Homework Assignment 8

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1.
   a. BC: For all x when $n = 1$, $x - 1 \mid x^1 - 1$ because the expressions are equivalent.
      IH: Assume that the claim holds for $n = k$ where $k \geq 0$.
      IS: $x^{k+1} - 1$ is the starting expression. Taking out the x: $x \times (x^k - 1) + x - 1$. Using the IH we know $x^k - 1 = (x - 1) * l$ for some integer l. Substituting: $x(x - 1)l + (x - 1)$. Combining terms: $(x - 1)(xl + 1)$. Since $xl + 1$ is an integer, the claim holds when $n = k + 1$.
   b. BC: For all x when $n = 1$, $1 + x \geq 1 + 1 \times x$ because the expressions are equivalent.
      IH: Assume that the claim holds for $n = k$ where $k \geq 1$.
      IS: $(1 + x)^{k+1} \geq 1 + (k + 1)x$ is the starting expression. Moving terms: $(1 + x)(1 + x)^k \geq x + (1 + kx)$. Replacing with the IH (at least): $(1 + x)(1 + kx) \geq x + (1 + kx)$. Subtracting $1 + kx$: $x(1 + kx) \geq x$. Dividing both sides with x: $1 + kx \geq 1$. kx can't be negative because both k and x are non-negative integers, so the claim holds when $n = k + 1$.
2. Let $n - 3 = 4 \times k$ where k is an integer. Since $4 \times (k + 1)$ is a multiple of 4 ($k + 1$ is an integer due to addition rules) and $4 \times (k + 1) = 4 \times k + 4$, $4 \mid n + 1$. $2 \mid n + 1$ because $4 \mid n + 1$ and $2 \mid 4$. Using similar logic as about let $n + 1 = 2l$ for some integer l. Since $2 \times (l - 1)$ is a multiple of 2 ($l - 1$ is an integer due to subtraction rules) and $2 \times (l - 1) = 2 \times l - 2$, $2 \mid n - 1$. With algebra: $n^2 - 1 = (n + 1)(n - 1)$. Since $4 \mid n + 1$, $2 \mid n - 1$, and $n^2 - 1$ is formed by multiplying the 2, $8 \mid n^2 - 1$.
3.
   a. BC: When $n = 1$, $1 < 2\sqrt{1}$. This simplifies to $1 < 2$, so the base case it true.
      IH: Assume that the claim holds for $n = k$ where $k \geq 1$.
      IS: $\displaystyle\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} < 2\sqrt{k+1}$. Expanding: $\displaystyle\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$. Applying the IH (at most) to the expression: $2\sqrt{k} + \dfrac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$. Multiplying and subtracting: $1 < 2(k + 1) - 2\sqrt{k(k+1)}$. Moving and reducing: $0 < 2(k + 1/2) - 2\sqrt{k(k+1)}$. Converting to a square root: $0 < 2\sqrt{(k + 1/2)^2} - 2\sqrt{k(k+1)}$. Simplifying: $k(k + 1) < (k + 1/2)^2$. Expanding: $k^2 + k < k^2 + k + 1/4$. This expression is always true (LHS=RHS+1/4) so the claim holds when $n = k + 1$.

b. BC: For all a and b when $n = 1$, $a - b \mid a^1 - b^1$ because they are the equivalent.
   IH: Assume that the claim holds for $n = k$ and either $n = k - 1$ or $k = 1$.
   IS: With algebra: $(a^k - b^k)(a + b) = a^{k+1} + a^k b - ab^k - b^{k+1}$. That means:
   $a^{k+1} - b^{k+1} = (a^k - b^k)(a + b) - (a^k b - ab^k)$. Factoring:
   $a^k b - ab^k = ab(a^{k-1} - b^{k-1})$. With the IH or the fact that $x_1^0 = x_2^0 = 1$,
   $a - b \mid a^k b - ab^k$. With the IH $a - b \mid a^k - b^k$. Since the subtraction of two multiple of
   $a - b$ is a multiple of $a - b$, $a - b \mid a^{k+1} - b^{k+1}$ and the claim holds when $n = k + 1$.

