A fast nonlinear conjugate gradient based method for frictional contact problems

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Outline

- Introduction.
- Formulation of frictional contact.
- TangCG algorithm.
- Numerical results of a Cattaneo shift problem.
- Conclusion.

Introduction

United States

France

Spain

China

Germany

England

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Introduction

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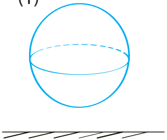

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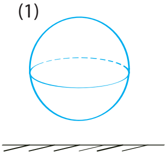

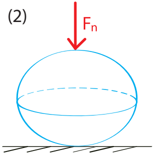

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Introduction

Pressure &
Contact area

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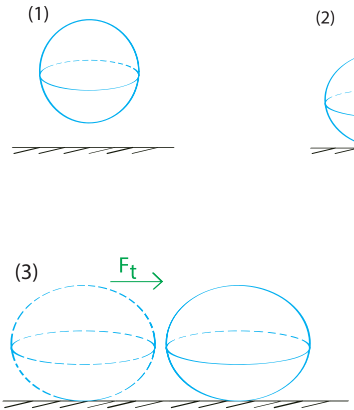

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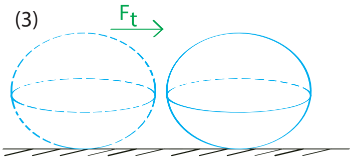

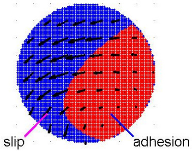

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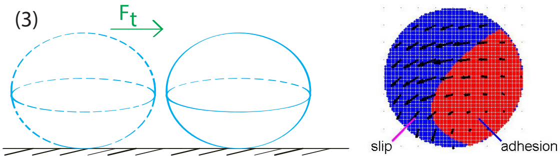

Frictional stress,
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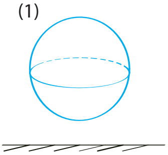

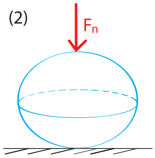

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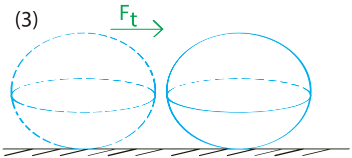

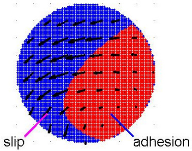

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Formulation of frictional contact

Kalker's numerical variational form (after discretization):

$$\min_{\mathbf{p}} \ \phi = \frac{1}{2}\mathbf{p}^T A\mathbf{p} + \mathbf{w}^T\mathbf{p}$$

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We aim at a faster solver.

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$$\begin{cases} ||\mathbf{p}_I|| = g_I \\ p_{Ix}s_{Iy} - p_{Iy}s_{Ix} = 0 \end{cases}$$

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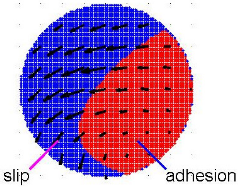

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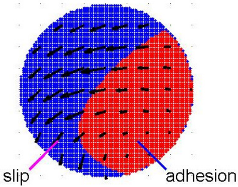
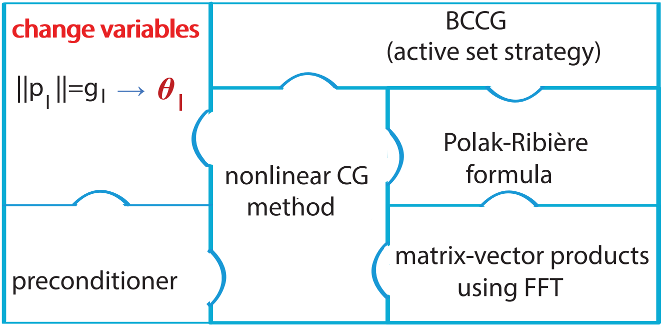


We solve for $\mathbf{p}$, H and S.

