# 2 HW Solutions

## 2.1 Problem 2.1

*Find extremals for the functional* $J = \int_0^2 [2x^2(t) + \dot{x}^2(t)]dt$ *with* $x(0) = 0$, $x(2) = 5$.

The Euler-Lagrange equation is $\frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial V}{\partial \dot{x}} = 0$, from which $\ddot{x} - 2x = 0$. Hence, $x(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$ and $c_i$ are such that the B.C. are satisfied: $c_1 + c_2 = 0$ and $c_1 e^{\sqrt{2}2} + c_2 e^{-\sqrt{2}2} = 5$

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## 2.2 Problem 2.6

*Find the optimal control for* $J = \frac{1}{2}\int_0^5 u^2(t)dt$ *subject to* $\ddot{x} = u$, $x(0) = 2$, $\dot{x}(0) = 2$, $x(5) = 0$, $\dot{x}(5) = 0$.

Rewrite the system in state-space as

$$\dot{x} = Ax + Bu, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad x(5) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

The Hamiltonian is $\mathcal{H}(x, u, \lambda) = \frac{1}{2}u^2 + \lambda^\top(Ax + Bu)$. The Euler-Lagrange conditions for an extremal become

$$\begin{aligned} u &= \arg\min_u \mathcal{H} = -B^\top \lambda \\ \dot{\lambda} &= -\frac{\partial \mathcal{H}}{\partial x} = -A^\top \lambda \\ \dot{x} &= \frac{\partial \mathcal{H}}{\partial \lambda} = Ax + Bu \end{aligned}$$

Hence,

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \underbrace{\begin{pmatrix} A & -BB^\top \\ 0 & -A^\top \end{pmatrix}}_{H} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad \begin{pmatrix} x \\ \lambda \end{pmatrix}(0) = \begin{pmatrix} 2 \\ 2 \\ \star \\ \star \end{pmatrix}, \quad \begin{pmatrix} x \\ \lambda \end{pmatrix}(5) = \begin{pmatrix} 0 \\ 0 \\ \star \\ \star \end{pmatrix}$$

where, $H$ is the Hamiltonian matrix and $\star$ denotes unspecified terms. This is a TPBVP and its solution can be obtained through the state transition matrix (STM) that describes solutions of the state-costate equations. That is, letting $w = (x; \lambda)$, we have $\dot{w} = Hw$ and, therefore,

$$w(t) = \Phi(t, t_0)w(t_0), \quad t_0 = 0, \quad \Phi(t, \tau) = e^{H(t-\tau)}$$

Suppressing the $t_0$-notation, we compute

$$\Phi(5) = e^{H5} = \begin{pmatrix} 1 & 5 & 20.833 & -12.5 \\ 0 & 1 & 12.5 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 \end{pmatrix}$$

Next, partition $\Phi(5)$ according to the state-costate vectors and solve for the unknown initial conditions $\lambda(0)$. We have that $x(5) = \Phi_{11}(5)x(0) + \Phi_{12}(5)\lambda(0)$, from which

$$\lambda(0) = \Phi_{12}^{-1}(5)x(5) - \Phi_{12}^{-1}(5)\Phi_{11}(5)x(0) = -\begin{pmatrix} -0.096 & -0.24 \\ -0.24 & -0.8 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.672 \\ 2.08 \end{pmatrix}$$

For the numerical computation of the solution, the MATLAB code is

```
t=[0:.01:5]';
H=[A -B*B';0*B*B' -A'];
p=expm(H*5);p11 = p(1:2,1:2);p12 = p(1:2,3:4);
```

```
S=ss(A,B,eye(2,2),zeros(2,1));
x0=[2;2;-(p12\p11)*[2;2]]
Sa=ss(-A',B,eye(2,2),zeros(2,1));
L=lsim(Sa,0*t,t,x0(3:4));plot(t,L),pause
u=-L*B; % lsim output: L rows are the costate transpose at each time instant
x=lsim(S,u,t,x0(1:2));
plot(t,x,t,u)
```

Not surprisingly, (the costate system is the homogeneous solution of a double integrator) the optimal input is an affine function of time (straight line).

