Transforming Manhattan Distance

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USACO Feb 2020: Triangles

Feb 2020 - Triangles

- Given a 300x300 grid with some points filled in
- Define "distance" to be $|x_1-x_2| + |y_1-y_2|$
- Count the number of equilateral triangles using points on the grid

SAMPLE INPUT:

```
3
*..
.*.
*..
```

SAMPLE OUTPUT:

```
1
```

Definition of Manhattan Distance

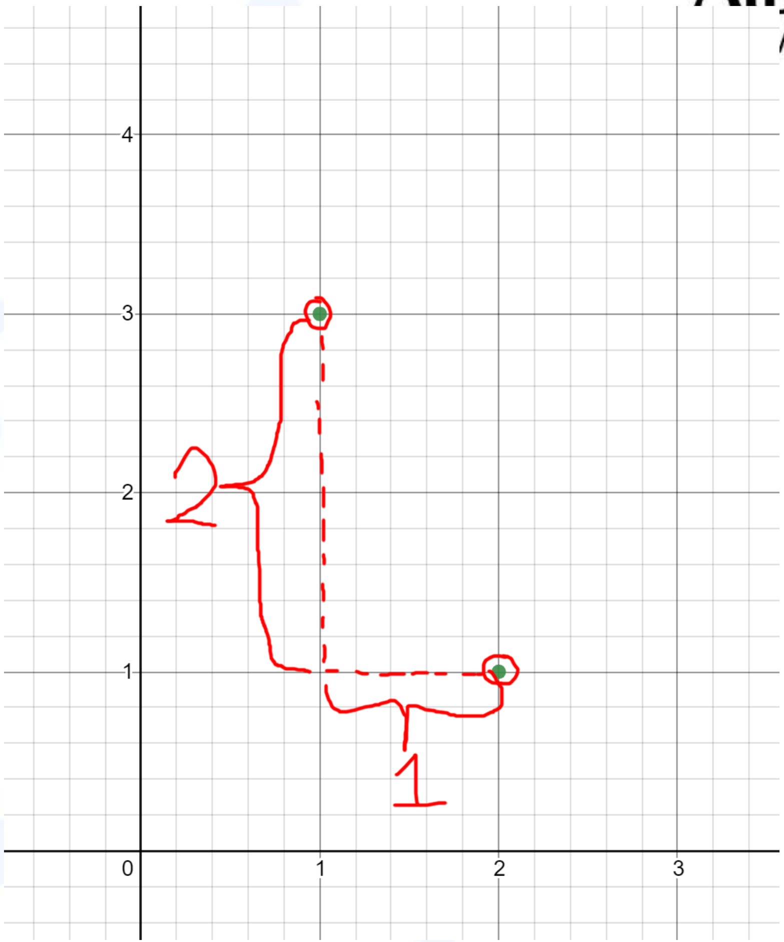
- The distance between (x_1, y_1) and (x_2, y_2) is $|x_1 - x_2| + |y_1 - y_2|$

Example:

Distance between (1, 3) and (2, 1)

