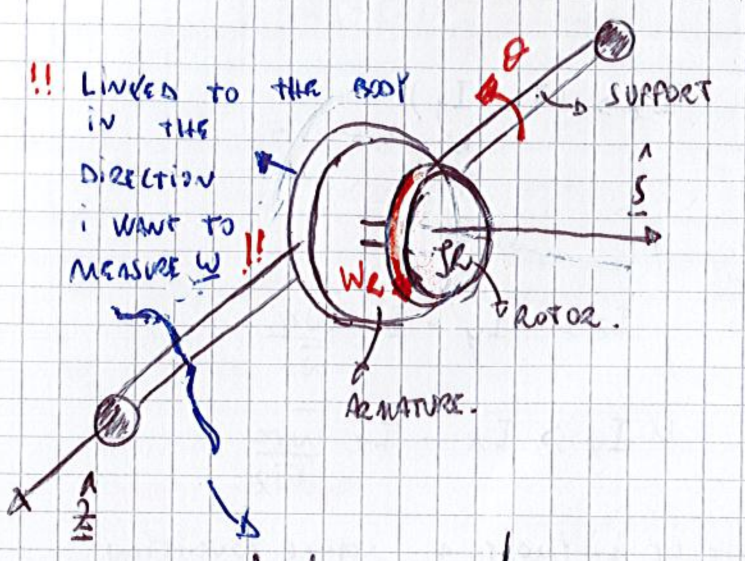
*→ SINGLE SPIN $\left(\dfrac{dT}{dt} < 0\right)$: $I_z > (I_x; I_y)$

*→ DUAL SPIN $\left(\dfrac{dT}{dt} < 0\right)$: $\begin{cases} I_z > I_y - I_r \dfrac{\overline{\omega}_r}{\overline{\omega}_z} \\ I_z > I_x - I_r \dfrac{\overline{\omega}_r}{\overline{\omega}_z} \end{cases}$

[(b) CONDITION WILL NOT BE ANYMORE A STABLE CONDITION FOR THE SPACECRAFT.]

→ " gyroscopes & gyroscopic effect."

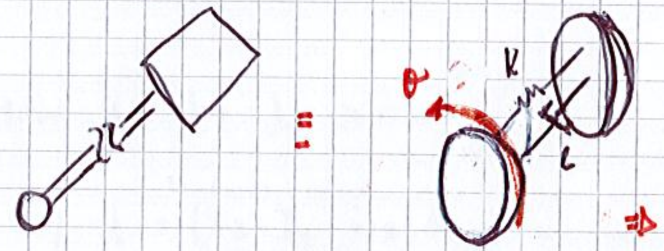
So how the double-spin dynamic (and stabilization) is used to measure an angular velocity:



1 → ROTOR ROTATING AROUND $\hat{s}$ AT CONSTANT VELOCITY W_R

1 → DEGREE OF FREEDOM IN THE SUPPORT. θ AROUND AXIS $\hat{z}$

Such appendage can be modeled as a mass-spring-damper system:



$\Rightarrow M_z = -K\theta - c\dot{\theta}$

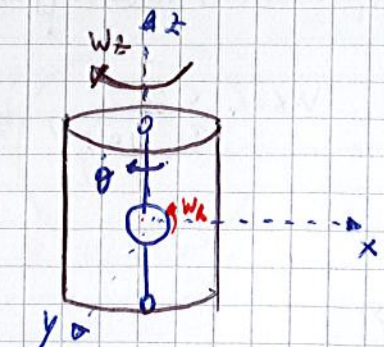
The total angular momentum of such system will be:

$$\begin{cases} \underline{H}_g = I_R\, W_R\, \hat{s} + J_z\, \dot{\theta}\, \hat{z} \\ J_z = I_R|_z + I_{armature}|_z + I_{support}|_z \end{cases}$$

Writing Euler's equations for the Gyroscope (the M_z term will contain the moments transmitted by the body)

$$\frac{d\underline{H}_g}{dt} + \underline{W} \times \underline{H}_g = \underline{M}$$

↳ Supposing that we want to measure W_x



$\Rightarrow \hat{z} = \hat{z}|_{bf}$

$\hat{s} = \hat{y}|_{bf}$

$$\Rightarrow \underline{W} = \begin{Bmatrix} W_x \\ W_y \\ W_z + \dot{\theta} \end{Bmatrix} ; \quad \underline{W}_R = \begin{Bmatrix} 0 \\ W_R \\ 0 \end{Bmatrix}$$

← Rotor is completely free inside the armature.

$$\frac{d}{dt}\begin{Bmatrix} 0 \\ I_R W_R \\ J_z \dot{\theta} \end{Bmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ W_x & W_y & W_z+\dot{\theta} \\ 0 & I_R W_R & J_z\dot{\theta} \end{vmatrix} = \underline{M}$$

$$\begin{Bmatrix} 0 \\ I_R \dot{W}_R \\ J_z \ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} J_z \dot{\theta} W_y - I_R W_R (W_z + \dot{\theta}) \\ J_z \dot{\theta}\, W_x \\ + W_x I_R W_R \end{Bmatrix} = \underline{M}$$

⇒ Euler's equations on the gyroscope:

$$\begin{cases} W_y J_z \dot{\theta} - I_R W_R (W_z + \dot{\theta}) = M_x \\ W_x J_z \dot{\theta} + I_R \dot{W}_R = M_y \\ J_z \ddot{\theta} + W_x I_R W_R = M_z \end{cases}$$

+ $M_z = -K\theta - c\dot{\theta}$ ← Generated by relative motion between the gyroscope and the satellite.

⇒ The third equation will become:

$J_z \ddot{\theta} + W_x I_R W_R = (-K\theta - c\dot{\theta})$

→ $J_z \ddot{\theta} + K\theta + c\dot{\theta} = -I_R W_R W_x$

($c=0$; $K=\bar{K}$)

The steady state solution will then become:

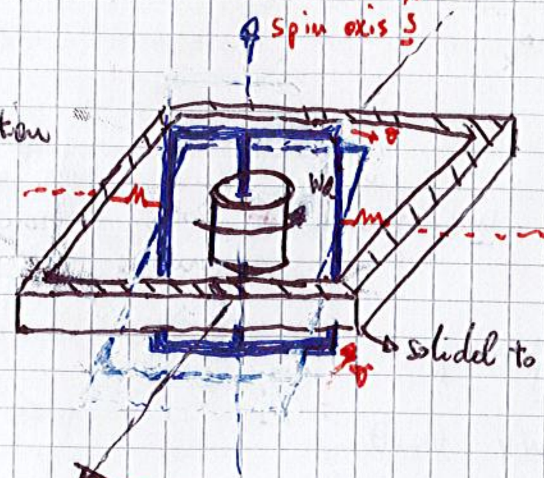
$$\bar{\theta} = -\frac{I_R W_R W_x}{\bar{K}} \rightarrow W_x = -\frac{\bar{K}\bar{\theta}}{I_R W_R}$$

$(c = \bar{c}\ ;\ K \approx 0)$

Steady state is represented by condition: $\dot{\theta} = \text{const}$

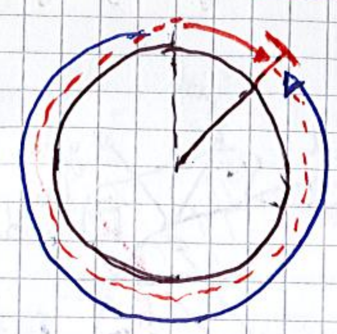
$$\Rightarrow \dot{\theta} = -\frac{I_R W_R W_K}{c} \quad \rightarrow \quad W_x = +\frac{c\,\dot{\theta}}{I_R W_R}.$$

In a more realistic representation →



↳ Natural evolution of such device is the LASER-GYROSCOPE.

⇒



x→ length run by "+" ray before interacting with "−" ray

$$c t^+ = 2\pi R + W t^+ \cdot R = R(2\pi + W t^+)$$

x→ length run by "−" ray before interacting with "+" ray

$$c t^- = 2\pi R - W t^- R = R(2\pi - W t^-)$$

↳ measuring time difference is possible to compute W

$$\begin{cases} t^+ = \dfrac{R \cdot 2\pi}{c - WR} \\ t^- = \dfrac{R \cdot 2\pi}{c + WR} \end{cases} \Rightarrow t^+ - t^- = R \cdot 2\pi \left(\frac{1}{c - WR} - \frac{1}{c + WR} \right)$$

$$\Delta t = 2\pi R \left(\frac{\cancel{c} + WR - \cancel{c} + WR}{c^2 - W^2 R^2} \right) = 2\pi R \left[\frac{2WR}{c^2 - W^2 R} \right]$$

$$= \frac{4\pi R^2 W}{c^2 - W^2 R} \approx \frac{4A R^2}{c^2} \rightarrow \boxed{W = \frac{\Delta t\, c^2}{4A}}$$

↳ Measuring the phase difference between the 2 rays.

+) $y^+ = A e^{J(Kx - \Omega t)}$

−) $y^- = A e^{J(Kx - \Omega t)}$

|| $K = \frac{2\pi}{\lambda}$; $\Omega = \frac{2\pi}{T}$

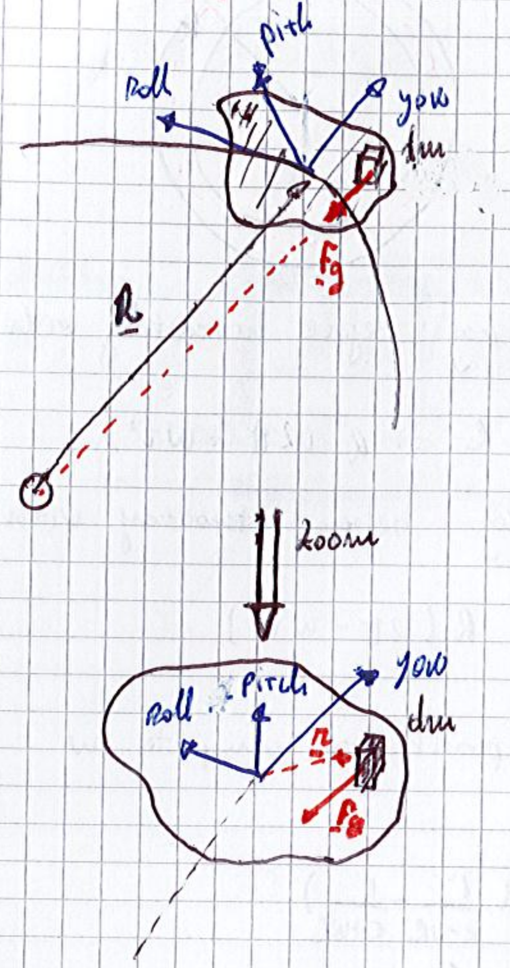
↳ $\frac{1}{T}$ is the frequency of the signal

when the 2 rays will meet-up:

$$y^+ + y^- = A \cdot [\, e^{JKx^+ - \Omega t^+} +$$

Ist perturbative effect

⇒ " gravity gradient."



