EEE 304, HW 3 **NAME__Solutions_____**

Problem 1:
Do Problems 7.30, 7.31 from the textbook.

7.30
The impulse response is the inverse Laplace of the transfer function:

$$y_c(s) = \frac{1}{s+1}x(s) = \frac{1}{s+1}$$

$$\Rightarrow y_c(t) = e^{-t}u(t)$$

$$y(n) = e^{-nT}u(nT) = \lambda^n u(n), \quad \lambda = e^{-T}$$

$$\Rightarrow y(z) = \frac{z}{z-\lambda}$$

For $w(n) = \delta(n)$, i.e., $w(z) = 1$,

$$w(z) = H(z)y(z) = H(z)\frac{z}{z-\lambda}$$

$$\Rightarrow H(z) = \frac{z-\lambda}{z} = 1 - \lambda z^{-1}$$

$$\Rightarrow H(e^{jw}) = 1 - \lambda e^{-jw}, \quad h(n) = \delta(n) - \lambda\delta(n-1)$$

7.31
Consider a test signal $x_c(t)=\exp(jwt)$, with $w < \pi/T$. Then, following the operations in Fig.P.7.31,

$$x(n) = e^{jwnT} = \left(e^{jwT}\right)^n$$

$$y(n) = \frac{1}{2}y(n-1) + x(n) = H(z)\big|_{z=e^{jwT}}\left(e^{jwT}\right)^n \text{ (since x is an exponential)}$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}}\Bigg|_{z=e^{jwT}}\left(e^{jwT}\right)^n = \frac{2}{2-e^{-jwT}}\left(e^{jwT}\right)^n$$

$$y_c(t) = Lowpass\left[\frac{2}{2-e^{-jwT}}e^{jwnT}\right] = \frac{2}{2-e^{-jwT}}e^{jwt} \quad \text{(because } w < \pi/T\text{)}$$

$$\Rightarrow H(jw) = \frac{2}{2-e^{-jwT}}; \text{ (since, for an LTI system with an exponential input } x_c(t) = e^{jwt},\ y_c(t) = H(jw)e^{jwt}\text{)}$$

Notice that from the last expression, the transfer function is H(s) = 2/(2-exp(-sT)) which is not a finite dimensional system. Since the book does not specify the amplitude of the lowpass, in the reconstruction we assumed that the low-pass has the correct amplitude to recover the signal (i.e., T). If we assume an amplitude of 1, the transfer function must be divided by T.

Problem 2:
Estimate the largest sampling interval T_s to allow perfect reconstruction of the signals (x*y denotes convolution)

1. $\frac{\sin^2 2t}{t^2}\sin 3t$ Using the shortcuts, $w_{max} = 2 + 2 + 3 = 7 \Rightarrow T_s < \frac{\pi}{7}$

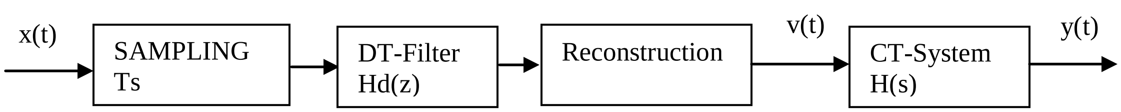
2. $\frac{\sin 2t}{t^2} * \sin 3t \quad w_{max} = \min(\infty, 3) = 3 \Rightarrow T_s < \frac{\pi}{3}$ (1/t is not bandlimited)

3. $\frac{\sin 3t}{2t}\frac{\sin 2t}{2t} \quad w_{max} = 3 + 2 = 5 \Rightarrow T_s < \frac{\pi}{5}$

4. $\frac{\sin 3t}{t} * \sin 2t \quad w_{max} = \min(3,2) = 2 \Rightarrow T_s = \frac{\pi}{2}$

> Note: For (2), the Fourier of 1/t is $F\left\{\frac{1}{t}\right\} = \frac{\pi}{j} sign(w)$. This is a consequence of Duality, which can be briefly stated as $FF = 2\pi R$, where, R denotes the reflection operation. Duality allows us to compute Fourier transforms for time functions that appear in the frequency column, e.g., 1/jw. (Verify!) Then, for example, $F\left\{\frac{1}{t}\frac{\sin t}{t}\right\} = \frac{1}{2\pi}\frac{\pi}{j} sign(w) * \text{pulse}(w, 1)$which is not bandlimited. A similar computation appears in pp.311 of the textbook, but with a typo in Eqn. 4.42 (the integrant should be X(n)).

Problem 3:
Suppose that a continuous time signal x(t) is bandlimited to 1kHz and it is pre-processed by DT system with ideal sampling and reconstruction. The output of the discrete system is then processed by a CT system with transfer function $H(s) = \frac{1}{s+1}$. Select a suitable sampling time Ts and find the discrete-time filter transfer function $H_d(z)$ so that $y(t) = x(t)$.



An exponential test input $x(t) = e^{jwt}$, with w $< 2000\pi$, will be reconstructed as $v(t) = H_d\left(e^{jwT_s}\right)e^{jwt}$, provided that $T_s < 0.5ms$. Then, after processing with the CT system, $y(t) = H(jw)H_d\left(e^{jwT_s}\right)e^{jwt}$. Hence, the DT filter should be such that $H(jw)H_d\left(e^{jwT_s}\right) = 1 \Rightarrow H_d\left(e^{jwT_s}\right) = jw + 1$. Or, $H_d(z) = \frac{\log z}{T_s} + 1$. This cannot be implemented by a finite dimensional system, but can be approximated by one (e.g., via a Taylor expansion of log).