EEE 582 HW #1 SOLUTIONS

1). We have the ODE

$$\dot{v} = \frac{c_a}{m}\bar{u} - \frac{c_f |v| v}{m} - g\sin\phi$$

We select to linearize about the steady-state where $v = 40$, $\dot{v} = 0$. We also assume that ϕ is just an external disturbance and we linearize about $\phi = 0$ but we do not consider it as an input. Under these conditions (utc)

$$v_0 = 40, \quad \bar{u}_0 : \quad 0 = \frac{c_a}{m}\bar{u}_0 - \frac{c_f v_0^2}{m} - g\sin 0$$

$$\Rightarrow \bar{u}_0 = \frac{c_f}{c_a} v_0^2 = 0.533$$

Define $y = v - v_0$, $u = \bar{u} - \bar{u}_0$, $x = y$. Then the standard lin. model of the linearized system becomes:

$$\dot{x} = -\frac{2c_f v_0}{m} x + \frac{c_a}{m} u \qquad \xrightarrow{\text{sub}} \qquad \dot{x} = -0.027x + 1u$$

$$y = 1 \cdot x + 0 \cdot u \qquad\qquad y = 1x + 0u$$

In MATLAB: $H = ss(-0.027, 1, 1, 0)$, $tf(H) \rightarrow \frac{1}{s + 0.027}$

2.) We have $\ddot{\theta} = bT - \epsilon\dot{\theta} - a\sin\theta$, where $b = \frac{T_{max}}{mL^2}$, $a = \frac{g}{L}$,

$\epsilon = \frac{c}{mL^2}$.

We linearize about $\dot{\theta} = 0$, $\theta = \pi$ and $T_0 = 0$.

We define $x = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} - \begin{pmatrix} \theta_0 \\ \dot{\theta}_0 \end{pmatrix}$, $y = (\theta - \theta_0)$, $u = T - T_0$.

Then, $\dot{x} = \begin{bmatrix} 0 & 1 \\ -a\cos\theta_0 & -\epsilon \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u$ $\overset{SUB}{\Rightarrow}$ $\dot{x} = \begin{bmatrix} 0 & 1 \\ 19.6 & -0.15 \end{bmatrix} x + \begin{bmatrix} 0 \\ 7.3 \end{bmatrix} u$

$y = [1 \quad 0] x + 0 \cdot u$ $\qquad$ $y = [1 \quad 0] x + 0u$

In MATLAB: $H = ss(A, B, C, D)$, $tf(H) \rightarrow \dfrac{7.27}{s^2 + 0.15s - 19.6}$

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3). We have $(m+M)\ddot{z} = F - c_c\dot{z} - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta$

$mL^2\ddot{\theta} = -c_p\dot{\theta} - mgL\sin\theta - mL\ddot{z}\cos\theta$

where z is the cart position ~~relative to the center of mass~~.

The equations are implicit in $\ddot{z}, \ddot{\theta}$, in the form

$Q(\theta)\dot{x} = f(x, u)$

where x is the linearization state $\begin{pmatrix} z \\ \dot{z} \\ \theta \\ \dot{\theta} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \pi \\ 0 \end{pmatrix}$

and u is the linearization input $F - 0$.

A reasonable output is the cart position, which is given by

$z + \frac{Lm}{m+M}\sin\theta$, so $y = z + \frac{Lm}{m+M}[\sin\theta - \sin\pi]$

The linearization approximations are: (easier to substitute)

$\cos\theta = -1 + HOT$, $\dot{\theta}^2\sin\theta = 0 + HOT$, $\sin\theta = -(\theta - \pi) + HOT$

where HOT are second order or higher.

Thus,

$$Q(\theta)\,\mathring{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -c_c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & mgL & -c_p \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u$$

where $$Q(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m+M & 0 & mL\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & mL\cos\theta & 0 & mL^2 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m+M & 0 & -mL \\ 0 & 0 & 1 & 0 \\ 0 & -mL & 0 & mL^2 \end{bmatrix}$$

This matrix is always invertible since $\det \begin{bmatrix} m+M & -mL \\ -mL & mL^2 \end{bmatrix} =$

$$= (m+M)mL^2 - m^2L^2 = m^2L^2\left(\frac{m+M}{m} - 1\right) = mML^2 > 0.$$

We substitute the numerical values in MATLAB

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.775 & 0 & -0.1375 \\ 0 & 0 & 1 & 0 \\ 0 & -0.1375 & 0 & 0.0688 \end{bmatrix}, \quad A = Q^{-1}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.05 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1.3489 & 0.01 \end{bmatrix}, \quad B = Q^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

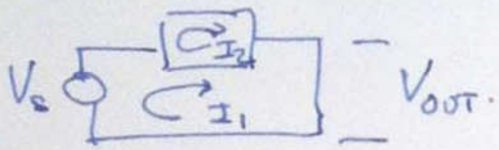
$$C = [1, 0, -0.177, 0], \quad D = 0$$

$$H = ss(A,B,C,D), \quad tf(H) \longrightarrow \frac{2s + 0.29}{s^3 + 0.126s^2 - 0.0146s}$$

4). Here, it is easier to perform Loop analysis instead of nodal because there are only two loops. (The extra nodes would produce algebraic equations that do not add states).

With the handout notation:

$$-V_s + R_1 I_1 + V_c + R_3 I_1 = 0$$



$$-V_c + R_2 I_2 + V_L = 0, \quad V_L = L\frac{dI_L}{dt}$$

where, the inductor current is $I_L = I_2$ and the capacitor voltage is $V_c = \frac{1}{C}\int I_c = \frac{1}{C}\int I_1 - I_2 \Rightarrow I_1 = C\frac{dV_c}{dt} + I_2$

We define the circuit states $x = \begin{pmatrix} V_c \\ I_L \end{pmatrix}$, for which we get.

$$\dot{x} = \begin{pmatrix} -\frac{1}{(R_1+R_3)C} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{pmatrix} x + \begin{pmatrix} \frac{1}{(R_1+R_3)C} \\ 0 \end{pmatrix} u \quad (u = V_s)$$

The output $y = V_{OUT} = R_3 I_1 = R_3\left[C\frac{dx_1}{dt} + x_2\right]$

$$= R_3 C\,[1, 0](Ax + Bu) + R_3[0 \;\, 1]\,x$$

$$= \left(R_3 C\,(1,0)A + R_3(0,1)\right)x + \frac{R_3}{R_1+R_3}u$$

Substituting, $\dot{x} = \begin{pmatrix} -9.091 & -1E4 \\ 14.2 & -1.42 \end{pmatrix} x + \begin{pmatrix} 9.091 \\ 0 \end{pmatrix} u$

$$y = \begin{pmatrix} -0.9091 & 0 \end{pmatrix} x + (0.9091)u$$

whose transfer function is $tf(ss(a,b,c,d)) = \dfrac{0.909s^2 + 1.29s + 1.29e5}{s^2 + 10.5s + 1.42e5}$

For a finite difference discretization, we let $\frac{dx}{dt} \approx \frac{x_{k+1} - x_k}{T}$

so the discrete-time state space representation becomes

$$x_{k+1} = (I + TA)\, x_k + TB\, u_k$$
$$y_k = Cx_k + Du_k$$

i.e., $\quad$ Hd = ss (I+T*A, T*B, C, D, T)

We try in matlab with, say, T = 1e-4, 1e-5 and observe that the DT response approximates the CT one, only for sufficiently small T (here 1e-5). If T is large, unstable responses are produced (i.e., $|\mathrm{eig}(I + TA)| > 1$).

Compare responses by using "bode(H, Hd)" or "step(H, Hd)".