Feedback control and state estimation

Master Lab in Autonomous driving - Motion planning and control

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Lecture objectives

- A general control structure. State- and output-feedback control

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- Pole placement, Linear Quadratic regulators

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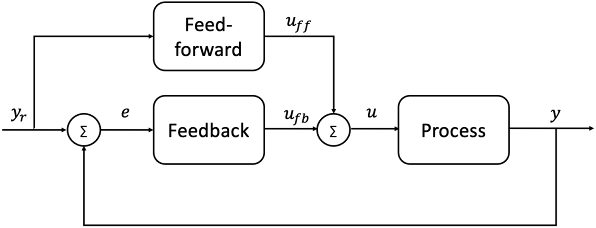
A general control structure



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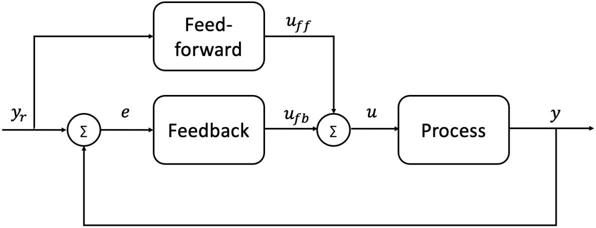

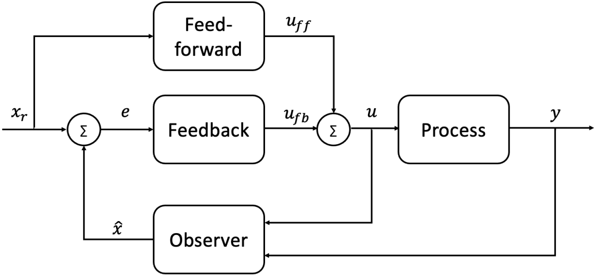

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How do we design K?

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Objective. Design the gain K in order to assign the poles of the closed-loop system

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K is to be designed such that the roots of $p(s)$ are placed as desired.

Pole placement. Example

Consider the double integrator

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$
$$y = [1,\ 0]x.$$

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$K_r = k_1$ leads to unitary steady state gain. That is, $y(t) = r,\ t \to \infty$.

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The feedback gain is given by

$$K = [p_1 - a_1 \; p_2 - a_2 \; \ldots \; p_n - a_n] \tilde{W}_r W_r^{-1},$$
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with

$$W_r = [B\ AB\ A^2B \ldots A^{n-1}B],$$

$$\tilde{W}_r = \begin{bmatrix} 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 1 & a_1 & \ldots & a_{n-1} \\ \vdots & & & & \\ 0 & 0 & \ldots & 0 & 1 \end{bmatrix}^{-1}$$

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Step 3. Calculate the characteristic polynomials and equate the coefficients of equal powers of s to the desired polynomials

$$p(s) = s^n + p_1 s_{n-1} + \dots + p_{n-1}s + p_n$$

.

The LQ problem

System:

$$S: \quad x^{+} = Ax + Bu \tag{1}$$

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$$P: \quad \min_{u(0:N-1)} V_N(x(0), u(0:N-1))$$

where the minimization is with respect to the sequence of control inputs

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The *objective* or *criterion* or *cost function* V_N is given by

$$\begin{aligned} V_N(x(0), u(0:N-1)) &= \sum_{i=0}^{N-1} (x^\top(i) Q x(i) + u^\top(i) R u(i)) + x^\top(N) P_f x(N) \\ &= \sum_{i=0}^{N-1} l(x(i), u(i)) + l_f(x(N)) \end{aligned} \tag{2}$$

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Remark 2: The first term $x^\top(0)Qx(0)$ in the objective is really redundant but is kept for notational convenience.

