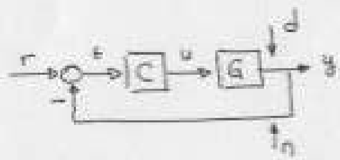
EEE 550 K. TSAKALIS

Notes on Performance Limitations of Feedback Control Systems

BASIC FEEDBACK LOOP:



r: reference signal, u: control input, y: output

ε: tracking error (unity feedback)

d: output disturbance, n: measurement noise

TYPICAL OBJECTIVE: "Keep the contribution of d to y and n to y small."

DEFINITIONS: Sensitivity transfer function:

$$S(s) = \frac{1}{1 + CG(s)}$$

Complementary sensitivity t.f.:

$$T(s) = \frac{CG(s)}{1 + CG(s)}$$

Loop Gain (Loop t.f) $L(s) = CG(s)$

d→y contribution: $\frac{y(s)}{d(s)} = S(s)$

n→y contribution $\frac{y(s)}{n(s)} = -T(s)$

Fundamental Limitation 1 $\boxed{S(s) + T(s) = 1}$ $(\forall s)$

Implication: S and T cannot be simultaneously small

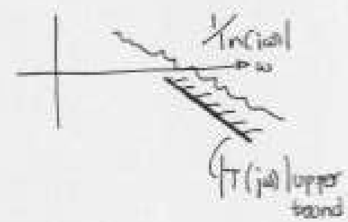
E.g. if $|S(j\omega)| < 0.01$ then $|T(j\omega)| = |1 - S(j\omega)|$

$\Rightarrow |T(j\omega)| \geq 1 - 0.01 = 0.99$

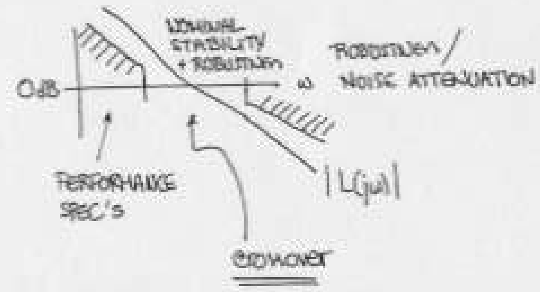
As a result, a frequency separation is assumed between the spectrum of d and the spectrum of n.

However both T and S can be large (bad designs)!

Translating the objective into S and T specs:



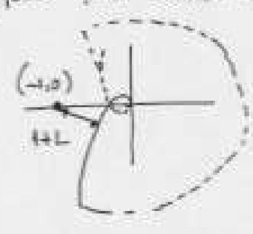
Notice that $|S(j\omega)|$ is small when $|L(j\omega)|$ is large and $|T(j\omega)|$ is small when $|L(j\omega)|$ is small. This observation reflects the classical "LOOP-SHAPING" OBJECTIVE



In the crossover region, the Nyquist plot comes near the critical point $(-1,0)$.

The minimum distance of the Nyquist plot from $(-1,0)$ is

$$|1+L(j\omega)| = \frac{1}{|S(j\omega)|}$$



High peaks of $S(j\omega)$ indicate:

- "strong" resonances in the closed loop. (System near instability ⇒ poles near $j\omega$-axis)
- Lack of robustness to modeling errors. ("small" perturbations in $G(s)$ cause small perturbations in $L(s)$ that can change the encirclement condition ⇒ instability)

It is therefore important that the controller accomplishes the transition between low-frequencies (high gain) and high-frequencies (low-gain) without causing excessive peaking in $S(j\omega)$.

- To avoid excessive peaks in $S(j\omega)$, $|L(j\omega)|$ should have a slope of -20db/dec around the crossover. This approximate rule-of-thumb, while not used here, should be kept in mind as an indication of a required minimum size of the crossover interval.

Further constraints on the behavior of $S(j\omega)$ are imposed by the location of RHP poles and zeros of $L(j\omega)$, relative to the crossover point (or interval). These constraints become crucial as the RHP poles and zeros approach (in magnitude) the crossover frequency. They are less important if the RHP zeros are much greater and the RHP poles are much smaller than the crossover.

It is important to emphasize that such constraints are fundamental in the sense that they exist independent of the controller. (Of course, a poor controller can amplify these effects!)

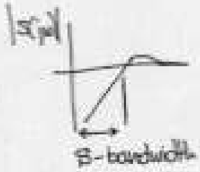
To investigate these fundamental performance limitations, observe first that any controller that stabilizes the closed-loop system must obey the following "interpolation" constraints:

- Any RHP zeros of $G(s)$ must also be zeros of $L(s)$ and $T(s)$. Any RHP poles of $G(s)$ must also be poles of $L(s)$ and zeros of $S(s)$.

This is due to the internal stability requirement that does not allow any cancellations in the RHP. Consequently,

- If z is a RHP zero of $G(s)$ then $T(z)=0$, $S(z)=1$
- If p is a RHP pole of $G(s)$ then $T(p)=1$, $S(p)=0$

Loosely speaking, the interpolation constraints imply that "uniform" disturbance attenuation (in an interval $[0,\omega_0]$) can be achieved only up to the nearest RHP zero. On the other hand, "uniform" noise attenuation (in an interval $[\omega_0,\infty]$) can be achieved only down to the nearest RHP pole. In a picture,



S-bandwidth $<$ |RHP zero|
(S_bw)

T-bandwidth $>$ |RHP pole|
(T_bw)

Notice that a 'good' controller should have S_bw $\approx$ T_bw. While the case S_bw $\gg$ T_bw is prevented by the constraint $S+T=1$, the case S_bw $\ll$ T_bw implies a "high" loop gain without a corresponding disturbance attenuation, or an unnecessarily large crossover interval. In practice $\frac{\text{S\_bw}}{\text{T\_bw}}$ should be between $0.1-1$, depending on RHP pole zero constraints. (Plants for which this ratio must be smaller than 0.1 are very hard to control and are not encountered frequently).

Returning to our Sensitivity-peaking problem, we need the following two preliminary results:

LEMMA 1 Let f be analytic and of bounded magnitude in $\operatorname{Re} s \geq 0$ and let $s_0 = \sigma_0 + j\omega_0 \in \mathbb{C}$ with $\sigma_0 > 0$. Then

$$f(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(j\omega)\,\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega$$

This is a Poisson integral formula, showing that the value of a bounded analytic function at a RHP point is completely determined by the point and the values of the function on the imaginary axis

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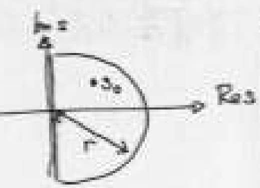
Proof Let C be a contour (Nyquist) enclosing s_0, as shown in the figure.



Cauchy's Integral formula gives $f(s_0) = \frac{1}{2\pi j}\int_C \frac{f(s)}{s-s_0}\,ds$

Also, since $-\bar{s}_0$ is not enclosed by C, $0 = \frac{1}{2\pi j}\int_C \frac{f(s)}{s+\bar{s}_0}\,ds$

Subtracting the two,

$$f(s_0) = \frac{1}{2\pi j}\int_C f(s)\,\frac{\bar{s}_0 + s_0}{(s-s_0)(s+\bar{s}_0)}\,ds$$

Splitting the integral into two parts ($j\omega$-axis and semicircle)

$$f(s_0) = \left.\frac{1}{\pi}\int_{-r}^{r} f(j\omega)\,\frac{\sigma_0}{(s_0-j\omega)(\bar{s}_0+j\omega)}\,d\omega\ \right\} I_1$$

$$\left. + \frac{1}{\pi j}\int_{-\pi/2}^{\pi/2} f(re^{j\theta})\,\frac{\sigma_0}{(re^{j\theta}-s_0)(re^{j\theta}+\bar{s}_0)}\,rje^{j\theta}d\theta\ \right\} I_2$$

As $r\to\infty$, $I_1 \to \frac{1}{\pi}\int_{-\infty}^{\infty} f(j\omega)\,\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega$

On the other hand,

$$|I_2| \le \frac{\sigma_0}{\pi}\sup_{Re\,s\ge 0}|f(s)|\cdot\frac{1}{r}\int_{-\pi/2}^{\pi/2}\frac{1}{|e^{j\theta}-\frac{s_0}{r}||e^{j\theta}+\frac{\bar{s}_0}{r}|}\,d\theta$$

The last integral converges as $r\to\infty$ (or simply it is bounded for large enough r since $|e^{j\theta}\pm\frac{s_0}{r}| \ge |e^{j\theta}| - |\frac{s_0}{r}| \ge 1 - \frac{|s_0|}{r} > \frac{1}{2}$ for $r > 2|s_0|$ and then $\int \frac{1}{|H|}\,d\theta \le 4\pi$.) Hence,

$$|I_2| \le \text{constant}\cdot\frac{1}{r} \Rightarrow I_2 \to 0 \text{ as } r\to\infty$$

and, therefore, as $r\to\infty$ $f(s_0)\to I_1$ which is the desired result.

Before stating and proving the second lemma, let us recall the following definitions (terminology)

- A function f, analytic in the RHP is "all-pass" if $|f(j\omega)| = 1\ \ \forall\omega$.

  Ex 1, $\frac{s-1}{s+1}$, $\frac{s^2-s+2}{s^2+s+2}$, $\frac{s-s_0}{s+\bar{s}_0}$ $(Re\,s_0>0)$

- A function f, analytic in the RHP is "min-phase" if it has no zeros in the RHP. $(Re\,s>0)$

  Ex. 1, $\frac{1}{s+1}$, $\frac{s+2}{s^2+s+1}$, $\frac{s}{s+1}$

It is stated, as a fact, that for each function f, analytic in the RHP, there exists a factorization

$$f(s) = f_{ap}(s)\cdot f_{mp}(s)$$

where f_{ap} is all-pass and f_{mp} is min-phase. This factorization is unique up to a sign.

<u>LEMMA 2</u> Let $S(s)$ be bounded analytic in the RHP with no zeros on the $j\omega$-axis or at ∞. Then, for every point $s_0 = \sigma_0 + j\omega_0$, $\sigma_0 > 0$,

$$\log|S_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\log|S(j\omega)|\,\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega$$

Proof Define $f(s) = \log S_{mp}(s)$. Then f is bounded analytic in the RHP. This follows from the properties of $S_{mp}(s)$ having no poles or zeros in the RHP.

Apply Lemma 1 to get

$$f(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(j\omega)\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega \qquad (1)$$

But $S_{mp}(s) = e^{f(s)} = e^{\operatorname{Re} f(s)}\, e^{j\operatorname{Im} f(s)}$

$$\Rightarrow |S_{mp}(s)| = e^{\operatorname{Re} f(s)}$$

$$\Rightarrow \log|S_{mp}(s)| = \operatorname{Re} f(s)$$

Taking Real parts of both sides of (1)

$$\log|S_{mp}(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|S_{mp}(j\omega)|\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega$$

from which the desired results follow since on the imaginary axis $|S_{ap}(j\omega)| = 1 \Rightarrow |S_{mp}(j\omega)| = |S_{mp}(j\omega)\, S_{ap}(j\omega)| = |S(j\omega)|$.

Notice that for the typical control problem, $G(s) \to 0$ as $s \to \infty$ $\Rightarrow S(s) \to 1$ as $s \to \infty$ while stabilization of the closed loop implies that $S(s)$ is analytic in the RHP. The only constraint for the application of Lemma 2 is that $L(s)$ should have no poles on the imaginary axis (poles of $L(s) \to$ zeros of $S(s)$). This is a minor technical condition which can be satisfied by a slight shift of the $j\omega$-axis, leaving the interpretation of the results largely unaffected — at least for the typical control problem.

With this background we may now state and prove the two main results:

THEOREM 1 (Waterbed Effect)

Suppose that the plant $G(s)$ has a RHP zero at z.
Also, define $M_1 = \max_{\substack{\omega \in [\omega_1,\omega_2] \\ \omega_1 \ge 0,\ \omega_2 > \omega_1}} |S(j\omega)|$, $M_2 = \sup_{\omega \in \mathbb{R}} |S(j\omega)|$

Then there exist positive constants c_1, c_2 depending only on ω_1, ω_2, z, such that

$$\boxed{c_1 \log M_1 + c_2 \log M_2 \ge \log|S_{ap}^{-1}(z)| \ge 0}$$

Proof Since z is a RHP zero of $G(s)$, it is also a zero of $L(s)$

$$\Rightarrow S(z) = 1 \Rightarrow S_{mp}(z) = S_{ap}^{-1}(z).$$

From Lemma 2 it follows that

$$\log|S_{mp}(z)| = \log|S_{ap}^{-1}(z)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\log|S(j\omega)|\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega$$
$$\le c_1\log M_1 + c_2 \log M_2$$

where
$$c_1 = \frac{1}{\pi}\left[\int_{\omega_1}^{\omega_2}\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega + \int_{-\omega_2}^{-\omega_1}\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega\right]$$

$$c_2 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_0}{\sigma_0^2+(\omega-\omega_0)^2}\,d\omega - c_1 = 1 - c_1$$

It remains to observe that $|S_{ap}(z)| \le 1$ by the maximum principle (or maximum modulus theorem) so

$$\log |S_{ap}^{-1}(z)| \ge 0.$$

Remarks: The "waterbed" effect appears if we attempt to "push" $S(j\omega)$ down in an interval since $S(z) = 1$ is fixed. It is applicable to non-min-phase plants only. Plants without LHP zeros can be stabilized with arbitrarily small $|S(j\omega)|$ in any finite interval and $\sup_{\omega \in \mathbb{R}} |S(j\omega)| \le 1 + \delta$ for any $\delta > 0$.

This effect becomes more pronounced if $G(s)$ has RHP poles and zeros near each other.

For example consider $G(s) = \frac{s-1}{(s+1)(s-p)}$ $p > 0$, $p \ne 1$

Since $S(p) = 0$, $S_{ap}(s)$ must contain $\frac{s-p}{s+p}$. Evaluated at 1, $|S_{ap}(1)| \le \left|\frac{1-p}{1+p}\right|$ while the rest of the all-pass factors are bounded by 1 (max principle). So the theorem gives

$$\log M_2 \ge \frac{1}{c_2} \log\left|\frac{1+p}{1-p}\right| - \frac{c_1}{c_2} \log M_1$$

that becomes very large as $p \to 1$.

As another example, let us consider the justification of the previously stated rule-of-thumb, that $S\_bw < |\text{RHP zero}|$.

Let us suppose that the plant $G(s)$ has a real RHP zero at $\sigma_0 > 0$ and that M_2 should be less than 2 (reasonable).

With $\omega_1 = 0$, $c_1 = \frac{1}{\pi} \int_{-\omega_2}^{\omega_2} \frac{\sigma_0}{\sigma_0^2 + \omega^2} d\omega = \frac{2}{\pi} \operatorname{atan}\left(\frac{\omega_2}{\sigma_0}\right)$

$\Rightarrow c_1 = 0.5$ for $\omega_2 = \sigma_0$. In other words M_1 can be less than $\frac{1}{2}$ only up to frequency $\omega_2 = \sigma_0$.

In a more general case, where the zero is at $\sigma_0 + j\omega_0$

$$c_1 = \frac{1}{\pi} \int_{-\omega_2}^{\omega_2} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega = \frac{1}{\pi} \left\{ \operatorname{atan}\left(\frac{\omega_2}{\sigma_0} - \frac{\omega_0}{\sigma_0}\right) + \operatorname{atan}\left(\frac{\omega_2}{\sigma_0} + \frac{\omega_0}{\sigma_0}\right) \right\}$$

Let $\omega_2 = \sqrt{\omega_0^2 + \sigma_0^2}$. Then $c_1 = \frac{1}{\pi} \left\{ \operatorname{atan}\left(\sqrt{1+x^2} - x\right) + \operatorname{atan}\left(\sqrt{1+x^2} + x\right) \right\}$

$x = \frac{\omega_0}{\sigma_0}$

After some lengthy calculations, we find that $\frac{dc_1}{dx} = 0$, i.e. $c_1(x) = \text{constant} = c_1(0) = 0.5$ with the same conclusion as before.

