# Homework Assignment 1

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1.
    a. $r \wedge \neg p$
    b. $p \wedge q \wedge r$
    c. $\neg p \implies \neg r$
    d. $p \wedge \neg q \wedge r$
    e. $(p \wedge q) \implies r$
    f. $r \iff (q \vee p)$
2.
    a. $\forall x \in P\, \exists y \in P, loves(x, y)$; $\exists x \in P\, \forall y \in P, \neg loves(x, y)$
    b. $\exists x \in P\, \forall y \in P, loves(x, y)$; $\forall x \in P\, \exists y \in P, \neg loves(x, y)$
    c. $\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}, 2 \mid x \implies x = 2 \times y$; $\exists x \in \mathbb{Z}\, \forall y \in \mathbb{Z}, 2 \mid x \wedge x \neq 2 \times y$
    d. $\exists P\, \forall q, P(q) = C(q)$; $\forall P\, \exists q, P(q) \neq C(q)$
    e. Let $P = \{p \in \mathbb{Z} \mid \forall x \in \mathbb{Z} \wedge (x = 1 \vee p = x \vee \neg(x \mid p)\}$ (the list of all primes)
    $\exists p \in P\, \forall x \in \mathbb{Z}, p \geq y \wedge p \leq y \times 2$; $\forall p \in P\, \exists x \in \mathbb{Z}, p < y \vee p > y \times 2$
3.
    a. This proposition forms a tautology

| $p$ | $q$ | $p \vee q$ | $\neg p \vee \neg q$ | $(p \vee q) \vee (\neg p \vee \neg q)$ |
|---|---|---|---|---|
| TRUE | TRUE | TRUE | FALSE | TRUE |
| TRUE | FALSE | TRUE | TRUE | TRUE |
| FALSE | TRUE | TRUE | TRUE | TRUE |
| FALSE | FALSE | FALSE | TRUE | TRUE |

    b. This proposition forms a tautology

| $p$ | $q$ | $p \wedge q$ | $p \implies q$ | $(p \wedge q) \implies (p \implies q)$ |
|---|---|---|---|---|
| TRUE | TRUE | TRUE | TRUE | TRUE |
| TRUE | FALSE | FALSE | FALSE | TRUE |
| FALSE | TRUE | FALSE | TRUE | TRUE |
| FALSE | FALSE | FALSE | TRUE | TRUE |

4.
    a. $2$ is a prime number, but $2 + 2 = 4$ is not prime. It can be represented as $4 = 2 \times 2$.

b. Let $m - n$ be even without loss of generality. It is known that $m + n = (m - n) + 2n$. Since $n$ is an integer, $2n$ is even. That way $m + n$ will be even if $m - n$ is even, or odd otherwise.

c. Let $x = 10 \wedge y = 0.2$, so $0.2 \leq 10$. Here $\lfloor x - y \rfloor = \lfloor 10 - 0.2 \rfloor = \lfloor 9.8 \rfloor = 9$. On the other hand $\lfloor x \rfloor - \lfloor y \rfloor = \lfloor 10 \rfloor - \lfloor 0.2 \rfloor = 10 - 0 = 10$. Since $9 \neq 10$, the proposition is false.

d. If $x$ is even, then $x^2 - x$ will be even because both $x^2$ and $x$ is even, and subtraction of $2$ even numbers results in an even number. Similarly If $x$ is odd, then $x^2 - x$ will be even because both $x^2$ and $x$ are odd, and subtraction of $2$ odd numbers results in an even number. Adding $x^2 - x$, which is an even number, to $3$, which is an odd number, will result in $x^2 - x + 3$ being an odd number due to addition rules.

e. If $m$ is even, $2$ is a factor of $m$ by definition. Since $m^7$ is $m$ multiplied by itself 7 times over, m is a factor of $m^7 . m \mid m^7 \wedge 2 \mid m \implies 2 \mid m^7$ because of a multiplication property. Obviously since $2 \mid m^7$, $m^7$ is even, by definition.

5. $d \mid a \implies a = d \times k$ for some integer $k$, so $ax = d \times k \times x$.
$d \mid b \implies b = d \times l$ for some integer $l$, so $by = d \times l \times y$.
$ax + by = (d \times k \times x) + (d \times l \times b)$ which simplifies to $d \times (k \times x + l \times b)$.
Since $ax + by$ is a multiple of $d$, $d \mid ax + by$.

6. Let $x - y = 2k + 1$ for some integer $k$, and $y - z = 2l$ for some integer $l$.
$x - z = (x - y) + (y - z) = (2k + 1) + 2l = 2(k + l) + 1$. Since both $k$ and $l$ are integers $k + l$ is an integer, and $x - z$ is odd.

