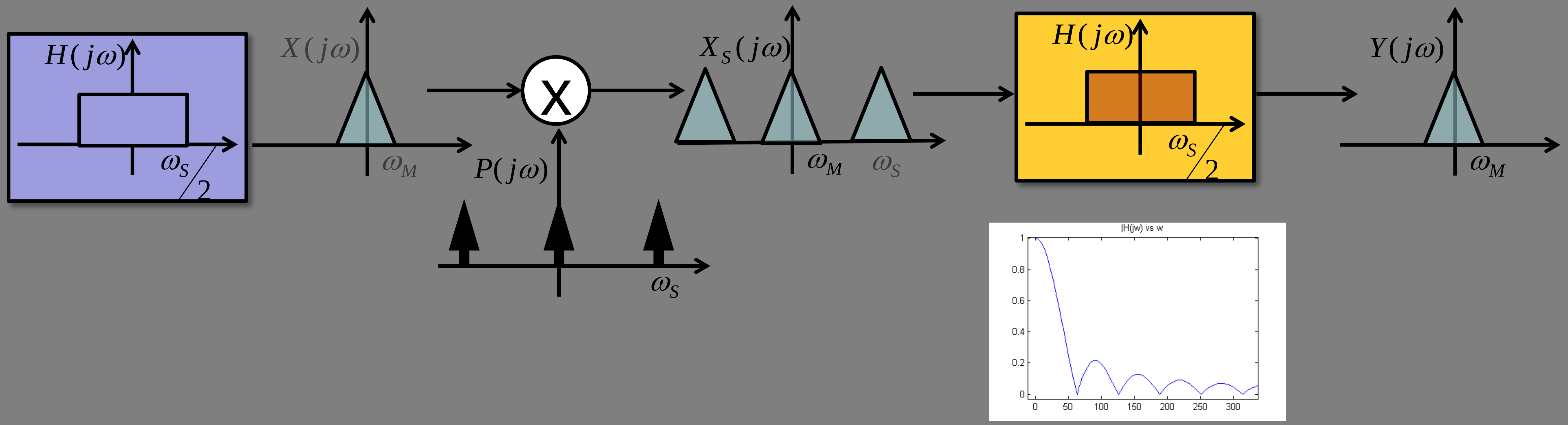
EEE304

Week 3: Sampling and Reconstruction



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Lecture 3.1: Sampling and Reconstruction

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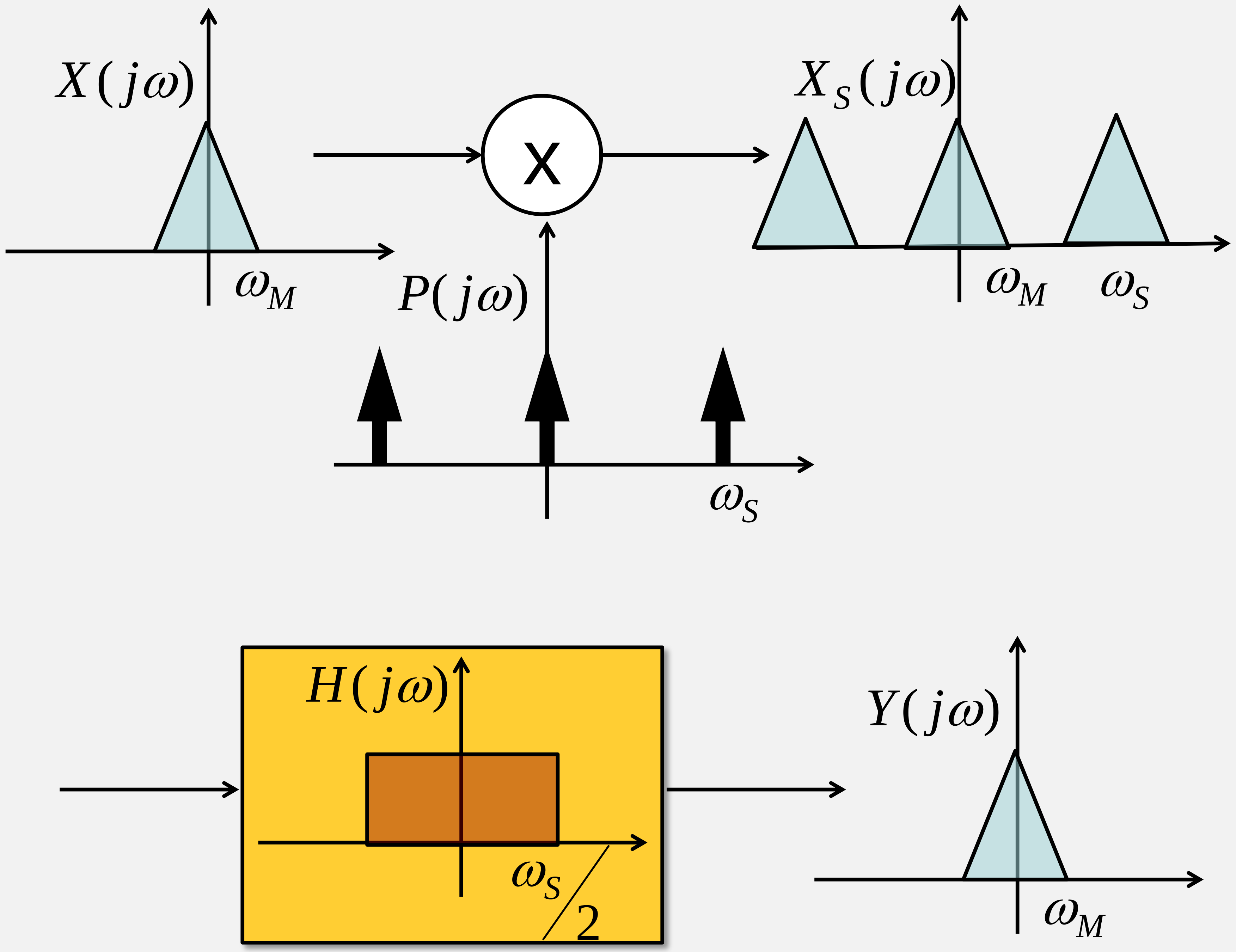
Sampling and reconstruction: Ideal case, Nyquist theorem

- A bandlimited signal is sampled by modulation with an impulse train.
- The result contains signal replicas centered at the sampling frequency harmonics.
- The original signal can be recovered by an ideal lowpass filter, provided that the Nyquist-Shannon sampling theorem holds (no aliasing):

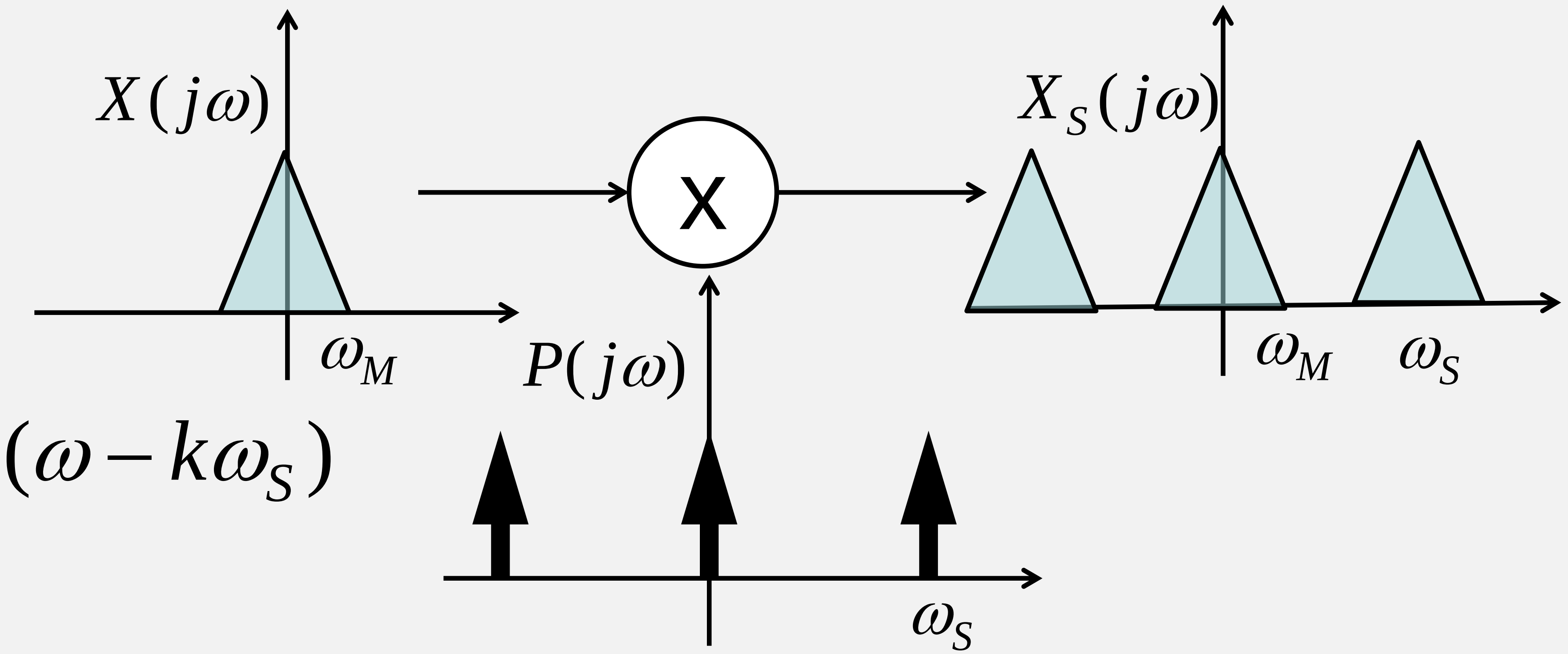
$$\omega_M < \frac{\omega_S}{2} = \frac{\pi}{T}$$



Sampling and reconstruction: Ideal case, Nyquist theorem. Derivations

- Fourier transform of impulse train (through Fourier Series expansion).

$$p(t) = \sum_n \delta(t - nT)$$



$=1/T$

$$FS: a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_s t}dt, \quad F\{e^{jk\omega_s t}\} = 2\pi\delta(\omega - k\omega_S)$$

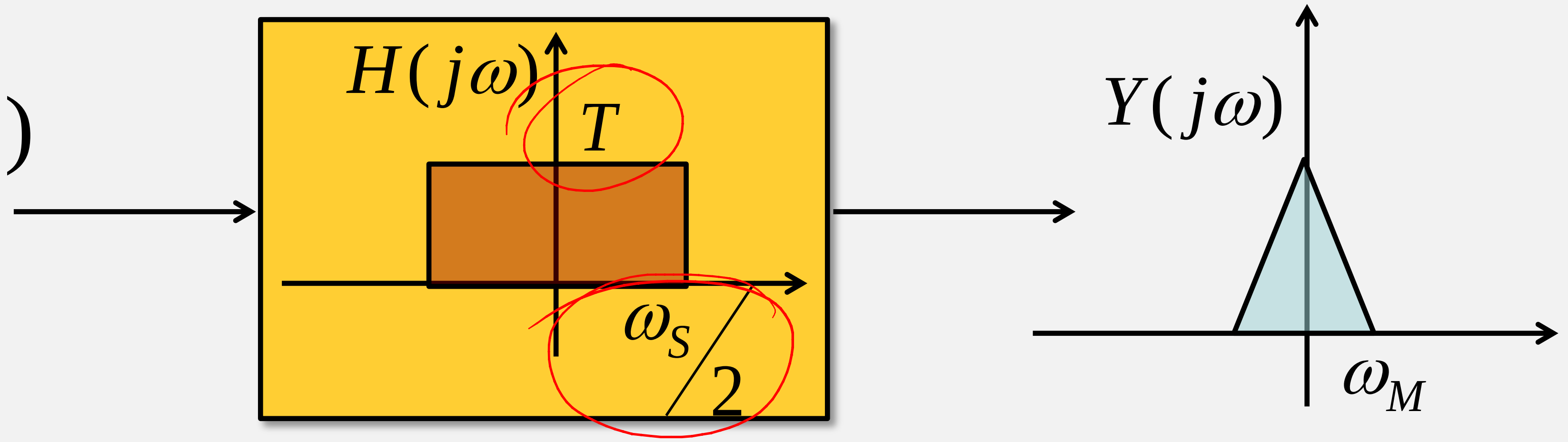
$$P(j\omega) = \frac{2\pi}{T}\sum_k \delta(\omega - k\omega_S); \quad \omega_S = \frac{2\pi}{T}$$

- Modulation property

$$X_S(j\omega) = \frac{1}{2\pi}X(j\omega) * P(j\omega) = \frac{1}{T}\sum_k X(j(\omega - k\omega_S))$$

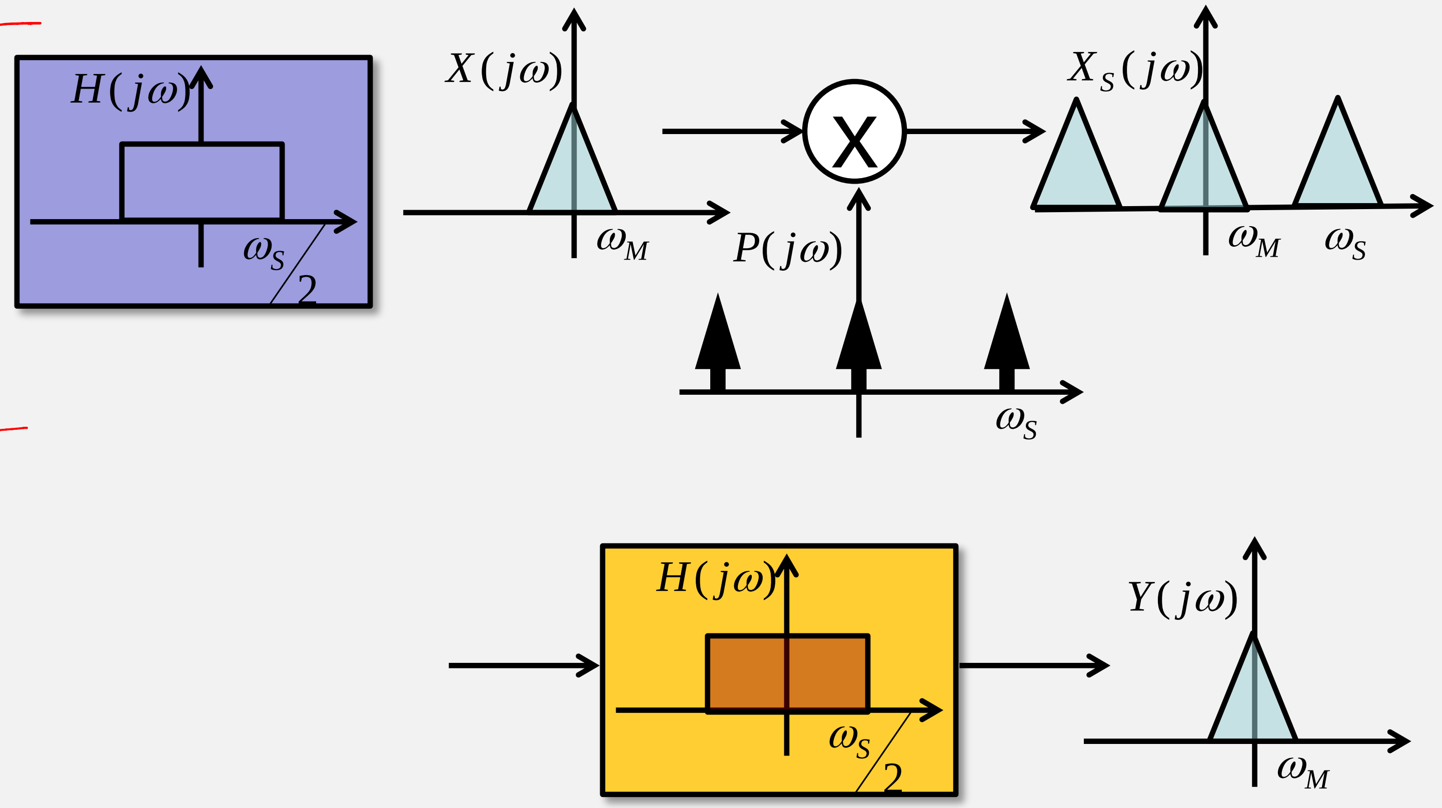
- Lowpass Filtering

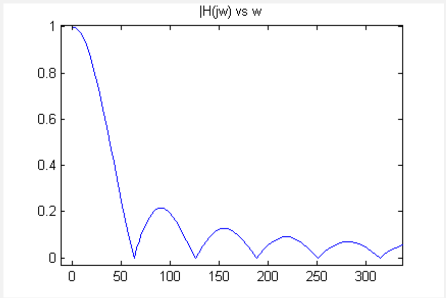
$$Y(j\omega) = H(j\omega)X_S(j\omega) = X(j\omega)$$



Sampling and reconstruction: Issues

- Nyquist rate and Nyquist frequency
- Aliasing and Anti-Aliasing Filters (AAF)
- Approximately bandlimited signals
- Non-ideal reconstruction: ZOH (DAC)
- Oversampling
- DT filtering and DSP





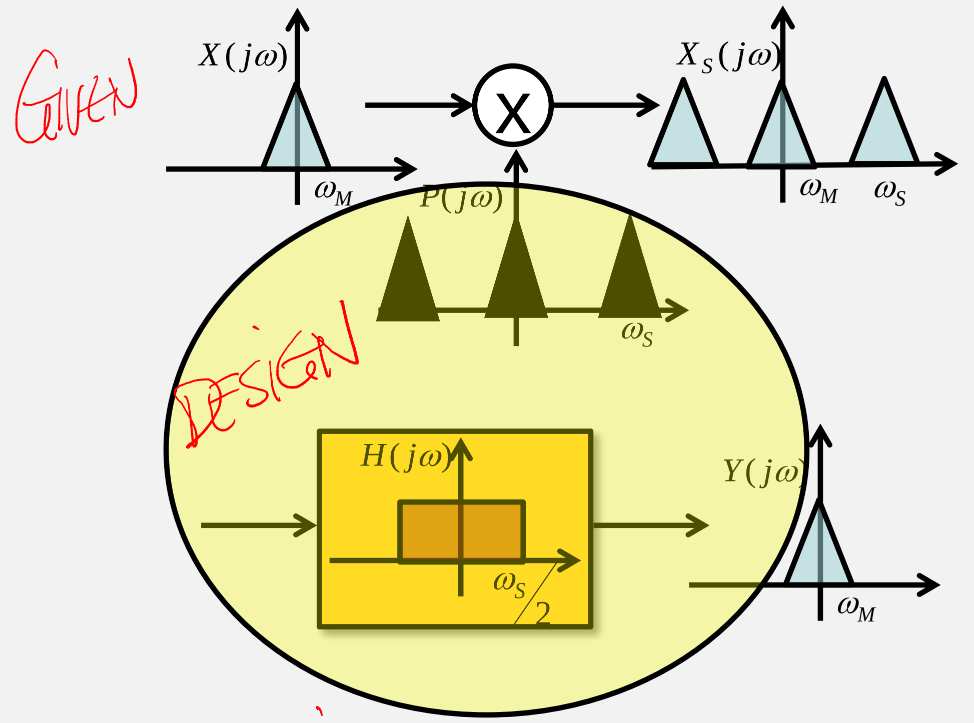
Nyquist rate and Nyquist frequency

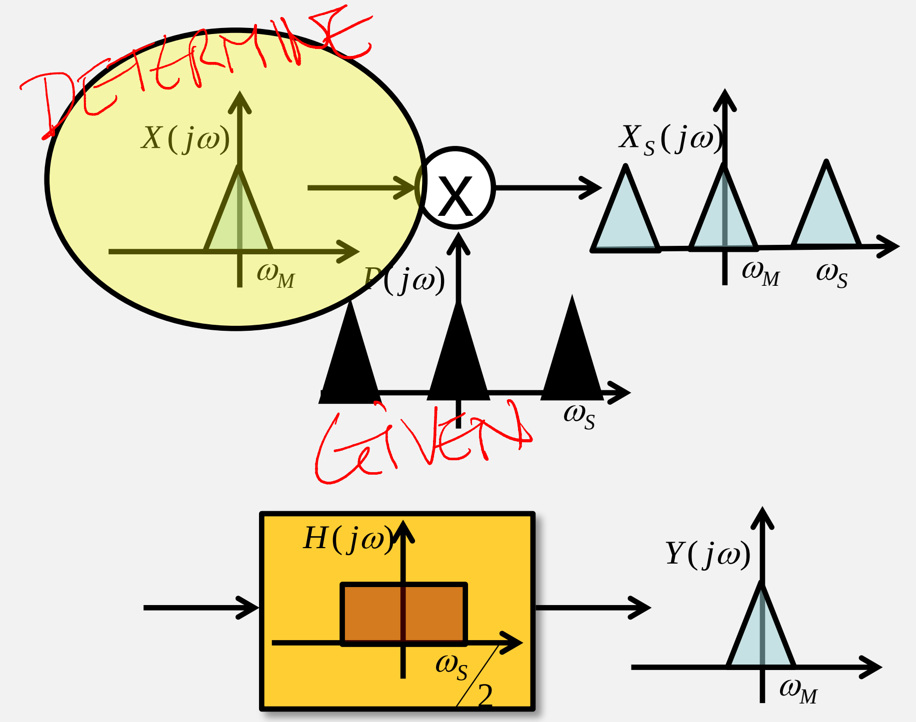
- Nyquist rate: Given a signal, bandlimited to ω_M, the Nyquist rate is the frequency of the sampling process that would allow perfect reconstruction

$$\omega_{NR} = 2\omega_M$$

- Nyquist frequency: Given a sampling process with frequency ω_S, the Nyquist frequency is the maximum frequency of an input signal that can be perfectly reconstructed.

$$\omega_{NF} = \frac{\omega_S}{2}$$

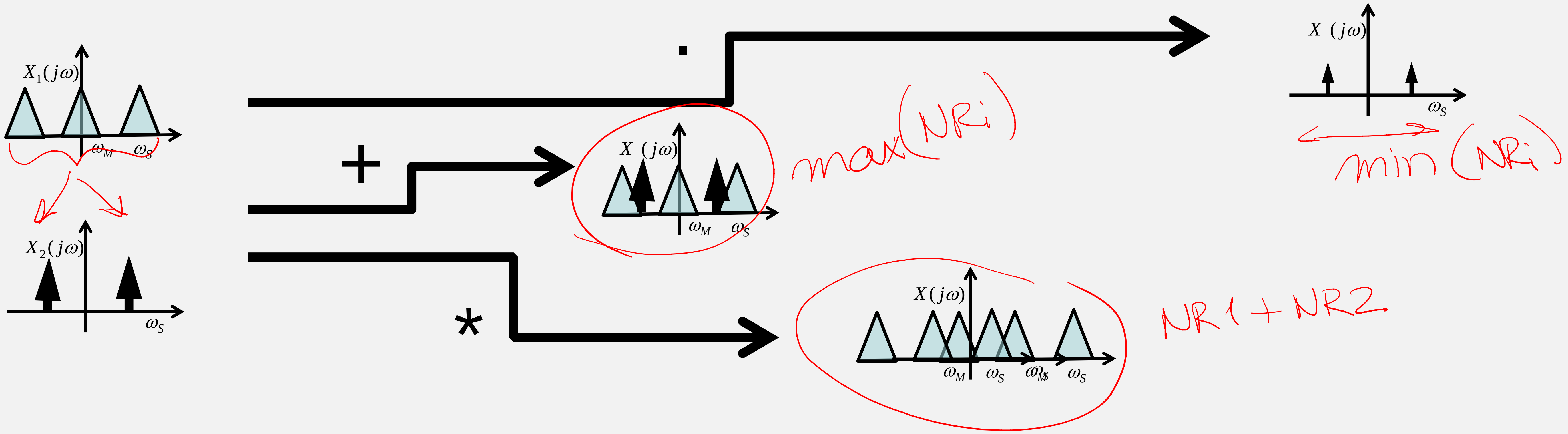




Estimates of Nyquist rates for composite signals

- Given a composite signal, e.g., $x_1(t) \pm x_2(t)$, $x_1(t) \cdot x_2(t)$, $x_1(t) * x_2(t)$, we want to estimate its Nyquist rate from the Nyquist rates of its components.