4. It has not been proven that the set of the first and last k horses overlap.
5. BC: Let n be the number of lines on the paper. When $n = 0$ the whole paper can be colored either of the two colors. That claim holds.
   IH: The claim holds for $n = j$ where $0 \leq j < k$.
   IS: The goal is to prove that after adding the k^{th} line, the claim holds. Adding the k^{th} line divided the paper into 2 sections. With the IH, the claim holds for both sides. The claim holds on the entire paper after flipping all the colors of 1 side. The flipped colored sections are the same relative to each other. The non-flipped colors remain the same. Neighboring sections of the boundary line used to be the same section. When 1 of those sections flipped, they are opposite. Every neighboring region is different for each other. The claim holds when there are k lines.
6. BC: When $n = 1$ there is no other nodes that must reach it, so the claim holds.
   IH: Assume when $n = j$ the claim holds for some $1 \leq j < k$.
   IS: Let A be the hub node, B the nodes that directly connect to A, C the nodes that must visit another node (in B) to get to A, and D the node getting added. By contradiction, assume that $D \notin A \vee B \vee C$. Because of this D can't directly connect to a node in A or B. Also D must have a path to a node in C, otherwise the previous statement would be violated or D could be A. Using the IH, the subgraph C must have an A. C's A could also be an A. A must directly connect to it, otherwise the node would be a B. All nodes in B can go though A to it. All nodes in C already satisfy this property. With the IH, Subgraph of C and D must have an A. That way $D \notin A \vee B \vee C$ with C's A. That claim holds.
7. BC: When $n = 3$ the polygon is a triangle. The sum of a triangles interior angles is 180. That is $180(3 - 1)$, so the claim holds.
   IH: The claim holds for $n = k$ where $k \geq 3$.
   IS: The claim should hold for $n = k + 1$. Take any polygon with $k + 1$ sides. Break it into a polygon of k sides and a triangle, by removing 1 vertex. With the IH, the polygon of k sides has an interior angle sum of $180(k - 2)$. The triangle has 180 interior angle sum. Since the lines match to form a closed polygon (the original), it has interior angle sum $180(k - 2) + 180$ or $180(k - 1)$ or $180((k + 1) - 2)$. The claim holds for $n = k + 1$.

8.

a. When $n = 4$, $k = 4$. Below is a list of all the exchanges a person has. The bullet is the person number, and in text is the list of all exchanges. Kim is 9 and Bob is 10. Neighboring people are in the same company, for example 1 and 2 (not 2 and 3).

1. $3,4,5,6,7,8,9,10 -> 8$
2. $NA -> 0$
3. $1,5,6,7,8,9,10 -> 7$
4. $1 -> 1$
5. $1,3,7,8,9,10 -> 6$
6. $1,3 -> 2$
7. $1,3,5,9,10 -> 5$
8. $1,3,5 -> 3$
9. $1,3,5,7 -> 4$
10. $1,3,5,7 -> 4$

b. BC: When $n = 0$, Bob (B)and Kim(K) exchange 0. K is distinct among herself.

IH: The claim holds when $n = k$ where $k \geq 0$.

IS: After adding a new 2 people, let the first person of each company connect to both (including K and B). Those new people will become K and B, so the claim holds.

9. There might not be a way to represent $k + 1$ as $i + j$ for $i, j < k + 1$ and non-negative. For example if $k + 1 = 1$, there are no non-negative integers < 1 that add to 1.

10. BC: When $n = 1$, n can be represented as 2^0.

IH: The claim holds for $n = j$ where $1 \leq j < k$.

IS: The goal is to prove that the claim holds for $n = k$. If k is odd, then it has the same representation as $k - 1$ with a $2^0 = 1$. Even numbers wouldn't include a 2^0 because it would make them odd. Otherwise, $k/2$ can be represented as a sum of powers of 2. Just multiply that by 2 (add 1 to every included exponent). That way the claim holds for $n = k$.

11. BC: When $n = 8$ the expression becomes $3^8 > 8^4$ or $6561 > 4096$, so the claim holds.

IH: The claim holds for $n = k$ where $k \geq 8$.

IS: The claim is $3^{k+1} > (k+1)^4$. Using the IH on the left side(at least): $3k^4 > (k+1)^4$. Dividing by k^4: $3 > \dfrac{k+1}{k}^4$. Reducing: $3 > \left(1 + \dfrac{1}{k}\right)^4$. Since the claim can always be reduced to this, the IH is applicable. According to the IH: $3 > \left(1 + \dfrac{1}{k-1}\right)^4$. Since $\dfrac{1}{k} < \dfrac{1}{k-1}$, $3 > \left(1 + \dfrac{1}{k}\right)^4$ holds. The claim holds for $n = k + 1$.