⇒ the gravity field is not uniform inside the body

Moment generated by the infinitesimal element of mass:

$$d\underline{M} = \underline{r} \times d\underline{F}_g$$

↳ at this stage is expressed in the LHLV reference frame

$$d\underline{F}_g = -G \frac{m_t \cdot dm}{\|\underline{R}+\underline{r}\|^3} (\underline{R}+\underline{r})$$

$$\Rightarrow \underline{M}_{gg} = -\int_{Body} \underline{r} \times \left[\frac{G\, m_t}{\|\underline{R}+\underline{r}\|^3} (\underline{R}+\underline{r}) \right] dm .$$

SINCE $\|\underline{R}\| >> \|\underline{r}\|$

THEN the therm $\|\underline{R}+\underline{r}\|^{-3}$ can be approximated through a Taylor serie ($\underline{r}=0$ centered)

$$\left[\ (1+x)^n \approx 1 + n x + \frac{n}{2}(n-1)\, x^2 + \dots \ \right]$$

$$\rightarrow \|\underline{R}+\underline{r}\|^{-3} = (R^2 + r^2 + 2\underline{R}\cdot\underline{r})^{-3/2}$$

$$= \left[R^2 \left(1 + \frac{r^2}{R^2} + 2\frac{\underline{R}\cdot\underline{r}}{R^2}\right)\right]^{-3/2} = R^{-3}\left(1 + \underbrace{\frac{r^2}{R^2}}_{\rightarrow 0 \text{ App 1.}} + 2\frac{\underline{R}\cdot\underline{r}}{R^2}\right)^{-3/2}$$

$$\Rightarrow \|\underline{R}+\underline{r}\|^{-3} \approx R^{-3}\Big(1 + \underbrace{2\frac{\underline{R}\cdot\underline{r}}{R^2}}_{x}\Big)^{-3/2}$$

↳ this is the term that has to be expanded in the serie. $\sim (1+x)^n$

$$\|\underline{R}+\underline{r}\|^{-3} \approx R^{-3}\left(1 + \left(-\frac{3}{\cancel{2}}\right)\cancel{2}\frac{\underline{R}\cdot\underline{r}}{R^2} + \dots\right)$$

$$\approx R^{-3}\left(1 - 3\frac{\underline{R}\cdot\underline{r}}{R^2}\right) \qquad \text{(App. 2)}$$

↳ therefore:
$$\underline{M}_{gg} = -\int_{Body} \underline{r} \times \left[\frac{1}{R^3}\left(1 - 3\frac{\underline{R}\cdot\underline{r}}{R^2}\right)\cdot G\, m_t\right](\underline{R}+\underline{r})\, dm .$$

$$\underline{M}_{gg} = -\frac{G m_t}{R^3}\int_{Body}\left(1 - 3\frac{\underline{R}\cdot\underline{r}}{R^2}\right)[\underline{r}\times(\underline{R}+\underline{r})]\, dm$$

$$= -\frac{G m_t}{R^3}\int_{Body}\left(1 - 3\frac{\underline{R}\cdot\underline{r}}{R^2}\right)[\underline{r}\times\underline{R} + \underbrace{\cancel{\underline{r}\times\underline{r}}}_{=0}]\, dm$$

$$= -\frac{G m_t}{R^3}\left[\underbrace{\int_{Body}\underline{r}\times\underline{R}\, dm}_{I^{st}} - 3\underbrace{\int_{Body}\frac{\underline{r}\cdot\underline{R}}{R^2}\cdot(\underline{r}\times\underline{R})}_{II^{nd}}\right]$$

↳ Performing this first integral in a reference system with origin in the center of mass of the body (*)

$$(I^{st}) \int_{Body}\underline{r}\times\underline{R}\, dm = -\int_{Body}\underline{R}\times\underline{r}\, dm = -\underline{R}\times\int_{Body}\underline{r}\, dm .$$

But wrt to the center of mass:

$$S_0 = \int_{Body}\underline{r}\, dm = \underline{0} \quad \Rightarrow \quad \int_{Body}\underline{r}\times\underline{R}\, dm = \underline{0} .$$

Therefore the expression of the gravity gradient perturbation become:

$$\underline{M}_{gg} = \frac{3 G m_T}{R^3} \int_{Body} (\underline{r} \cdot \underline{R}) \cdot [\underline{r} \times \underline{R}] \, dm.$$

(*) For each reference system considered with the origin in the center of mass sys. integral (I^{ST}) is equal to zero. (EVEN IF IT'S NOT THE P.I. R.F.)

$$(C-O) = \frac{1}{M} \int_{Body} \overbrace{(P-O)}^{\underline{r}} dm \quad \leftarrow \text{center of mass definition.}$$

$$(P-O)\Big|_{PI} = \underline{\underline{A}} \cdot (P-O)\Big|_{G \,(\text{generic})} \Rightarrow (C-O) = \frac{1}{M} \int_{Bod} (P-O)\Big|_{PI} dm$$

$$= \frac{1}{M} \underline{\underline{A}} \cdot \int (P-O)\Big|_{G} dm.$$

But if $C \equiv O \Rightarrow (C-O) = \underline{0}$

$$\Rightarrow \frac{1}{M} \int (P-C)\Big|_{PI} dm = \frac{1}{M} \underline{\underline{A}} \cdot \int (P-C)\Big|_{G} dm = \underline{0}.$$

(a)
→ Performing this integration using the LHLV reference frame:

$$\Rightarrow \underline{r} = x \hat{i} + y \hat{j} + z \hat{k}$$
$$\underline{R} = R \cdot \hat{i}$$

($\hat{i} \to$ yaw ; $\hat{j} \to$ roll ; $\hat{k} \to$ pitch)

$$\underline{M}_{gg} = \frac{3 G m_T}{R^3} \int_{Body} \underline{R} \cdot \underline{r} \left[\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ R & 0 & 0 \end{vmatrix} \right]$$

$$= \frac{3 G m_T}{R^3} \int_{Body} R \cdot x \, [\, 0 \hat{i} + R z \hat{j} - R y \hat{k} \,] =$$

$$= \frac{3 G m_T}{R^3} \int_{Body} \begin{Bmatrix} 0 \\ R z \\ -R y \end{Bmatrix} R x \, dm = \frac{3 G m_T}{R^{\cancel{5}\,3}} \cancel{R^2} \cdot \int_{Body} \begin{Bmatrix} 0 \\ x z \\ x y \end{Bmatrix} dm.$$

→ M_{gg}, LHLV r.f.

$$\underline{M}_{gg} = 3 \frac{G m_T}{R^3} [\, I_{xz} \hat{j} - I_{xy} \hat{k} \,].$$

$\hat{j} \to$ roll axis

$\hat{k} \to$ Pitch axis

$$I_{xy} = \int_{Body} xy \, dm$$

$$I_{xz} = \int_{Body} xz \, dm$$

integrals to be performed into LHLV reference frame

≈ considerations: No component of the gravity-gradient moment acts in the yaw direction since in that point the force of gravity acts with no arm.

(b)
→ Performing the same integration in the P.I. reference frame.

$$\Rightarrow \underline{R} = R \, (c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k})$$
$$\underline{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\left(c_1 = \frac{\underline{R}|_{PI} \cdot \hat{i}}{\|\underline{R}\|}, \quad c_2 = \frac{\underline{R}|_{PI} \cdot \hat{j}}{\|\underline{R}\|}, \quad c_3 = \frac{\underline{R}|_{PI} \cdot \hat{k}}{\|\underline{R}\|} \right)$$

therefore:

$$\underline{M}_{gg} = 3 \frac{\overbrace{G m_T}^{\mu_T}}{R^5} \int_{Body} (\underline{r} \cdot \underline{R}) \cdot [\, \underline{r} \times \underline{R} \,] \, dm$$

$$= 3 \frac{\mu_T}{R^5} \int_{Body} R \cdot (c_1 x + c_2 y + c_3 z) \, R \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ c_1 & c_2 & c_3 \end{vmatrix} dm.$$

$$= 3 \frac{\mu_T}{R^{\cancel{5}\,3}} \cancel{R^2} \cdot \int_{Body} (c_1 x + c_2 y + c_3 z) \begin{Bmatrix} -c_2 z + c_3 y \\ -c_3 x + c_1 z \\ -c_1 y + c_2 x \end{Bmatrix}$$

All the terms after the integral will be null if mixed $\int_{Body} xz \, dm = 0$

$$= 3 \frac{\mu_T}{R^3} \int_{Body} \begin{Bmatrix} c_1 c_3 \overset{=0}{\cancel{xz}} - c_2 c_3 z^2 + c_2 c_3 y^2 + \dots \\ - c_1 c_3 x^2 + c_1 c_3 z^2 + \dots \\ - c_1 c_2 y^2 + c_1 c_2 x^2 + \dots \end{Bmatrix} dm.$$

$$\underline{M}_{gg} = -3\,\frac{\mu_T}{R^3} \int_{Body} \begin{Bmatrix} c_2 c_3 (z^2 - y^2) \\ c_1 c_3 (x^2 - z^2) \\ c_1 c_2 (y^2 - x^2) \end{Bmatrix} dm.$$

⇒ from the definitions of inertia moments in the PI reference frame:

- $I_z = \int_{Body} x^2 + y^2 \, dm$; $I_y = \int_{Body} z^2 + x^2 \, dm$; $I_x = \int_{Body} z^2 + y^2 \, dm$

⇒ $\int_{Body} z^2 - y^2 \, dm = I_y - I_z.$

⇒ $\int_{Body} x^2 - z^2 \, dm = I_z - I_x$

⇒ $\int_{Body} y^2 - x^2 \, dm = I_x - I_y$

→ $\underline{M}_{gg}$ P.I. r.f.

$$\underline{M}_{gg} = +3\,\frac{\mu_T}{R^3} \begin{Bmatrix} (I_z - I_y)\, c_2 c_3 \\ (I_x - I_z)\, c_1 c_3 \\ (I_y - I_x)\, c_1 c_2 \end{Bmatrix}$$

where: $c_1 = \frac{\underline{R} \cdot \hat{\underline{i}}}{\|\underline{R}\|}$; $c_2 = \frac{\underline{R} \cdot \hat{\underline{j}}}{\|\underline{R}\|}$; $c_3 = \frac{\underline{R} \cdot \hat{\underline{k}}}{\|\underline{R}\|}$

$\{\hat{\underline{i}}, \hat{\underline{j}}, \hat{\underline{k}}\}$ → principal inertia axis.

obs 1 → If a body has a symmetry axis ⇒ $I_i = I_j$ will receive a $\overset{\text{NULL}}{\vee}$ moment only along $\hat{\underline{k}}$ direction. ⇒ $M_k = 0$

In fact. $-3\frac{\mu_T}{R^3}(I_i - I_j)\, c_1 c_2 = M_K$

obs 2 → CIRCULAR EQUATORIAL ORBIT
ONE ROTATION PER ORBIT ALONG $\hat{\underline{h}}$

⇒ $\left\{ \underline{W}|_{PI} = \underline{W}|_{LHLV} = \begin{Bmatrix} 0 \\ 0 \\ W \end{Bmatrix} \quad n = \frac{2\pi}{T_{orbit}}. \right.$

⇒ $\forall t$ $\{LHLV\} \equiv \{PI\}$

⇒ $c_1 = \hat{\underline{R}} \cdot \hat{\underline{i}} = \hat{\underline{R}} \cdot \hat{\underline{yaw}} = 1$

$c_2 = \hat{\underline{R}} \cdot \hat{\underline{j}} = \hat{\underline{R}} \cdot \hat{\underline{roll}} = 0$

$c_3 = \hat{\underline{R}} \cdot \hat{\underline{k}} = \hat{\underline{R}} \cdot \hat{\underline{pitch}} = 0.$

⇒ $\underline{M}_{gg} = \underline{0}.$

II^{nd} perturbative effect → " Magnetic field's ⊕ spacecraft dipoles "

[EARTH'S MAGNETIC FIELD AT HIGH ORBITS IS CONCEPTUALLY SIMILAR TO A DIPOLE'S MAGNETIC FIELD WITH AN AXIS INCLINED BY $\sim 10^0$ WITH RESPECT TO GEOGRAPHICAL (N-S) AXIS

depending on the year.

