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The shadow of a black hole & the image of an accretion disk

A modern rethinking of Luminet (1979)

In this tutorial we rethink the derivations of Luminet (1979, A&A 75, 228) and we study, using a modern computer-based approach, the image of an accretion disk around a non-rotating black hole (BH).

Purposes and cautionary notes

The tutorial has been inspired by the "Interstellar" movie and the simulations displayed there. We will cover ideas, ranging from classical mechanics to spherical trigonometry, to numerical techniques. Knowledge of General Relativity is only necessary to solve the questions of Sect. 2 (but one can just use the results provided as granted and skip this part).

The tutorial, necessarily, simplifies a lot of what is shown in "Interstellar". As a start, we consider a non-rotating (symmetric) black hole, instead of a rotating one and we generally assume the observer is very distant. For a deeper understanding on the "Interstellar" simulations see James et al. (2015, Class. Quantum Grav. 32 065001) and James et al. (2015, Am. J. Phys. 83, 486).

1. Classical orbits around a massive objet

A. Consider a central mass M and a small particle of mass m orbiting about it. Show that the angular momentum is conserved and that its modulus can be written in polar coordinates as

$$L = mr^2\dot{\varphi}\,.$$

Deduce from the conservation of L (as a vector) that the motion is on a plane.

Solution. A central force, such as the force of gravity, has vanishing torque, since $\vec{\tau} = \vec{r} \times \overrightarrow{F}$ and $\vec{r} \parallel \overrightarrow{F}$. Since $d\overrightarrow{L}\big/dt = \vec{\tau}$, we immediately find $\overrightarrow{L}$ = constant. Note that the constance of the angular momentum implies that the motion is on a plane, because $\overrightarrow{L} = m\vec{r} \times \vec{v}$ and therefore $\overrightarrow{L} \perp \vec{r}$. Also, if we use polar coordinates we can easily verify that the tangential velocity is $v_\varphi = r\dot{\varphi}$. Since the radial velocity $\dot{r}$ is parallel to $\vec{r}$, for the computation of the modulus of the angular momentum we need to take into account only v_φ, which is perpendicular to $\vec{r}$. We find them immediately the result above.

B. Show that the total energy is conserved and that can be written in polar coordinates as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}\,.$$

Solution. If we separate again the velocity in a radial component $\dot{r}$ and a tangential one $r\dot{\varphi}$, we can write the kinetic energy as

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2r^2m} \ .$$

C. The last two terms of this equation can be taken as an **effective potential**

$$V(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \ .s$$

so that the energy conservation takes the form

$$E = \frac{1}{2}m\dot{r}^2 + V(r) \ .$$

Use GeoGebra (or any other software) to make a plot of $U(R)$, using for example unit values for all constants, and deduce the properties of the orbits depending on the sign of E (note that the first term, $m\dot{r}^2/2$, is always positive).

Solution. Given the shape of $V(r)$, if $E < 0$ the orbits will be limited in a specific range of radii (the closed interval for which $V(r) \leq E$); vice-versa, if $E \geq 0$ the orbits will have a minimum value of r (corresponding two the perihelion distance), but will be unbounded at large radii.

2*. Differential equation for a light ray around a BH

From now on we will use a natural unit system where $G = c = 1$. Note that in these units the mass of the black hole is equivalent to a length. A non-rotating black hole can be described in terms of the Schwarzschild metric, which takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ .$$

A. To derive the equations of motions (*geodetic equations)* one can use a simple variational method: define

$$\mathcal{L} = g_{\alpha\beta}\dot{x}_\alpha\dot{x}_\beta = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

and show that the Euler-Lagrage equations, together with the condition $L = 0$ (null geodetic), give for a light ray moving along the plane $\theta = \pi/2$

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) = \frac{1}{b^2} \ .$$

Solution. First, note that since the metric has spherical symmetry, we can consider the "equatorial" orbit $\theta = \pi/2$. The Euler-Lagrange equations take the simple form

$$\frac{\partial \mathscr{L}}{\partial q_i} = \frac{d}{d\lambda}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) .$$