7. The contrapositive of the proposition is if $r^{1/t}$ is rational then $r$ is rational. Since $(r^{1/t})^t$ is a rational number multiplied by itself $t$ times, so $(r^{1/t})^t$ is a rational number. $(r^{1/t})^t = r$ when simplified, so $r$ is rational.

8. Since $4 \mid n - 3$, $4 \mid n - 3 + 4$ or $4 \mid n + 1$.
$4 \mid n + 1 \wedge 2 \mid 4 \implies 2 \mid n + 1$, so $2 \mid n - 1$ because $n - 1 = (n + 1) - 2$.
$n^2 - 1 = (n + 1) \times (n - 1)$ . Since $4 \mid n + 1$ and $2 \mid n - 1$: $8 \mid n^2 + 1$.

9.

a. $n^3 - n = (n - 1)(n)(n + 1)$. $n^3 + n$ is a multiple of $3$ consecutive numbers. There are $3$ cases for the remainder of $n/3$:

$0$: $3 \mid n$

$1$: $n$ is $1$ greater than a multiple of $3$, so $3 \mid n - 1$.

$2$: $n$ is $1$ less than a multiple of $3$, so $3 \mid n + 1$.

In all $3$ cases one of the factors of $n^3 + n$, as described above, is a multiple of $3$. Therefore, regardless of the remainder of $n/3$, $3 \mid n^3 + n$ for all integers $n$.

b. $2x^2 - 4x + 3 > 0$ becomes $2(x^2 - 2x) + 3 > 0$ when factoring out the $2$. Next it can be factored into a square becoming: $2((x-1)^2 - 1) + 3 > 0$. Taking the $-1$ out, the inequality would become $2(x-1)^2 + 1 > 0$. Since a square can't become negative, the least is can be is $0$. In that case, $1$ (or greater) will always be $> 0$.

10. If $p$ and $q$ equals $1$, then the expression would become $1 + 1 \leq 1 + 1 \times 1$ (which is true). $1$ is the smallest number that both $p$ and $q$ can be since they are positive integers. Next let $p$ increase by one without loss of generality. The left side going up by $1$, and the right side going up by $q$. Since $q \geq 1$ by definition, the right side will continue to be $\geq$ to the left side. In order to reach any integer pair $(p, q)$, you can keep increasing $p$ or $q$ by $1$ to reach the desired numbers. That way $p + q \leq 1 + pq$ will be true for all integers.

11.

a. When $A = \{1,2\}, B = \{1,3\}, C = \{2,3\}$
$A \cap B \cap C = \varnothing$ because the $3$ sets share no common elements
$A \cup B \cup C = \{1,2,3\}$ and its cardinality is $3$
$|A| + |B| + |C| = 2 + 2 + 2 = 6$
The proposition is false, $3 \neq 6$

b. Let $x \in A \cap B$. Since $A \cap (B - C) = \varnothing$ and $x \in A, x \notin B - C$. $x \in B$, so the only way for $x \notin B - C$ is for $x \in C$. That way $(\forall x \in A \cap B) \in C \implies A \cap B \subseteq C$.

c. Without loss of generality let $x \in A \cap C$. $x \in A \implies x \in A \cup B$.
$x \in C \implies x \in C \cup D$. Since $x$ is in both the left and right sides of the intersection, $x \in (A \cup B) \cap (C \cup D)$. Every element in $(A \cap B) \cup (C \cap D)$ is in $(A \cup B) \cap (C \cup D)$, so $(A \cap B) \cup (C \cap D) \subseteq (A \cup B) \cap (C \cup D)$.

d. For every element in $A$ there should be $|B|$ elements in $A \times B$. We know that there is at least one element in $A$ because $A \neq \varnothing$. Since that is the case, if $A \times B = \varnothing$ then there has to be $0$ for every element in $A$. This means that $|B| = 0$ or $B = \varnothing$.

e. Let $A = \{1\}$, $B = \{2\}$, and $A \cup B = \{1,2\}$. $\mathscr{P}(A) = \{\varnothing, \{1\}\}$ and $\mathscr{P}(A) = \{\varnothing, \{2\}\}$. Furthermore, $\mathscr{P}(A) \cup \mathscr{P}(B) = \{\varnothing, \{1\}, \{2\}\}$ and $\mathscr{P}(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$. Since $\{1,2\} \in \mathscr{P}(A \cup B)$ but $\{1,2\} \notin \mathscr{P}(A) \cup \mathscr{P}(B)$, so the proposition is false.