Vehicle modeling

Master Lab in Autonomous driving - Motion planning and control

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Dipartimento di Ingegneria "Enzo Ferrari"

Università di Modena e Reggio Emilia

Lecture objectives

- Lateral and longitudinal vehicle modeling

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- Kinematic and dynamic models

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- Formulation of control and simulation oriented models

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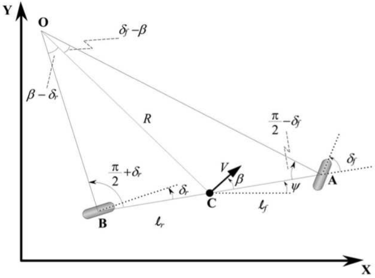
- Lateral and longitudinal vehicle modeling
- Kinematic and dynamic models
- Formulation of control and simulation oriented models
- Analysis of the basic properties of the vehicle models

Kinematic model of the lateral motion

Objective. Calculating the position variables from the speed variables, *without considering the forces generating them*.



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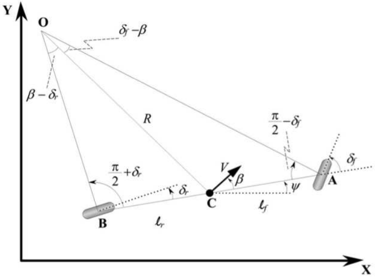

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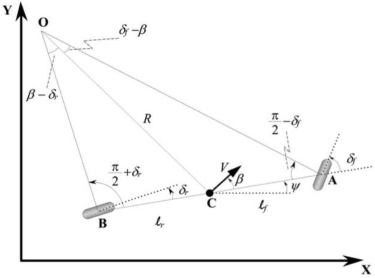

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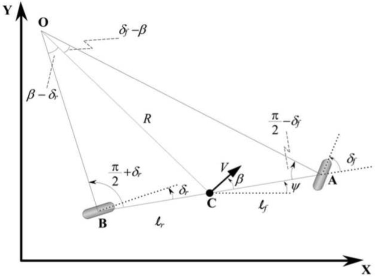

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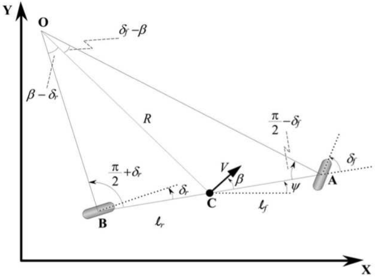

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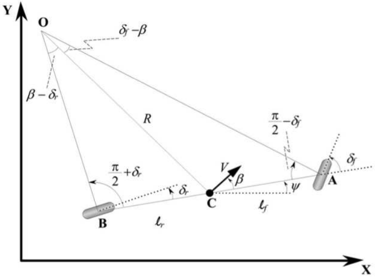

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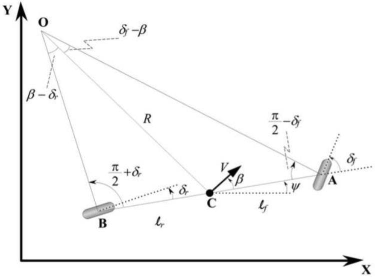

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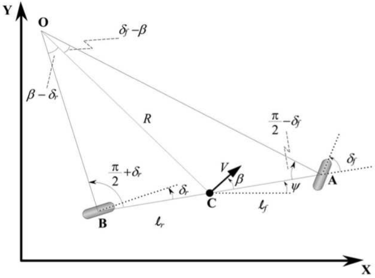

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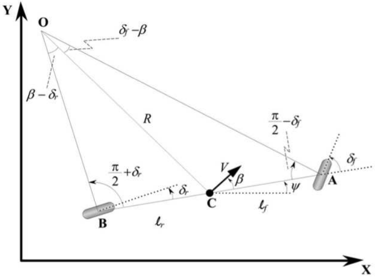

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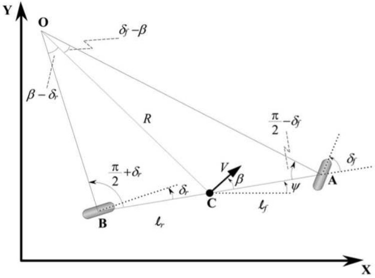

