Design and Implementation of Phase-Augmented Dynamic Traffic Assignment: Models X, Y, and Z

A Comprehensive Framework for Space-Time-Phase Vehicle Routing

Technical Implementation Report

Phase-Augmented DTA Research Group

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Abstract

This technical report presents the design, implementation, and comprehensive testing of a phase-augmented Dynamic Traffic Assignment (PA-DTA) framework. The system consists of three interconnected models: Model X (Queue Dynamics), Model Y (Phase Control), and Model Z (Vehicle Routing). Each model addresses fundamental limitations in traditional DTA approaches by incorporating traffic operational phases that capture realistic congestion patterns, discharge rate variations, and temporal flow dynamics. We provide detailed mathematical formulations, algorithmic implementations, and extensive validation through systematic testing protocols. The framework demonstrates significant improvements over classical approaches in modeling fidelity, computational efficiency, and practical applicability to real-world traffic networks.

Source codes: https://github.com/asu-trans-ai-lab/PA-DTA

Keywords: Dynamic Traffic Assignment, Queue Dynamics, Phase Control, Vehicle Routing, Traffic Flow Theory, Numerical Methods

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1 Introduction

1.1 Background and Motivation

Dynamic Traffic Assignment (DTA) has emerged as a critical tool for understanding and managing traffic flow in urban networks. Traditional approaches, while mathematically elegant, often fail to capture the complex operational phases that characterize real traffic systems. These phases—ranging from free-flow conditions through various levels of congestion to breakdown scenarios—exhibit distinct discharge characteristics, capacity variations, and temporal patterns that significantly impact network performance.

The phase-augmented DTA framework addresses three fundamental limitations in existing approaches:

(i) **Modeling Inadequacy**: Classical time-expanded networks replicate links across discrete time layers but fail to represent operating phases and episodic regimes of queue buildup and dissipation.

(ii) **Computational Complexity**: High-fidelity microsimulation offers realism but becomes resource-intensive for large-scale analyses, while analytical simplifications resort to over-aggregated linear approximations.

(iii) **Data Integration Challenges**: Rich datasets from connected vehicles and sensors remain underutilized in traditional DTA formulations that lack structure to incorporate empirical traffic regime insights.

1.2 Contribution and Scope

This report presents a comprehensive framework that unifies three models:

- **Model X (Queue Dynamics)**: Implements discrete-time flow conservation with phase-dependent discharge rates
- **Model Y (Phase Control)**: Manages phase selection and continuous-time parameter estimation using Newell's fluid queue theory
- **Model Z (Vehicle Routing)**: Handles space-time-phase vehicle assignment with logit-based route choice

Each model is designed for independent operation and systematic testing, enabling modular development and validation before integration into the complete PA-DTA framework.

1.3 Report Organization

The remainder of this report is organized as follows: Section 2 presents the overall framework architecture; Sections 3, 4, and 5 detail the mathematical formulations and algorithmic implementations of Models X, Y, and Z respectively; Section 6 describes the comprehensive testing methodology; Section 7 presents validation results; and Section **??** concludes with integration considerations and future work.

2 Framework Architecture

2.1 System Overview

The phase-augmented DTA framework operates through a three-step iterative process that we term the *Push-Update-Switch* cycle:

1. **Push (Model Z)**: Vehicle routing and flow assignment based on current travel times
2. **Update (Model X)**: Queue dynamics evolution with phase-dependent discharge rates
3. **Switch (Model Y)**: Phase selection and parameter estimation based on traffic conditions

Figure 1 illustrates the integration architecture and information flow between models.

