Characterization and calibration of volume-to-capacity ratio in volume-delay functions on freeways based on a queue analysis approach

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ABSTRACT

In the iterative traffic assignment process of travel demand models, the performance of road systems is primarily evaluated via volume-delay functions (VDFs). This paper attempts to systematically examine different ways of representing volume-to-capacity (*v/c*) ratios in the Bureau of Public Roads (BPR) function and propose a peak period-based calibrating framework from a freeway bottleneck modeling perspective. Based on a transient queueing state analysis, the proposed framework allows us to estimate the congestion period during which both speed and flow drop due to oversaturation. By characterizing the volume term in the *v/c* ratio as the queued demand after a bottleneck, we hope to show the connection between the speed-flow fundamental diagram with BPR curves through a well-defined demand over capacity ratio. The characterization of *v/c* ratios helps us estimate the hour-to-period factors that are commonly used to generate period capacity before assignments. We systematically calibrate different methods using both speed and volume measurements, in 15-min resolution, from freeways in the Phoenix metro area.

Keywords: Volume delay function; Bureau of Public Roads; Volume-to-capacity ratio; Queued demand; congestion period

1. INTRODUCTION

In the travel demand model, the performance of traffic systems is evaluated via traffic assignment for assessing the impacts of transportation improvement projects. The fundamentally important volume-delay functions (VDFs) have been used as the building blocks to account for the effects of traffic flow on roadway segments' capacities.

The Bureau of Public Roads (BPR) function is one of the most widely used VDFs in practice. However, how to interpret and estimate coefficients of the function still calls for a systematic analytical approach in a theoretically sound fashion. The challenge in calibrating BPR comes from a lack of mathematically rigorous definitions for its kernel volume-to-capacity (*v/c*) ratio; furthermore, its underlying long-term planning resolution is different from the operational perspective of traffic flow theories. In particular, the oversaturated part of the BPR function is not empirically observed in a typical 15-min resolution, where both speed and flow drop due to congestion. As inputs of static traffic assignment (STA) are usually provided in daily or peak period traffic demand, an hour-to-period factor is also required to convert hourly capacity to period capacity. It is also important to carefully examine the relationship between the conversion factors and congestion measures for each facility type (FT), using various types of traffic detector data.

This paper attempts to develop a theoretically consistent and practically effective framework for a data-driven VDF calibration process. We hope to shed new light on the following aspects:

1. A better concept of the *v/c* ratio in the BPR function by bridging the gap between the different temporal resolution of the demand-supply relation.
2. A peak-period demand-oriented calibration framework that closely connects traffic flow measures and queue dynamics (e.g. bottleneck, evolutions, and capacity drop).
3. A data-driven process that allows estimating period-based demand flows for oversaturated traffic conditions.

2. LIMITATIONS OF BPR FUNCTIONS

The BPR function has components of the free-flow speed u_f and a normalized delay function expressed in terms of the *v/c* ratio (Spiess, 1990, Mtoi and Moses, 2014).

$$u = \frac{u_f}{1 + \alpha\left(\frac{v}{c}\right)^{\beta}} \tag{1}$$

where c is the capacity of a link (or a roadway segment), α determines the ratio of speed at volume v to that at free-flow, and β determines the sensitivity that speed u drops from free-flow speed u_f. The existing literature (Horowitz, 1991) has discussed various challenges in applying the BPR function. Dowling et al. (1998) summarized that the function underestimates speed at $v/c \leq 1$, and overestimates speed when $v/c \geq 1$. The research community has developed some alternative VDFs to overcome degenerate situations when the *v/c* ratio is extremely large or small, e.g. Davidson's function (Davidson, 1966), Conical function (Spiess, 1990), and Akçelik function (Akçelik, 1991). Nonetheless, the BPR function is still the most widely used VDF in practice, given its minimal input requirements and simple mathematical form. Its differentiable and strictly monotonous curve is also important for developing efficient traffic equilibrium algorithms.

Before developing a calibration framework, let us examine some important questions about the BPR functions, from both theoretical and practical perspectives.

2.1 What is the meaning of the "capacity" is in the *v/c* ratio?

The concept of "capacity" in the *v/c* ratio originates from "practical capacity" proposed in the Highway Capacity Manual (HCM) in the 1950s. It is defined as the maximum volume that can pass a given point on a link during a specified period without the oversaturated conditions causing unreasonable delay under prevailing traffic conditions. In HCM 1965, a range of levels of service (LOSs) was developed to quantify congestion. Six LOSs are designated from A to F; LOS A represents the free-flow state and LOS F means breakdown. The "ultimate capacity" that is widely used today is defined as the maximum volume when the road link is under LOS E (Branston, 1976; HCM2010). "Ultimate capacity" can be easily estimated and has a consistent meaning across all FTs (Horowitz, 1991).

2.2 What is the meaning of the *v/c* ratio in the BPR function?

If the "c" in the *v/c* ratio means "ultimate capacity", many statements about the *v/c* could lead to different interpretations. As stated by Marshall (2018), one has to distinguish clearly different time durations with a similar value of v/c ≥ 1, for example, severe congestion for 3-4 hours vs. moderate congestion with a 30-min period at v/c ≥ 1. Marshall (2018) also examined possible capacity breakdown periods where the volumes can exceed "ultimate capacity" for only a few minutes and collapse into the state with the discharge rates lower than the capacity.

The research community also acknowledges different perspectives within traffic flow models and the BPR function, and one of the major challenges is that the ratio of *v/c* ≥ 1 might not be empirically observed or directly derived from the common speed-volume relation in a 15-min resolution. That is, the speed-flow relation plotted using field data has a parabolic U-shape, while the fitting of the BPR- function requires the monotonously decreasing function that extends the *v/c* ratio on the x-axis to cover oversaturated conditions in which *v/c* ≥ 1 (Huntsinger and Rouphail, 2011; Moses et al., 2013; Kucharski and Drabicki, 2017).

2.3 What is the relationship between the *v/c* ratio and the hour-to-period factors?

Since ultimate capacity is typically expressed in terms of hourly capacity, another question is how to convert hourly capacity to the daily/period capacity. In travel demand models, peak periods are defined as time windows where travel demands (OD matrices) are pre-specified and STA will be performed in the following:

- AM (6:00am-9:00am)
- MD (9:00am-2:00pm)
- PM (2:00pm-6:00pm)

Practitioners define the hour-to-period factor θ as follows (Benson et al. 1988; Jung, 2003):

$$\theta = \frac{\text{Total volume of a period}}{\text{Highest hourly volume}} = \frac{v_{\text{period}}}{v_{\max}} \tag{2}$$

Since the conversion factors could greatly impact the results of assignment checks (Travel Model Improvement Program, 2010), it is important to explore the linkage between the factors and congestion measures.

3. DERIVATION OF THE V/C RATIO IN BPR FUNCTIONS IN LITERATURE

Following the study by Mtoi and Moses (2014), we partition the VDF coordinate plane into three regimes, with the speed at capacity denoted as u_c.

Regime A: Observed flow rate with undersaturated state *v/c* ≤ 1 and uninterrupted free speed $u \geq u_c$
Regime B: Observed reduced flow rate with saturated state *v/c* ≤ 1 and reduced speeds $u \leq u_c$.
Regime C: Unobserved but derived "demand" volume with oversaturated state *v/c* ≥ 1 with reduced speeds $u \leq u_c$

Fig. 1 uses the Greenshield model and the BPR function to illustrate these three regimes. According to the traffic flow model, the observed flow falls into regime B. Comparatively, the oversaturated part of the BPR function falls in regime C. As a result, the challenge to be addressed in this research is how to map the speed-flow measurements as shown as point *m* from observable regime B to unobservable or derived point *n* in regime C (Akcelik, 2003).