- The exact answer is to find the maximum frequency of the FT of the composite signal $X_1(j\omega) \pm X_2(j\omega)$, $\frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$, $X_1(j\omega) \cdot X_2(j\omega)$, respectively.



Estimates of Nyquist rates for composite signals

- A quick conservative estimate can be derived by examining the worst case:

$$NR\{x_1(t) \pm x_2(t)\} = NR\{X_1(j\omega) \pm X_2(j\omega)\} \le \max\left(NR\{X_1(j\omega)\}, NR\{X_2(j\omega)\}\right)$$

$$NR\{x_1(t) \cdot x_2(t)\} = NR\left\{\frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)\right\} \le NR\{X_1(j\omega)\} + NR\{X_2(j\omega)\}$$

$$NR\{x_1(t) * x_2(t)\} = NR\{X_1(j\omega) \cdot X_2(j\omega)\} \le \min\left(NR\{X_1(j\omega)\}, NR\{X_2(j\omega)\}\right)$$

Estimates of Nyquist rates for composite signals

- The estimate is conservative because cancellations can eliminate parts of the signal that affect the maximum frequency content.

$NR(\sin t) = 2\,\text{rad/s}$

- Examples:

$$NR\{\sin(t)+\sin(2t)\} \le \max(2,4) = 4 \text{ (rad/s; for sampling time, } T = 2\pi/4)$$

NOTE: $\dfrac{\pi}{\text{max frequency}} = \dfrac{2\pi}{\text{Nyquist Rate}}$

$$NR\{\sin(t)-\sin(t)\} \le \max(2,2) = 2 \text{ (but actual exact answer is 0)}$$

$T = \infty$

$$NR\{\sin(t)\cos(2t)\} \le 2+4 = 6 \text{ rad/s}$$

Conservative Upper Bound

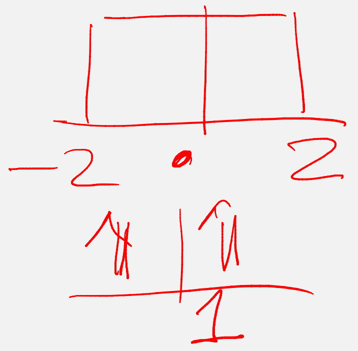
$$NR\left\{\frac{\sin(t)}{t}\cos(t)\right\} \le 2+2 = 4$$

Estimates of Nyquist rates for composite signals

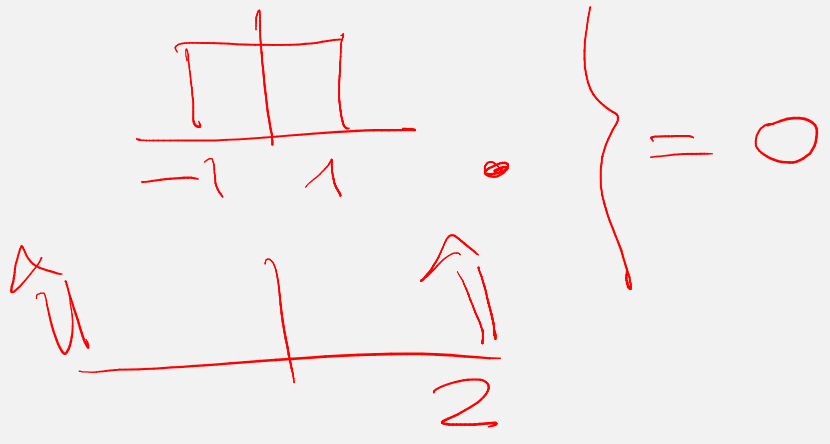
- Examples (cont.):

$$NR\left\{\frac{\sin(t)}{t}\cdot\frac{\cos(t)}{t}\right\}\le 2+\infty=\infty \quad \text{(second signal is not bandlimited)}$$

$$NR\left\{\frac{\sin(2t)}{t}*\cos(t)\right\}\le \min(4,2)=2$$



$$NR\left\{\frac{\sin(t)}{t}*\cos(2t)\right\}\le \min(2,4)=2 \text{ (but actual exact answer is 0)}$$



$$NR\left\{\frac{\sin(t)}{t}*e^{-2t}u(t)\right\}\le \min(2,\infty)=2$$

- Implications of the last expression on filtering (ideal vs. practical)

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Lecture 3.2: Aliasing and anti-aliasing filters

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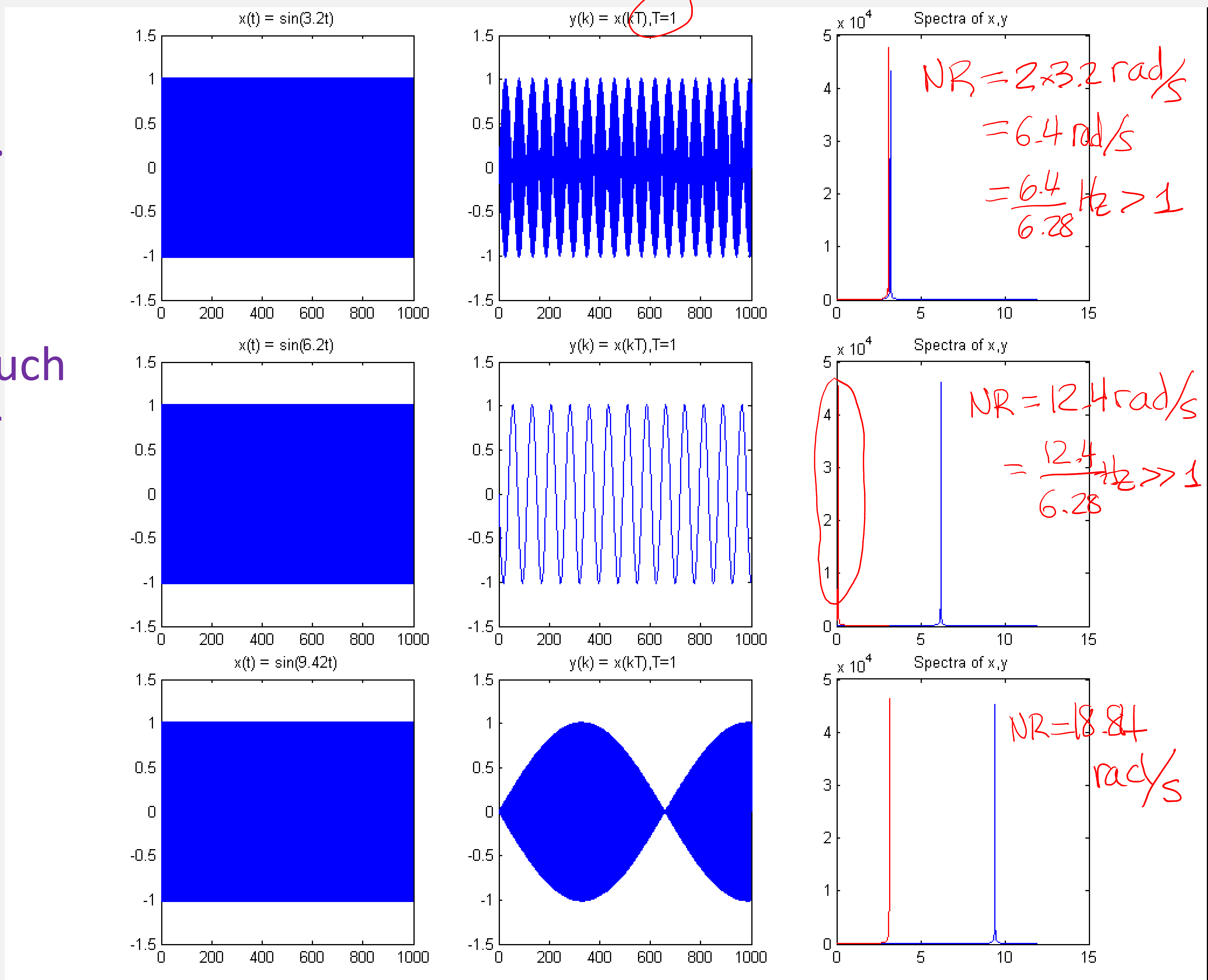
Aliasing and Anti-Aliasing filters

1Hz Sampling

- Aliasing is caused by the overlapping spectra of the shifted replicas of X(jω).

- A standard aliasing example is the translation of a single frequency. The sampling and reconstruction with a much lower sampling rate can lead to rather “unexpected” results.

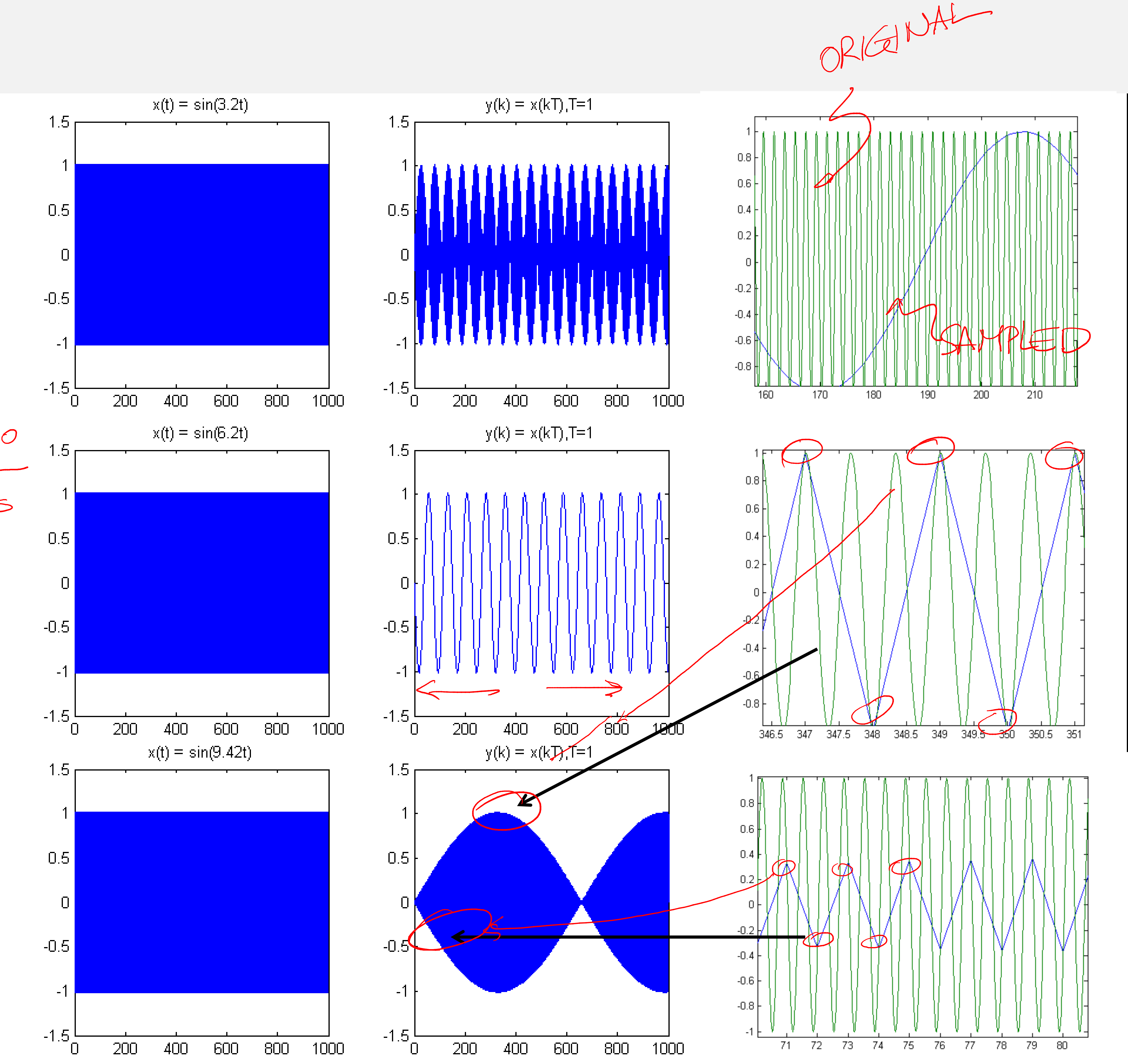
- The examples shown are
  - sin(3.2t), T = 1
  - sin(6.2t), T = 1
  - sin(9.42t), T = 1



Aliasing

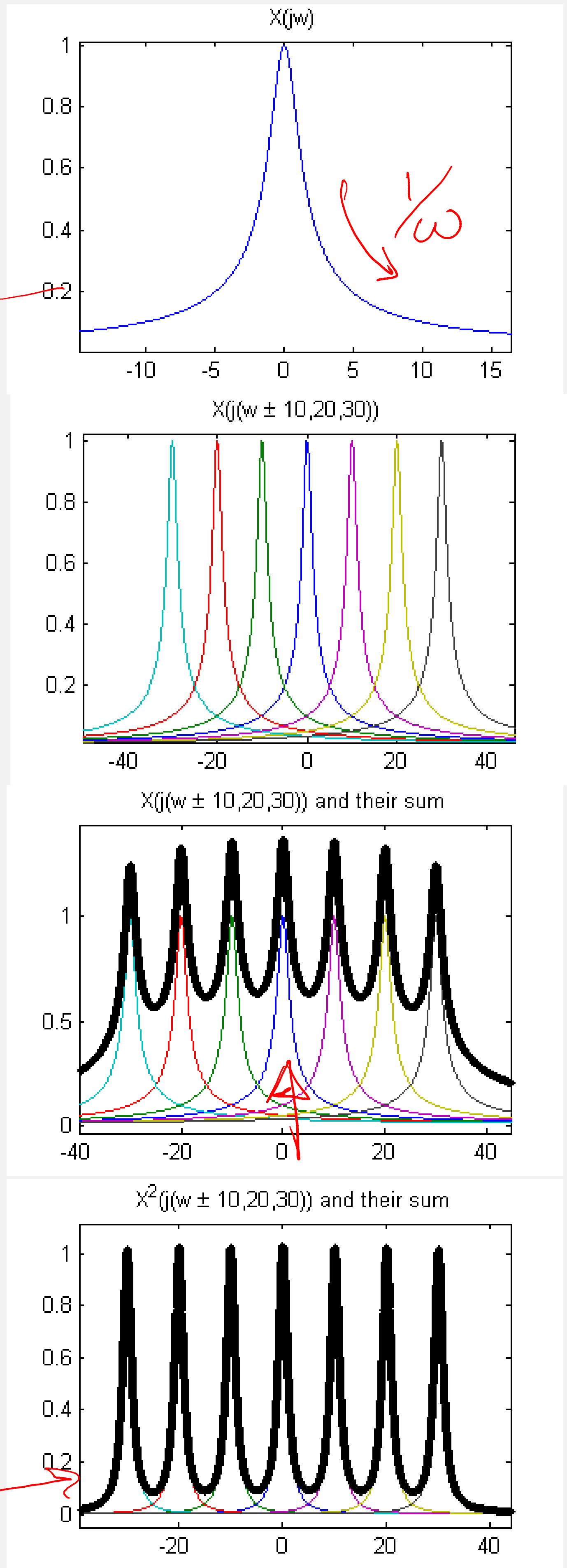
- The results are of course predictable by following the sampling and reconstruction sequence:
  - 0^{th} replica frequencies: $0 \pm \omega_0$ ω_0
  - 1^{st} replica frequencies: $\omega_S \pm \omega_0$ ω_0 ω_s
  - Etc.

- Here, Nyquist frequency = π/1=3.14, sampling frequency = 2π/1=6.28
  - 6.28 – 3.2 = 3.08 rad/s
  - 6.28 – 6.2 = 0.08 T=1
  - -6.28 + 9.42 = 3.137, 2x(6.28)-9.42 = 3.146



Aliasing and band-limited signals

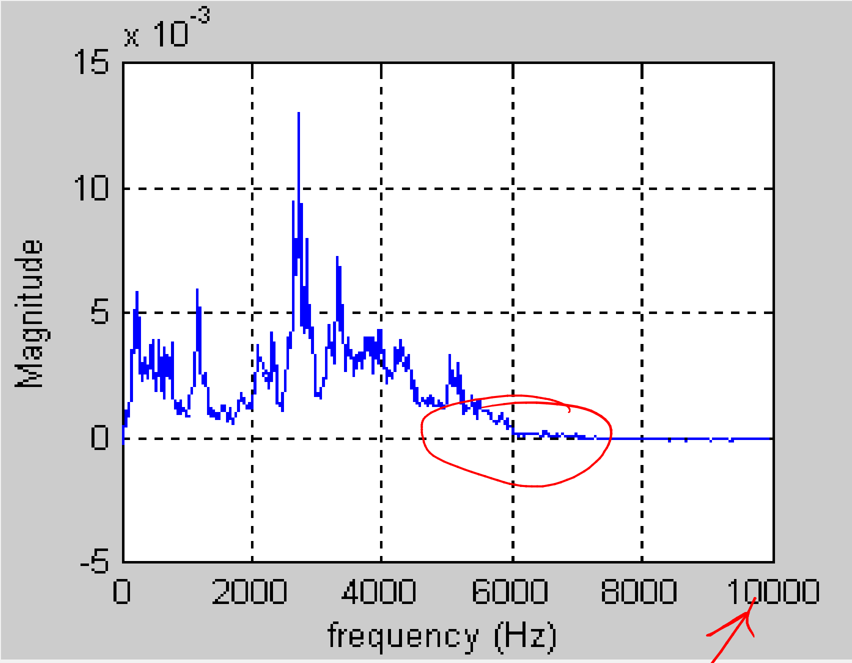
- Real-life signals are not strictly bandlimited. Their decaying "tails" can also change the sampled signal significantly, for example: $$X(j\omega)=\left(\frac{1}{\sqrt{\omega^2+1}}\right)$$
  - The sum of the first 7 terms is already significantly different. In fact, the series at frequency 0 diverges: $\sum_n 1\big/\sqrt{(10n)^2+1} \approx \sum_n 1/n = \infty$
  - Real-life signal FTs decay faster than 1/ω, but to minimize aliasing we need Anti-Aliasing Filters (AAF). These are low-pass filters 1st, 2nd or higher order depending on the expected frequency content of the signal.
  - If we use the same frequency response as a lowpass filter, the sum is now very close to the individual replicas $F(j\omega)X(j\omega)=\left(\frac{1}{\sqrt{\omega^2+1}}\right)^2$. (Notice that FX is distorted relative to the original signal X, but at least it is predictably so.)



Aliasing and Anti-Aliasing filters

- Antialiasing filters do not completely prevent aliasing (some high frequency signal goes through), but at least the amplitude is small and, because of rolloff, the summation of all the high-frequency replicas converges rapidly.
- Ideally, the AAF is a lowpass with cutoff $\omega_s/2$, the Nyquist frequency.
  - In practice, their design will tradeoff aliasing, low-frequency distortion, complexity, and power consumption. (Recall the filter design problem in Lecture 2.1)
  - One other possibility is to sample much faster with a low quality analog AAF and then use a high quality digital AAF to reduce the sampling rate.

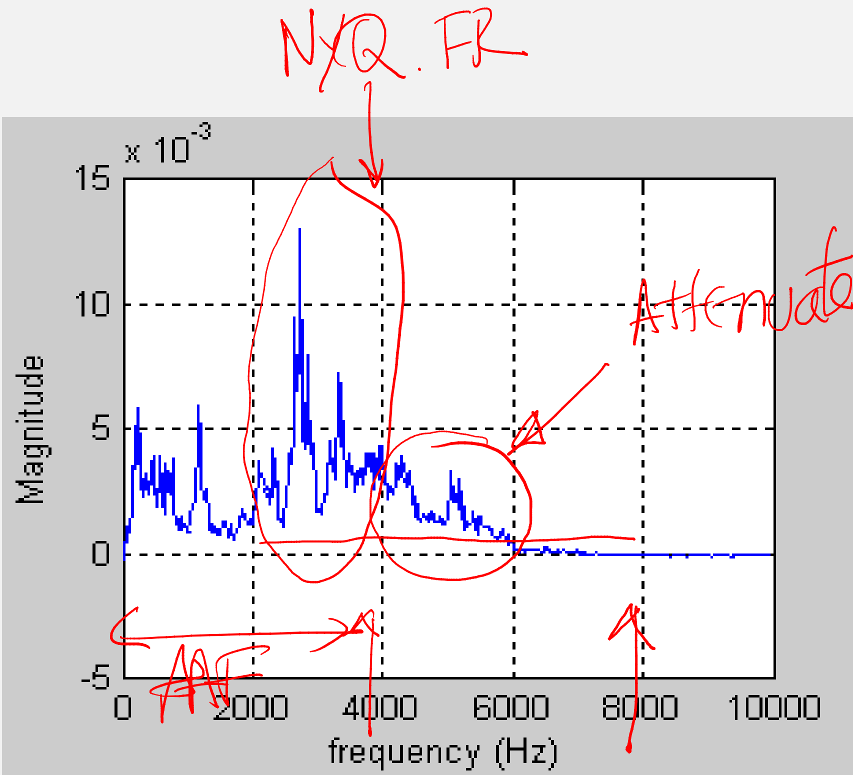
Example



$f_s \geq 20\,kHz$

- Suppose that the frequency spectrum of a vibration signal is shown in the figure. How fast we should sample to capture all the information contained in the signal?
- The highest signal frequency is around 7kHz, so we need to sample faster than 14kHz.
- An Anti-Aliasing Filter with cutoff frequency a little below 7kHz would help guarantee that a perturbation would not cause aliasing
- Operations: → AAF 7kHz → SAMPLING 14kHz → Analysis
- (if needed, Reconstruction Lowpass 14KHz, amplitude T = 1/14000)

Example



- Suppose that the frequency spectrum of a vibration signal is shown in the figure. Is it possible to sample and monitor the spectral peaks around 3kHz but with a computer that can only support sampling rates up to 8kHz?
- To analyze the spectral peaks around 3kHz we need to sample faster than 6kHz (plus any margin desirable) so the computer has sufficient capabilities, at least in principle. From the given spectrum, there are components in the signal with frequencies higher than Nyquist (4kHz) so we need to use a good (e.g., 4^{th} order Butterworth) Anti-Aliasing Filter with cutoff frequency below 4kHz, but above 3kHz to include the interesting portion of the signal. (The analysis should account for the signal distortion at 3kHz.)
- Operations: → AAF, 3-4kHz → SAMPLING 6-8kHz → Analysis
- (if needed, Reconstruction Lowpass 3-4KHz, amplitude T = 1/3000-1/4000)

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Lecture 3.3: Approximately Bandlimited Signals: Sampling of Noisy Data

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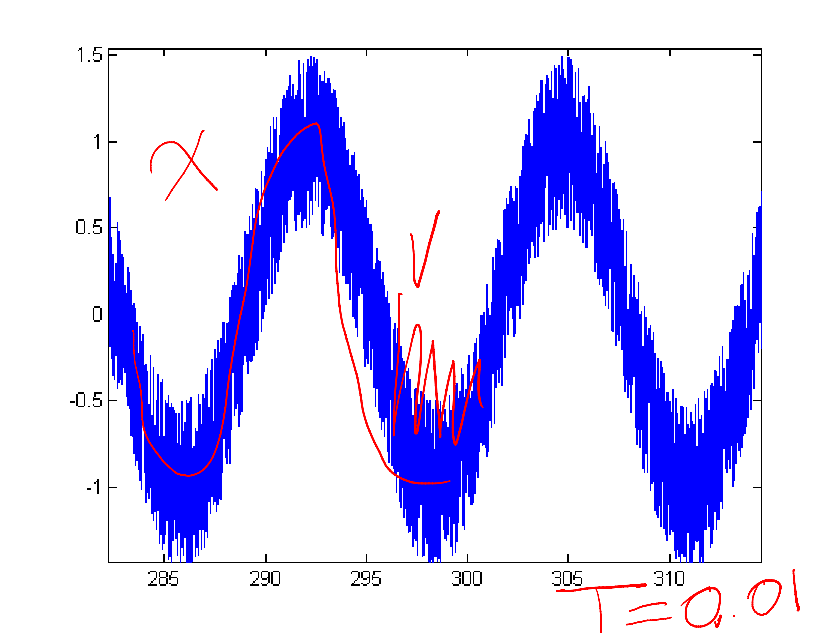
Motivating example

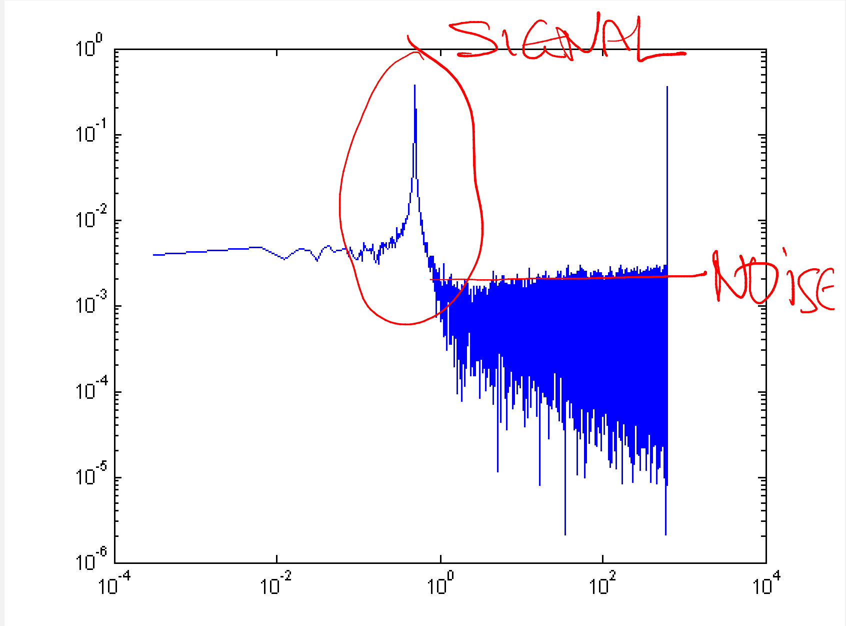
- For a practical version of sampling, we can consider a signal “x” as an ideal component, bandlimited, with additive noise that is bounded by a given threshold.