= 2 + 1

= **3**



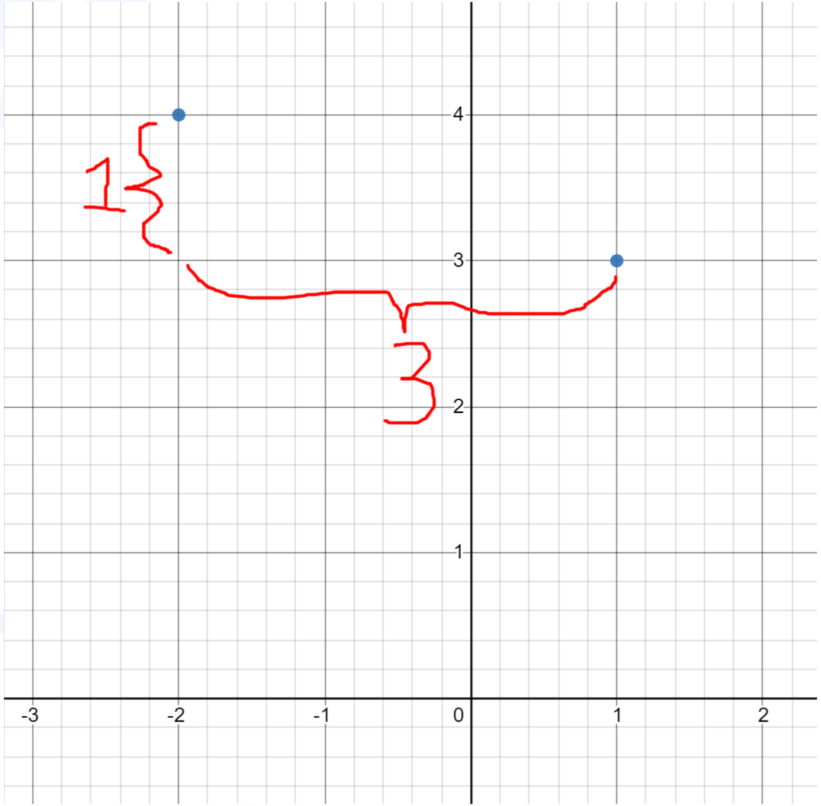
Cool Property: Transformation

- Manhattan distance is the sum of the differences in x and y coordinate
- After transforming $(x,y) \rightarrow (x-y, x+y)$, Manhattan distance is the maximum of the difference in x or y coordinate

Cool Property: Transform points (x,y) → (x-y,x+y)

(1, 3) → (-2, 4)
(2, 1) → (1, 3)

Manhattan distance
= max(|4 - 3|, |-2 - 1|)
= **3**



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- Get rid of the absolute values:
  - $|x_1 - x_2| + |y_1 - y_2|$ is the same as the maximum of these four cases:
    - $(x_1 - x_2) + (y_1 - y_2)$
    - $(x_1 - x_2) - (y_1 - y_2)$
    - $-(x_1 - x_2) + (y_1 - y_2)$
    - $-(x_1 - x_2) - (y_1 - y_2)$

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    - $|x_1 - x_2| + |y_1 - y_2|$ is the same as the maximum of these four cases:
        - $(x_1 - x_2) + (y_1 - y_2) = x_1 + y_1 - x_2 - y_2$
        - $(x_1 - x_2) - (y_1 - y_2) = x_1 - y_1 - x_2 + y_2$
        - $-(x_1 - x_2) + (y_1 - y_2) = -x_1 + y_1 + x_2 - y_2$
        - $-(x_1 - x_2) - (y_1 - y_2) = -x_1 - y_1 + x_2 + y_2$

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- Get rid of the absolute values:
  - $|x_1 - x_2| + |y_1 - y_2|$ is the same as the maximum of these four cases:
    - $(x_1 - x_2) + (y_1 - y_2) = x_1 + y_1 - x_2 - y_2$
    - $(x_1 - x_2) - (y_1 - y_2) = x_1 - y_1 - x_2 + y_2$
    - $-(x_1 - x_2) + (y_1 - y_2) = -x_1 + y_1 + x_2 - y_2$
    - $-(x_1 - x_2) - (y_1 - y_2) = -x_1 - y_1 + x_2 + y_2$
  - Let $\mathbf{P_1 = x_1 - y_1}$ and $\mathbf{Q_1 = x_1 + y_1}$
  - Let $\mathbf{P_2 = x_2 - y_2}$ and $\mathbf{Q_2 = x_2 + y_2}$

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- Get rid of the absolute values:
  - $|x_1 - x_2| + |y_1 - y_2|$ is the same as the maximum of these four cases
    - $(x_1 - x_2) + (y_1 - y_2)$ = $x_1 + y_1 - x_2 - y_2$ = **$P_1 - P_2$**
    - $(x_1 - x_2) - (y_1 - y_2)$ = $x_1 - y_1 - x_2 + y_2$ = **$Q_1 - Q_2$**
    - $-(x_1 - x_2) + (y_1 - y_2)$ = $-x_1 + y_1 + x_2 - y_2$ = **$-Q_1 + Q_2$**
    - $-(x_1 - x_2) - (y_1 - y_2)$ = $-x_1 - y_1 + x_2 + y_2$ = **$-P_1 + P_2$**
  - Let $P_1 = x_1 - y_1$ and $Q_1 = x_1 + y_1$
  - Let $P_2 = x_2 - y_2$ and $Q_2 = x_2 + y_2$