Also observe that when $\sigma_0 \to 0$ then $S(j\omega_0) \to 1$ so that $S\_bw < |\omega_0|$ becomes a "hard" constraint due to the interpolation conditions.

THEOREM 2 (Area Formula)

Suppose that the relative degree of $L(s)$ is at least 2, and $L(s)$ has no poles on the imaginary axis. Then

$$\boxed{\int_0^\infty \log|S(j\omega)|\,d\omega = \pi(\log e)\sum \operatorname{Re} p_i}$$

where p_i are the RHP poles of $L(s)$.

(Relative degree = deg. denominator − degree numerator.
Alternatively it is $\max n : |L(s)s^n| < \infty$ as $s \to \infty$)

Proof In Lemma 2 take $\omega_0 = 0$ to get

$$\log|S_{mp}(\sigma_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \log|S(j\omega)|\,\frac{\sigma_0}{\sigma_0^2+\omega^2}\,d\omega$$

S real-analytic

$$\Rightarrow \int_0^\infty \log|S(j\omega)|\,\frac{\sigma_0^2}{\sigma_0^2+\omega^2}\,d\omega = \frac{\pi}{2}\sigma_0\log|S_{mp}(\sigma_0)|$$

Taking limits of both sides as $\sigma_0 \to \infty$:

$$\text{i)}\ \left|\int_0^\infty \log|S(j\omega)|\,\frac{\sigma_0^2}{\sigma_0^2+\omega^2}\,d\omega - \int_0^\infty \log|S(j\omega)|\,d\omega\right|$$

$$= \left|\int_0^\infty \log|S(j\omega)|\,\frac{\omega^2}{\sigma_0^2+\omega^2}\,d\omega\right|.$$ But $|S(j\omega)| \to 1$ as $\omega \to \infty$

In fact, $|S(j\omega)| \le 1 + |T(j\omega)| \le e^{|T(j\omega)|} \xRightarrow{\text{monotonicity of log}}$

$$\log|S(j\omega)| \le |T(j\omega)| \le \frac{K_1}{\omega^2} \text{ for some constant } K_1$$

by the relative degree assumption.

For the other side, $|S(j\omega)| \ge 1 - |T(j\omega)|$. For large ω $(\omega > \Omega_0)$, $|T(j\omega)| < 1$ and $\log|S(j\omega)| \ge -2|T(j\omega)|$.
(For example, since $|T(j\omega)| \le \frac{K_1}{\omega^2} \Rightarrow |T(j\omega)| < \frac{1}{2}$ for $\omega > \Omega_0 \ge \sqrt{\frac{K_1}{2}}$. Then, we claim that for $x \in [0, \frac{1}{2}]$ $\ln(1-x) \ge -2x$, or $\ln(1-x) + 2x \ge 0$. This is true for $x=0$ and $\frac{d}{dx}\left(\ln(1-x)+2x\right) = 2 - \frac{1}{1-x} \ge 0$ for $x \in [0, \frac{1}{2}]$ which proves the claim.)

Thus, $-2|T(j\omega)| \le \log|S(j\omega)| \le |T(j\omega)|$ $\quad \omega > \Omega_0$

$$\Rightarrow \left|\log|S(j\omega)|\right| \le \frac{2K_1}{\omega^2} \quad \omega > \Omega_0.$$

Now,
$$\left|\int_0^\infty \log|S(j\omega)|\,\frac{\omega^2}{\sigma_0^2+\omega^2}\,d\omega\right| \le \int_0^{\Omega_0} \left|\log|S(j\omega)|\right|\frac{\Omega_0^2}{\sigma_0^2+\Omega_0^2}\,d\omega + \int_{\Omega_0}^\infty \frac{2K_1}{\omega^2}\,d\omega$$

Invoking our technical condition that $L(s)$ has no poles on the $j\omega$-axis, $\log|S(j\omega)|$ is bounded, say, by K_2. Then the first integral is bounded by $\frac{K_2\Omega_0^3}{\sigma_0^2+\Omega_0^2}$ and the second by $\frac{2K_1}{\Omega_0}$. Letting $\sigma_0 = \Omega_0^2$ and $\Omega_0 \to \infty$ the right hand side approaches 0, implying

$$\int_0^\infty \log|S(j\omega)|\,\frac{\sigma_0^2}{\sigma_0^2+\omega^2}\,d\omega \to \int_0^\infty \log|S(j\omega)|\,d\omega$$

as $\sigma_0 \to \infty$.

2). For the right-hand side, let $S = S_{ap} S_{mp}$ where

$$S_{ap}(s) = \prod_i \frac{s - p_i}{s + \bar{p}_i}$$

Using the previous bounding procedure, it follows that

$$\lim_{\sigma\to\infty} \sigma \log|S(\sigma)| = 0, \Rightarrow \lim_{\sigma\to\infty} \sigma \log|S_{mp}(\sigma)| = \lim_{\sigma\to\infty}(-\sigma \log|S_{ap}(\sigma)|) = \lim_{\sigma\to\infty} \sigma \log|S_{ap}^{-1}(\sigma)|$$

So, it remains to show that

$$\lim_{\sigma\to\infty} \frac{\sigma}{2} \log|S_{ap}^{-1}(\sigma)| = \sum \operatorname{Re} p_i.$$

and thus it suffices to show $\left(\log|S_{ap}^{-1}(\sigma)| = \sum_i \log\left|\frac{\sigma+\bar{p}_i}{\sigma-p_i}\right|\right)$

$$\lim_{\sigma\to\infty} \frac{\sigma}{2} \log\left|\frac{\sigma+\bar{p}_i}{\sigma-p_i}\right| = \operatorname{Re} p_i$$

let $p_i = x + jy$:

$$\frac{\sigma}{2}\log\left|\frac{\sigma+\bar{p}_i}{\sigma-p_i}\right| = \frac{\sigma}{2}\log\left|\frac{1+x/\sigma - jy/\sigma}{1-x/\sigma-jy/\sigma}\right| = \frac{\sigma}{4}\log\frac{(1+x/\sigma)^2+(y/\sigma)^2}{(1-x/\sigma)^2+(y/\sigma)^2}$$

$$= \frac{\sigma}{4}\left\{\log\left[(1+x/\sigma)^2+(y/\sigma)^2\right] - \log\left[(1-x/\sigma)^2+(y/\sigma)^2\right]\right\}$$

Bring in the series expansion $\log(1+x) = x - \frac{x^2}{2} + \cdots$

(a formal bounding procedure is possible but tedious)

$$= \frac{\sigma}{4}\left\{2\frac{x}{\sigma} + 2\frac{x}{\sigma} + \cdots\right\} = x + \text{HOT}$$

Higher order terms in $\frac{1}{\sigma}$

which proves the theorem by letting $\sigma\to\infty$.

Notice the appearance of $\log_e$ to ensure the validity for log with different base.

REMARKS The area formula is due to Bode (1945). In contrast to the waterbed effect (applicable to non-min-phase systems), the area formula applies in general. By itself, it does not imply a peaking phenomenon, only an area conservation.

However, one can infer a type of peaking phenomenon from the area formula when another type of constraint is imposed, namely controller bandwidth.

For example, suppose $|GC| = |L| < \frac{k}{\omega^2}$ as it is often the case in practice. This is one way of saying that the loop bandwidth is less than $\omega_1 = \sqrt{K}$. Then, for $\omega > \omega_1$

$$|S| \le \frac{1}{1-|L|} < \frac{\omega^2}{\omega^2 - K}. \quad \text{Hence} \int_{\omega_1+\epsilon}^{\infty} \log|S|\,d\omega \le \int_{\omega_1+\epsilon}^{\infty} \log\frac{\omega^2}{\omega^2-K}\,d\omega$$

$\epsilon > 0$.

The last integral is finite (e.g. use a series expansion of log). Hence the possible positive area over $[\omega_1+\epsilon, \infty)$ is limited. Thus, if $|S|$ is made smaller over $[0, \frac{\omega_1}{2}]$, then $|S|$ must necessarily become larger somewhere in $[\frac{\omega_1}{2}, \omega_1+\epsilon]$. Loosely speaking, with a loop bandwidth constraint, the waterbed effect applies even to minimum-phase plants.

REFERENCE : Doyle, Francis, Tannenbaum, "Feedback Control Theory" Macmillan, NY. 1992

EEE 550 PROBLEM STATEMENTS

PR 2.9 Show that the function $f(s) = \begin{cases} 1 & s \text{ on the unit circle} \\ s & \text{otherwise} \end{cases}$
is continuous at $s=1$ but not continuous at other points on the unit circle. (Calculate $\lim_{\epsilon \to 0} (\rho \pm \epsilon) e^{j\phi}$).

PR 2.11 Show that the function $f(s) = |s|^2$ is continuous at all finite points.

PR 2.12 Show that the function $f(s) = 1/s$ is continuous at all points except $s=0$.

PR 2.20 Prove that $f(s) = s^2$ satisfies the Cauchy-Riemann differential equations.

PR. 2.23 Show that $f(s) = s|s|$ has a derivative only at the origin.

PR. 3.3 Consider the function $w = s^2$. Using the rectangular coordinates $s = \sigma + j\omega$, sketch the w-plane loci for appropriate values of σ and ω, such that the w-plane is a transformation of

a) the top half of the s-plane $(\omega > 0)$
b) the bottom half of the s-plane
c) the right half of the s-plane
d) the left half of the s-plane

PR. 3.19 Determine how rectangular coordinates in the left half of the s-plane are transformed by the function $w = \frac{s-1}{s+1}$

PR 3.20 Determine how polar coordinates in the s-plane are transformed by the function $w = \frac{s-1}{s+1}$

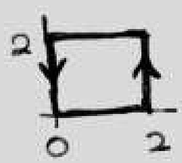
PR 4.7 Perform each of the following integrations:

a) $\int_C \frac{s}{|s|}\, ds$, b) $\int_C \frac{\operatorname{Re} s}{|s|}\, ds$, c) $\int_C \frac{\operatorname{Im} s}{|s|}\, ds$

for the path of radius R shown in the figure: 

PR 4.11 In the proof of the Cauchy integral theorem, where do we use the condition that the function shall be regular on C ?

PR. 4.12 By performing the integration over each side of the square,



check the validity of the Cauchy integral theorem using the function $f(s) = s^2$.

PR. 4.13 Let C denote the unit circle and C_1 the top half of the unit circle. Prove that

$$\int_C f(s)\,ds = \int_{C_1} [\, f(s) - f(-s)\,]\,ds$$

PR. 4.16 Evaluate each of the following integrals in a counter-clockwise sense around a simple path C consisting of a circle of radius 1, centered at $s = j$:

a) $\int_C \frac{s^2-1}{s^2+1}\, ds$ b) $\int_C \frac{s^2-1}{(s+1)^2}\, ds$

c) $\int_C \frac{s^2+1}{s^2-1}\, ds$ d) $\int_C \frac{s^2+j}{(s+j)^2}\, ds$

PR 5.2 Investigate the convergence properties of:

a) $\sum_{n=1}^{\infty} n s^n$ b) $\sum_{n=0}^{\infty} 2^n s^n$

c) $\sum_{n=0}^{\infty} n! s^n$ c) $\sum_{n=0}^{\infty} n^{\sqrt{n}} s^n$

*

PR 5.5 If each of the series of constants

$$\sum_{k=0}^{\infty} a_k, \quad \sum_{k=0}^{\infty} b_k, \quad \sum_{k=0}^{\infty} c_k$$

converges and if $c_k = a_k + b_k$, prove that

$$\sum_{k=0}^{\infty} = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

PR. 5.6 With the notation of PR. 5.5, assume that $\sum_{k=0}^{\infty} c_k$ converges. Show, by a counter example, that it is not necessarily true that $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge

PR 5.4 Determine the regions of convergence and uniform convergence of the series:

a) $\sum_{n=0}^{\infty} e^{ns}$ b) $\sum_{n=0}^{\infty} \left(\frac{s-1}{s+1}\right)^n$

PR 5.17 Use the Cauchy Integral formula for the n-th derivative of a function $f(s)$ to prove that for a Taylor series $f(s) = \sum_{n=0}^{\infty} a_n (s-s_1)^n$, the coefficients obey the inequality $|a_n| \leq \frac{M_r}{r^n}$, where $M_r = \max|f(s)|$ on $|s-s_1| = r <$ [radius of convergence].

PR 5.19 Given a Laurent series $\sum_{k=-\infty}^{\infty} a_k s^k$ which converge for $R_1 < |s| < R_2$, determine its region of uniform convergence.

PR 5.20 For the function $\dfrac{s^2+s+3}{s^3+2s^2+s+2}$ obtain the following expansions, and establish the region of convergence in each case.

a) Taylor expansion about $s=0$

b) Taylor expansion about $s=-1$

c) Laurent expansions (two of them) about $s=0$

d) Laurent expansion about each singular point.

PR 5.25 For each of the following functions, locate the singular points, and identify whether they are poles (and of what order) or essential singularities (and of what kind):

a) $\dfrac{e^s}{s}$ b) $e^{1/s}$ c) $e^{-1/s}$

d) $\dfrac{s^2}{(s^2+1)^2}$ e) $\sin s$ f) $\dfrac{1}{\sinh s}$

PR 5.27 Find the residues at the indicated singular points for the following:

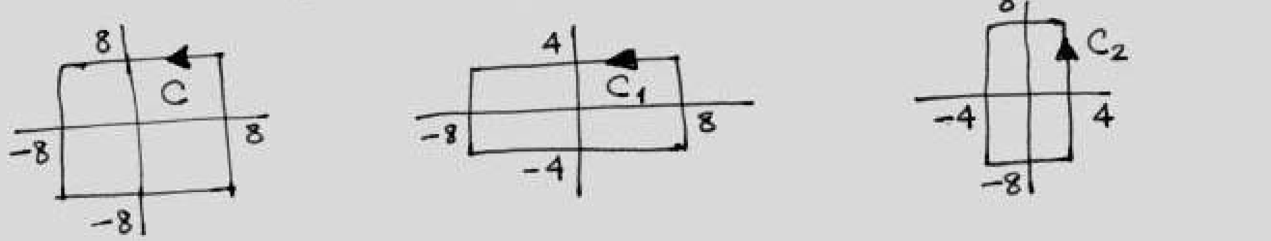
a) $\dfrac{\sin s}{s^3}$ at $s=0$ b) $\dfrac{1}{s^3-s^2}$ at $s=1$

c) $\dfrac{1-e^{-2s}}{s^4}$ at $s=0$ d) $\dfrac{\cos s}{\sin^2 s}$ at $s=\pi$

e) $\dfrac{e^{2s}}{(s-1)^2}$ at $s=1$ f) $\dfrac{\tan s}{(1-e^s)^2}$ at $s=0$

PR 5.29 a) Use the method of residues to evaluate the integral $\int_C \frac{s\,ds}{1-e^s}$, where C is the rectangular path shown in the figure below.

b) Let I_0 designate the answer to part (a). In terms of I_0, what is the above integral if the contour is changed to C_1? What if it is changed to C_2?



