A restarted Induced Dimension Reduction method to approximate eigenpairs of large unsymmetric matrices

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Introduction

In this work we are interested in solving the eigenvalue problem for large and unsymmetric matrices $A \in \mathbb{C}^{n\times n}$, i.e. find $\lambda \in \mathbb{C}$, and $\mathbf{x} \neq \mathbf{0} \in \mathbb{C}^n$ such that:

$$A\mathbf{x} = \lambda\mathbf{x}. \qquad (1)$$

We construct a standard Hessenberg decomposition based on the Induced Dimension Reduction method (IDR). This decomposition has two main advantages. First, the computational efficiency since IDR is a short-recurrence method. Second, the IDR polynomial is used as a filter to refine the eigenvalues obtained.

Background

- The IDR(s) was introduced for solving linear systems in [3].
- IDR(s) creates residual vectors in the nested and shrinking subspaces $\mathcal{G}_j$ defined as

$$\mathcal{G}_j \equiv (A - \mu_j I)(\mathcal{G}_{j-1} \cap P^{\perp}) \quad j = 1, 2, \ldots$$

  where $P \in \mathbb{C}^{n\times s}$ and $\mathcal{G}_0 \equiv \mathbb{C}^n$; in order to extract the approximated solution.
- First IDR(s) method to solve Eq. (1) was proposed by M. H. Gutknecht and J.-P. M. Zemke [2]. The work we present here is an extension of [1].

An IDR(s)-Hessenberg decomposition

Every vector $\mathbf{w}_{i+1}$ in $\mathcal{G}_{j+1}$ can be written as:

$$\mathbf{w}_{i+1} = (A - \mu_{j+1} I)\left(\mathbf{w}_i - \sum_{\ell=1}^{s} c_\ell \mathbf{w}_{i-\ell}\right),$$

with $\{\mathbf{w}_\ell\}_{\ell=i-s}^{i} \in \mathcal{G}_j$ and the coefficients c_ℓ are obtained from:

$$(P^T[\mathbf{w}_{i-s}, \mathbf{w}_{i-s+1}, \ldots, \mathbf{w}_{i-1}])\mathbf{c} = P^T\mathbf{w}_i.$$

Setting $W_k = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k]$, and $A\mathbf{w}_{i-\ell} = W_{i-\ell+1}\mathbf{H}_{i-\ell}$,:

$$A\mathbf{w}_i = W_{i+1}\left(\begin{bmatrix} \mathbf{0} \\ -\mu_{j+1}\begin{bmatrix} \mathbf{c} \\ -1 \end{bmatrix} \\ 1 \end{bmatrix} + \sum_{\ell=1}^{s} c_\ell \mathbf{H}_{i-\ell}\right) = W_{i+1}\mathbf{H}_{i+1}.$$

Applying Eq. above for $i = 1, 2, \ldots, m$, we obtain a Hessenberg decomposition that we call it **IDR factorization**:

$$AW_m = W_{m+1}\bar{H}_m = W_m H_m + \mathbf{w}_{m+1}\mathbf{e}_m^*.$$

For stability, we orthogonalize every set of vectors created in $\mathcal{G}_j$.

Filtering the Ritz values

To refine the spectral information obtained from the IDR factorization:

1) IDR calculations include a polynomial whose roots are μ_j. We exploit this fact by selecting μ_j to minimize the norm of the IDR polynomial in the area where the unwanted eigenvalues are localized.

