Modeling and dynamical systems

Master Lab in Autonomous driving - Motion planning and control

Paolo Falcone

Dipartimento di Ingegneria "Enzo Ferrari"

Università di Modena e Reggio Emilia

Lecture objectives

- Illustrating the physical modeling and modeling from data paradigms

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- Examples of physical modeling

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- Response of dynamical systems

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- Illustrating the physical modeling and modeling from data paradigms
- Examples of physical modeling
- Response of dynamical systems
- Fundamental properties of dynamical systems

The modeling problem

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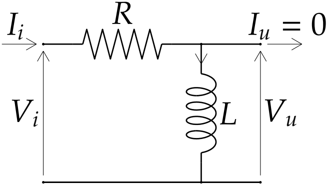
- ***Cons.*** Data is necessary. Experiments for data collection needs to be carefully designed. A change in the system params need the complete modeling procedure to be redone.

Examples of physical models. RL circuit



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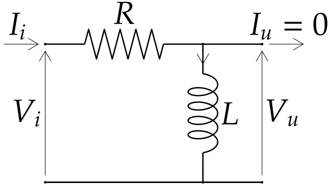

The variation of the concatenated flux $\phi_c(t) = LI_i(t)$ is equal to the voltage $V_u(t)$ applied to the inductance.

$$\frac{d}{dt}[\phi_c(t)] = V_u(t) \qquad \rightarrow \qquad L\,\frac{d}{dt}[I_i(t)] = V_i(t) - R\,I_i(t)$$

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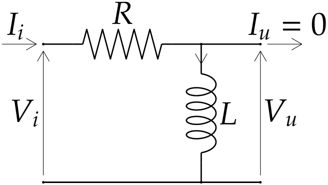

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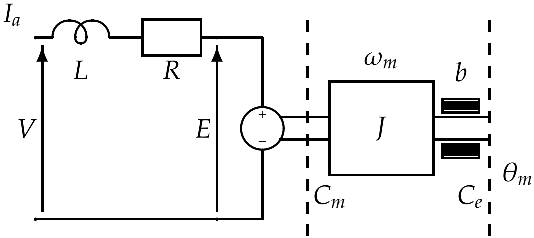
By setting $x(t) = I_i(t)$ (state variable), $u(t) = V_i(t)$ (input variable) and $y(t) = I_i(t)$ (output variable), the following dynamical model (ODE) ***in the state-space*** can be derived

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

with

$$A = -\frac{R}{L}, \;\; B = \frac{1}{L}, \;\; C = 1.$$

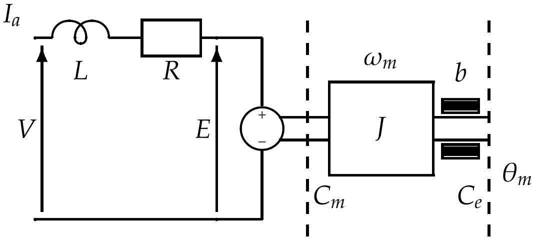
Examples of physical models. DC motor



Examples of physical models. DC motor

The system is described by the following two differential equations:

$$\begin{cases} L\dot{I}_a &= -RI_a - K_e\,\omega_m + V \\ J\dot{\omega}_m &= K_e I_a - b\,\omega_m - C_e \end{cases}$$



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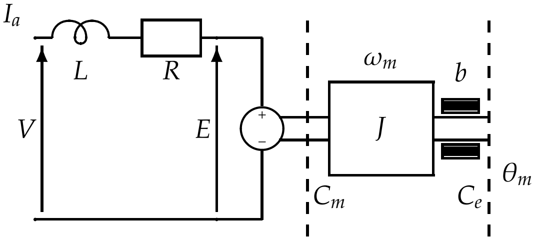

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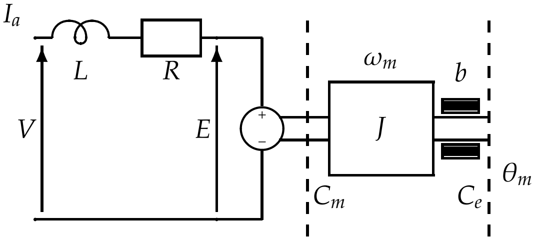