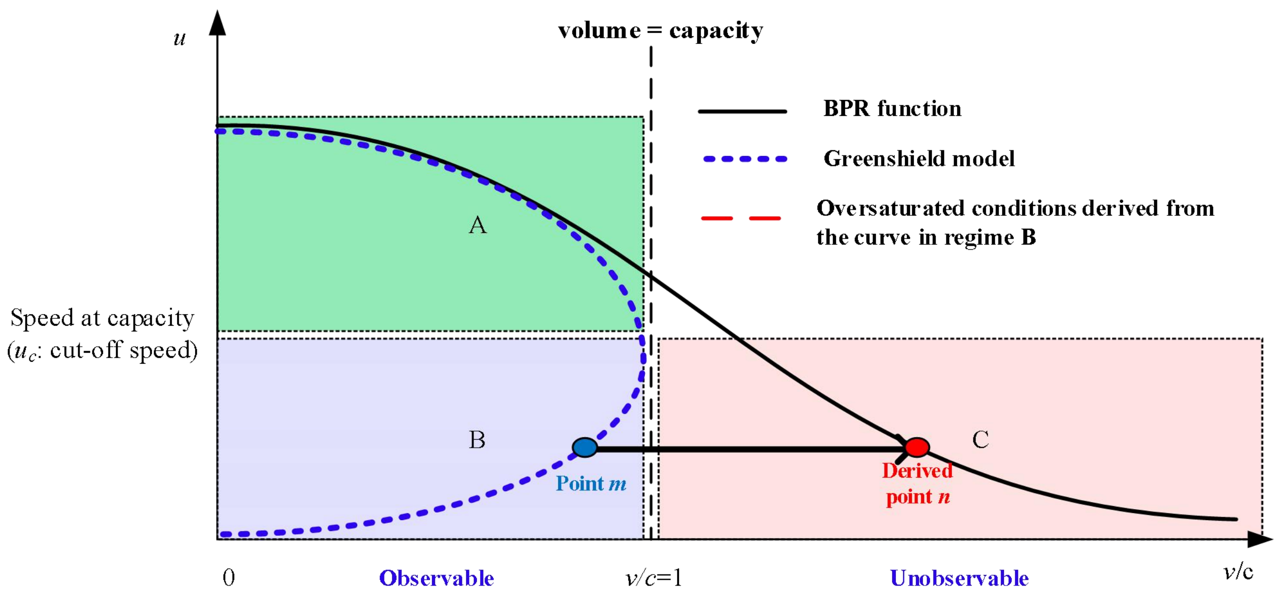


Figure 1 Three regimes in speed vs. *v/c* ratio coordinate plane

3.1 Volume-based method (VBM)

The VBM is adapted from the Florida Standard Urban Transportation Modeling Structure (FSUTMS) (Moses et al., 2013). The following equation is used to determine the derived flow v^* in regime C from observed flow v in regime B, as shown in **Fig.1**:

$$v^* = \begin{cases} c + (c - v) & \text{if } u \leq u_c \\ v & \text{if } u > u_c \end{cases} \tag{3}$$

When $u > u_c$, the *v/c* ratio is $\frac{v^*}{c} = \frac{v}{c}$. When $u \leq u_c$, the *v/c* ratio is $\frac{v^*}{c} = 2 - \frac{v}{c}$.

3.2 Density-based method (DBM)

The DBM proposes an effective way to use density measurements to cover regimes A and C and connect fundamental diagrams with the BPR function (Kucharski and Drabicki, 2017). The flow v^* is derived from density k and critical density k_c:

$$v^* = c\frac{k}{k_c} \tag{4}$$

where v^* exceeds capacity c and falls in regime C. The *v/c* ratio is then $\frac{v^*}{c} = \frac{k}{k_c}$

It is interesting to show that the above mapping can be also viewed as a shrink of capacity from c to c^*, when $u \leq u_c$:

$$\frac{k}{k_c} = \frac{v/u}{k_c} = \frac{v}{uk_c} = \frac{v}{c^*}$$

Then we have

$$c^* = uk_c \leq u_c k_c = c$$

4. QUEUED DEMAND-BASED CALIBRATION FRAMEWORK

After preparing speed-flow observations, we group data collected from different links into different VDF types which are combinations of **facility types (FT)** (e.g. HOV, freeway, arterials) and **area types (AT)** (e.g. CBD, outlying CBD, mixed urban). Then we perform the following two modules.

1. *Calibration:* For each VDF type
    a. *Traffic stream model calibration:* Calibrate the coefficients of the traffic flow model, i.e. free-flow speed, ultimate capacity, and critical density.
    b. *Volume delay function calibration:* For each peak period, map measures from observable regime B to unobservable regime C. Calibrate the coefficients (i.e.. alpha and beta in BPR function) for a given peak period.
    c. *Hour-to-period conversion:* For each peak period, calibrate the hour-to-period conversion factors and period capacity.
2. *Evaluation:* Compare estimation with the observations based on calibrated BPR function and traffic steam model.

Our proposed framework has some characteristics different from other methods.

1. We use observable speed-flow measurements and the macroscopic traffic stream model to calibrate key parameters of free-flow speed and ultimate capacity in the BPR function, while the ultimate capacity corresponds to the cut-off speed calibrated from the traffic flow model.
2. We also hope to distinguish the congestion period and the peak period from a queuing perspective, through clear identification of the congestion period defined through the calibrated cut-off speed from the traffic flow model.
3. We uniquely compute the demand volume (for the peak hour) based on flow observations directly. Under congested conditions, the demand volume includes all the flow within the

congestion period. Thus, the data point n in regime C (**Fig.1**) is now observable other than being mapped from other measurements.

4. By establishing the equivalency of v/c ratio under both peak hour and peak period as shown in Eq. (15) in a late section, we can now compare the observed and derived total volume over the peak period, which leads to a more systematic assessment for different methods.

Fig. 2 summarizes the main steps of our calibration procedure.