*→ Conservative field ⇒ $\exists V$: $\underline{B} = \nabla V$

*→ V → MAXIMUM → at the NORTH/SOUTH POLES

MINIMUM → at the EQUATOR $\approx \frac{1}{2} B|_{poles}$

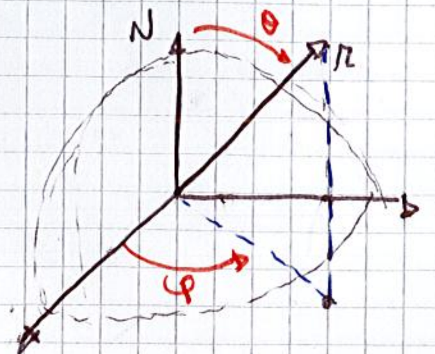
!! CAN BE CONSIDERED A PERFECT DIPOLE'S FIELD ONLY AT $h > 7000$ km ⇒ AT LOWER ORBITS THE INTERACTIONS WITH SUN'S MAGNETIC FIELDS BECOME RELEVANT.]

→ Potential function.

$$V = V(r,\theta,\varphi) = R_{earth} \left\{ \sum_{n=1}^{K} \left(\frac{R_{earth}}{r}\right)^{n+1} \sum_{m=0}^{n} \left[g_n^m \cos(m\varphi) + h_n^m \sin(m\varphi) \right] P_n^m(\theta) \right\}$$

$$\underline{B}(r,\theta,\varphi) = \nabla V = \begin{vmatrix} -\frac{\partial V}{\partial r} \\ -\frac{1}{r}\frac{\partial V}{\partial \theta} \\ -\frac{1}{r \sin\theta}\frac{\partial V}{\partial \varphi} \end{vmatrix}$$

to be performed in spherical coordinates.



$$V(r,\theta,\varphi) = R_\oplus \left\{ \sum_{n=1}^{K} \left(\frac{R_\oplus}{r}\right)^{n+1} \sum_{m=0}^{n} \left[g_n^m \cos(m\varphi) + h_n^m \sin(m\varphi) \right] P_n^m(\theta) \right\}.$$

this potential form correspond to Gauss-Normalization form for $P_n^m(\theta)$ polynomia.

*→ (g_n^m, h_n^m) → VARIATE EACH YEAR (SPERIMENTAL)

		YEAR 2014	
n	m	g_n^m	h_n^m
1	1	...	...

*→ P_n^m → NORMALIZED ACCORDING TO SMITH

$$\Rightarrow P_n^m: \int_0^\pi [P_n^m(\theta)]^2 \sin\theta \, d\theta = \frac{2(2-\delta_m^0)}{2n+1}$$

$$\left(\delta_m^0: \quad \delta_m^0 = 0 \quad m \neq 0; \quad \delta_m^0 = 1 \quad m = 0 \right)$$

The potential function can be reconstructed also using other normalizations

*→ $P^{n,m}$ → NORMALIZED ACCORDING TO GAUSS

$$P_n^m = S_{n,m} P^{n,m} \; ; \quad S_{n,m} = \left[\frac{(2-\delta_m^0)(n-m)!}{(n+m)!} \right] \frac{(2n-1)!!}{(n-m)!}$$

*→ $(g^{n,m}, h^{n,m})$ → MUST BE RE-SCALED ACCORDING TO THIS NEW POLYNOMIA'S NORMALIZATION.

$$g^{n,m} = S_{n,m}\, g_n^m$$

$$h^{n,m} = S_{n,m}\, h_n^m$$

$S_{0,0} = 1$

$S_{n,0} = S_{n-1,0} \dfrac{(2n-1)}{n} \quad (n \geq 1)$

$S_{n,m} = S_{n,m-1} \left[\dfrac{(\delta_m^1 + 1)(n-m)+1}{(n+m)} \right]^{1/2} \quad (m \geq 1)$

* → $P^{n,m}$ → ANALYTICAL FORMULATION IS THE FOLLOWING:

$$P^{0,0} = 1$$

$$P^{n,n} = \sin\theta \; P^{n-1,n-1}$$

$$P^{n,m} = \cos\theta \; P^{n-1,m} - K^{n,m} \; P^{n-2,m}$$

where

$$K^{n,m} = 0 \qquad n = 1$$

$$K^{n,m} = \frac{(n-1)^2 - m^2}{(2n-1)(2n-3)} \qquad \text{per } n > 1$$

⇓ Computing the gradient.

→ MAGNETIC FIELD IN SPHERICAL REFERENCE FRAME.

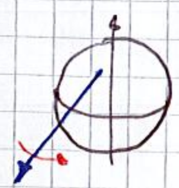
$$B_r = -\frac{\partial V}{\partial r} = \sum_{n=1}^{k} \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g^{n,m} \cos(m\varphi) + h^{n,m} \sin(m\varphi)\right) P^{n,m}(\theta)$$

$$B_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}^{k} \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g^{n,m} \cos(m\varphi) + h^{n,m} \sin(m\varphi)\right) \frac{\partial P^{n,m}(\theta)}{\partial \theta}$$

$$B_\varphi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} = -\frac{1}{\sin\theta} \sum_{n=1}^{k} \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{n} m\left(-g^{n,m} \sin(m\varphi) + h^{n,m} \cos(m\varphi)\right) P^{n,m}(\theta)$$

WHERE:

* r → distance from the center of Earth.
* θ → COLATITUDE ⇒  $\pi/2$ - Equatorial angle.
* φ → EAST LONGITUDE FROM GREENWICH.



DIPOLE MODEL.

Is therefore obtained by arresting the serie of potential at the I^{ST} term.

$$\Rightarrow V(r,\theta,\varphi) = R_{earth} \sum_{n=1}^{\cancel{k}=1} \left(\frac{R}{r}\right)^{n+1} \sum_{m=0}^{n} \left(g^{n,m} \cos(m\varphi) + h^{n,m} \sin(m\varphi)\right) P^{n,m}(\theta)$$

↳ therefore the potential function becomes:

$$V(r,\theta,\varphi) = R_{earth} \left\{ \left(\frac{R_{eart}}{r}\right)^2 \cdot \left[g^{1,0} P^{1,0}(\theta) + \left(g^{1,1}\cos(\varphi) + h^{1,1}\sin\varphi\right) P^{1,1}(\theta)\right]\right\}$$

$$= \frac{R_{earth}^3}{r^2} \cdot \left[g^{1,0}\cos(\theta) + g^{1,1}\cos(\varphi)\sin\theta + h^{1,1}\sin(\varphi)\sin\theta\right]$$

↳ Magnetic field so becomes:

$$B_r = -2 \left(\frac{R_{earth}}{r}\right)^3 \left[g^{1,0}\cos(\theta) + g^{11}\cos(\varphi)\sin\theta + h^{1,1}\sin(\varphi)\sin\theta\right]$$

$$B_\theta = \left(\frac{R_{earth}}{r}\right)^3 \left[-g^{1,0}\sin\theta + g^{11}\cos(\varphi)\cos\theta + h^{11}\sin\varphi\cos\theta\right]$$

$$B_\varphi = \left(\frac{R_{earth}}{r}\right)^3 \left[-g^{11}\sin\varphi\sin\theta + h^{11}\cos\varphi\sin\theta\right].$$

↳ In a generic reference frame (EARTH-CENTERED)

with the equatorial plane, x axis not coincident with greenwich.

$\{1, 2, 3\} =$

$$B_1 = [(B_r\cos(\delta) + B_\theta\sin(\delta)]\cos\alpha - B_\varphi\sin\alpha$$

$$B_2 = [B_r\cos(\delta) + B_\theta\sin(\delta)]\sin\alpha + B_\varphi\cos\alpha$$

$$B_3 = [B_r\sin(\delta) - B_\theta\cos(\delta)]$$

Where: → $\delta = \pi/2 - \theta$

→ $\alpha = \varphi + \alpha_G$

↳ GREENWICH MERIDIAN RIGHT ASCENSION.

$DP_{0,0}$ 1 $\quad P_{00} = 1$

$DP_{n,n}$: $\quad P_{nn} = \sin\theta \cdot P_{n-1,n-1}$

$P_{n,m} = \cos\theta \; P_{n-1,m} - K_{n,m} \; P_{n-2,m}$

$DP_{00} = 0$

$DP_{n,n} = \cos\theta \; P_{n-1,n-1} + DP_{n-1,n-1} \cdot \sin\theta$

$DP_{n,m} = -\sin\theta \cdot P_{n-1,m} + \cos\theta \; DP_{n-1,m} - K_{n,m} \; DP_{n-2,m}$

n	m	
1	0	$-\sin\theta \cdot P_{0,0} + \cos\theta \; \cancel{DP_{0,0}}^{=0} = 0$
1	1	$\cos\theta \; P_{0,0} + DP_{0,0} \cdot \sin\theta$
2	0	$-\sin\theta \cdot P_{1,0} + \cos\theta \; \underset{DP_{1,0}}{DP_{1,0}} - K_{n,m} \cdot \cancel{DP_{0,0}}^{=0}$
2	1	$-\sin\theta \cdot P_{1,1} + \cos\theta \; DP_{1,1} - \cancel{K_{n,m}}^{=0} \; DP$
2	2	$\cos\theta \; P_{1,1} + \sin\theta \cdot DP_{1,1}$

$$B_r = \sum_{n=1}^{k} \left(\frac{R}{r}\right)^{n+2} \cdot (n+1) \cdot \left[\sum_{m=0}^{n} \left(g^{n,m} \cos(m\varphi) + h^{n,m} \sin(m\varphi) \right) \cdot P^{n,m}(\theta \right.$$

B_θ =

$$B_\theta = -\sum_{n=1}^{k} \left(\frac{R}{r}\right)^{n+2} \sum_{m=0}^{k} \left[\sum_{m=0}^{n} \left(\quad \right) \cdot \frac{\partial P^{n,m}}{\partial \theta} \right]$$

$$B_\varphi = -\frac{1}{\sin\theta} \sum_{n=1}^{k} \left(\frac{R}{R}\right)^{n+2} \sum_{m=0}^{n} m \left(-g^{n,m} \sin m\varphi + h^{n,m} \cos(m \right.$$

$$P^{n,m}(\theta).$$

→ " Stability of a single spin satellite UNDER GRAVITY-GRADIENT-TORQUE."

