Homework Assignment 6

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1. Base Case: When $n = 1$ the equation is $1 \times 1! = 2! - 1$, which is true.

   Induction: When $n = k + 1$ and assume that it works for k when $k > 0$.

   LHS $= \sum_{i=1}^{k+1} (i \times i!)$

   $= \sum_{i=1}^{k} (i \times i!) + (k+1) \times (k+1)!$

   Due to the assumption: $= (k+1)! - 1 + (k+1) \times (k+1)!$

   $= (k+1)! \times (k+2) - 1$

   $= (k+2)! - 1$

   This proves that the claim holds for $n = k + 1$.

2. Base Case: When $n = 0$ the claim is $5 \mid 3^1 + 2^1$, which is true.

   Induction: When $n = k + 1$ and assume that it works for k when $k \geq 0$.

   $3^{3n+4} + 2^{n+2}$

   $3^3 \times 3^{3n+1} + 2 \times 2^{n+1}$

   $27 \times 3^{3n+1} + 2 \times 2^{n+1}$

   $25 \times 3^{3n+1} + 2 \times (2^{n+1} + 3^{3n+1})$

   With the induction: $5 \mid 2 \times (2^{n+1} + 3^{3n+1})$

   $5 \mid 25 \times 3^{3n+1}$ because $5 \mid 25$

   This proves that the claim holds for $n = k + 1$.

3. Base Case: When $n = 0$ the claim is $0 \times 1 \times 2 = 0 \times 1 \times 2 \times 3 \div 4$, which is true because both sides of the equation evaluate to 0.

   Induction: When $n = k + 1$ and assume that it works for k when $k \geq 0$. LHS:

   $\sum_{i=0}^{k+1} i(i+1)(i+2)$

   $\sum_{i=0}^{k} i(i+1)(i+2) + (k+1)(k+2)(k+3)$

   With the induction: $k(k+1)(k+2)(k+3) \div 4 + (k+1)(k+2)(k+3)$

   $k(k+1)(k+2)(k+3) \div 4 + 4(k+1)(k+2)(k+3) \div 4$

   $(k+4)(k+1)(k+2)(k+3) \div 4 = (k+1)(k+2)(k+3)(k+4) \div 4$

   This proves that the claim holds for $n = k + 1$.

4. Base Case: When $n = 1$ the claim is $\frac{1}{1 \times 2} = 1 - \frac{1}{1+1}$, and $1/2 = 1/2$.

   Induction: When $n = k + 1$ and assume that it works for k when $k > 0$. LHS:

   $\sum_{i=1}^{k+1} \frac{1}{i \times (i+1)}$

   $\sum_{i=1}^{k} \frac{1}{i \times (i+1)} + \frac{1}{(k+1)(k+2)}$

   With the induction: $1 - \frac{1}{k+2} + \frac{1}{(k+1)(k+2)}$

   $1 - \frac{k+1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$

   $1 - \frac{k+2}{(k+1)(k+2)}$

   $1 - \frac{1}{k+1}$

   This proves that the claim holds for $n = k + 1$.
5. Since m is an integer, $n^2 + 1$ must be even. Furthermore, n^2 is odd so n is odd. Let $n = 2k + 1$ for some integer k. Expanding: $4k^2 + 4k + 2 = 2m$. Dividing by 2: $m = 2k^2 + 2k + 1 = k + (k+1)^2$. m *can* be represented as the sum of 2 squares.
6. For the sake of contradiction, let there be an ordering such that there is no person with There can't be more than 2 girls sitting together, otherwise one of the girls will be neighboring 2 girls. At least 2 boys must separate clusters of girls, because if only 1 boy was he would then be surrounded by girls. The minimum amount of clusters is $\lceil 25/2 \rceil$ which is 13. There then have to be $\geq 13 \times 2 = 26$ boys dividing all the clusters of girls. Since $26 > 25$, there is a contradiction with the number of boys. No such ordering exists.
7. Base Case: When $n = 2$ the claim is $1 + \frac{1}{4} < 2 - \frac{1}{2}$, and which is clearly true.

   Induction: When $n = k + 1$ and assume that it works for k when $k \geq 2$.

   $\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k+1}$

   $\frac{1}{(k+1)^2} < (2 - \frac{1}{k+1}) - \sum_{i=1}^{k} \frac{1}{i^2}$

   With the induction (RHS - at most): $\frac{1}{(k+1)^2} < (2 - \frac{1}{k+1}) - (2 - \frac{1}{k})$

$$\frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1}$$

$$\frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$$

$$(k+1)^2 > k(k+1)$$

$$k+1 > k$$

This proves that the claim holds for $n = k + 1$.

8. Base Case: When $n = 1$ the claim is $1 + 1/2^1 \geq 1 + 1/2$,which is clearly true.

   Induction: When $n = k + 1$ and assume that it works for k when $k > 0$. LHS:

$$\sum_{i=1}^{2^{k+1}} 1/i \geq 1 + \frac{k+1}{2}$$

$$\sum_{i=1}^{2^k} 1/i + \sum_{i=2^k+1}^{2^{k+1}} 1/i \geq 1 + \frac{k+1}{2}$$

With the induction (LHS - at least): $(1 + \frac{k}{2}) + \sum_{i=2^k+1}^{2^{k+1}} 1/i \geq 1 + \frac{k+1}{2}$

$$\sum_{i=2^k+1}^{2^{k+1}} 1/i \geq \frac{1}{2}$$

$$\frac{1}{2^k+1} + \ldots + \frac{1}{2^{k+1}} \geq \frac{1}{2}$$

Since $2^{k+1} = 2 \times 2^k = 2^k + 2^k$, $2^{k+1} - 2^k = 2^k$. That way there are 2^k fractions being added. Every fraction is $\geq \frac{1}{2^{k+1}}$ so LSH $\geq \frac{1}{2^{k+1}} \times 2^k$

$\frac{2^k}{2^{k+1}} \geq \frac{1}{2}$ because $\frac{2^k}{2^{k+1}} = \frac{1}{2}$.

This proves that the claim holds for $n = k + 1$.