PR 8.16 Use contour integration to evaluate the following integrals:

a) $\int_{-\infty}^{\infty} \frac{\sin y}{1+y^2}\,dy$ b) $\int_{-\infty}^{\infty} \frac{e^{jy}}{1+y^2}\,dy$ c) $\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^2}$

PR 8.26 Let a rational function $F(s)$ have at least a second-order zero at infinite s. This means that the degree of the denominator must be at least 2 greater than the degree of the numerator. Use an appropriate theorem to prove that the summation of the residues over all the poles is zero.

PR. 9.2 Check the function

$$f(t) = \begin{cases} 0 & |t| > 1 \\ -1 & -1 < t < 0 \\ 1 & 0 < t < 1 \end{cases}$$

in the Fourier integral theorem. That is, find $\mathbf{F}(j\omega)$ and then recover $f(t)$ from the inversion integral.

PR. 10.45 Obtain inverse transforms for the following:

a) $F(s) = \dfrac{6s^2+s-1}{s^3+s}$

b) $F(s) = \dfrac{4s^2+16s+16}{s^3+5s^2+9s+5}$

c) $F(s) = \dfrac{3s^3+8s^2+9s+4}{s^4+5s^3+9s^2+7s+2}$

Pr 2.9 $f(s) = \begin{cases} 1 & s \text{ on U.C.} \\ s & \text{otherwise} \end{cases}$

1. Continuity at $s_0 = 1$.

$f(s_0) = 1$,

Given any $\epsilon > 0$ we want to find $\delta(s_0)$ s.t.

$|s - s_0| < \delta \Rightarrow |f(s) - f(s_0)| < \epsilon$.

CASE 1: s on U.C. $\Rightarrow f(s) = 1 \Rightarrow f(s) - f(s_0) = 0$

$\therefore$ the inequality is trivially satisfied with no restrictions on δ.

CASE 2: s not on U.C., Then $|f(s) - f(s_0)| = |s - 1|$

$= |s - s_0|$. $\therefore$ Taking $\delta = \epsilon$,

$|s - s_0| < \delta = \epsilon \Rightarrow |f(s) - f(s_0)| < \epsilon$.

$\therefore$ f is continuous at $s = 1$. QED

2. Discontinuity at s_0 on U.C $s_0 \neq 1$.

We should show the following.

$\exists\, \epsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, s_\delta$ s.t. $|s_\delta - s_0| < \delta$

and $|f(s_\delta) - f(s_0)| > \epsilon$.

Alternatively, we want to show that there exists a sequence of s_δ's (say $s_\delta(k)$) converging to s_o ($s_\delta(k) \xrightarrow[k\to\infty]{} s_o$) s.t. $|f(s_\delta) - f(s_o)| > \epsilon$ for some fixed ϵ.

Note that, again, $f(s_o) = 1$. (i.e. s_δ must NOT be chosen on U.C. otherwise the result would be inconclusive).

Now, given any $\delta > 0$ pick s_δ not on U.C. and s.t.

$$|s_\delta - s_o| < \delta_1 \leq \delta$$

Note that such an s_δ always exists. For example, if $s_o = 1 \cdot e^{j\phi_o}$, let $s_\delta = (1-\delta_1)\, e^{j\phi_o}$ $\delta_1 < 1$ or $s_\delta = (1+\delta_1)\, e^{j\phi_o}$ and verify the above inequality.

Clearly, since $\delta_1 > 0$, $s_\delta \notin$ U.C. Hence

$$|f(s_\delta) - f(s_o)| = |s_\delta - 1| = |s_\delta + s_o - s_o - 1| \geq$$

$$\geq |s_o - 1| - |s_\delta - s_o|$$

$$> |s_o - 1| - \delta_1$$

Since $s_o \neq 1$ we have that $|s_o - 1| = \mu > 0$ (a fixed number)

Pick δ_1 to be smaller than $\mu/2$, i.e.,

$$\delta_1 \triangleq \min(\delta, \mu/2)$$

Then $|f(s_\delta) - f(s_o)| > \mu/2 \triangleq \epsilon$. proving that f is discontin. at s_o on U.C. other than 1.

OEA

Pr 2.11 $\quad f(s) = |s|^2 = \sigma^2 + \omega^2 \quad$ where $\quad s = \sigma + j\omega$

Hence
$$\begin{aligned}
|f(s) - f(s_0)| &= |\sigma^2 + \omega^2 - \sigma_0^2 - \omega_0^2| \\
&\le |\sigma - \sigma_0||\sigma + \sigma_0| + |\omega - \omega_0||\omega + \omega_0| \\
&\le |\sigma - \sigma_0||\sigma - \sigma_0 + 2\sigma_0| + |\omega - \omega_0||\omega - \omega_0 + 2\omega_0| \\
&\le |\sigma - \sigma_0|^2 + |\sigma - \sigma_0||2\sigma_0| + |\omega - \omega_0|^2 + |\omega - \omega_0||2\omega_0|
\end{aligned}$$

Now observe that $|s - s_0| < \delta \Rightarrow |\sigma - \sigma_0 + j(\omega - \omega_0)| < \delta$

or, $(\sigma - \sigma_0)^2 + (\omega - \omega_0)^2 < \delta^2$

$\Rightarrow |\sigma - \sigma_0| < \delta$ and $|\omega - \omega_0| < \delta$

Hence, $|s - s_0| < \delta \Rightarrow |f(s) - f(s_0)| < \delta^2 + \delta|2\sigma_0| + \delta^2 + \delta|2\omega_0|$
$$< 2\delta^2 + 2\delta(|\sigma_0| + |\omega_0|)$$

Although $|\sigma_0| + |\omega_0|$ can be left as is, it "looks better" if we express it in terms of s_0. For this purpose, note that $|\sigma_0| + |\omega_0| \le |s_0| + |s_0| \quad (|\sigma_0| \le |s_0|, \ |\omega_0| \le |s_0|)$

That is, $|s - s_0| < \delta \Rightarrow |f(s) - f(s_0)| < 2\delta^2 + 4\delta|s_0|$

we want to pick $\delta = \delta(\epsilon, s_0)$ s.t. $|f(s) - f(s_0)| < \epsilon$.

Hence, it suffices to pick δ s.t. $2\delta^2 + 4\delta|s_0| \le \epsilon$

or, pick δ s.t. $\quad 2\delta^2 \le \epsilon/2$
$$4\delta|s_0| \le \epsilon/2$$

Thus, if we define $\delta \equiv \min\left(\frac{\sqrt{\epsilon}}{2}, \frac{\epsilon}{8|s_0|}\right)$ the result follows. $\quad$ QED

Pr 2.12 $\qquad f(s) = \frac{1}{s}$.

When $s \to 0$ $\quad \lim f(s)$ does not exist

($e.g.$ $s = \rho e^{j\phi}$, $\rho \to 0$ etc.)

Hence $f(s)$ cannot be continuous at $s = 0$.

Next let $s_0 \neq 0$ and form $|f(s) - f(s_0)|$.

$$|f(s) - f(s_0)| = \left|\frac{1}{s} - \frac{1}{s_0}\right| = \left|\frac{s_0 - s}{s\, s_0}\right|$$

Hence, for $|s - s_0| < \delta$

$$|f(s) - f(s_0)| < \frac{\delta}{\underbrace{|s|}_{=|s-s_0+s_0| \geq |s_0| - |s-s_0| > |s_0| - \delta}|s_0|} < \frac{\delta}{(|s_0| - \delta)|s_0|}$$

$\therefore$ It suffices to choose δ : $\frac{\delta}{(|s_0|-\delta)|s_0|} \leq \epsilon$ (Note $|s_0| > 0$)

or $\quad \delta < \epsilon(|s_0| - \delta)|s_0|$

Choosing $\delta \leq \frac{|s_0|}{2}$, it suffices to have $\delta \leq \epsilon \frac{|s_0|^2}{2}$

Thus, letting $\delta = \min\left(\frac{|s_0|}{2}, \epsilon\frac{|s_0|^2}{2}\right)$ the result follows.

QED

REM Observe that, as expected, as $|s_0| \to 0$, $\delta \to 0$ and in the limit $|s_0| = 0$, $\delta = 0$ which shows our inability to conclude that f is continuous at $s = 0$.

Pr. 2.20 $\quad f(s) = s^2 = u + jv \quad ; \quad s = \sigma + j\omega.$

$$s^2 = \sigma^2 - \omega^2 + j\,2\sigma\omega.$$

$$\therefore \begin{cases} u = \sigma^2 - \omega^2 \\ v = 2\sigma\omega. \end{cases}$$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial \sigma} = 2\sigma \ , \ \dfrac{\partial v}{\partial \omega} = 2\sigma \quad \therefore \quad \dfrac{\partial u}{\partial \sigma} = \dfrac{\partial v}{\partial \omega} \\ \dfrac{\partial u}{\partial \omega} = -2\omega \ , \ \dfrac{\partial v}{\partial \sigma} = 2\omega \quad \therefore \quad \dfrac{\partial u}{\partial \omega} = -\dfrac{\partial v}{\partial \sigma} \end{array}\right\} CR \checkmark$$

✓ QED

Pr. 2.21 $\quad f(s) = \dfrac{1}{s} = u + jv.$

$$\frac{1}{s} = \frac{\bar{s}}{|s|^2} = \frac{\sigma - j\omega}{\sigma^2 + \omega^2} \quad \therefore \begin{cases} u = \dfrac{\sigma}{\sigma^2+\omega^2} \\ v = -\dfrac{\omega}{\sigma^2+\omega^2}. \end{cases}$$

$$\frac{\partial u}{\partial \sigma} = \frac{1}{\sigma^2+\omega^2} - \frac{2\sigma^2}{(\sigma^2+\omega^2)^2} = \frac{\sigma^2+\omega^2-2\sigma^2}{(\sigma^2+\omega^2)^2} = \frac{\omega^2-\sigma^2}{(\sigma^2+\omega^2)^2}$$

$$\frac{\partial v}{\partial \omega} = \frac{-1}{\sigma^2+\omega^2} - \frac{(-\omega)\cdot 2\omega}{(\sigma^2+\omega^2)^2} = \frac{-\sigma^2-\omega^2+2\omega^2}{(\sigma^2+\omega^2)^2} = \frac{-\sigma^2+\omega^2}{(\sigma^2+\omega^2)^2}$$

$$\therefore \boxed{\frac{\partial u}{\partial \sigma} = \frac{\partial v}{\partial \omega}} \checkmark$$

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial \omega} = -\dfrac{2\sigma\omega}{(\sigma^2+\omega^2)^2} \\ \dfrac{\partial v}{\partial \sigma} = -\dfrac{-2\sigma\omega}{(\sigma^2+\omega^2)^2} \end{array}\right\} \quad \therefore \boxed{\frac{\partial u}{\partial \omega} = -\frac{\partial v}{\partial \sigma}} \checkmark \quad \Big\} CR.$$

where the various partial derivatives exist, provided that $\sigma^2+\omega^2 \neq 0 \Leftrightarrow s \neq 0$ —

QED

Pr 2.23 $\quad f(s) = s|s|$.

A. $f'(s)|_{s=0}$ exists.

1. form the differential quotient.

$$\left.\frac{f(s+\Delta s) - f(s)}{\Delta s}\right|_{s=0} = \left.\frac{(s+\Delta s)\,|s+\Delta s| - s|s|}{\Delta s}\right|_{s=0} =$$

$$= \frac{\Delta s\,|\Delta s|}{\Delta s} = |\Delta s|$$

2. take limits.

$$\lim_{|\Delta s|\to 0} \frac{f(s+\Delta s) - f(s)}{\Delta s} = \lim_{|\Delta s|\to 0} |\Delta s| = 0. \qquad \text{QED}$$

B. $f'(s)|_{s\neq 0}$ does not exist.

$$\frac{f(s+\Delta s) - f(s)}{\Delta s} = \frac{(s+\Delta s)\sqrt{s\bar{s} + \Delta s\,\bar{s} + \overline{\Delta s}\,s + |\Delta s|^2} - s|s|}{\Delta s}$$

$$= \frac{(s+\Delta s)|s|\sqrt{1 + \frac{\Delta s}{s} + \frac{\overline{\Delta s}}{\bar{s}} + \left|\frac{\Delta s}{s}\right|^2} - s|s|}{\Delta s}$$

$$= \frac{s|s|\left(\sqrt{(\ )} - 1\right)}{\Delta s} + |s|\sqrt{(\cdot)}$$

NOTE :

1) $\frac{\Delta s}{s} + \frac{\overline{\Delta s}}{\bar{s}} = 2\,\mathrm{Re}\left(\frac{\Delta s}{s}\right)$

2) $\sqrt{1+\epsilon} = 1 + \frac{\epsilon}{2} + O(\epsilon^2)$ where $\frac{O(\epsilon^2)}{\epsilon} \to 0$ as $\epsilon \to 0$

Although we cannot take limits at this point (since we do not expect the limit to exist) we can "emulate" the process.

That is, choose two paths for Δs as follows

1. $\Delta s = \rho e^{j0} \quad \rho \to 0$
2. $\Delta s = \rho e^{j\pi/2} \quad \rho \to 0$

Substituting and taking limits as $\rho \to 0$ we have

$$1. \lim_{\rho \to 0} \frac{f(s + \Delta s_1) - f(s)}{\Delta s_1} = \lim_{\rho \to 0} \frac{s|s|\left(\sqrt{1 + \frac{\rho}{s} + \frac{\rho}{\bar{s}} + \frac{\rho^2}{|s|^2}} - 1\right)}{\rho} + \lim_{\rho \to 0} |s| \sqrt{\cancel{(\ )}}^{\,1}$$

as $\rho \to 0$

$$\sqrt{1 + (\ )} \to 1 + \frac{\rho}{2}\left(\frac{1}{s} + \frac{1}{\bar{s}}\right) \qquad = \lim_{\rho \to 0} \frac{s|s|\left(+\frac{\rho}{2}\left(\frac{\bar{s}+s}{|s|^2}\right)\right)}{\rho} + |s|$$

$$= +\frac{1}{2|s|}\left(|s|^2 + s^2\right) + |s|$$

$$= \boxed{\frac{3}{2}|s| + \frac{1}{2}\frac{s^2}{|s|}}$$

$$2. \lim_{\rho \to 0} \frac{f(s + \Delta s_2) - f(s)}{\Delta s_2} = \lim_{\rho \to 0} \frac{s|s|\left(\sqrt{1 + \rho\left(\frac{e^{j\pi/2}}{s} + \frac{e^{-j\pi/2}}{\bar{s}}\right) + \frac{\rho^2}{|s|^2}} - 1\right)}{\rho e^{j\pi/2}}$$

$$+ \lim_{\rho \to 0} |s| \sqrt{1 + \rho(\ )}$$

$$= \lim_{\rho \to 0} \frac{s|s|\left(\frac{\rho}{2}\left(\frac{e^{j\pi/2}\bar{s} + e^{-j\pi/2}s}{s\bar{s}}\right)\right)}{\rho e^{j\pi/2}} + |s|$$

$$= \frac{|s|}{2} + |s| + \frac{s^2}{2|s|} e^{-j\pi} = \boxed{\frac{3}{2}|s| + \frac{1}{2}\frac{s^2}{|s|}(-1)}$$

If $f(s)$ were differentiable at any $s \neq 0$ the two limits should be the same, i.e.,

$$\frac{3}{2}|s| + \frac{1}{2}\frac{s^2}{|s|} = \frac{3}{2}|s| + \frac{1}{2}\frac{s^2}{|s|}(-1)$$

$$\overset{s\neq 0}{\Longleftrightarrow} \quad s^2 = -s^2 \Longleftrightarrow s^2 = 0$$

But $s^2 = \left(R^2 e^{2j\phi}\right) = 0 \Longleftrightarrow R = 0$

contradicting the assumption $s \neq 0$.