In such conditions Euler's equations will be:

$$\underline{\underline{I}} \, \dot{\underline{w}} + \underline{w} \times (\underline{\underline{I}} \, \underline{w}) = \underline{M}_{gg}$$

Scalar view:

$$\begin{cases} I_x \dot{w}_x + (I_z - I_y) w_z w_y = 3 \dfrac{\mu_T}{R^3} (I_z - I_y) c_2 c_3 \\ I_y \dot{w}_y + (I_x - I_z) w_x w_z = 3 \dfrac{\mu_T}{R^3} (I_x - I_z) c_1 c_3 \\ I_z \dot{w}_z + (I_y - I_x) w_x w_y = 3 \dfrac{\mu_T}{R^3} (I_y - I_x) c_2 c_1 \end{cases}$$

↳ Writing rotation with respect to the LHLV frame separating contributions:

$$\underline{w}_B = \underline{w}_0 + \underline{w}_{B,0}$$

angular velocity of the body with respect to the orbital (LHLV) frame in PI r.f.

angular velocity of rotation of {r0} with respect to inertial frame {rI} expressed in body (PI) r.f.

angular velocity of the orbital reference frame expressed in body (PI) frame.

$$\Rightarrow \underline{w} = \underbrace{\underline{\underline{A}}_{LHLV2PI} \cdot \begin{Bmatrix} \dot\alpha_x \\ \dot\alpha_y \\ \dot\alpha_z \end{Bmatrix}}_{\underline{w}_{B0}} + \underbrace{\underline{\underline{A}}_{LHLV2PI} \begin{Bmatrix} 0 \\ 0 \\ n \end{Bmatrix}}_{\underline{w}_0}$$

$$\underline{\underline{A}}_{LHLV2PI} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \xrightarrow{lin} \underline{\underline{A}}^{lin}_{LHLV2PI} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix}$$

$$\Rightarrow \underline{w} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} \dot\alpha_x \\ \dot\alpha_y \\ \dot\alpha_z + n \end{Bmatrix} \quad (i)$$

($\{\dot\alpha_x ; \dot\alpha_y ; \dot\alpha_z\}$ represent the angular velocity of the body with respect to the orbit r.f. (LHLV)) while computing the $\{c_1, c_2, c_3\}$ coefficients;

$$c_1 = \hat{i} \cdot \hat{R} \quad ; \quad c_2 = \hat{j} \cdot \hat{R} \quad ; \quad c_3 = \hat{k} \cdot \hat{R} .$$

so making this product in the P.I. reference frame:

$$\hat{R}\big|_{PI} = \underline{\underline{A}}_{LHLV2PI} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow c_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}^T \left(\underline{\underline{A}}_{LHLV2PI} \cdot \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \right)$$

this means to select the I^{st} row of $\left(\underline{\underline{A}}_{LHLV2PI} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \right)$ element.

$$\Rightarrow \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = A_{LHLV}^{LIN} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} 1 & \alpha_z & -\alpha_y \\ -\alpha_z & 1 & \alpha_x \\ \alpha_y & -\alpha_x & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} . \quad (ii)$$

$$(ii) \quad \begin{cases} c_1 = 1 \\ c_2 = -\alpha_z \\ c_3 = \alpha_y \end{cases}$$

$$(i) \quad W = \underline{\underline{A}}_{LHLV2PI} \begin{Bmatrix} \dot\alpha_x \\ \dot\alpha_y \\ \dot\alpha_z + n \end{Bmatrix} = \begin{Bmatrix} \dot\alpha_x + \alpha_z \dot\alpha_y - (\dot\alpha_z + n)\alpha_y \\ -\alpha_z \dot\alpha_x + \dot\alpha_y + (\dot\alpha_z + n)\alpha_x \\ \alpha_y \dot\alpha_x - \alpha_x \dot\alpha_y + (\dot\alpha_z + n) \end{Bmatrix}$$

Substituting everything into Euler's equation:

$$(i.der) \quad \dot{W} = \begin{Bmatrix} \ddot\alpha_x + \dot\alpha_z \dot\alpha_y + \alpha_z \ddot\alpha_y - \ddot\alpha_z \alpha_y - \dot\alpha_z \dot\alpha_y - n \dot\alpha_y \\ \ddot\alpha_y - \dot\alpha_z \dot\alpha_x - \alpha_z \ddot\alpha_x + \ddot\alpha_z \alpha_x + \dot\alpha_z \dot\alpha_x + n \dot\alpha_x \\ \dot\alpha_y \dot\alpha_x - \dot\alpha_x \dot\alpha_y + \alpha_y \ddot\alpha_x - \alpha_x \ddot\alpha_y + \ddot\alpha_z . \end{Bmatrix}$$

Hp^* supposing a circular orbit $\Rightarrow \frac{dn}{dt} = 0$

OR ANYWAY A REALLY SMALL TIME VARIATION OF THE ORBIT WITH RESPECT TO THE ATTITUDE OF THE SATELLITE.

Hp^{**}: SMALL ROTATIONS $\Rightarrow$ $\alpha_i \cdot \alpha_j \to 0$

$\dot\alpha_i \dot\alpha_j \to 0$

$\ddot\alpha_i \alpha_j \to 0$

Therefore:

$$\begin{cases} I_x \ddot\alpha_x - I_x n \dot\alpha_y + (I_z - I_y)(\dot\alpha_y + n\alpha_x)(\dot\alpha_z + n) = -3\frac{\mu_t}{R^3}(I_z - I_y)\alpha_z\alpha_y \\ I_y \ddot\alpha_y + I_y n \dot\alpha_x + (I_x - I_z)(\dot\alpha_x - n\alpha_y)(\dot\alpha_z + n) = 3\frac{\mu_t}{R^3}(I_x - I_z)\alpha_y \\ I_z \ddot\alpha_z + (I_y - I_x)(\dot\alpha_y + n\alpha_x)(\dot\alpha_x + n\alpha_y) = -3\frac{\mu_t}{R^3}(I_y - I_x)\alpha_z . \end{cases}$$

Always retaining negligible double products $\alpha_i \alpha_j \to 0$ ....

$$\begin{cases} I_x \ddot\alpha_x + [I_z - I_y - n I_x]\dot\alpha_y + [I_z - I_y] n^2 \alpha_x = 0 \\ I_y \ddot\alpha_y + [I_x - I_z + n I_y]\dot\alpha_x - [I_x - I_z] n^2 \alpha_y = 3\frac{\mu_t}{R^3}(I_x - I_z)\alpha_y \\ I_z \ddot\alpha_z = -3\frac{\mu_t}{R^3}(I_y - I_x)\alpha_z . \end{cases}$$

thanks to Kepler's law: ---> Kepler's law.

$$\sqrt{\frac{\mu}{a^3}}\, \Delta t = E - e \sin E$$

$\Rightarrow$ circular orbit: $\sqrt{\frac{\mu}{R^3}}\, T = 2\pi \to \frac{2\pi}{T} = n = \sqrt{\frac{\mu}{R^3}}$ $\quad \Rightarrow n^2 = \frac{\mu_t}{R^3}$

$$\Rightarrow \begin{cases} I_x \ddot\alpha_x + [I_z - I_y - n I_x]\dot\alpha_y + [I_z - I_y] n^2 \alpha_x = 0 \\ I_y \ddot\alpha_y + [I_x - I_z + n I_y]\dot\alpha_x - [I_x - I_z] n^2 \alpha_y = 3 n^2 (I_x - I_z)\alpha_y \\ I_z \ddot\alpha_z = -3 n^2 (I_y - I_x)\alpha_z \end{cases}$$

$$\begin{cases} I_x \ddot\alpha_x + [I_z - I_y - n I_x]\dot\alpha_y + [I_z - I_y] n^2 \alpha_x = 0 \\ I_y \ddot\alpha_y + [I_x - I_z + n I_y]\dot\alpha_x - 4[I_x - I_z] n^2 \alpha_y = 0 \\ I_z \ddot\alpha_z + 3 n^2 [I_y - I_x]\alpha_z = 0 \end{cases}$$

obs$_1$ → III^{rd} equation is decoupled from the other two

obs$_2$ → Stability is hereby evaluated for this only condition of CIRCULAR ORBIT

immobile P.I. r.f. with respect to LHLV r.f. $\longrightarrow \begin{Bmatrix} \dot\alpha_x \\ \dot\alpha_y \\ \dot\alpha_z \end{Bmatrix} = \underline{0} \Rightarrow \begin{cases} (I_z - I_y) n^2 \alpha_x = 0 \\ -4 n^2 (I_x - I_z) n^2 \alpha_y = 0 \\ 3 n^2 (I_y - I_x)\alpha_z = 0 . \end{cases} \Rightarrow \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix} = \underline{0}$

[SPACECRAFT is in EQUILIBRIUM IF AND ONLY IF

AT EACH TIME THE PI r.f. is COINCIDENT WITH THE LHLV r.f.]

obs 1

SINCE the third equation is decoupled

THEN It's enough to look at it to evaluate the stability
(!! the equation is the one of the harmonic oscillator!)

III^{rd})

$$I_z \ddot{\alpha}_z + 3n^2 (I_y - I_x)\alpha_z = 0$$

$$\Big\downarrow \mathcal{L}$$

$$s^2 I_z \alpha_z(s) + 3n^2 (I_y - I_x)\alpha_z(s) = 0$$

$$[I_z s^2 + 3n^2 (I_y - I_x)]\alpha_z(s) = 0$$

STABLE $\Longleftrightarrow 3\frac{I_y - I_x}{I_z} n^2 > 0 \Longrightarrow I_y > I_x$

obs 2

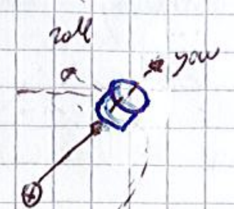
SINCE the Equilibrium condition is $\begin{Bmatrix}\alpha_x\\ \alpha_y\\ \alpha_z\end{Bmatrix} = \underline{0}$ (LHLV r.f. ≡ PI r.f.)