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- The distance between (x_1, y_1) and (x_2, y_2) is $|x_1 - x_2| + |y_1 - y_2|$
- Get rid of the absolute values:
  - $|x_1 - x_2| + |y_1 - y_2|$ is the same as the maximum of these four cases = **max(|P_1-P_2|, |Q_1-Q_2|)**
    - $(x_1 - x_2) + (y_1 - y_2)$ = $x_1 + y_1 - x_2 - y_2$ = **$P_1 - P_2$**
    - $(x_1 - x_2) - (y_1 - y_2)$ = $x_1 - y_1 - x_2 + y_2$ = **$Q_1 - Q_2$**
    - $-(x_1 - x_2) + (y_1 - y_2)$ = $-x_1 + y_1 + x_2 - y_2$ = **$-Q_1 + Q_2$**
    - $-(x_1 - x_2) - (y_1 - y_2)$ = $-x_1 - y_1 + x_2 + y_2$ = **$-P_1 + P_2$**
  - Let $P_1 = x_1 - y_1$ and $Q_1 = x_1 + y_1$
  - Let $P_2 = x_2 - y_2$ and $Q_2 = x_2 + y_2$

The Transformation

- Let $\mathbf{P_1 = x_1 - y_1}$ and $\mathbf{Q_1 = x_1 + y_1}$
- Let $\mathbf{P_2 = x_2 - y_2}$ and $\mathbf{Q_2 = x_2 + y_2}$
- Thus, if we transform points from $(x, y) \rightarrow (x\text{-}y, x\text{+}y)$, the Manhattan Distance is now equal to the largest difference in P-coordinate or Q-coordinate

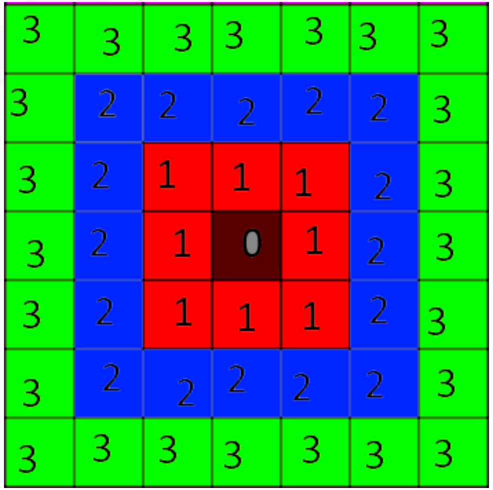
What does it look like?

- $(x, y) \rightarrow (x-y, x+y)$ can also be visualized on the complex plane:
  - (x, y) is x+yi
  - (x+yi)*(1+i) = (x-y)+(x+y)i
  - Multiplying by (1+i) is the same transformation!

- Multiplying by (1+i) is the same as rotating by 45 degrees counterclockwise and scaling by sqrt(2)

Back to USACO "Triangles"