Alternatively, one can take advantage of the specific form of equations to find the explicit solution: $\lambda_1(t) = c_3$, $\lambda_2(t) = -c_3 t + c_4$, $u(t) = -\lambda_2(t)$, etc., $x_2(t) = \cdots$, $x_1(t) = \cdots$. Then, the constants $c_i$ are found by solving the four-equation, four-unknown linear system.

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## 2.3 Problem 2.7

*Find the optimal control for $J = \int_0^{t_f} [x^2(t) + u^2(t)]dt$ subject to $\dot{x} = -x + u$, $x(0) = 5$, $x(t_f) = 0$, and with $t_f$ free.*

The Hamiltonian is $\mathcal{H}(x, u, \lambda) = x^2 + u^2 + \lambda(-x + u)$. The Euler-Lagrange conditions for an extremal become

$$\begin{aligned} u &= \arg\min_u \mathcal{H} = -\frac{1}{2}\lambda \\ \dot{\lambda} &= -\frac{\partial \mathcal{H}}{\partial x} = -2x + \lambda \\ \dot{x} &= \frac{\partial \mathcal{H}}{\partial \lambda} = -x + u \end{aligned}$$

Hence,

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}}_{H} \begin{pmatrix} x \\ \lambda \end{pmatrix}, \quad \begin{pmatrix} x \\ \lambda \end{pmatrix}(0) = \begin{pmatrix} 5 \\ \star \end{pmatrix}, \quad \begin{pmatrix} x \\ \lambda \end{pmatrix}(t_f) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where, $H$ is the Hamiltonian matrix and $\star$ denotes unspecified terms.

Notice that, due to the free final time, the additional boundary condition is $\mathcal{H}^*(t_f) = 0$ Substituting the optimal input in the Hamiltonian, $\mathcal{H}_* = x^2 - \lambda x - \frac{1}{4}\lambda^2$. Evaluating at $t_f$, $x(t_f) = 0$, so $\mathcal{H}_* = -\frac{1}{4}\lambda(t_f)^2 = 0$.

Thus, the extremals satisfy the state-costate equations: $w = (x; \lambda)$, $\dot{w} = Hw$ and, therefore, $w(t) = \Phi(t, t_0)w(t_0)$. The state transition matrix $\Phi$ is always nonsingular and an equivalent expression of the solution is $w(t) = \Phi^{-1}(t_f, t)w(t_f) = \Phi(t, t_f)w(t_f)$.

The problem now is that $w(t_f) = 0$ making the entire extremal trajectory equal to zero. The reason for this is that we can obtain smaller values of the cost $J$ as we allow $t_f \to \infty$, hence, the minimum does not exist. Let us analyze this situation further and gain some insight.

- Consider the fixed final time problem $\min J_T = \int_0^T V(x, u)\,dt$, with $x(T) = 0$. It follows that $J_T$ is a decreasing function of $T$. (Suppose not; then $J_{T1} < J_{T2}$ for some $T1 < T2$. Since $x = 0$ is an equilibrium point, and $V(x, u) \geq 0$, we can define a policy in the T2 interval as the T1-policy for $t < T1$ and zero afterwards. Since $x(T1) = 0$, then the rest of the solution is $u = 0$, $x = 0$ for which $V(0, 0) = 0$, so the corresponding cost is $J'_{T2} = J_{T1} < J_{T2}$. This contradicts the optimality of the original solution in the T2-interval.) Hence, unless there are multiple minima, the minimum does not occur in finite time.

- Alternatively, consider the fixed final time problem $\min J_T = (x^\top F x)(T) + \int_0^T x^\top P x + u^\top R u\,dt$ with free final state. Its solution is $u = -R^{-1}B^\top P x$, where $-\dot{P} = A^\top P + PA + Q - PBR^{-1}B^\top P$, $P(T) = F$.

The minimum $J_T = x^\top(0)P(0)x(0)$. Letting $F = \rho I$, $\rho \to \infty$ we approximate the problem of the fixed final state $x(T) = 0$.