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By setting $x = \begin{bmatrix} I_a \\ \omega_m \end{bmatrix}$, $u = V, d = C_e,\ y = \omega_m$, we obtain the following state-space model

$$\dot{x} = Ax + Bu + B_d d, \quad y = Cx,$$

with $A = \begin{bmatrix} -\frac{R}{L} & -\frac{K_e}{L} \\ \frac{K_e}{J} & -\frac{b}{J} \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_d = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ C = [0\ 1]$

Response of linear dynamical systems

The solution of $\dot{x}(t) = Ax(t) + Bu(t)$ with initial condition $x(t_0)$ is

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}Bu(s)ds \tag{1}$$

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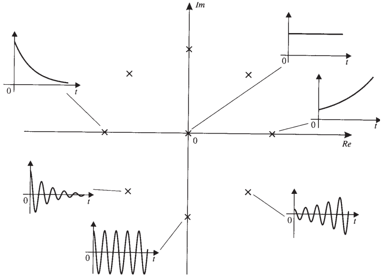

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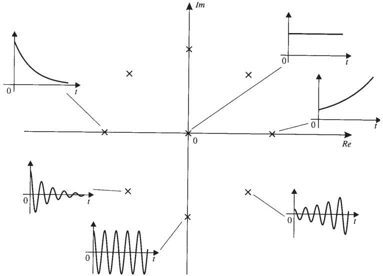
Real eigenvalues corresponds to aperiodic responses.



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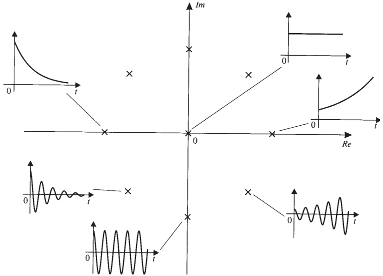
The response depends on the eigenvalues of the matrix A.

Real eigenvalues corresponds to aperiodic responses.

Conjugate complex eigenvalues corresponds to pseudo-periodic responses.

Since the system is linear, the *superposition principle holds*.



Stability of continuous-time dynamical systems

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Definition (*Stability*)

The trajectory $\tilde{x}(t)$ is stable if

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 :\ \|x_0 - \tilde{x}_0\| \le \delta,\ \Rightarrow\ \|x(t) - \tilde{x}(t)\| \le \varepsilon.$$

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Definition (*Asymptotic stability*)

The trajectory $\tilde{x}(t)$ is *asymptotically stable* if

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and $\lim_{t\to\infty} \|x(t) - \tilde{x}(t)\| = 0$.

Stability of LTI systems

The linear, discrete time system

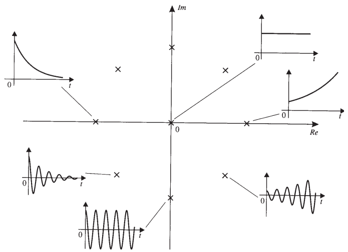
$$\dot{x} = Ax \tag{2}$$

is asymptotically stable (solutions converge to the origin) if and only if the eigenvalues of A lie in the left half-plane .

The position of the eigenvalues affect the behavior of the system.

Real poles in the lhp contribute with aperiodic stable responses.

Conjugate complex poles in the lhp contribute with pseudo-periodic stable responses.



Response of LDS. Real poles

Example. Step response of the following differential equation:

$$a\,\dot{y}(t) + b\,y(t) = c\,u(t)$$

$$A = -\frac{b}{a},\; B = \frac{C}{a},\; C = 1.$$

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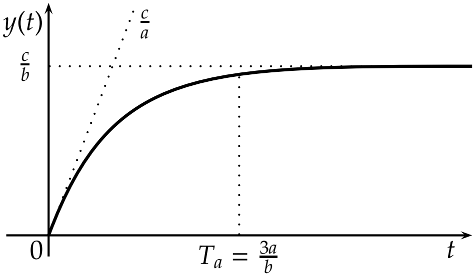