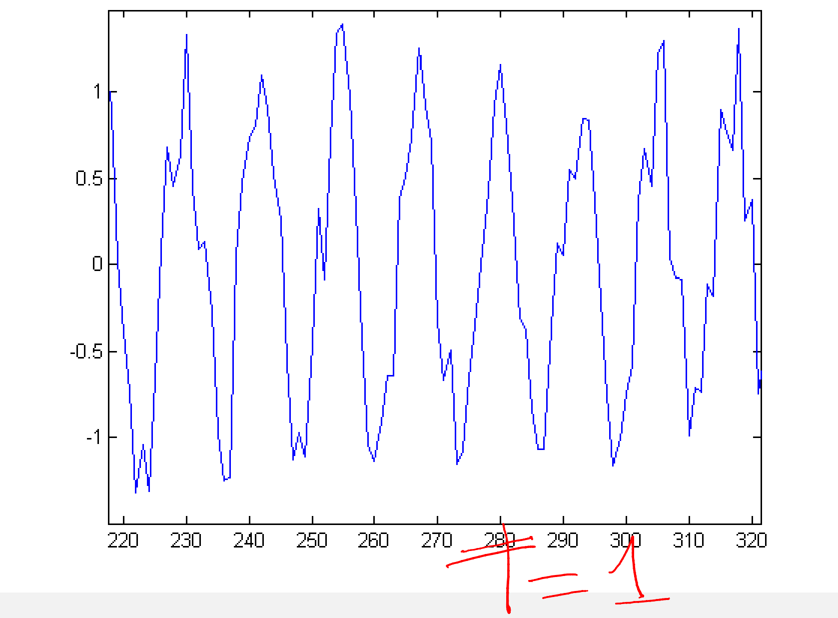
$$x_v(t) = x(t) + v(t); \quad x(t) = \sin(0.5t); \quad |v(t)| < 0.5$$

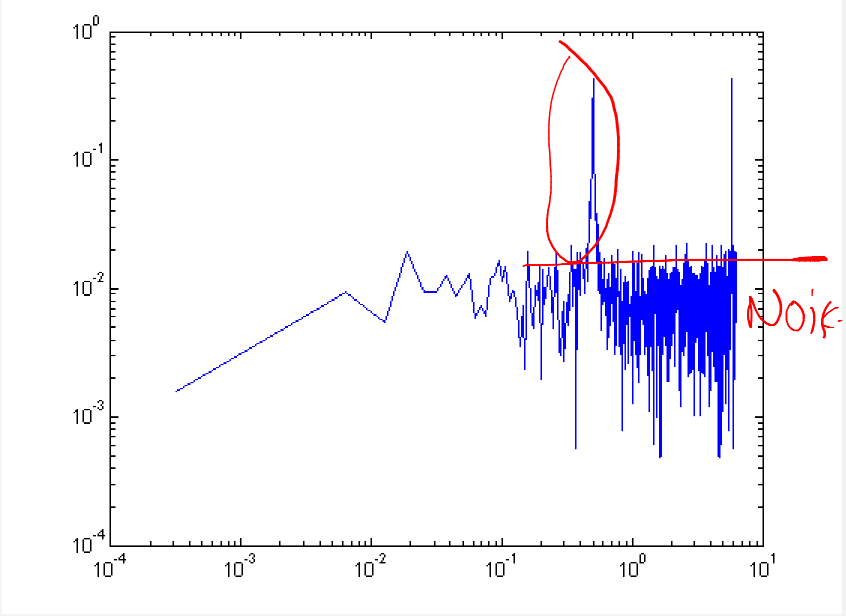
- The noise signal does not satisfy the sampling theorem. Still, the FFT result agrees with intuition separating signal from noise with T = 0.01 and 1. (Why?)

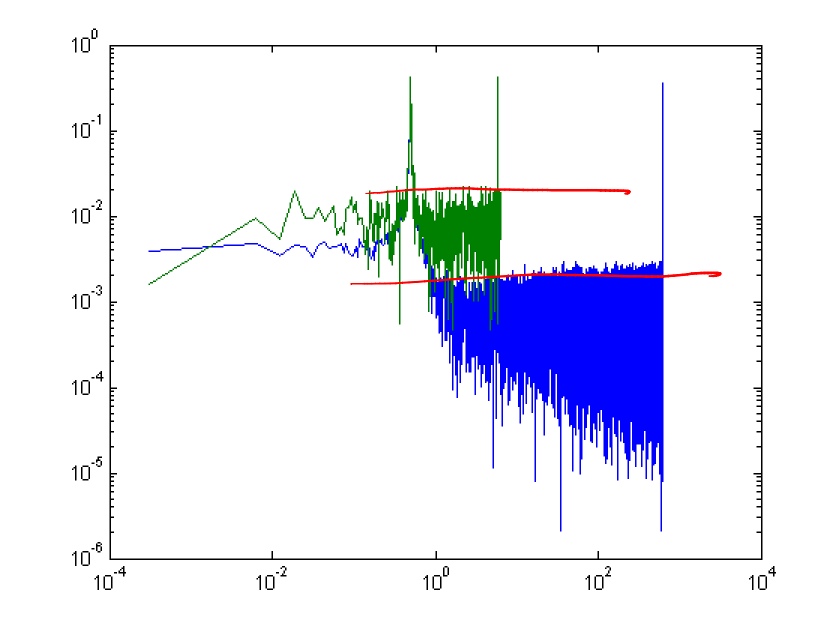
- Our ultimate objective here is to characterize and quantify the relationship between signal and noise in a way that justifies the sampling of noisy signals that, strictly speaking, violate the sampling theorem.











Motivating example framework

- The main difficulty in the noisy signal case is to take advantage of the properties of noise to develop tighter estimates of its propagation through sampling, rather than the generic ones of plain replicas.

- To analyze and explain our observations in this example, we need to recall background on the computation of the transforms and Parseval's theorem.

- In our framework we have a bandlimited signal x perturbed by noise that is not bandlimited but small in some sense. As it truns out, a convenient and practically meaningful metric is the energy or the RMS value of a signal

$$RMS(x) = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1}|x(nT)|^2}$$

The former is appropriate for signals with finite energy on the entire domain. The latter provides a natural extension to power signals and it is meaningful when the signal is stationary, so its properties do not change with time.

FT, FS, DTFS, and FFT

- Much of the following discussion focuses on the use of FFT as a computational tool for the analysis. The importance of the FFT is that its computation can be implemented in an extremely efficient way (~N logN operations) so that any approximation with an FFT is very appealing.

- For an energy signal, we can consider that it is 0 outside that interval of measurement and take FT as usual. The transform can be approximated by a finite sum which in turn becomes the DT Fourier Series

- CT-FT $X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{0}^{(N-1)T}x(t)e^{-j\omega t}dt\cong\sum_{n=0}^{N-1}x(nT)e^{-j\omega nT}T$, DT-FS $X(k)=\sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi n}{N}k}$

- For a power signal, the sampled signal spectrum is

$$X_s(j\omega)=F\left\{x(t)\sum_n\delta(t-nT)\right\}=\int\left[x(t)\sum_n\delta(t-nT)\right]e^{-j\omega t}dt$$

$$=\int\left[\sum_n x(nT)\delta(t-nT)e^{-j\omega nT}\right]dt=\sum_n x(nT)e^{-j\omega nT}\int\delta(t-nT)dt=\sum_n x(nT)e^{-j\omega nT}$$

sampling at nT

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FT, FS, DTFS, and FFT

- Here, given a finite set of measurements n = 1…N, it makes sense to consider x(t) as a periodic signal. Then, when the last transform is sampled in frequency, we recover the well known FFT expression

$$\frac{1}{T}X(j\omega) \sim X_s\left(jk\frac{2\pi}{TN}\right) = \sum_n x(nT)e^{-j\frac{2\pi n}{N}k} = FFT\{x(nT)\}; \quad F\{x\}|_{\omega=k\omega_0} \cong FFT\{x(nT)\}T$$

Thus, both formulations lead to similar connections between CT and DT spectra.

- In relation to the FT we have that for an interval D = NT

$$\frac{1}{NT}F\{x\}|_{w=kw_0} = \frac{1}{D}F\{x\}|_{w=kw_0} \cong \frac{1}{N}FFT\{x(nT)\} = \{a_k\}_{DTFS}$$

where ak are the coefficients of the DT Fourier Series. So, N, the number of points is indeed a suitable normalizer for the FFT in order to compare signals and properties when the intervals or the number of points change.

Eg. Sample for 1 sec at 1 kHz → N = 1000

1 sek at 100 Hz → N = 100

Quantitative description of signal and noise properties

- We recap with the observation that the noise signal has some invariant properties, one of which is its size (originally defined in a suitable discretization or in CT):

$$RMS(v) = \sqrt{\frac{1}{N}\sum_{n=0}^{N-1} |v(nT)|^2} = d$$

This is a quantity we reasonably expect to be invariant "regardless" of the size or location of the sample.

- Then Parseval states that the energy in time and frequency domain, (or in the periodic case, the average power) will be the same.

$$\underline{\frac{1}{N}\sum_{n=0}^{N-1} |v(nT)|^2 = \sum_{n=0}^{N-1} |a_n|^2} = \underline{\sum_{n=0}^{N-1} \left|\frac{FFT(v)}{N}\right|^2} = d^2$$

- We now invoke the concept of a "white random signal", that is, we assume that a noise signal will have no particular structure, i.e., its <u>spectrum will be "flat"</u>.

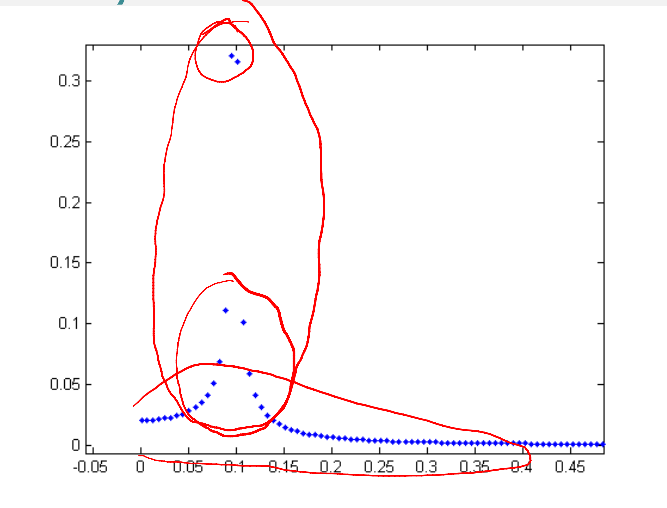
$$|a_n| = \left|\frac{FFT(v)}{N}\right| = \frac{d}{\sqrt{N}}$$

Quantitative description of signal properties

- Thus, the normalized FFT is affected by sampling. Keeping the length of the interval the same, its coefficients will increase as we sample fewer points. In a way, when we sample fewer points out of a noise signal, the invariant average FFT coefficient increases, accumulating the energy of the points that were discarded.

- Finally, we bring in the analogous properties for the signal x. Here it makes sense to look at deterministic, bandlimited signals, e.g., sinusoids and ask when they fall below the noise level. Taking the FFT of a sinusoid we find that the corresponding coefficient will be ½ if the frequency is part of the FFT frequencies $\frac{2\pi n}{N}$. Otherwise the energy is distributed to adjacent frequencies with the two closest ones receiving most of the energy (see figure). In general, for a sinusoid with amplitude A, the FFT peak is around 0.5A-0.3A.

- Note: In any case, Parseval holds and the norm of the vector of the FFT coefficients, normalized by N, is $A/\sqrt{2}$.

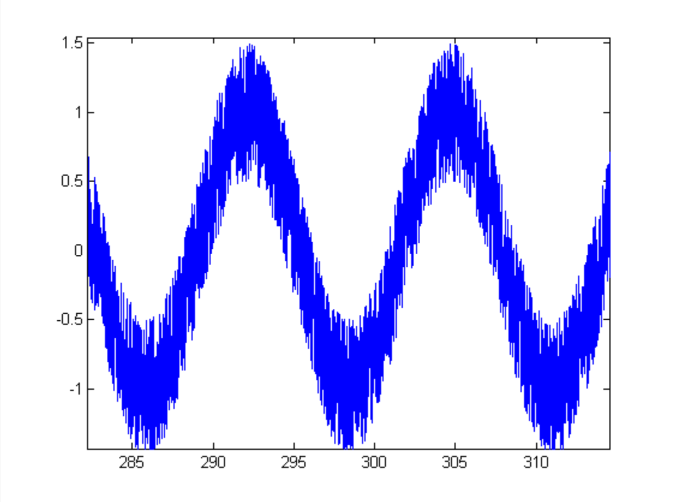


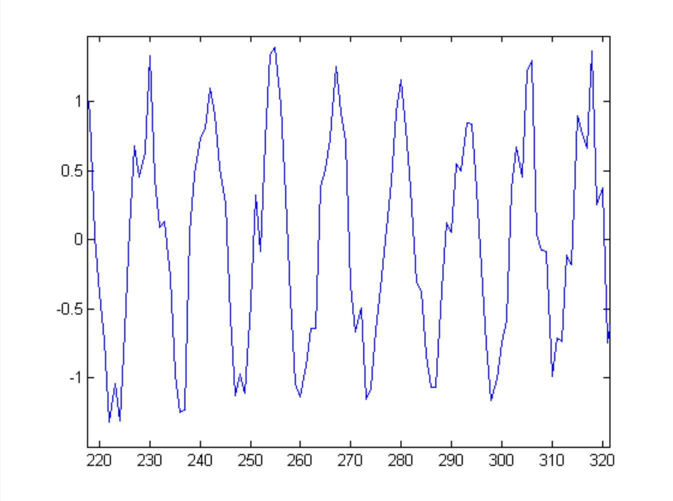
Motivating Example, revisited

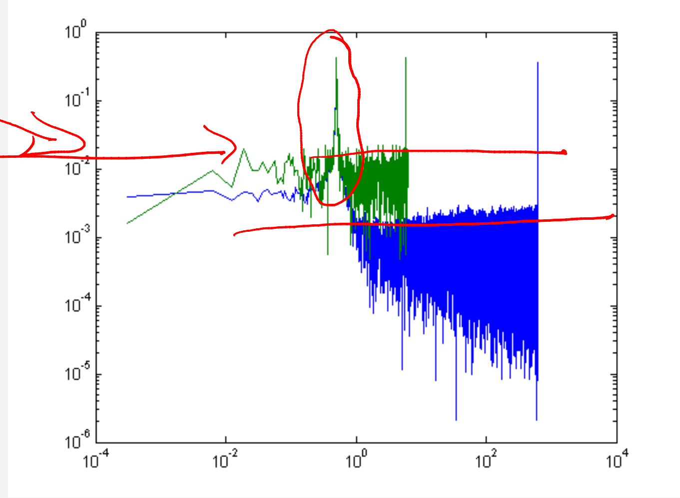
- The first plot shows a “dense” sampling of the sinusoid with random noise (T = 0.01). This can be part of an initial experiment to assess the noise properties. The noise amplitude is 0.5 and its RMS value is 0.28 ($RMS(x) = \sqrt{\frac{1}{3}d^2}$).

- The FFT clearly shows the sinusoid peak and the noisy background around 1-2e-3. This agrees with our estimate of $RMS(x)/\sqrt{N}, \quad N = 1e5$

- We now sample that signal with T = 1, so N=1e3. We observe that the peak is nearly the same but the noise level has increased to 1-2e-2, a factor of 10 difference, which is the root of the ratio of points. The sinusoid peaks are the same, around 0.36, (thanks to the FFT normalization) and even though the Signal to Noise Ratio (SNR) has been reduced, the signal is still more than an order of magnitude above the noise and can be easily detected.

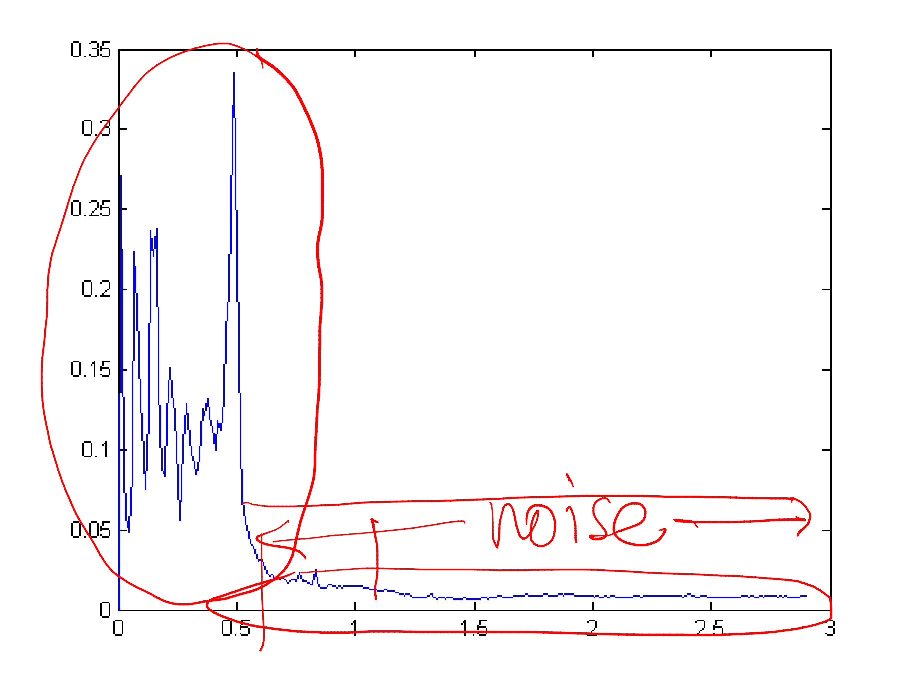






Applications

- Applications of this analysis in the sampling of noisy signals can vary. One example is the sampling time selection given a SNR level.
  - Suppose that the frequency spectrum of a vibration signal is shown in the figure. It is sampled at 6kHz, 512 points. Can we reduce the sampling frequency keeping the same interval andwithout compromising the detection of the 0-500Hz peaks?



For a bandlimited signal, the peak at 500Hz could be detected and analyzed by 1kHz sampling. In our case, the signal is not strictly bandlimited, but it becomes very small after 500Hz. If we consider it as noise, we could sample 4x slower and only double the level of noise. That would still be well-below all other peaks below 500Hz so that looks as a feasible scenario.

Applications

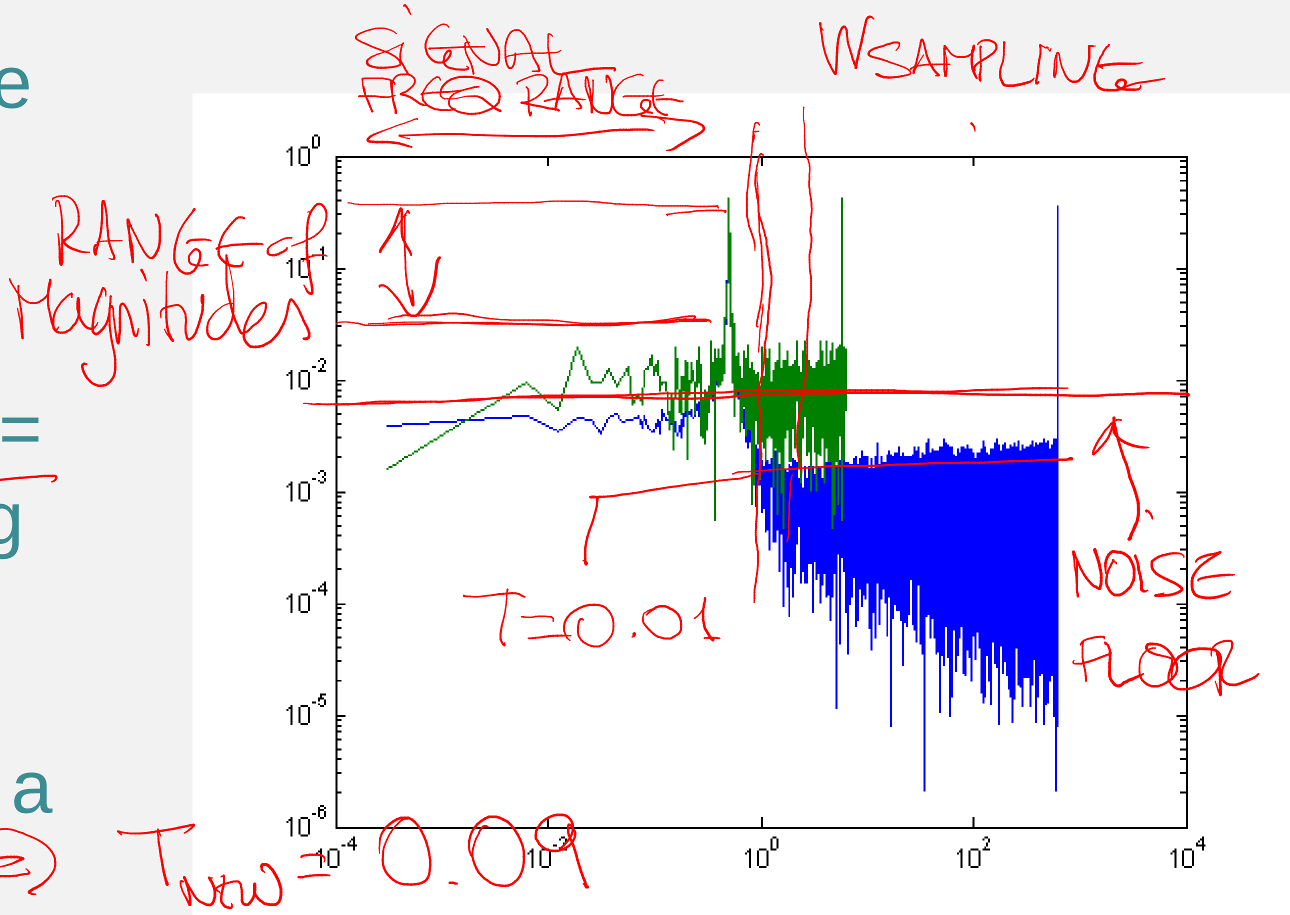
- Other applications include the determination of the RMS noise level for a given sampling rate. In all cases the theme is that the sampling rate is essentially determined by the Nyquist rate of the deterministic part of the signal, subject to a constraint on the reduction of the SNR due to sampling.

