Induced Dimension Reduction method for solving linear matrix equations

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Induced Dimension Reduction method (IDR(s))

IDR(s) [2] is a Krylov subspace method proposed originally to solve systems of linear equations,

$$A\mathbf{x} = \mathbf{b}, \tag{1}$$

where the coefficient matrix A is large, sparse, and nonsymmetric. IDR(s) is based on IDR theorem.

In this work, we generalized the IDR(s) method to solve linear matrix equations,

$$\sum_{j=1}^{k} A_j X B_j^T = C. \tag{2}$$

Generalization of the IDR(s) theorem

We propose the following generalization of the IDR(s) theorem valid in any finite-dimesional linear subspace.

Corollary 1 *Let $\mathcal{A}$ be any linear operator over a finite dimensional subspace $\mathcal{D}$ and $\mathcal{I}$ the identity operator over the same subspace. Let $\mathcal{S}$ be any (proper) subspace of $\mathcal{D}$. Define $\mathcal{G}_0 \equiv \mathcal{D}$, if $\mathcal{S}$ and $\mathcal{G}_0$ do not share a nontrivial invariant subspace of the operator $\mathcal{A}$, then the sequence of subspace $\mathcal{G}_j$, defined as*

$$\mathcal{G}_j \equiv (\mathcal{I} - \omega_j\mathcal{A})(\mathcal{G}_{j-1} \cap \mathcal{S}) \quad j = 0, 1, 2 \ldots,$$

with ω_j's nonzero scalars, have the following properties,

1. $\mathcal{G}_{j+1} \subset \mathcal{G}_j$, for $j \geq 0$ and

2. dimension$(\mathcal{G}_{j+1})$ < dimension$(\mathcal{G}_j)$ unless $\mathcal{G}_j = \{\mathbf{0}\}$.