Hence the two limits are not equal for any $s \neq 0$ implying that $\lim_{\Delta s \to 0} \frac{f(s+\Delta s) - f(s)}{\Delta s}$ does not exist for any $s \neq 0$. Hence $f(s)$ is not differentiable at any $s \neq 0$

QED

PR. 3.3

$$w = s^2 \quad \therefore \quad \begin{array}{l} u = \sigma^2 - \omega^2 \\ v = 2\sigma\omega \end{array}$$

1. TOP HALF OF THE S-PLANE

$\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}^+$

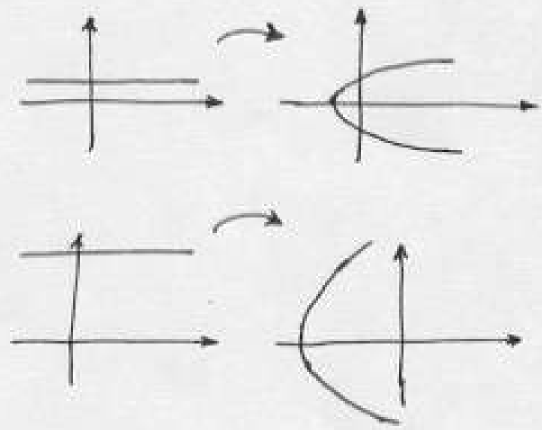
$u(v) = \frac{1}{4\omega^2} v^2 - \omega^2$ (Parabola)

extremum : $\frac{du}{dv} = \frac{2}{4\omega^2} v = 0$

@ $v_* = 0$

$u(v_*) = -\omega^2$

$\frac{d^2u}{d^2v} = \frac{1}{2\omega^2} > 0 \downarrow$ as $\omega \uparrow$



2. BOTTOM HALF OF THE S-PLANE $\sigma \in \mathbb{R}$, $\omega \in \mathbb{R}^-$

Similar picture as in 1 since $u(v)$ depends only on ω^2.

3. RIGHT-HALF OF THE S-plane $\sigma \in \mathbb{R}^+$, $\omega \in \mathbb{R}$

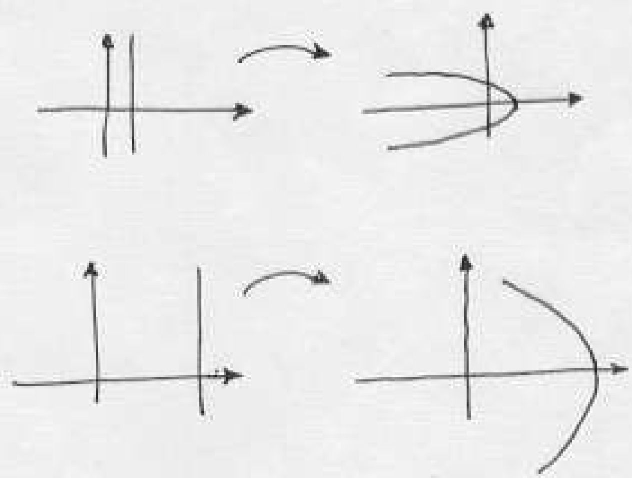
$u(v) = -\frac{1}{4\sigma^2} v^2 + \sigma^2$

extremum @ $v_* = 0$

$u(v_*) = \sigma^2$

$\frac{d^2u}{d^2v} = -\frac{1}{2\sigma^2} < 0$

$\left|\frac{d^2u}{d^2v}\right| \downarrow$ as $\sigma \uparrow$



4. LEFT HALF OF THE s-plane $\sigma \in \mathbb{R}^-$, $\omega \in \mathbb{R}$

Similar picture as in 3 since $u(v)$ depends only on σ^2

Pr. 3.19

$$W = \frac{s-1}{s+1} \quad ; \quad \sigma \in \mathbb{R}^-, \ \omega \in \mathbb{R}$$

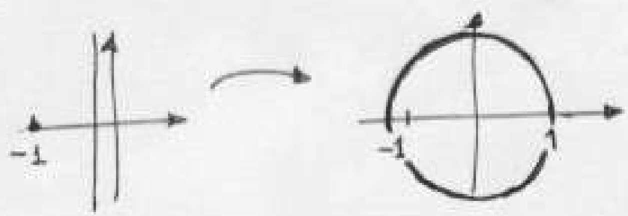
From 3.34, $\quad v^2 + \left(u - \frac{\sigma}{\sigma+1}\right)^2 = \left(\frac{1}{\sigma+1}\right)^2$

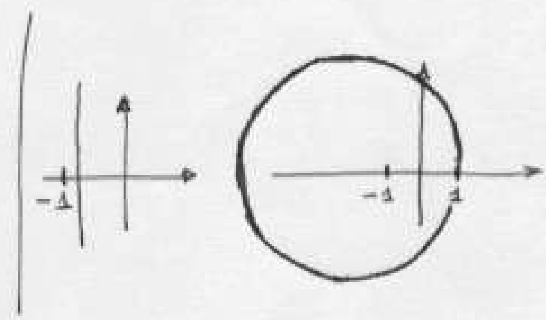
$\therefore$ circles with center $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \sigma/\sigma+1 \\ 0 \end{pmatrix}$

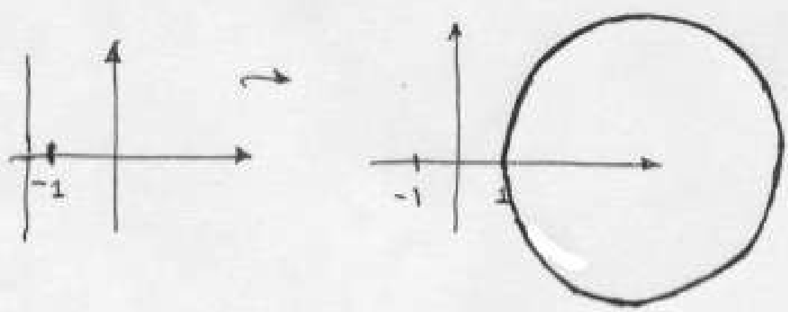
and radius $\frac{1}{\sigma+1}$

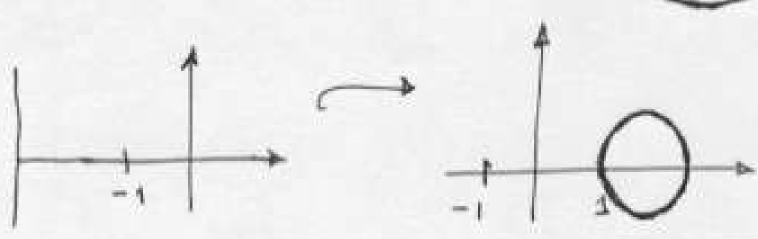
Note the singularity @ $\sigma = -1$. Also note that all the circles pass through the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ i.e. $0^2 + \left(1 - \frac{\sigma}{\sigma+1}\right)^2 = \left(\frac{1}{\sigma+1}\right)^2$

Thus;









Pr. 3.20 $\qquad w = \dfrac{s-1}{s+1} \qquad s = \rho e^{j\theta}, \quad w = u + jv$

$$\therefore \quad \rho e^{j\theta} = -\frac{w+1}{w-1} = -\frac{(u+1)+jv}{(u-1)+jv}$$

Taking complex conjugates

$$\rho e^{-j\theta} = -\frac{(u+1)-jv}{(u-1)-jv}$$

Multiply $\rho e^{j\theta}$ by $\rho e^{-j\theta}$

$$\rho^2 = \frac{(u+1)^2 + v^2}{(u-1)^2 + v^2}$$

$$\Rightarrow \quad \rho^2 (u-1)^2 + \rho^2 v = [(u-1)+2]^2 + v^2$$

$$\Rightarrow \quad (\rho^2-1)(u-1)^2 - 4(u-1) + (\rho^2-1)v^2 = 4$$

$$\Rightarrow \quad \Big[\underbrace{(u-1)^2 - \frac{4}{\rho^2-1}(u-1)}_{\text{COMPLETE THE SQUARES}} \Big] + v^2 = \frac{4}{\rho^2-1}$$

$\pm \beta^2$ s.t.

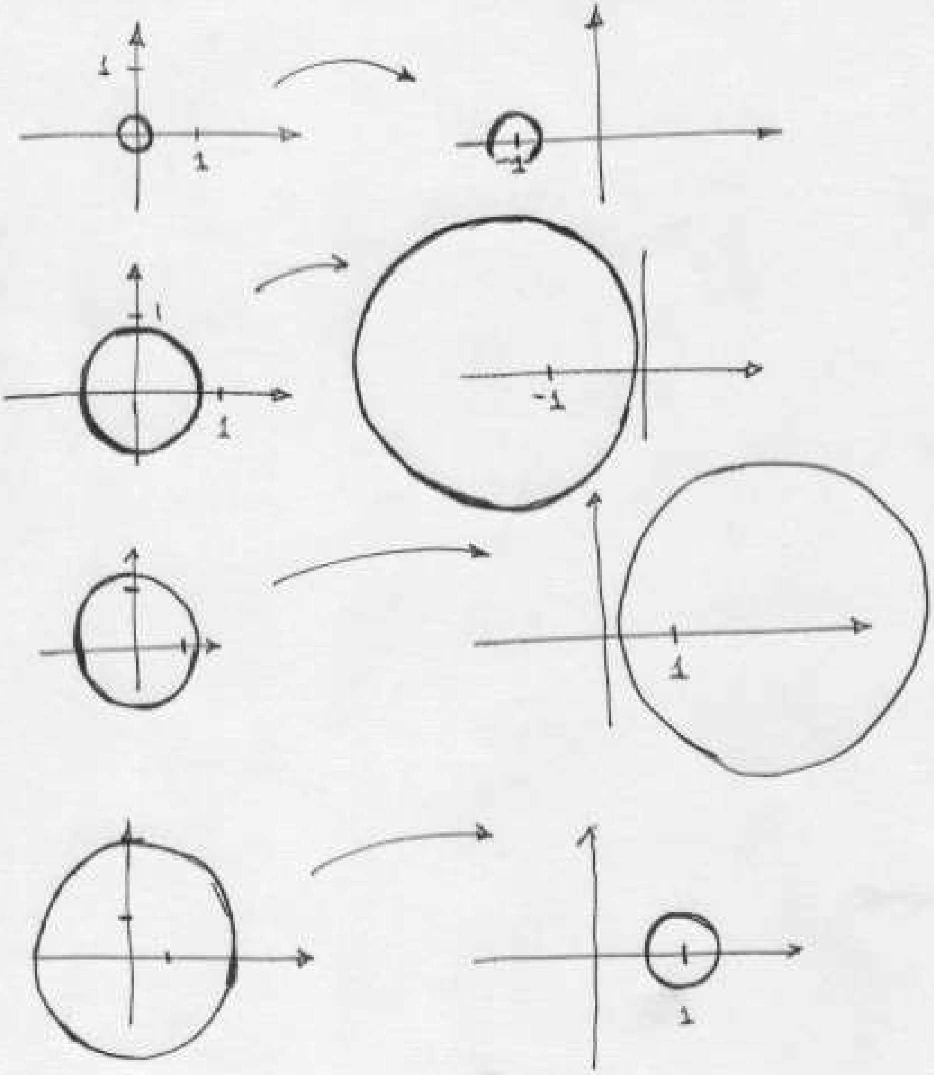
$$[\quad] = [(u-1) - \beta]^2 - \beta^2 \quad \therefore \beta = \frac{2}{\rho^2-1}$$

$$\Rightarrow \quad \Big[u - 1 - \frac{2}{\rho^2-1} \Big] + v^2 = \left(\frac{2\rho}{\rho^2-1} \right)^2$$

$$\Rightarrow \quad \Big[u - \frac{\rho^2+1}{\rho^2-1} \Big] + v^2 = \left(\frac{2\rho}{\rho^2-1} \right)^2$$

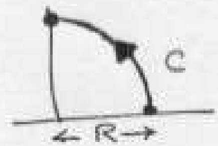
which is a circle with center $\begin{pmatrix} \frac{\rho^2+1}{\rho^2-1} \\ 0 \end{pmatrix}$ and radius $\frac{2\rho}{\rho^2-1}$. (observe the similarity with 3.35)

Thus



PR. 4.7

a.) $\int_C \frac{s}{|s|}\, ds$



On the path C, $s = Re^{j\theta}$, $ds = jRe^{j\theta}\, d\theta$.

$$\therefore \int_C \frac{s}{|s|}\, ds = \int_C \frac{Re^{j\theta}}{R} jRe^{j\theta} d\theta = \int_0^{\pi/2} jR\, e^{j2\theta} d\theta$$

$$= \frac{jR}{2j} e^{j2\theta}\Big|_0^{\pi/2} = \frac{jR}{2j}\left[e^{j\pi} - e^{j0}\right] = \boxed{-R}$$

NOTE: $\frac{s}{|s|}$ is continuous everywhere except 0 and nowhere differentiable.

1. $s = re^{j\theta}$, $\frac{s}{|s|} = e^{j\theta}$ $\therefore \lim_{|s|\to 0} \frac{s}{|s|}$ does not exist.

2. $$\left| \frac{s_0+\Delta s}{|s_0+\Delta s|} - \frac{s_0}{|s_0|} \right| = \left| \frac{s_0}{|s_0|}\left(\frac{1}{\left|1+\frac{\Delta s}{s_0}\right|} - 1\right) + \frac{\Delta s}{|s_0+\Delta s|} \right| \le$$

$$\le \left|\frac{s_0}{|s_0|}\right| \left| \frac{1 - \left|1+\frac{\Delta s}{s_0}\right|}{\left|1+\frac{\Delta s}{s_0}\right|} \right| + \left| \frac{\Delta s}{|s_0+\Delta s|} \right|$$

$$\le \frac{\left|1 - \left|1+\frac{\Delta s}{s_0}\right|\right|}{|1| - \left|\frac{\Delta s}{s_0}\right|} + \frac{|\Delta s|}{|s_0| - |\Delta s|}$$

$$\left|\frac{\Delta s}{s_0}\right| = \left|1 - 1 + \frac{\Delta s}{s_0}\right| \ge \left| |1| - \left|1+\frac{\Delta s}{s_0}\right| \right| \qquad \le 2\,\frac{\left|\frac{\Delta s}{s_0}\right|}{1 - \left|\frac{\Delta s}{s_0}\right|} < \epsilon \quad \text{for } |\Delta s| < \delta$$

$$\delta = \min\left(\frac{|s_0|}{2}, \frac{\epsilon |s_0|}{4}\right)$$

3. Differential quotient: $\underset{s\neq 0}{}$ $\dfrac{f(s+\Delta s)-f(s)}{\Delta s} =$

$$= \left(\frac{s+\Delta s}{|s+\Delta s|} - \frac{s}{|s|} \right) / \Delta s$$

$$= \frac{1}{|s+\Delta s|} + \frac{s}{|s|} \left(\frac{1 - \left|1 + \frac{\Delta s}{s}\right|}{\left|1+\frac{\Delta s}{s}\right| \Delta s} \right)$$

Take limits as 1) $\Delta s \to 0$, Δs REAL
2) $\Delta s \to 0$ Δs imaginary.