- Suppose that in our earlier example, we would like to select the sampling rate to detect sinusoids up to frequency 1 rad/s and amplitudes 0.1-1 with 12dB SNR.

  A

  The first constraint is the Nyquist rate of the signals (2rad/s, T = 3.14). The smallest peaks would be 0.3A (including a smearing effect) so around 0.03. Including a 12dB SNR (4x), the noise should be smaller than 7.5e-3. The current level, found by sampling at T=0.01 has yielded noise at 2.5e-3. The margin is a factor of 3, implying that T can be increased by a factor of 9. ⇒ $T_{new} = 0.09$

  $\sqrt{N}$



- Note: The estimate of SNR is facilitated by the reasonable assumption that the noise is uncorrelated with all other signals, hence orthogonal, implying $RMS^2(x+v) = RMS^2(x) + RMS^2(v)$

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Lecture 3.4: ZOH reconstruction

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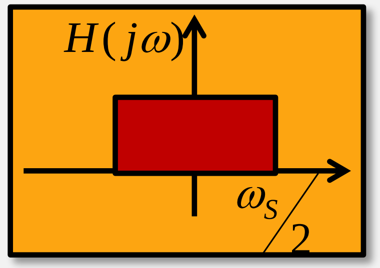
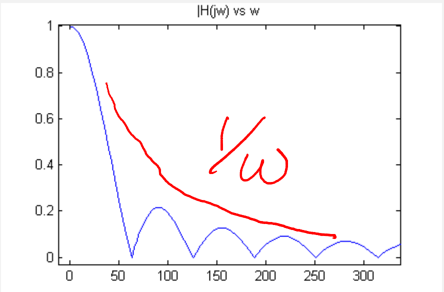
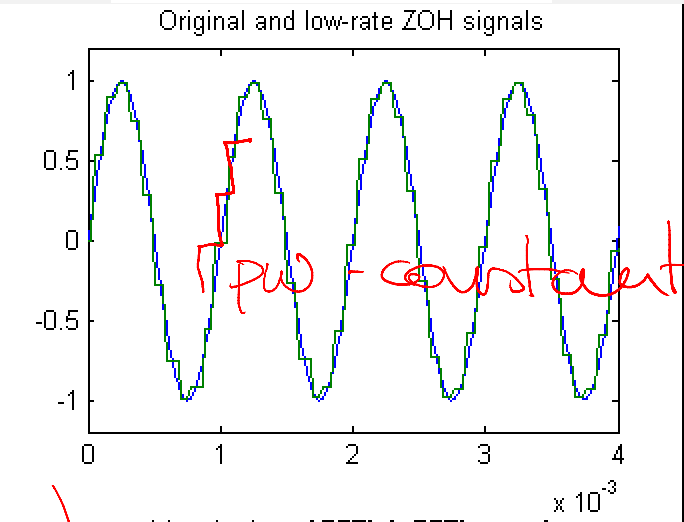
Non-Ideal reconstruction: ZOH (DAC)

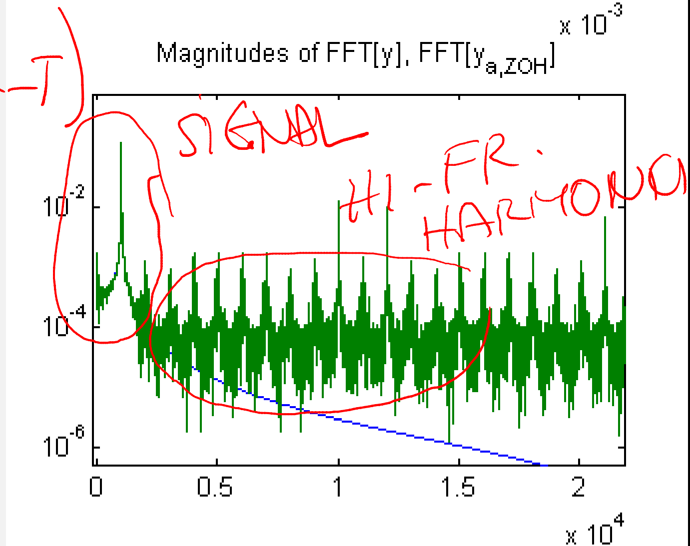
- ZOH reconstruction is a typical approach with computers and DAC.
- As a filter, the ZOH attenuates as a 1st order system. This means that a portion of the energy in the high frequency replicas goes through.
- For example, the ZOH reconstruction of the sinusoid contains discontinuities that are seen in the FFT of the signal as high frequency harmonics.

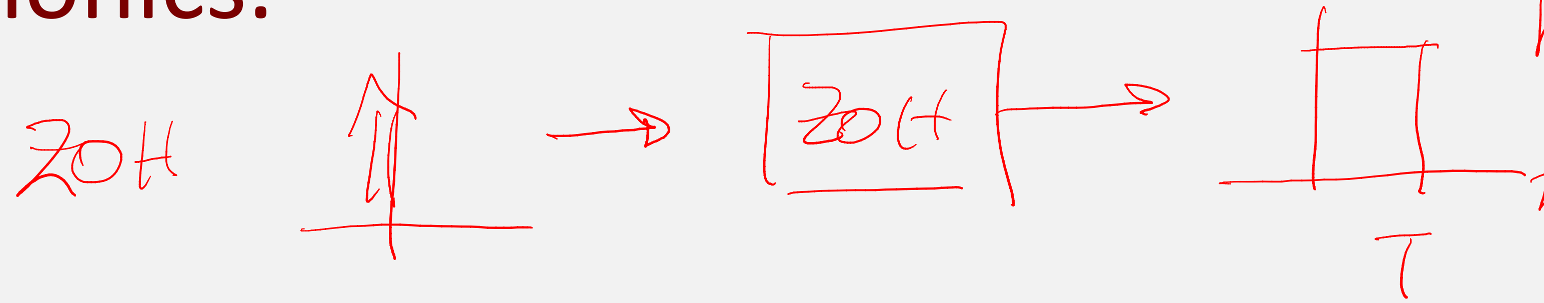


ZOH





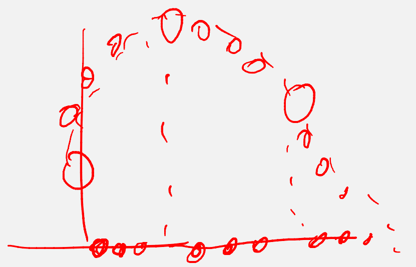




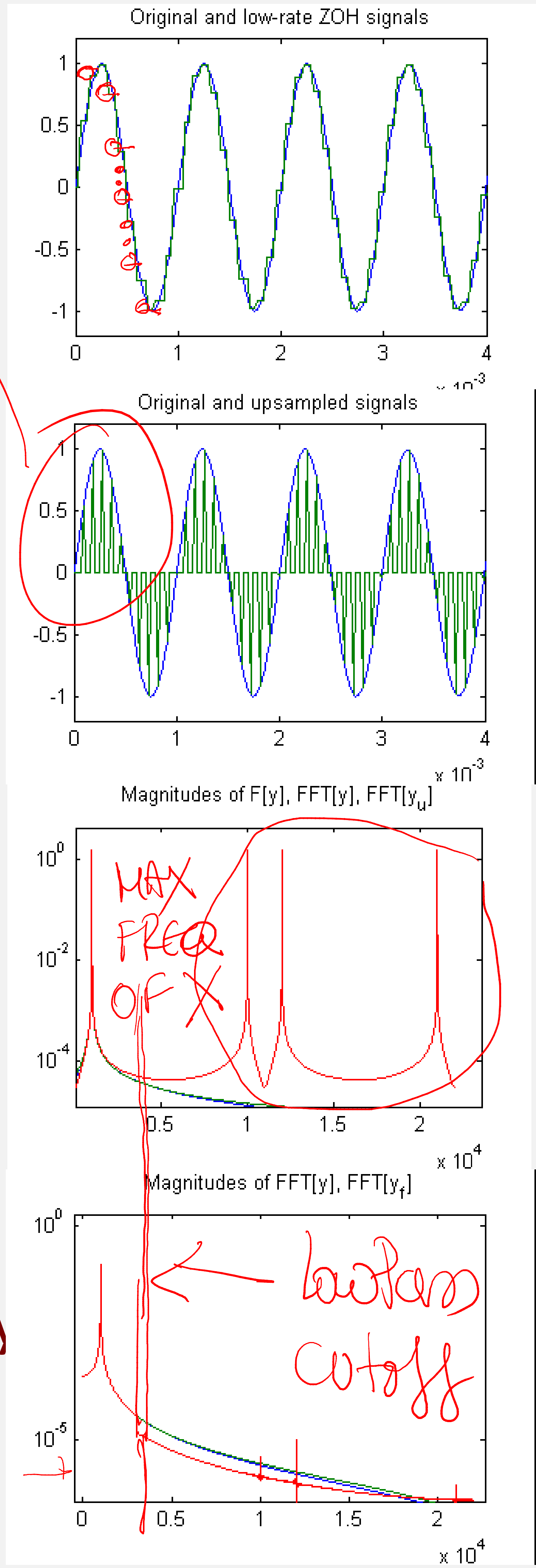
$$h(t) = u(t) - u(t-T)$$

$$H(s) = \frac{1-e^{-sT}}{s}$$

DAC Oversampling



- A remedy for the shortcomings of ZOH reconstruction is the use of oversampling.
  - While the original signal remains the same, we can interpolate a higher sampling rate signal and reconstruct it with a higher rate DAC (shorter spacing between samples, hence smaller discontinuities, at higher frequencies).
  - This technique is implemented with an “up-sampling” step (i.e., interlace zeros between samples) and filtering with a high order digital filter
  - The “up-sampled” signal contains replicas of the sampled signal spectrum (here, a 4x up-sampling is shown) at the new, higher, sampling frequency.
  - After filtering, the other replicas are attenuated in the extended frequency range while the reconstruction signal occurs in the flatter portion of the ZOH filter (less distortion)

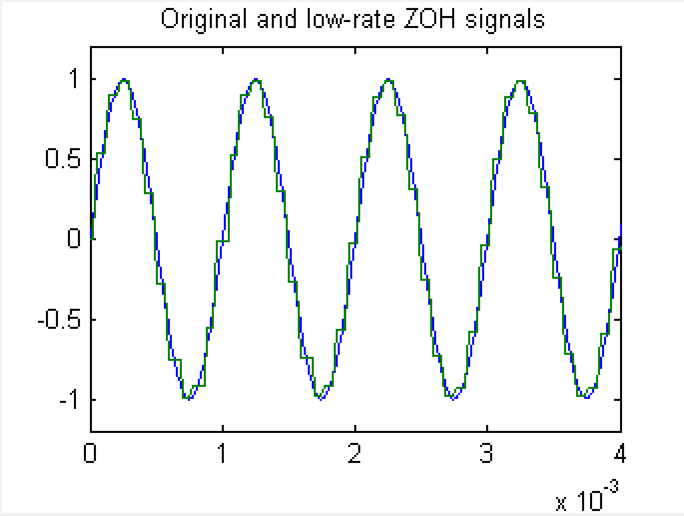


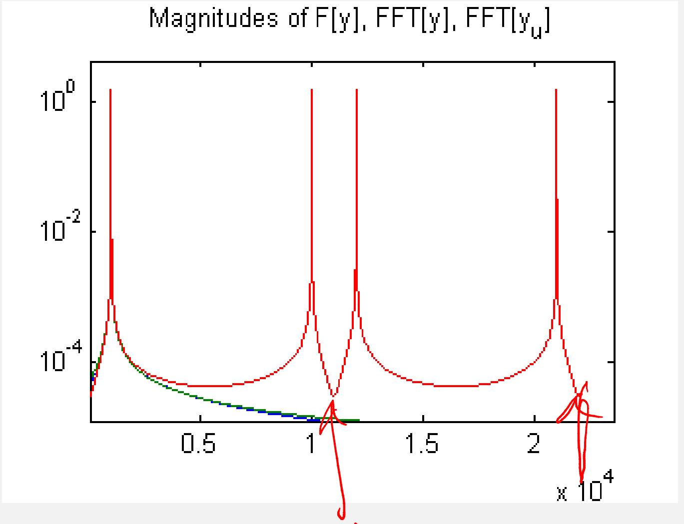
DAC Example: Reconstruction error

- Suppose we sample a 1kHz signal at 11kHz and then reconstruct it with a DAC (i.e., ZOH). We want to find the distortion of the signal and the contribution of the high frequency harmonics.
  - The Fourier transform and Fourier Series coefficients of the signal are given below:

$$X(j\omega)=\pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right),\quad \omega_0=2\pi 1000,\ \ FS: a_k=\tfrac{1}{2}, k=\pm 1$$

  - The sampled signal has a Fourier transform that can be found as the convolution with the F-T of the impulse train of the sampling sequence (Lecture 3.1). This transform has impulses of amplitude T=1/11e3 and located around the sampling frequency harmonics $n\,{}^{2\pi}\!/_{T} \pm 1kHz$





ω_s $2\omega_s$

DAC Example : Reconstruction error

- Its ZOH reconstruction yields a train of weighted impulses:

$$\underbrace{H_{ZOH}(j\omega)}\,\underbrace{X_S(j\omega)} = \boxed{\frac{1-e^{-j\omega T}}{j\omega} X_S(j\omega)}$$

- It is of interest here to compute the magnitudes of the coefficients so that we can determine the contribution of the various components on the reconstructed signal:

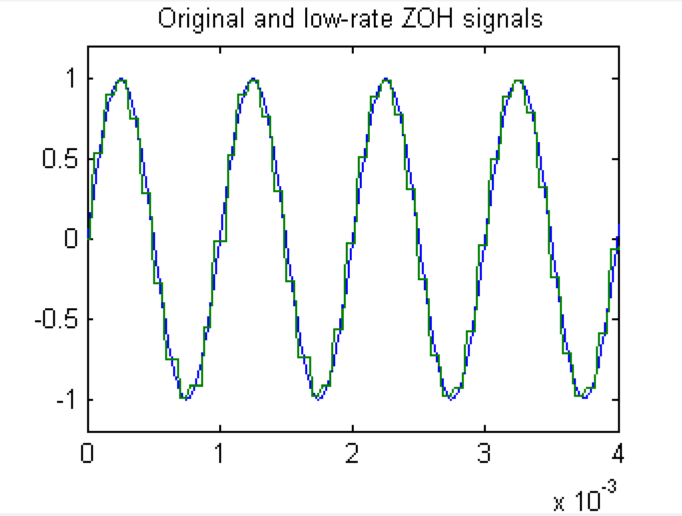
$$\rightarrow \pm 1kHz, \quad |a_k|^2 = 0.973/4$$

SHOULD BE 1 if no distortion occurs

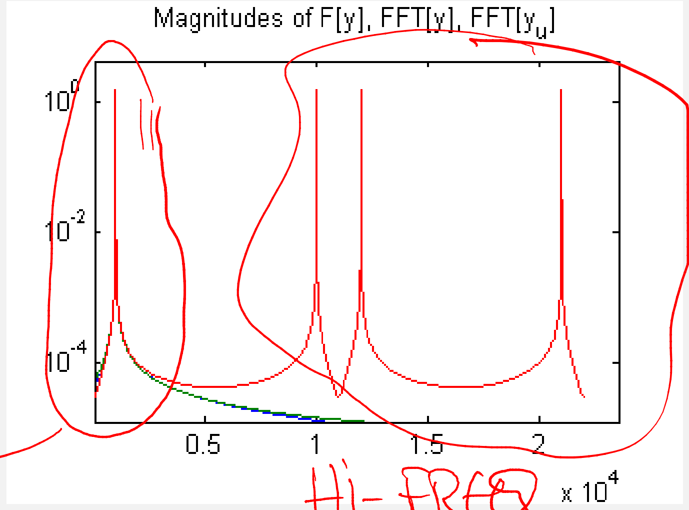
$$11 \pm 1kHz \quad |a_k|^2 = \underline{0.0068}/4, 0.0097/4$$

$$22 \pm 1kHz \quad |a_k|^2 = \underline{0.0018}/4, 0.0022/4$$

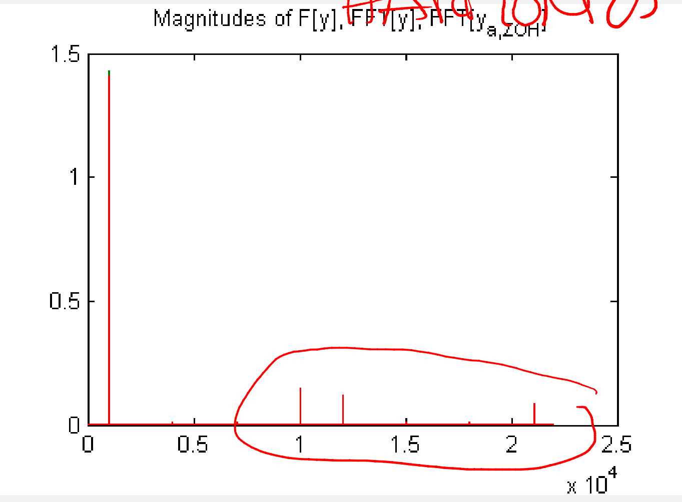
- The 0th harmonic represents a distorted version of the original signal while the rest of the harmonics represent the reconstruction error, including the discontinuities.



ORIGINAL SIG



HI-FREQ HARMONICS



DAC Example : Reconstruction error

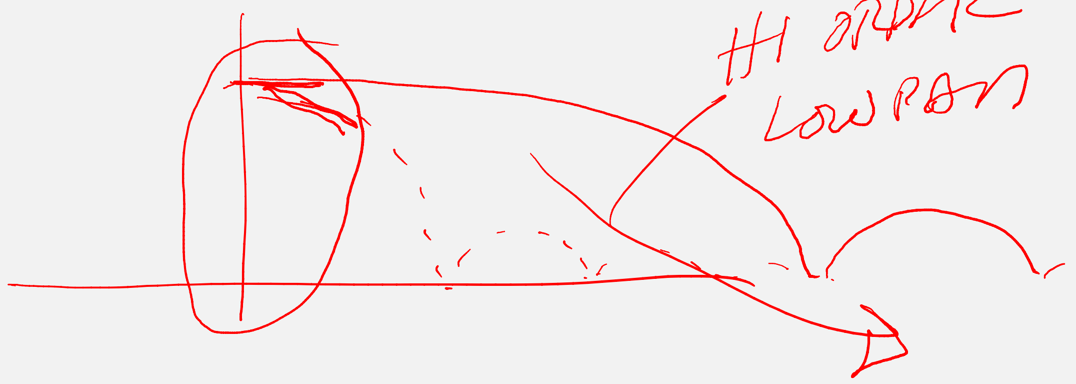
- From Parseval's theorem, we can compute the contribution of each component to average power of the reconstructed signal:
- Original signal average power = 2(¼) = ½
- The two 0^{th}-harmonic components contain 97.3% of the original signal power. For the distortion, however, we need to compute the corresponding coefficient of the error transfer function

$$E(j\omega) = X(j\omega) - H_{ZOH}(j\omega)X_S(j\omega) = \left[1 - \left(\frac{1-e^{-j\omega T}}{j\omega}\right)\right]\left\{\pi\left(\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right)\right\} + H.F.T.$$

$$\Rightarrow |b_k|^2 = 0.08/4 \Rightarrow \text{Distortion} = 8\%\ (28\%\ \text{RMS})$$

- The 4 components around the 1^{st} harmonic contain 1.65% of the original power. This is a high frequency error contribution, introduced by the ZOH. (Similarly for the other harmonics.)

DAC Example: Oversampling



- With similar computations we can analyze the benefits of oversampling in DAC reconstruction of audio signals (excluding "perception" issues).
- The original signal is sampled at 44.1kHz and the highest audio frequency to be reconstructed is 20kHz. If reconstructed with a 44.1kHz DAC, its distortion is computed by the error coefficient of the transfer function at 20kHz

$$\frac{1-e^{-j\omega T}}{j\omega}, \quad \omega = 2\pi 20000, \quad T = 1/44100 \Rightarrow |a_k|^2 = 0.482/4 \quad \text{Error t.f.: } 1-\left(\frac{1-e^{-j\omega T}}{j\omega}\right) \Rightarrow |b_k|^2 = 1.28/4$$

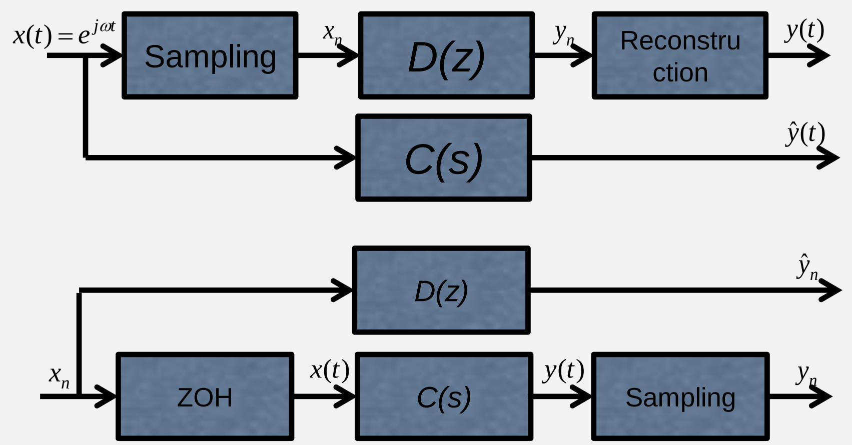
- Next, suppose that we perform an up-sampling process by 16x (i.e., we up-sample and filter in software and reconstruct with a 16x DAC (176kHz). Reevaluating the transfer function magnitudes $T = 1/44100/16 \Rightarrow |a_k|^2 = 0.997/4, \quad |b_k|^2 = 0.008/4$
- That is, due to phase difference, the standard reconstruction can distort the highest frequency by 128% (113% RMS), but with 16x oversampling, this figure drops to 0.8% (9% RMS).