Batch solution

Repeated use of $x^+ = Ax + Bu$ gives

$$\begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix} x(0) + \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix} \tag{3}$$

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or, with a more compact notation,

$$\boldsymbol{x} = \Omega x(0) + \Gamma \boldsymbol{u} \tag{4}$$

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The LQ criterion (2) can now be written

$$\begin{aligned} V_N(x(0), \boldsymbol{u}) &= x^\top(0) Q x(0) + \boldsymbol{x}^\top \bar{Q} \boldsymbol{x} + \boldsymbol{u}^\top \bar{R} \boldsymbol{u} \\ &= x^\top(0) Q x(0) + (\Omega x(0) + \Gamma \boldsymbol{u})^\top \bar{Q} (\Omega x(0) + \Gamma \boldsymbol{u}) + \boldsymbol{u}^\top \bar{R} \boldsymbol{u} \\ &= \boldsymbol{u}^\top (\Gamma^\top \bar{Q} \Gamma + \bar{R}) \boldsymbol{u} + 2 x^\top(0) \Omega^\top \bar{Q} \Gamma \boldsymbol{u} + x^\top(0)(Q + \Omega^\top \bar{Q} \Omega) x(0) \end{aligned} \tag{5}$$

where $\bar{Q} = \operatorname{diag}(Q, \dots, Q, P_f)$ and $\bar{R} = \operatorname{diag}(R, \dots, R)$.

Dynamic programming solution to the LQ problem

Sequence of optimal control laws (*control policy*):

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\begin{aligned}
u^0(k;x) &= K(k)x, \quad k = 0, \ldots, N-1 \\
K(k) &= -(R + B^\top P(k+1)B)^{-1} B^\top P(k+1) A
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Riccati equation:

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Optimal cost-to-go (from time k to time N):

$$V^0_{k\to N}(x) = x^\top P(k)x$$

Infinite horizon LQ control

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The latter equation is called the *algebraic Riccati equation*. The optimal cost is given by

$$V^0_\infty(x(0)) = x^\top(0)Px(0)$$

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$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} r.$$

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The eigenvalues of the closed-loop system are the roots of the characteristic polynomial

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That is, if the system is *reachable* and *observable* the control and observer dynamics can be arbitrarily assigned (*separation principle*).

The Kalman filter

System model:

$$x(k+1) = Ax(k) + Bu(k) + w(k), \quad x(0) \sim \mathcal{N}(x_0, P_0), w \sim \mathcal{N}(0, Q)$$
$$y(k) = Cx(k) + v(k), \quad v \sim \mathcal{N}(0, R)$$

The Kalman filter

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$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k), \quad x(0) \sim \mathcal{N}(x_0, P_0), w \sim \mathcal{N}(0, Q) \\ y(k) &= Cx(k) + v(k), \quad v \sim \mathcal{N}(0, R) \end{aligned}$$

State estimator:

Correction: $\hat{x}(k|k) = \hat{x}(k|k-1) + L(k)[y(k) - C\hat{x}(k|k-1)], \ \hat{x}(0|0) = x_0$

Prediction: $\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$

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Kalman filter gain:

$$L(k) = P(k)C^\top [CP(k)C^\top + R]^{-1}$$

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State estimation error covariance update (note that $P(k) \widehat{=} P(k|k-1)$):

Estimation error: $P(k|k) = P(k) - P(k)C^\top[CP(k)C^\top + R]^{-1}CP(k),\ P(0|0) = P_0$

Prediction error: $P(k+1) = AP(k|k)A^\top + Q$

Stationary Kalman filter

System:

$$\begin{aligned} x^+ &= Ax + Bu + w, \quad w \sim \mathcal{N}(0, Q) \\ y &= Cx + v, \quad v \sim \mathcal{N}(0, R) \end{aligned}$$

If the pair (C, A) is observable and $Q, R \succ 0$, then the Kalman filter gain $L(k)$ and the prediction error covariance $P(k)$ converge to the solution of the (filtering) algebraic Riccati equation

$$\begin{aligned} L &= PC^\top [CPC^\top + R]^{-1} \\ P &= APA^\top - APC^\top [CPC^\top + R]^{-1} CPA^\top + Q \end{aligned}$$