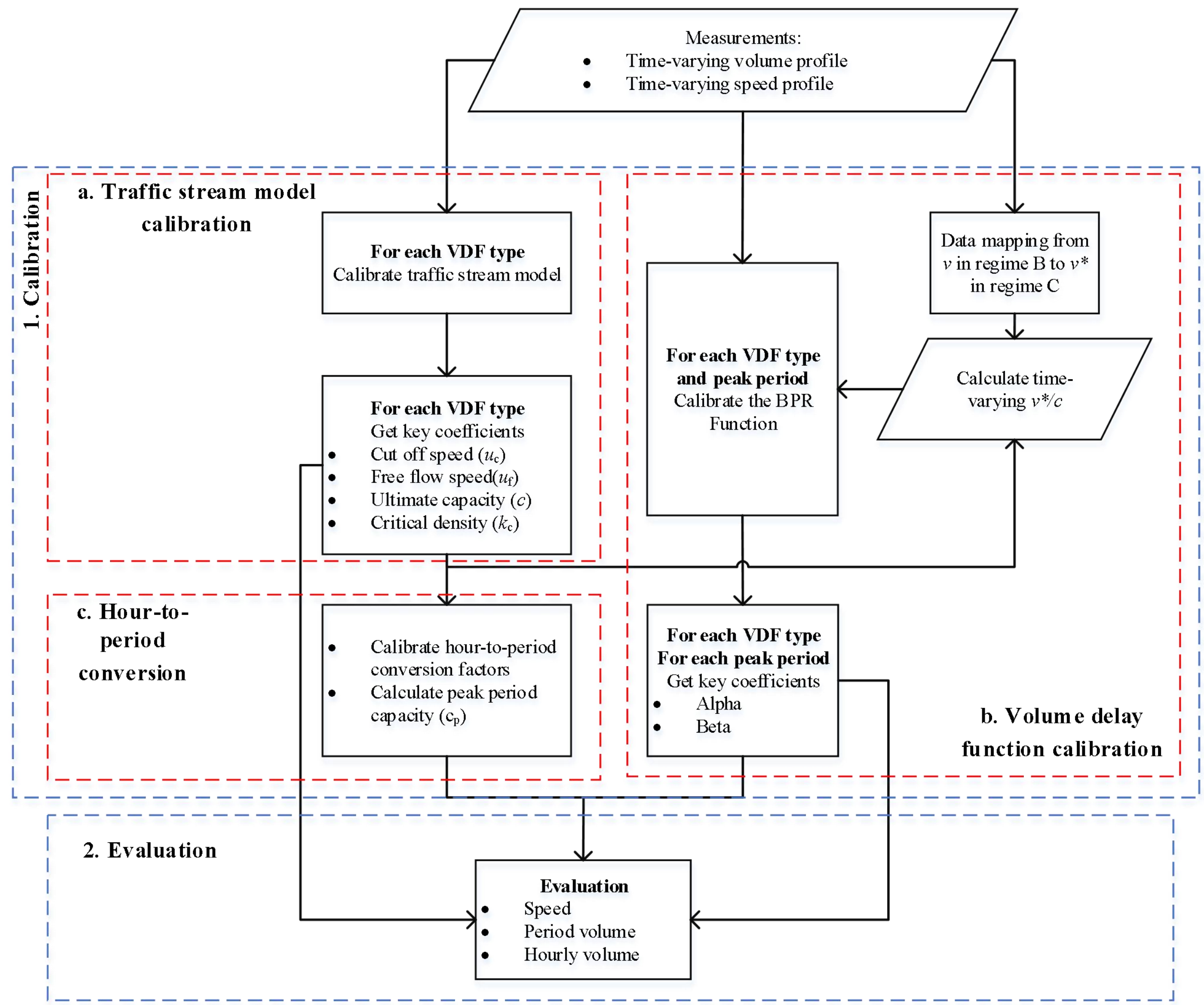


Figure 2 The procedure of calibration

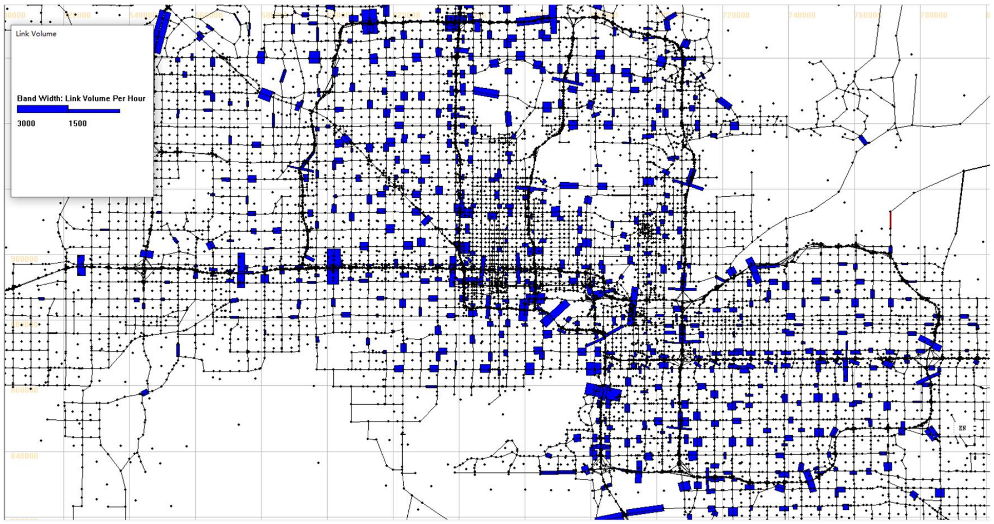
The experiments in this study are conducted based on a 2018 network in the Phoenix metropolitan region consisting of 21,945 nodes and 52,846 directional links from the Maricopa Association of Governments (MAG) travel demand model. Two speed-volume datasets are maintained and provided by the MAG. The data collected from different links are grouped into different VDF types.

Dataset 1 (85 sensors and 3M records): Two months' data on freeway's HOV and general-purpose lanes at 15-min intervals. The data are collected from January 1st to February 29th, 2020 before the COVID-19 pandemic.

Dataset 2 (1,736 sensors and 457K records): Data in two representative weekdays with 15-min intervals in 2018-2019 (only data collected on freeways are used in the following analysis).

Fig.3 shows the volume and speed profile of one representative day at 16:00 from Dataset 2 using the open-source NEXTA GUI data hub: [https://github.com/xzhou99/dtalite_software_release.](https://github.com/xzhou99/dtalite_software_release)

(a)Observed link volume



(b)Observed speed profile

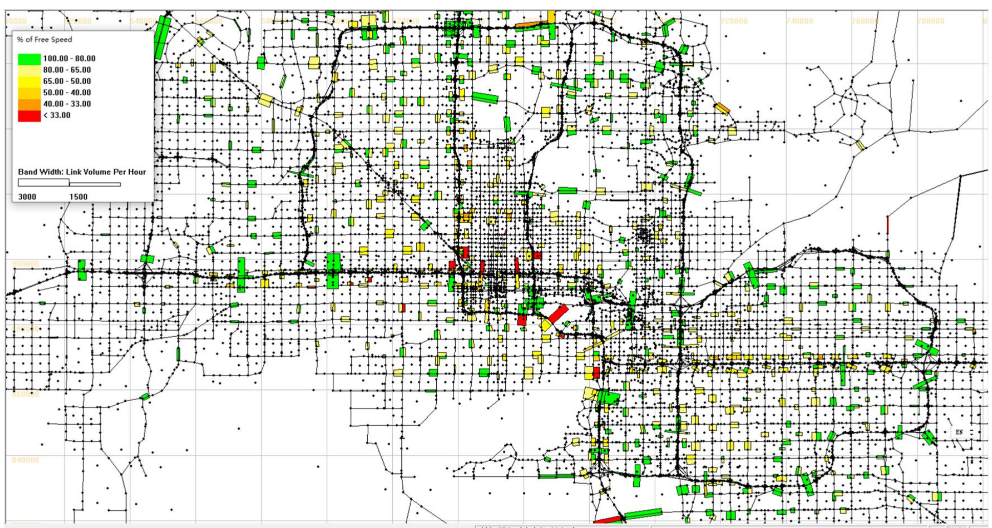


Figure 3 Observed link volume and speed profile in Maricopa County

This section focuses on the details of (a) traffic flow model calibration, (b) volume delay function calibration, and (c) Hour-to-period conversion.

4.1 Macroscopic traffic stream model calibration

The speed-density, flow-density, and speed-flow relationship, known as the fundamental diagrams, are recognized as the foundations of traffic flow theory. Before calibrating the BPR function, we first need to estimate (*a*) free-flow speed u_f, (*b*) critical density k_c, (*c*) ultimate capacity c, and (*d*) cut-off speed u_c. We adopted the three-dimensional (S3) model (Cheng et al. 2020[a]) to estimate the parameters. The projections of the S3 model on the speed-density plane, speed-flow plane, and flow-density plane can simultaneously capture three fundamental diagrams.

The S3 model has the following speed-density relationship:

(speed-density)

$$u = \frac{u_f}{[1 + \left(\frac{k}{k_c}\right)^m]^{\frac{2}{m}}} \tag{5}$$

where u is the speed and k is the density. The term m is introduced to control the smoothness or flatness of the curves for different planes across the feasible range of traffic congestion conditions. The unique characteristic of the S3 model is that the speed-flow and flow-density relationships can be derived analytically. According to the conservative law $v = ku$, we have

(flow-density)

$$v = \frac{ku_f}{[1 + \left(\frac{k}{k_c}\right)^m]^{\frac{2}{m}}} \tag{6}$$

Further, $u^m = \frac{{u_f}^m}{[1+(\frac{k}{k_c})^m]^2}$. Then, we have $\frac{k}{k_c} = \sqrt[m]{\sqrt{\frac{{u_f}^m}{u^m}} - 1}$. Then,

(flow-speed)

$$v = uk_c \sqrt[m]{\sqrt{\frac{{u_f}^m}{u^m}} - 1} \tag{7}$$

Then we can derive ultimate capacity and cut-off speed when the critical density is reached:

1. Ultimate capacity: When k_c is reached, the ultimate capacity c reaches

$$c = v_{\max} = \frac{k_c u_f}{2^{\frac{2}{m}}} \tag{8}$$

2. Cut-off speed: When k_c is reached, the cut-off speed u_c reaches

$$u_c = \frac{u_f}{2^{\frac{2}{m}}} \tag{9}$$

Consider that one has N observations of the flow-speed records of a VDF type

$$(q_i, u_i),\ i = 1,2,\ldots,N$$

there are the density-speed records (k_i, u_i), $i = 1,2,\ldots,N$, where $k_i = q_i/u_i$. Then we can find the best fit **speed-density** function by calibrating coefficients u_f, k_c, and m.

$$\min Z = \sum_{i=1,2,..,P} \left\| u_i - \widehat{u}(k_i; u_f, k_c, m) \right\| \tag{10}$$

where $\widehat{u}(k_i; u_f, k_c, m)$ is the estimated speed for the N observations.

It should be noted that to reach the right-hand-side boundary of speed-flow scatters that correspond to capacity as maximum flow rates, the quantile regression can be used instead of a pure least-squares regression. We separate the speed-axis into several sections along 10 miles/hour intervals. In each section, we only use the data points with a density larger than 90% quantile (Dervisoglu et al. 2009).

Fig. 4 and **Fig.5** depict the speed–flow, speed–density, and flow–density relationships based on Dataset 1 and Dataset 2, respectively. DF type (1,1) means freeways in the CBD area based on MAG's definitions. **Table 1** shows the calibrated parameters for different VDF types including freeways (FT=1) in the CBD area (AT=1), Outlying CBD (AT=2), and mixed urban (AT=3) in the MAG modeling network.