The optimal cost depends both on T and F through the solution of the Riccati. Rewriting the Riccati as a forward-in-time ODE, we get (using the same symbol $P$) $\dot{P} = A^\top P + PA + Q - PBR^{-1}B^\top P$, $P(0) = F$, for which the optimal solution is $J_T = x(0)^\top P(T)x(0)$ and $u(t) = -R^{-1}B^\top P(T-t)x(t)$. We can explore the behavior of $J_T$ by solving this ODE, for different initial conditions $F$. In our case, $\dot{P} = -2P + 1 - P^2$, $P(0) = F$, for which the optimal solution is easily computed in SIMULINK. The steady state value is the positive root of $0 = -2P + 1 - P^2$, i.e., $\sqrt{2} - 1$. For $F > \sqrt{2} - 1$ the solution decays monotonically to the steady-state value. This implies that for any given $F$ (large enough), the minimum $J_T$ with respect to $T$ will occur at infinity.

Thus, we conclude that the optimal control is the limiting solution for the infinite interval

$$\begin{aligned} t_f &= \infty \\ P &= \sqrt{2} - 1 \\ u &= -Px \end{aligned}$$

and $\min J = x(0)^\top Px(0) = 25(\sqrt{2} - 1) = 10.355$.

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## 2.4 Problem 2.11

*Find the optimal control for* $J = \frac{1}{2}(x(t_f) - x_f)^\top F(x(t_f) - x_f) + \frac{1}{2}\int_0^{t_f} x^\top Qx + u^\top Ru\,dt$ *subject to*

$$\dot{x} = Ax + Bu, \quad A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad R = 4$$

*for four different cases.*

**Case 1:** $t_f = 2$, $x(2) = [4; 6]$. Here, the terminal cost $(F, x_f)$ is irrelevant. The Hamiltonian is $\mathcal{H}(x, u, \lambda) = \frac{1}{2}(x^\top Qx + u^\top Ru) + \lambda^\top(Ax + Bu)$. The Euler-Lagrange conditions for an extremal become

$$\begin{aligned} u &= \arg\min_u \mathcal{H} = -R^{-1}B^\top\lambda \\ \dot{\lambda} &= -\frac{\partial \mathcal{H}}{\partial x} = -A^\top\lambda - Qx \\ \dot{x} &= \frac{\partial \mathcal{H}}{\partial \lambda} = Ax + Bu \end{aligned}$$

Hence, letting $w = (x; \lambda)$, we have $\dot{w} = Hw$, where

$$H = \begin{pmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{pmatrix}, \quad w(0) = \begin{pmatrix} 1 \\ 2 \\ \star \\ \star \end{pmatrix}, \quad w(2) = \begin{pmatrix} 4 \\ 6 \\ \star \\ \star \end{pmatrix}$$

Thw solution of this TPBVP can be obtained through the state transition matrix (STM) that describes solutions of the state-costate equations $w(t) = e^{Ht}w(0)$. From this, we compute the initial condition for the costates as $\lambda(0) = \Phi_{12}^{-1}[x(2) - \Phi_{11}x(0)] = [4.8958;\ 2.7143]$. The computation is similar to the procedure listed in Problem 2.6:

```
t=[0:.01:2]';
Q=[1 0;0 2];R=4;
H=[A -B*inv(R)*B';-Q -A'];
p=expm(H*2);p11 = p(1:2,1:2);p12 = p(1:2,3:4);
S=ss(A,B,eye(2,2),zeros(2,1));
```

```
x0=[1;2;p12\([4;6]-p11*[1;2])]
% x-lambda are coupled; produce u by solving the entire state-costate eqn.
Sh=ss(H,[B;B],[0*B' -inv(R)*B'],0);
u=lsim(Sh,0*t,t,x0);
x=lsim(S,u,t,x0(1:2)); % verify solution
plot(t,x,t,u)
```

**Case 2:** $t_f$ free, $x(t_f) = [4;6]$. Here, the terminal cost $(F, x_f)$ is again irrelevant. With the same setup as in Case 1, we now have the additional condition $\mathcal{H}(t_f) = 0 = \frac{1}{2}x^\top Qx - \frac{1}{2}\lambda^\top BR^{-1}B^\top\lambda + \lambda^\top Ax|_{t=t_f} = 44 - 1.125\lambda_2^2(t_f) - 8\lambda_2(t_f) + 6\lambda_1(t_f)$.

