Homework Assignment 12

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1. Yes, consider a graph G of all living people where siblings are connected. Consider a connected component of siblings within G. If a person in that component has an odd number of siblings there must be an even number of people in the component. That means the number of people with an odd number of siblings in that component is even (multiplying and odd and even number). Since this would apply to all components with people in them that have and odd degree, the answer to the question asked must be yes.
2. A tree is a connected acyclic graph. That means every component of the forest G is a tree. As proven in class a tree has $n - 1$ edges. Using these facts G has $n - c$ edges.
3. Let P be the longest path in graph G and u be the node where is ends. Since u has 2 other connections, they must connect back to P, otherwise P could be longer. Let v_1, v_2 be nodes that directly connects to u and are on P. If the distance from v_1 or v_2 to u through P is odd the cycle is even. Otherwise the distance from v_1 to v_2 through P is even and from v_1 to v_2 through u is 2 so the cycle is even.
4. If there are n edges there is one cycle: Consider a tree with $n - 1$ edges. Adding another edge to it from u to v will make exactly one cycle because there is already exactly one path from u to v.

   If there is one cycle that are n edges: By contrapositive if there are not n edges, the isn't exactly one cycle. If there are less a tree has no cycles. Otherwise by question 5 having more edges would result in 2 or more cycles.
5. Consider forest F to be a subgraph of G. G has $m - (n - 1)$ more edges than F and F is acyclic. Since every additional edge in G connects already connected nodes (that how F is chosen) adding another path to connect the nodes will add at least 1 more cycle. For each of the $m - (n - 1)$ edges there are at least $m - (n - 1)$ cycles in G.
6. BC: When $n = 1$ there is 1 node one G, so there aren't any cycles.

   IH: Assume that the claim holds for $n = k$ where $k \geq 1$.

   IS: Let u be the node getting added to G. Let H be the directed acyclic version of G with the same connections as before u was added. For every undirected edge with u, let the directed edge in H go toward u. Every cycle must have an entry and an exit from a node, since u has no exits u can't be part of a cycle in H. There are no cycles in H without u because H is acyclic for $n = k$. That way H is acyclic for $n = k + 1$ and the claim holds.

7. Let p be if edge e is in every spanning tree for a connected graph G, and q be true if the removal of e disconnects G.

   $p \implies q$: The contrapositive is if removing e leaves G connected, e is not in every spanning tree of G. Combining the ST of the components e would connect with another edge, would create another tree ($n - 1$ edges). Since it reaches every node in G, it is a ST of G. That way there would exist a ST that doesn't contain e.

   $q \implies p$: With contradiction if e was not included, the ST wouldn't span all nodes.