Homework Assignment 7

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1.
    a. $\binom{17}{12}$
    b. $\binom{11}{6}$
    c. $1x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + 1x^0y^6$
    d. $\binom{18}{5} \times 3^5 \times (-2)^{13}$
    e. True
2. There are 2 kinds of r sized multisets that can be chosen from M:

    Case 1 - Including a_1: In this case there are $r - 1$ items which can be from $k - 1$ categories. This is a sticks and crosses problem: $\binom{k+r-3}{r-1}$.

    Case 2 - Excluding a_1: In this case there are r items which can be from $k - 1$ categories. This is a sticks and crosses problem: $\binom{k+r-2}{r}$.

    Adding the 2 different cases together gives $\binom{k+r-3}{r-1} + \binom{k+r-2}{r}$.
3. Choosing a k sized committee from n people is $\binom{k}{n}$. The choosing the chair from that group is $\binom{k}{1}$ or k. That is $k\binom{k}{n}$. Another way to form a committee is as follows. Choose the committee chair in $\binom{n}{1}$ or n. Next pick the other $k - 1$ people in the committee from $n - 1$ people in $\binom{k-1}{n-1}$ ways. That is $n\binom{k-1}{n-1}$. Because both expressions solve the same problems, $k\binom{k}{n} = n\binom{k-1}{n-1}$ for integers n and k.
4. Both sides of the equation are trying to solve the same problem in the about question for all committee sizes. Add the number of ways to form a committee of size k for all k from 1 to n: $\sum_{k=1}^{n} k\binom{n}{k}$. Another way to solve the problem is pick the committee chair in $\binom{n}{1}$ or n.

Each of the remaining people can either be in or out of the committee: 2^{n-1}. That makes $n2^{n-1}$. Since $\sum_{k=1}^{n} k\binom{n}{k}$ and $n2^{n-1}$ both solve the same problem, they are equivalent.

5. Again, the problem of this question is similar to the above question. The difference is there are 2 committee chairs. Add all the combinations when a committee of size k for all k from 1 to n: $\sum_{k=2}^{n} k(k-1)\binom{n}{k}$. Another way to solve the problem is to pick the 2 committee chairs in $\binom{n}{2}$ or $n(n-1)$. After that each of the other $n-2$ people can be in or out making: $n(n-1)2^{n-2}$. Since $\sum_{k=2}^{n} k(k-1)\binom{n}{k}$ and $n(n-1)2^{n-2}$ both solve the same problem, they are equivalent.
6. How many ways can you choose 2 groups of size m and k from n people, such that the groups are disjoint? One way to sole this problem is to choose m people from n, then choose the k people from $n-m$: $\binom{n}{m}\times\binom{n-m}{k}$. Another way is the choose the group of size k first: $\binom{n}{k}\times\binom{n-k}{m}$. Since both $\binom{n}{m}\times\binom{n-m}{k}$ and $\binom{n}{k}\times\binom{n-k}{m}$ solve the same problem, they are equivalent expressions.
7. $2^{2n}=\sum_{i=0}^{2n}\binom{2n}{i}$ was proven in class with the binomial theorem. $0<n<2n$ because n is a positive integer. That way, $\binom{2n}{n}$ must be in the summation. Because $0<n<2n$, there has to be other additions in the summation. That means $\binom{2n}{n}<2^{2n}$. Due to power arithmetic $2^{2n}=(2^2)^n=4^n$, so $\binom{2n}{n}<4^n$ and the claim is true.
8. There are n “bins” for the numbers to differ by >1, because $2n/2=n$. With the PHP there is no way to put $n+1$ items in n bins without putting multiple in one bin. That means that at least 1 pair of the $n+1$ numbers must differ by 1.

9. Divide the $2x2$ square into 4 $1x1$ squares. A square's diagonal is the longest line within it, and a $1x1$ square's is $\sqrt{2}$. That means any 2 points in the same $1x1$ square will be $\geq \sqrt{2}$ apart. With the PHP, at least 1 pair of points must be in the same $1x1$ square.
10. When dividing a number by n there are n different remainders: $0,1,...,n-1$. Since $n+1$ integers are picked and with the PHP, there is at least 1 remainder that is repeated. Let r be 1 of the remainders that are repeated from the $n+1$ integers. That way $x_i = n \times k + r$ and $x_j = n \times l + r$ from some integers k and l. $x_i - x_j = (n \times k + r) - (n \times l + r)$. $x_i - x_j = n \times (k - l)$. Because of integer subtraction $k - l$ is an integer. Since $x_i - x_j$ can be represented as a multiple of n, $n \,|\, x_i - x_j$.