Zero-one laws: A quantitative insight

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'Every mathematician believes that he is ahead of the others. The reason none state this belief in public is because they are intelligent people.' - Andrey Kolmogorov

Introduction

Zero-one laws are theorems which state that events of a certain type occur with probability 0 or 1.

Such events include tail events. A tail event T is associated with a sequence of independent events (A_n). By changing a finite number of occurrences concerning the events A_n, it doesn't change the occurrence of the event T.

For an analogy, one can think of a sequence of real numbers converging; the tail of the sequence dictates the behaviour.

Borel's zero-one law

Given a sequence of events (A_n), the event A_∞ is the event that infinitely many of the A_n occur. A_∞ is a tail event.

$$A_\infty = \lim_{k\to\infty} \bigcup_{i=k}^{\infty} A_i = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i \equiv \forall\, k \;\exists\, i \geq k \;\; A_i.$$

Theorem (Borel's zero-one law: Qualitative version)

Let (A_n) be a sequence of independent events; then $\mathbb{P}(A_\infty)$ can only assume only the values zero or one.

Borel's zero-one law in logical form

$$
\begin{aligned}
& \left[\mathbb{P}(A_\infty) = 0 \,\vee\, \mathbb{P}(A_\infty) = 1\right] \\
\Leftrightarrow\quad & \neg\left[\mathbb{P}(A_\infty) \in (0,1)\right] \\
\Leftrightarrow\quad & \neg\left[\mathbb{P}(A_\infty) > 0 \,\wedge\, \mathbb{P}(A_\infty) < 1\right] \\
\Leftrightarrow\quad & \neg\left[\lim_{k\to\infty} \mathbb{P}\left(\bigcup_{i=k}^{\infty} A_i\right) > 0 \,\wedge\, \lim_{k\to\infty} \mathbb{P}\left(\bigcup_{i=k}^{\infty} A_i\right) < 1\right] \\
\Leftrightarrow\quad & \neg\Big[\exists\varepsilon > 0\, \forall l \in \mathbb{N}\; \mathbb{P}\left(\bigcup_{i=l}^{\infty} A_i\right) \geq \varepsilon \\
& \qquad \wedge \exists\delta > 0\, \exists k \in \mathbb{N}\; \mathbb{P}\left(\bigcup_{i=k}^{\infty} A_i\right) \leq 1 - \delta\Big] \\
\Leftrightarrow\quad & \neg\Big[\exists\varepsilon > 0\, \forall l \in \mathbb{N}\, \exists n \in \mathbb{N}\; \mathbb{P}\left(\bigcup_{i=l}^{n} A_i\right) \geq \varepsilon \\
& \qquad \wedge \exists\delta > 0\, \exists k \in \mathbb{N}\, \forall m \in \mathbb{N}\; \mathbb{P}\left(\bigcup_{i=k}^{m} A_i\right) \leq 1 - \delta\Big]
\end{aligned}
$$

Borel's zero-one law in metastable form

$$\Leftrightarrow \quad \neg\Big[\exists \varepsilon > 0\, \forall l \in \mathbb{N}\, \exists n \in \mathbb{N}\ \mathbb{P}\big(\bigcup_{i=l}^{n} A_i\big) \geq \varepsilon \\ \wedge \exists \delta > 0\, \exists k \in \mathbb{N}\, \forall m \in \mathbb{N}\ \mathbb{P}\big(\bigcup_{i=k}^{m} A_i\big) \leq 1 - \delta\Big]$$

$$\Leftrightarrow \quad \neg\Big[\exists \varepsilon > 0\, \exists g : \mathbb{N} \to \mathbb{N}\, \forall l \in \mathbb{N}\ \mathbb{P}\Big(\bigcup_{i=l}^{l+g(l)} A_i\Big) \geq \varepsilon \\ \wedge \exists \delta > 0\, \exists k \in \mathbb{N}\, \forall m \in \mathbb{N}\ \mathbb{P}\big(\bigcup_{i=k}^{m} A_i\big) \leq 1 - \delta\Big]$$

$$\Leftrightarrow \quad \neg\Big[\exists \delta, \varepsilon > 0\, \exists k \in \mathbb{N}\, \exists g : \mathbb{N} \to \mathbb{N}\, \forall l, m \in \mathbb{N} \\ \mathbb{P}\Big(\bigcup_{i=l}^{l+g(l)} A_i\Big) \geq \varepsilon \wedge \mathbb{P}\big(\bigcup_{i=k}^{m} A_i\big) \leq 1 - \delta\Big]$$

$$\Leftrightarrow \quad \forall \delta, \varepsilon > 0\, \forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists l, m \in \mathbb{N} \\ \Big[\mathbb{P}\Big(\bigcup_{i=l}^{l+g(l)} A_i\Big) < \varepsilon \vee \mathbb{P}\big(\bigcup_{i=k}^{m} A_i\big) > 1 - \delta\Big]$$

Borel's zero-one law in a quantitative statement

Theorem (Borel's zero-one law: Quantitative version)

Let (A_n) *be a sequence of independent events. Then for all* $\delta, \varepsilon > 0$, k *a natural number and* $g : \mathbb{N} \to \mathbb{N}$, *we have*

$$\exists\, l \leq h^{(\alpha)}(k) \left[\mathbb{P}\left(\bigcup_{i=l}^{l+g(l)} A_i \right) < \varepsilon \vee \mathbb{P}\left(\bigcup_{i=k}^{m} A_i \right) > 1 - \delta \right],$$

where

$$\begin{aligned} \alpha &:= \max\left\{ \left\lceil \frac{-\ln \delta}{\varepsilon} \right\rceil - 1, 0 \right\}, \\ \tilde{g}(n) &:= n + g(n), \\ h(n) &:= \tilde{g}(n) + 1, \\ m &:= \tilde{g} \circ h^{(\alpha)}(k). \end{aligned}$$

