

**Problem 1.** (Short questions from Linear Algebra)

1. Let  $\|\cdot\|$  be a matrix norm; is it always true that  $\|AB\| \leq \|A\| \|B\|$ ?

No, the norm must be consistent. Induced norms are always consistent. Also the Frobenius norm (sum of entries squared) is consistent. The maximum absolute element norm is not consistent.

2. If the matrices A and B commute, show that  $e^{A+B} = e^A e^B$ .

Write the power series for the two sides and regroup terms. In particular we have

$$e^{A+B} = I + (A+B) + \frac{(A+B)(A+B)}{2!} + \dots = I + (A+B) + \frac{A^2+B^2+AB+BA}{2} + \dots$$

$$e^A e^B = \left( I + A + \frac{A^2}{2!} + \dots \right) \left( I + B + \frac{B^2}{2!} + \dots \right) = I + (A+B) + \frac{A^2+B^2}{2!} + AB + \dots$$

Equality requires  $AB=BA$ .

3. Find the minimum norm solution of  $[1,0,1]x = 1$ , where  $x$  in  $\mathbf{R}^3$

$$x_{MN} = [1 \ 0 \ 1]'([1 \ 0 \ 1] * [1 \ 0 \ 1]')^{-1}(1) = [\frac{1}{2} \ 0 \ \frac{1}{2}]'$$

4. Let Q be an orthogonal matrix. Show that  $\|Qx\| = \|x\|$

$$\|Qx\|^2 = x'Q'Qx = x'x = \|x\|^2 \text{ (since } Q'Q=I)$$

**Problem 2.** Let  $X(t) = e^{At}$  be a fundamental matrix for the differential equation

$$\dot{x} = Ax$$

that is, a square matrix whose columns are linearly independent solutions of the differential equation. Show that this matrix can be used to transform the system

$$\begin{array}{lcl} \dot{x} = Ax + Bu & \text{into} & \dot{\tilde{x}} = \bar{B}(t)u \\ y = Cx & & y = \bar{C}(t)\tilde{x} \end{array}$$

$$x = X\tilde{x} \Rightarrow \dot{X}\tilde{x} + X\dot{\tilde{x}} = AX\tilde{x} + X\dot{\tilde{x}} = AX\tilde{x} + Bu \Rightarrow \dot{\tilde{x}} = (X^{-1}B)u,$$

$$y = Cx = (CX)\tilde{x}$$

**Problem 3.** Find the state transition matrix for

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = aI + N \Rightarrow e^{At} = e^{alt} e^{Nt}$$

$$e^{Nt} = I + Nt + \frac{N^2 t^2}{2} + \dots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} t + 0 \Rightarrow e^{At} = \begin{bmatrix} e^{-2t} & te^{-2t} & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix} \text{ Problem}$$

4. Under what conditions is  $\Phi(t, \tau) = \exp\left(\int_{\tau}^t A(s)ds\right)$  the state transition matrix of  $\dot{x} = A(t)x$ ?

We show that the assumed STM satisfies the ODE and the IC:  $\Phi(t, t) = I$

Expanding the exponential and taking derivatives

$$\begin{aligned} \frac{d}{dt} \Phi(t, \tau) &= 0 + A(t) + \frac{A(t) \int A(s)ds}{2} + \frac{\int A(s)ds A(t)}{2} + \dots = A(t) \left\{ I + \int A(s)ds + \dots \right\} \\ &= A(t) \exp\left(\int A(s)ds\right) = A(t) \Phi(t, \tau) \end{aligned}$$

Provided that  $\frac{A(t) \int A(s)ds}{2} = \frac{\int A(s)ds A(t)}{2}$  (i.e., the matrix A commutes with its integral).