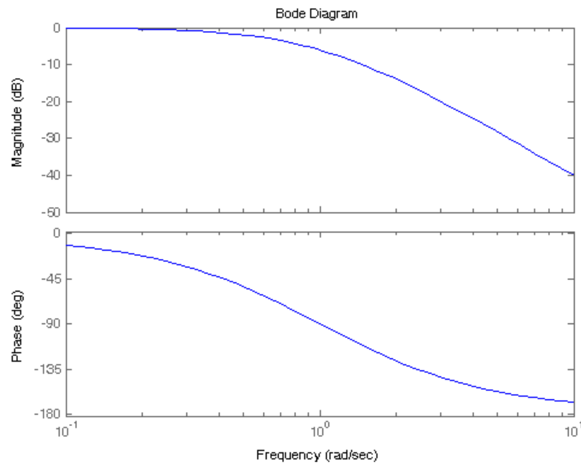
EEE 481 Test Name: ____SOLUTIONS____

Problem 1:
Consider the following system with transfer function

$$P(s) = \frac{1}{(s+1)^2}$$

whose Bode plot is shown in the figure.



1. Design a PID so that the closed loop crossover is at 7rad/s and the phase margin is 50^{o}.
2. Select a method and the sampling rate and discretize the PID.

For a crossover of 7rad/s, choose the pseudo-differentiator time constant at T=1/70, i.e., one order of magnitude higher bandwidth. The compensator transfer function has the form

$$C(s) = K\frac{(\tau_z s+1)^2}{s(Ts+1)}$$

The numerator time constant is computed to satisfy the angle condition at crossover:

$$\angle P(j\omega_c) + \angle C(j\omega_c) = -180 + 50 \Rightarrow -163.7 + (-90 + \angle(j\tau_z\omega_c + 1)^2 - \angle(j0.1+1)) = -130$$
$$\Rightarrow \tan^{-1}(7\tau_z) = 129.5/2 \Rightarrow \tau_z = 0.3$$

Using this value in C(s), the overall gain K is computed to satisfy the gain condition at crossover.

$$|P(j\omega_c)||\angle C(j\omega_c)| = 1 \Rightarrow (0.02)K(0.77) = 1 \Rightarrow K = 65$$

The final compensator is

$$C(s) = 65\frac{(0.3s+1)^2}{s(s/70+1)}$$

To discretize this compensator we select the sampling rate according to the “6-samples/rise-time” rule: The rise time is approximately 2/BW and BW is 1.5*crossover (or approximately equal to the crossover). So the sampling time is selected as 2/(1.5*7)/6 = 1/31.5. For this sampling time we cannot use the forward Euler discretization which requires a sampling time smaller than (2pole-TC) for all poles. In this case, the F-Euler sample time limit is 2/70 or 1/35. Thus, using the backward Euler,

$$C(z) = 65\left.\frac{(0.3s+1)^2}{s(s/70+1)}\right|_{s=(1-z^{-1})/Ts} = 65\frac{(0.3\frac{z-1}{z}31.5+1)^2}{\frac{z-1}{z}31.5(\frac{z-1}{z}31.5/70+1)} = \frac{155.4z^2 - 281.1z + 127.1}{z^2 - 1.31z + 0.31}$$

Problem 2:
Repeat Problem 1 for a crossover frequency of 0.2rad/s.

For the crossover at 0.2, the plant phase is only -22.6 deg so only the integral part of the controller is needed. With $C(s) = \dfrac{K}{s}$, the total loop angle is -112.6 that meets the 50-deg phase margin requirement. The plant gain at the same frequency is approximately 1 (0dB) so K =0.2 is a reasonable approximation.

$$C(s) = \frac{0.2}{s}$$

To discretize this compensator we select the sampling rate according to the "6-samples/rise-time" rule: The rise time is approximately 2/BW and BW is 1.5*crossover (or approximately equal to the crossover). So the sampling time is selected as 2/(1.5*0.2)/6 = 1.1. In this case, the simpler F-Euler is acceptable since the integrator does not pose any constraints on sample time selection

$$C(z) = \left.\frac{0.2}{s}\right|_{s=(z-1)/Ts} = \frac{0.2}{\frac{z-1}{1.1}} = \frac{0.22}{z \;\; -1}$$

In the case of the B-Euler, the controller turns out to be the same except that it is advanced by one step.

$$C(z) = \left.\frac{0.2}{s}\right|_{s=(1-z^{-1})/Ts} = \frac{0.2}{\frac{z-1}{1.1z}} = \frac{0.22z}{z-1}$$

Problem 3:
An analog filter with the transfer function $\dfrac{s+10}{(s+1)(s+100)}$ is to be replaced by a computer. Determine an appropriate sampling time and transfer function of the discretized filter using forward and backward Euler approximations.

The filter bandwidth is approximately 1 so a sampling time of ~ 2/BW/6 = 0.3 is a starting point. For the F-Euler method the sampling time is larger than 2TC of the fast pole and filter is not stable. For the B-Euler method the discrete filter is

$$G(z) = \frac{\frac{z-1}{0.3z}+10}{(\frac{z-1}{0.3z}+1)(\frac{z-1}{0.3z}+100)} = \frac{0.00767z^2 - 0.00744z}{z^2 - 0.802z + 0.0248}$$

The F-Euler will work provided that the sample time is less than 2/100, say 0.01. For this sample time

$$G(z) = \frac{\frac{z-1}{0.01}+10}{(\frac{z-1}{0.01}+1)(\frac{z-1}{0.01}+100)} = \frac{0.01z - 0.009}{z^2 - 0.99z}$$

Problem 4:
Compute the transfer function of the system with state space representation

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k \end{aligned} \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -0.5 & 0.2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0.1 \quad 2]$$

The transfer function is $G(z) = C(zI - A)^{-1}B = [0.1 \quad 2]\begin{bmatrix} z & -1 \\ 0.5 & z-0.2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \dfrac{2z+0.1}{z^2 - 0.2z + 0.5}$