IDR(s) for matrix equations

In a similar way as in [3], we rewrite Equation (2) as

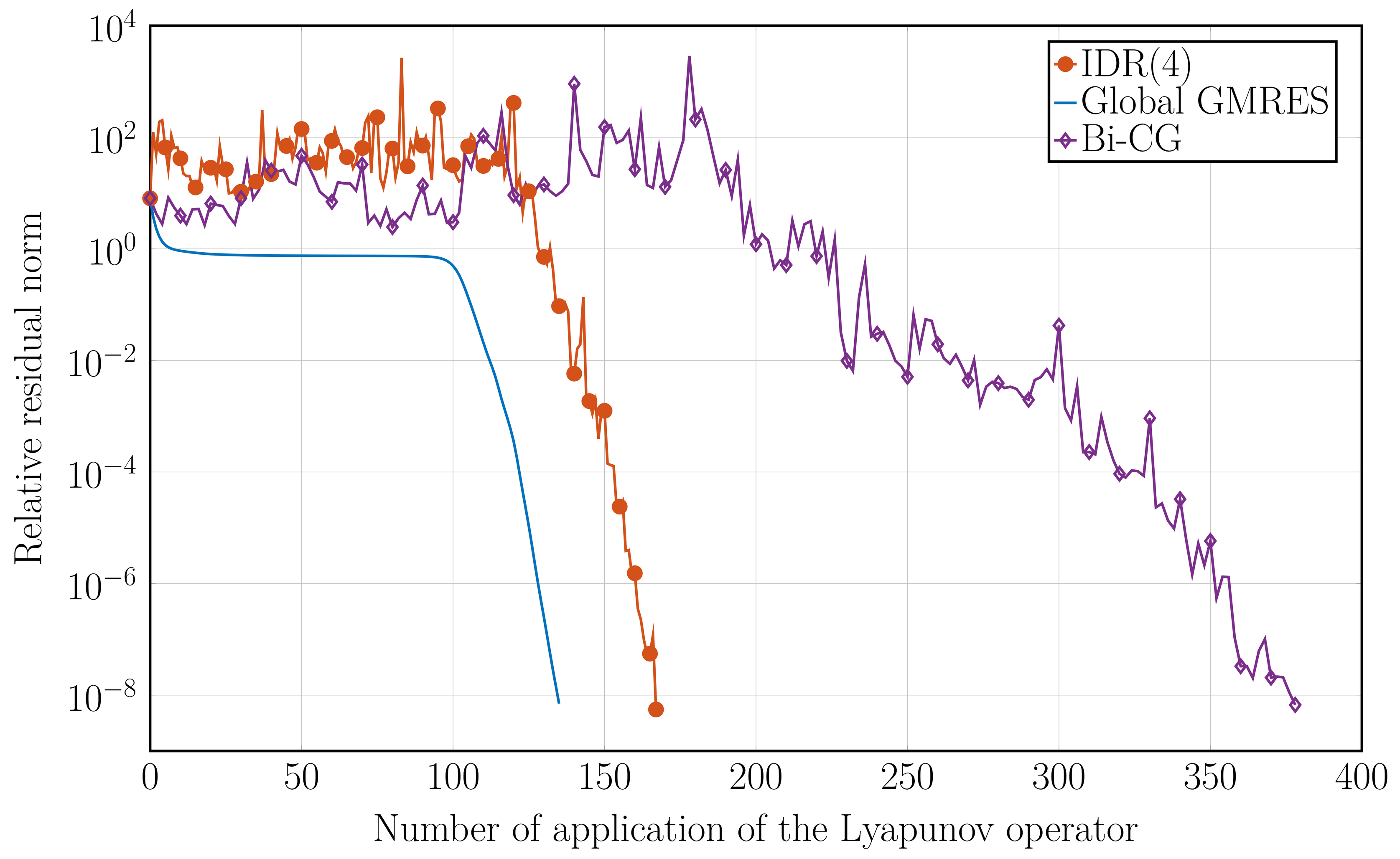
$$\mathcal{A}(X) = C, \tag{3}$$

using the proposed generalization of the IDR(s) theorem and changing the matrix-vector product by the application of operator $\mathcal{A}(X)$, and the Euclidean inner product by the Frobenius inner product, we obtain an IDR(s) algorithm to solve linear matrix equations.

Example 1: Following figure shows the evolution of the residual norms of IDR(4) and other Krylov methods for solving the Lyapunov equation,

$$AX + XA^T = C, \tag{4}$$

where A is the matrix CDDE6 from NEP Collection.



Preconditioning

A basic iterative method is proposed as preconditioner for the Sylvester equation,

$$AX + XB = C. \tag{5}$$

We apply few step of the following fixed-point iteration as preconditioner,

$$AX_{k+1} = -X_k B + C, \tag{6}$$

or whenever is not possible to invert or solve a block linear systems efficiently with the matrix A, we use an inexact version of (6),

$$MX_{k+1} = -X_k B + C, \tag{7}$$

where $M \approx A$.