EEE304

Lecture 3.5: Other Classic Sampling Problems

Arizona State University

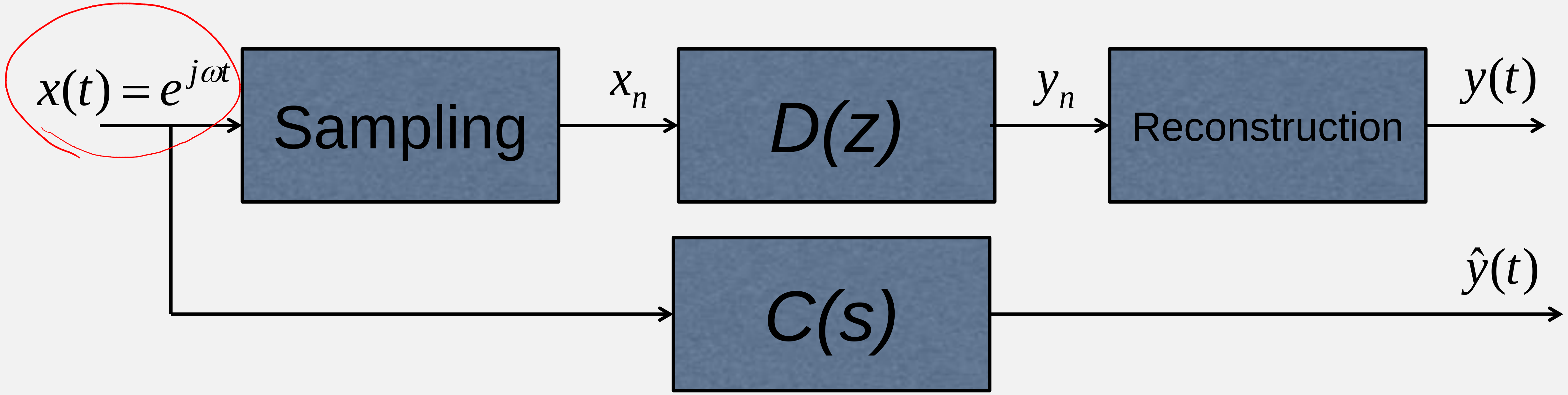
DT-CT filter "equivalence"



- DT digital filters replacing CT analog filters
  - Implementation, flexibility, consistency, power requirements, maintenance
  - Experience and specifications are available mostly in CT terms
- CT and DT (sampled-data) filters cannot be strictly equivalent, but we can define an approximation for two classes of problems:
  - Given a bandlimited signal within the Nyquist frequency of the sampling system, determine H(z) or H(s) so that the CT output is identical to the ideal-filter reconstructed DT output.
  - Given a sampled-data signal (pw constant) determine H(z) or H(s) so that the sampled CT output is identical to the DT output.

DT-CT filter "equivalence"

- Bandlimited signals: Matching the frequency response of the two filters



– Given D(z), find equiv. C(s)

– Given C(s) find equiv. D(z)

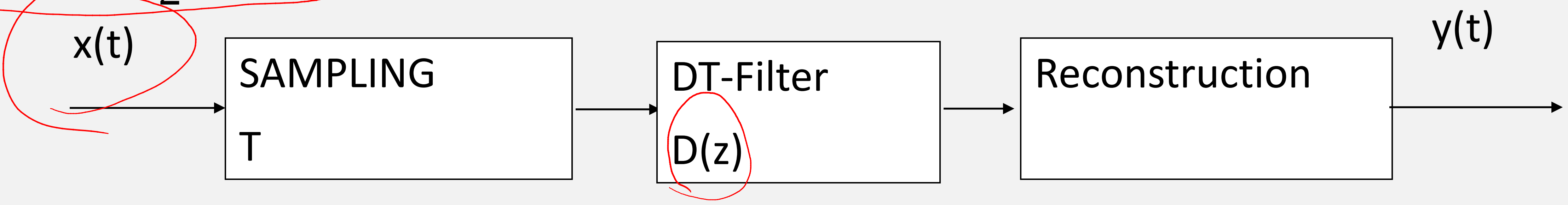
- Sampling x(t) we get $x_n = x(nT) = e^{j\omega nT} = \left(e^{j\omega T}\right)^n = \left(e^{j\Omega}\right)^n = z^n$
- Then $y_n = D\left[e^{j\Omega}\right]\left(e^{j\Omega}\right)^n$
- After ideal reconstruction, $y(t) = REC\left\{D\left[e^{j\Omega}\right]e^{j\Omega n}\right\} = REC\left\{D\left[e^{j\Omega}\right]e^{j\omega Tn}\right\} = D\left[e^{j\Omega}\right]e^{j\omega t}$

REC

- Matching the frequency responses $\boxed{D\left(e^{j\omega T}\right) = C(j\omega)}$

Sampling and DT processing

- Suppose that a CT, bandlimited signal x(t) is processed by DT system with sampling time T and ideal reconstruction. The sampled signal is processed by a DT system such that $y(n)=\frac{1}{2}y(n-1)+\frac{1}{2}x(n-1)$. Find the equivalent CT system frequency response.



- The transfer function of the system is obtained by taking the $\mathcal{Z}$-transform of both sides of the difference equation: $D(z)=\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} \Rightarrow D(e^{j\omega T})=\frac{\frac{1}{2}}{e^{j\omega T}-\frac{1}{2}}$

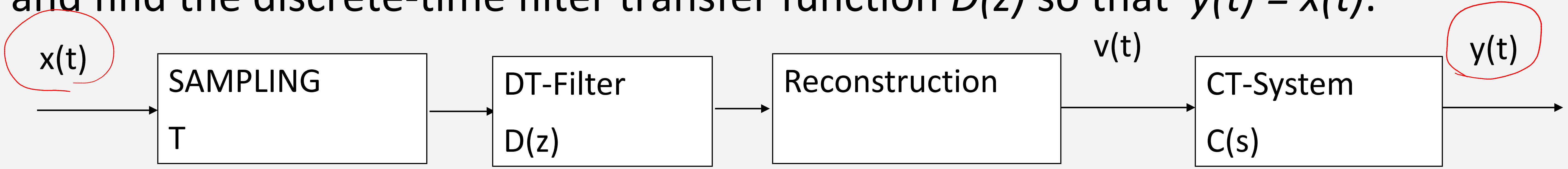
$\omega T = \Omega$

- Matching the frequency responses, $C(j\omega)=\frac{\frac{1}{2}}{e^{j\omega T}-\frac{1}{2}}$
- Approximation using FE yields: $C(s)=\left.\frac{0.5}{z-0.5}\right|_{z=1+sT}=\frac{0.5}{sT+0.5}$

$e^{sT}=1+sT+\cancel{H.O.T.}$

DT-CT filtering example

- Suppose that a continuous time signal x(t) is bandlimited to 100Hz and it is pre-processed by DT system with ideal sampling and reconstruction. The output of the discrete system is then processed by a CT system with transfer function $C(s)=\dfrac{1}{(10s+1)}$. Select a suitable sampling time T and find the discrete-time filter transfer function *D(z)* so that *y(t) = x(t)*.



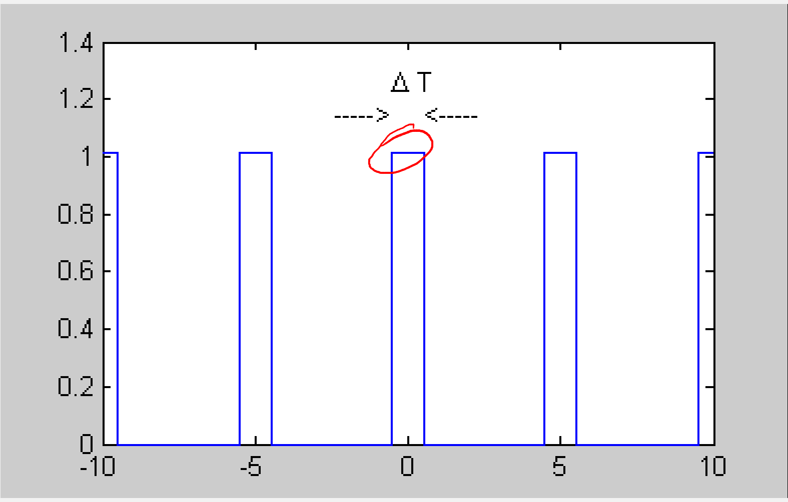
- The sampling rate should be faster than twice the maximum signal frequency (200Hz) so T < 0.005s or 5ms. For example, without any other constraints, following the rule of thumb "an order of magnitude faster than Nyquist", T = 0.5ms is a suitable choice.
- Matching the frequency responses, $C(j\omega)D(e^{j\omega T})=1 \Rightarrow D(e^{j\omega T})=10j\omega+1$, or, $D(z)=1+10\log(z)/T$
- Approximation using FE yields: $D(z)=10s+1\big|_{s=\frac{z-1}{T}}=\dfrac{10z-(1-T)}{T}$ $\cdot\left(\frac{1}{z}\text{ for realization}\right)$

Time-sharing sampling example

- Suppose that a CT, ω_M -bandlimited signal x(t) is sampled by a periodic square wave with frequency 1/T and duty cycle Δ. Find conditions on T and Δ to guarantee that no aliasing occurs.

$$p(t) = \sum_n s_\Delta(t - nT); \quad \text{where } s_\Delta \text{ is a pulse of duty cycle } \Delta$$

$$P(j\omega) = 2\pi \sum_k a_k \delta(\omega - k\omega_S); \quad \omega_S = \tfrac{2\pi}{T},\ a_k = \mathcal{FS}\{s_\Delta\}$$

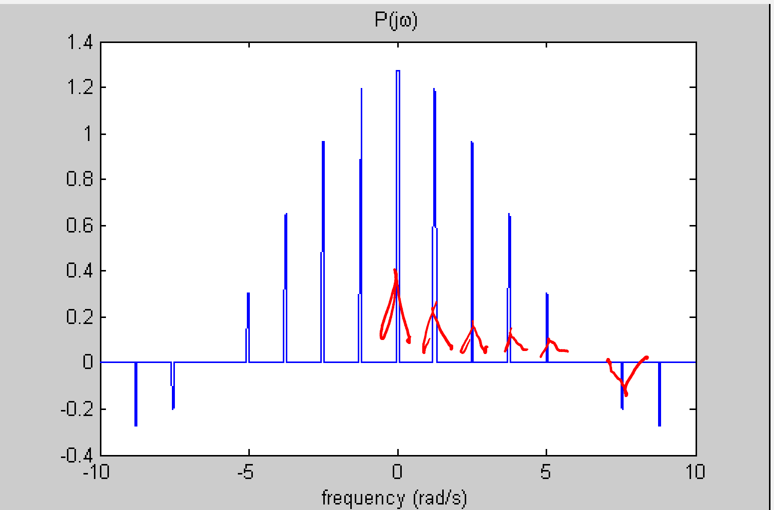


- The Fourier transform of p(t) is found from the Fourier Series coefficients of the square wave. From the tables (or direct calculation) we find

$X_s(j\omega) = \frac{1}{2\pi} P(j\omega) * X(j\omega)$ $P(j\omega)$

$$a_k = \frac{\sin\left(k \frac{2\pi}{T} \cdot \frac{\Delta T}{2}\right)}{k\pi} = \frac{\sin(k\pi\Delta)}{k\pi}$$

- In the plots, we graph the sampling sequence p(t) for $T = 5, \Delta = 0.2$
- It follows that P(jω) is a weighted impulse train, with $\omega_0 = 1.26$

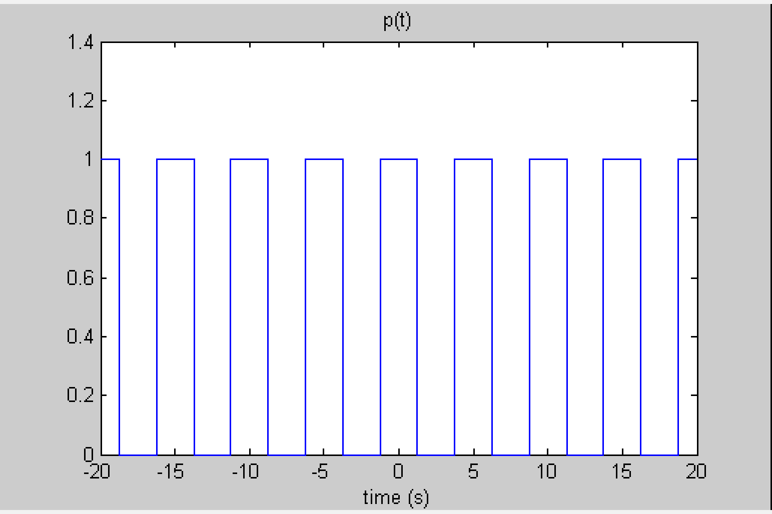


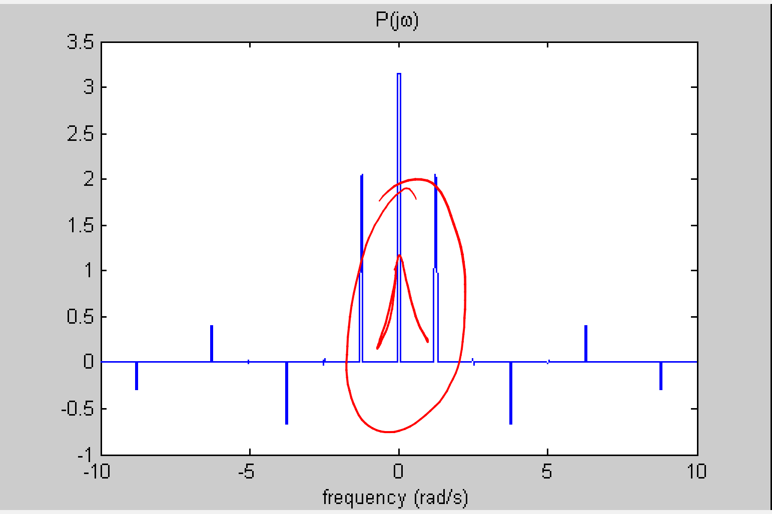
Time-sharing sampling example

- To ensure that no aliasing occurs, we should have $\boxed{\omega_M < \frac{\pi}{T} = 0.628}$, regardless of Δ.
- Reconstruction can be performed as usual, with a lowpass filter of Bandwidth $\frac{\omega_S}{2} = \frac{\pi}{T}$.
- One special case occurs for 50% duty cycle, $\Delta = 0.5$. Here, all the even-k impulses are zero. If, in addition, we create a zero-mean square wave (i.e., alternating between 1 and -1, instead of 1 and 0), then the k = 0 impulse will also be zero. This implies that the no-aliasing condition is now

$$\boxed{\omega_M < \frac{2\pi}{T} = 1.26}$$

- The reconstruction of such a sampled signal would require an extra “frequency shifting” operation (demodulation) to move the first harmonic replicas to zero, before the usual lowpass filtering operation.





Bandpass sampling example

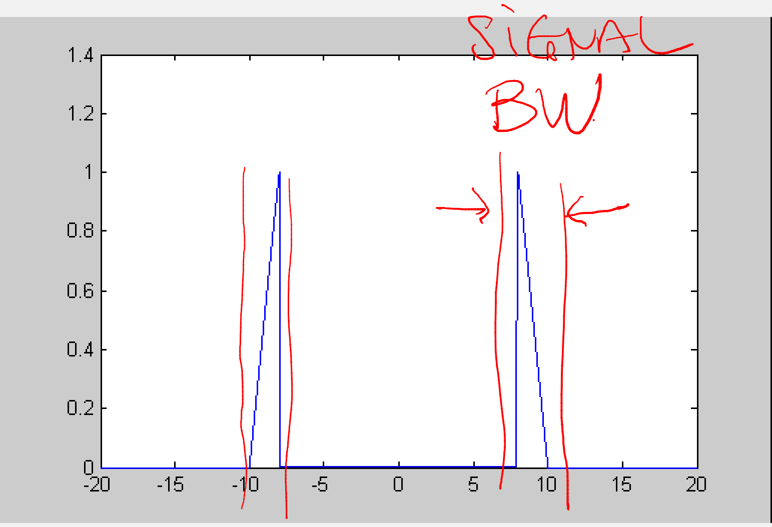
- Suppose that a CT bandlimited signal x(t) has its energy concentrated in a frequency band, as shown below. Such signals are referred to as “bandpass” signals. Bandpass signals with energy in the intervals $\pm[\omega_{\min}, \omega_{\max}]$ can be sampled and reconstructed at a frequency higher than twice their bandwidth, that is, $2(\omega_{\max} - \omega_{\min})$, instead of the much more conservative $2\omega_{\max}$. One additional condition for this is $\omega_{\min} > (\omega_{\max} - \omega_{\min})$. We are asked to select the sampling frequency so that there is no aliasing between the replicas and the reconstruction can be achieved with an ideal bandpass filter in the same interval.

- With a sampling frequency that satisfies the above condition,

$$P(j\omega) = \frac{2\pi}{T}\sum_k \delta(\omega - k\omega_S); \quad \omega_S = \frac{2\pi}{T} > 2(\omega_{\max} - \omega_{\min})$$

we have that the lower frequencies of the replicas will be located at

$$k\omega_S + \omega_{\min}, \quad k\omega_S - \omega_{\min}, \quad k = 0, \pm 1, \pm 2, \ldots$$



ω_{min} ω_{max}

Generic Sampling $\omega_s > 2\omega_{MAX}$

Bandpass sampling example

- Consider the integers n-1, n such that the shifted replicas are below and above the center replica. Then the no aliasing condition is

$$(n-1)\omega_S - \omega_{\min} < \omega_{\min} \quad \wedge \quad \omega_{\max} < n\omega_S - \omega_{\max}$$

$$\text{Define: } \mathrm{BW} = \omega_{\max} - \omega_{\min}, \quad \boxed{\overline{\omega}_* = \omega_* / BW}. \ \text{ Then,}$$

all freqs

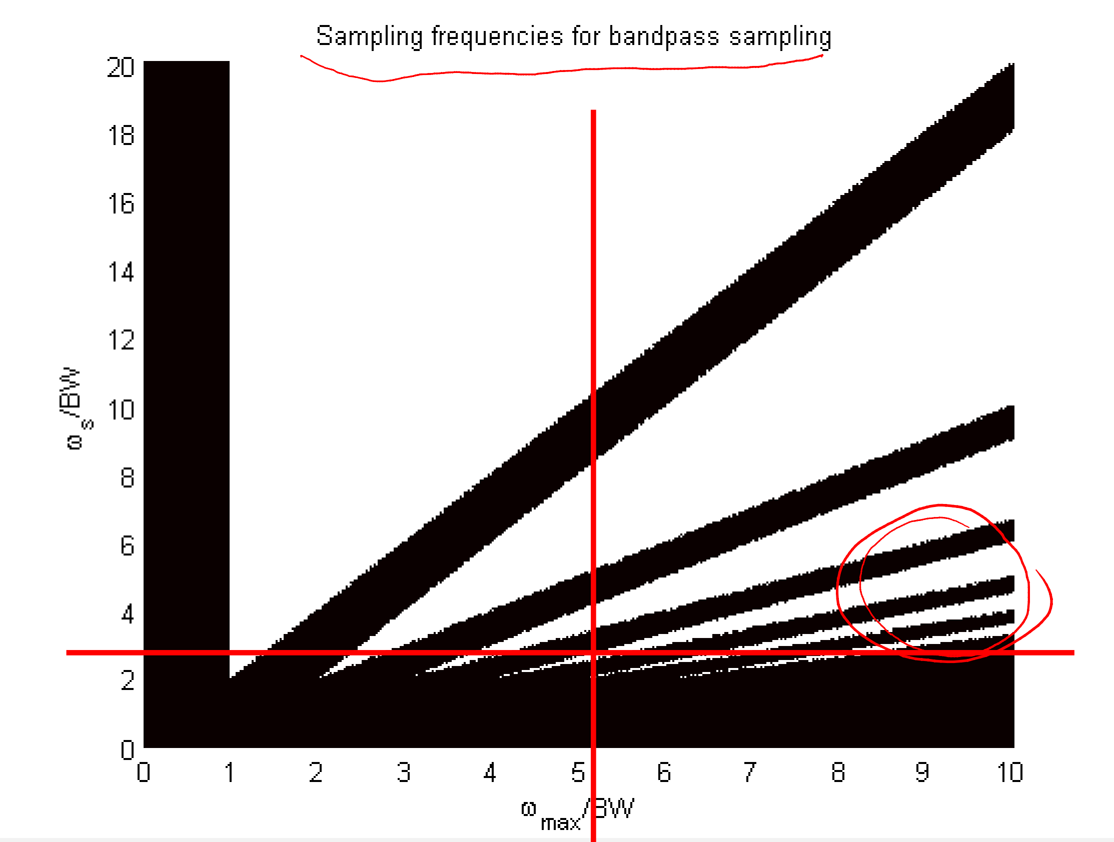
$$\left\{\begin{matrix} 2\overline{\omega}_{\max} < n\overline{\omega}_S < 2\overline{\omega}_{\max} + \overline{\omega}_S - 2 \\ n < [\overline{\omega}_{\max}] \end{matrix}\right\} \text{ Bandpass Sampling Conditions}$$

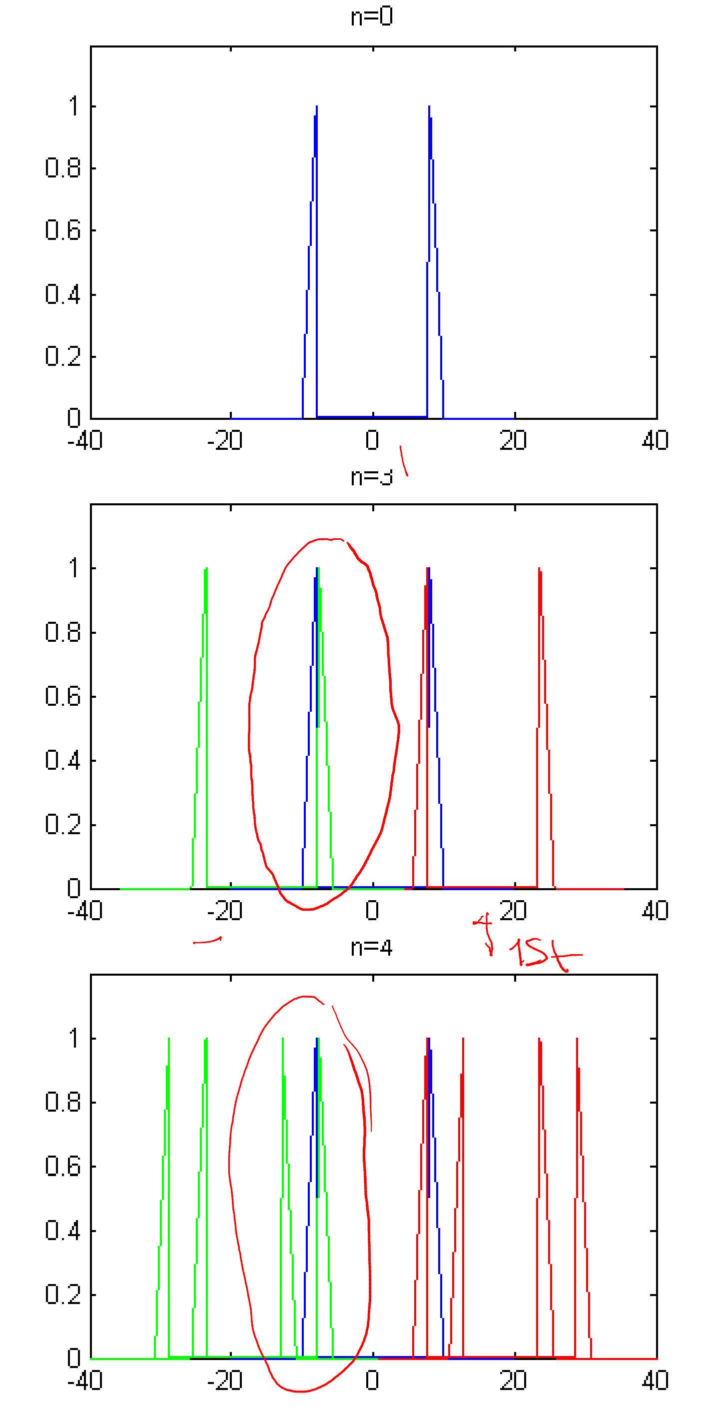
- Plotting the intervals of normalized frequencies where the conditions are satisfied for some n we obtain the adjacent graph.