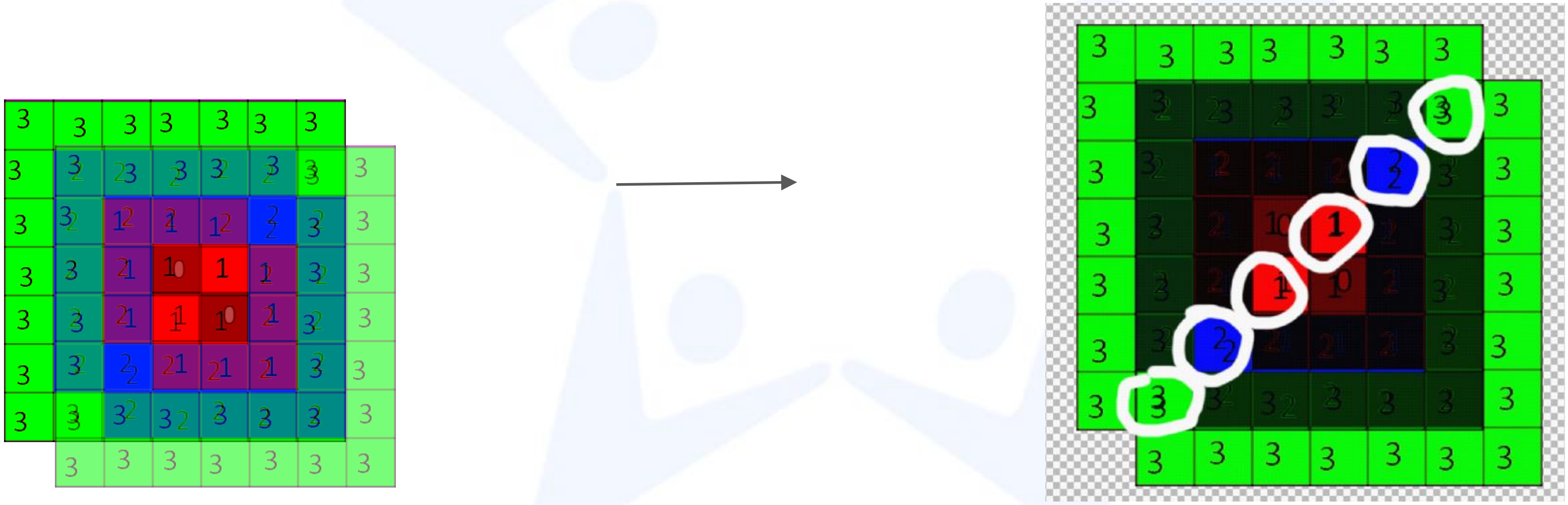
- Imagine we've already applied the transformation; then the distance from a fixed point to each other point looks like this:



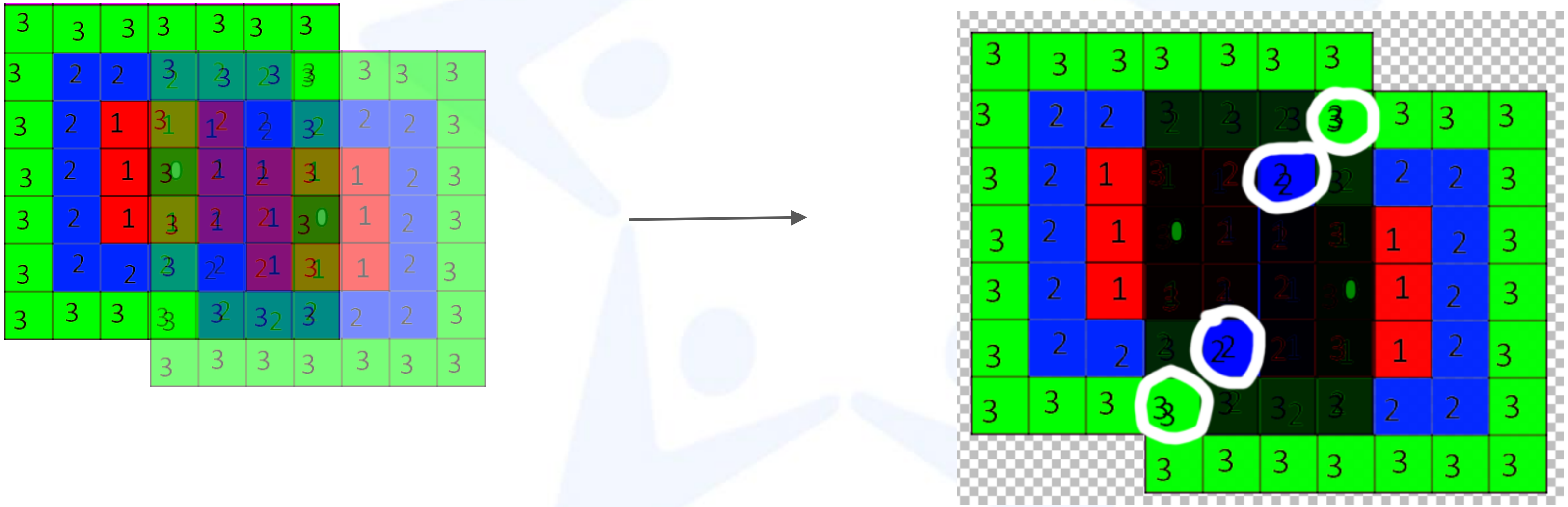
USACO "Triangles"

So, given two fixed points, which points are equidistant to both?