Response of LDS. Conjugate complex poles

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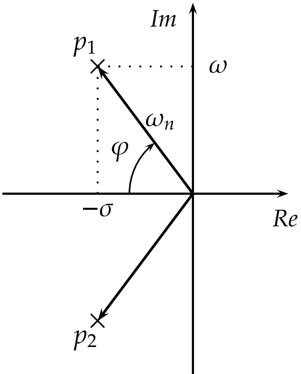
The response to a unitary step is

$$y(t) = 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \sin(\omega\, t + \varphi)$$



$$\omega := \omega_n \sqrt{1-\delta^2}$$
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Step response. Second order system

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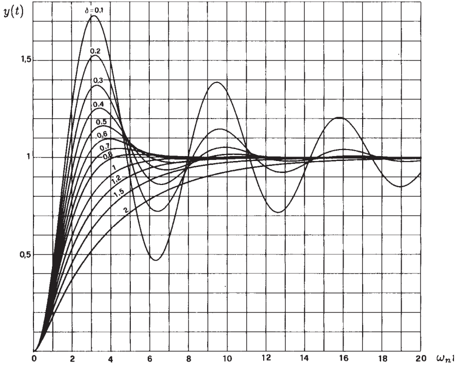

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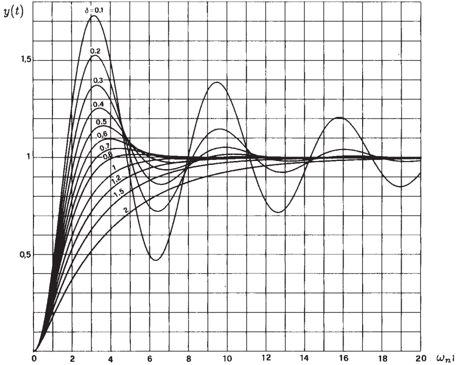

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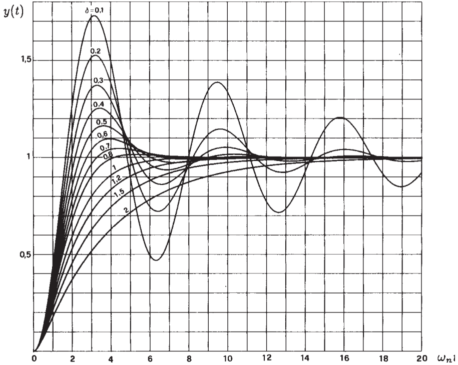

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- With $\delta = 0$, the response tends to an undamped sinusoid.

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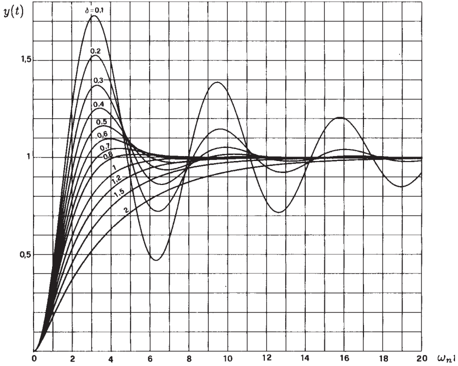

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- Large σ $\rightarrow$ large ω_n. Fast decaying exp.

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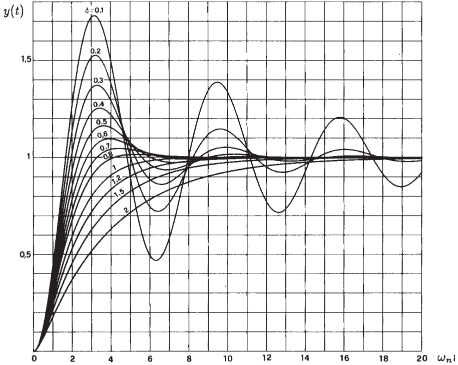

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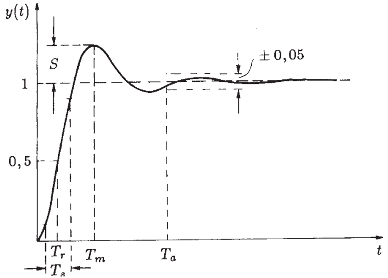
- With $\delta = 0$, the response tends to an undamped sinusoid.
- Large $\sigma \rightarrow$ large ω_n. Fast decaying exp.
- Small $\varphi \rightarrow$ large δ. Fast decaying exp.