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The vehicle motion is described by the following ODEs system

$$
\begin{aligned}
\dot{X} &= V\cos(\psi+\beta),\\
\dot{Y} &= V\sin(\psi+\beta),\\
\dot{\psi} &= \frac{V\cos\beta}{l_f+l_r}\left(\tan\delta_f-\tan\delta_r\right),\\
\beta &= \tan^{-1}\left(\frac{l_f\tan\delta_r+l_r\tan\delta_f}{l_f+l_r}\right).
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A nonlinear model in the state space

$$\dot{x}=f(x,u),$$

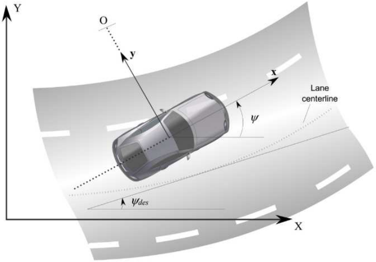
can be obtained by setting

$$x=\begin{bmatrix} X\\ Y\\ \psi\end{bmatrix},\ u=\begin{bmatrix} V\\ \delta_f\\ \delta_r\end{bmatrix}$$

Lateral vehicle dynamics

Write the Newton's law along the y axis

$$m\ddot{y} = -\underbrace{V_x\dot{\psi}}_{\text{centripetal acceleration}} + F_{yf} + F_{yr}.$$



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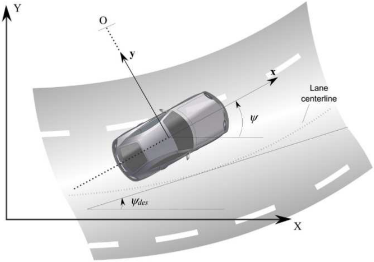
$$I_z\ddot{\psi} = l_f F_{yf} - l_r F_{yr}.$$



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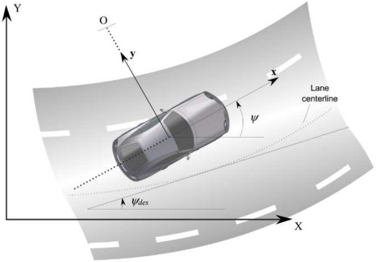

Write the lateral tire forces as

$$\begin{aligned} F_{yf} &= 2C_{\alpha f}(\delta - \theta_{Vf}), \\ F_{yr} &= -2C_{\alpha r}\theta_{Vr}, \end{aligned}$$

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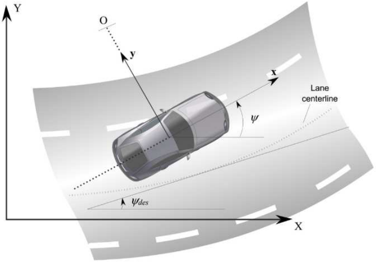

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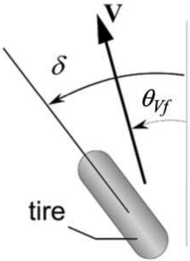
where (small) tire slip angles can approximated as

$$\theta_{Vf} = \frac{\dot{y} + l_f\dot{\psi}}{V_x}, \quad \theta_{Vr} = \frac{\dot{y} - l_r\dot{\psi}}{V_x}.$$



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The resulting, speed (V_x) dependent state space model is

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2C_{\alpha f}+2C_{\alpha r}}{mV_x} & 0 & -V_x - \frac{2C_{\alpha f}l_f-2C_{\alpha r}l_r}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2C_{\alpha f}l_f-2C_{\alpha r}l_r}{I_zV_x} & 0 & -\frac{2C_{\alpha f}l_f^2+2C_{\alpha r}l_r^2}{I_zV_x} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2C_{\alpha f}}{m} \\ 0 \\ \frac{2l_fC_{\alpha f}}{I_z} \end{bmatrix} \delta$$

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Remark. The model has been derived under the assumption of linear tire forces. This assumption holds for small tire slip angles. More accurate tire forces reveals tire force saturations for large slip angles.

Lateral vehicle dynamics with road-aligned reference frame

Assume to know the road geometry (curvature radius).

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$$\ddot{e}_1 = (\ddot{y} + V_x \dot{\psi}) - \frac{V_x^2}{R} = \ddot{y} + V_x \underbrace{(\dot{\psi} - \dot{\psi}_{des})}_{\dot{e}_2}.$$

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By combining with the Newton's law....