TangCG algorithm

Main components of TangCG:



TangCG algorithm

♠ Change variables in slip area

Inspired by:

In slip area: $||\mathbf{p}_I|| = g_I$

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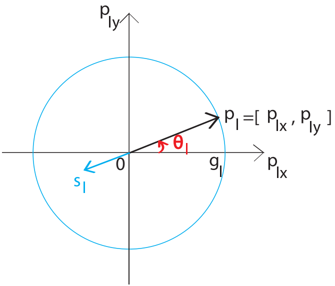

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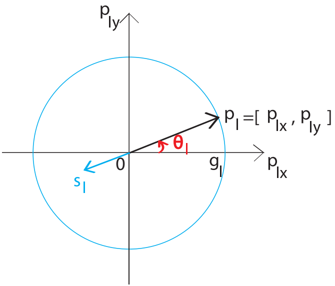

- Conventional variables
  p_{Ix}, p_{Iy}

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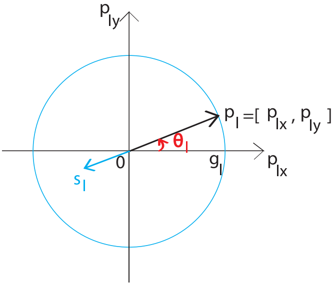

- Conventional variables

  $p_{lx}, p_{ly} \Rightarrow \quad \theta_l$.

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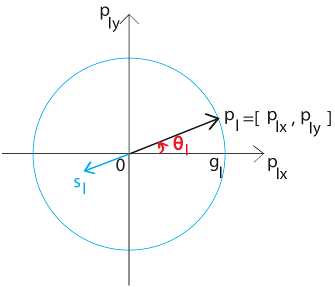

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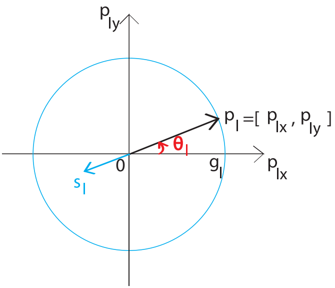

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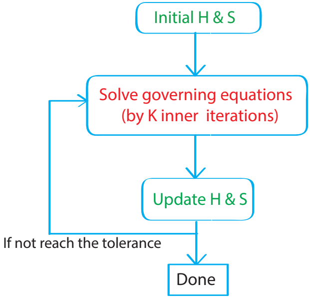

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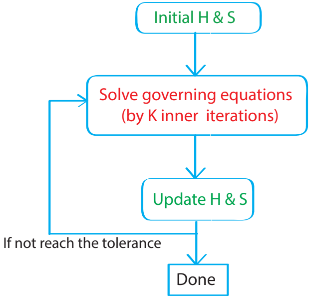

The governing equations:

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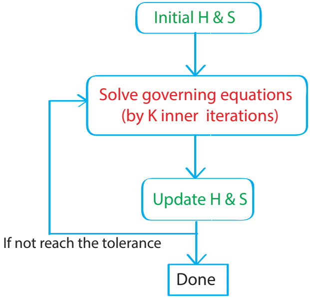

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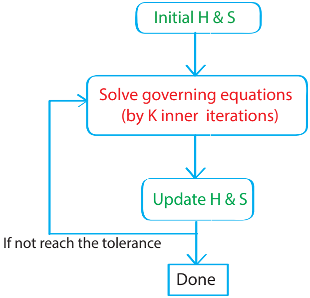

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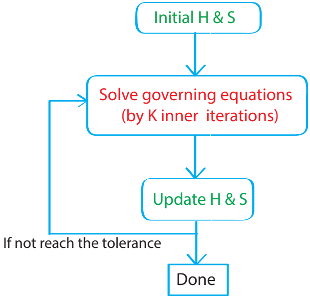

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- Not necessary to generate J^k explicitly.
- Involves computing the matrix-vector products with a BTTB matrix A.
- Apply the fast Fourier transform (FFT) technique, the complexity is reduced to $\mathcal{O}(n \log n)$ from $\mathcal{O}(n^2)$.