1. $$\lim_{\substack{\Delta s \to 0 \\ \Delta s \text{ real}}} D.Q. = \frac{1}{|s|} + \frac{s}{|s|} \lim_{\substack{\Delta s \to 0 \\ \Delta s = \overline{\Delta s}}} \frac{1 - \sqrt{\left(1+\frac{\Delta s}{s}\right)\left(1+\frac{\overline{\Delta s}}{\bar{s}}\right)}}{\Delta s}$$

$$= \frac{1}{|s|} + \frac{s}{|s|} \lim_{\Delta s \to 0} \frac{1 - 1 - \frac{\Delta s}{2}\left(\frac{1}{s} + \frac{1}{\bar{s}}\right)}{\Delta s}$$

$$= \frac{1}{|s|} + \frac{s}{|s|^2} \cdot \frac{1}{2}\left(\bar{s}+s\right)$$

2. $$\lim_{\substack{\Delta s \to 0 \\ \Delta s \text{ imag.}}} D.Q = \frac{1}{|s|} + \frac{s}{|s|} \lim_{\substack{\Delta s \to 0 \\ \Delta s = -\overline{\Delta s}}} \frac{1 - \sqrt{\left(1+\frac{\Delta s}{s}\right)\left(1+\frac{\overline{\Delta s}}{\bar{s}}\right)}}{\Delta s}$$

$$= \frac{1}{|s|} + \frac{s}{|s|} \lim_{\Delta s \to 0} \frac{1 - 1 - \frac{\Delta s}{2}\left(\frac{1}{s} - \frac{1}{\bar{s}}\right)}{\Delta s}$$

$$= \frac{1}{|s|} + \frac{s}{|s|^2} \cdot \frac{1}{2}\left(\bar{s} - s\right)$$

for the two limits to be equal $(s \neq 0)$ we must have

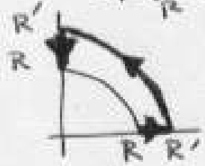
$$\bar{s} + s = \bar{s} - s \quad \text{or} \quad s = -s \Rightarrow s = 0 \text{ (contradiction)}$$

Hence, $\frac{s}{|s|}$ is nowhere differentiable.

This implies that the above integral cannot be evaluated by integrating on a different path and using Cauchy's thm.

Furthermore, the integral along a different path with the same end points will, in general, yield a different value.

E.g. compute $\int_{C'} \frac{s}{|s|}$ where $C' =$  $(= -2R)$

$\int_{C''} \frac{s}{|s|}$ $\quad C'' =$  $(= R' - 2R)$

etc.

b) $$\int_C \frac{\mathrm{Re}\, s}{|s|} ds = \int_C \frac{R\cos\theta}{R} Rje^{j\theta} d\theta$$

$$= Rj \int_0^{\pi/2} \frac{e^{j\theta} + e^{-j\theta}}{2} e^{j\theta} d\theta$$

$$= \frac{Rj}{2}\left[\int_0^{\pi/2} e^{2j\theta} d\theta + \int_0^{\pi/2} d\theta\right]$$

$$= \boxed{-\frac{R}{2} + \frac{R\pi j}{4}}$$

c) $$\int_C \frac{\mathrm{Im}\, s}{|s|} ds = \int_C \frac{R\sin\theta}{R} \cdot Rje^{j\theta} d\theta$$

$$= \frac{R}{2} \int_0^{\pi/2} (e^{j\theta} - e^{-j\theta}) e^{j\theta} d\theta$$

$$= \boxed{-\frac{R}{2j} - \frac{R\pi}{4}}$$

Note As expected $\int_C \frac{s}{|s|} ds = -R = \int_C \frac{\mathrm{Re}\, s}{|s|} ds + j\int_C \frac{\mathrm{Im}\, s}{|s|} ds = -\frac{R}{2} + \frac{R\pi j}{4} + j\left(-\frac{R}{2j} - \frac{R\pi}{4}\right)$

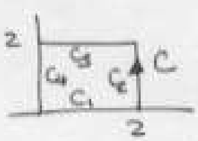
PR 4.11 Analyticity was used to approximate $f(s)$ by a constant + linear function within an arbitrarily small residual, over any of the —sufficiently small— triangles i.e.

$$f(s) = f(s_0) + f'(s_0)(s-s_0) + m(s)(s-s_0).$$

where $m(s) < \epsilon$. Since the theorem has been shown to hold for constant and linear functions, all that is left is to account for the contribution of the residual term $m(s)(s-s_0)$.

Note that since we cannot determine a priori the point of expansion (s_0), other than $s_0 \in R$, analyticity is assumed throughout R.

PR. 4.12 s^2 is everywhere analytic, so it must be $\int_C s^2 = 0$



Since the path C is easier described in cartesian coordinates we choose the same for the evaluation of the integral, i.e., $s = \sigma + j\omega$. Thus,

$$\begin{aligned}\int_C s^2 ds = &\int_{C_1} (\sigma^2 - \omega^2 + j2\sigma\omega)\, d(\sigma + j\omega) \quad ; C_1: d\omega = 0,\ \omega = 0\\ &+ \int_{C_2} (\sigma^2 - \omega^2 + j2\sigma\omega)\, d(\sigma + j\omega) \quad ; C_2: d\sigma = 0,\ \sigma = 2\\ &+ \int_{C_3} (\sigma^2 - \omega^2 + j2\sigma\omega)\, d(\sigma + j\omega) \quad ; C_3: d\omega = 0,\ \omega = 2\\ &+ \int_{C_4} (\sigma^2 - \omega^2 + j2\sigma\omega)\, d(\sigma + j\omega) \quad ; C_4: d\sigma = 0,\ \sigma = 0\end{aligned}$$

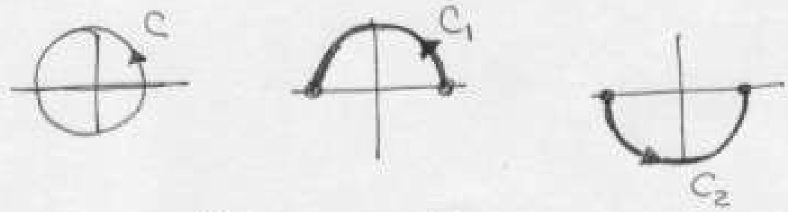
$$= \int_0^2 \sigma^2 d\sigma + \int_0^2 (4-\omega^2+j4\omega)\, j\, d\omega + \int_2^0 (\sigma^2-4+j4\sigma)\, d\sigma + \int_2^0 -\omega^2 j\, d\omega$$

$$= \left(\tfrac{8}{3}\right) + \left(8j - \tfrac{8j}{3} - 8\right) + \left(-\tfrac{8}{3} + 8 - 8j\right) + \left(j\tfrac{8}{3}\right)$$

$$= 0.$$

QED

Pr 4.13



$$\int_C f(s)\, ds = \int_{C_1} f(s)\, ds + \int_{C_2} f(s)\, ds$$

Observe that when $s \in C_2$ then $s' = -s \in C_1$, $ds' = -ds$

Thus, changing the variable of integration in the second integral,

$$\int_{C_2} f(s)\, ds = \int_{C_1} f(-s')(-ds') = \int_{C_1} -f(-s')\, ds'.$$

$$\Rightarrow \int_C f(s)\, ds = \int_{C_1} [f(s) - f(-s)]\, ds \qquad \text{QED}$$

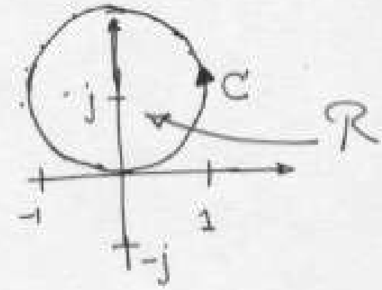
Next, if $f(s)$ is analytic on and inside C, $\int_C f(s)\,ds = 0$

$$\Rightarrow \int_{C_1} [f(s) - f(-s)]\, ds = 0$$

If, in addition $f(s)$ is odd $\left(f(s) = -f(-s)\right)$ then

$$0 = \int_{C_1} [f(s) - f(-s)]\, ds = 2\int_{C_1} f(s)\, ds \Rightarrow \int_{C_1} f(s)\, ds = 0 \qquad \text{QED}$$

Pr 4.16



a). $\int_C \frac{s^2-1}{s^2+1}\,ds$; $\frac{s^2-1}{s^2+1}$ has poles at $+j, -j$.

$\therefore$ $\frac{s^2-1}{s^2+1}$ is analytic in the (doubly connected) region

$R - R'$; $R' =$ 

$$\Rightarrow \int_C \frac{s^2-1}{s^2+1}\,ds = \int_{C'} \frac{s^2-1}{s^2+1}\,ds \quad \text{where } s \in C' \Leftrightarrow s = \rho e^{j\theta} + j, \quad 0 < \rho \ll 1.$$

$$= \int_{C'} \frac{s^2+1-2}{s^2+1}\,ds$$

$$= \underbrace{\cancel{\int_{C'} ds}}_{0} - 2\int_{C'} \frac{1}{s^2+1}\,ds$$

$$= -2\int_{C'} \frac{1}{(s+j)(s-j)}\bigg|_{s=\rho e^{j\theta}+j} j\rho e^{j\theta}\,d\theta$$

$$= -2\int_0^{2\pi} \frac{1}{2j+\rho e^{j\theta}} \cdot \frac{j\rho e^{j\theta}}{\rho e^{j\theta}}\,d\theta$$

$$= -2j\int_0^{2\pi} \frac{1}{2j+\rho e^{j\theta}}\,d\theta$$

$$= -2j \int_0^{2\pi} \left[\frac{1}{2j} + f(\rho,\theta) \right] d\theta$$

where $f(\rho,\theta) = -\dfrac{\rho e^{j\theta}}{2j + \rho e^{j\theta}}$; $|f(\rho,\theta)| \leq \dfrac{\rho}{2-\rho} < \rho$ for $\rho < 1$.

$$\therefore \int_C \frac{s^2-1}{s^2+1} ds = -2j \cdot \frac{1}{2j} \int_0^{2\pi} d\theta + \bar{f}(\rho,\theta) \; ; \; \bar{f}(\rho,\theta) < \rho \cdot 2\pi$$

$$= -2\pi + \bar{f}(\rho,\theta)$$

$$\Rightarrow \left| \int_C \frac{s^2-1}{s^2+1} ds + 2\pi \right| = |\bar{f}(\rho,\theta)| < 2\pi\rho$$

and, since ρ is an arbitrarily small positive constant,

$$\boxed{\int_C \frac{s^2-1}{s^2+1} ds = -2\pi}$$

b), c), d) The path C does not contain any singularities of the respective functions.

((b) poles @ $-1, -1$, (c) poles @ $+1, -1$, (d) poles @ $-j, -j$)

Hence, since in all cases the functions are rational, they are all analytic in and on C, implying that the respective integrals are all zero.

EEE 550 K. TSAKALIS HW#3 SOLUTIONS

PR 5.2

a). $\sum n s^n$: ROOT TEST $\rightarrow \overline{\lim} \sqrt[n]{n} = 1$

$\Rightarrow$ convergence for $|s| < 1$

b) $\sum 2^n s^n$: ROOT TEST $\rightarrow \overline{\lim} \sqrt[n]{2^n} = 2$

$\rightarrow$ convergence for $|s| < 1/2$

c) $\sum n! s^n$: RATIO TEST $\rightarrow \lim \frac{(n+1)!}{n!} = \lim n+1 = \infty$

$\Rightarrow$ no region of convergence can be concluded

In fact the series converges for $|s| = r$ only if $r = 0$.

Prf. Let $r > 0$ and pick $s = r$ s.t. $|s| = r$. Then

RATIO TEST $\rightarrow \lim \frac{(n+1)!}{n!} \cdot r = \lim (n+1) r = \infty$ DIVERGENT SERIES

d) $\sum n^{\sqrt{n}} s^n$: ROOT TEST $\rightarrow \overline{\lim} \sqrt[n]{n^{\sqrt{n}}} = \overline{\lim}\, n^{\frac{\sqrt{n}}{n}}$

$= \overline{\lim} \left(e^{\log n}\right)^{\frac{1}{\sqrt{n}}} = \overline{\lim}\, e^{\frac{\log n}{\sqrt{n}}}$

$= 1 \quad \left(\frac{\log n}{\sqrt{n}} \rightarrow 0 \text{ as } n \to \infty \right)$

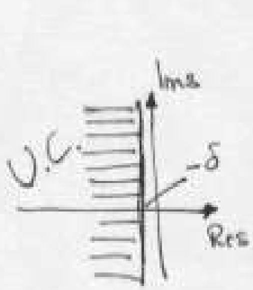
$\therefore$ Convergence for $|s| < 1$.

PR 5.4

a) $\sum e^{ns}$ $\quad s = \sigma + j\omega$

$\sum e^{ns} = \sum e^{n\sigma} \cdot e^{jn\omega}$; convergence for $\sigma < 0$ or $\mathrm{Re}\, s < 0$.

$|e^{ns}| < M_n(\sigma)$; $M_n(\sigma) = e^{n\sigma}$



$\therefore |e^{ns}| < M_n = \lambda^n$ for $(0 < \lambda < 1)$ for all s s.t. $e^{\mathrm{Re}\, s} \le \lambda$.

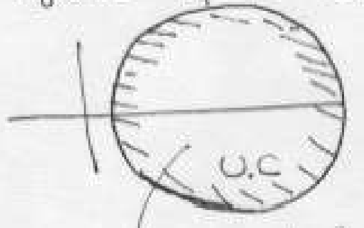
$\Rightarrow$ UNIF. CONV. for s : $\mathrm{Re}\, s \le -\delta < 0$

b) $\sum \left(\frac{s-1}{s+1}\right)^n = \sum (w)^n$ convergence for $|w| < 1$

unif. conv. for $|w| \le \rho < 1$

(e.g. via an M-Test)

From eqn 3.35 & Fig. 3.20 b, the corresponding regions of s are $\operatorname{Re} s > 0$ (convergence)



$$\left(\operatorname{Re} s - \frac{1+\rho^2}{1-\rho^2}\right)^2 + \operatorname{Im}^2 s \le \left(\frac{2\rho}{1-\rho^2}\right)^2$$

(unif. conv.)

CENT: $\frac{1+\rho^2}{1-\rho^2}$

RAD: $\frac{2\rho}{1-\rho^2}$

PR. 5.5 Let $A = \sum_0^\infty a_k$, $B = \sum_0^\infty b_k$. Then $\forall \epsilon > 0$

$\exists N_1(\epsilon), N_2(\epsilon)$: $\left|\sum_0^{N_1'} a_k - A\right| < \epsilon/2$, $\left|\sum_0^{N_2'} b_k - B\right| < \epsilon/2$

$N_1' > N_1$ $\quad N_2' > N_2$

Define $N(\epsilon) = \max(N_1, N_2)$. Then for any $N' > N$

$$\left|\sum_0^{N'} c_k - A - B\right| = \left|\left(\sum_0^{N'} a_k - A\right) + \left(\sum_0^{N'} b_k - B\right)\right|$$

$$\le \left|\sum_0^{N'} a_k - A\right| + \left|\sum_0^{N'} b_k - B\right| < \epsilon$$

Hence, $\sum_0^N c_k$ converges as $N \to \infty$ and the limit is $\sum_0^\infty c_k = A + B$

or $\sum_0^\infty c_k = \sum_0^\infty a_k + \sum_0^\infty b_k$. QED

PR. 5.6 1. $c_k = 0 = 1 + (-1)$; $a_k = 1$, $b_k = -1$

$\sum c_k = 0$ (converges) ; $\sum a_k$, $\sum b_k$ diverge.

2. $c_k = \frac{(-1)^{n+1}}{n+1}$ $\sum c_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ converges

$\sum a_k = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} \cdots$, $\sum b_k = -(0 + \frac{1}{2} + 0 + \frac{1}{4} + \ldots)$ diverge.