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$$+ \begin{bmatrix} 0 \\ -\frac{2C_{\alpha f}l_f-2C_{\alpha r}l_r}{mV_x} - V_x \\ 0 \\ -\frac{2C_{\alpha f}l_f^2+2C_{\alpha r}l_r^2}{I_zV_x} \end{bmatrix} \dot{\psi}_{des}$$

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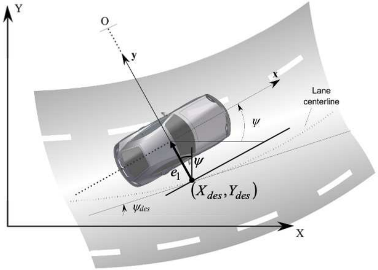
$$\dot{x} = A(V_x)x + B(V_x)u + B_d(V_x)d.$$

Transformation into global coordinates

Objective. Calculating the global coordinates $X,\ Y$ from $e_1,\ e_2$.



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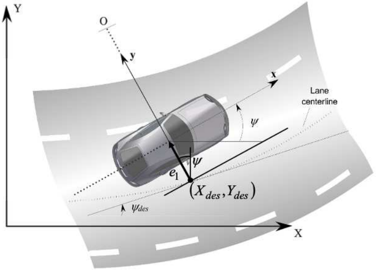

The global coordinates are calculated from the coordinates of the desired path and the lateral deviation e_1

$$X = X_{des} - e_1 \sin\psi,$$
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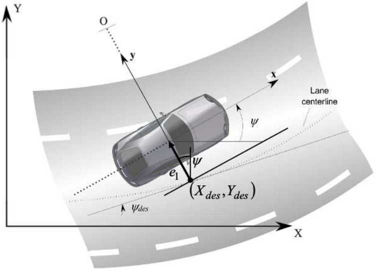

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$$Y = Y_{des} + e_1 \cos \psi$$

Using

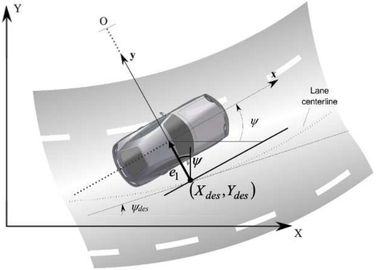
$$X_{des} = \int_0^t V \cos \psi_{des} dt,$$
$$Y_{des} = \int_0^t V \sin \psi_{des} dt,$$
$$\psi = e_2 + \psi_{des}$$

Transformation into global coordinates

Objective. Calculating the global coordinates X, Y from e_1, e_2.

The global coordinates are calculated from the coordinates of the desired path and the lateral deviation e_1

$$X = X_{des} - e_1 \sin\psi,$$
$$Y = Y_{des} + e_1 \cos\psi$$



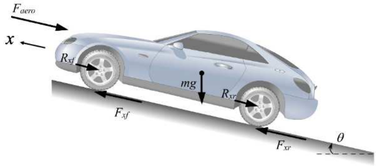
Using

$$X_{des} = \int_0^t V \cos\psi_{des} dt,$$
$$Y_{des} = \int_0^t V \sin\psi_{des} dt,$$
$$\psi = e_2 + \psi_{des}$$

the global coordinates are written as

$$X = \int_0^t V \cos\psi_{des} dt - e_1 \sin(e_2 + \psi_{des}),$$
$$Y = \int_0^t V \sin\psi_{des} dt + e_1 \cos(e_2 + \psi_{des}).$$

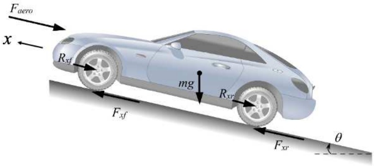
Longitudinal vehicle dynamics



Longitudinal vehicle dynamics

Notation

1. F_{xf}, F_{xr} front and rear longitudinal tire forces,



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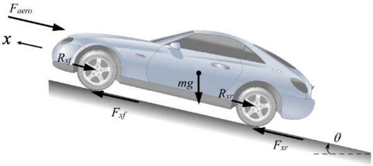

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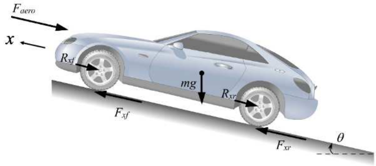