One procedure to solve this problem is to begin by fixing $t_f$, evaluate the optimal solution, check the above transversality condition at $t_f$, and then adjust $t_f$ and repeat. Alternatively, we could also simply solve the the fixed-time optimal control problem $J_T$ and minimize $J_T$ with respect to $T$. Performing the evaluation, we find that the transversality condition is satsified at several times: 0.759, 2.098, 5.2495,... These are points where the necessary conditions for extremals are satisfied. For identifying the minimum, we need to evaluate the cost. Plotting $J_T$ as a function of $T$ we indeed find several local minima and maxima with the one at $t_f = 0.759$ yielding the least cost, $J_{0.759} = 31.53$.

The following function is useful in performing the necessary computations:

```
function [ER,J]=p211(tf);
t=[0:.01:tf]';
A=[0 1;-2 0];B=[0;3];
xf=[4;6];x0=[1;2];
Q=[1 0;0 2];R=4;
H=[A -B*inv(R)*B';-Q -A'];
p=expm(H*tf);p11 = p(1:2,1:2);p12 = p(1:2,3:4);
xx0=[x0;p12\(xf-p11*x0)];
% x-lambda are coupled; produce u by solving the state-costate eqn.
Sh=ss(H,[B;B],[eye(4,4);0*B' -inv(R)*B'],zeros(5,1));
U=lsim(Sh,0*t,t,xx0);
u=U(:,5);Lf=U(length(U),3:4)';
ER=xf'*Q*xf/2-Lf'*B*inv(R)*B'*Lf/2+Lf'*A*xf;
S=ss(A,B,eye(2,2),zeros(2,1));
x=lsim(S,u,t,x0(1:2));
%plot(t,x,t,u);
J=(norm(x(:,1))^2+norm(x(:,2))^2*2+norm(u)^2*4)/2*.01;
```

**Case 3:** $t_f = 2$, $x(t_f) = [\star;6]$, $F = 0$. Here, $\left[\frac{\partial S}{\partial x} - \lambda\right]^\top_{t=t_f}\delta x_f = 0$ implying that $\lambda_1(t_f) = 0$. Again, we express the state-costate equations as $\dot{w} = Hw$, where

$$H = \begin{pmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{pmatrix}, \quad w(0) = \begin{pmatrix} 1 \\ 2 \\ \star \\ \star \end{pmatrix}, \quad w(2) = \begin{pmatrix} \star \\ 6 \\ 0 \\ \star \end{pmatrix}$$

The rest of the problem is completely analogous to Case 1, except that a different partition of the state transition matrix (rows 2-3) is needed to compute the unknown initial conditions for $\lambda(0)$.

**Case 4:** $t_f$ free, $x(t_f)$ on $[4; -5t_f + 15]$, $F = \mathrm{diag}(3,5)$, $x_f = [4;6]$. Now, the boundary conditions for the state-costate equations $\dot{w} = Hw$ become

$$w(0) = \begin{pmatrix} 1 \\ 2 \\ \star \\ \star \end{pmatrix}, \quad w(2) = \begin{pmatrix} 4 \\ -5t_f + 15 \\ \star \\ \star \end{pmatrix}$$

and $0 = \mathcal{H}(t_f) + \left[\frac{\partial S}{\partial x} - \lambda\right]_{t=t_f}^{\top} [0; -5] = \mathcal{H}(t_f) + [F(x(t_f) - x_f) - \lambda]_{t=t_f}^{\top} [0; -5] = 0$, yielding

$$\mathcal{H}(t_f) + 5(-5t_f + 15 - 6) - \lambda_2(t_f)(-5) = 0$$

This is a straightforward extension of Case 2, with the difference that the final states are functions of $t_f$ instead of fixed constants. That presents no additional difficulty in the procedure outlined in Case 2.