Numerical examples

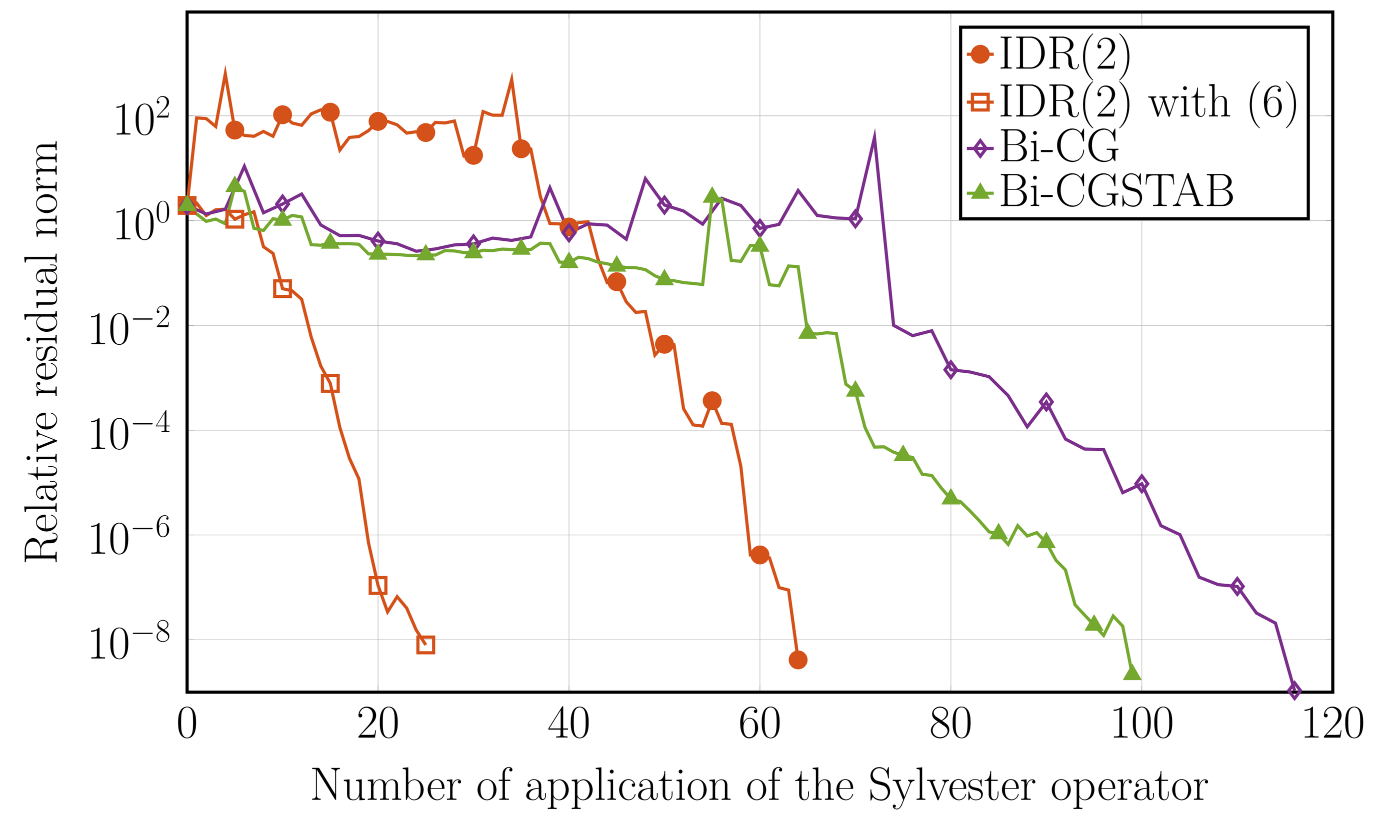
Example 2: In this example, we solve a Sylvester equation,

$$AX + XD = C,$$

obtained from the discretization of,

$$\frac{du}{dt} - \epsilon\frac{d^2u}{dx^2} + \omega\frac{du}{dx} = 0, \qquad 0 \leq t \leq 1, \quad u_{t_0} = 1.$$