THEN The STABILITY AROUND PITCH ($\hat{z}$) axis is guaranteed if and only if
The Minor inertia axis (minor between the other 2) is coincident with the yaw axis

PITCH-STABILITY $\Longleftrightarrow$ [I^{st} condition. $I_y > I_x$] ; $\begin{cases}\hat{x} \equiv \widehat{yaw}\\ \hat{y} \equiv \widehat{roll}\end{cases} \Rightarrow$ [$I_{yaw} < I_{roll}$]

($\underline{w} = n\,\hat{h}|_{PI}$)

⇒ STABLE (PITCH AXIS)



⇒ UNSTABLE (PITCH AXIS)

I^{st} + II^{nd})

$$\begin{cases} I_x \ddot{\alpha}_x + (I_z - I_y - I_x)\, n\, \dot{\alpha}_y + (I_z - I_y)\, n^2 \alpha_x = 0 \\ I_y \ddot{\alpha}_y + (I_x + I_y - I_z)\, n\, \dot{\alpha}_x + 4n^2 (I_z - I_x)\alpha_y = 0 \end{cases}$$

$$K_x \triangleq \frac{I_z - I_y}{I_x} \quad ; \quad K_y \triangleq \frac{I_z - I_x}{I_y} \xrightarrow[\hat{y} \equiv \widehat{roll},\ \hat{x} \equiv \widehat{yaw}]{LHLV \equiv PI} \boxed{K_x = K_{yaw} \quad K_y = K_{roll}}$$

$$\begin{cases} \ddot{\alpha}_x + (K_y - 1)\, n\, \dot{\alpha}_y + K_y n^2 \alpha_x = 0 \\ \ddot{\alpha}_y + (1 - K_z)\, n\, \dot{\alpha}_x + 4n^2 K_z \alpha_y = 0. \end{cases}$$

$$\Big\downarrow \mathcal{L}$$

$$\begin{cases} s^2 \alpha_x(s) + (K_y - 1)\, n s\, \alpha_y(s) + K_y n^2 \alpha_x(s) = 0 \\ s^2 \alpha_y(s) + (1 - K_z)\, n s\, \alpha_x(s) + 4n^2 K_z \alpha_y(s) = 0 \end{cases}$$

$$\begin{bmatrix} s^2 + n^2 K_y & ; & (K_y - 1)\, n s \\ (1 - K_z)\, n s & & s^2 + 4n^2 K_z \end{bmatrix} \begin{Bmatrix} \alpha_x(s) \\ \alpha_y(s) \end{Bmatrix} = \underline{0}$$

$$\det\left(\begin{bmatrix} \cdot & ; & \cdot \\ \cdot & & \cdot \end{bmatrix}\right) = s^4 + s^2 (n^2 K_y + 4n^2 K_z) + 4n^4 K_z K_y - (-1 + K_z - K_z K_y + K_y)\, n^2 s^2$$

$$= s^4 + [\cancel{K_y} + 4K_z + 1 - K_z + K_z K_y - \cancel{K_y}]\, n^2 s^2 + 4n^4 K_z K_y$$

$$= s^4 + (3K_z + 1 + K_z K_y)\, n^2 s^2 + 4n^4 K_z K_y$$

$$\cancel{s^2_{1,2} = \frac{n^2 (3K_z + 1 + K_z K_y) \pm \sqrt{(3K_z + 1 + K_z K_y)^2 - 16 K_z K_y\, n^4}}{2}}$$

$$\cancel{s^2_{1,2} = \frac{n^2}{2}\left[3K_z + 1 + K_z K_y \pm \sqrt{9K_z^2 + 1 + K_z^2 K_y^2 + 6K_z^2 K_y + 6K_z + 2K_z K_y - 16 K_z K_y}\right]}$$

$$\cancel{= \frac{n^2}{2}\left[3K_z + 1 + K_z K_y \pm \sqrt{9K_z^2 + 1 + K_z^2 K_y^2 + 6K_z^2 K_y + 6K_z - 12 K_z K_y}\right]}$$

$$s^4 + n^2 s^2 (1 + 3K_r + K_r K_y) + 4n^4 K_r K_y = 0.$$

$s^2 + bs + c = 0$ in order to get stability must be that:

$$\begin{cases} c > 0 & (1) \\ \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2} < 0 & (2) \\ b^2 - 4c > 0 & (3) \end{cases}$$

in fact (2): $s^2_{1,2} = \dfrac{-(1 + K_r + K_r K_y) \pm \sqrt{(1 + K_r + K_r K_y)^2 - 16n^2 K_r K_y}}{2}$

in order to have IMAGINARY (stable) POLES (roots)

$$s_{1,2} = \pm\sqrt{\frac{-b + \sqrt{b^2 - 4ac}}{2}} \qquad s_{3,4} = \pm\sqrt{\frac{-b - \sqrt{b^2 - 4ac}}{2}}$$

(3) In order to have 4 COMPLETELY IMAGINARY ROOTS ($s_{1,2} \cup s_{3,4}$)

$s^2_{1,2}$ must be real and negative.

(the square root of an imaginary number:

$$\sqrt{a + jb} = \pm(a^2 + b^2)^{1/4} \cdot \sqrt{e^{i\theta}} \qquad (\theta = \tan^{-1}(b/a))$$

always gives 2 poles with positive real part)

(1) Since $a = 1$ in the formula

$b \pm \sqrt{b^2 - 4ac}$ if $c < 0 \Rightarrow b \pm \underbrace{\sqrt{b^2 + \dots}}_{> b}$

$\Rightarrow$ condition (2) cannot be satisfied.

$\Rightarrow$ (1) $K_r \cdot K_y > 0$

(3) $n^4 (1 + 3K_r + K_r K_y)^2 - 4 \cdot 4n^4 K_r K_y > 0$

$$(1 + 3K_r + K_r K_y)^2 > 16 K_r K_y$$

(1) + (3) Automatically satisfy (2) $\Rightarrow$ (that is the real condition) since are easier to verify.

$$\Rightarrow \begin{cases} K_r K_y > 0 & (I^{st}) \\ 1 + 3K_r + K_r K_y > 4\sqrt{K_r K_y} \;\cup\; 1 + 3K_r + K_r K_y < -4\sqrt{K_r K_y} & (II^{nd}) \end{cases}$$

!! The presence of a gravity gradient perturbation reduces the stability region

in fact: with: $\underline{M}_{gg} = \underline{0}$; $\begin{cases} \text{CIRCULAR ORBIT} \\ \underline{w} = n\,\underline{h} \\ \forall t \;\; \text{LHLV} \equiv \text{PI} \end{cases}$

The only required condition was (I^{st}) $K_r K_y > 0$. !!

that means: $I_z > (I_y, I_x)$ ∨

$I_z < (I_y, I_x)$

and NO CONDITION BETWEEN I_x, I_y.

To evaluate (II^{nd}) condition:

$K_y = 0 \Rightarrow 1 + 3K_r = 0 \Rightarrow K_r = -1/3$

$K_y = -1 \Rightarrow 1 + 3K_r - K_r = 4\sqrt{-K_r} \Rightarrow 1 + 2K_r = +4\sqrt{-K_r}$

$K_y = K_r \Rightarrow 1 + 3K_r + K_r^2 = -4\sqrt{K_r^2} \Rightarrow 1 + 3K_r + K_r^2 = -4K_r.$

$K_r^2 + 7K_r + 1 = 0.$

a) PITCH STABILITY $\Leftrightarrow$ $I_{roll} > I_{yaw}$ $\Leftrightarrow$ $K_r > K_y$

E.g. III)

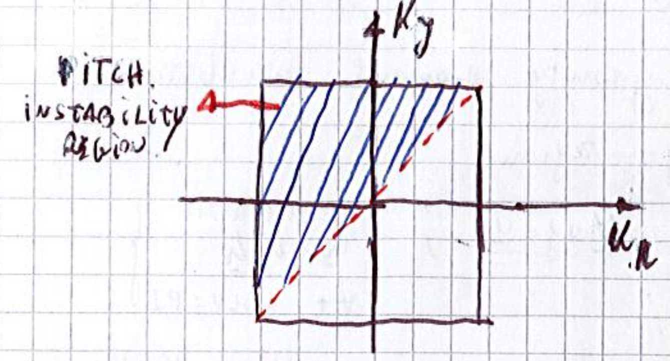
in fact: $K_{yaw} = \dfrac{I_z - I_y}{I_x} = \dfrac{I_p - I_{roll}}{I_{yaw}}$; $K_{roll} = \dfrac{I_z - I_x}{I_y} = \dfrac{I_p - I_{yaw}}{I_{roll}}.$

$$\Rightarrow K_{roll} - K_{yaw} = \frac{I_p - I_{yaw}}{I_{roll}} + \frac{I_p - I_{roll}}{I_{yaw}} = \frac{1}{I_{roll} I_{yaw}} \left[I_p I_{yaw} - I_{yaw}^2 - I_p I_{roll} + I_{roll}^2 \right]$$

$$= \frac{1}{I_{roll} I_{yaw}} \left[I_p (I_{yaw} - I_{roll}) + I_{roll}^2 - I_{yaw}^2 \right]$$

$$= \underbrace{\frac{(I_{yaw} - I_{roll})}{I_{roll} I_{yaw}}}_{< 0 \; (\text{Hp: } I_{roll} > I_{yaw})} \cdot \Big[\underbrace{I_p - (I_{roll} + I_{yaw})}_{< 0 \; \text{prop: } (I_k \leq (I_i + I_j))} \Big] > 0$$

⇒ it's demonstrated that $I_{roll} > I_{yaw}$ (CONDITION NECESSARY TO HAVE PITCH STABILITY) implies that $K_{roll} > K_y$ ⊛



retta: $K_y = 1 K_R \Rightarrow \frac{K_y}{K_R} = 1$

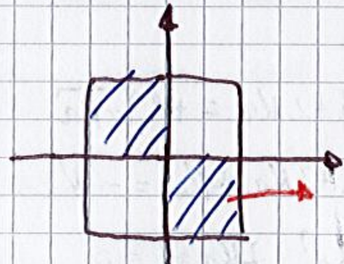
tutti i valori di $K_R > K_y$ stanno sotto tale retta.

b) YAW AND ROLL STABILITY

Eq II) + I)

$$\begin{cases} K_R K_y > 0 \\ 1 + 3K_R + K_R K_y > 4\sqrt{K_R K_y} \quad \cup \quad 1 + 3K_R + K_y K_R < -4\sqrt{K_R K_y} \end{cases}$$

⋆→ ($K_R K_y > 0$) automatically exclude the II^{nd} and IV^{th} quadrants.



⋆→ ($1 + 3K_R + K_R K_y > 4\sqrt{K_R K_y}$)

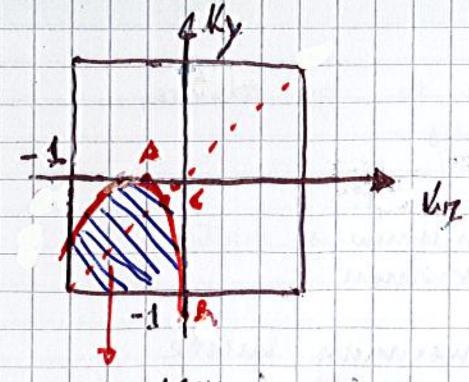
curve will pass for: ($K_y = 0$; $K_R = -\frac{1}{3}$) ...