- The selection of sampling rate is no longer simple and can be very sensitive.

Here :

$$\omega_{\max} = 10, \omega_{\min} = 8, \omega_S = 5.2$$





Bandpass sampling example

- As a practical application, consider the sampling of a 25kHz signal plus 5kHz guard bands transmitted at 10.7MHz. Then the max n = [10730/30] = 357. Choosing n = 300, the normalized sampling rate limits are

$$2 \times 357.67 < 300\overline{\omega}_S < 2 \times 357.67 + \overline{\omega}_S - 2$$

$$2.3844 < \overline{\omega}_S, \quad \overline{\omega}_S < 2.3857$$

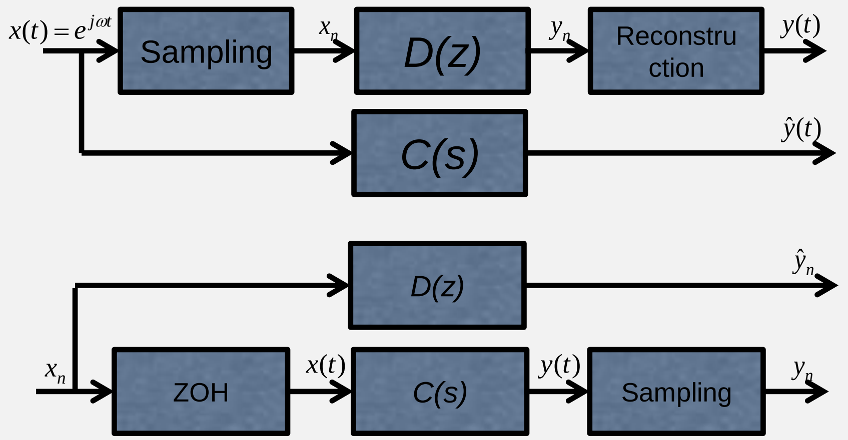
- In other words, even though the actual signal Nyquist rate is 21.4MHz, we can sample at 71.55kHz to be able to reconstruct the 30kHz signal.
- Also notice that, for this choice, the tolerance in the sampling frequency is in the 4th decimal of the multiplier, which translates to 18Hz.
- The theoretical minimum is 2.0037 x BW, which is very close to the sampling rate of the net signal, but with virtually no tolerances.

R.G. Vaughan, N.L. Scott, D.R. White, The theory of bandpass sampling, IEEE TSP, 39, 9, pp. 1973-1984, Sep. 1991

EEE304

Lecture 4.1: DT vs CT Filtering

Arizona State University

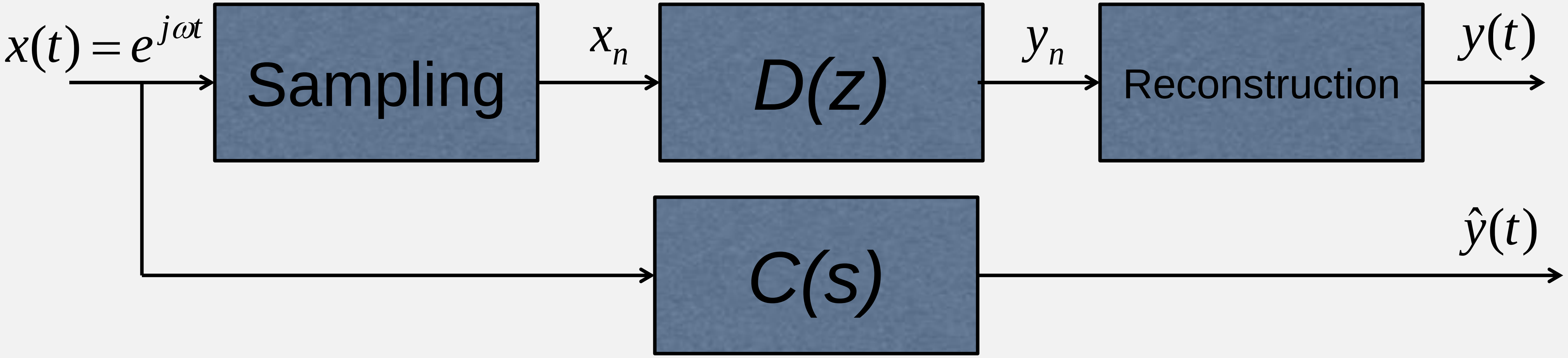


DT-CT filter “equivalence”

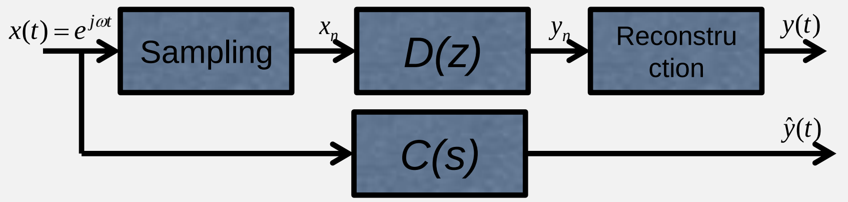
- DT digital filters replacing CT analog filters
  - Implementation, flexibility, consistency, power requirements, maintenance
  - Experience and specifications are available mostly in CT terms
- CT and DT (sampled-data) filters cannot be strictly equivalent, but we can define an approximation for two classes of problems:
  - Given a bandlimited signal within the Nyquist frequency of the sampling system, determine H(z) or H(s) so that the CT output is identical to the ideal-filter reconstructed DT output.
  - Given a sampled-data signal (pw constant) determine H(z) or H(s) so that the sampled CT output is identical to the DT output.

DT-CT filter "equivalence"

- Bandlimited signals: Matching the frequency response of the two filters



- Sampling x(t) we get $x_n = x(nT) = e^{j\omega nT} = \left(e^{j\omega T}\right)^n = \left(e^{j\Omega}\right)^n = z^n$
- Then $y_n = D\left[e^{j\Omega}\right]\left(e^{j\Omega}\right)^n$
- After ideal reconstruction, $y(t) = REC\left\{D\left[e^{j\Omega}\right]e^{j\Omega n}\right\} = REC\left\{D\left[e^{j\Omega}\right]e^{j\omega Tn}\right\} = D\left[e^{j\Omega}\right]e^{j\omega t}$
- Matching the frequency responses $D\left(e^{j\omega T}\right) = C(j\omega)$



DT-CT filter "equivalence:" FE

- Frequency response matching filters $D(z)\big|_{z=e^{sT}} = C(s)$. The transformation $z=e^{sT}$ is not finite-dimensional and should be approximated.
- One possibility is the so-called Forward Euler $z=1+sT$ (the first two terms of the Taylor series expansion). The FE approximates the frequency responses of DT/CT filters for low frequencies such that |sT| << 1.
- The FE transformation can also be motivated by the forward Euler approximation of the derivative

$$sX(s) \leftrightarrow \frac{dx}{dt} \approx \frac{x_{n+1}-x_n}{T} \leftrightarrow \frac{z-1}{T} X(z)$$

- This approach yields a straightforward iterative solution of general nonlinear differential equations. Higher order variants find extensive applications in numerical analysis.

$$s = \frac{z-1}{T}$$

DT-CT filter "equivalence:" Example FE1

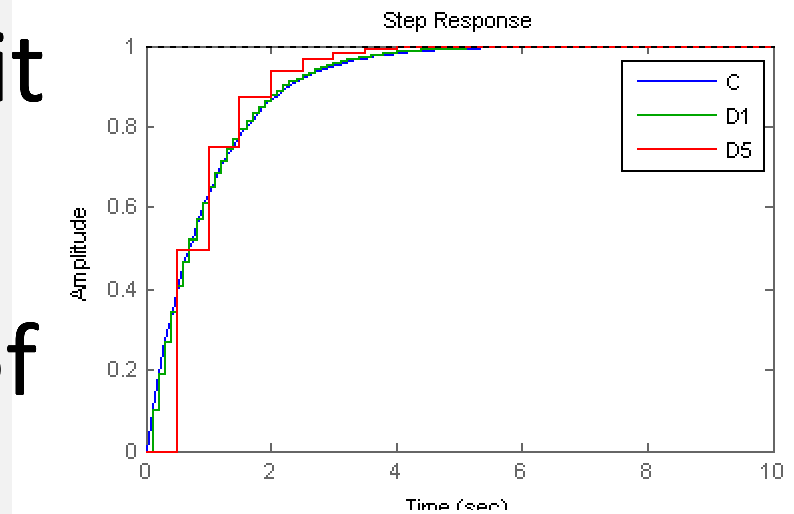
- The Forward Euler discretization of the CT transfer function $C(s) = \dfrac{1}{\tau s + 1}$

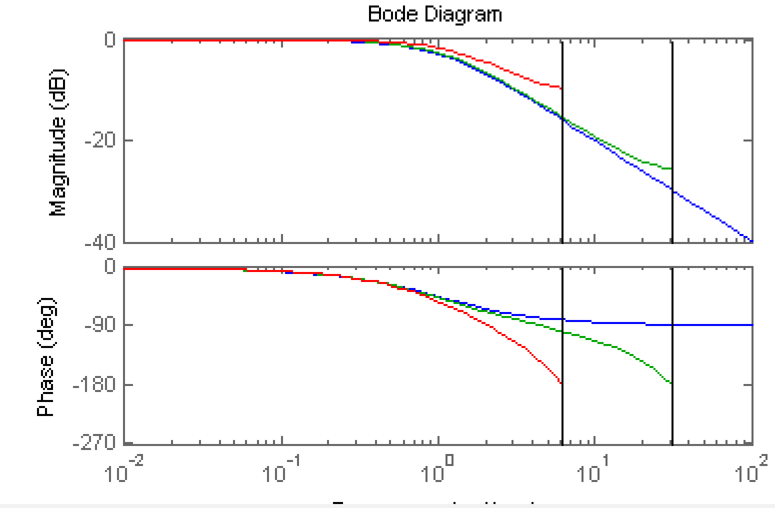
$$D(s) = \frac{T/\tau}{z - \left(1 - T/\tau\right)}$$

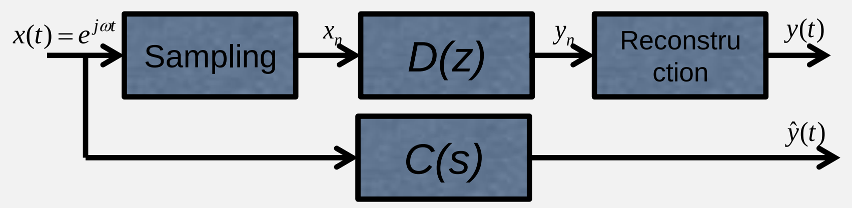
- The approximation is well-behaved for $T/\tau << 1$
- As T increases, the discretization gets naturally worse but it even becomes unstable when $T/\tau > 2$
- FE yields quick answer and preserves the relative degree of the transfer function.
- Its main limitation is that the sampling rate must be faster than the fastest system pole.

In the examples :

$\tau = 1, \quad T = 0.1, 0.5$







DT-CT filter "equivalence:" Tustin

- A better approximation to achieve frequency response matching is the so-called Tustin transformation

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} = \frac{1+sT/2}{1-sT/2} \Leftrightarrow s = \frac{2}{T}\cdot\frac{z-1}{z+1}$$

  This transformation provides a good approximation up to frequencies close to the Nyquist frequency (π/T), e.g., up to 1/3-Nyquist.

- Tustin belongs to the general class of "bilinear" transformations. It is also a special case of the Padé rational approximations of the delay.
- "Pre-warped" versions of Tustin allow the exact matching at some specific frequency of interest.
- MATLAB: c2d(C,T,'Tustin'), d2c(D,'Tustin')

$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter "equivalence:" Example T1

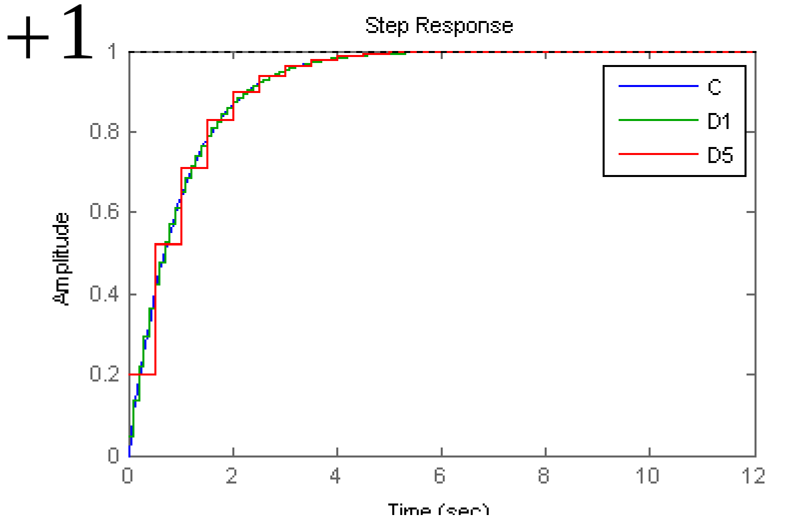
In the examples:

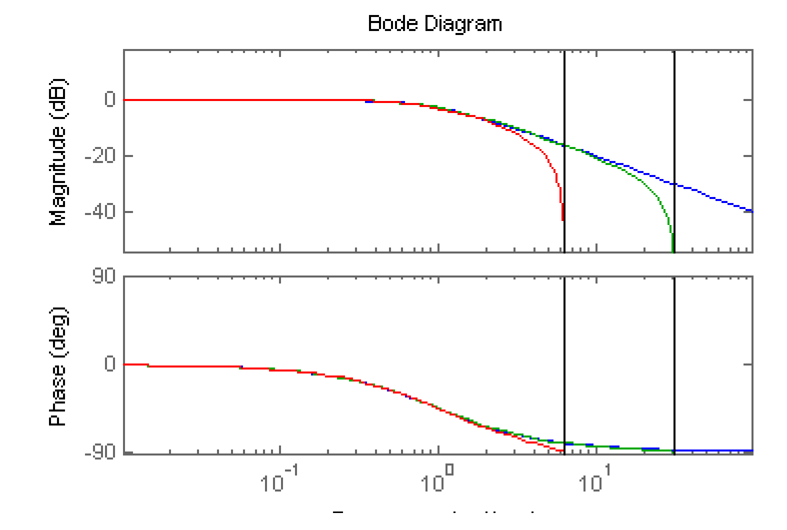
$\tau = 1, \quad T = 0.1, 0.5$

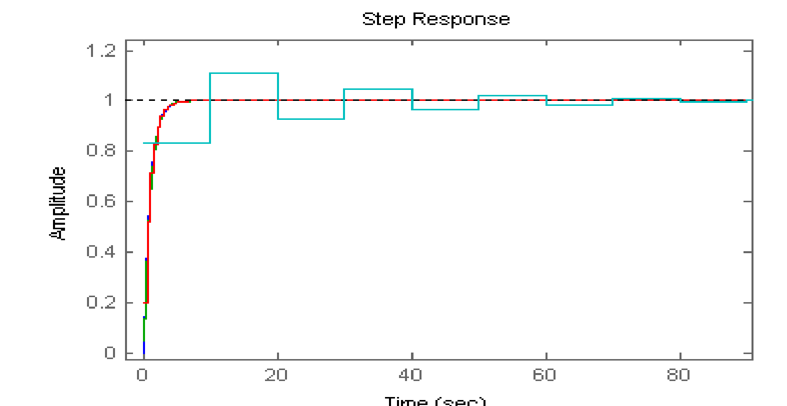
- The Tustin discretization of the CT transfer function $C(s) = \dfrac{1}{\tau s + 1}$

$$D1(s) = \frac{0.04762\,\mathrm{z} + 0.04762}{\mathrm{z} - 0.9048}; T = 0.1, \quad D5(s) = \frac{0.2\,\mathrm{z} + 0.2}{\mathrm{z} - 0.6}; T = 0.5$$

- The approximation is always well-behaved and interpolates closely the step response samples.
- As T increases, the approximation gets worse but only since the Nyquist frequency decreases. Even for T = 10, the response is "reasonable".
- Tustin equivalents always contain direct throughput (not strictly causal). That may present a problem in certain applications.

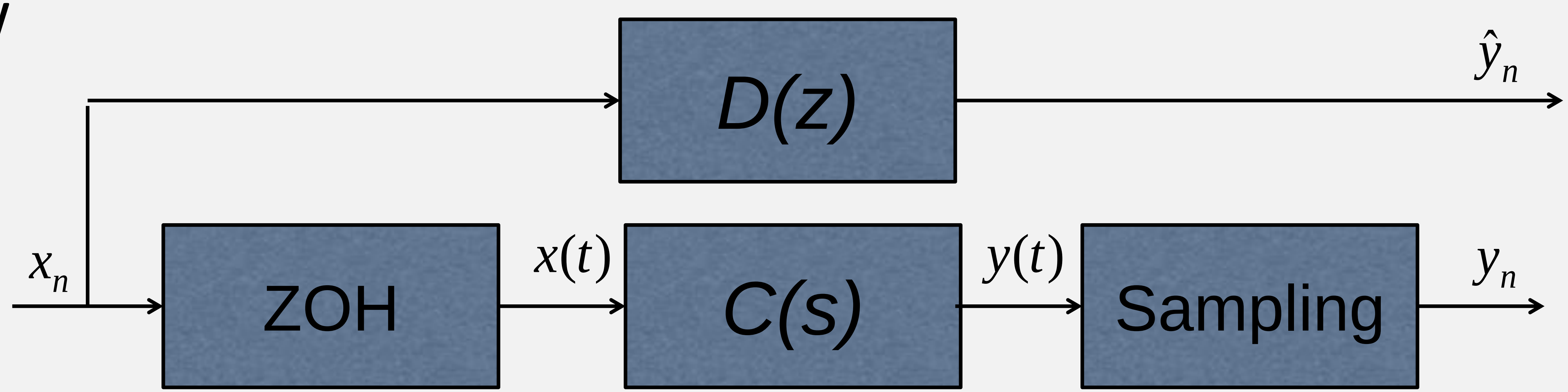






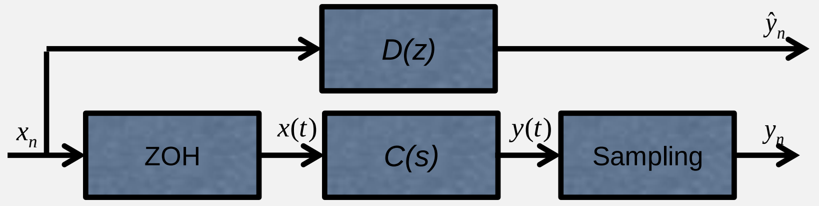
DT-CT filter "equivalence": ZOH

- Piece-wise constant signals: Matching the sampled step response of the two filters is a problem arising frequently in sampled-data systems, computer controlled systems etc. The setting for this problem is shown below



- Here, the input is assumed to be a sampled signal, that is reconstructed using a ZOH. On the other end, we only try to match the sampled responses. Such a matching can be exact and we refer to D(z) as the ZOH-equivalent of C(s).

DT-CT filter "equivalence": ZOH



$$D(z) = \frac{z-1}{z}\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{1}{s}\cdot C(s) \right\}\Big|_{t=nT} \right\}$$

- Computing the ZOH-equivalent discretization is fairly tedious. Consider the example $C(s) = \dfrac{1}{s+1}$, to illustrate the steps.
  1. Compute the step response of C: $y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s}\cdot\frac{1}{s+1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s} + \frac{-1}{s+1} \right\} = (1-e^{-t})u(t)$
  2. Sample the step response: $y_n = y(nT) = (1-e^{-nT})u(nT) = (1-\lambda^n)u_n; \quad \lambda = e^{-T}$
  3. Find the $\mathcal{Z}$-transform of y_n: $Y(z) = \mathcal{Z}\{y_n\} = \frac{z}{z-1} - \frac{z}{z-\lambda}; \quad \lambda = e^{-T}$
  4. Compute the corresponding t.f.: $D(z) = \frac{z-1}{z}Y(z) = 1 - \frac{z-1}{z-\lambda} = \frac{1-\lambda}{z-\lambda}; \quad \lambda = e^{-T}$
- Notice that, in contrast to FE and Tustin, the ZOH-equivalent is not multiplicative, i.e., cannot be performed as a combination of factors:

$$\mathcal{ZOH}\{C_1(s)C_2(s)\} \neq \mathcal{ZOH}\{C_1(s)\}\mathcal{ZOH}\{C_2(s)\}$$

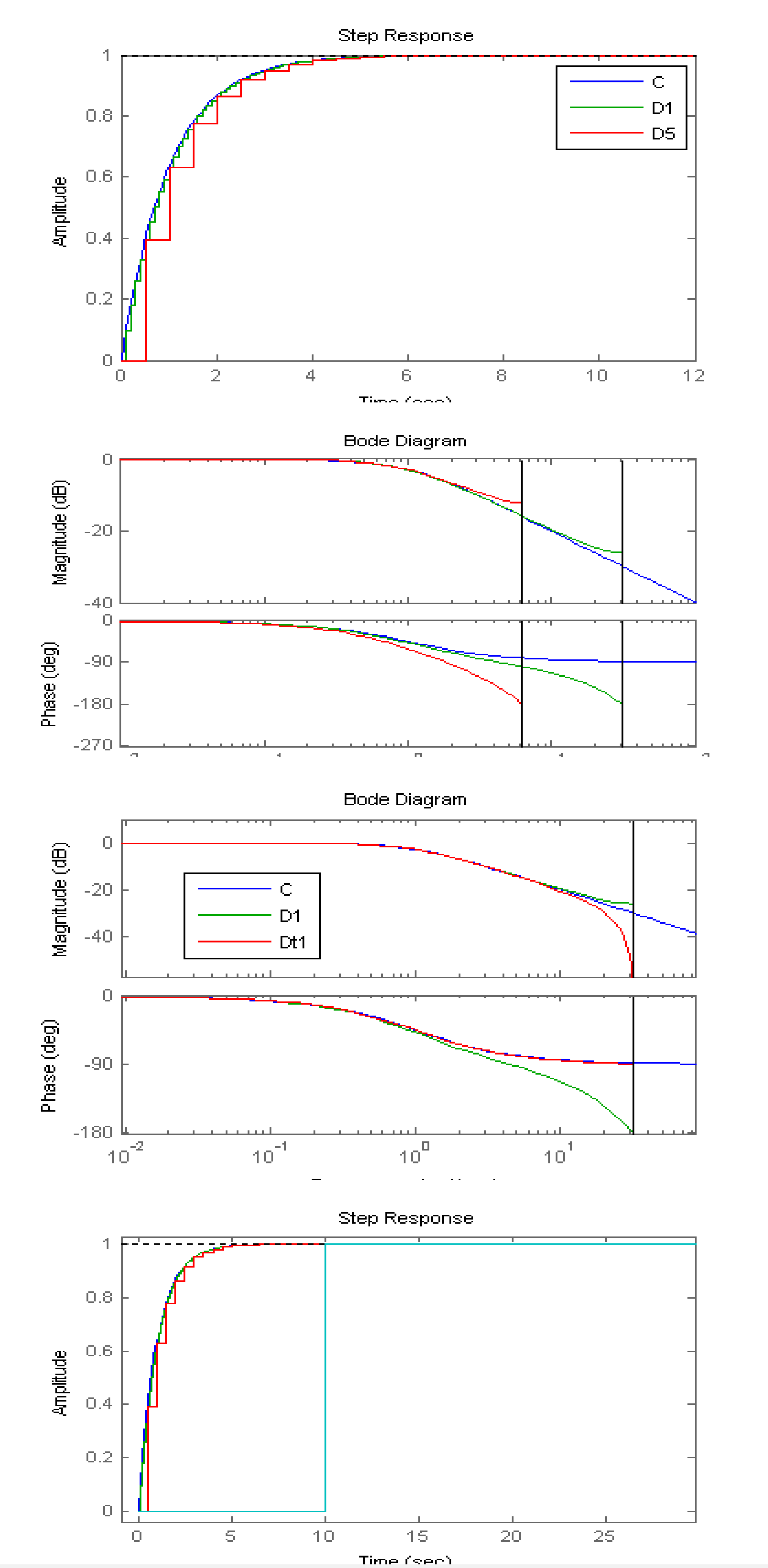
$$D(z)=\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\Big|_{t=nT}\right\}$$