In our case, since L does not depend on t explicitly, we have

$$\frac{\partial \mathscr{L}}{\partial \dot{t}} = -2\left(1 - \frac{2M}{r}\right)\dot{t} = \text{const} \equiv C_1 .$$

Similarly, since L does not depend on ϕ, we have

$$\frac{\partial \mathscr{L}}{\partial \dot{\phi}} = 2r\dot{\phi} = \text{const} \equiv C_2 .$$

Inserting these results in the null condition $\mathscr{L} = 0$, we find

$$\frac{\dot{r}^2}{\dot{\phi}^2} = \frac{C_1^2}{C_2^2}r^4 - \left(1 - \frac{2M}{r}\right)r^2 .$$

The result than follows by recognizing that $\dot{r}/\dot{\phi} \equiv dr/d\phi$ and calling $C_2/C_1 \equiv b$.

3. General properties of the orbits and BH shadow

A. In the previous equation consider the motion of a photon far away from the mass M and interpret the meaning of b as the **impact parameter**. (Hint: consider the quantity $r^2 d\phi/dr$ and show that it is just b).

Solution. Call $\ell \equiv L/m$ the angular momentum for unit mass. It is easy to show that $\ell = r^2\dot{\phi} = bv\sin\alpha$, where α is the angle between the radius $\vec{r}$ and the light ray direction $\vec{v}$. The identity then follows immediately.

B. Note the similarities of the two grey-shaded equations: a constant (E in one case, $1/b^2$ in another case) is equal to the sum of a positive quantity and a term acting as an *effective potential*. In the last case, the effective potential is

$$V(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) .$$

C. Given all the discussions above, an orbit can go through all points r for which $V(r) \leq 1/b^2$.

D. Consider this potential. Plot it and show that it has a global maximum (for positive radii) at $r = 3M$. Find the corresponding value $V(3M)$. If $1/b^2 < V(3M)$, not all values of r can be touched by the photon: that its, the light ray reaches a minimum radius (periastron), then r increases again. Show that this happens for $b > b_c = 3\sqrt{3}M \approx 5.197\,M$.

E. Light rays with $b < b_c$ will be captured by the black hole. If we follow the light backwards, we can then deduce that there will be a **shadow** in the images of objects behind the black hole of radius b_c.

F. Consider the supermassive black hole at the center of the Milky Way. Its mass is approximately $4.3 \times 10^6\,\mathrm{M}_\odot$ and it is $8.3\,\mathrm{kpc}$ far away from us. Compute the angular size of the shadow of this object.

Solution. The shadow has an apparent angular size $b_c/d \approx 26.6 \times 10^{-6}$ arcsec.

G. The shadow has been "imaged" at a wavelength of $\lambda = 1.3$ mm using the Event Horizon Telescope (EHT). As an order of magnitude, the resolution of this telescope is the diameter of the observing wavelength divided by Earth's diameter. Compare the resolution of the EHT with the apparent size of the black hole in the our galaxy.

Solution. The resolution is $\lambda/D \approx 21 \times 10^{-6}$ arcsec.

4. Periastron and impact parameter

A. Light ray with $b = b_c$ will be travel along an unstable circular orbit. Show that the radius of this orbit it $3M$. Compare with the situation of classical orbits around a point mass of Sect. 1.

Solution. This result is obvious, given the fact that $V(3M)$ is a maximum.

B. Show that for there is a simple relationship between the impact parameter b of non-captured orbits (i.e. with $b > b_c$) and the periastron P, given by

