EEE 586, TEST 1 NAME: SOLUTIONS

Closed-Book, 1 sheet of formulae allowed

Problem 1

1. Determine whether the function $f : \mathbf{R} \mapsto \mathbf{R},\ f(x) = \cos|x|$ is
   A. differentiable,
   B. locally Lipschitz continuous,
   C. globally Lipschitz continuous.

The only potential problem is at the origin, where left and right limits of the derivative of $|x|$ are not equal. However, since cos is an even function, $f(x)$ differentiable everywhere. The derivative of f is -sin(x) which is bounded, therefore it is locally and globally lipschitz.

2. Determine whether the following functions are locally positive definite, semi-definite, or indefinite

$$\begin{aligned} f_1(x_1, x_2) &= \frac{x_1^2}{x_1^4 + 1} + x_2^2 \\ f_3(x_1, x_2) &= x_1^2 x_2^4 \\ f_4(x_1, x_2) &= (x_1 + x_2)^2 + (x_1 - x_2)^2 \end{aligned}$$

f_1 is locally pdf: for small x it is approximately $x_1^2 + x_2^2$.

f_3 is locally psdf: it is always zero on the lines $x_1 = 0$ and $x_2 = 0$.

f_4 is locally pdf: the transformation $(x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2)$ is invertible; then we can write $|[x_1 + x_2, x_1 - x_2]|^2 = x^\top A^\top Ax$, where A, and hence $A^\top A$, are full rank.

Problem 2

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -5x_1 - x_2 \end{aligned}$$

Use a suitable Lyapunov function to show that the origin is a UAS stable equilibrium.

The only equilibrium is the origin. We select $V(x) = x^\top Px$ where P is the solution of the Lyapunov equation $A^\top P + PA = -I$, with $A = [01; -5 - 1]$. We find $P = [3.10.1; 0.10.6]$ which is positive definite, hence the origin is UAS.

Problem 3

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -5x_1^5 - x_2^3 \end{aligned}$$

Show that the origin is a UAS stable equilibrium.

The linearization has zero eigenvalues and it is inconclusive.

We select $V(x) = \frac{5}{6}x_1^6 + \frac{1}{2}x_2^2$. Then, $\dot{V} = 5x_1^5x_2 - 5x_1^5x_2 - x_2^4 = -x_2^4 \le 0$. Hence, 0 is US. Next, applying LaSalle's theorem, $\dot{V} = 0$ on $\{x_2 = 0\}$, where the solutions of the ODE satisfy $\dot{x}_2 = 0 = -5x_1^5 - x_2^3 = -5x_1^5$. Hence, the only solution in $\dot{V} = 0$ is the origin, which, by LaSalle, is UAS.

Problem 4

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - (x_1^2 + x_2^2 - 1)x_2\end{aligned}$$

1. Find and characterize all the equilibrium points.
2. Show that the system has a periodic orbit.

1. The only equilibrium is the origin. The linearization matrix is $[0,\ 1; -1,\ 1]$, whose eigenvalues are $0.5 \pm j\sqrt{3}/2$. It is an unstable focus.

2. Using the Poincare-Bendixson theorem, we select the level set function to cancel the cross terms created by $\dot{x}_1$. $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, yielding $\dot{V} = -(|x|^2 - 1)x_2^2$. This implies that $\dot{V} \leq 0$ outside the circle $\{x : |x|^2 = 1\}$. Hence, trajectories that start inside a level set $M = \{x : V(x) \leq C\}$, whose boundary is contained in the set $\{x : \dot{V}(x) \leq 0\} = \{x : |x|^2 \geq 1\}$, never leave M. This, together with the type of the equilibrium establish the existence of a periodic orbit.