Exam Two

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1.
   a. False: A node in a graph of 6 vertices can have a maximum degree of $6 - 1 = 5$.
   b. False: A tree must be connected.
   c. True: The connect component of u must have sum of degrees to be even (each edge adds 2 to to sum of degrees). Since u and v have odd degrees, they are connected.
2. Expanding $Pr[A \mid B]$ is $\dfrac{Pr[A \cap B]}{Pr[B]}$ or $\dfrac{Pr[A \cap B]}{3/10}$, so $Pr[A \cap B] = \dfrac{2}{3} \times \dfrac{3}{10}$ or $\dfrac{1}{5}$.

   Expanding $Pr[B \mid A]$ is $\dfrac{Pr[A \cap B]}{Pr[A]}$. Substituting: $\dfrac{1/5}{2/5}$ or $\dfrac{1}{2}$. $Pr[B \mid A] = \dfrac{1}{2}$.
3. Var$[3X] = \mathbf{E}[9X^2] - \mathbf{E}[3X]^2$. Since Var$[X] = \mathbf{E}[X^2] - \mathbf{E}[X]^2$ or $5 = \mathbf{E}[X^2] - 16$ so $E[X^2] = 21$. Substituting: Var$[3X] = (9 \times 21) - (3 \times 4)^2$ or Var$[3X] = 45$.
4. Let P_5 be the event that the fifth position is selected, and H the event that a heads shows. The question is asking $Pr[P_5 \mid H]$ or $\dfrac{Pr[P_5 \cap H]}{Pr[H]}$. $Pr[H] = \dfrac{1}{10} \times (\dfrac{1}{10} + \dfrac{2}{10} \ldots + \dfrac{10}{10})$ or $\dfrac{1}{100} \times (1 + 2 + \ldots + 10)$ or $\dfrac{1}{100} \times \dfrac{10 \times 11}{2}$ or $\dfrac{11}{20}$. Substituting into $Pr[P_5 \mid H]$:
   $\dfrac{\frac{1}{10} \times \frac{1}{2}}{\frac{11}{20}}$ or $\dfrac{1}{11}$. $Pr[P_5 \mid H] = \dfrac{1}{11}$.
5. The events are independent if $Pr[B \mid A] = Pr[B]$. Expanding $Pr[B \mid A]$: $\dfrac{Pr[A \cap B]}{Pr[A]}$.

   There is no way to both numbers to be odd and add to 7. The sum of 2 odds is an even. That way $Pr[A \cap B] = 0$. Since $Pr[A] \neq 0$ and $Pr[B] \neq 0$, they aren't independent.
6. Given any pair of point that sum to degree at least $n - 1$, there are 3 cases.
   The points are directly connected: The points are clearly connected.
   The points connect to the same vertex: Points are still connected.
   Other: This case can't happen. There are $n - 2$ other points on the graph and together the pair of points have a $n - 1$ degree.

   Since every pair of point must be connected, by definition G is connected.