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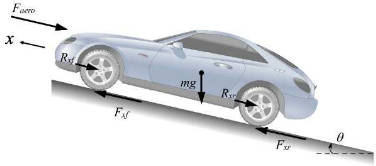

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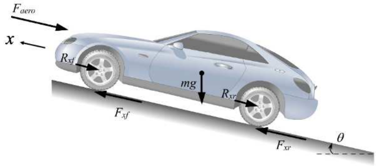

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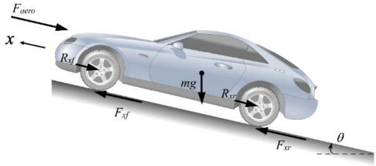

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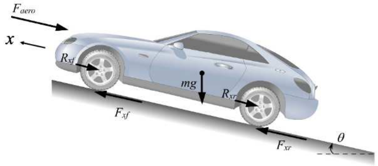

Write the Newton's law along the vehicle longitudinal axis

$$m\ddot{x} = F_{xf} + F_{xr} - F_{aero} - R_{xf} - R_{xr} - mg\sin\theta.$$

Longitudinal vehicle dynamics

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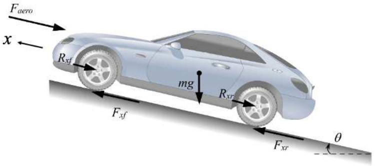

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Air drag force

$$F_{aero} = \frac{1}{2}\rho C_d A_F (V_x + V_{wind})^2,$$
$$A_F = 1.6 + 0.00056(m - 765),$$
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Longitudinal vehicle dynamics

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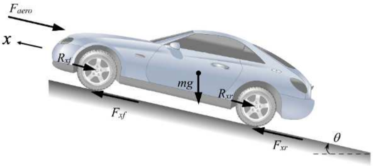

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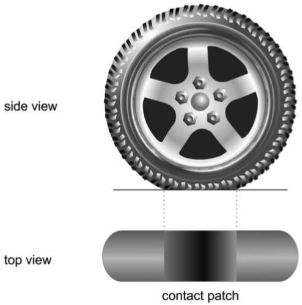
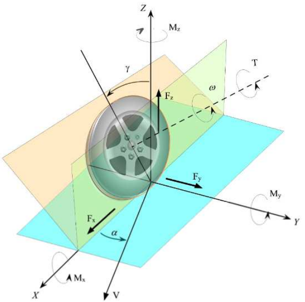
$$\begin{aligned} F_{x\star} &= C_{\sigma\star}\sigma_{x\star}, \\ \sigma_{x\star} &= \frac{r_w\omega_{w\star} - V_x}{r_w\omega_{w\star}},\ \star \in \{f,\ r\} \end{aligned}$$

Tire forces

The interaction between the tire contact patch and the road generates a number of forces and moments





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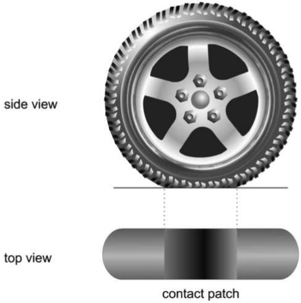

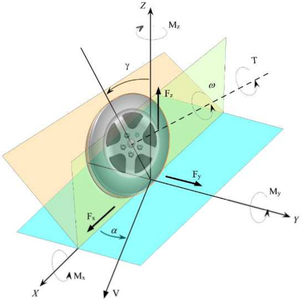

In this course we are interested in modeling the forces F_x, F_y, as function of the vehicle states and control input.

$$F_{x_\star} = f_x(\alpha_\star, \sigma_\star, \mu_\star, F_{z_\star}),$$
$$F_{y_\star} = f_y(\alpha_\star, \sigma_\star, \mu_\star, F_{z_\star}).$$

Tire forces
Longitudinal tire forces

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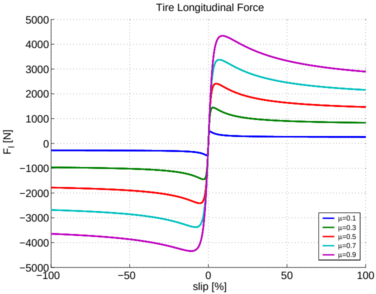

The forces have been plotted for a constant normal force F_z and varying friction coefficient μ.

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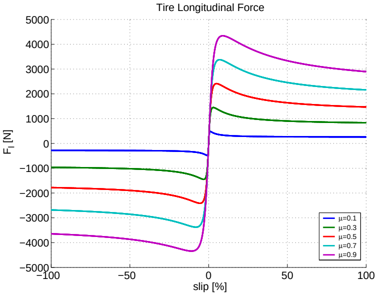

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These are *static* tire forces characteristics. Indeed, tire dynamics are fast and can be neglected in our applications.

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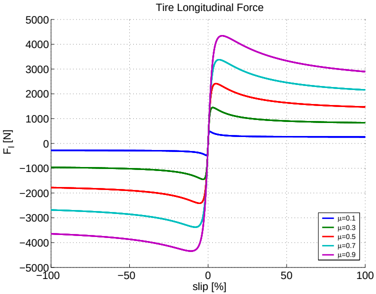

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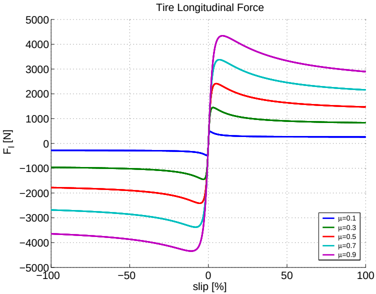

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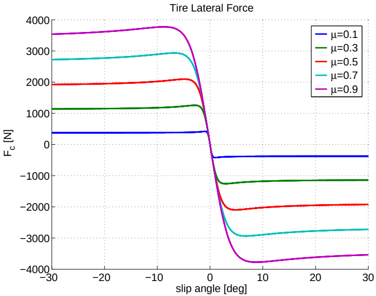

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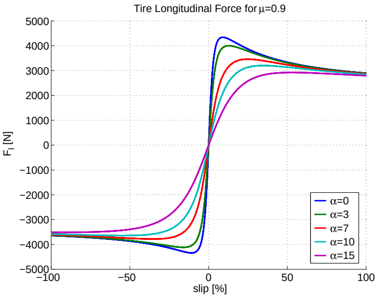

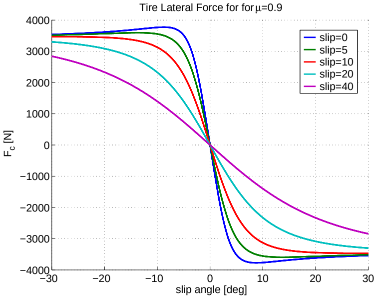

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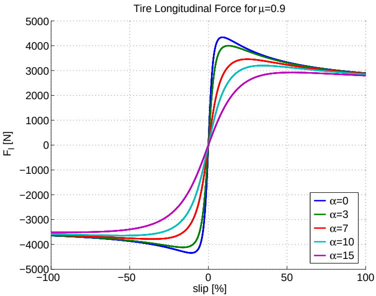

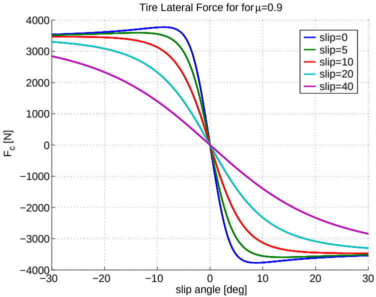

Physical modeling of the tire forces can be very much involving.

Although useful physical tire models exist (Brush model, LuGre friction model, Dugoff's model), the *semi-empirical Pacejka's model* is the most widespread.

The magic formula

The Pacejka's tire model relies on functions, which are "shaped" to resemble the tire forces.

$$Y(X) = D \sin\left(C \arctan\left(B\Phi(X)\right)\right) + S_v$$

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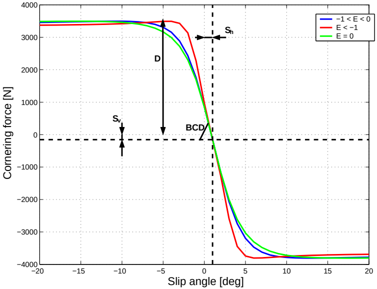

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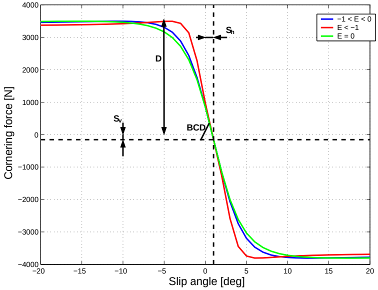

