EEE 582 HW #2 SOLUTIONS

#5) All solutions are of the form $x_0 + \mathcal{N}(A)$, where x_0 is a particular sol'n and $\mathcal{N}(A)$ is the null space of A.

A particular sol'n can be derived in more or less systematic ways. Eg. arbitrarily define $x_3 = x_4 = 0$ & solve

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 = 1,\ x_1 = -1$ (but we must have initially observed that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is nonsingular).

$\mathcal{N}(A)$ can be defined in terms of vectors whose span is the space, i.e., two (in this case) independent solutions of $Ax = 0$. [Here, $\dim(\mathcal{N}(A)) + \dim(\mathcal{R}(A)) = \dim(\text{Domain}) = 4$]

$\Rightarrow \mathcal{N}(A) = \text{span}\left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right]$. (Matlab produces an orthonormal basis)

$\Rightarrow$ All solutions are of the form

$$x = \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{x_0} + \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}}_{N} \xi$$

$\xi \in \mathbb{R}^2$.

The min norm solution can be found by solving $\min \| N\xi + x_0 \|$, or simply by using the formula

$$x_{MN} = A^T (AA^T)^{-1} b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{pmatrix}$$

#6) A has rank 2 $\Rightarrow$ its null space is trivial $\Rightarrow$ the minimizer is unique. It can be computed by $x_{LS} = (A^TA)^{-1}A^Tb \Rightarrow$

$$x_{LS} = \frac{1}{21} \begin{pmatrix} 2 & -5 \\ -5 & 23 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 8 \\ 1 \end{pmatrix}.$$

#7) We have $eig(A) = 1, 1, 2$ so for $\lambda = 1$

$$N(\lambda I - A) = N\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \dim N(\lambda I - A) = 2$$

$\Rightarrow$ 2 Jordan blocks correspond to $\lambda = 1 \Rightarrow$

$$J = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}.$$

For the rest of the computations we need to find the eigenvector matrix: $E = \begin{pmatrix} 1 & 0 & 0.577 \\ 0 & 1 & -0.577 \\ 0 & 0 & 0.577 \end{pmatrix}$ so $A = EJE^{-1}$

$$A^{10} = EJ^{10}E^{-1} = \begin{bmatrix} 1 & 0 & 1023 \\ 0 & 1 & -1023 \\ 0 & 0 & 1024 \end{bmatrix}$$

$$e^{At} = Ee^{Jt}E^{-1} = \begin{bmatrix} 1 & 0 & 0.577 \\ 0 & 1 & -0.577 \\ 0 & 0 & 0.577 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^t & \\ & & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1.732 \end{bmatrix}$$

$$= \begin{pmatrix} e^t & 0 & e^{2t}/\sqrt{3} \\ 0 & e^t & -e^{2t}/\sqrt{3} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{3} \end{pmatrix} = \begin{bmatrix} e^t & 0 & e^{2t} - e^t \\ 0 & e^t & e^t - e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

#8) $\sigma^2 = eig(A^TA) = eig(A^2) = eig^2(A) \Rightarrow \sigma = |\lambda|$.

#9) Verify that $\begin{pmatrix} 1 & 0 \\ b^T & 1 \end{pmatrix} \begin{pmatrix} 1 + ab^T & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 + b^Ta \end{pmatrix}$

Taking determinants of both sides

$$(1)\det(I + ab^T)(1) = \det(1 + a^Tb)$$

So for a scalar $a^Tb \Rightarrow \det(1 + ab^T) = 1 + a^Tb$.