7. Let H_2 be the event that heads is flipped twice, and F the event that the fair coin is pick. The question is asking $Pr[F \mid H_2]$ or $\frac{Pr[F \cap H_2]}{Pr[H_2]}$. $Pr[H_2]$ is $Pr[H_2 \cap F] + Pr[H_2 \cap \overline{F}]$ or $\frac{1}{2} \times \frac{1}{2}^2 + \frac{1}{2} \times \frac{1}{4}^2$ or $\frac{5}{32}$. Substituting $Pr[F \mid H_2] = \frac{\frac{1}{8}}{\frac{5}{32}} = \frac{4}{5}$.
8. If the path contains k nodes so far there are at least $\delta(G) - k$ possible next nodes. That way there must be a path for every vertex that has length $\delta(G)$. Given a path of length at least $\delta(G)$ (as proven earlier), there are at most $\delta(G) - 1$ vertices for the next node that will result in a cycle (with a little 'tail') of less or equal to $\delta(G) - 1 + 1 = \delta(G)$. That means there will always be ≥ 1 vertex that will continue the search for a cycle. Eventually the nodes will run out an a cycle with $\geq \delta(G) + 1$ length will be revealed.
9. Let $\mathbf{E}[X]$ be the expected number of friend groups of size m. Let $\mathbf{E}[X_i]$ be the expected value of if a group of m people is a friend group such that $\mathbf{E}[X] = \binom{n}{m}\mathbf{E}[X_i]$. $E[X_i] = p^{m-1+m-2+...+1+0}$ or $p^{m(m-1)\div 2}$. That way $\mathbf{E}[X] = \binom{n}{m}p^{m(m-1)\div 2}$.
10. Any vertex of degree x increase the number of leaves by $x - 1$ to a tree. Adding the vertex takes away 1 leaf node and add x more. A tree starts out with 1 leaf (the root). The means the number of leafs is $4 \times 2 + 6 \times 3 + 1 \times 4 + 8 \times 5$ or $8 + 18 + 4 + 40$ or 70 leafs.
11. Let numbers $1 - 4$ represent the 4 colors. Let $\mathbf{E}[X_i]$ be if color i is picked such that $\mathbf{E}[X]$ or number of different colored balls is $\sum_{i=1}^{4} \mathbf{E}[X_i]$ or $4 \times \mathbf{E}[X_i]$ since all color have the same probability of being picked. $\mathbf{E}[X_i]$ is $1 - (3/4)^4$ or $1 - 81/256$ or $175/256$. That means that $\mathbf{E}[X] = 4 \times 175/256$ or $175/64$ or about 2.73.
12. Let N_2 be the event that the product is not divisible by 2, and N_5 the event that it is not divisible by 5. The question is asking $Pr[\overline{N_2} \cap \overline{N_5}]$. By solving $Pr[N_2 \cup N_5]$, the question can be easily solved ($1 - Pr[N_2 \cup N_5]$). With the union rule, $Pr[N_2 \cup N_5] = Pr[N_2] + Pr[N_5] - Pr[N_2 \cap N_5]$. Substituting clear probabilities: $\frac{8}{9}^n + \frac{5}{9}^n - \frac{4}{9}^n$. That way the final answer is $1 - \frac{8}{9}^n - \frac{5}{9}^n + \frac{4}{9}^n$.

13. BC: If $n = 2$ then there are 2 nodes with 1 connection. Both are leave so the claim holds.
    IH: Assume the claim hold when $n = k$ for some $k \geq 2$.
    IS: Adding the new edge and vertex to a non-leaf node will $+1$ to the number of leave. Otherwise the vertex is added to a leaf resulting in $-1 + 1 = 0$ change. Using the IH, the number of leaves is ≥ 2 and adding another vertex doesn't decrease it, so the claim holds.

Bonus 1: Let S_5 be the event that a 5 is rolled, F_5 be the event that 5 is rolled before 7, and C the event that we must continue rolling (doesn't get a 5 or 7). The question is asking $Pr[F_5]$. Expanding the scenarios of a roll: $Pr[F_5] = Pr[S_5 \cap F_5] + Pr[C \cap F_5]$ or $4/36 + 26/36 \times Pr[F_5]$. Moving expressions: $10/36 \times Pr[F_5] = 4/36$ or $10 \times Pr[F_5] = 4$ or $Pr[F_5] = 2/5$.

Bonus 2: There are 2 cases for how the mathematicians share languages.

All pairs of people share a language: Given any person, they must share a language with $9 - 1 = 8$ other people. Since he/she can only speak ≤ 3 languages and by PHP, there are at least 4 people that speak a common language.

Otherwise: Take a pair of people that don't share any languages. All the other $9 - 2 = 7$ people must share at least 1 language with 1 person in the pair. That means 1 person in the pair must share at least 1 language with $\lceil 7/2 \rceil = 4$ other people. Since that person can speak a max of 3 languages and by PHP, 2 of the 4 people speak the same language as the person in the pair speaks. That is $2 + 1 = 3$ people that share a language.