Homework Assignment 10

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1. $\Delta(G) \leq n - 1$. This is true because every vertex can connect to at most any other vertex ($n - 1$). In class we proved $2m = \sum_{v \in V} deg(v)$. Using both expressions $\Delta(G) \times n \leq \sum_{v \in V} deg(v)$. Substituting: $2m \leq n^2 - n$.
2. There are n different amount of friends in the group. There can't be a person with 0 friends and $n - 1$ friends in the same group. Using the PHP, the claim is proven.
3. Since each node has $deg() = r$, and $2m = \sum_{v \in V} deg(v)$. Replacing: $2m = r \times n$.
4. If graph G is connected, the claim holds for obvious reasons. Otherwise, let G has multiple connected components. Let u be a vertex in G. u will be directly connected to all vertices in $\bar{G}$, other then the connected component. Let v be a vertex in u's connected component in G. v is connected to the same vertices as u, for the same reasons. So u and v must be in the same connected component. u is connected to all nodes in $\bar{G}$, so $\bar{G}$ is connected.
5. For every vertex u there are $n - 1$ other possible (direct) connections. At least $n/2$ of those connections must be made. Then there is at least $n/2 + 1$ in u's connected component. Let v be another vertex u is not yet connected to u. There must be $n/2 + 1$ in its connected component but since there are only n nodes, 1 of the connections will overlap. That way all n nodes are G are connected.
6. Let u be any odd degree vertex G. Let H be a subgraph of G, such that H is the connected component of u. Since H is a connected component, every vertex in H maintains all its connections. In class we have proven for any graph the number of odd degree vertices is even. That way there is at least 1 other vertex in H that has an odd degree. That way, by definition there is a path from u to another odd degree vertex.
7. For the sake of contradiction assume P and Q don't share any vertices. Let R be the shortest path from a vertex on P to a vertex on Q. R can't contain any other vertices from P or Q, otherwise it won't be the shortest path. Since P and Q are disjoint, the length of R is ≥ 1. Let R_1 be R's endpoint on P, and R_2 be R's endpoint on Q. The longer of the paths from R_1 to P's endpoints is $k/2$. Similarly for Q, the longer path is $k/2$. That means one *path* from P's endpoint to Q's endpoint is $k/2 + len(R) + k/2$ or at least $k/2 + 1 + k/2$. Since $k + 1 > k$ there exists a path longer than k in the graph, which is a contradiction to the question.