TangCG algorithm

One iteration on the slip element *l*:

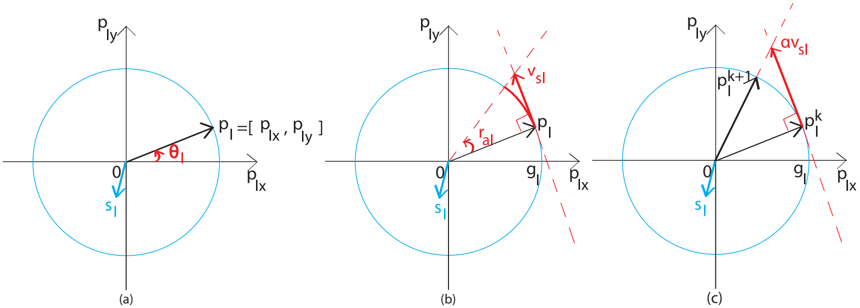

Figure: (a) current iterate. (b) residual and search direction. (c) update.

TangCG algorithm

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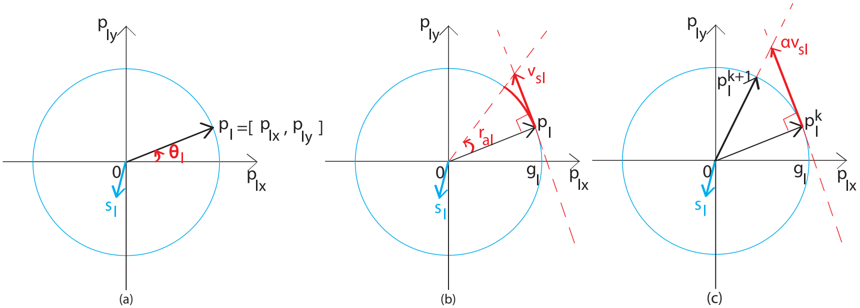

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- Use what approach?

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Numerical results of a Cattaneo shift problem

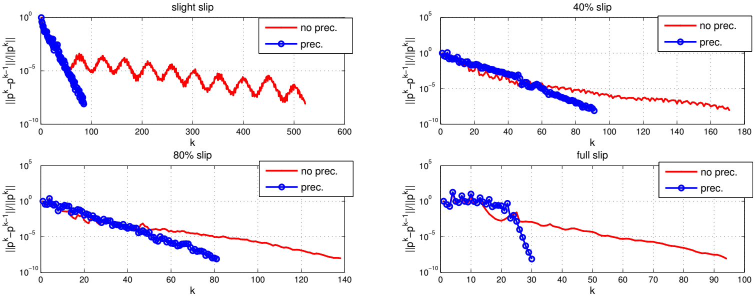
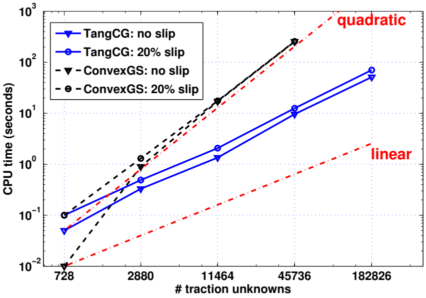
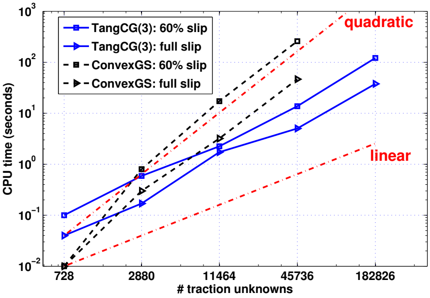


Figure: The convergence behavior of TangCG with and without prec.