A generalization of Borel's zero-one law

Arthan and Oliva study a generalized version of the Borel-Cantelli lemmas, namely the Erdös-Rényi theorem in [1].
Similarly, we can give a generalization of Borel's zero-one law:

Theorem

Let (A_n) be a sequence of pairwise independent events. Then for all $\delta, \varepsilon > 0$, k natural number and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists l \leq h^{(\beta(2k))}(1) \left[\mathbb{P}\left(\bigcup_{i=l}^{l+g(l)} A_i \right) < \varepsilon \vee \mathbb{P}\left(\bigcup_{i=k}^{m} A_i \right) > 1 - \delta \right],$$

where

$$\begin{aligned} \tilde{g}(n) &:= n + g(n), \\ \omega(n) &:= \tilde{g} \circ h^{(\beta(n))}(1), \\ \beta(n) &:= \max\{\lceil \tfrac{n}{\varepsilon} \rceil - 1, 0\}, \\ h(n) &:= \tilde{g}(n) + 1, \\ m &:= \max\left\{\omega(2k), \lceil -\tfrac{\ln \delta}{\ln 2} \rceil + 4\right\}. \end{aligned}$$

Moving away from Borel's zero-one law

Tail events of the form A_∞ can be presented in a more abstract way:

$$T := \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} B(n,k) \equiv \forall n \, \exists k \geq n \;\, B(n,k),$$

where the events $B(n,k)$ satisfies the following four properties:

1. if $k < n$, then $B(n,k) = \emptyset$;
2. if $n \leq n'$ and $l' \leq l$, then $B(n', l') \subseteq B(n,l)$;
3. $B(n,m) \cup B(m+1,k) \subseteq B(n,k)$;
4. if $k < m$, then $B(n,k)$ and $B(m,l)$ are independent.

We take $B(n,k) = \bigcup_{i=k}^{n} A_i$ to recover the event A_∞.

A framework for more general zero-one laws

Theorem (A more generalized quantified zero-one law)

Let the abstract event $B(n,k)$ satisfy the previous four properties. Then for all $\delta, \varepsilon > 0$, k a natural number and $g : \mathbb{N} \to \mathbb{N}$,

$$\exists l \leq \left(h^{(e-1)}(k) + 1\right) \left[\mathbb{P}\left(B\left(l, l + g(l)\right)\right) < \varepsilon \vee \mathbb{P}\left(B\left(k, m\right)\right) > 1 - \delta.\right],$$

where

$$\begin{aligned} e &:= \left\lceil \frac{1}{\delta\varepsilon} \right\rceil + 1, \\ \tilde{g}(n) &:= n + g(n), \\ h(n) &:= \tilde{g}(n+1), \\ m &:= h^{(e+1)}(k). \end{aligned}$$

Examples

Examples where the framework may apply:

- **Rademacher sequences:** Given a sequence of real numbers (a_n) and a sequence of independent and identically distributed random variables (ϵ_n) taking values in $\{-1, 1\}$, does the sum

  $$\sum_{n=1}^{\infty} \epsilon_n a_n$$

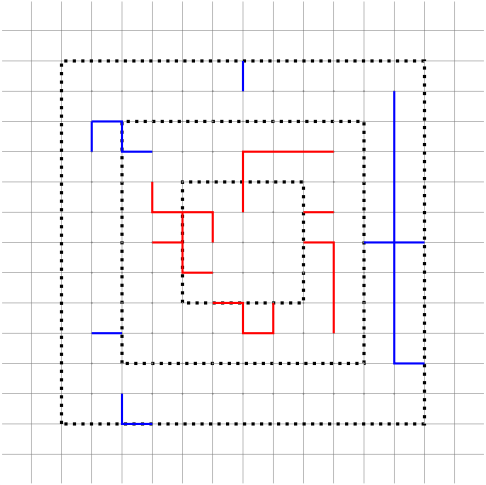
  almost surely converge?

- **Bond percolation:** Given a random process on the edges of a lattice, almost surely does there exist an infinitely connected component?

Percolation: N-squares

Let $B(n,k)$ be the statement

'there exists no filled path from the n-square to k-square'.



Behind the framework

Theorem (Kolmogorov's zero-one law)

Let (A_n) *be a sequence of independent events and denote by* $\mathcal{B}^{(n)} = \mathcal{B}(A_{n+1}, A_{n+2}, \ldots)$ *the smallest* σ*-algebra which contains all the sets* A_k *with subscript* $k \geq n+1$*. Suppose that* T *is an event which belongs to* $\mathcal{B}^{(n)}$ *for all* n*, then either* $\mathbb{P}(T) = 0$ *or* $\mathbb{P}(T) = 1$*.*

Summary

For tail events, we can finitize the associated zero-one law statement and give a uniform bound (independent of the sequence of the event) to say probabilistically if we are close to one case or the other.

Not only are finitizations interesting in their own right, but they can be used to extract numerical rates from existential proofs that use 0-1 laws as a lemma.

From this, it is natural to finitize more complicated zero-one laws, such as Kolmogorov's zero-one law and Hewitt-Savage's zero-one law.

Thank you!

References

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