Homework Assignment 4

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1. The contrapositive of this statement is $n \mid a \vee n \mid b \implies n \mid ab$. Let $n \mid a$ without loss of generality. Then $a = n \times k$ for some integer k. So, $ab = n \times k \times b$. Since $k \times b$ is an integer, $n \mid ab$.
2.
    a. Let irrational number be i and the rational number be $\frac{r_1}{r_2}$ for some integers r_1, r_2. For the sake of contradiction let the sum of the 2 numbers be rational so $i + \frac{r_1}{r_2} = \frac{s_1}{s_2}$ for some integers s_1, s_2. Solving for i: $i = \frac{s_1}{s_2} - \frac{r_1}{r_2}$, becomes $i = \frac{s_1 r_2 - s_2 r_1}{s_2 r_2}$. Here, an irrational number is being represented in rational terms, so the sum must be irrational.

    b. Let's assume that $\sqrt{1}3$ is rational, so $\sqrt{1}3 = \frac{p}{q}$ for some integers p, q. Squaring both sides: $13 = \frac{p^2}{q^2}$, and removing the fraction: $13q^2 = p^2$. Since $13 \mid p^2$, 13 is prime, and all the prime factors from p^2 must come from p, $13 \mid p$. This means $p = 13k$ so $p^2 = 169k^2$. Substituting back: $13q^2 = 169k^2$, $q^2 = 13k^2$. For the same reasons as above $13 \mid q$, so p and q are both multiples of 13 and share a common factor. Since p and q have to be relatively prime, this is a contradiction to the definition of rational so $\sqrt{1}3$ is prime.
3. The incorrect assumption is that since $x - 3$ is not in the set, then it is a multiple of 3. $x - 3$ won't be in X because it only contains natural numbers. It is possible for other integers not to be divisible by 3.
4. There are 3 possible remainders when dividing a positive integer by 3: $0,1,2$. If given an integer with a negative remainder, add 3 to that remainder so we are only dealing with those 3 remainders. Let s be the sum of the remainders of the chosen numbers, x_1, x_2, x_3. If $3 \mid s$ then the 3 numbers are valid because: Let $x_i = 3k_i + r_i$ so the sum is $3(k_1 + k_2 + k_3) + r_1 + r_2 + r_3$. Since k_i is an integer, $r_1 + r_2 + r_3$ must be a multiple of 3 for the sum to be a multiple of 3.

    Case 1: 3 or more numbers have the same remainder: the sum of those numbers will be divisible by 3. This is because, $r_1 = r_2 = r_3$ so $s = r_1 \times 3$. This means $3 \mid s$.

    Case 2: There are $2,2,1$ integers for each remainder (in any order): take one x_i from each remainder so that $r_1 = 0, r_2 = 1, r_3 = 2$ (in any order). Here, $s = 3$ and $3 \mid 3$.

5. $(a-b)(a+b) = a^2 - ab + ab - b^2 = a^2 - b^2$. Let's put that into the equation. $(a-b)(a+b) + 2b^2 = c^2$. Proving $2\,|\,a \vee 2\,|\,b \vee 2\,|\,c$ will prove $2\,|\,abc$. There are 2 cases:

   Case 1: $2\,|\,a-b$: Here, $(a-b)(a+b)$ must be a multiple of 2 because one of its factors is. That way the left side the of equation is a multiple of 2 so $2\,|\,c^2$. Since the multiplication of 2 odd numbers is an odd number, $2\,|\,c$.

   Case 2: $2 \nmid a-b$: The subtraction of 2 odd numbers or 2 even numbers results in an even number. Since $a-b$ is odd, either a or b must be even. In other words $2\,|\,a \vee 2\,|\,b$
   Regardless of the case, $2\,|\,a \vee 2\,|\,b \vee 2\,|\,c$, so $2\,|\,abc$.

6. Let $X = \{x_1, x_2, \ldots x_n\}$. For the sake of contradiction let $\forall x \in X, x < \bar{x}$. Because of this we know $\sum_{i=1}^{n} x_i < n \times \bar{x}$. Dividing both sides by n: $(\sum_{i=1}^{n} x_i)/n < \bar{x}$. By definition the average of X is the sum of all elements divided by the number of elements, so we can replace the left side of the equation with $\bar{x}$. This gives $\bar{x} < \bar{x}$ which will always be false.