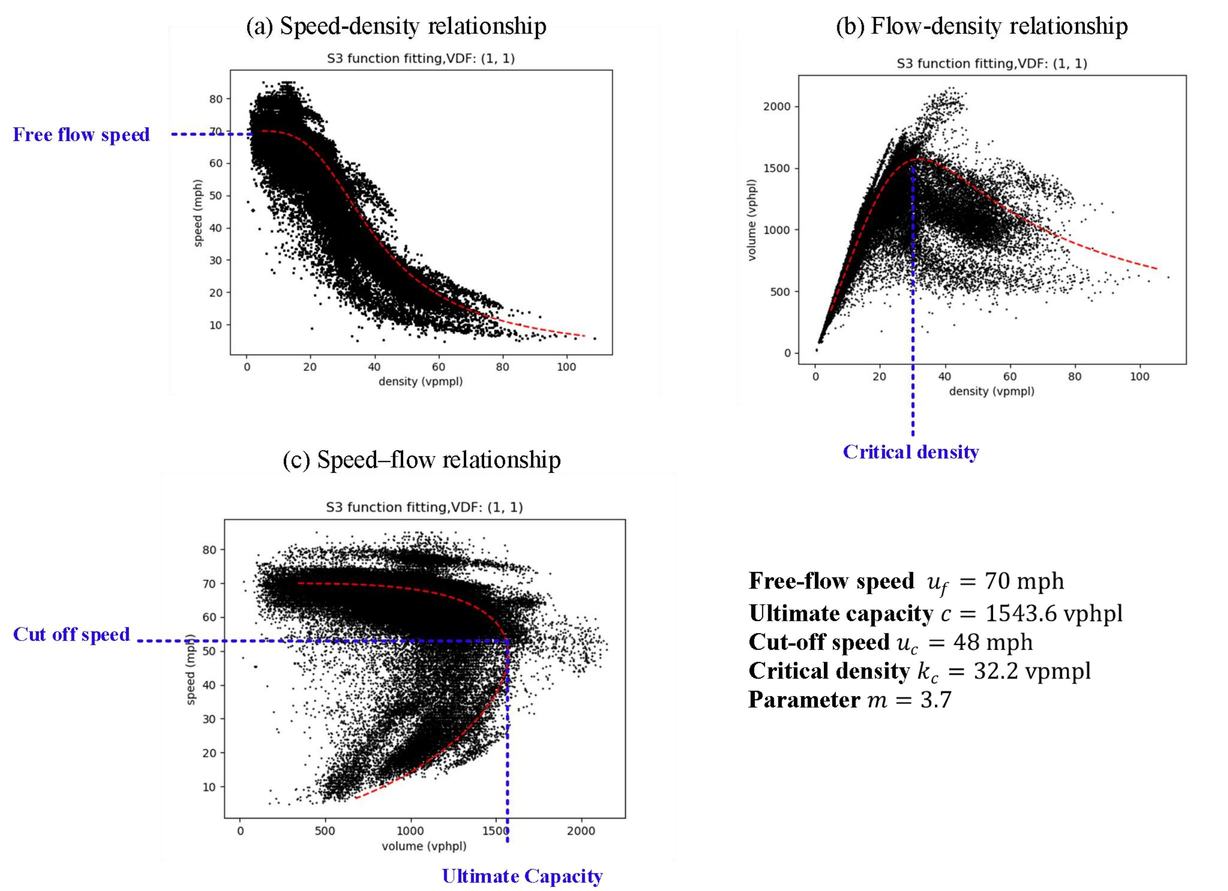


Figure 4 Calibration of S3 model to show flow-speed-density relationships using data collected from freeways in the CBD area (FT=1, AT=1) of the MAG modeling network (Dataset 1)

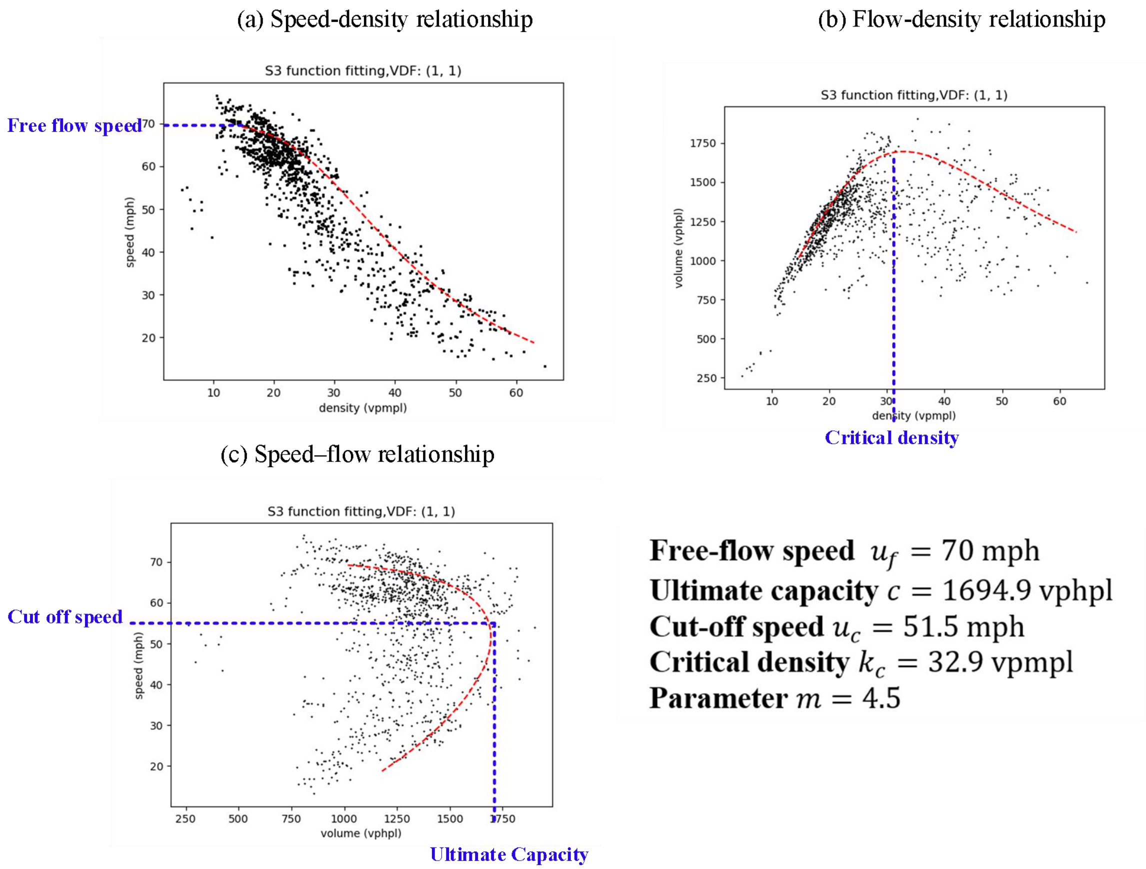


Figure 5 Calibration of S3 model to show flow-speed-density relationships using data collected from freeways in the CBD area (FT=1, AT=1) of the MAG modeling network (Dataset 2)

Table 1 Calibrated parameters of the S3 model for different VDF types and datasets

FT	AT	Dataset	u_c	c	k_c	u_f	m
Freeway	CBD	1	48.95	1570.54	32.08	70.00	48.95
Freeway	Outlying CBD	1	50.61	1714.87	33.89	69.38	50.61
Freeway	Mixed urban	1	56.27	1679.39	29.84	71.00	56.27
Freeway	CBD	2	51.45	1694.94	32.94	70.00	4.50
Freeway	Outlying CBD	2	52.07	1848.10	35.49	70.00	4.69
Freeway	Mixed urban	2	58.90	2063.80	35.04	70.89	7.48

4.2 Volume-delay function calibration

Our method borrows key insights from Huntsinger and Rouphail (2011) to distinguish volume and demand: (a) **Volume** is defined as the flow rate of vehicles during a time interval. (b) **Demand** includes the queue discharge rate at the bottleneck and vehicles in the queue, which is termed as **queued demand** in this paper. Both freeways and intersections have recurring bottlenecks where the discharge rate is constrained and queuing occurs upstream of bottlenecks during rush hours. As discussed below, if *v* in the *v/c* ratio is treated as measured volume, then it is impossible to have $v/c \geq 1$ as the volume cannot exceed capacity. However, it is theoretically consistent to consider the queued demand as *v*, which could be larger than the supplied capacity during a given time interval.

In this paper, the *v/c* ratio is not viewed as **an instantaneous state** of a road link. Instead, we attempt to use this ratio to reflect the **(cumulative)** supply-demand relationship of a link **during a peak period**. Thus, "$v/c \geq 1$" implies that demand v is larger than capacity c for a certain time period. From a bottleneck modeling perspective, the *v/c* ratio could be related to the congestion duration for all v vehicles to be discharged at a constant ultimate capacity c.

