Introduction:

In this report, we investigate whether dropping a test mass down the 4km deep mine at the equator is a feasible method of measuring the depth. To do so, we calculated the ideal fall time, the fall time with a varying gravitational acceleration, and with drag. We also considered the Coriolis force to determine whether the mass would hit the side of the wall before it hit the bottom. Finally, we investigated the effect that a non-uniform density in the Earth had on the fall time and motion of the test mass.

All objects at the earth's surface fall with identical accelerations (~9.8 m/s^2). To calculate the fall time, neglecting drag, we first solved the general freefall equation, a quadratic which relies on initial velocity, change in distance, and acceleration, for time. We then plotted the velocity and position and found the fall time by calculating where the position of the test mass was 4km and finding the time at that point.

Drag, or air resistance, is opposed to gravity and is caused by the object coming into contact with air molecules. This force is proportional to the velocity of the object, so we must use a differential equation to incorporate it.

Although the test mass falls straight down, the Earth's rotation causes the walls of the shaft to move in relation to the test mass. This could be a problem for our experiment, as the test mass hitting the walls of the shaft adds time to the fall time, which can lead to inaccurate data. The Coriolis force is a force that describes the movement of the Earth in relation to the test mass as a force on the test mass. This is how we will be evaluating whether the ball will hit the sides of the shaft during our test.

Calculation of Fall Time:

First, we calculated the theoretical fall time using the freefall equation:

$$0 = \Delta x + v_o t + \frac{1}{2} a_o t^2 \qquad 1$$

and solving for time. The fall time from this approach was 28.6 seconds.

The second approach used a second order differential equation:

$$\frac{d^2x}{dt^2} = g - \alpha v^\gamma \qquad 2$$

For now, we can ignore the second term, as alpha, the drag coefficient, is set to zero. We can see that the acceleration is equal to the gravitational acceleration, 9.8 m/s^2. From this approach, we found the fall time to also be 28.6 seconds, confirming our theoretical estimation from before.

However, gravity is dependent on the mass of the object. By going deep into the earth, the gravity affecting the test mass changes slightly. To account for this, we used the equation, where g_0 is the gravity at the surface, and R_e is the radius of the Earth:

$$g(r) = g_0\left(\frac{R_e - |r|}{R_e}\right) \qquad 3$$

By inserting E.3 into E.2 in place of the g term, we can calculate the fall time to be 28.6 seconds. There is a difference of about 0.002 seconds between this fall time and without a variable g.

As the tunnel has air in it, we must include the drag coefficient. To find the correct alpha, we used the knowledge that the terminal velocity should be ~50m/s^2. We used E.2 over many values of alpha and plotted them against the maximum (terminal) velocity generated. From this, we estimated the correct value of alpha to be 0.004. Using this alpha, we found a fall time of 84.3 seconds.

Varying the gravity by the position did not result in a noticeable change in the fall time; however, this is to be expected as 4km is only about 0.06% of the Earth's radius, so the reduction in gravitational acceleration would be minimal. However, we can see that including drag increases the fall time by an approximate factor of 3.

Feasibility of Depth Measurement Approach:

We calculated the Coriolis forces using the vector equation, where m is the mass of the object and omega is the Earth's rotation rate:

$$F_c = \left(+2m\Omega v_y\right)\hat{\imath} + (-2m\Omega v_x)\hat{\jmath} + 0\hat{k} \qquad 4$$

Note that î, ĵ, and k̂ indicate forces in the x, y, and z direction respectively. To start, we neglected the drag force. By adding E.4 to E.2 and calculating the x position, we can determine that the test mass will hit the wall of the shaft before it hits the bottom, at a time of 21.9 seconds and position of 2352 meters down.

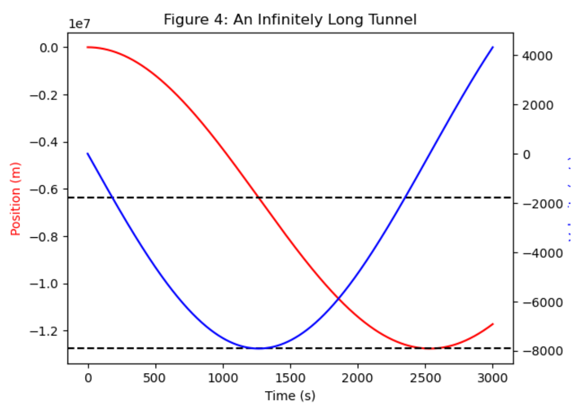
When we include the drag force again, the transverse travel increases by 500%. The time at which the test mass first hits the wall of the shaft is 29.7 seconds, and at 1296 meters. Including drag does not change the result that the test mass will hit the side of the shaft. As mentioned previously, this will impact the final calculations of the length of the shaft. Therefore, I recommend not proceeding with this technique, as it will not produce accurate data.

Calculation of Crossing Times for the Homogenous and Non-Homogenous Earth:

We will now consider the case of a pole-to-pole mineshaft. In reality, this cannot exist as the outer core is liquid, but we will imagine it is. For this case, the varying gravity is very important, as the test mass will traverse the entirety of the earth. However, the Earth does not have a uniform density- the core is approximately 4 times more dense than the mantle. We will be considering several different models of the density of the earth, based off of the following equation, where ρ_n is the normalizing constant, or the estimated density which makes our models reflect the earth:

$$\rho(r) = \rho_n \left(1 - \frac{(R_e - r)^2}{R_e^2}\right)^n \tag{5}$$

We used n = 0, 1, 2, and 9 for testing values, where n = 0 is the homogeneous density Earth we had assumed before. We used the time until the test mass reached the center, or the 'crossing time' to determine the effect of a non-homogenous earth. In particular, the crossing time for n = 0, and a ρ_n of 5494 kg/m^3, is 1267.2 seconds, while for n = 9, and a ρ_n of 135,545 kg/m^3, the crossing time is 944.2 seconds, which is ~3/4 of the n = 0 value. We can see that as the density concentration increases, the time to the center is less, as the acceleration of the test mass will be higher. We also considered a lunar shaft from pole-to-pole and found that the density for a homogenous moon is 3341 kg/m^3, and the crossing time is 1624.9 seconds.



We then considered a theoretical situation of an object orbiting around a point mass with the mass of the Earth or Moon, at the radius of the Earth or Moon, respectively. For the Earth, this orbital time is about 5069.4 seconds. For the moon, the orbital time is about 6500.5 seconds. For both of these cases, the orbital time is approximately four times the crossing time. In Figure 4, we can see that the movement of the test mass is periodic. Therefore, we can conclude that the orbital time is equal to the time it takes the test mass to fall through the earth, turn around, and return to its original position. In both instances, the only force acting on the object is the gravitational force of the earth, and both motions are periodic.

Discussion and Future Work:

Through our investigation, we determined that the proposed method of dropping a test mass into the mineshaft to measure the depth is not recommended, as the mass will hit the wall before it hits the bottom, distorting the data. We also investigated the impact of a non-homogenous and homogenous Earth on crossing time.

We used several simplifications during our investigation. For example, we estimated the Earth to be a perfect sphere, when the Earth is slightly flattened on the poles, so would not have the same volume as a perfect sphere. Furthermore, as the test mass traverses deeper into the shaft, the air density would increase in the shaft, making the air resistance increase, where we assume a constant air density/air resistance factor. To make these calculations more realistic, we recommend further work into these topics.