Numerical results and conclusion





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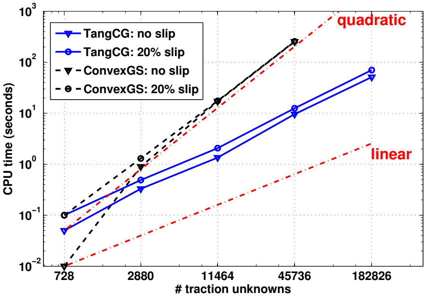
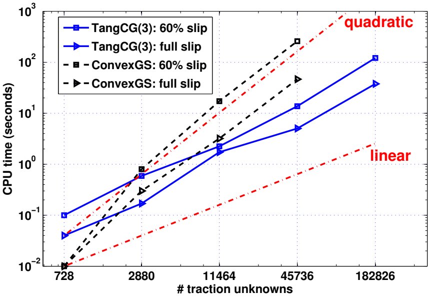
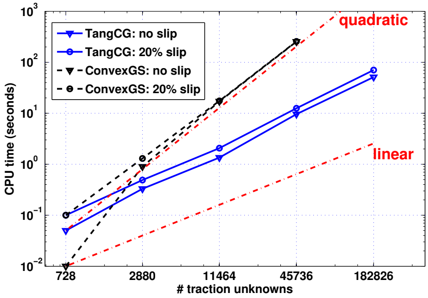
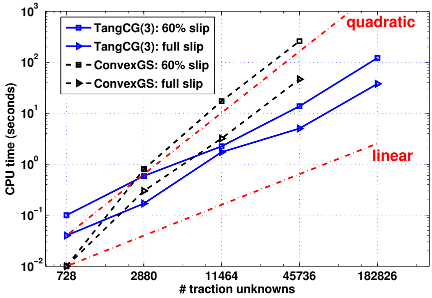




Conclusion about TangCG with prec.:

Numerical results and conclusion

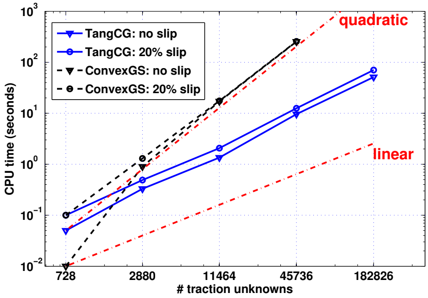


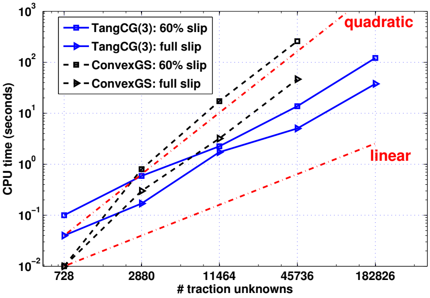


Conclusion about TangCG with prec.:

- robust: it works well for different situation of slip.

Numerical results and conclusion



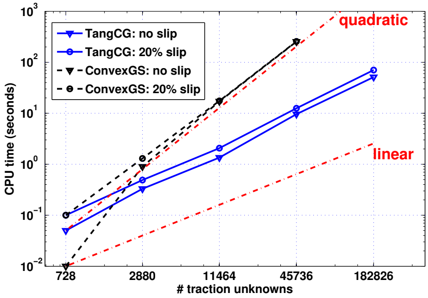


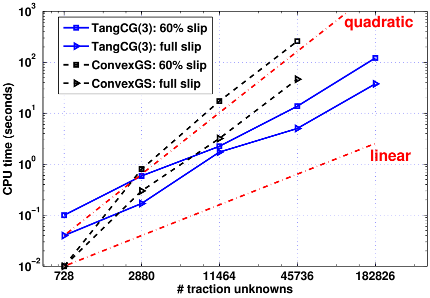
Conclusion about TangCG with prec.:

- robust: it works well for different situation of slip.
- fast: it reduces the computational time dramatically (around $\mathcal{O}(n^{1.7})$).

Delft University of Technology

Numerical results and conclusion





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Thank you for your attention!