To characterize the oversaturation degree of a peak period, we can only focus on a "peak hour" within the period. In the existing studies, a peak hour is defined as the hour with the highest volume during a peak period. However, in this paper, we define the peak hour as the hour containing the lowest speed during the peak period. In the example illustrated in **Fig. 6(a)**, the peak period AM is from 6:00 to 9:00. The lowest speed u_{min} happens within 8:00-8:15. The peak hour is from 7:45 to 8:45 including data collected from 7:45-8:00, 8:00-8:15, 8:15-8:30, and 8:30-8:45. We denote the volume within the peak hour as D_{h}.

Next, we consider a **congestion period** from t_0 and t_1 **containing** the peak hour. The total volume D within the congestion period is viewed as the queued demand for the peak hour's capacity under oversaturated conditions, which implies when $L = t_1 - t_0 \geq 1$ hour, $D \geq D_{\mathrm{h}}$ and D becomes the **queued demand** for the peak hour. Conversely, D_{h} is served as the demand.

There are different ways to define the congestion period (Cheng et al, 2020[b]). In this paper, we first find the lowest speed u_{min} and then extend the range of the congestion period until the speed is larger than cut off speed u_c, as shown in **Fig.6 (b).** This method is based on the assumption that the speed evolution only has one dip during the peak period. In this case, D is the summation of all volumes under the cut-off speed including both discharges and queues during the peak period. Then D/c means the congestion duration to service all D vehicles if the system has discharge rate c. When c is the ultimate capacity of the link, D/c is the **lower bound** of the congestion duration, because the actual queue discharge rates are less than the ultimate capacity.

To achieve the mapping from regime B to regime C, we define the **highest** speed **within the peak hour** as u_{min}'. v^* , which can be obtained using the following:

$$v^* = \begin{cases} D_{\mathrm{h}} & \text{if } u_{\mathrm{min}}' > u_c \\ D & \text{if } u_{\mathrm{min}}' \leq u_c \end{cases} \tag{11}$$

The defined *v/c* ratio can be interpreted and observed more intuitively.

$$\frac{v^*}{c} = \begin{cases} \dfrac{D_{\mathrm{h}}}{c} & \text{if } u_{\mathrm{min}}' > u_c \\ \dfrac{D}{c} & \text{if } u_{\mathrm{min}}' \leq u_c \end{cases} \tag{12}$$

Based on the above formula, we reformulate the speed-flow fundamental diagram as an estimation problem with speeds as a monotonically decreasing function of v^*/c ratios.

Consider P links attributing to a VDF type. Each link has a flow-speed observation during a peak period, i.e. $(v_i^*, \overline{u_i})$, $i = 1,2,\ldots,P$. v_i^* is the demand on link i during the peak period, and $\overline{u_i}$ is the mean-speed of link i during the peak period. One finds the best fit representing the **speed-flow** function and calibrates α and β using non-linear least squares.

$$\min Z = \sum_{i=1,2,..,P} \left\| \overline{u_i} - \hat{u}(v_i{}^*; u_f, \alpha, \beta) \right\| \tag{13}$$

where $\hat{u}(v_i{}^*; u_f, \alpha, \beta)$ is the estimated speed for the P observations.

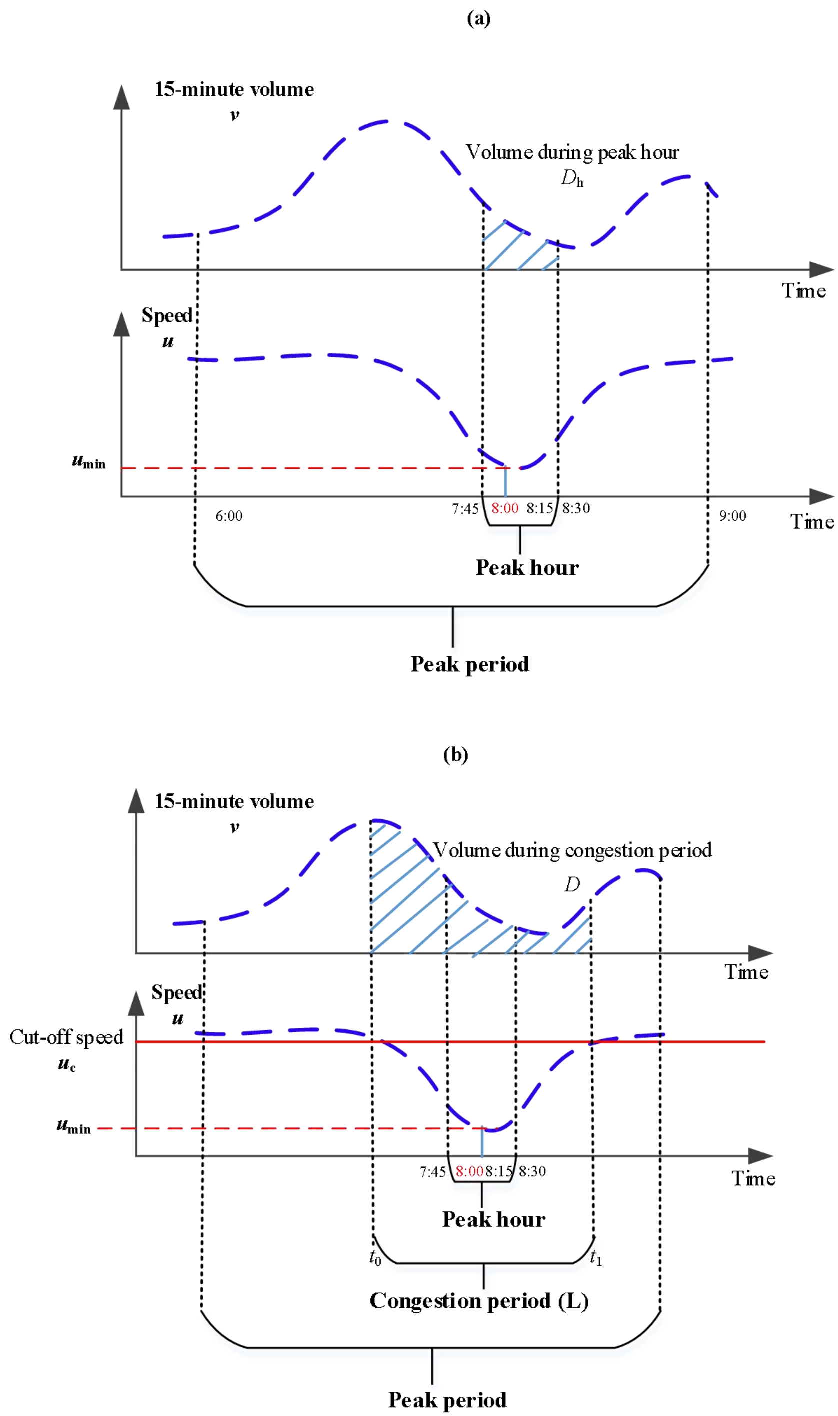


Figure 6 Illustration of peak hour, congestion periods and peak period

4.3 Hour-to-period conversion

To convert hourly capacity c to period capacity c_{period}, we use the following formula with hour-to-period factor θ.

$$c\theta = c_{\text{period}} \tag{14}$$

Let v_{period} be the total volume of the peak period. We now assume the following equivalency of v/c ratio between peak hour and peak periods.

$$\frac{v^*}{c} = \frac{v_{\text{period}}}{c_{\text{period}}} \tag{15}$$

Being different from Eq. (2), we can estimate θ for each link

$$\theta = \frac{\text{Total volume of a period}}{\textbf{Queued demand}\text{ of a peak hour}} = \frac{v_{\text{period}}}{v^*} \tag{16}$$

We can also calculate period capacity c_{period} for all links of a VDF type that is expressed by $\overline{\theta}$.

4.4 Comparison of different methods

In this subsection, we compare our proposed queued demand-based method (QBM) with the two reviewed existing methods: volume-based method (VBM) and density-based method (DBM). **Table 2** shows the differences.

