Exam One

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1.
   a. True
   b. True
   c. False
   d. True
   e. True
   f. True
   g. False
2.
   a. Let $n^3 - 5 = 2k + 1$ for some integer k. Adding 5: $n^3 = 2(k+3)$. Since $k+3$ is an integer, n^3 is even. Since odd numbers multiply into odd numbers, n has to be even.
   b. When $n = 3$ the claim fails. $3^2 - 1 = 9 - 1 = 8$. Since $2 \mid 8$, 8 is not prime.
   c. Let $x = 2$ and $y = 1/2$. Since $(2^{1/2})^2 = 2^{1/2 \times 2} = 2$ and $\sqrt{2}^2 = 2$, $2^{1/2} = \sqrt{2}$. In class we have proven that $\sqrt{2}$ is irrational.
3.
   a. $\dfrac{8!}{2!2!}$
   b. $\dbinom{n}{k}$
   c. 3^{100}
   d. $\dbinom{10}{3} \times 2^7$
   e. $\dbinom{28}{24}$
4. For contradiction assume that $13\sqrt{2}$ is rational or $13\sqrt{2} = p/q$ for some integers p and q. Since 13 is an integer we can divide by 13: $\sqrt{2} = p/13q$. Since both 13 and q are integers, $13q$ is an integer. This would mean $\sqrt{2}$ is rational, but its we have previously proven otherwise. This is a contradiction, so $13\sqrt{2}$ must be irrational.
5. Everyone gets 1: $8 \times 7 \times 6$. Pick the candy: $\dbinom{5}{3} + \dbinom{5}{4}$. Order the children: $3!$. In general: $8 \times 7 \times 6 \times (\dbinom{5}{3} + \dbinom{5}{4}) \times 3!$

6.

a. $\binom{22}{20}$

b. $\sum_{i=0}^{4} \binom{21-i}{1} = 21 + 20 + \ldots + 17 = 95$

c. Every child must have at least 5 cookies. $20 - 4 = 16$, 4 can't be given to a child and it be fair. $\binom{7}{5}$ when each child must of have 5. If all cookies are given to 1 child they would have 10. So the answer is $\binom{7}{5} - 3$.

7. BC: When $n = 1$, $1 = 0 + 1$. ($f_2 = f_0 + f_1 = 0 + 1$).
IH: The claim is true when $n = k$.
IS: Prove that the claim holds for $n = k + 1$. LHS: $f_1 + f_2 + \ldots f_{2(k+1)-1}$. Expanding: $(f_1 + f_2 + \ldots f_{2k-1}) + f_{2k+1}$. Using the IH: $f_{2k} + f_{2k+1}$. Using the property of the (Fibonacci) numbers: $f_{2k} + f_{2k+1} = f_{2(k+1)}$. The claim holds for $n = k + 1$.

8. If one of the 3 integers is 0, multiplying it with anything will get 0 (non-negative). Otherwise the numbers are either negative or positive. Due to the PHP, there are at least 2 numbers with the same sign. Multiplying those 2 numbers will get a non-negative number.

9. Let the 2 numbers be a/b and c/d. Making the denominators the same: ad/bd and bc/bd. Since the numbers are different, $|ad - bc| \geq 1$. That means $|2ad - 2bc| \geq 2$. Matching the denominators gives: $2ad/2bd$ and $2bc/2bd$. Since the nominators defer by at least 2, there is another nominator between the numbers.

10. Consider the problem of how many way are there to: pick a committee from n people, from which that committee there should be a chair and an optional substitute chair. To solve this problem let k be an instance of a committee size. Choosing a committee is $\binom{n}{k}$. Picking a committee chair can be done in k ways. The substitute chair can be picked in $k - 1$ ways, but $+1$ if there is no substitute chair. That way one answer to this question is: $\sum_{k=0}^{n} k^2 \binom{n}{k}$.

Another way to answer the question is as follows. Pick the chair and substitute in $n(n-1)$ ways then the rest of the $n - 2$ people can either be in or out of the committee: $n(n-1)2^{n-2}$. Also, if there is no substitute then $n2^{n-1}$. Combining the terms:

$n(n-1)2^{n-2} + n2^{n-1}$. Because both $\sum_{k=0}^{n} k^2 \binom{n}{k}$ and $n(n-1)2^{n-2} + n2^{n-1}$ are solutions to the same problem, the expression are equivalent.

11. The number of combinations of at least 3 increasing students is: $3! \times \binom{5}{3} - 2 \times \binom{5}{4}$. Simplifying $6 \times 10 - 2 \times 5 = 60 - 10 = 50$. The number of permutation is $5!$ or 120. The answer is $120 - 50$ or 70 combinations.
12. $\{2,3,6\}$
13. The inductive Hypothesis doesn't say $p' - 1 = q' - 1$, because one or more of the two expression could be 0 (not positive)
14. There are 15 combinations of (a, b) that will be in T's power-set. There are 3 shared elements so it is $3 \times (6 - 1)$.
15. BC: When $n = 2$: The last person (a man) is behind the first person (woman), claim holds.
    IH: For all $n = k$, the claim holds.
    IS: Trying to prove the claim holds when $n = k + 1$. Do this is the same as adding a person as the second to last person in the new line. There are 2 cases:
    Case 1: Adding a woman: This will mean there will be the recently added women right in front of the last man.
    Case 2: Adding a man: This will be the same as adding a man into the last place. Using the IH, the k sized line works. After that, adding a man to the end of the line doesn't change the reason a k sized line works.

Bonus: $2017!$