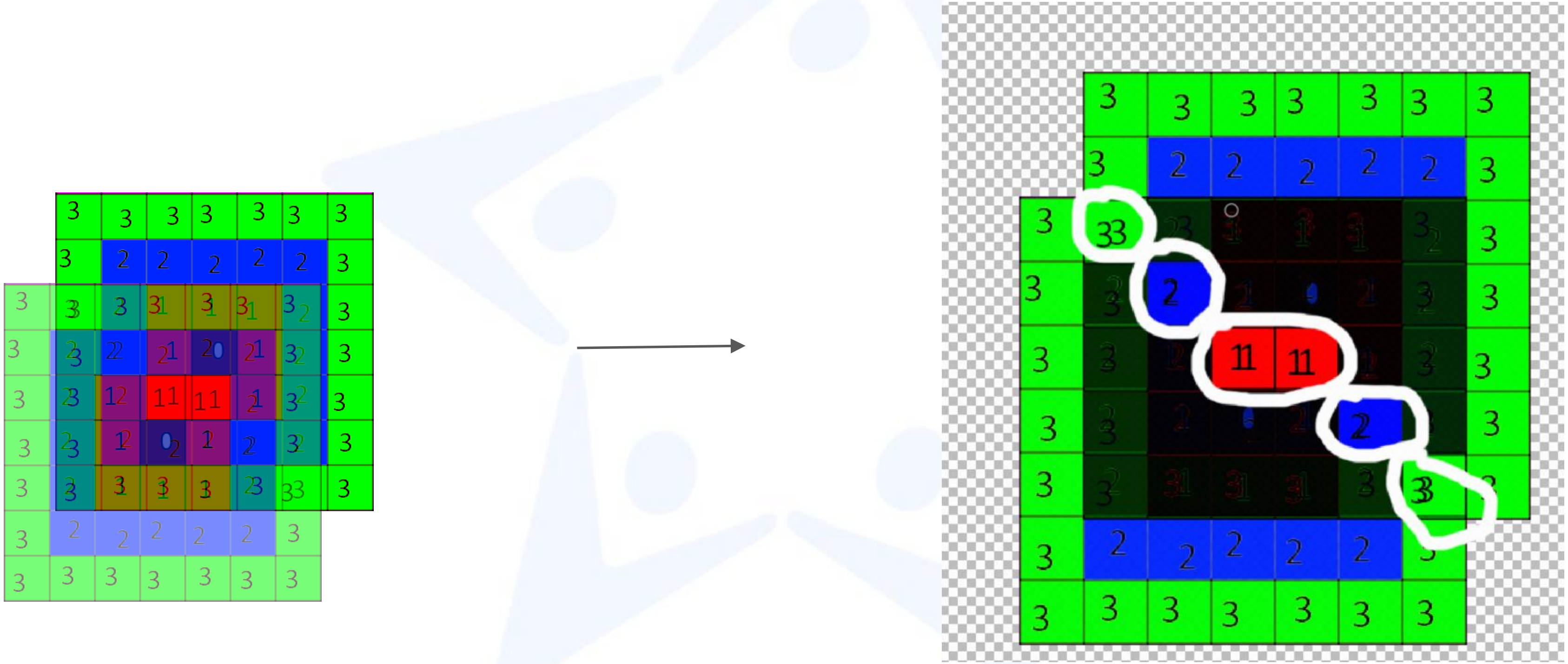
When two cells overlap, if they aren't the same color, then darken it:



Another case



Another case



The observation!

- Given two fixed points, the points that are equidistant to both fixed points lie on a line with slope = -1, 0, 1, or vertical line
- Thus, all equilateral triangles contain a vertical, horizontal, or a 45 degree line!