EEE 550 HW #4 SOLUTIONS

K. TSAKALIS

PR 5.17 For the Taylor series expansion coefficients.

$$a_k = \frac{1}{2\pi j}\int_C \frac{f(z)}{(z-s_1)^{k+1}}\,dz = \frac{f^{k}(s_1)}{k!}$$

Taking $C = \{ s \,|\, |s-s_1| = r \}$ with $r <$ radius of convergence, C is entirely contained in the region of convergence of the Taylor series, implying that $f(z)$ is analytic inside and on C. Hence, with C being a closed and bounded subset of $\mathbb{C}$ ($\Rightarrow$ compact) $\sup_{z\in C} |f(z)| = \max_{z\in C} |f(z)| \triangleq M_r$ is well defined & finite. In turn,

$$|a_k| = \frac{1}{2\pi}\left|\int_C \frac{f(z)}{(z-s_1)^{k+1}}\,dz\right| \qquad \Rightarrow \quad |z-s_1| = r,\ z = re^{j\theta}+s_1,\quad dz = rje^{j\theta}d\theta,\quad \theta\in[0,2\pi]$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(z)|}{r^{k+1}}\cdot r\,d\theta$$

$$\le \frac{M_r}{r^k}$$

QED

Pr 5.19 Let $f(s) = \sum_{k=-\infty}^{\infty} a_k s^k$ be a Laurent series which converges (absolutely) in $R_1 < |s| < R_2$. We want to find a region $\mathcal{R}$ s.t. $\forall \epsilon > 0 \ \exists N(\epsilon) : n > N \Rightarrow$

$$\left| \sum_{k=-N}^{N} a_k s^k - f(s) \right| < \epsilon \quad \forall s \in \mathcal{R}.$$

Expressing f as

$$f(s) = \underbrace{\sum_{k=0}^{\infty} a_k s^k}_{f_a(s)} + \underbrace{\sum_{k=1}^{\infty} a_{-k} s^{-k}}_{f_b(s)}$$

f_a converges (absolutely) in $|s| < R_2$ & f_b converges (abs) in $|s| > R_1$.

Let $s \in \{ s \mid |s| \le R_2' - \delta \}$ where $\delta > 0$ and $R_2' < R_2$.

Then $\hat{f}_a(s) \triangleq \sum_{k=0}^{\infty} a_k s^k \left(\frac{R_2'}{R_2' - \delta} \right)^k \triangleq \sum_{k=0}^{\infty} a_k z^k$ converges abs.

(since $|z| = |s| \frac{R_2'}{R_2' - \delta} \le R_2'$.)

This implies that $a_k s^k \left(\frac{R_2'}{R_2' - \delta} \right)^k$ is bounded, unif. in s,

i.e., $\exists M : \left| a_k s^k \left(\frac{R_2'}{R_2' - \delta} \right)^k \right| < M \quad \forall |s| \le R_2' - \delta$.

Pf: Since $\sum_{k=0}^{\infty} a_k z^k$ is absolutely convergent, $|a_k z^k| \to 0$ as $k \to \infty$. Therefore, for N large enough, $|a_k z^k| < \epsilon, \ \forall k > N$. Hence,

$$|a_k z^k| < \max \{ |a_0|, |a_1 z|, \cdots |a_N z^N|, \epsilon \} \triangleq M.$$

Hence, for $|s| \le R_2' - \delta$, $|a_k s^k| < M \cdot \left[\frac{R_2' - \delta}{R_2'} \right]^k$

and $\sum_{k=0}^{\infty} a_k s^k$ converges uniformly in $\{ |s| \le R_2' - \delta \}$

by the Weierstrass' M-test. And since both δ & R_2' were arbitrary (other than $\delta > 0$, $R_2' < R_2$), the region of unif. convergence can be written as $|s| \le R_2 - \delta'$.

Similarly, $f_b(s)$ converges uniformly in a region $|s| \geq R_1 + \delta'$. (Here, there is no need to make a distinction between the two δ's)

Thus $\forall \epsilon > 0 \quad \exists \; N_1(\epsilon), N_2(\epsilon)$ s.t.

$$\left.\begin{aligned} \left| \sum_{k=0}^{N_1} a_k s^k - f_a(s) \right| &< \epsilon/2 \\ \left| \sum_{k=1}^{N_2} a_{-k} s^{-k} - f_b(s) \right| &< \epsilon/2 \end{aligned}\right\} \begin{array}{l} \text{for all} \\ R_1+\delta' \leq |s| \leq R_2 - \delta' \end{array}$$

(N_1, N_2 may (will) depend on δ' but are independent of s)

and, taking $N = \max(N_1, N_2)$

$$\left| \sum_{k=-N}^{N} a_k s^k - f(s) \right| < \epsilon \,, \quad \text{for all } s : \; R_1+\delta' \leq |s| < R_2 - \delta'$$

Therefore, for a Laurent series converging in $R_1 < |s| < R_2$, the region of uniform convergence is of the form

$$R_1 + \delta \leq |s| \leq R_2 - \delta$$

where δ is a positive constant.

Obviously, δ must be s.t. $\delta < \frac{R_2 - R_1}{2}$ so that the two inequalities can be simultaneously satisfied.

Pr 5.20

$$f(s) = \frac{s^2+s+3}{s^3+2s^2+s+2} = \frac{s^2+s+3}{(s+2)(s+j)(s-j)}$$

$$= \frac{1}{s+2} + \frac{-(\frac{1}{2}j)}{s-j} + \frac{(\frac{1}{2}j)}{s+j}$$

a). Taylor Expansion about $s=0$

Expanding each term of the PFE about 0

$$\frac{1}{s+2} = \frac{\frac{1}{2}}{1-(-\frac{1}{2}s)} = \frac{1}{2}\left(1 - \frac{1}{2}s + \frac{1}{4}s^2 - \frac{1}{8}s^3 + \frac{1}{16}s^4 + \dots\right); \ |s|<2$$

$$\frac{-\frac{1}{2}j}{s-j} = \frac{(-\frac{1}{j})(-\frac{1}{2}j)}{1-(\frac{s}{j})} = \frac{1}{2}\left(1 + \frac{s}{j} + \frac{s^2}{j^2} + \frac{s^3}{j^3} + \frac{s^4}{j^4} + \dots\right); \ |s|<1$$

$$\frac{\frac{1}{2}j}{s+j} = \frac{(\frac{1}{j})(\frac{1}{2}j)}{1-(-\frac{s}{j})} = \frac{1}{2}\left(1 - \frac{s}{j} + \frac{s^2}{j^2} - \frac{s^3}{j^3} + \frac{s^4}{j^4} + \dots\right); \ |s|<1$$

NOTE: COEFF's are complex conjugates

Collecting terms:

$$f(s) = \frac{3}{2} - \frac{1}{4}s - \frac{7}{8}s^2 - \frac{1}{16}s^3 + \frac{33}{32}s^4 + \dots \qquad |s|<1$$

The same result could be obtained by long division, but without the insight of how the coefficients can be expressed as a function of K.

b) Taylor series about $s=-1$.

$$f(s) = \frac{1}{(s+1)+1} + \frac{-\frac{1}{2}j}{(s+1)-(1+j)} + \frac{\frac{1}{2}j}{(s+1)-(1-j)}$$

Working as before,

$$\frac{1}{1-(-(s+1))} = 1 - (s+1) + (s+1)^2 - (s+1)^3 + \dots \; ; \; |s+1|<1$$

$$\frac{(-\frac{1}{2}j)\left(\frac{-1}{1+j}\right)}{1-\left(\frac{s+1}{1+j}\right)} = \frac{j}{2(1+j)}\left[1 + \left(\frac{s+1}{1+j}\right) + \left(\frac{s+1}{1+j}\right)^2 + \left(\frac{s+1}{1+j}\right)^3 + \dots\right] ; |s+1|<\sqrt{2}$$

$$\frac{\frac{1}{2}j}{(s+1)-(1-j)} = \dots .$$ (complex conjugate coeff of the above ; f is real analytic!)

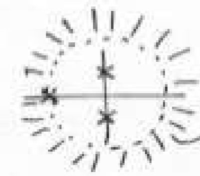
Collecting terms

$$f(s) = \Big[1 + \underbrace{\mathrm{Re}\left(\frac{j}{1+j}\right)}_{1/2}\Big] + \Big[-1 + \underbrace{\mathrm{Re}\left(\frac{j}{(1+j)^2}\right)}_{1/2}\Big](s+1) + \Big[1 + \overbrace{\mathrm{Re}\left(\frac{j}{(1+j)^3}\right)}^{1/4}\Big](s+1)^2 + \dots \; ; |s+1|<1$$

$$= \frac{3}{2} - \frac{1}{2}(s+1) + \frac{5}{4}(s+1)^2 + \dots \; ; \; |s+1|<1$$

c) Laurent expansions about $s = 0$

①  $1 < |s| < 2$

②  $|s| > 2.$

①. For the first term of the PFE, use its Taylor expansion from part (a).

$$\frac{1}{s+2} = \frac{1}{2} - \frac{1}{4}s + \frac{1}{8}s^2 - \frac{1}{16}s^3 + \cdots \quad ; \ |s| < 2$$

For the second and third terms find the Laurent expansion:

$$\frac{-\frac{1}{2}j}{s-j} = \frac{-\left(\frac{1}{2}\right)\left(\frac{j}{s}\right)}{1 - \frac{j}{s}}$$

$$= -\frac{j}{2s}\left(1 + \frac{j}{s} + \frac{j^2}{s^2} + \frac{j^3}{s^3} + \cdots\right) \ ; \ \left|\frac{j}{s}\right| < 1$$

$$\Leftrightarrow |s| > 1$$

$$= \left(-\frac{j}{2}\right)\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2} + \frac{j}{2}\cdot\frac{1}{s^3} - \frac{1}{2}\frac{1}{s^4} + \cdots \ ; \ |s| > 1$$

$$\frac{\frac{1}{2}j}{s+j} = \overline{\left(-\frac{j}{2}\right)}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2} + \overline{\left(\frac{j}{2}\right)}\cdot\frac{1}{s^3} - \frac{1}{2}\frac{1}{s^4} + \cdots \ ; \ |s| > 1$$

$$f(s) = \frac{1}{2} - \frac{1}{4}s + \frac{1}{8}s^2 - \frac{1}{16}s^3 + \cdots$$

$$+ \frac{1}{s^2} - \frac{1}{s^4} + \cdots \qquad ; \quad 1 < |s| < 2$$

②. Laurent exp. about $s=0$, $|s|>2$.

We only need the expansion of the first term of the PFE :

$$\frac{1/s}{1-(-2/s)} = \frac{1}{s}\left(1 - \frac{2}{s} + \frac{4}{s^2} - \frac{8}{s^3} + \dots\right) \quad |s|>2$$

$$= \frac{1}{s} - \frac{2}{s^2} + \frac{4}{s^3} - \frac{8}{s^4} + \dots$$

Using the results of part C.1 for the other two terms,

$$f(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{4}{s^3} - \frac{9}{s^4} + \dots \quad ; |s|>2$$

---

d). Laurent expansions about each singular pt.

①. About $s=-2$; $0<|s+2|<\sqrt{5}$

$\frac{1}{s+2}$ DONE !

$$\frac{-\frac{1}{2}j}{s-j} = \frac{-\frac{1}{2}j}{(s+2)-(2+j)} = \frac{\frac{j}{2(2+j)}}{1-\left(\frac{s+2}{2+j}\right)}$$

$$= \frac{j}{2(2+j)}\left[1 + \left(\frac{s+2}{2+j}\right) + \left(\frac{s+2}{2+j}\right)^2 + \dots\right] \quad ; |s+2|<\sqrt{5}$$

$$= \frac{j}{2(2+j)} + \left(\frac{j}{2(2+j)^2}\right)(s+2) + \frac{j}{2(2+j)^3}(s+2)^2 + \dots$$

$$\frac{\frac{1}{2}j}{s+j} = \left(\overline{\frac{j}{2(2+j)}}\right) + \left(\overline{\frac{j}{2(2+j)^2}}\right)(s+2) + \left(\overline{\frac{j}{2(2+j)^3}}\right)(s+2)^2 + \dots \quad ; |s+2|<\sqrt{5}$$

$$f(s) = \frac{1}{s+2} + 0.2 + 0.16\,(s+2) + 0.088\,(s+2)^2 + \cdots ;$$

$$0 < |s+2| < \sqrt{5}.$$

---

② About $s = j$; $0 < |s-j| < 2$

$-\dfrac{\frac{1}{2}j}{s-j}$: DONE !

$$\frac{\frac{1}{2}j}{s+j} = \frac{\frac{1}{2}j}{(s-j)+2j} = \frac{1/4}{1-\left(-\frac{s-j}{2j}\right)}$$

$$= \frac{1}{4}\left[1 - \frac{s-j}{2j} + \frac{(s-j)^2}{(2j)^2} + \cdots\right] ; |s-j| < 2$$

$$\frac{1}{s+2} = \frac{1}{(s-j)+(2+j)} = \frac{1/(2+j)}{1-\left(-\frac{s-j}{2+j}\right)}$$

$$= \frac{1}{2+j}\left[1 - \frac{s-j}{2+j} + \frac{(s-j)^2}{(2+j)^2} + \cdots\right] ; |s-j| < \sqrt{5}.$$

$$f(s) = \frac{\left(-\frac{1}{2}j\right)}{s-j} + \left(\frac{1}{4} + \frac{1}{2+j}\right) + \left(\frac{-1}{8j} + \frac{-1}{(2+j)^2}\right)(s-j) + \left(\frac{-1}{16} + \frac{1}{(2+j)^3}\right)(s-j)^2$$

$$+ \cdots ; 0 < |s-j| < 2$$

$$= (-0.5j)\cdot\frac{1}{s-j} + (0.65 - 0.2j) + (-0.12 + 0.285j)(s-j) + (-0.0465 - 0.088j)$$

$$(s-j)^2 + \cdots ; 0 < |s-j| < 2.$$

③ about $s=-j$; $0<|s+j|<2$

Complex conjugate coeff. of d-2

$$f(s) = 0.5j\frac{1}{s+j} + (0.65+0.2j) + (-0.12-0.285j)(s+j)$$
$$+ (-0.0465+0.088j)(s+j)^2 + \dots \quad ; \quad 0<|s+j|<2.$$

Pr 5.25

a) $f(s) = \dfrac{e^s}{s}$: singular at 0, ∞

$s\,f(s) = e^s$ has a Taylor exp. about 0

$\Rightarrow$ 0 is a pole of order 1

To identify the type of singularity at ∞ consider

$f(\frac{1}{s}) = \dfrac{e^{1/s}}{1/s} = s\,e^{1/s}$ around 0.

It's Laurent expansion is $s + 1 + \frac{1}{2s} + \dots + \frac{1}{(k+1)!\,s^k} + \dots$

whose principal part has an infinite number of terms.

Hence, $f(\frac{1}{s})$ has an essential singularity of the first kind at 0

$\Rightarrow$ f(s) has an essential singularity of the 1st kind at ∞

b). $f(s) = e^{1/s}$; f(s) has an essential singularity of the 1st kind at 0 (as in part a)

c). $f(s) = e^{-1/s}$; as in (a) and (b) f(s) has an essential singularity of the 1st kind at 0 d). $f(s) = \frac{s^2}{(s^2+1)^2} = \frac{s^2}{(s+j)^2(s-j)^2}$

$f(s)$ has poles of order 2 at $s=j$ and $s=-j$

($(s+j)^2 f(s)$ has a Taylor expansion about $-j$ etc.)

e). $f(s) = \sin s$ has a singularity at ∞ only. To identify its type consider $\sin(\frac{1}{s})$ at 0. $f(\frac{1}{s})$ has an isolated singularity at 0 and its Laurent expansion is $\frac{1}{s} - \frac{1}{3!}\frac{1}{s^3} + \frac{1}{5!}\cdot\frac{1}{s^5} + \cdots$ having an infinite number of terms.