Table 2 Comparison between QBM, VBM, and DBM

	QBM	**VBM**	**DBM**
Volume v	N/A	Mean volume of a link during the peak period	N/A
Speed u	Mean speed of a link during the peak period	Mean speed of a link during the peak period	Mean speed of a link during the peak period
Derived flow v*	$\begin{cases} D_{\text{h}} & \text{if } u'_{\min} > u_c \\ D & \text{if } u'_{\min} \le u_c \end{cases}$	$\begin{cases} c + (c - v) & \text{if } u \le u_c \\ v & \text{if } u > u_c \end{cases}$	$c\dfrac{k}{k_c}$
v*/c ratio	$\begin{cases} \dfrac{D_{\text{h}}}{c} & \text{if } u'_{\min} > u_c \\ \dfrac{D}{c} & \text{if } u'_{\min} \le u_c \end{cases}$	$\begin{cases} \dfrac{v}{c} & \text{if } u > u_c \\ 2 - \dfrac{v}{c} & \text{if } u \le u_c \end{cases}$	$\dfrac{k}{k_c}$
Hour-to-period factor for each link and peak period	$\dfrac{v_{\text{period}}}{v^*}$	$\dfrac{v_{\text{period}}}{v_{\max}}$	$\dfrac{v_{\text{period}}}{v_{\max}}$
Hour-to-period factor for each VDF and peak period	Given a peak period, average the hour-to-period factors of links, if the link attributes to the VDF type	Given a peak period, average the hour-to-period factors of links, if the link attributes to the VDF type	Given a peak period, average the hour-to-period factors of links, if the link attributes to the VDF type
Hourly capacity	Ultimate capacity c	Ultimate capacity c	Ultimate capacity c
Period capacity for each link and peak period	$\dfrac{v_{\text{period}}}{v^*}c$	$\dfrac{v_{\text{period}}}{v_{\max}}c$	$\dfrac{v_{\text{period}}}{v_{\max}}c$
Period capacity for each VDF and peak period	(Hour-to-period factor for the VDF type and peak period)×(Ultimate capacity of the VDF type)	(Hour-to-period factor for the VDF type and peak period)×(Ultimate capacity of the VDF type)	(Hour-to-period factor for the VDF type and peak period)×(Ultimate capacity of the VDF type)

# of data points for calibration	Each link i per period has a $v_i{}^*$ and $\overline{u_i}$	Each link i per period has a $v_i{}^*$ and $\overline{u_i}$	Each link i per period has a $v_i{}^*$ and $\overline{u_i}$
Unit of v*/c	Hour (or minute)	No unit	No unit
Economic meaning	The lower bound of congestion duration	N/A	N/A

5. RESULTS

Estimate speed using calibrated *v/c* ratio

After the derived v*/c ratio is plotted against the speed, we can calibrate the BPR functions using the three aforementioned methods. **Fig.7** compares the BPR curves calibrated by different methods using Datasets 1 and 2 respectively. The figures show that the curves calibrated by QBM have a larger *v/c* ratio compared to the curves generated by VBM and DBM. **Fig.7** only displays the calibrated results of morning peak periods 6:00-9:00, when FT=1 (freeway) and AT=1 (CBD).

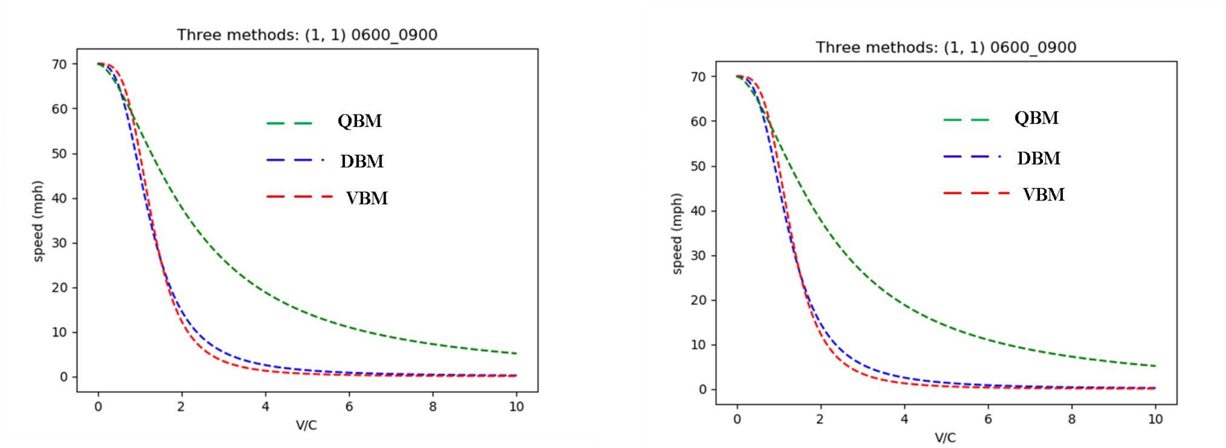


Figure 7 Comparison of calibrated curves of peak periods AM using two datasets when FT=1 and AT=1.

Furthermore, **Fig.8** shows the calibrated curves with scatters. The mean absolute percentage error of speed (MAPE_S) is used to evaluate the effectiveness of the calibration for each link:

$$MAPE\_S_i = \frac{\left|\overline{u_i} - \widehat{u}(v_i{}^*; u_f, \alpha,\beta)\right|}{\widehat{u}(v_i{}^*; u_f, \alpha,\beta)}\% \tag{17}$$

Table 3 reports the average MAPE_S of different VDF types based on three different methods. We find that DBM can find the best parameters (alpha and beta) to minimize the MAPE_S.

Dataset 1 | Dataset 2

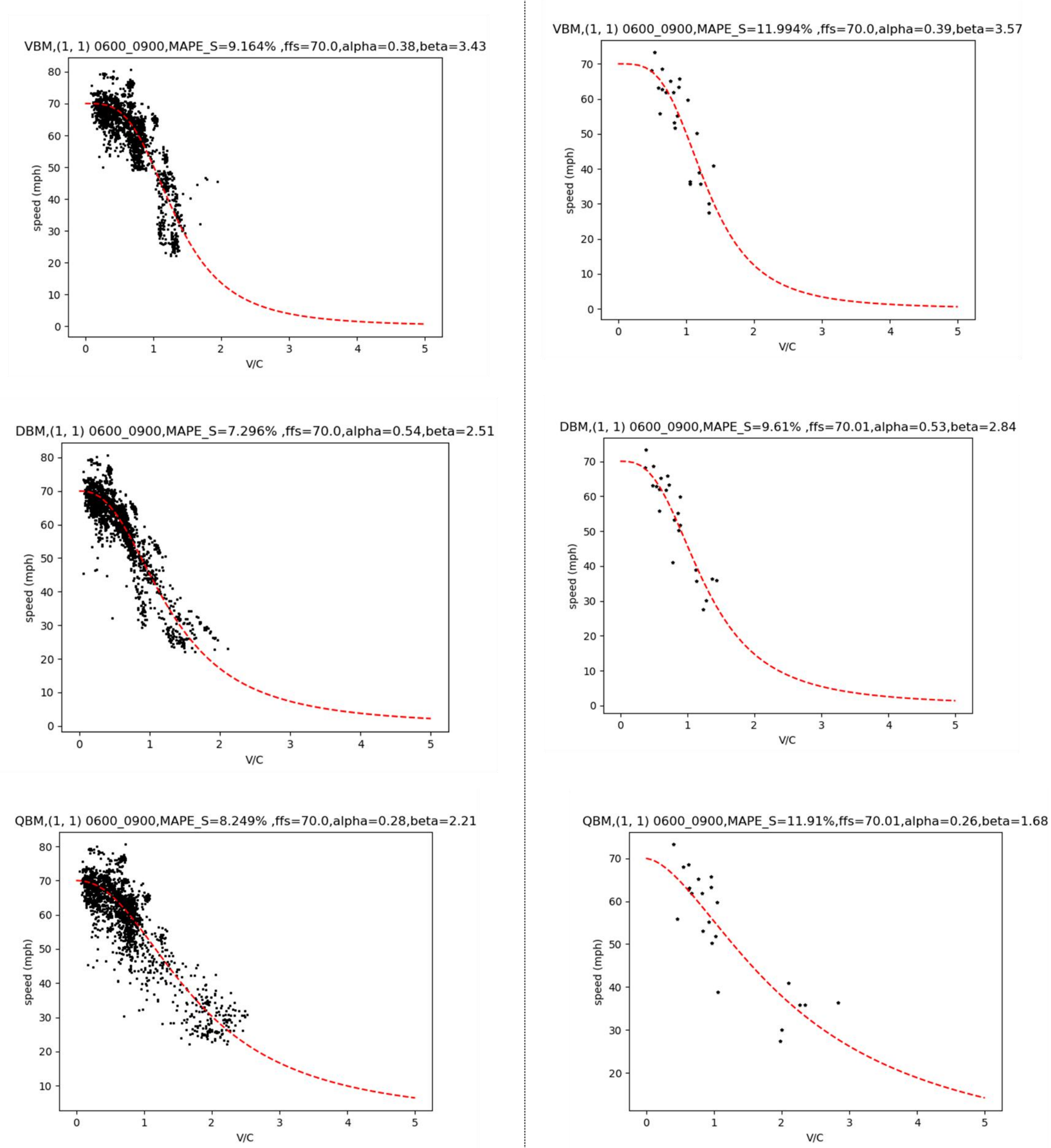


Figure 8 Calibrated curves of using two datasets with peak period =AM, FT=1, and AT=1.