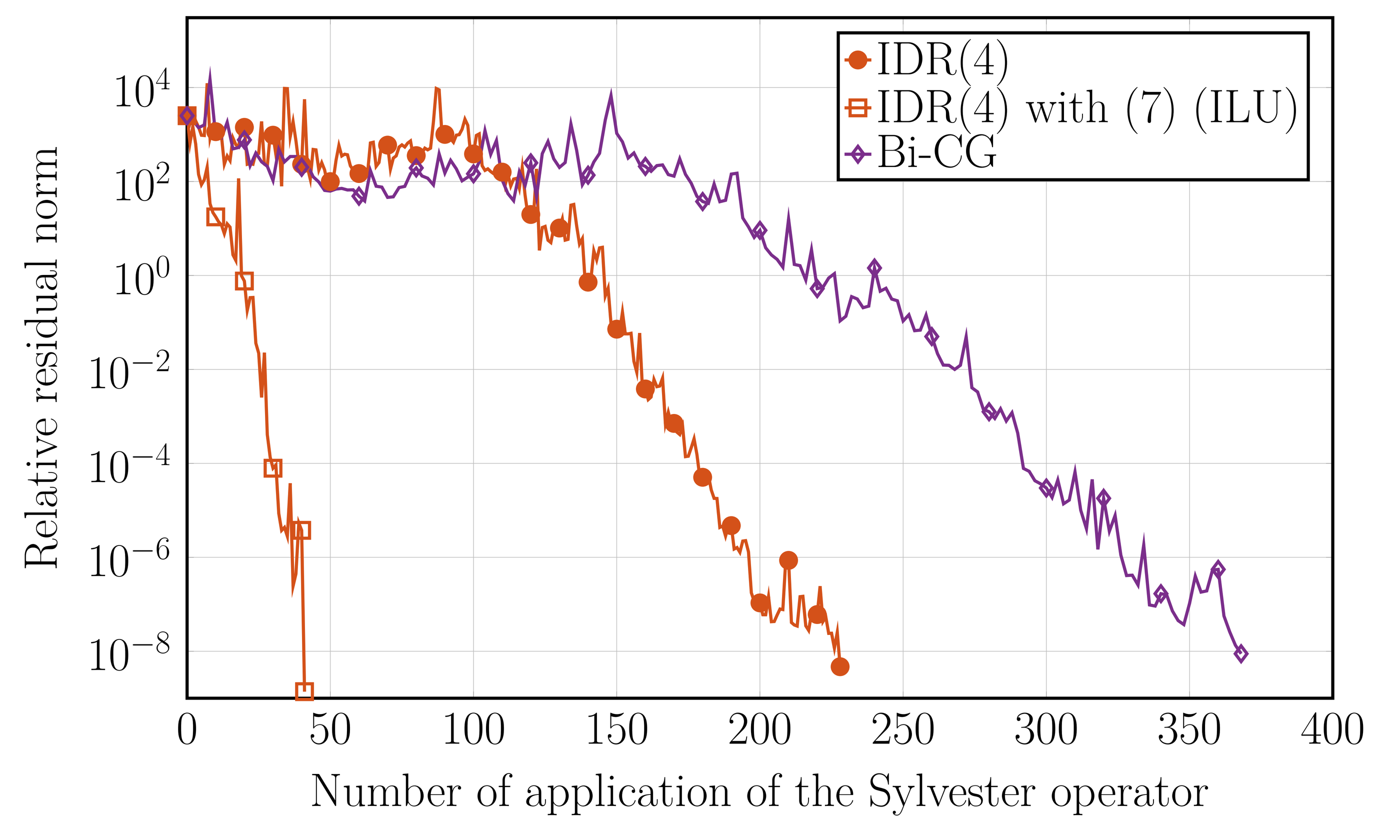
Matrix A represents the discretization in space, and matrix D represents the discretization in time using backward Euler.



Example 3: We solve a Sylvester equation from discretization of:

$$-\epsilon\triangle u + \mathbf{b}^T\nabla u - ru = f,$$

with Dirichlet boundary condition in the unit cube, and different values of r.



Code availability and contact

IDR(s) code is available in Matlab, Python, Julia, and Fortran 90.
Web: http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html
E-mail: {R.A.Astudillo, M.B.vanGijzen}@tudelft.nl

References

[1] R. Astudillo and M. B van Gijzen. Induced Dimension Reduction method for solving linear matrix equations. Delft University of Technology, DIAM TR-05, 2015.

[2] P. Sonneveld and M. B. van Gijzen. IDR(s): a family of simple and fast algorithms for solving large nonsymmetric linear systems. *SIAM J. Sci. Comput.*, 31(2):1035–1062, 2008.

[3] M. Hochbruck and G. Starke. Preconditioned Krylov Subspace Methods for Lyapunov Matrix Equations. *SIAM J. Math. Anal. Appl.*, 16(1):156–171, 1995.