$\therefore$ $f(s)$ has an essential singularity of the first kind at ∞

f) $f(s) = \frac{1}{\sinh s} = \frac{2}{e^s - e^{-s}}$ Differentiable everywhere except when $e^s - e^{-s} = 0$

$\therefore$ singularities @ $s = k\pi j$ $K = 0, \pm 1, \pm 2.$

$\therefore$ $f(s)$ has poles of order 1 at $s = k\pi j$

($(s-k\pi j) f(s)$ has a Taylor exp. about $k\pi j$)

To check the point @ ∞, consider $f(\frac{1}{s})$ around 0.

$f(\frac{1}{s})$ has singularities @ $\frac{1}{s} = k\pi j$ or $s = 1/k\pi$

and therefore 0 is not an isolated singularity

$\therefore$ $f(s)$ has an essential singularity of the 2nd kind at ∞

PR. 5.27

a) $\dfrac{\sin s}{s^3}$ pole of order 2 at $s=0$

$$\text{Res} = \frac{d}{ds}\left(\frac{s^2 \sin s}{s^3}\right)\Bigg|_{s=0} = \frac{s\cos s - \sin s}{s^2}\Bigg|_0 =$$

$$= \frac{-s\sin s + \cos s - \cos s}{2s}\Bigg|_0 = -\frac{\sin s}{2}\Bigg|_0 = \underline{\underline{0}}$$

b) $\dfrac{1}{s^2(s-1)}$ pole of order **1** at $s = 1$

$$\text{RES} = \frac{s-1}{s^2(s-1)}\Bigg|_{s=1} = \frac{1}{s^2}\Bigg|_1 = \underline{\underline{1}}$$

c) $\dfrac{1-e^{-2s}}{s^4}$ pole of order 3 at $s=0$

$$\text{RES} = \frac{1}{2!}\frac{d^2}{ds^2}\left(\frac{s^3(1-e^{-2s})}{s^4}\right)\Bigg|_{s=0} = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1-e^{-2s}}{s}\right)\Bigg|_0 .$$

$$= \frac{1}{2}\frac{d}{ds}\left(\frac{2e^{-2s}s - (1-e^{-2s})}{s^2}\right)\Bigg|_0 = \frac{1}{2}\left(\frac{(2e^{-2s} - 4se^{-2s} - 2e^{-2s})s^2 - [2e^{-2s}s - (1-e^{-2s})]2s}{s^4}\right)\Bigg|_0$$

$$= \frac{1}{2}\left(\frac{-4s^2e^{-2s} - 4se^{-2s} + 2 - 2e^{-2s}}{s^3}\right)\Bigg|_0.$$

$= \ldots$ SUCCESSIVE L'HOSPITAL RULES

$\frac{d}{ds}$ NUM $\mapsto 8e^{-2s}s^2$, $\frac{d}{ds}$ DEN $\rightarrow 3s^2 \mapsto \frac{1}{2}\cdot\frac{8}{3}e^{-2s}\Big|_0 = \underline{\underline{\frac{4}{3}}}$

A simpler alternative, in this case, is to consider a Taylor expansion of e^{-2s} and take a "sufficient" number of terms (at any stage).

e.g.
$$\frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1-e^{-2s}}{s}\right)\Big|_0 = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1-\left[1-2s+\frac{4s^2}{2}-\frac{8s^3}{6}+HOT\right]}{s}\right)\Big|_0$$

$$= \frac{1}{2}\frac{d^2}{ds^2}\left(2-2s+\frac{4}{3}s^2+H.O.T\right)\Big|_0 = \frac{1}{2}\frac{d}{ds}\left(-2+\frac{8}{3}s+H.O.T\right)$$

$$= \frac{1}{2}\left(\frac{8}{3}+H.O.T\right)\Big|_0 = \boxed{\frac{4}{3}}$$

or,
$$\frac{1}{2}\;\frac{-4s^2e^{-2s}-4se^{-2s}+2-2e^{-2s}}{s^3}\Big|_0 =$$

$$= \frac{1}{2}\;\frac{-4s^2(1-2s)-4s\left(1-2s+\frac{4s^2}{2}\right)+2-2\left(1-2s+\frac{4s^2}{2}-\frac{8s^3}{6}\right)}{s^3}\Big|_0$$
$+HOT.$

$$= \frac{1}{2}\left(\frac{8}{3}+HOT\right)\Big|_0 = \boxed{\frac{4}{3}}$$

---

d) $\dfrac{\cos s}{\sin^2 s}$ pole of order 2 at $s=\pi$

$$RES = \frac{d}{ds}\left[\frac{(s-\pi)^2\cos s}{\sin^2 s}\right]\Bigg|_\pi = \left[\frac{2(s-\pi)\cos s\,\sin s-(s-\pi)^2\sin^2 s-2\cos^2 s\,(s-\pi)^2}{\sin^3 s}\right]\Bigg|_\pi$$

$$= \underbrace{\frac{s-\pi}{\sin s}\Big|_\pi}_{-1}\cdot\underbrace{\cos s\Big|_\pi}_{-1}\cdot\frac{2\sin s-2\cos s\,(s-\pi)}{\sin^2 s}\Bigg|_\pi+\underbrace{\frac{(s-\pi)^2\sin^2 s}{\sin^3 s}\Bigg|_\pi}_{0}$$

$$= 2 \cdot \frac{\sin s - \cos s\,(s-\pi)}{\sin^2 s}\Bigg|_{\pi}$$

$$= 2 \; \frac{\cos s - \cos s + \sin s\,(s-\pi)}{2 \sin s \cos s}\Bigg|_{\pi}$$

$$= \frac{s-\pi}{\cos s}\Bigg|_{\pi} = \boxed{0}$$

---

e) $\dfrac{e^{2s}}{(s-1)^2}$ pole of order 2 at $s=1$

$$\text{RES} = \frac{d}{ds}\left(\frac{(s-1)^2 e^{2s}}{(s-1)^2}\right)\Bigg|_{s=1} = \frac{d}{ds}\left(e^{2s}\right)\Big|_{1} = 2e^{2s}\Big|_{1} = \boxed{2e^2}$$

---

f) $\dfrac{\tan s}{(1-e^{-s})^2}$ pole of order 1 at 0

$$\text{RES} = \frac{s \tan s}{(1-e^{-s})^2}\Bigg|_{s=0} = \frac{s \sin s}{\cos s\,(1-e^{-s})^2}\Bigg|_{0} = \frac{1}{\cos s}\Big|_{0} \frac{s \sin s}{(1-e^{-s})^2}\Bigg|_{0}$$

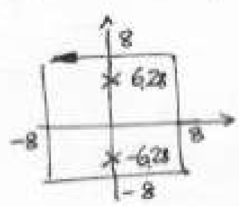
$$\overset{\text{L'HOSPITAL}}{=} \frac{\sin s + s\cos s}{2(1-e^{-s})\underbrace{e^{-s}}_{\to 1}}\Bigg|_{0} \overset{\text{L'HOSPITAL}}{=} \frac{1}{2}\left(\frac{\overbrace{\cos s + \cos s}^{\to 2} - \overbrace{s \sin s}^{\to 0}}{\underbrace{e^{-s}}_{\to 1}}\right)\Bigg|_{0} =$$

$$= \boxed{1}$$

PR 5.29

$$\int_C \underbrace{\frac{s}{1-e^s}}_{f(s)}\, ds = 2\pi j \sum_k \underbrace{d_k}_{\text{RESIDUES IN } C}$$

Singularities of $f(s)$: poles of order 1 at $s = 2k\pi j$, $k = \pm 1, \pm 2$ $(k \neq 0)$

a) C: 

Evaluate residues at $2\pi j$, $-2\pi j$:

1). $2\pi j$: $\text{RES} = \left.\frac{(s-2\pi j)s}{1-e^s}\right|_{s=2\pi j} = \left.\frac{2s-2\pi j}{-e^s}\right|_{2\pi j} = \underline{-2\pi j}$

2) $-2\pi j$: $\text{Res} = \left.\frac{(s+2\pi j)s}{1-e^s}\right|_{-2\pi j} = \left.\frac{2s+2\pi j}{-e^s}\right|_{-2\pi j} = \underline{+2\pi j}$

$$\to I_0 = \int_C \frac{s}{1-e^s}\, ds = 2\pi j\,(2\pi j - 2\pi j) = \underline{0}$$

b)

$C_1 =$ 

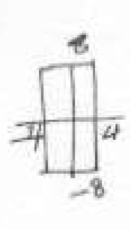
$$I_1 = \int_{C_1} \frac{s}{1-e^s}\, ds = I_0 - 2\pi j \underbrace{\sum_{k'} d_{k'}}$$

RES at singularities in C but not in C_1

$$= I_0 - 2\pi j\,(\underbrace{2\pi j - 2\pi j}_{0})$$

$$= I_0 = \underline{0}$$

(Also note that C_1 encloses no singularities of $f(s)$)

c) $C_2 =$ 

$$I_2 = \int_{C_2} \frac{s}{1-e^s}\, ds = I_0 - 2\pi j \sum_{k''} d_{k''} \quad (\text{As in part b})$$

$$= I_0 = \underline{0}$$

Note: All integrals are zero but their expressions are different!

EEE 550 K. TSAKALIS HW # 5 SOLUTIONS

8.16a). $\int_{-\infty}^{\infty} \frac{\sin y}{1+y^2}\, dy = \int_{-\infty}^{\infty} \frac{e^{jy} - e^{-jy}}{(1+y^2)\, 2j}\, dy$

$$= \int_{-\infty}^{\infty} \frac{e^{jy}}{2(1+y^2)} \frac{dy}{j} - \int_{-\infty}^{\infty} \frac{e^{-jy}}{2(1+y^2)} \frac{dy}{j}$$

$$\overset{z=jy}{=} \int_{C} \frac{e^{z}}{2(1-z^2)}\, dz - \int_{C} \frac{e^{-z}}{2(1-z^2)}\, dz$$

NOTE: Both integrals exist since for $y \in \mathbb{R}$ $|e^{jy}| \le 1$ and $\frac{1}{1+y^2}$ is abs. integrable.

Letting $H(z) = \frac{1}{2(1-z^2)}$, $H(Re^{j\theta}) \underset{R\to\infty}{\to} 0$ unif. in θ

From Jordan's thm. $\int_{C_2} \frac{e^{z}}{2(1-z^2)}\, dz \underset{R\to\infty}{\to} 0$

$\int_{C_1} \frac{e^{-z}}{2(1-z^2)}\, dz \underset{R\to\infty}{\to} 0$

Thus $\int_{C+C_2} \frac{e^{z}}{2(1-z^2)}\, dz = 2\pi j \sum_{C+C_2} \text{RES} = 2\pi j \frac{e^{-1}}{2(1-(-1))}$

$$= \frac{\pi j}{2} e^{-1}$$

$\int_{C+C_1} \frac{e^{-z}}{2(1-z^2)}\, dz = -2\pi j \sum_{C+C_1} \text{RES} = -2\pi j \frac{e^{-1}}{2(1+1)}(-1) \leftarrow \frac{z-1}{1-z}$

$$= \frac{\pi j}{2} e^{-1}$$

$\therefore \int_{-\infty}^{\infty} \frac{\sin y}{1+y^2}\, dy = \int_{C} \frac{e^{z}}{2(1-z^2)}\, dz - \int_{C} \frac{e^{-z}}{2(1-z^2)}\, dz = \left(\frac{\pi j e^{-1}}{2} - \int_{C_2} \cdots\right) - \left(\frac{\pi j e^{-1}}{2} - \int_{C_1} \cdots\right)$

Taking Limits as $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{\sin y}{1+y^2}\, dy = 0.$$

as expected since the integrand is odd.

In fact, since we have established that the integrals exist, we need not distinguish between the integral and its principal value, i.e. $\int_{-\infty}^{\infty} \frac{\sin y}{1+y^2}\, dy = PV \int_{-\infty}^{\infty} \frac{\sin y}{1+y^2}\, dy$.

b) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{jy}}{1+y^2}\, dy$ (as before, the integral exists and will be equal to its principal value)

$$= \int_C \frac{e^z}{1-z^2} \frac{dz}{j} \quad ; \text{ in the limit as } R \to \infty$$

$$= \int_{C+C_2} \frac{e^z}{1-z^2} \frac{dz}{j} \quad ; \text{ Jordan Thm., } H(z) = \frac{1}{1-z^2}$$

C_2 

$$= 2\pi j \sum_{C+C_2} \text{RES} \left(\frac{e^z}{j(1-z^2)} \right)$$

$$= 2\pi \sum_{C+C_2} \text{RES} \left(\frac{e^z}{1-z^2} \right) = 2\pi \frac{e^{-1}}{1+1} = \boxed{\pi e^{-1}}$$

c) $\displaystyle\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^2}$ (Integral exists, equal to its PV.)

$$= \frac{1}{j} \int_C \frac{dz}{(1-z^2)^2} \quad ; \text{ in the limit as } R \to \infty$$

$$= \frac{1}{j} \int_{C+C_2} \frac{dz}{(1-z^2)^2} \quad ; \quad H(z) = \frac{1}{(1-z^2)^2} \; ; \; RH(Re^{j\theta}) \xrightarrow[R\to\infty]{} 0$$

unif. in θ.

$$= \frac{1}{j} \left(2\pi j \sum_{C+C_2} \text{RES} \left(\frac{1}{(1-z)^2(1+z)^2} \right) \right)$$

$$= 2\pi \frac{d}{dz} \left(\frac{1}{(1-z)^2} \right) \Big|_{z=-1} = 2\pi \left(\frac{-2}{(z-1)^3} \right) \Big|_{z=-1} = 2\pi \left(\frac{1}{4} \right)$$

$$\boxed{= \frac{\pi}{2}}$$

PR 8.26 Suppose $F(s)$ has only finite plane singularities and $RF(Re^{j\theta}) \to 0$ as $R \to \infty$ unif. in θ.

Then, with C being the circle $Re^{j\theta}$

$$\left| \int_C F(s)\, ds \right| \le \int_0^{2\pi} |F(Re^{j\theta})| \, R |je^{j\theta}| \, d\theta$$

$$\le R \cdot \max_\theta |F(Re^{j\theta})| \cdot 2\pi$$

$\therefore$ as $R \to \infty$, $\left| \int_C F(s)\, ds \right| \to 0$.

But $\int_C F(s) ds = 2\pi j \sum_C \text{RES}(F)$

since all singularities of F are in the finite plane, $\exists\, R_0$ s.t. $R > R_0 \Rightarrow C$ encloses all singularities of F.

$$\Rightarrow \quad \int_C F(s)\,ds = 2\pi j \underbrace{\sum \text{Res}(F)}_{\text{at all singularities}} \qquad ; \quad R > R_0$$

Letting $R \to \infty$, the left hand-side approaches zero while the right hand side is a constant. Hence

$$\sum \text{Res}\, F = 0$$

All that remains is to verify that the given class of functions satisfies the above assumptions:

1. Since F is Rational, it has only a finite ~~place~~ number of singularities. Hence all its singularities, except a possible singularity at ∞, can be enclosed by a circle $R_0 e^{j\theta}$ for some R_0.