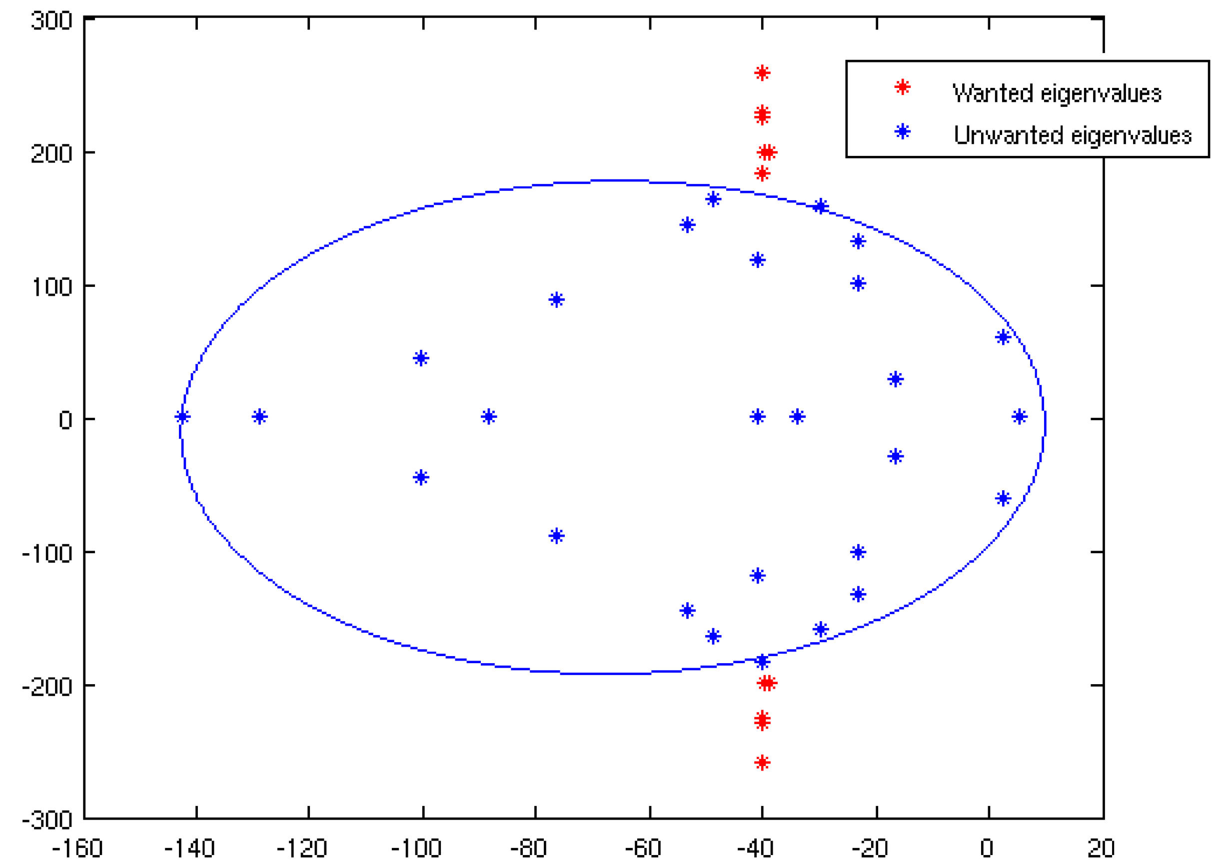

Figure 1: Select μ_j to minimize the norm of the IDR polynomial in the ellipse which encloses the unwanted eigenvalues.

This is achieved by choosing μ_j as the Chebyshev nodes on the interval $[a, b]$, where a and b are the foci of the ellipse that encloses the unwanted portion of the spectrum.

2) We implement the implicit restarting by D.C. Sorensen [4].

Numerical tests

- *Matrix AF23560 from Matrix Market (12 LM eigenvalues)*

Method	Restarts	Time (sec.)	Max. difference
IRAM($k=13$,$p=26$)	16	1.76	*
IDR($s=13$,$m=26$)	7	0.47	5.08e-07

- *Discretized 2D Schrödinger equation (16 SM eigenvalues)*

Method	Restarts	Time (sec.)	Max. difference
IRAM($k=17$,$p=34$)	9	10.71	*
IDR($s=17$,$m=34$)	9	9.08	3.73e-06

References

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