DT-CT filter “equivalence:” Example Z1

- The ZOH discretization of the CT transfer function $C(s)=\dfrac{1}{\tau s+1}$

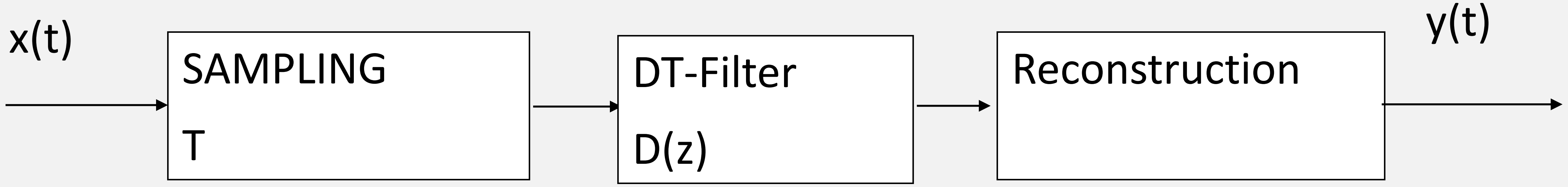
$$D1(s)=\frac{1-e^{-0.1}}{z-e^{-0.1}};T=0.1,\quad D5(s)=\frac{1-e^{-0.5}}{z-e^{-0.5}};T=0.5$$

- The approximation is always well-behaved and lags behind the CT response by ½ sample time on the average.
- Magnitude approximation is good, even past ½ -Nyquist but phase deviates rapidly even at 1/10-Nyquist (see comparison with Tustin-equivalent).
- As T increases, the approximation gets worse but only since the Nyquist frequency decreases. Even for T = 10, the response is “reasonable” (one-step, “dead-beat”).
- In MATLAB, c2d(C,T,’zoh’), d2c(C,’zoh’). (Default)



Sampling and DT processing

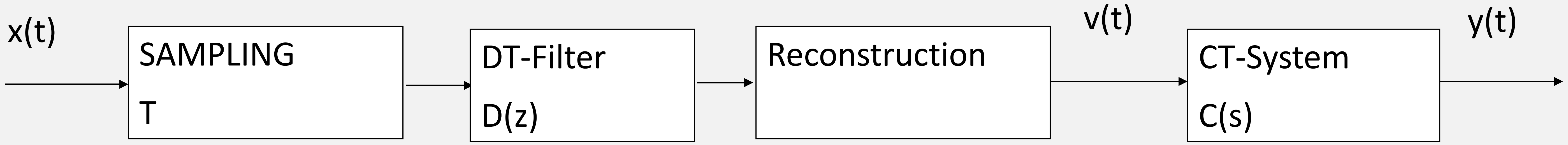
- Suppose that a CT, bandlimited signal x(t) is processed by DT system with sampling time T and ideal reconstruction. The sampled signal is processed by a DT system such that $y(n)=\frac{1}{2}y(n-1)+\frac{1}{2}x(n-1)$. Find the equivalent CT system frequency response.



- The transfer function of the system is obtained by taking the $\mathcal{Z}$-transform of both sides of the difference equation: $D(z)=\dfrac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} \Rightarrow D(e^{j\omega T})=\dfrac{\frac{1}{2}}{e^{j\omega T}-\frac{1}{2}}$
- Matching the frequency responses, $C(j\omega)=\dfrac{\frac{1}{2}}{e^{j\omega T}-\frac{1}{2}}$
- Approximation using FE yields: $C(s)=\left.\dfrac{0.5}{z-0.5}\right|_{z=1+sT}=\dfrac{0.5}{sT+0.5}$
- Approximation using Tustin yields: $C(s)=\left.\dfrac{0.5}{z-0.5}\right|_{z=\frac{1+sT/2}{1-sT/2}}=\dfrac{-s\frac{T}{3}+\frac{2}{3}}{sT+\frac{2}{3}}$

DT-CT filtering example

- Suppose that a continuous time signal x(t) is bandlimited to 100Hz and it is pre-processed by DT system with ideal sampling and reconstruction. The output of the discrete system is then processed by a CT system with transfer function $C(s)=\frac{1}{(10s+1)}$. Select a suitable sampling time T and find the discrete-time filter transfer function *D(z)* so that *y(t) = x(t)*.



- The sampling rate should be faster than twice the maximum signal frequency (200Hz) so T < 0.005s or 5ms. For example, without any other constraints, following the rule of thumb "an order of magnitude faster than Nyquist", T = 0.5ms is a suitable choice.
- Matching the frequency responses, $C(j\omega)D(e^{j\omega T})=1 \Rightarrow D(e^{j\omega T})=10j\omega+1$, or, $D(z)=1+10\log(z)/T$
- Approximation using FE yields: $D(z)=10s+1\big|_{s=\frac{z-1}{T}}=\frac{10z-(1-T)}{T}$
- Approximation using Tustin yields: $D(z)=10s+1\big|_{s=\frac{2(z-1)}{T(z+1)}}=\frac{20(z-1)+T(z+1)}{T(z+1)}$

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Lecture 4.2: DT vs CT Filtering: More Examples

Arizona State University

$$s = \frac{z-1}{T}$$

DT-CT filter "equivalence:" Example FE2

- To illustrate the implication of the FE sampling rate constraint, let us consider the system

$$C(s) = \frac{1}{(s+1)(0.01s+1)}$$

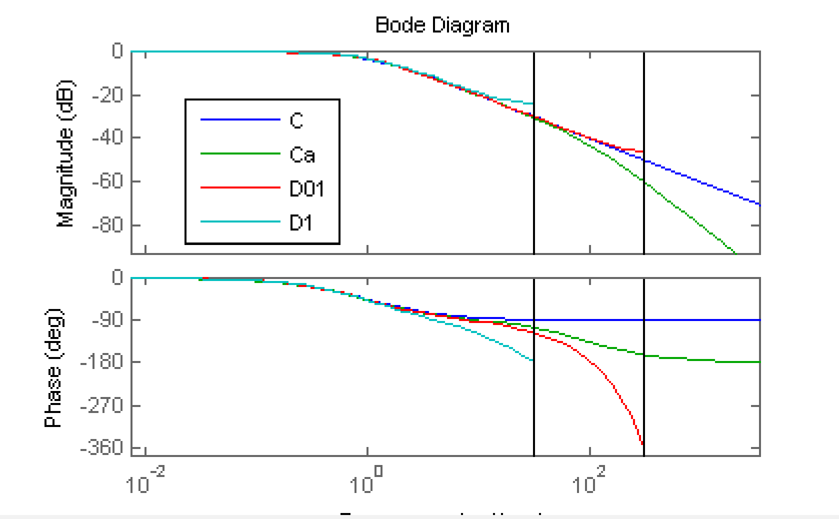
Such characteristics arise in systems that are composed of slow and fast subsystems, e.g., mechanical and electrical components.

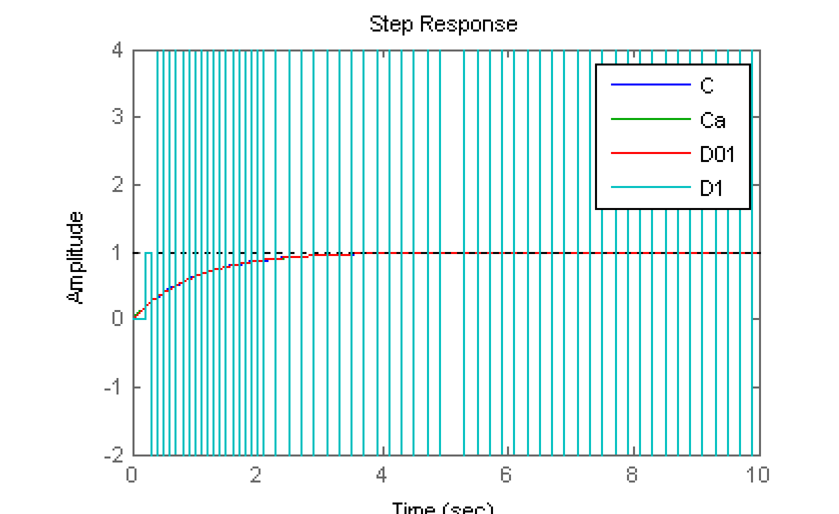
- PFE yields $C(s) = \frac{1.01}{(s+1)} + \frac{-0.0101}{(0.01s+1)}$, indicating that $C(s) \approx \frac{1}{(s+1)}$, something that is also verified by the step response.

- The stability constraint for FE discretization of this transfer function is $T/0.01 < 2$. For T = 0.1 the discretization is unstable, even though the dominant part of the system is well approximated.

In the examples:

$\tau = 1, 100, \quad T = 0.01, 0.1$





$$s = \frac{z-1}{T}$$

DT-CT filter "equivalence:" Example FE3

- Consider the discretization of the CT transfer function $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$ using FE transformation $s = (z-1)/T$ for T = 0.1, 0.5

$$D1(s) = \frac{0.01}{z^2 - 1.98z + 0.99}; T = 0.1, \quad D5(s) = \frac{0.25}{z^2 - 1.9z + 1.15}; T = 0.5$$

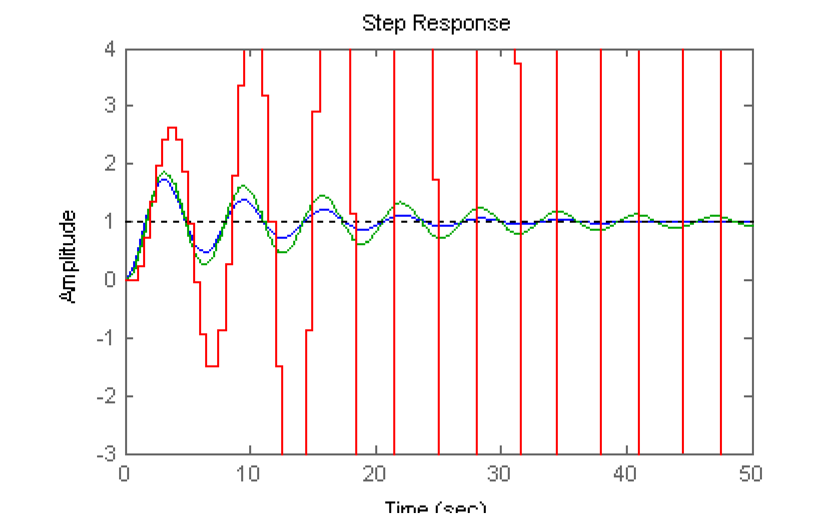
In the examples: $T = 0.1, 0.5$

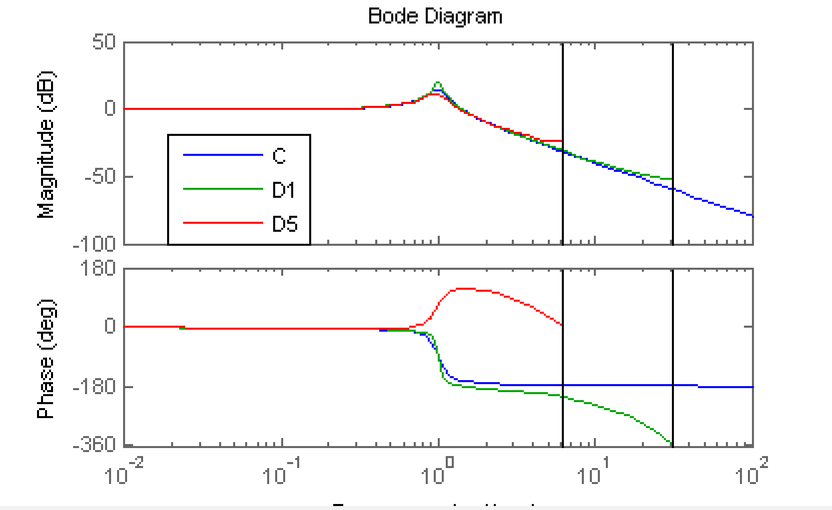
- We observe that the resonance peak is over-estimated for T = 0.1 and under-estimated for T = 0.5.

- For T = 0.5 the DT "equivalent" is unstable. Here, the sampling time does not satisfy the general condition for complex poles

$$\left|1 + poles[C(s)] \times T\right| < 1, \quad \text{or,} \quad \left|\left(1 + \mathrm{Re}[p_i]T\right) + \left(\mathrm{Im}[p_i]T\right)\right| < 1$$

- This condition eventually can be written as $|p_i|T < 2\cos(\angle p_i)$





$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter “equivalence:” Example T2

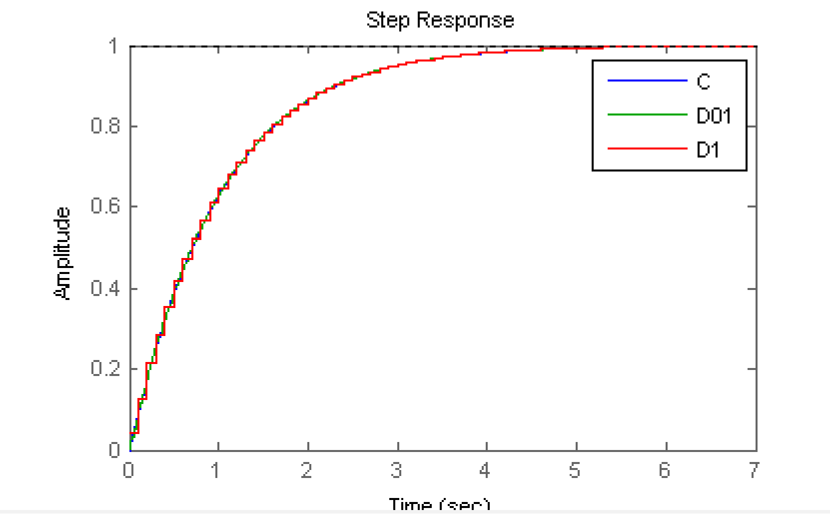
- Returning to the example with fast and slow dynamics, $C(s) = \frac{1}{(s+1)(0.01s+1)}$
- We compute the Tustin discretization for T = 0.01 and 0.1

In the examples:

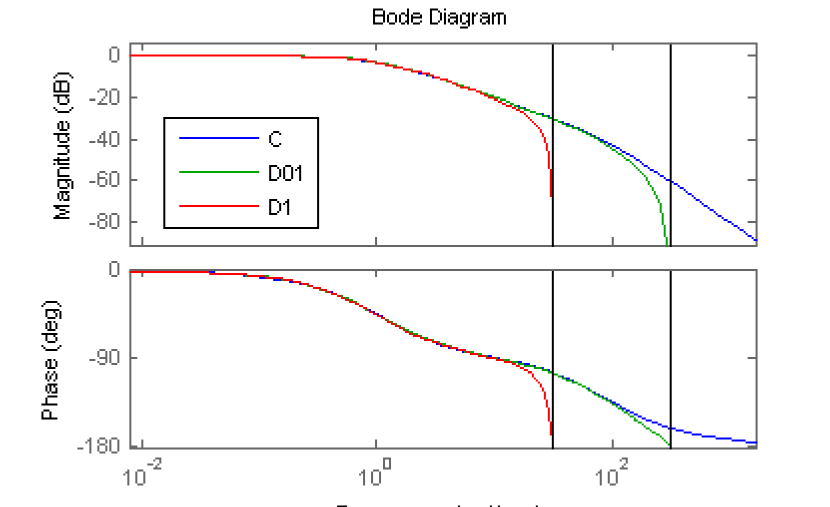
$\tau = 1, 100, \quad T = 0.01, 0.1$

$$D01(s) = \frac{0.001658\,z^2 + 0.003317\,z + 0.001658}{z^2 - 1.323\,z + 0.33}; T = 0.01,$$

$$D1(s) = \frac{0.03968\,z^2 + 0.07937\,z + 0.03968}{z^2 - 0.2381\,z - 0.6032}; T = 0.1$$



- We observe that the discretization approximates the CT system very well up to the usual 1/3-Nyquist frequency.
- The fast pole, appearing at 100 rad/s has a small effect on the Tustin-discretized transfer functions, as it is intuitively expected. (But a formal comparison with the 1st order transfer function is not straightforward)



$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter “equivalence:” Example T3

- The Tustin discretization of the CT transfer function $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$

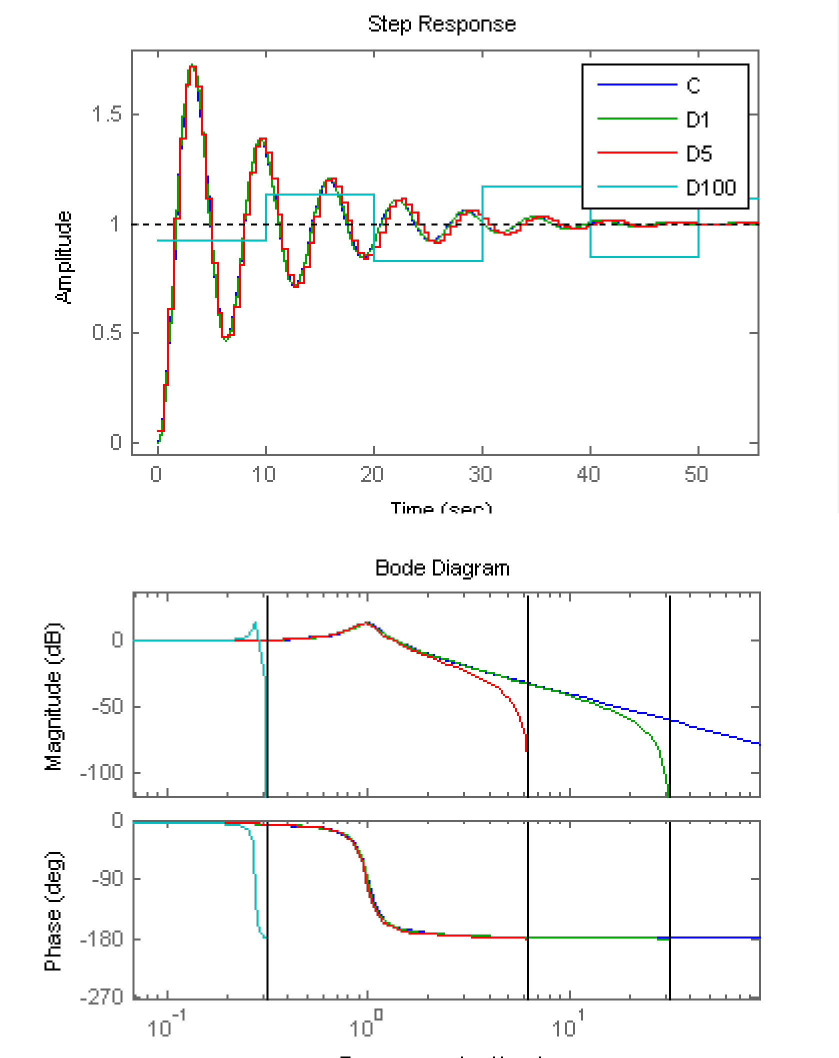
$$D1(s) = \frac{0.002469\,z^2 + 0.004938\,z + 0.002469}{z^2 - 1.97\,z + 0.9802}; T = 0.1,$$

$$D5(s) = \frac{0.05618\,z^2 + 0.1124\,z + 0.05618}{z^2 - 1.685\,z + 0.9101}; T = 0.5$$

In the examples :

$T = 0.1, 0.5, 10$

- The approximation is always well-behaved and interpolates closely the step response samples.
- As T increases, the approximation gets worse but only since the Nyquist frequency decreases. Even for T = 10, the response is stable, although far from the CT response.



$$D(z)=\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\Big|_{t=nT}\right\}$$

DT-CT filter “equivalence:” Example Z2

In the examples:

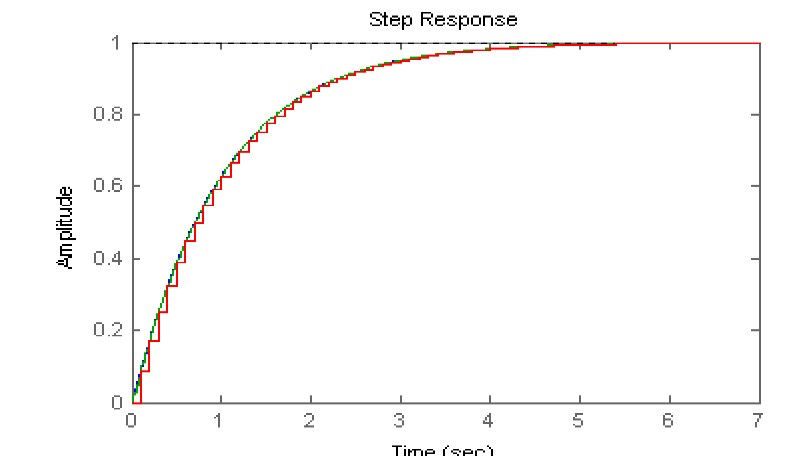
$$\tau=1,100,\quad T=0.01,0.1$$

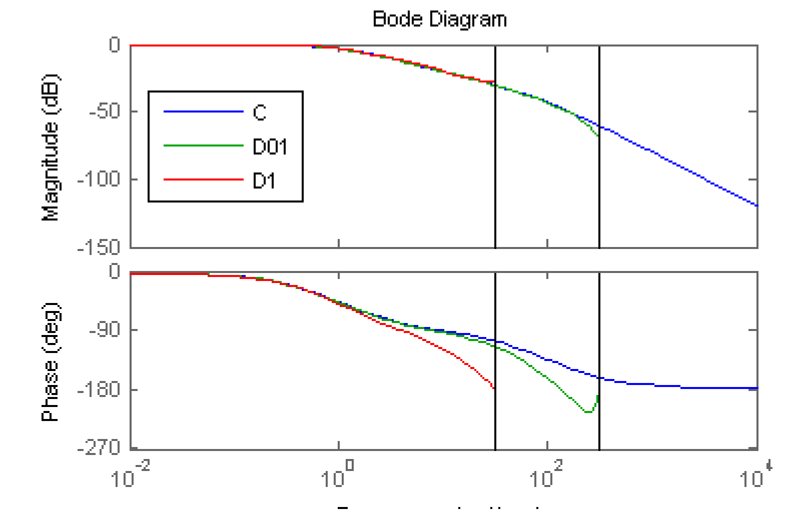
- For the transfer function $C(s)=\dfrac{1}{(s+1)(0.01s+1)}$
- We compute the ZOH discretization for T = 0.01 and 0.1