Table 3 MAPE for speed for different VDF types and peak periods

Dataset 1					
FT	AT	Period	PES(QBM)	PES(VBM)	PES(DBM)
Freeway	CBD	6:00-9:00	8.00%	9.40%	**7.30%***
Freeway	CBD	9:00-14:00	6.00%	5.80%	**4.60%***
Freeway	CBD	14:00-18:00	14.00%	9.90%	**8.10%***
Freeway	Outlying CBD	6:00-9:00	11.00%	9.40%	**9.00%***
Freeway	Outlying CBD	9:00-14:00	7.00%	5.90%	**6.10%***

Freeway	Outlying CBD	14:00-18:00	15.00%	10.90%	**9.90%***
Freeway	Mixed Urban	6:00-9:00	9.00%	7.80%	**6.50%***
Freeway	Mixed Urban	9:00-14:00	7.00%	7.00%	**6.10%***
Freeway	Mixed Urban	14:00-18:00	8.00%	7.50%	**6.70%***
Mean			9.40%	8.20%	**7.10%***
Dataset 2					
FT	AT	Period	PES(QBM)	PES(VBM)	PES(DBM)
Freeway	CBD	6:00-9:00	12.00%	12.00%	**9.60%***
Freeway	CBD	9:00-14:00	6.00%	6.90%	**5.20%***
Freeway	CBD	14:00-18:00	17.00%	16.30%	**13.10%***
Freeway	Outlying CBD	6:00-9:00	11.00%	**9.30%***	9.70%
Freeway	Outlying CBD	9:00-14:00	5.00%	4.20%	**3.80%***
Freeway	Outlying CBD	14:00-18:00	12.00%	11.70%	**9.50%***
Freeway	Mixed Urban	6:00-9:00	13.00%	8.60%	**7.70%***
Freeway	Mixed Urban	9:00-14:00	4.00%	3.60%	**3.30%***
Freeway	Mixed Urban	14:00-18:00	7.00%	7.20%	**6.30%***
Mean			9.67%	8.87%	**7.58%***

* implies the best MAPE_S in each row

Estimate peak period volume using the calibrated *v/c* ratio

Note that if we calculate the period capacity for each link and peak period, the QBM is the only method that satisfies the following intuitive equation:

$$\textbf{(period capacity)} \times \textbf{(}v^*/c\textbf{ ratio)} = \textbf{(total volume of the peak period)} \tag{18}$$

For each link and peak period, we apply the following mean absolute percentage error of period volume (MAPE_PV) to measure whether the equation is satisfied or not.

$$MAPE\_PV = \frac{\left| v_{\text{period}} - \frac{v^*}{c} c_{\text{period}} \right|}{\frac{v^*}{c} c_{\text{period}}} \tag{19}$$

Furthermore, we compare the total volume of the peak periods with the "(period capacity)× (v*/c ratio)" in **Fig. 9** using the three methods, respectively. If we calculate the **period capacity for each link individually,** Eq. (18) will be satisfied in QBM, as shown in **Fig.9 (a) and (b)**. However, it is not strictly satisfying when we use VBM or DBM**.**

In the experiments, we calculate the **average** MAPE_PV for links attributing to a VDF type. For a given peak period, all links of the same VDF type share the same period capacity. **Table A.1** in the appendix reports the calibrated capacity and conversion factors. As shown in **Table 4**, the QBM gets lower MAPE_PV than VBM and DBM in Dataset 1. However, the QBM does not perform well in Dataset 2 like in Dataset 1. DBM works better for MAPE_PV.

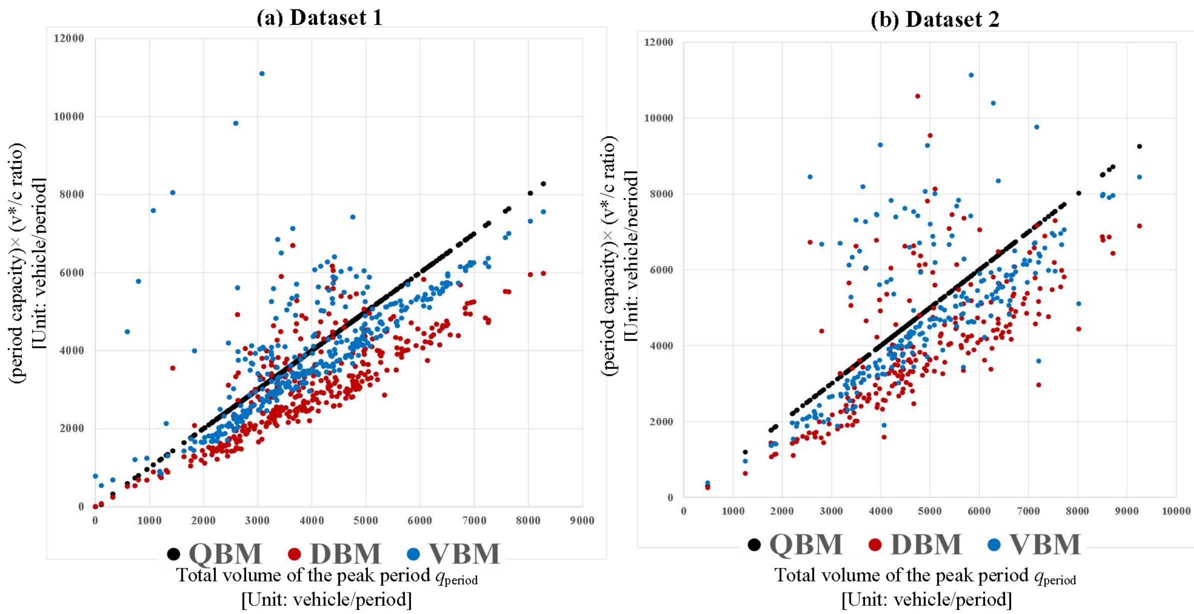


Figure 9 Total volume of the peak period vs. (period capacity) × (v*/c ratio). Each link has its own period capacity.

Table 4 MAPE for period volume for different VDF types and peak periods

Dataset 1					
FT	AT	Peak period	MAPE (QBM)	MAPE (VBM)	MAPE (DBM)
Freeway	CBD	6:00-9:00	**13.30%***	13.70%	45.00%
Freeway	CBD	9:00-14:00	**6.00%***	18.70%	57.50%
Freeway	CBD	14:00-18:00	26.00%	**16.90%***	33.70%
Freeway	Outlying CBD	6:00-9:00	28.60%	**21.20%***	34.70%
Freeway	Outlying CBD	9:00-14:00	**8.40%***	21.10%	48.80%
Freeway	Outlying CBD	14:00-18:00	31.00%	**20.80%/***	29.20%
Freeway	Mixed Urban	6:00-9:00	**8.40%***	21.50%	46.00%
Freeway	Mixed Urban	9:00-14:00	**3.10%***	19.80%	52.50%
Freeway	Mixed Urban	14:00-18:00	**2.60%***	12.00%	38.00%
Average			**14.20%***	18.40%	42.80%
Dataset 2					
FT	AT	Peak period	MAPE (QBM)	MAPE (VBM)	MAPE (DBM)
Freeway	CBD	6:00-9:00	31.70%	30.72%	**17.64%***
Freeway	CBD	9:00-14:00	**15.08%***	35.46%	15.31%
Freeway	CBD	14:00-18:00	51.78%	**26.81%***	27.42%
Freeway	Outlying CBD	6:00-9:00	22.50%	45.02%	**21.76%***
Freeway	Outlying CBD	9:00-14:00	**8.86%***	48.35%	15.56%
Freeway	Outlying CBD	14:00-18:00	33.52%	29.11%	**19.00%***
Freeway	Mixed Urban	6:00-9:00	**21.54%***	47.02%	29.86%
Freeway	Mixed Urban	9:00-14:00	**11.91%***	46.78%	25.26%

Freeway	Mixed Urban	14:00-18:00	21.86%	32.79%	**20.47%***
Average			24.31%	38.01%	**21.36%***
* means the best MAPE_PV in each row					

Estimate hourly volume using the calibrated *v/c* ratio

We use the following steps to evaluate the effectiveness of each method, as shown in **Fig. 10**

1. Using the calibrated BPR function, we calculate the speed u_{BPR}. We start from point A, then go to point B, and then find point C:

$$u_{\mathrm{BPR}} = \frac{u_f}{1+\alpha\left(\frac{v_{\mathrm{period}}}{c_{\mathrm{period}}}\right)^{\beta}} \tag{20}$$