2. $\deg \text{num}(F) \leq \deg \text{den}(F) - 2$

   $\therefore$ F has no singularity at ∞

   and, in fact, $R\,F(Re^{j\theta}) \to 0$ as $R \to \infty$ unif. in θ. To show this, consider a general form for $F(s)$

$$F(s) = K \cdot \frac{s^m + b_1 s^{m-1} + \cdots}{s^n + a_1 s^{n-1} + \cdots} \qquad ; \quad n - m \geq 2$$

$$\therefore \quad |F(s)|\Big|_{s = Re^{j\theta}} \leq |K| \cdot \frac{R^m + |b_1| R^{m-1} + \cdots}{R^n - |a_1| R^{n-1} - \cdots} \qquad ;$$

R large enough for the denom. to make sense

$$\therefore \left|F(s)\right|\Big|_{s=Re^{j\theta}} \le \frac{B\cdot R^{m}}{R^{n} - AR^{n-1}} \quad ; \quad B = |\kappa|(m+1)\cdot\max(1, |b_1|, \ldots) \quad A = n\cdot\max(|a_1|, \ldots)$$

$$\le \frac{B}{R(R-A)} \quad ; \quad R \ge 1.$$

$$\le \frac{2B}{R^2} \quad ; \quad R > 2A \ , \ R > 1$$

Note: $R > 2A \Rightarrow R - A = \left(\frac{R}{2} - A\right) + \frac{R}{2} > \frac{R}{2}$

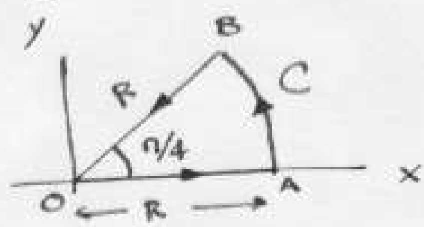
Hence, $R\left|F(Re^{j\theta})\right| \le \frac{2B}{R}$; $R > \max(2A, 1)$

and as $R \to \infty$ $\lim R\left|F(Re^{j\theta})\right| = 0.$ QED

OTHER PROBLEMS.

- SHOW that $\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

Let C :



By Cauchy's Thm,

$$\int_C e^{jz^2} dz = 0$$

or $$\int_{OA} e^{jz^2} dz + \int_{AB} e^{jz^2} dz + \int_{BO} e^{jz^2} dz = 0$$

Now, on OA $z = x$

AB $z = Re^{j\theta}$ $\theta \in [0, \pi/4]$

BO $z = re^{j\pi/4}$ $r \in [R \to 0]$

$$\therefore \int_0^R e^{jx^2} dx + \int_0^{\pi/4} e^{jR^2 e^{j2\theta}} + \int_R^0 e^{jr^2 e^{j\pi/2}} e^{j\pi/4} dr = 0$$

$$\therefore \int_0^R (\cos x^2 + j \sin x^2)\, dx = e^{j\pi/4} \int_0^R e^{-r^2} dr - \int_0^{\pi/4} e^{jR^2(\cos 2\theta + j \sin 2\theta)} jRe^{j\theta} d\theta.$$

For the second integral in the RHS,

$$\left| \int \right| \leq \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R \, d\theta$$

$$= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin\phi} d\phi$$

$$\leq \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \phi/\pi} d\phi \qquad ; \sin\phi \geq \frac{2\phi}{\pi} \text{ for } \phi \in [0, \pi/2]$$

$$= \frac{\pi}{4R} \left(1 - e^{-R^2}\right)$$

$$\to 0 \text{ as } R \to \infty.$$

The first integral in the RHS becomes, as $R \to \infty$

$$e^{\pi j/4} \underbrace{\int_0^\infty e^{-r^2} dr}_{\text{erf}} = \frac{(1+j)}{\sqrt{2}} \int_0^\infty e^{-r^2} dr.$$

To compute the last integral, note that

$$\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta \qquad ; \text{ polar coordinates } \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$$

$$= \int_0^{\pi/2} d\theta \left(\frac{1}{2}\underbrace{\int_0^\infty e^{-r^2}dr^2}_{1/2}\right) = \frac{\pi}{4}$$

$$\therefore \quad \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Substituting,

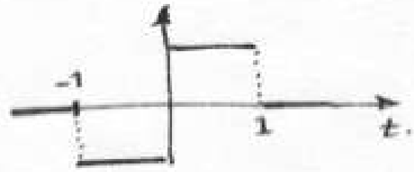
$$\int_0^\infty (\cos x^2 + j\sin x^2)\,dx = (1+j)\cdot\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

and equating real and imaginary parts

$$\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Pr 9.2

$f(t)$ 

1. f is PW continuous & has bounded support ($f \neq 0$ on a bounded interval) $\rightsquigarrow$ abs. Integrable.

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1}^{0} (-1)\cdot e^{-j\omega t}\, dt + \int_{0}^{1} (1)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{j\omega}\left(2 - e^{j\omega} - e^{-j\omega}\right)$$; NOTE: $F(0) = 0 = \int_{-\infty}^{\infty} f(1)\, dt$

Inversion integral

$$\hat{f}(t) = \frac{1}{2\pi}\, PV \int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\, PV \int_{-\infty}^{\infty} \frac{2 - e^{j\omega} - e^{-j\omega}}{j\omega}\, e^{j\omega t}\, d\omega.$$

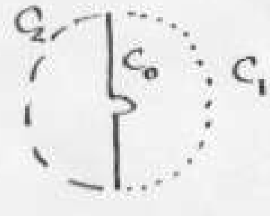
$$= \frac{1}{2\pi}\cdot\frac{1}{j}\cdot\int_{C} \frac{2 - e^{z} - e^{-z}}{z}\, e^{zt}\, dz$$

$$= \frac{1}{2\pi}\cdot\frac{1}{j}\int_{C_0} \frac{2 - e^{z} - e^{-z}}{z}\, e^{zt}\, dz$$; $C_0 =$

Integrals are equal as the radius of the indentation approaches zero since $\lim F(j\omega) = 0$ at 0 — (finite)

$$= \frac{1}{2\pi j}\left\{ \int_{C_0} \frac{2}{z} e^{zt}\, dz - \int_{C_0} \frac{1}{z} e^{z(t+1)}\, dz - \int_{C_0} \frac{1}{z} e^{z(t-1)}\, dz \right\}.$$

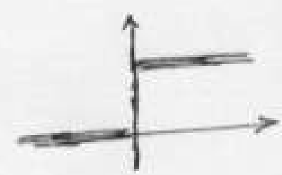
$\Rightarrow$ Use Jordan's lemma with $H(z) = \frac{1}{z}$



- $$\frac{1}{2\pi j} \int_{C_0} \frac{2}{z} e^{zt}\, dz = \begin{cases} \frac{1}{2\pi j} \int_{C_0+C_2} \frac{2}{z} e^{zt}\, dz & ; t>0 \\ \frac{1}{2\pi j} \int_{C_0+C_1} \frac{2}{z} e^{zt}\, dz & ; t<0 \end{cases}$$

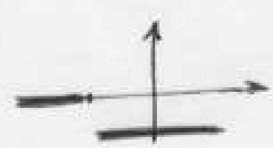
$$= \sum_{C_0+C_2} \text{RES} = 2 \ ; t>0$$

$$-\sum_{C_0+C_1} \text{RES} = 0 \ ; t<0$$

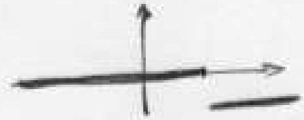


- Similarly for the rest

$$-\frac{1}{2\pi j} \int_{C_0} \frac{1}{z} e^{z(t+1)}\, dz = \begin{cases} -1 & ; t>-1 \\ 0 & ; t<-1 \end{cases}$$



$$-\frac{1}{2\pi j} \int_{C_0} \frac{1}{z} e^{z(t-1)}\, dz = \begin{cases} -1 & ; t>1 \\ 0 & ; t<1 \end{cases}$$



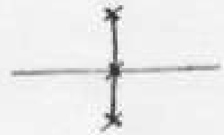
Adding the pieces,

$$\hat{f}(t) = \begin{cases} 0 & |t| > 1 \\ -1 & -1<t<0 \\ 1 & 0<t<1 \end{cases} = f(t).$$

If needed, a different approach should be used to compute $\hat{f}(t)$ at the discontinuity points (e.g. as in section 9.2).

PR 10_45

a) $F(s) = \dfrac{6s^2+s-1}{s^3+s} = \dfrac{6s^2+s-1}{s(s^2+1)}$

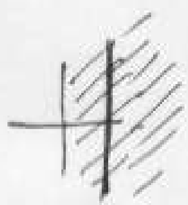


Possible ROC's : ① $\operatorname{Re}s > 0$
② $\operatorname{Re}s < 0$.

Inversion formula: $f(t) = \dfrac{1}{2\pi j}\displaystyle\int_{Br} F(s)\, e^{st}\, ds$

$$= \begin{cases} -\sum \text{RES}_{\text{RIGHT}} & ; t<0 \\ +\sum \text{RES}_{\text{LEFT}} & ; t>0 \end{cases}$$

① ROC



Br

$$f(t) = \begin{cases} 0 & ; t<0 \\ \sum_{\text{LEFT}} \text{RES} & ; t>0 \end{cases}$$

RESIDUES: $\text{RES}_0 = \left.\dfrac{6s^2+s-1}{s^2+1} e^{st}\right|_{s=0} = -1$

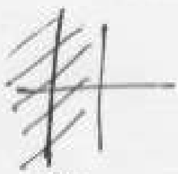
$$\text{RES}_j = \left.\frac{(6s^2+s-1)e^{st}}{s(s+j)}\right|_{s=j} = \frac{(-6+j-1)\,e^{jt}}{j\,(2j)}$$

$$= -\frac{-7+j}{2} e^{jt} = \frac{7-j}{2} e^{jt}$$

$$\text{RES}_{-j} = \frac{7+j}{2} e^{-jt}$$

$$\therefore \; f(t) = \begin{cases} 0 \; ; & t<0 \\ -1 + \underbrace{\frac{7-j}{2} e^{jt} + \frac{7+j}{2} e^{-jt}}_{\mathrm{Re}\left((7-j)e^{jt}\right)} \; ; & t>0 \end{cases}$$

② ROC



Br.

$$f(t) = \begin{cases} -\sum_{\text{RIGHT}} \text{Res} \; ; & t<0 \\ 0 \; ; & t>0 \end{cases}$$

$$= \begin{cases} 1 - \mathrm{Re}\left((7-j)e^{jt}\right) \; ; & t<0 \\ 0 \; ; & t>0 \end{cases}$$

---

b) $$F(s) = \frac{4s^2+16s+16}{s^3+5s^2+9s+5} = \frac{4s^2+16s+16}{(s+1)(s+2+j)(s+2-j)}$$

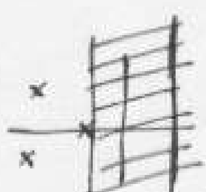
POSSIBLE ROC's

① $\mathrm{Re}\,s > -1$

② $\mathrm{Re}\,s > -2$, $\mathrm{Re}\,s < -1$

③ $\mathrm{Re}\,s < -2$.

① ROC



Br

$$f(t) = \begin{cases} 0 \; ; & t<0 \\ \sum_{\text{LEFT}} \text{Res} \; ; & t>0 \end{cases}$$

$$\mathrm{Res}_{-1} = \frac{4s^2+16s+16}{(s+2+j)(s+2-j)} e^{st}\Big|_{s=-1} = \frac{4-16+16}{(1+j)(1-j)} e^{-t}$$

$$= 2e^{-t}$$

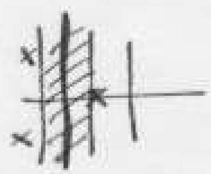
$$\mathrm{RES}_{-2+j} = \frac{4s^2+16s+16}{(s+1)(s+2+j)} e^{st}\Big|_{s=-2+j} = (1-j)\, e^{(-2+j)t}$$

$$\mathrm{RES}_{-2-j} = (1+j)\, e^{(-2-j)t}$$

$$\Rightarrow f(t) = \begin{cases} 0 & ; t<0 \\ 2e^{-t} + 2\,\mathrm{Re}\{(1-j)\, e^{(-2+j)t}\} \end{cases}$$

---

② ROC :

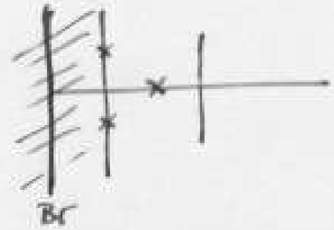


Br

$$f(t) = \begin{cases} -\sum\limits_{\text{RIGHT}} \mathrm{RES} & ; t<0 \\ \sum\limits_{\text{LEFT}} \mathrm{RES} & ; t>0 \end{cases}$$

$$= \begin{cases} -2e^{-t} & ; t<0 \\ 2\,\mathrm{Re}\{(1-j)\, e^{(-2+j)t}\} & ; t>0 \end{cases}$$

③ ROC :



$$f(t) = \begin{cases} -\sum_{\text{RIGHT}} \text{RES} & ; t<0 \\ 0 & ; t>0 \end{cases}$$

$$= \begin{cases} -2e^{-t} - 2\,\text{Re}\left\{(1-j)\, e^{(-2+j)t}\right\} & ; t<0 \\ 0 & ; t>0 \end{cases}$$

---

c) $$F(s) = \frac{3s^3+8s^2+9s+4}{s^4+5s^3+9s^2+7s+2} = \frac{3s^3+8s^2+9s+4}{(s+1)^3(s+2)}$$

$$= \frac{(3s^2+5s+4)(s+1)}{(s+1)^3(s+2)}$$

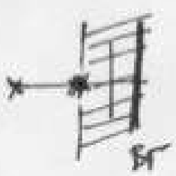
pole of order 2

POSSIBLE ROC's : ① $\text{Re}\,s > -1$

② $-2 < \text{Re}\,s < -1$

③ $\text{Re}\,s < -2$

① $$f(t) = \begin{cases} 0 & ; t<0 \\ \sum_{\text{LEFT}} \text{Res} & t>0 \end{cases}$$



$$\text{RES}_{-2} = \left.\frac{3s^2+5s+4}{(s+1)^2}\, e^{st}\right|_{s=-2} = 6e^{-2t}$$

$$\text{RES}_{-1} = \left.\frac{d}{ds}\, \frac{3s^2+5s+4}{s+2}\, e^{st}\right|_{s=-1}$$

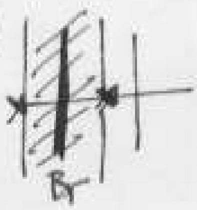
$$= \frac{(6s+5)\,e^{st} + (3s^2+5s+4)\,te^{-st}}{s+2} - \frac{(3s^2+5s+4)\,e^{st}}{(s+2)^2}\Bigg|_{s=-1}$$

$$= -e^{-t} + 2te^{-t} - 2e^{-t}$$

$$= -3e^{-t} + 2te^{-t}$$

$$\Rightarrow \quad f(t) = \begin{cases} 0 & ;\ t<0 \\ 6e^{-2t} - 3e^{-t} + 2te^{-t} & ;\ t>0 \end{cases}$$

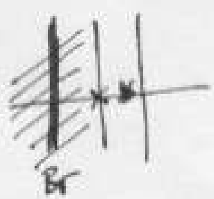
②



$$f(t) = \begin{cases} -\sum\limits_{\text{RIGHT}} \text{RES} & ;\ t<0 \\ +\sum\limits_{\text{LEFT}} \text{RES} & ;\ t>0 \end{cases}$$

$$= \begin{cases} 3e^{-t} - 2te^{-t} & ;\ t<0 \\ 6e^{-2t} & ;\ t>0 \end{cases}$$

③



$$f(t) = \begin{cases} -\sum\limits_{\text{RIGHT}} \text{RES} & ;\ t<0 \\ 0 & ;\ t>0 \end{cases}$$

$$= \begin{cases} 3e^{-t} - 2te^{-t} - 6e^{-2t} & ;\ t<0 \\ 0 & \ t>0. \end{cases}$$