EEE 587 HW 3 Solutions

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%  PROBLEM 3.8

>> a=[0 1 0;0 0 1;-5 -7 -10];
>> b=[0 0 4]';
>> Q=diag([1 1 1]);r=1;
>> K=lqr(a,b,Q,r)

K =

  3.5078e-001  8.3407e-001  2.6894e-001

>> [K,P]=lqr(a,b,Q,r)

K =

  3.5078e-001  8.3407e-001  2.6894e-001


P =

  1.9490e+000  1.3075e+000  8.7695e-002
  1.3075e+000  2.6925e+000  2.0852e-001
  8.7695e-002  2.0852e-001  6.7235e-002

>> eig(a-b*K)

ans =

 -1.0117e+001
 -4.7958e-001 +6.3477e-001i
 -4.7958e-001 -6.3477e-001i



>> Q=diag([10 1 1]);r=1;
>> [K,P]=lqr(a,b,Q,r)

K =

  2.1504e+000  2.4462e+000  4.1086e-001


P =

  1.2081e+001  6.7730e+000  5.3759e-001
  6.7730e+000  7.3021e+000  6.1156e-001
  5.3759e-001  6.1156e-001  1.0272e-001

>> eig(a-b*K)

ans =

 -1.0117e+001
 -7.6308e-001 +8.7298e-001i
 -7.6308e-001 -8.7298e-001i

>> % Higher gain control
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```
>> Q=diag([1 1 1]);r=10;
>> [K,P]=lqr(a,b,Q,r)

K =

  3.9380e-002  1.0400e-001  3.0218e-002


P =

  2.0301e+000  1.3741e+000  9.8449e-002
  1.3741e+000  3.0618e+000  2.6000e-001
  9.8449e-002  2.6000e-001  7.5544e-002

>> eig(a-b*K)

ans =

 -9.3896e+000
 -3.6566e-001 +6.4465e-001i
 -3.6566e-001 -6.4465e-001i

>> % Lower gain control

%-----------------------------------------------

%  PROBLEM 3.9

>> a=[0 1;1 1];b=[1 1;0 1];Q=diag([2,4]);r=diag([.5 .25]);
>> [K,P]=lqr(a,b,Q,r)

K =

  1.2887e+000 -4.2666e-001
  1.7240e+000  5.0787e+000


P =

  6.4433e-001 -2.1333e-001
 -2.1333e-001  1.4830e+000

>> % u=-Kx
>> eig(a-b*K)

ans =

 -1.8345e+000
 -5.2569e+000

>>

%  PROBLEM 4.2

% [m,t]=tvric([0 1;-2 -4],[0;.5],diag([4 6]),[.02],5,zeros(2,2));
% steady-state solution
%  2.0357e-001 -1.7894e-001
% -1.7894e-001  2.2117e+000
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% plot(5-t,m)