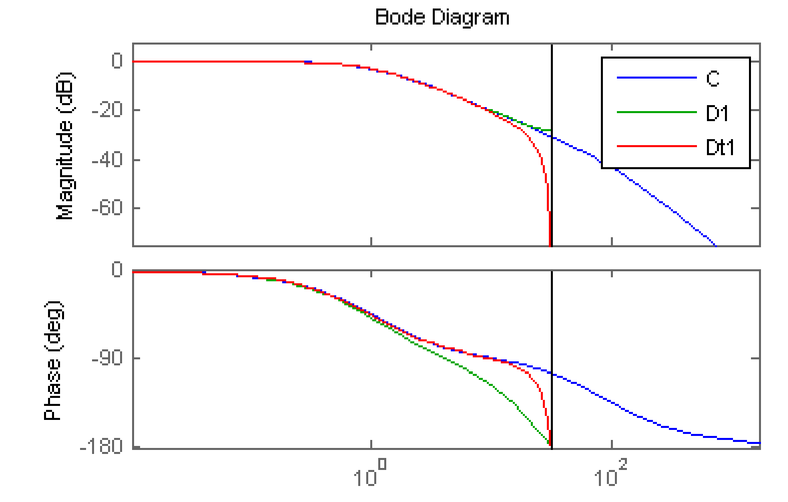
$$D01(s)=\frac{0.003666\,z+0.002624}{z^2-1.358\,z+0.3642};T=0.01,$$

$$D1(s)=\frac{0.08602\,z+0.009135}{z^2-0.9049\,z+4.108e-5};T=0.1$$

- We observe that the discretization approximates the CT system very well up to the 1/10-Nyquist frequency (phase deviations).
- Comparison with the Tustin-discretized system illustrates the phase lag of the ZOH-discretization.







$$D(z)=\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\Big|_{t=nT}\right\}$$

DT-CT filter “equivalence:” Example Z3

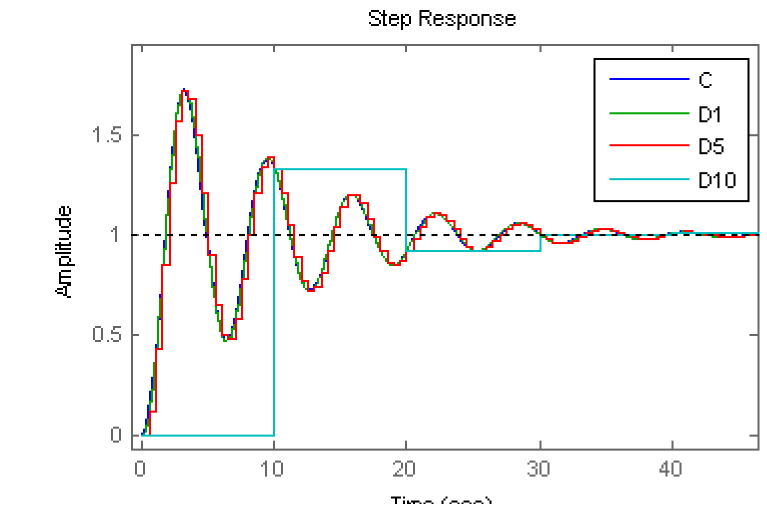
In the examples : $T = 0.1, 0.5, 10$

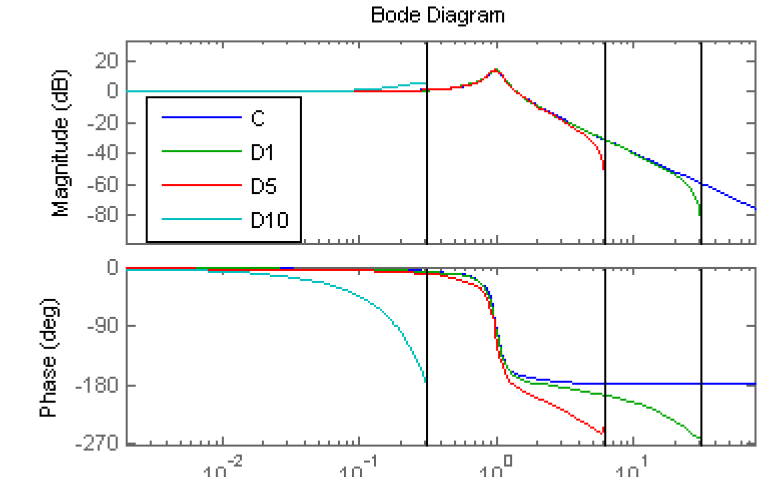
- The ZOH discretization of $C(s)=\dfrac{1}{s^2+0.2s+1}$

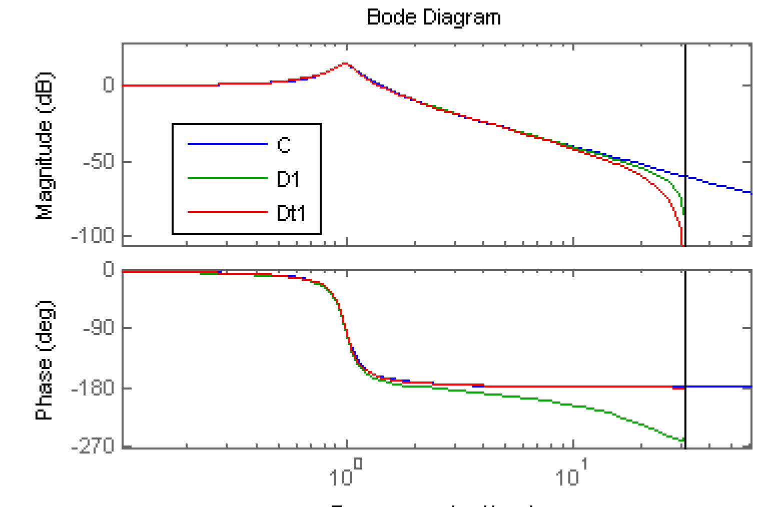
$$D1(s)=\frac{0.004963\,z+0.00493}{z^2-1.97\,z+0.9802};\, T=0.1,$$

$$D5(s)=\frac{0.1185\,z+0.1145}{z^2-1.672\,z+0.9048};\, T=0.5$$

- The approximation is always well-behaved and matches the CT step response samples at the sampling instants.
- As T increases, the approximation gets worse but only since the Nyquist frequency decreases. Even for T = 10, the response is stable, although far from the CT response.
- Comparison with Tustin-discretization shows similar (or better) magnitude but much worse phase approximation.







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Lecture 4.3: DT vs CT Filtering: More Examples

Arizona State University

$$s = \frac{z-1}{T}$$

DT-CT filter "equivalence:" Example FE2

- To illustrate the implication of the FE sampling rate constraint, let us consider the system

$$C(s) = \frac{1}{(s+1)(0.01s+1)}$$

Such characteristics arise in systems that are composed of slow and fast subsystems, e.g., mechanical and electrical components.

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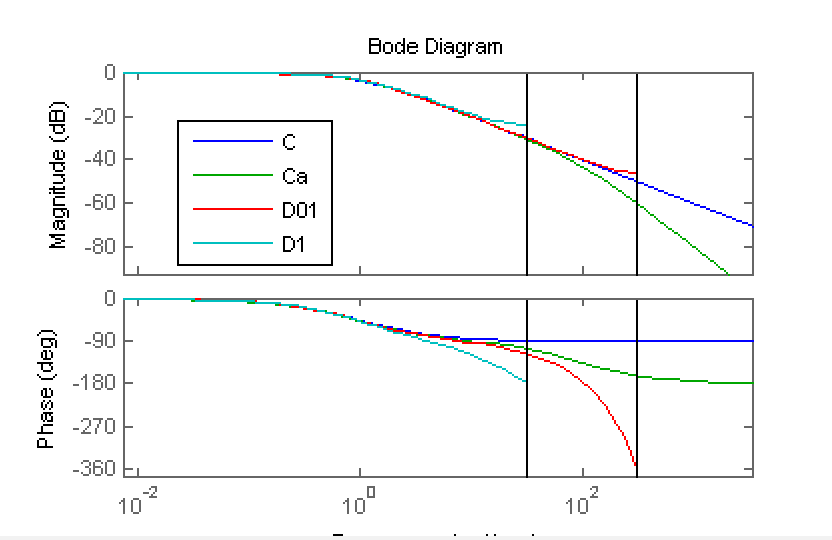
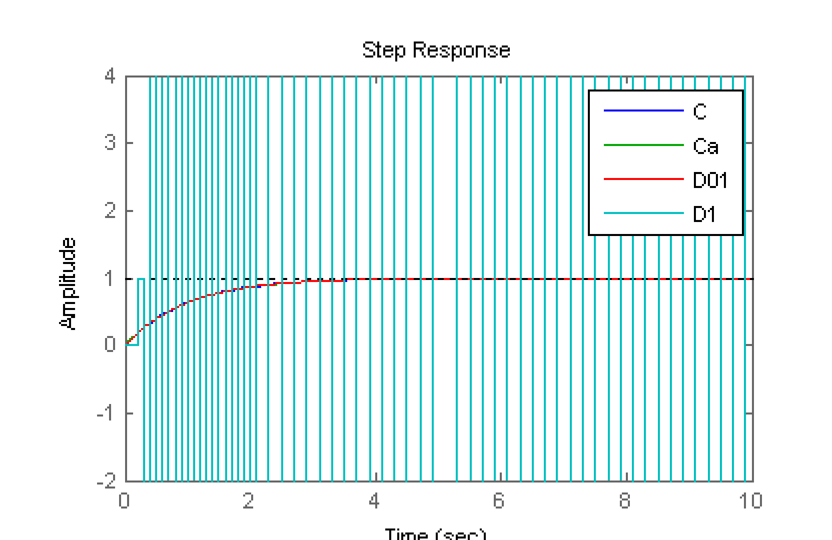
- The stability constraint for FE discretization of this transfer function is $T/0.01 < 2$. For T = 0.1 the discretization is unstable, even though the dominant part of the system is well approximated.

In the examples:

$\tau = 1, 100, \quad T = 0.01, 0.1$





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DT-CT filter "equivalence:" Example FE3

- Consider the discretization of the CT transfer function $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$ using FE transformation $s = (z-1)/T$ for T = 0.1, 0.5

$$D1(s) = \frac{0.01}{z^2 - 1.98z + 0.99}; T = 0.1, \quad D5(s) = \frac{0.25}{z^2 - 1.9z + 1.15}; T = 0.5$$

In the examples: $T = 0.1, 0.5$

- We observe that the resonance peak is over-estimated for T = 0.1 and under-estimated for T = 0.5.
- For T = 0.5 the DT "equivalent" is unstable. Here, the sampling time does not satisfy the general condition for complex poles

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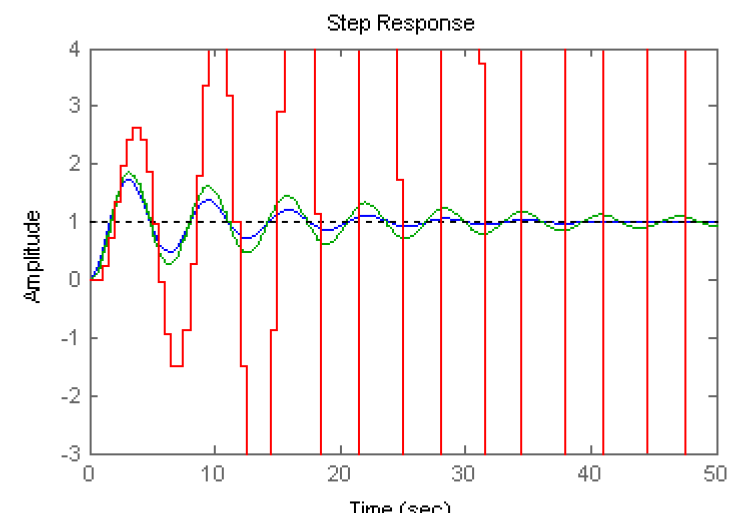
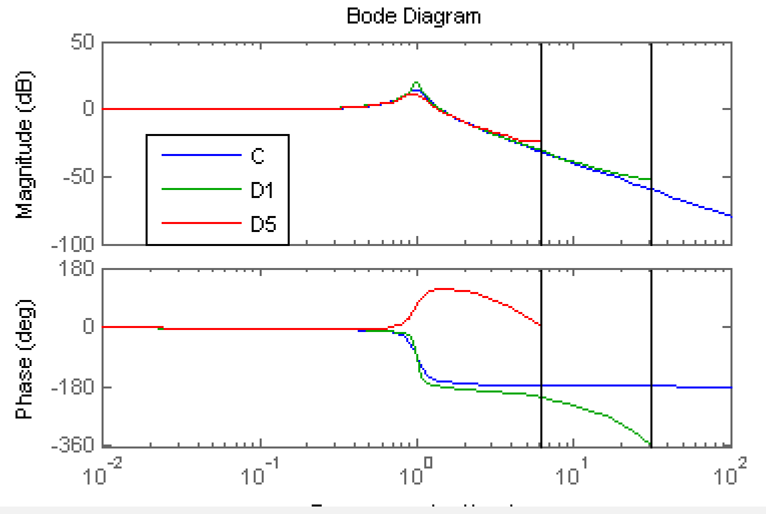
- This condition eventually can be written as $|p_i|T < 2\cos(\angle p_i)$





$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter "equivalence:" Example T2

- Returning to the example with fast and slow dynamics, $C(s) = \dfrac{1}{(s+1)(0.01s+1)}$
- We compute the Tustin discretization for T = 0.01 and 0.1

$$D01(s) = \frac{0.001658\,z^2 + 0.003317\,z + 0.001658}{z^2 - 1.323\,z + 0.33};\, T = 0.01,$$

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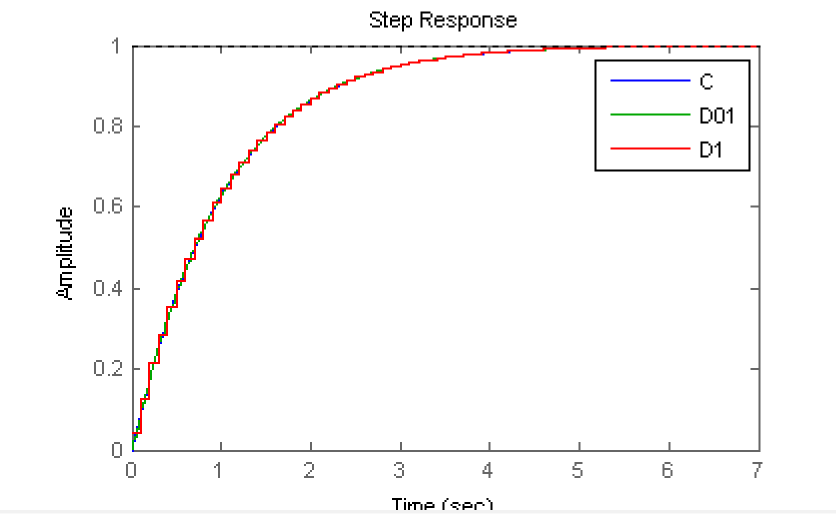

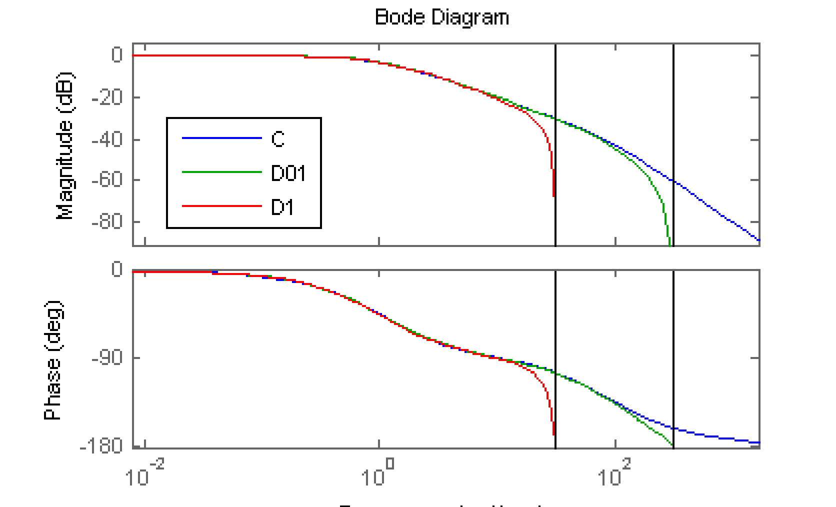

$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter "equivalence:" Example T3

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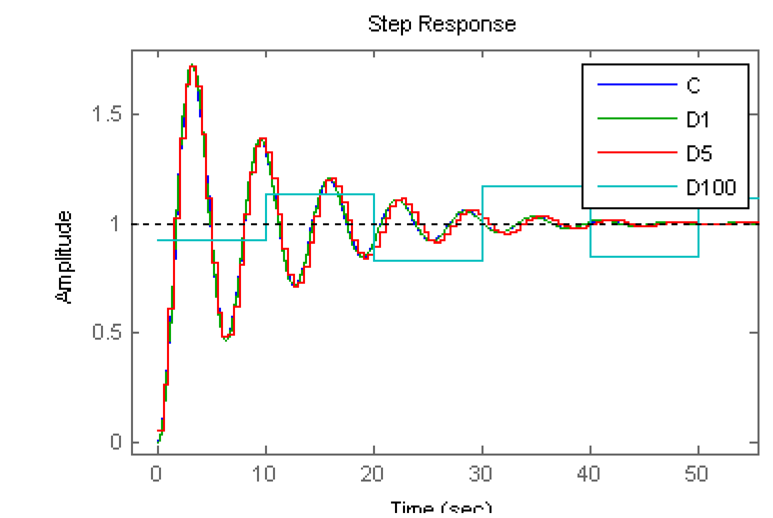
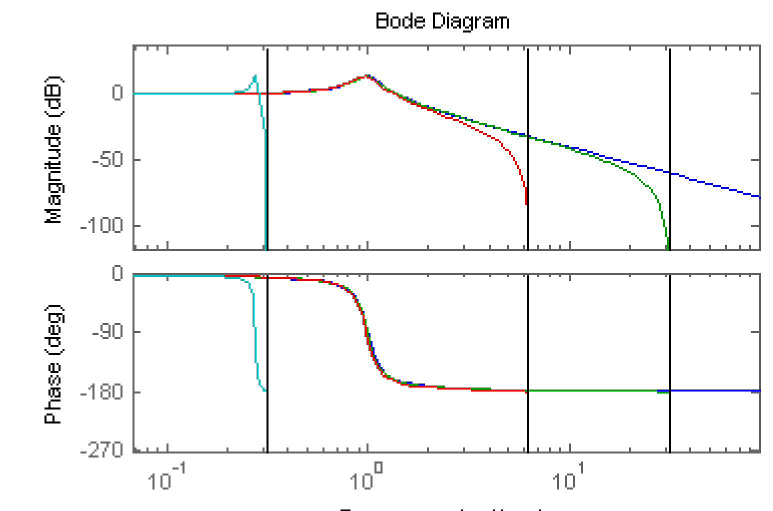
In the examples :

$T = 0.1, 0.5, 10$





$$D(z) = \frac{z-1}{z}\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{1}{s}\cdot C(s) \right\}\Big|_{t=nT} \right\}$$

DT-CT filter "equivalence:" Example Z2

In the examples:

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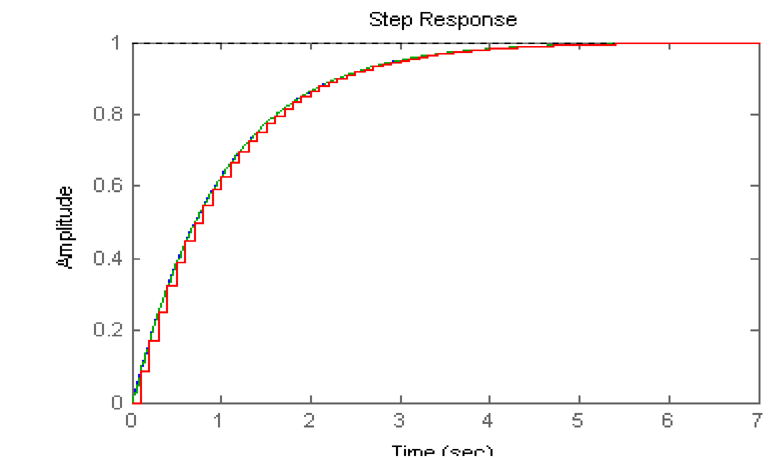

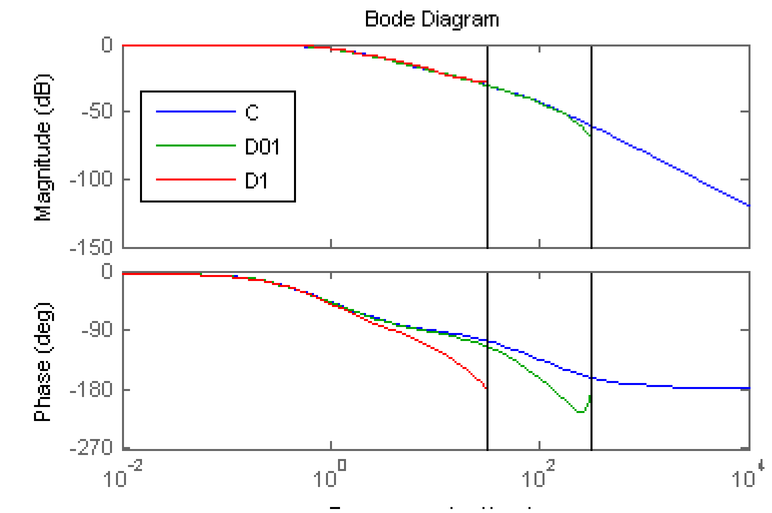

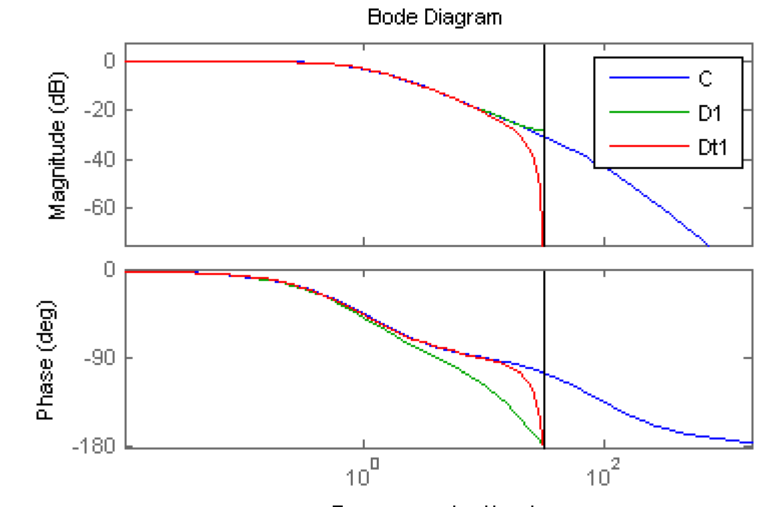

$$D(z) = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\Big|_{t=nT}\right\}$$

DT-CT filter "equivalence:" Example Z3

In the examples : $T = 0.1, 0.5, 10$

- The ZOH discretization of $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$

$$D1(s) = \frac{0.004963\,z + 0.00493}{z^2 - 1.97\,z + 0.9802};\ T = 0.1,$$

$$D5(s) = \frac{0.1185\,z + 0.1145}{z^2 - 1.672\,z + 0.9048};\ T = 0.5$$

- The approximation is always well-behaved and matches the CT step response samples at the sampling instants.
- As T increases, the approximation gets worse but only since the Nyquist frequency decreases. Even for T = 10, the response is stable, although far from the CT response.
- Comparison with Tustin-discretization shows similar (or better) magnitude but much worse phase approximation.