$K_y = 0 \Rightarrow 1 + 3K_R = 0 \Rightarrow K_R = -1/3$ A

$K_y = -1 \Rightarrow 1 + 3K_R - K_R = -4\sqrt{K_R} \Rightarrow 1 + 4K_R^2 + 4K_R = 1 \Rightarrow K_R \approx -0{,}051$ B

$K_y = K_R \Rightarrow 1 + 3K_R + K_R^2 = -4K_R \Rightarrow 1 + 7K_R + K_R^2 = 0$

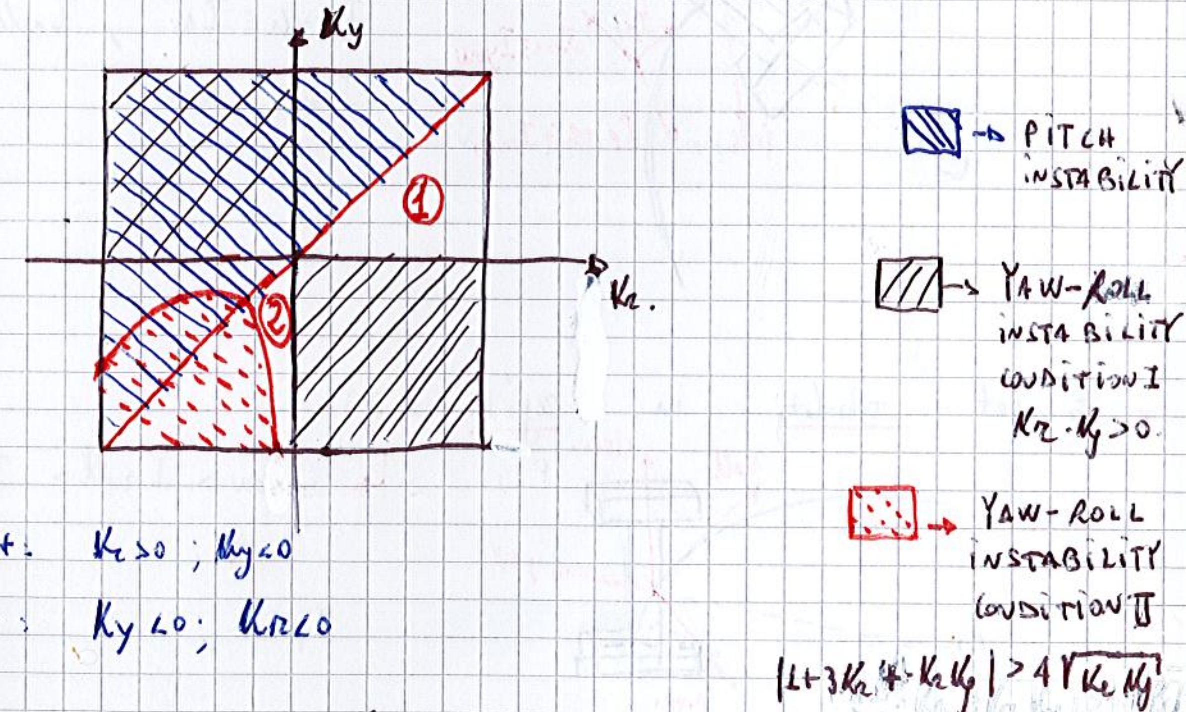
$$K_R = \frac{-7 \pm \sqrt{49 - 4}}{2} = \frac{-7 \pm \sqrt{45}}{2} \approx \begin{cases} -0{,}291 & C \\ -13 \end{cases}$$



YAW/ROLL INSTABILITY (thanks to condition II)

F STABILITY FOR SINGLE SPIN-SATELLITE ROTATING AT ONE ROTATION PER ORBIT UNDER THE EFFECT OF GRAVITY GRADIENT.

$$K_R = \frac{I_{pitch} - I_{yaw}}{I_{roll}} \quad ; \quad K_{yaw} = \frac{I_{pitch} - I_{roll}}{I_{yaw}}$$



I^{st} quadrant: $K_R > 0$; $K_y > 0$

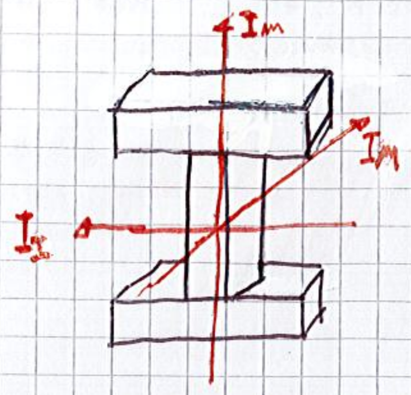
III^{rd} quadrant: $K_y < 0$; $K_R < 0$

therefore there are only 2 stable regions 1 and 2

1 → $\begin{cases} K_R > 0 \; ; \; K_y > 0 \Rightarrow I_{pitch} > I_{yaw} \cup I_{pitch} > I_{roll} \\ K_R > K_y \overset{⊛}{\Rightarrow} I_{roll} > I_{yaw} \end{cases}$ ⇒ $I_{pitch} > I_{roll} > I_{yaw}$

2 → $\begin{cases} K_R < 0 \; ; \; K_y < 0 \Rightarrow I_{pitch} < (I_{yaw}, I_{roll}) \\ K_R > K_y \overset{⊛}{\Rightarrow} I_{roll} > I_{yaw} \\ |1 + 3K_R + K_R K_y| > 4\sqrt{K_R K_y} \end{cases}$ ⇒ $I_{roll} > I_{yaw} > I_{pitch}$ LIMITED TO ADDITIONAL CURVE

* therefor a generic body in order to be stabilized

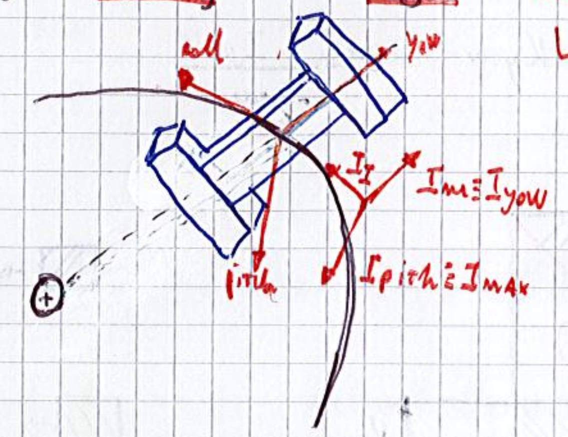


$I_m \rightarrow$ minimum inertie moment.

$I_M \rightarrow$ maximum inertie moment

$I_I \rightarrow$ intermidiate inertie moment.

* To get stability in region (1)



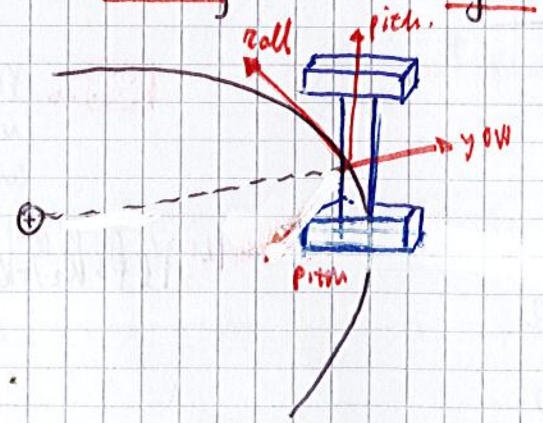
↳ $I_{pitch} > I_{roll} > I_{yaw}$

⇓

$I_{pitch} \equiv I_M$, $I_{roll} \equiv I_I$

$I_{yaw} \equiv I_m$

* To get stability in region (2)



↳ $I_{yaw} > I_{roll} > I_{pitch}$.

⇓

I_{pitch}

III^{rd} perturbation effect → " Solar radiation torque "

Momentum transmitted by radiation due to $\left\{\begin{array}{l} \text{DIRECT SOLAR RADIATION} \\ \text{SOLAR RADIATION REFLECTED BY EARTH (ALBEDO)} \\ \text{PROPER EARTH RADIATION (IR)} \end{array}\right.$

Cause of different optical properties of the external surfaces of the spacecraft can generate torques on the spacecraft.

→ solar pressure: DIRECT SOLAR RADIATION

$$P_s = \frac{\text{mean solar flux}}{\text{speed of light}} \quad [N/m^2] \quad \left(\propto \frac{1}{r^2}\right)$$

$$P_{\odot} = \frac{L_{\odot}}{4\pi c r^2}$$ where.
- $c = 300\,000\ \text{Km/s}$
- $L_{\odot} = 3{,}846 \times 10^{26}$ Watt.

$$P_{\oplus} = \frac{L_{\odot}}{4\pi c r_{\oplus}^2}$$ → Solar pressure radiating Earth.

$$\Rightarrow \quad P_{\odot} = P_{\oplus} \cdot \left(\frac{r_{\oplus}}{r}\right)^2$$

→ Earth's Albedo

$$Al = \frac{\text{energy reflected by Earth}}{\text{total energy incident Earth}} \quad ; \quad \langle Al \rangle = 0{,}34.$$

(medium by definition, depending on seasons)

→ I.R. proper Earth pressure.

Depends on Altitude and might be interpolated by the data:

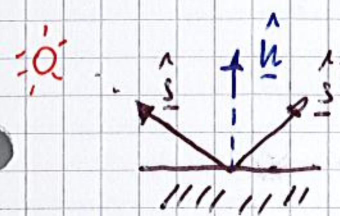
h [Km]	P_{SR} [W/m²]
	⋮

→ FOR EACH MATERIAL, OPTICAL PROPERTIES:

ρ_s → specularly reflected ρ_d → diffused.
ρ_a → absorbed

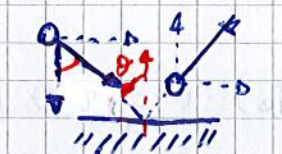
⇒ Kirchoff : $\rho_s + \rho_a + \rho_d = 1$
($\forall$ material)

Reflected radiation



" Force due to reflected radiation acts in the $(-\hat{n})$ direction. "

MODEL: elastic collision, impulsive force.
⇒ momentum transmitted along $\hat{n}$ direction

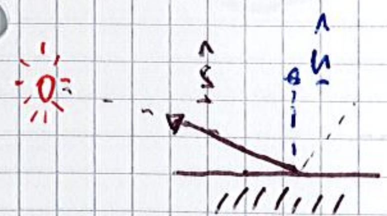


$\underline{v}_{in}^{L} = v_x \hat{i} - v_y \hat{j}$
$\underline{v}_{out}^{L} = v_x \hat{i} + v_y \hat{k}$
$(\hat{i} = \hat{t},\ \hat{j} = \hat{n})$

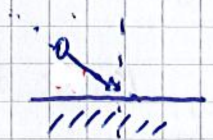
$\Delta \underline{q} = \Delta \underline{q}_L + \Delta \underline{q}_{spacecraft} = \underline{0}$

$\Delta \underline{q} = m_L \cdot (\underline{v}_{in}^{L} - \underline{v}_{out}^{L}) + (\underline{v}_0^{SP} - \underline{v}_{el}^{SP})\, m_{SP}$
$= 2 m_L \cdot v^L \cos\theta\, \hat{n} + m_{SP} \cdot \underline{v}_N^{SP} = \underline{0}$
$\Rightarrow v_N^{SP} = -2 \frac{m_L}{m_{SP}} \cdot v^L \cos\theta \cdot \hat{n}.$

Absorbed radiation.