function [m,t]=tvric(A,B,Q,R,tf,M);
dt=1e-3;t=0:dt:tf;t=t';
% a different dt may be required for different problems
m=vector(M);m=(t*0+1)*m';
for i=2:length(t);
    disp(t(i))
    M=M-(A*M+M*A'+M*Q*M-B*inv(R)*B')*dt;
    % rem: "-" for solving backwards in time from M(tf)
    m(i,:)=(vector(M))';
end
```

PROBLEM 43: CROSS-TERMS + TRACKING

Consider two cases: 1. $\mathring{x}_n = Ax_n + Bu_n$

2. x_n; u_n arbitrary

where x_n, u_n define the reference trajectory to be tracked.

CASE 1 Let $\tilde{x} = x - x_n$, $\tilde{u} = u - u_n$. Then

$$\mathring{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

$$\min J = \frac{1}{2}\|\tilde{x}(t_f)\|_F + \frac{1}{2}\int \underbrace{[\tilde{x}; \tilde{u}]\,\bar{Q}\begin{bmatrix}\tilde{x}\\ \tilde{u}\end{bmatrix}}_{\left\|\begin{smallmatrix}\tilde{x}\\ \tilde{u}\end{smallmatrix}\right\|_{\bar{Q}}}$$

where $\|x\|_Q \triangleq \sqrt{x^T Q x}$

$$\bar{Q} = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

Now, $H = \frac{1}{2}\left\|\begin{smallmatrix}\tilde{x}\\ \tilde{u}\end{smallmatrix}\right\|_{\bar{Q}} + \lambda^T(A\tilde{x} + B\tilde{u})$.

$$\tilde{u} = \underset{\tilde{u}}{\operatorname{argmin}}\, H(\tilde{x}, \tilde{u}, \lambda) = -R^{-1}S^T\tilde{x} - R^{-1}B^T\lambda.$$

$$\mathring{\lambda} = -\frac{\partial H}{\partial \tilde{x}} = -Q\tilde{x} - S\tilde{u} - A^T\lambda$$

Sub. the optimal $\tilde{u}$ and get

$$\begin{pmatrix}\mathring{x}\\ \mathring{\lambda}\end{pmatrix} = \begin{pmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ -Q + SR^{-1}S^T & -A^T + SR^{-1}B^T \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix}$$

from which, as usual $P = \mathrm{Ric}\left[A - BR^{-1}S^T, B, Q - SR^{-1}S^T, R\right]$.

For finite time problems (non constant P in general) Ric should be interpreted as the Riccati differential eqn.

Its boundary conditions are $P(t_f) = F(t_f)$

Alternatively, define $\tilde{u} = u_\Delta + u_c$ and express

$$\|\tilde{x}, \tilde{u}\|_{\bar{Q}} = \tilde{u}^T R \tilde{u} + \tilde{x}^T Q \tilde{x} + 2\tilde{x}^T S \tilde{u}$$

in the form $u_\Delta^T R u_\Delta + \tilde{x}^T Q_\Delta \tilde{x}$. For this,

$$(\tilde{u} - u_c)^T R (\tilde{u} - u_c) = \tilde{u}^T R \tilde{u} + u_c^T R u_c - 2u_c^T R \tilde{u}.$$

Choose $2\tilde{x}^T S = -2u_c^T R \Rightarrow u_c = -R^{-1} S^T \tilde{x}$.

Then, $\tilde{x}^T Q \tilde{x} = u_c^T R u_c + \tilde{x}^T Q_\Delta \tilde{x}$ or, $Q_\Delta = Q - SR^{-1}S^T$.

Moreover,
$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + B\tilde{u} = A\tilde{x} + Bu_\Delta + Bu_c \\ &= (A - BR^{-1}S^T)\tilde{x} + Bu_\Delta\end{aligned}$$

Now we have the usual problem

$$\min \int \tilde{x}^T Q_\Delta \tilde{x} + u_\Delta^T R u_\Delta$$
$$\text{s.t. } \dot{\tilde{x}} = (A - BR^{-1}S^T)\tilde{x} + Bu_\Delta.$$

Its solution requires $R > 0$, $Q_\Delta = Q - SR^{-1}S^T \geq 0$ and $[(A - BR^{-1}S^T), B]$ to be controllable. The last, is equivalent to $[A, B]$ controllable, since controllability is invariant under feedback ($A - BR^{-1}S^T$ is simply state feedback on (A, B) with $u = kx$, $k = R^{-1}S^T$)

CASE 2 Here using the original coordinates,

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ -Q + SR^{-1}S^T & -A^T + SR^{-1}B^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} + \begin{pmatrix} Bu_n + BR^{-1}S^T x_n \\ -SR^{-1}S^T x_n \end{pmatrix}$$

optimal input: $u_{opt} - u_n = -R^{-1}S^T(x - x_n) - R^{-1}B^T\lambda$.

The Riccati for the feedback is as before, but the optimal input now contains an additional contribution $v(x_n, u_n)$.