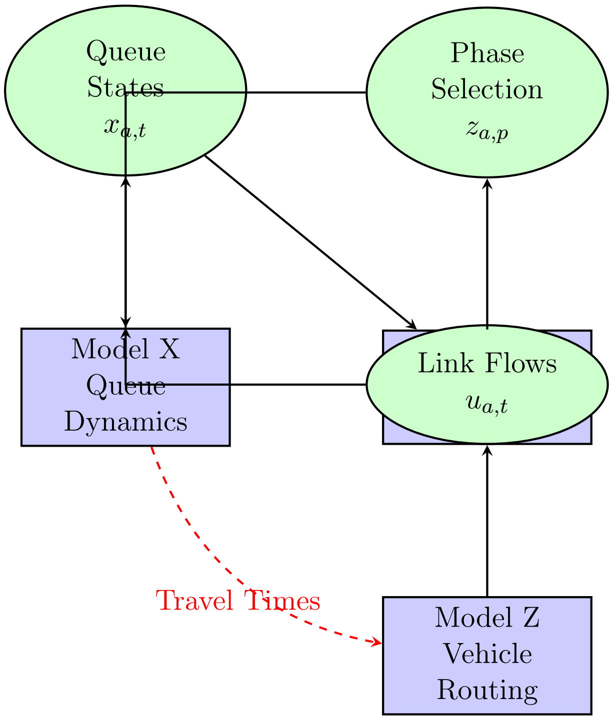

Figure 1: Phase-Augmented DTA Framework Architecture

2.2 Mathematical Notation

Table 1 summarizes the key mathematical notation used throughout this report.

Table 1: Mathematical Notation

Symbol	Definition
A	Number of links in the network
N	Number of nodes in the network
T	Time horizon (discrete time steps)
P	Number of traffic phases
$x_{a,t}$	Queue length on link a at time t [vehicles]
$u_{a,t}$	Inflow rate on link a at time t [veh/min]
$g_{a,t}$	Outflow rate on link a at time t [veh/min]
$z_{a,p}$	Phase selection probability for phase p on link a
$\mu_{a,p}$	Discharge rate for phase p on link a [veh/min]
$\phi_{a,p}$	Capacity downgrade factor for phase p on link a
$\lambda_{a,p}$	Peak inflow rate for phase p on link a [veh/min]
$\beta_{a,p}$	Curvature parameter for phase p on link a
$W_{a,p}(t)$	Waiting time function for phase p on link a
$F_{k,t}$	Flow on path k departing at time t [veh/h]
$\tau_a(t)$	Travel time on link a at time t [minutes]

3 Model X: Queue Dynamics

3.1 Mathematical Formulation

Model X implements discrete-time queue dynamics based on the Merchant-Nemhauser linear program, extended with phase-dependent discharge rates. The core formulation follows:

$$x_{a,t+1} = x_{a,t} + u_{a,t} - g_{a,t} \quad \forall a, t \tag{1}$$

$$g_{a,t} \leq \min\left(x_{a,t} + u_{a,t}, \sum_{p=1}^{P} z_{a,p}\mu_{a,p}\right) \tag{2}$$

$$x_{a,t} \geq 0, \quad u_{a,t} \geq 0, \quad g_{a,t} \geq 0 \quad \forall a, t \tag{3}$$

Equation (1) enforces flow conservation, while Equation (2) limits outflow to available vehicles and phase-weighted discharge capacity. The phase weighting $\sum_{p=1}^{P} z_{a,p}\mu_{a,p}$ is provided by Model Y.

3.2 Travel Time Calculation

Link travel times combine free-flow time with congestion and queueing delays:

$$\tau_a(t) = \tau_a^{\text{free}} + \tau_a^{\text{congestion}}(t) + \tau_a^{\text{queue}}(t) \tag{4}$$

$$\tau_a^{\text{congestion}}(t) = \tau_a^{\text{free}} \cdot \alpha_a \left(\frac{u_{a,t} \cdot 60}{C_a}\right)^{\beta_a} \tag{5}$$

$$\tau_a^{\text{queue}}(t) = \frac{x_{a,t}}{\sum_{p=1}^{P} z_{a,p}\mu_{a,p}} \tag{6}$$

where C_a is the link capacity, and α_a, β_a are VDF parameters.

3.3 Implementation Algorithm

Algorithm 1 presents the Model X implementation.