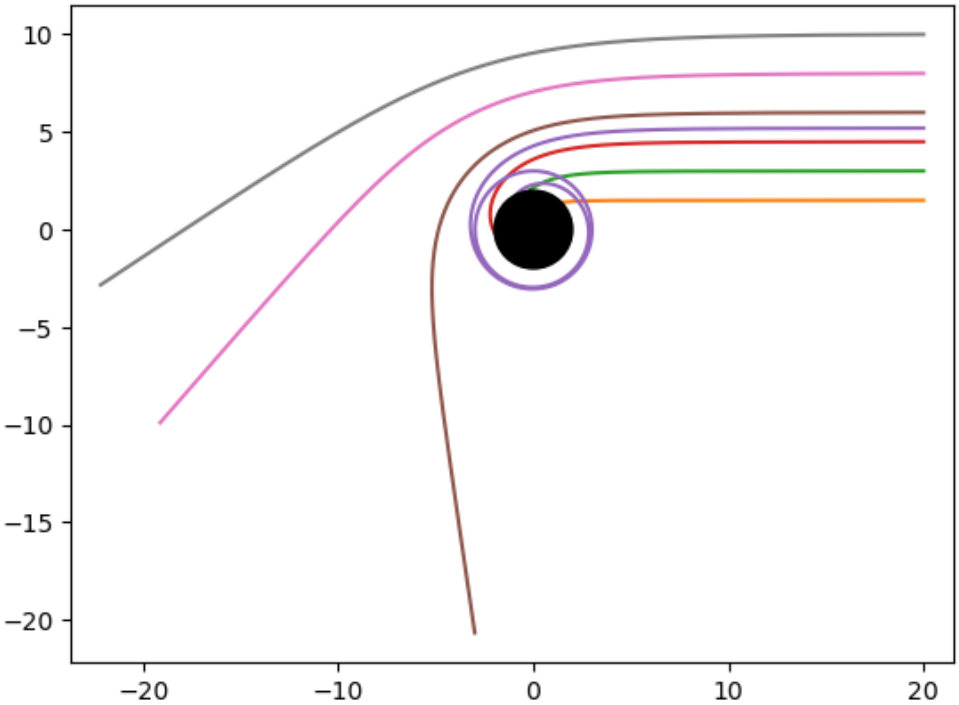
$$b^2 = P^3/(P - 2M)\ .$$

Solution. Just rearrange $V(P) = 1/b^2$.

C. [Optional] To invert this relation, i.e. to write P as a function of b, one can either use numerical methods (using for example Newton's method) or simple algebra.[Hint: look for algebraic "Cardano-like" solutions of a 3rd degree polynomial; use the "depressed cubic" formula, i.e. polynomials of the form $x^3 + px + q$. Show that for $b > b_c$ the equation admits three real solutions.]

Solution. We need to solve $P^3 - b^2 P + 2b^2 M = 0$ for P. Therefore, in the standard notation for third degree equations, we have $p = -b^2$ and $q = 2b^2 M$. The discriminant is $\Delta = -(4p^3 + 27q^2) = -4b^4(27M^2 - b^2)$. Therefore, if $b > b_c = 3\sqrt{3}M$ the discriminant is positive and the equation has a three real roots. Since the product of these solutions must be $-q < 0$, there must be a negative solution and two positive ones. The solutions can be found using various techniques (see, e.g., the Wikipedia page for "Cubic equation").

5. Numerical integration of the orbits



A. Call $u \equiv 1/r$ and convert the equation for r above into an equation for u. Show that one finds

$$\left(\frac{du}{d\phi}\right)^2 = 2Mu^3 - u^2 + \frac{1}{b^2} \ .$$

Solution. Just rewrite the differential as $du = -dr/r^2$ and use it in the equation at point 2A.

B. Write a simple program to display the solutions of this equation, i.e., the orbits followed by a light ray with a given impact parameter. Note that for $b \leq b_c$ the $u(\phi)$ function is monotonic, while for $b > b_c$ there is a periastron P and therefore the function attains a minimum $u = 1/P$ and is symmetric around that point. [Hint: there are various ways of implementing this point. The most naïve one is to use the Euler's method, starting with $u = 0$ and with a relatively small step $d\phi$; other more advanced options include using the Runge-Kutta order-four integrator.]

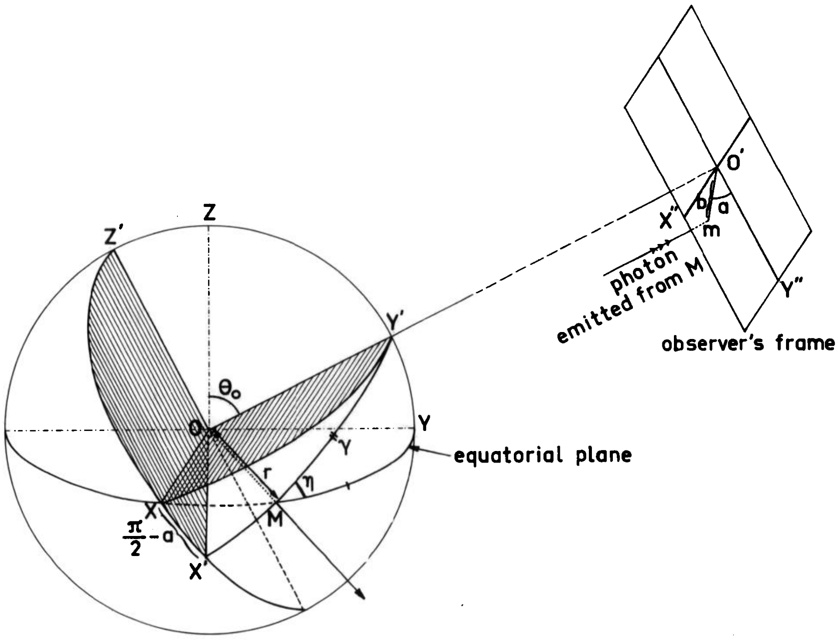
6*. Projection of an accretion disk

We want now to display how an accretion disk would be seen by a distant observer. To this purpose, we imagine the accretion disk to be along the XY "equatorial" plane, and the observer to be above it by an angle θ_0 (see figure above). The purpose of this section is to map, for each "ending point" of a photon on the equatorial plane, the point(s) on the equatorial plane where a photon originates. In other words, we map the photon backwards, as usually done in gravitational lensing.

A. Consider a point m with polar coordinates b and α on the observer frame. Show that b can be identified as the impact parameter discussed above.

B. Show, using symmetry considerations, that the orbit of a photon observed in (b, α) will be entirely on the plane containing O, O', and the point m itself.

Since we are able to integrate the solution (see point 5B above), we need to know the extremes of the integration in ϕ. To this purpose, we can use the angle γ (see, in the figure above, the arc connecting M with Y'). This is the angle on the plane



$OO'm$ between the line OO' and the line OM, obtained as the intersection of $OO'm$ with XY. To obtain this angle we need to use some spherical trigonometry. To this purpose, we will use Napier's rules for right spherical triangles.

C. Consider the right spherical triangle $MX'X$. Show that the angle on X is $\pi/2 - \theta_0$ and that the side $X'M$ is $\pi/2 - \gamma$.

D. Show that the application of Napier's rules lead to the expression

$$\cos\alpha = \cot\theta_0 \cdot \cot\gamma \ .$$

Solution. The first set of Napier's rules state that, in a sequence of parts of a right spherical triangle, the sine of the middle part is equal to the product of the tangents of the adjacent parts. In our case, therefore

$$\sin(\pi/2 - \alpha) = \tan(\pi/2 - \theta_0) \cdot \tan(\pi/2 - \gamma)$$

from which the expression above follows immediately.

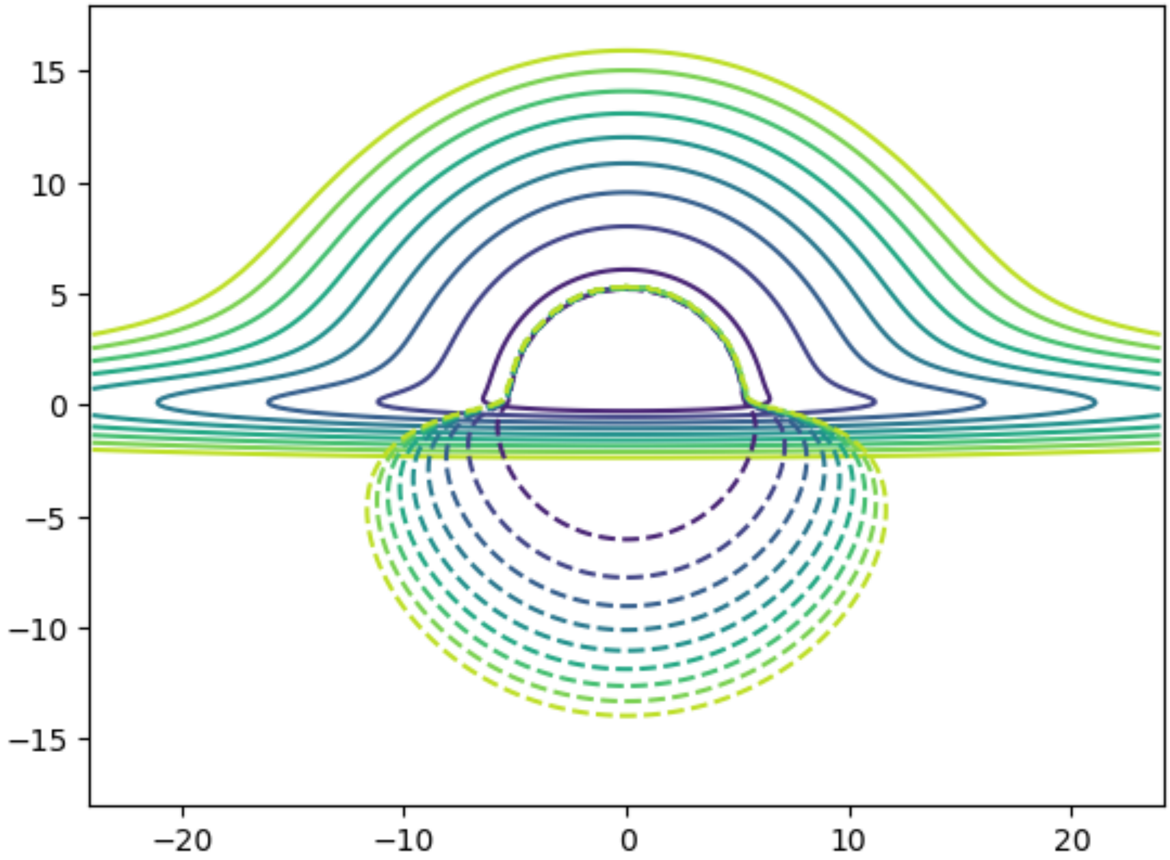
E. Use the trigonometric identity

$$\cos\gamma = \frac{\cot\gamma}{\sqrt{1 + \cot^2\gamma}}$$

to rewrite the previous equation as

$$\cos\gamma = \frac{\cos\alpha}{\sqrt{\cos^2\alpha + \cot^2\theta_0}} \ .$$

F. Write a program to show the observed shape an accretion disk around a black hole. Display the images of circles around the black hole by computing, for each impact parameter b, the orbit of a light ray from the observer (taken at infinity, thus with $u = 0$) integrated up to the angle γ. Repeat the same exercise up to $\gamma + \pi$ to obtain the counter-images for light rays turning once around the black hole. Repeat the exercise for various values of θ_0 (the figure below is for $\theta_0 = 87°$).



G. The time-averaged flux emitted by a black hole has been estimated by Page & Thorne (1974, ApJ 191, 499). The surface brightness can be expressed as a function of $x = r/M$ as

$$F(x) \propto \frac{1}{x^{5/2}\,(x-3)}\left[\sqrt{x} - \sqrt{6} + \frac{\sqrt{3}}{2}\left(\frac{(\sqrt{x}+\sqrt{3})(\sqrt{6}-\sqrt{3})}{(\sqrt{x}-\sqrt{3})(\sqrt{6}+\sqrt{3})}\right)\right] .$$

Produce an image of the observed accretion disk using this expression in the previous point.

H. The expression above should be corrected by a "relativistic dimming" factor associated to the redshift of the photons. Call z the redshift: its expression is

$$1 + z = \frac{1 + \sqrt{Mb^2/r^3}\,\sin\theta_0 \sin\alpha}{1 - 3M/r} \ .$$

The flux observed is then given by $F(x)/(1+z)^4$. Note how the $(1+z)^{-4}$ factor is the same present in the "cosmological dimming".