2. Calculate hourly volume v_{BPR} in the S3 model corresponding to u_{BPR}. For each link, we start from point C, then go to point D, and then find point E:

$$v_{\mathrm{BPR}} = u_{\mathrm{BPR}} k_c \sqrt[m]{\sqrt{\frac{{u_f}^m}{{u_{\mathrm{BPR}}}^m}} - 1} \tag{21}$$

3. Calculate mean absolute percentage error of hourly volume (MAPE_HV) between observation v_{OBS} and v_{BPR}. We then compare point E with point F to calculate MAPE_HV:

$$MAPE\_HV = \frac{|v_{\mathrm{OBS}} - v_{\mathrm{BPR}}|}{v_{\mathrm{BPR}}} \tag{22}$$

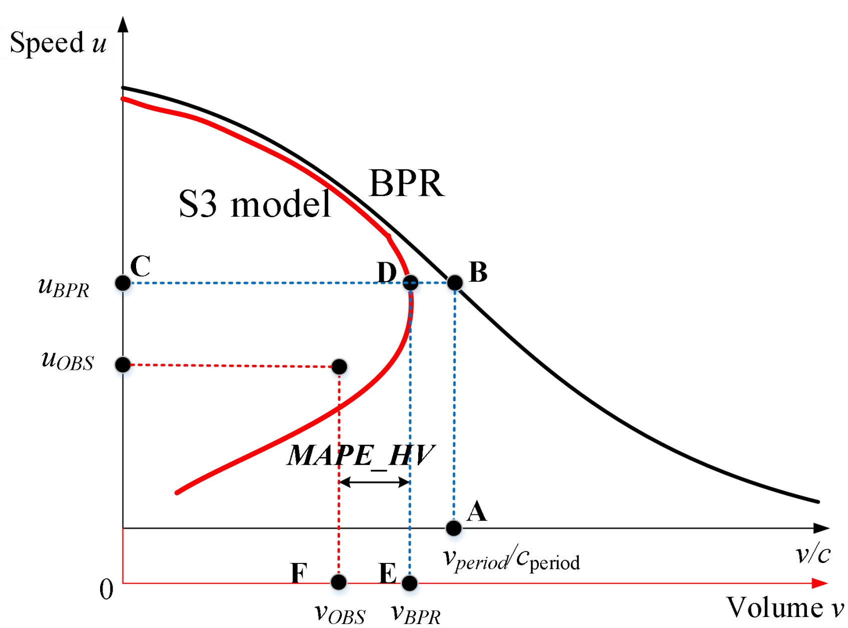


Figure 10. Illustration of the error function for assignment checking

Table 5 reports the MAPE_HV for different VDF types and peak periods when all links of the same VDF type share the same period capacity. As shown in **Table 5**, the QBM reaches lower MAPE_HV than VBM and DBM in Dataset 1 with two-month data. In Dataset 2 with more sensor spatial coverage, DBM performs quite well.

Table 5 MAPE for hourly volume for different VDF types and peak periods. Each VDF has its own period capacity.

Dataset 1					
FT	AT	Period	MAPE_HV (QBM)	MAPE_HV (VBM)	MAPE_HV (DBM)
Freeway	CBD	6:00-9:00	31.50%	37.55%	**30.67%***
Freeway	CBD	9:00-14:00	**26.94%***	36.62%	30.01%
Freeway	CBD	14:00-18:00	31.35%	32.24%	**26.44%***
Freeway	Outlying CBD	6:00-9:00	39.97%	41.19%	**38.06%***
Freeway	Outlying CBD	9:00-14:00	**30.16%***	34.27%	30.78%
Freeway	Outlying CBD	14:00-18:00	**30.28%***	32.84%	31.63%
Freeway	Mixed Urban	6:00-9:00	36.40%	42.74%	**35.18%***
Freeway	Mixed Urban	9:00-14:00	**20.43%***	47.37%	64.49%
Freeway	Mixed Urban	14:00-18:00	**21.59%***	30.17%	21.72%
Average			**29.85%***	37.22%	34.33%
Dataset 2					
FT	AT	Period	MAPE_HV (QBM)	MAPE_HV (VBM)	MAPE_HV (DBM)
Freeway	CBD	6:00-9:00	22.95%	23.62%	**21.88%***
Freeway	CBD	9:00-14:00	**19.80%***	24.86%	22.53%
Freeway	CBD	14:00-18:00	25.97%	25.95%	**25.36%***
Freeway	Outlying CBD	6:00-9:00	31.08%	34.22%	**28.71%***
Freeway	Outlying CBD	9:00-14:00	**28.66%***	36.68%	31.74%
Freeway	Outlying CBD	14:00-18:00	30.73%	31.54%	**28.06%***
Freeway	Mixed Urban	6:00-9:00	37.81%	36.93%	**32.98%***
Freeway	Mixed Urban	9:00-14:00	**42.77%***	45.82%	43.23%
Freeway	Mixed Urban	14:00-18:00	**31.12%***	34.62%	31.24%
Average			30.10%	32.69%	**29.52%***

* means the best PEHV in each row

Tables 6 summarizes the average MAPE for speed, period volume, and hourly volume. The values are calculated by averaging the MAPEs of different VDF types and peak periods. The beneficial finding is that the QBM works relatively well to reproduce the period-based and hourly volumes compared with other methods. The DBM can find the best fit for speed.

Table 6 MAPE for different VDF types and peak periods. Each VDF has its own period capacity.

		QBM	VBM	DBM
Average MAPE for speed	Dataset 1	9.40%	8.20%	**7.10%***
	Dataset 2	9.67%	8.87%	**7.58%***
Average MAPE for period volume	Dataset 1	**14.20%***	18.40%	42.80%
	Dataset 2	24.31%	38.01%	**21.36%***
Average MAPE for hourly volume	Dataset 1	**29.44%***	36.89%	33.57%
	Dataset 2	**29.10%***	32.67%	29.53%
Total average		**19.35%***	23.84%	23.66%

* means the best MAPE in each row

6. CONCLUSIONS AND FUTURE RESEARCH

This research proposed a queue-theoretic modeling framework for several key parameters in the BPR function. As the v/c ratio is widely used to measure the degree of congestion, it is important to systematically understand and calibrate the v/c ratio for planning applications. By characterizing the volume term in the *v/c* ratios as the queued demand after a bottleneck, this paper shows the connection between the fundamental diagrams with BPR curves through a well-defined demand-over-capacity ratio. The characterization of *v/c* ratios helps us estimate the hour-to-period factors that are commonly used to generate period capacity before assignments.

Using freeway detector data in the Phoenix metro area in Arizona, we systematically compared our proposed method with the existing volume-based method (VBM) and density-based method (DBM). Based on the calibration results from two real-world data sets, different methods have their pros and cons. The QBM can be used to better relate the v/c ratio with the observed peak period volume. Future research could investigate the application of this approach in the traffic assignment stage and integrate it into a travel demand estimation framework using multiple data sources. (Wu et al., 2018).

APPENDIX

Table A.1 Calibrated conversion factor and for different VDF types and peak periods

Dataset 1						
FT	AT	Peak period	θ (Others)	θ (QBM)	c_{period} (Others)	c_{period} (QBM)
Freeway	CBD	6:00-9:00	2.57	2.76	4041	4336
Freeway	CBD	9:00-14:00	4.37	4.96	6870	7797
Freeway	CBD	14:00-18:00	3.31	2.95	5195	4640
Freeway	Outlying CBD	6:00-9:00	2.48	2.44	4247	4176
Freeway	Outlying CBD	9:00-14:00	4.40	4.89	7544	8388
Freeway	Outlying CBD	14:00-18:00	3.30	2.90	5661	4981
Freeway	Mixed Urban	6:00-9:00	2.40	2.84	4039	4767
Freeway	Mixed Urban	9:00-14:00	4.31	5.19	7233	8724

Freeway	Mixed Urban	14:00-18:00	3.33	3.72	5586	6252
Dataset 2						
FT	AT	Peak period	θ (Others)	θ (QBM)	c_{period} (Others)	c_{period} (QBM)
Freeway	CBD	6:00-9:00	2.72	2.52	4603	4278
Freeway	CBD	9:00-14:00	4.66	5.32	7895	9023
Freeway	CBD	14:00-18:00	3.29	2.21	5579	3750
Freeway	Outlying CBD	6:00-9:00	2.55	2.64	4711	4872
Freeway	Outlying CBD	9:00-14:00	4.54	5.18	8384	9566
Freeway	Outlying CBD	14:00-18:00	3.37	2.89	6229	5346
Freeway	Mixed Urban	6:00-9:00	2.52	2.91	5209.85	6002.28
Freeway	Mixed Urban	9:00-14:00	4.45	5.67	9186.97	11700.50
Freeway	Mixed Urban	14:00-18:00	3.48	3.55	7172.01	7325.69

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