Step response. Second order system

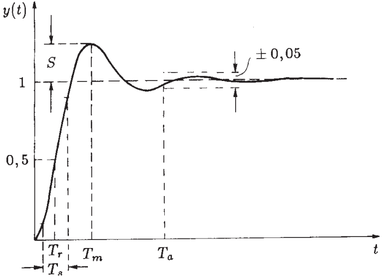
Main parameters of the step response



Step response. Second order system

Main parameters of the step response

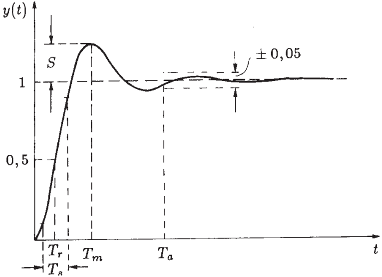
- *Maximum overshoot S*



Step response. Second order system

Main parameters of the step response

- *Maximum overshoot S*
- *Delay time T_r*



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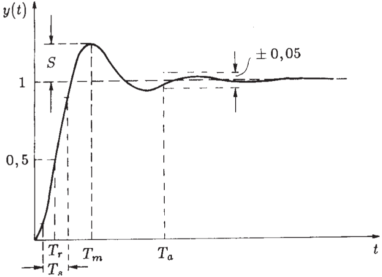
- *Delay time T_r*
- *Rise time T_s*



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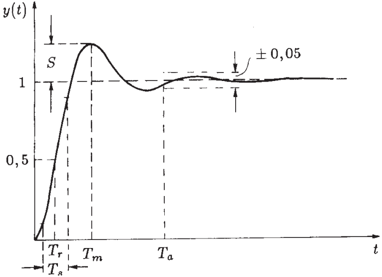
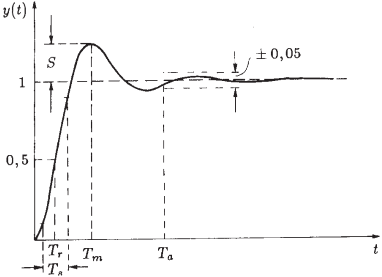
- *Delay time T_r*
- *Rise time T_s*
- *Settling time T_a*



Step response. Second order system

Main parameters of the step response

- *Maximum overshoot S*
- *Delay time T_r*
- *Rise time T_s*
- *Settling time T_a*
- *Instant of the maximum overshoot T_m*



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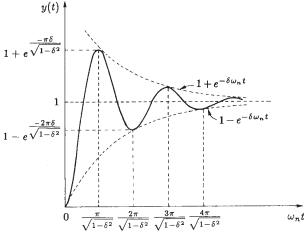

Step response. Second order system

A number of parameters of the step response can be related to δ and ω_n (hence to σ and ω).

- *Peaks* and *peak times*. It can be shown that

$$y(t)\Big|_{\substack{\max\\ \min}} = 1 - (-1)^n \, e^{\frac{-n\pi\delta}{\sqrt{1-\delta^2}}}$$

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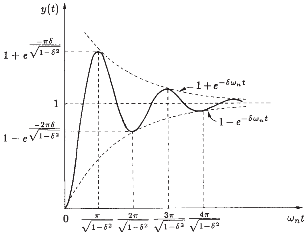

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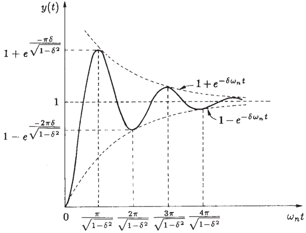

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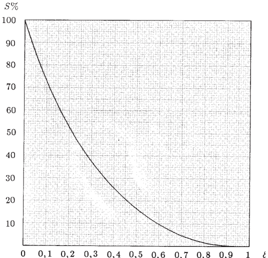

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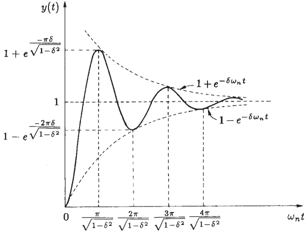

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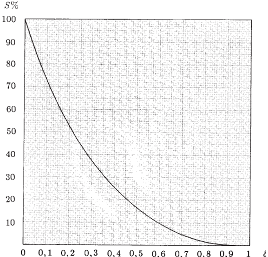

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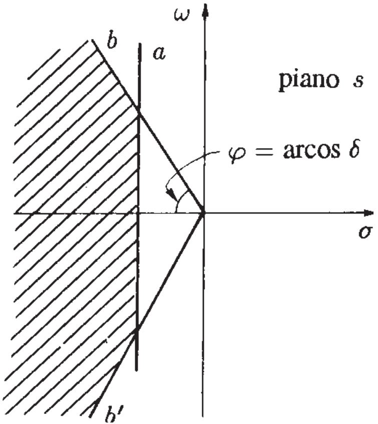

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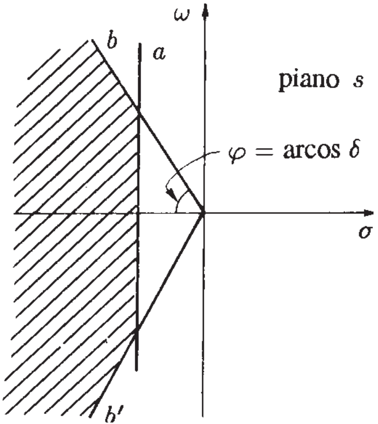

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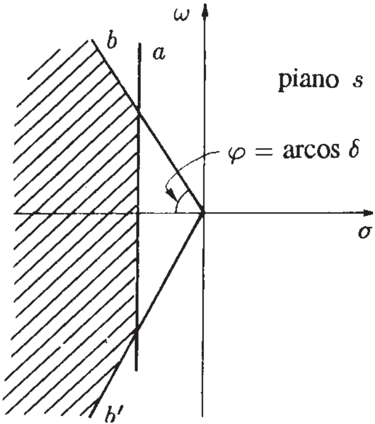
$$\boxed{T_a = \frac{3}{\delta\,\omega_n} = \frac{3}{|\sigma|}}$$



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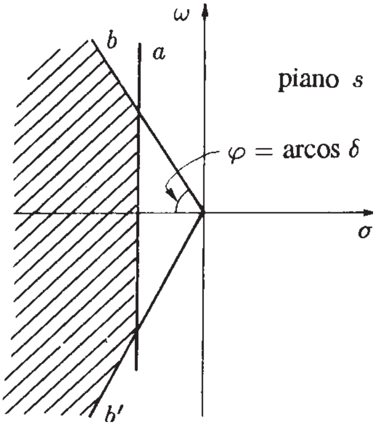

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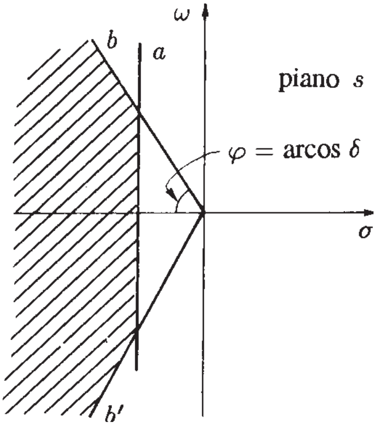

Step response. Second order system

Effect of the ***damping coefficient*** δ.

Step response. Second order system

Effect of the ***damping coefficient*** δ. Vary δ, while $\omega_n = \text{const}$.

$0 < \delta < 1$

$\frac{\pi}{2} > \varphi > 0$

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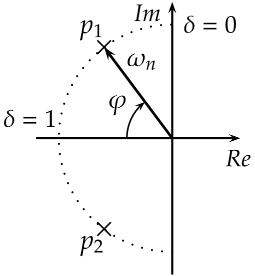

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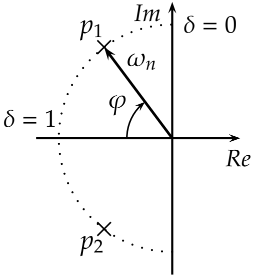
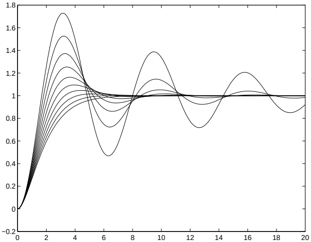


$\delta \in [0.1,\ 0.2,\ \ldots,\ 1]$



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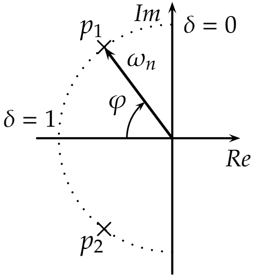

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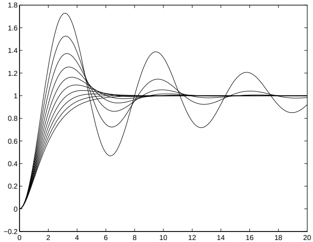

Recall that

$$y(t) = 1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \sin(\omega\, t + \varphi)$$

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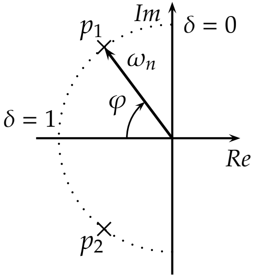

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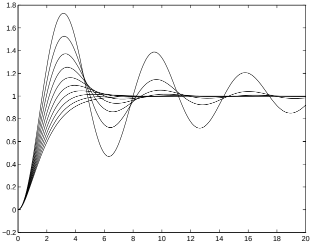

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Effect of the ***natural frequency*** ω_n.

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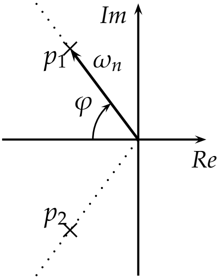

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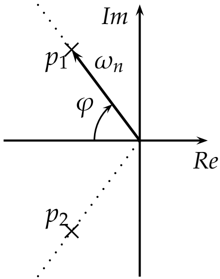
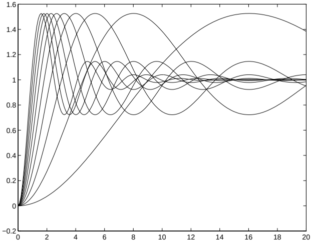


$\omega_n \in [0.2,\ 0.4,\ \ldots,\ 2]$



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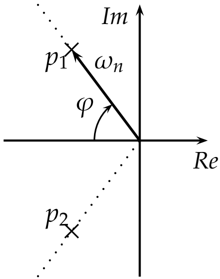

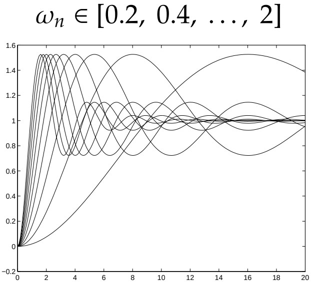

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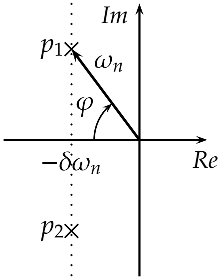

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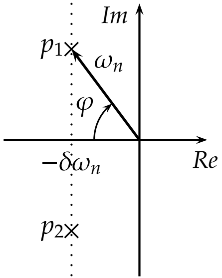

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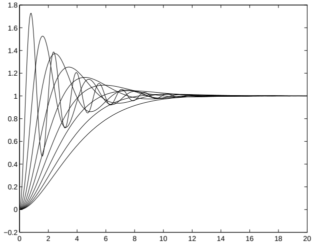

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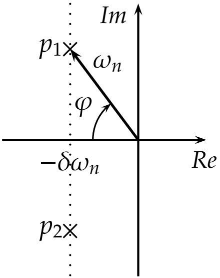

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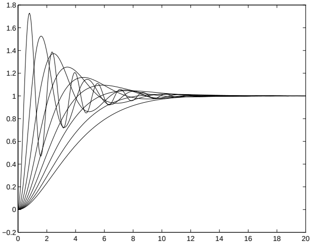

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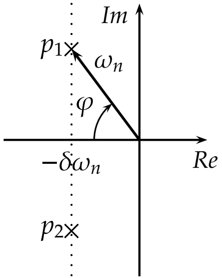

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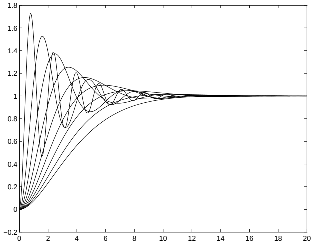

Recall that $\boxed{T_a = \dfrac{3}{\delta\omega_n}}$. <u>The poles move along a locus of constant T_a</u>

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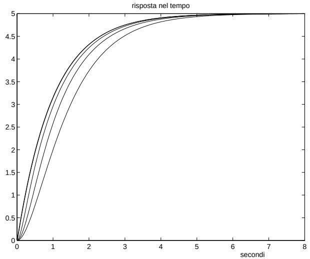

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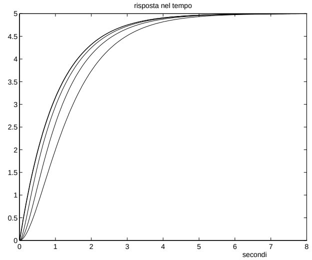

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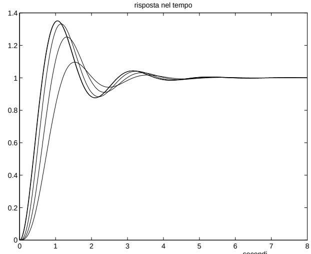

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Assume that the control signal u is piecewise constant (h is the sampling interval):

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$$x(k+1) = e^{Ah}x(k) + \left(\int_0^h e^{As} B_c \, ds \right) u(k) = Ax(k) + Bu(k),$$

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A compact version of the discrete state-space model in the case $z = y$ is

$$\begin{aligned} x^+ &= Ax + Bu \\ y &= Cx \end{aligned}$$

Response of discrete-time systems

The response of

$$x(k+1) = Ax(k) + Bu(k)$$
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to an input signal $[u(0),\ u(1), \dots, u(k-1)]$, from an initial state $x(0)$, can be calculated by recursively applying the state update equation

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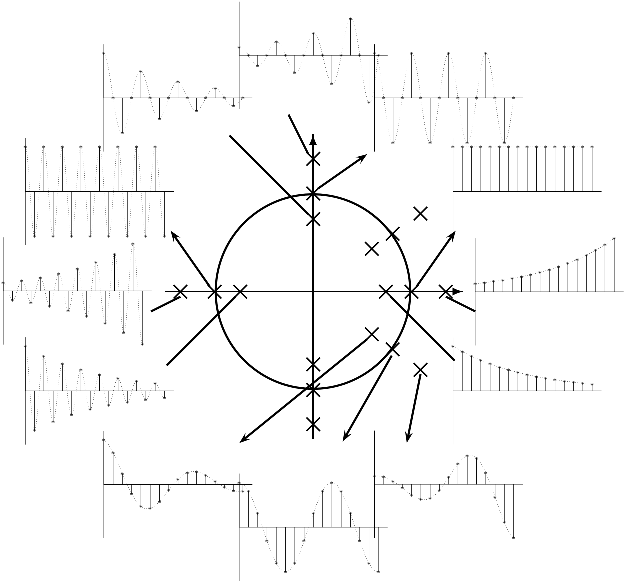
Example. Consider the system $x^+ = 0.1x,\ x(0) = 1$. How does the response look like?

Response of discrete-time systems



Observability

A linear, discrete time system

$$\begin{aligned} x^+ &= Ax \\ y &= Cx \end{aligned}$$

is observable if for some N, any $x(0)$ can be determined from $\{y(0), y(1), \ldots, y(N-1)\}$.

The system is observable if and only if any of the following, equivalent, conditions hold:

- The matrix $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank n
- The matrix $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has rank n for all $\lambda \in \mathbf{C}$

Note. A weaker condition is *detectability*, which requires that any unobservable modes are strictly stable.

Controllability

A linear, discrete time system

$$x^+ = Ax + Bu$$

is controllable if it is possible to steer the system from any state x_0 to any state x_f in finite time.

The system is controllable if and only if any of the following, equivalent, conditions hold:

- The matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ has full rank n
- The matrix $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ has rank n for all $\lambda \in \mathbf{C}$

Note. A weaker condition is *stabilizability* which requires that any uncontrollable modes are strictly stable.