⇒



" Absorbed radiation generates a force acting in the $-\hat{s}$ direction "

$\Delta \underline{q} = \Delta \underline{q}_L + \Delta \underline{q}_{spacecraft} = \underline{0}$
$= -m_L(-\hat{s})\, v^L + m_L \cdot 0 - 0 + q_0^{SP}$
$\Rightarrow \underline{v}_N^{SP} = -\frac{m_L}{m_{SP}} v^L \hat{s}$

Diffused radiation.

" Can always be divided into 2 contributions: one perpendicular to the surface and the other directed as $-\hat{s}$. "

$$d\underline{F}_{abs} = -(P \cdot \rho_a \cos\theta) \cdot \hat{\underline{S}}\, dA.$$

$$d\underline{F}_{spec} = -2 P \rho_s \cos^2\theta\, \hat{\underline{N}}\, dA.$$

$$d\underline{F}_d = P \rho_d \left(-\frac{2}{3} \cos\theta\, \hat{\underline{N}} - \cos\theta\, \hat{\underline{S}}\right) dA$$

Force acting on i-th surface:

$$d\underline{F}_i = -P\left[(1-\rho_s)\hat{\underline{S}} + 2\left(\rho_s \cos\theta + \frac{1}{3}\rho_d\right)\hat{\underline{N}}\right]\cos\theta\, dA$$

in fact:

$$\rho_d = 1 - \rho_s - \rho_a$$

$$d\underline{F}_d = P \cdot (1-\rho_s-\rho_a) \cdot \left(-\frac{2}{3}\cos\theta\, \hat{\underline{N}} - \cos\theta\, \hat{\underline{S}}\right) dA.$$

Torque acting on the spacecraft:

i_th surface

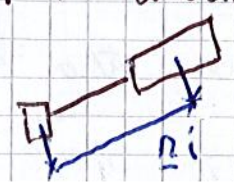
$$\underline{M}_i = \int_{S_i} \underline{r}_i \times d\underline{F}_i\, ds$$

decomposing in plane / cylindrical surface

$$\underline{F}_i = -A P\left[(1-\rho_s)\hat{\underline{S}} + 2\left(\rho_s \cos\theta + \frac{1}{3}\rho_d\right)\hat{\underline{N}}\right]$$

$$\underline{M} = \sum \underline{r}_i \times \underline{F}_i$$

($\underline{r}_i$: directed from the center of mass of the spacecraft to the center of the surface i)



Geometrical consideration

* → FLAT PANEL

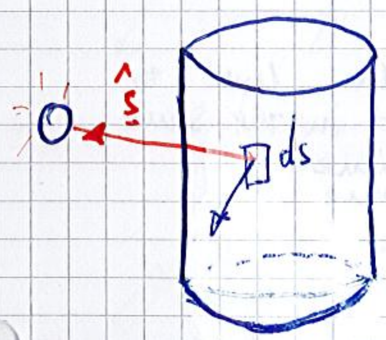
$$\underline{F}_i = -A P\left[(1-c_s)\hat{\underline{S}} + 2\left(c_s \cos\theta + \frac{1}{3}c_d\right)\hat{\underline{N}}\right]\cos\theta$$

$$(\theta = \cos^{-1}(\hat{\underline{S}} \cdot \hat{\underline{N}}))$$

* → CYLINDRICAL SURFACE

$$\underline{F}_{lateral} = PA \sin\theta \left\{\left[\left(1+\frac{1}{3}c_s\right)+\frac{\pi}{6}c_d\right]\cdot\hat{\underline{S}} + \left[-\frac{4}{3}c_s - \frac{\pi}{6}c_d\right]\cos\theta\, \hat{\underline{N}}\right\}$$

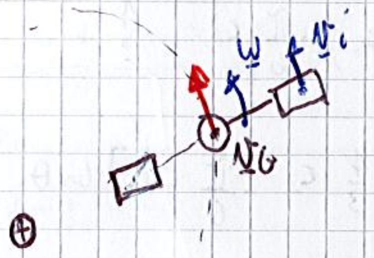
$$\underline{F}_{base} = PA \cos\theta \left[(1-c_s)\cdot\hat{\underline{S}} + 2\left(c_s\cos\theta + \frac{1}{3}c_d\right)\cdot\hat{\underline{N}}\right]$$



IVth perturbative effect. → " Aerodynamic torque "

(MAIN EFFECT FOR $h \leq 400\,km$)

MODEL: air particles that hit the external surface of a spacecraft totally transfer their kinetic energy to the spacecraft.



$\underline{v}_G$ → orbital velocity

$\underline{v}_i$ → velocity with respect to the inertial reference frame of the i-th surface. (CENTER OF THE SURFACE).

⇒ $\underline{v}_i = \underline{v}_G + \underline{w} \times \underline{r}_i$

this computation have to be performed in the same reference frame

$$\underline{v} = \underline{v}_G + \underline{\underline{A}}^T (\underline{w} \times \underline{r}_i)$$

Already defined in the inertial reference frame.

$\underline{w}$ and $\underline{r}_i$ defined in the principal inertia reference-frame.

force acting on i-th surface.

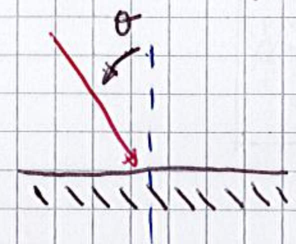
$$\begin{cases} d\underline{F}_i = \frac{1}{2} C_D \, \rho \, \|\underline{v}\|^2 (\hat{\underline{N}}_i \cdot \underline{v}) \, \underline{u}_v \, dA \\ \{\hat{\underline{N}}_i\}_{in} = \underline{\underline{A}}^T \{\hat{\underline{N}}_i\}_{p.i.} \end{cases}$$

$$\{d\underline{F}_i\}_{p.i.} = \underline{\underline{A}} \{d\underline{F}_i\}_{in}$$

SENSOR 1) SUN SENSOR

[ALL THE SUN SENSORS ARE BASED ON MATERIALS THAT PRODUCES AN ELECTRIC SIGNAL WHEN IRRADIATED.]

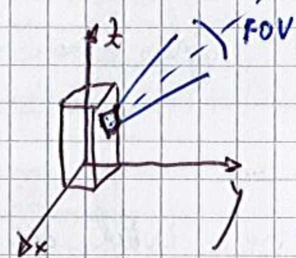
therefore the output (a current I) is proportional to the angle of incidence of the radiation



⇒ $\begin{cases} I = I_0 \cos\theta \\ I_0 = \alpha \cdot S \cdot W \end{cases}$

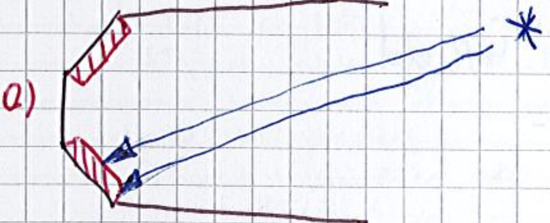
surface, power of radiation

↳ Each sensor has a define FOV

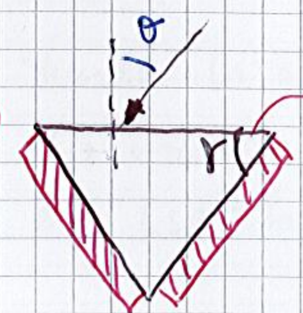


Sun presence sensors ⇒ only capable to detect if the Sun is in the (FOV) of the sensor.

a)



b)



→ CRITICAL ANGLE FOR INTERNAL TOTAL REFRACTION

$n_1 \sin\theta_1 = n_2 \sin\theta_2$ → $1 \cdot \sin\theta_1 = n_2 \sin\theta_2$

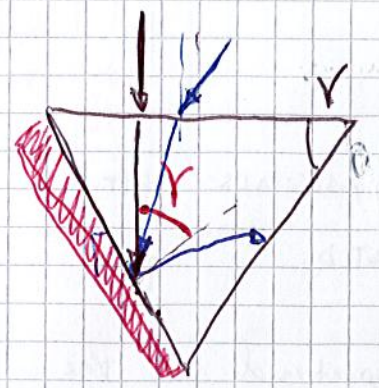
→ $\sin\gamma = n$

IF $\theta > \gamma$ ⇒ NO OUTPUT

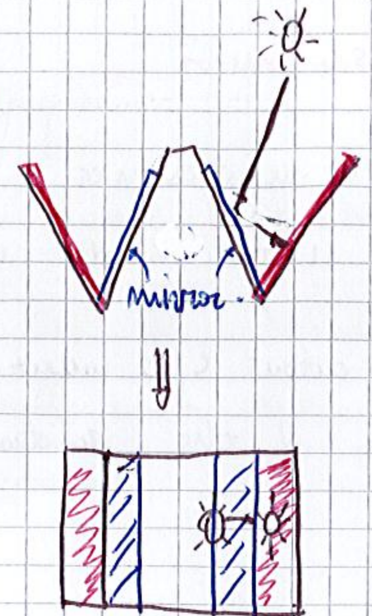
IF $\theta < \gamma$ ⇒ OUTPUT PROPORTIONAL TO θ.

Better.

consider the problem of the double refracted ray.



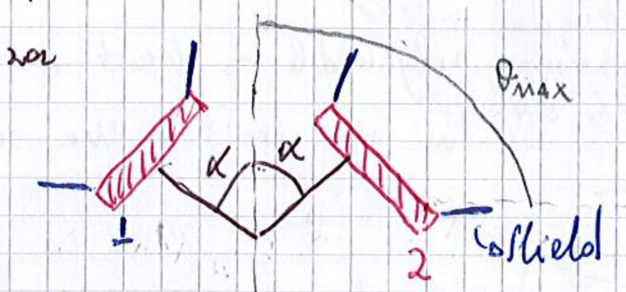
c)



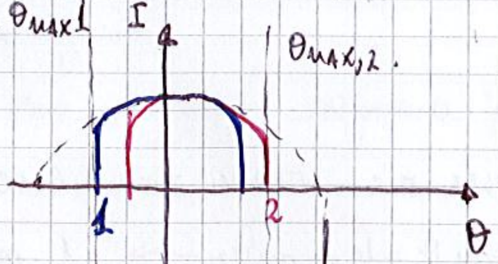
Analog and digital Sun sensors

!! cosine law cannot distinguish positive angles from negative angles !!

⇒ solution is put cells inclined w.r.t. the central axis of the sensor

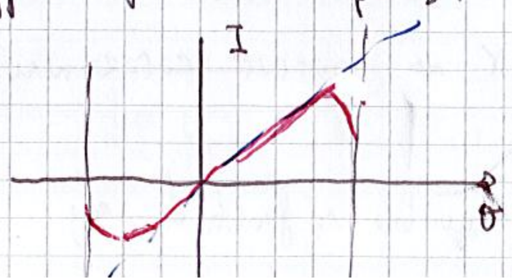


Now :



output for a common S.S.

Taking the difference of the 2 outputs.



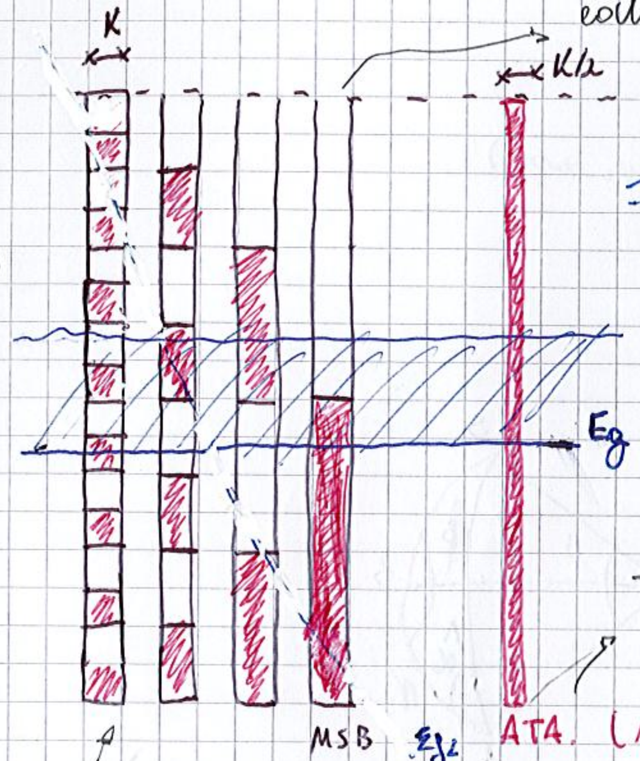
Sun perceived by the sensor will be $\begin{cases} \theta+\alpha \to \text{panel 2} \\ \theta-\alpha \to \text{panel 1} \end{cases}$

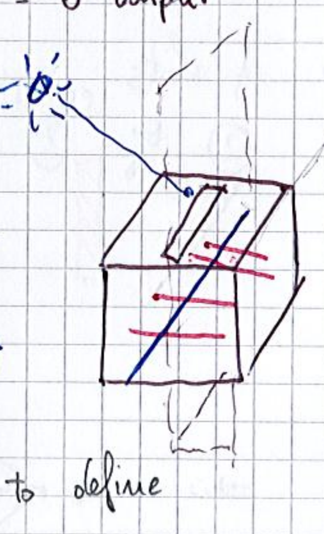
$$\begin{cases} I = I_0 \cos(\theta-\alpha) + I_0 \cos(\theta+\alpha) \\ \cos(\theta-\alpha) - \cos(\theta+\alpha) = \cdots = 2 \sin\theta \sin\alpha = K \sin\theta . \end{cases}$$

!! the output is sinusoidal as depending on the sign of θ. !!

Digital sun sensor can provide directly the θ-measure in binary number

each stripes gives a current as output.





to define

- if one sensible strip is illuminated ⇒ the value of the Bit is set to 1 $\left(I_{strip\,i} \lessgtr I_{ate} \right)$
- if one NOT sensible strip is illuminated ⇒ the value of the Bit is set to 0

Threshold (ATA) ⇒ $I_{ATA} = \alpha \frac{JK}{2} \cos\theta$. (J: GRID SPACING OF THE LSB)

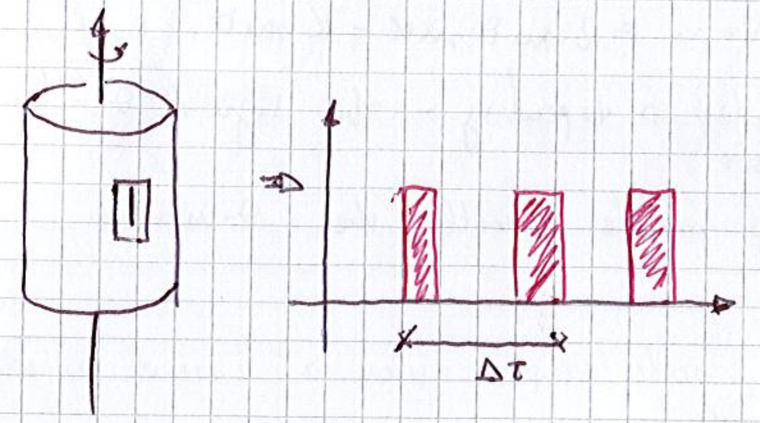
Bit i-th (strip i-th)

e.g. LSB ⇒ $I_{LSB} = \alpha \frac{xK}{2} \cos\theta$. (x: PORTION OF THE LSB ILLUMINATED)

IF $I_{LSB} > I_{ATA}$ ⇒ LSB=1
LSB: MORE than half illuminated

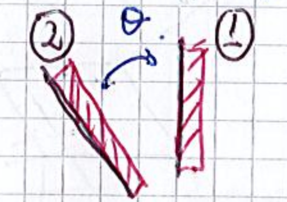
ELSEIF $I_{LSB} < I_{ATA}$ ⇒ LSB=0.

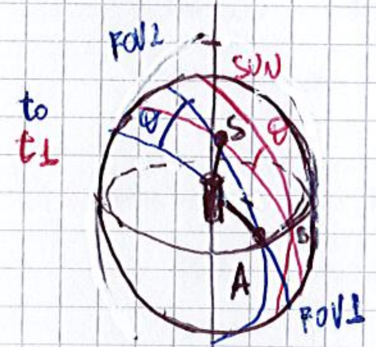
Precise sun sensor in single spin satellites



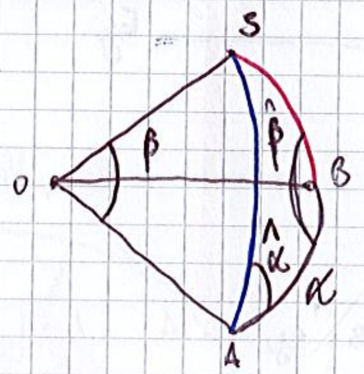
Signal produced at every rotation of the spacecraft.

Using a Pair (2) of slits





$\Rightarrow$



$\Rightarrow \alpha = AB = W \cdot t \quad (W = 2\pi / T_{rot})$

$\hat{\alpha} = 90^\circ$ (orthogonal to the equator)

$\hat{\beta} = 90^\circ - \theta$ (θ: angle at which sun is observed)

-- spherical triangle

$$\begin{cases} \cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma \cdot \cos\hat{\alpha} \\ \cos\hat{\alpha} = -\cos\hat{\beta}\cos\hat{\gamma} + \sin\hat{\beta}\sin\hat{\gamma}\cos\alpha \end{cases}$$

$+$

$$\frac{\sin\hat{\alpha}}{\sin\alpha} = \frac{\sin\hat{\beta}}{\sin\beta} = \frac{\sin\hat{\gamma}}{\sin\gamma}$$

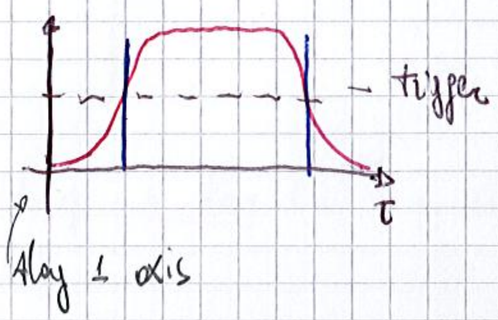
$\Rightarrow \cos\beta = \cos\alpha$ ...

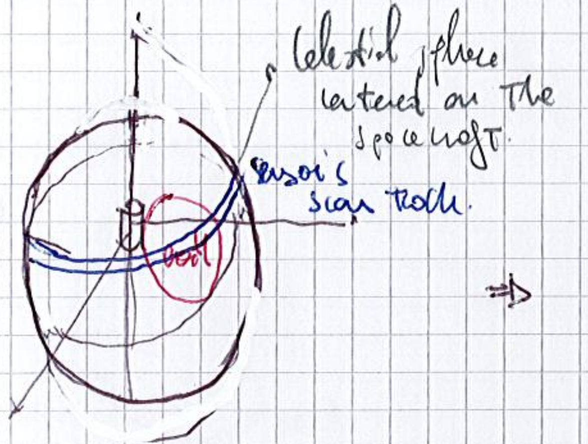
Sensor 2) Horizon Sensor.

IR → sensor , ORBITING AROUND a planet.

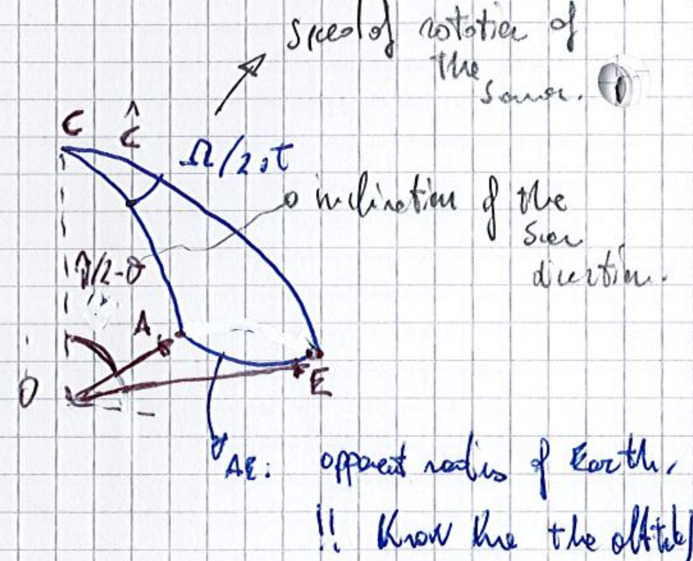
Assumption: • Planet is a sphere.

• Sensor is scanning along 2 axis





⇒



$\overline{CA} = \gamma$ (-latitude $+\pi/2$ = colatitude of the satellite)

$\hat{c} = \dfrac{\Omega_{scan} \cdot \Delta T}{2}$

$AE = \rho$

⇒