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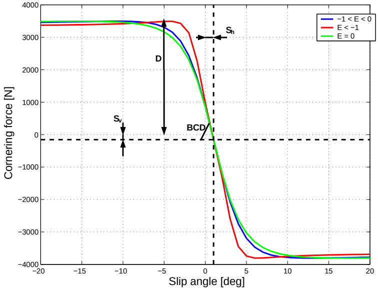

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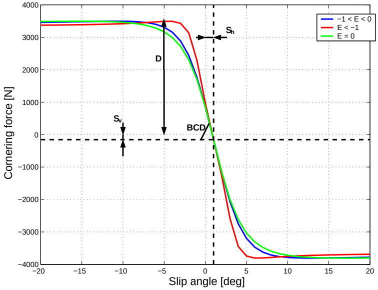

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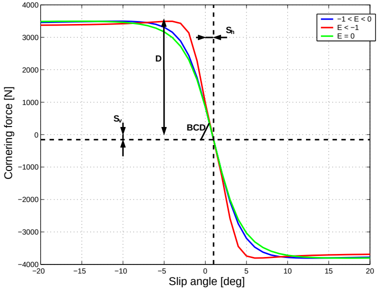

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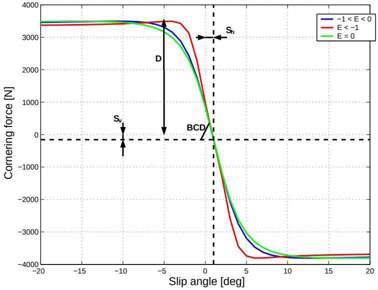

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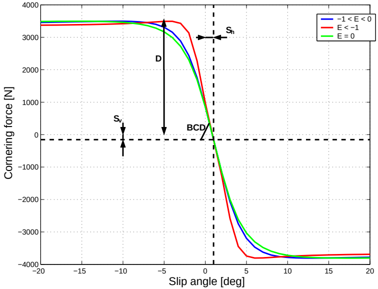

- Φ is defined as:

$$\Phi(X) = (1 - E)(X + S_h) + (E/B)\arctan\left(B(X + S_h)\right),$$

where E is the *curvature factor* and S_h is the *horizontal shift*

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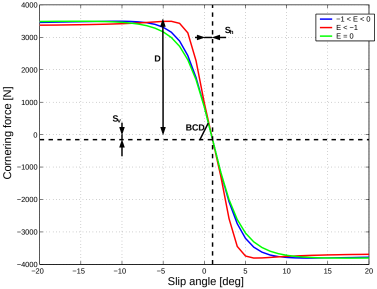

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The parameters in the magic formula are calibrated on experimental data.

Longitudinal tire forces and wheels dynamics

In order to calculate F_x we need the tire longitudinal slip. This depends on the wheels speed.

$$m\ddot{x} = F_{xf} + F_{xr} - F_{aero} - R_{xf} - R_{xr} - mg\sin\theta,$$

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Remarks.

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\sigma_\star &= \begin{cases} \dfrac{r_w\omega_\star}{V_{x_\star}} - 1 \text{ if } V_{x_\star} > r_w\omega_\star,\ V_{x_\star} \neq 0 \text{ for braking} \\ 1 - \dfrac{v_{x_\star}}{r_w\omega_\star} \text{ if } V_{x_\star} < r_w\omega_\star,\ \omega_\star \neq 0 \text{ for driving,} \end{cases} \\
J_{w,\star}\dot{\omega}_{\star,\bullet} &= -F_{l_{\star,\bullet}} r_w - b\cdot\omega_{\star,\bullet} - T_{b_{\star,\bullet}} + T_{t_{\star,\bullet}}, \\
&\star \in \{f,\ r\},\ \bullet \in \{l,\ r\}
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- $V_{x_\star}$ should be the wheel velocity. You can approximate it with the vehicle speed.

Longitudinal tire forces and wheels dynamics

In order to calculate F_x we need the tire longitudinal slip. This depends on the wheels speed.

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- $T_{b_{\star,\bullet}}$, $T_{t_{\star,\bullet}}$ can be seen as *control inputs* to the system.