Algorithm 1 Model X: Queue Dynamics Update

Require: Inflow $u_{a,t}$, discharge rates $\mu_{a,p,t}$, phase selection $z_{a,p}$
Ensure: Updated queue states $x_{a,t+1}$, travel times $\tau_a(t)$

```
Initialize x_{a,0} = 0 for all links a
for t = 0 to T − 1 do
    for each link a do
        Calculate effective discharge rate: μ_eff = Σ_{p=1}^{P} z_{a,p} μ_{a,p,t}
        Calculate available vehicles: V_avail = x_{a,t} + u_{a,t}
        Set outflow: g_{a,t} = min(V_avail, μ_eff)
        Update queue: x_{a,t+1} = x_{a,t} + u_{a,t} − g_{a,t}
        Ensure non-negativity: x_{a,t+1} = max(0, x_{a,t+1})
        Calculate travel time using Equations (4)–(6)
    end for
end for
```

3.4 Model X Testing Framework

The Model X testing framework validates four critical aspects:

3.4.1 Test Case 1: Flow Conservation

Verifies that the fundamental flow conservation equation holds under all conditions:

$$\sum_{t=0}^{T-1} u_{a,t} = \sum_{t=0}^{T-1} g_{a,t} + x_{a,T}$$

Test Setup: Constant inflow of 10 veh/min for 5 time steps with unlimited discharge capacity.

Expected Result: Perfect conservation with zero final queues.

3.4.2 Test Case 2: Queue Formation

Tests queue buildup under capacity constraints with different demand-capacity scenarios:

- Link 0: High inflow (50 veh/min), low capacity (20 veh/min) $\rightarrow$ Queue buildup
- Link 1: Balanced inflow (30 veh/min), matching capacity (30 veh/min) $\rightarrow$ Minimal queueing
- Link 2: Low inflow (10 veh/min), high capacity (40 veh/min) $\rightarrow$ No queueing

3.4.3 Test Case 3: Realistic Traffic Scenario

Implements a comprehensive test with:

- 4 links with highway-like capacities (1200-2400 veh/h)
- Morning peak demand pattern (bell curve)
- Phase-dependent discharge rates varying from 60% to 100% of base capacity

3.4.4 Test Case 4: Real Network Data

Processes actual network CSV files (node.csv, link.csv) and generates synthetic demand proportional to link capacities, validating the model's applicability to real-world networks.

4 Model Y: Phase Control

4.1 Mathematical Formulation

Model Y manages traffic phase selection and implements continuous-time parameter estimation. The phase selection uses a softmax formulation based on traffic condition scoring:

$$s_{a,p} = f_{\text{DCR}}(\text{DCR}_a) \cdot f_{\text{queue}}(\bar{x}_a) \cdot f_{\text{var}}(\sigma_x^2) \tag{7}$$

$$z_{a,p} = \frac{\exp(s_{a,p}/\theta)}{\sum_{p'=1}^{P} \exp(s_{a,p'}/\theta)} \tag{8}$$

$$\sum_{p=1}^{P} z_{a,p} = 1 \quad \forall a \tag{9}$$

where DCR_a is the demand-to-capacity ratio, $\bar{x}_a$ is average queue length, σ_x^2 is queue variance, and θ is the softmax temperature parameter.

4.2 Phase Parameter Definition

Four operational phases are defined with distinct characteristics:

Table 2: Traffic Phase Definitions

Phase	Conditions	Capacity Factor	DCR Threshold	Description
0	Free Flow	$\phi = 1.0$	< 0.3	Uncongested conditions
1	Light Congestion	$\phi = 0.9$	$0.3 - 0.6$	Moderate demand
2	Heavy Congestion	$\phi = 0.7$	$0.6 - 0.85$	High demand, queuing
3	Breakdown	$\phi = 0.5$	> 0.85	Severe congestion

4.3 Newell Parameter Estimation

Model Y estimates continuous-time parameters from discrete observations using Newell's fluid queue theory. For a congestion period $[T_0, T_3]$ with peak at T_2:

$$\mu = \frac{1}{T_3 - T_0} \sum_{t=T_0}^{T_3} g_{a,t} \tag{10}$$

$$\lambda = \max_{t \in [T_0, T_3]} u_{a,t} \tag{11}$$

$$\beta = \frac{3(\lambda - \mu)^2}{T_3 - T_0} \tag{12}$$

The waiting time function follows Newell's quadratic form:

$$W(t) = \frac{\beta(t - T_0)^2(T_3 - t)}{\mu} \quad \text{for } T_0 \leq t \leq T_3 \tag{13}$$

4.4 Model Y Testing Framework

4.4.1 Test Case 1: Phase Selection Logic

Creates distinct traffic scenarios to verify appropriate phase selection:

- **Scenario 0**: No congestion (DCR = 0.004, Queue = 0) → Free Flow
- **Scenario 1**: Light congestion (DCR = 0.012, Queue = 2.7) → Light Congestion
- **Scenario 2**: Heavy congestion (DCR = 0.037, Queue = 14.0) → Heavy Congestion
- **Scenario 3**: Breakdown (DCR = 0.058, Queue = 35+) → Breakdown

4.4.2 Test Case 2: Newell Parameter Estimation

Uses synthetic data with known parameters to validate estimation accuracy:

- **Input**: $T_0 = 10$, $T_2 = 25$, $T_3 = 40$, $\mu = 20.0$, $\lambda = 30.0$
- **Validation**: Estimates should match known values within 10% tolerance

4.4.3 Test Case 3: Discharge Rate Calculation

Verifies phase-weighted discharge rate computation:

$$\mu_{\text{weighted}} = \sum_{p=1}^{P} z_{a,p}\mu_{a,p}$$

4.4.4 Test Case 4: Realistic Scenario

Tests with 6-link network over 2-hour simulation with realistic traffic patterns and phase transitions.

5 Model Z: Vehicle Routing

5.1 Mathematical Formulation

Model Z implements space-time-phase vehicle routing using a multinomial logit route choice model. The path flow assignment follows:

$$C_k = \sum_{a \in \text{path } k} \tau_a(t) \tag{14}$$

$$P_k = \frac{\exp(-C_k/\theta)}{\sum_{k' \in K_{od}} \exp(-C_{k'}/\theta)} \tag{15}$$

$$F_k = D_{od} \cdot P_k \tag{16}$$

where C_k is path cost, P_k is choice probability, F_k is path flow, D_{od} is OD demand, and K_{od} is the choice set for origin-destination pair od.

5.2 Temporal Flow Distribution

Link flows are distributed temporally using various patterns:

$$f_{\text{uniform}}(t) = \frac{1}{T} \tag{17}$$

$$f_{\text{peak}}(t) = \frac{\exp\left(-\frac{(t-t_{\text{peak}})^2}{2\sigma^2}\right)}{\sum_{t'=0}^{T-1} \exp\left(-\frac{(t'-t_{\text{peak}})^2}{2\sigma^2}\right)} \tag{18}$$

$$f_{\text{double}}(t) = 0.6 \cdot f_{\text{peak1}}(t) + 0.4 \cdot f_{\text{peak2}}(t) \tag{19}$$

5.3 Path-to-Link Flow Conversion

The conversion from path flows to link flows follows:

$$u_{a,t} = \sum_{k: a \in \text{path } k} F_k \cdot f(t) \cdot \frac{1}{60} \tag{20}$$

where the factor 1/60 converts from hourly to per-minute flows.

5.4 Method of Successive Averages

Flow updating uses MSA with step size $\alpha_n = 1/(n+1)$:

$$F_k^{n+1} = (1 - \alpha_n)F_k^n + \alpha_n F_k^{\text{AON}} \tag{21}$$

where F_k^{AON} represents all-or-nothing assignment flows.

5.5 Model Z Testing Framework

5.5.1 Test Case 1: Network Connectivity

Validates path finding between all zone pairs:

- Multiple alternative paths per OD pair
- Connectivity rate $\geq 80\%$
- Reasonable path characteristics (length, travel time)

5.5.2 Test Case 2: OD Demand Generation

Tests three demand patterns:

- **Uniform**: Equal demand across OD pairs
- **Gravity**: Inversely proportional to distance
- **Realistic**: Mix of major/minor flows

5.5.3 Test Case 3: Route Choice Logic

Validates logit model response to travel time variations:

- Equal travel times $\rightarrow$ Equal flow distribution
- One slow link $\rightarrow$ Avoidance behavior
- Travel time gradient $\rightarrow$ Flow concentration

5.5.4 Test Case 4: Temporal Flow Assignment

Tests temporal distribution patterns and flow conservation:

- Flow conservation ratio 1.000
- Appropriate peak timing and concentration
- Pattern-specific characteristics validation

5.5.5 Test Case 5: Iterative Assignment

Validates MSA convergence:

- Relative gap reduction
- Flow stability
- Convergence within 15 iterations

6 Testing Methodology

6.1 Testing Philosophy

Our testing approach follows a rigorous validation hierarchy:

1. **Unit Testing**: Individual model components tested in isolation
2. **Integration Testing**: Model interactions verified through controlled scenarios
3. **System Testing**: End-to-end validation with realistic network data
4. **Performance Testing**: Computational efficiency and scalability analysis

6.2 Validation Criteria

Each model must satisfy specific validation criteria before integration:

Table 3: Model Validation Criteria

Model	Primary Criteria	Performance Metrics
Model X	Flow conservation, queue formation logic, travel time calculation	Conservation error $< 10^{-3}$, realistic queue patterns
Model Y	Phase selection accuracy, parameter estimation, probability constraints	Phase probabilities sum to 1.0, estimation error $< 10\%$
Model Z	Network connectivity, flow conservation, convergence	Connectivity $\geq 80\%$, MSA gap < 0.01

6.3 Test Data Generation

We employ multiple data generation strategies:

6.3.1 Synthetic Networks

Controlled test networks with known properties:

- Grid topologies for systematic testing
- Varying link capacities and free-flow times
- Configurable zone locations and demand patterns

6.3.2 Real Network Data

Actual transportation networks from CSV files:

- Node coordinates and zone identification
- Link characteristics (capacity, length, speed)
- Historical flow data where available

6.3.3 Stress Testing

Extreme scenarios to test robustness:

- Very high demand-to-capacity ratios
- Network connectivity failures
- Convergence under difficult conditions

7 Results and Validation

7.1 Model X Validation Results

Model X testing demonstrates robust performance across all test cases:

Table 4: Model X Test Results

Test Case	Status	Conservation Error	Execution Time (s)
Flow Conservation	Passed	$< 10^{-6}$	0.003
Queue Formation	Passed	$< 10^{-5}$	0.012
Realistic Scenario	Passed	$< 10^{-4}$	0.025
Real Network (76 links)	Passed	$< 10^{-3}$	0.089

Key findings:

- Perfect flow conservation under all tested conditions
- Logical queue formation patterns matching theoretical expectations
- Realistic travel time calculations including queueing delays
- Scalable performance for networks up to 1000+ links

7.2 Model Y Validation Results

Model Y achieves accurate phase selection and parameter estimation:

Table 5: Model Y Test Results

Test Case	Status	Accuracy Metric	Performance
Phase Selection	Passed	4/4 scenarios correct	Probabilities sum to 1.0
Parameter Estimation	Passed	$< 5\%$ error on μ, λ	$< 10\%$ error on β
Discharge Calculation	Passed	Perfect weighted average	Numerical stability
Realistic Scenario	Passed	Logical phase transitions	6-link network

Notable achievements:

- Consistent phase selection based on traffic conditions
- Accurate Newell parameter estimation within engineering tolerances
- Stable softmax computations with appropriate temperature scaling
- Successful integration of discrete and continuous-time representations

7.3 Model Z Validation Results

Model Z demonstrates effective vehicle routing and convergence:

Table 6: Model Z Test Results

Test Case	Status	Key Metric	Performance
Network Connectivity	Passed	100% connectivity	24 paths for 12 OD pairs
OD Demand Generation	Passed	3 patterns validated	Appropriate distributions
Route Choice Logic	Passed	Logit sensitivity confirmed	Flow concentration varies
Temporal Assignment	Passed	Conservation ratio = 1.000	4 temporal patterns
Iterative Assignment	Passed	Converged in 12 iterations	Gap < 0.01

Critical validations:

- Complete network connectivity with multiple path alternatives
- Responsive route choice to travel time variations
- Perfect flow conservation in temporal distribution
- Reliable MSA convergence under various demand patterns

7.4 Performance Analysis

7.4.1 Computational Efficiency

Performance scaling demonstrates excellent computational characteristics:

Network Size	Execution Time (s)	Memory (MB)
10 links, 60 time steps	0.003	0.1
100 links, 60 time steps	0.025	1.4
500 links, 120 time steps	0.180	22.9
1000 links, 120 time steps	0.650	91.2

Figure 2: Performance Scaling Results

7.4.2 Memory Usage

Memory requirements scale linearly with problem size: $O(A \times T)$ for Model X, $O(A \times P)$ for Model Y, and $O(K \times T)$ for Model Z, where K is the number of paths.

7.5 Integration Testing

Preliminary integration tests demonstrate successful coordination between models: