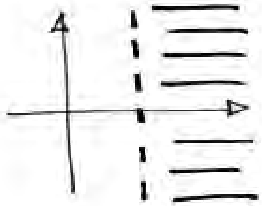
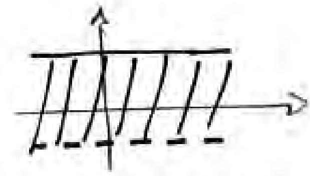
EEE 550 HW #1 SOLUTIONS

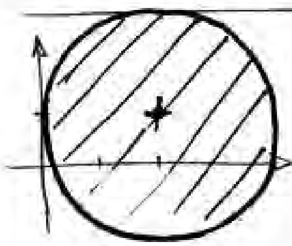
1.6.9

i) 

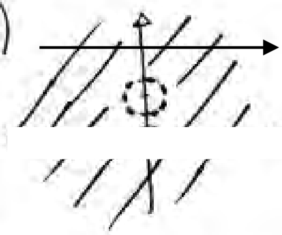
open
connected
domain
region

ii) 

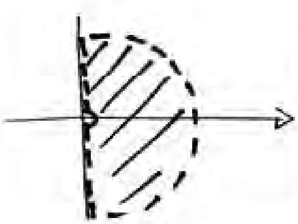
connected, region

iii) 

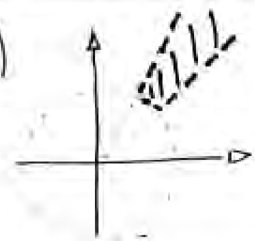
connected
region
closed region
bounded

iv) 

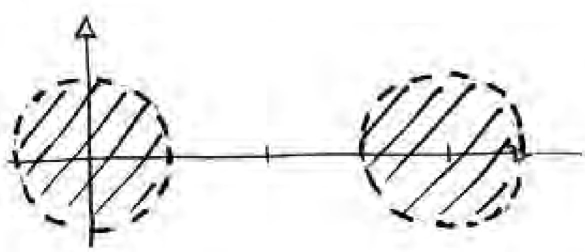
open
connected
domain
region

v) 

open
connected
domain
region
bounded

vi) 

open
connected
domain
region

vii 

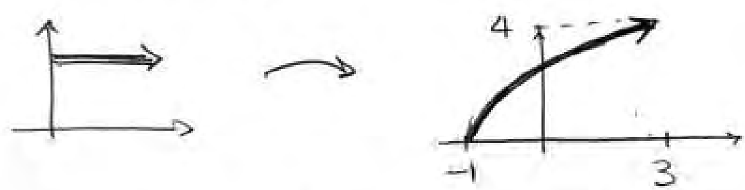
open
bounded

2.3.3.

1) $y \in (0,1)$, $x=0 \Rightarrow u=-y^2$, $v=0$

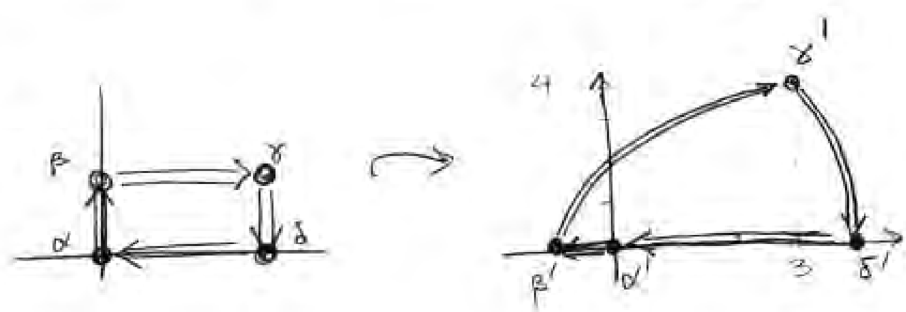
2) $x \in (0,2)$, $y=0 \Rightarrow u=x^2$, $v=0$

3) $x \in (0,2)$, $y=1 \Rightarrow u=x^2-1$, $v=2x \Rightarrow u=\frac{v^2}{4}-1$

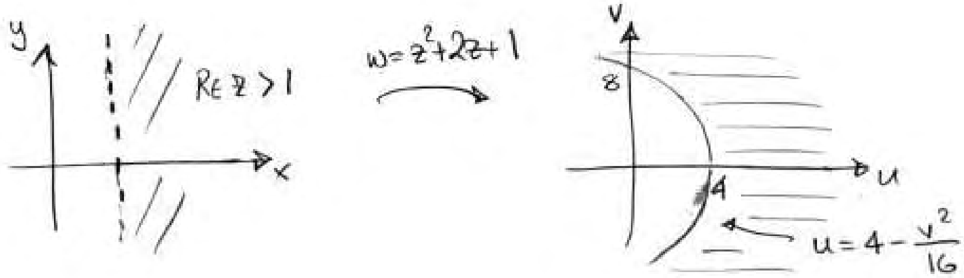


4) $x=2$, $y \in (0,1) \Rightarrow u=4-y^2$, $v=4y \Rightarrow u=4-\frac{v^2}{16}$

Overall,



2.3.13

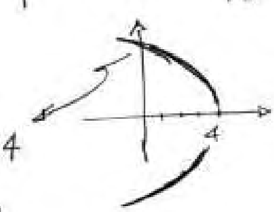


$$w = x^2 - y^2 + j\,2xy + 2x + j2y + 1$$

$$= \left(x^2 - y^2 + 2x + 1\right) + j\left(2y + 2xy\right)$$

$$= \underbrace{\left[(x+1)^2 - y^2\right]}_{u} + j\underbrace{\left[2y(x+1)\right]}_{v}$$

Boundary $x = 1$ maps to $\left.\begin{array}{l} u = 4 - y^2 \\ v = 4y \end{array}\right\}\ u = 4 - \frac{v^2}{16}$



$\frac{v^2}{16} = 4$

$\frac{v}{4} = 2$

$v = 8$

Any $x > 1$ maps to $u = (1+x)^2 - y^2$

$$v = 2y(x+1) \Rightarrow y = \frac{v}{2(x+1)} \Rightarrow y^2 = \frac{v^2}{4(x+1)^2}$$

$$\Rightarrow u = (1+x)^2 - \frac{v^2}{4(x+1)^2}$$

For the same v, $u(x) - u(1) = (1+x)^2 - \frac{v^2}{4(x+1)^2} - 4 + \frac{v^2}{4^2}$

$$= \underbrace{\left[(1+x)^2 - 4\right]}_{>0} + \underbrace{\frac{v^2}{4(x+1)^2}}_{>0}\underbrace{\left[\frac{(x+1)^2}{4} - 1\right]}_{>0} > 0$$

$$\Rightarrow u(x) > u(1)$$

$\Rightarrow w$ to the right of $u = 4 - \frac{v^2}{16}$

2.4.6b

$$w = \frac{z+1}{z^2+1}$$

Use Thm 2.4 : $\left.\begin{array}{l} w = z \text{ continuous} \\ w = 1 \text{ continuous} \end{array}\right\}$ $z+1$ continuous

$w = z^2 = zz$ continuous $\rightarrow$ z^2+1 continuous

$\Rightarrow$ $\frac{z+1}{z^2+1}$ continuous for $z^2+1 \neq 0$

At $z^2+1 = 0$, $z = \pm j$ $\lim_{\Delta z \to 0} \frac{j+\Delta z+1}{(j+\Delta z)^2+1} = \lim_{\Delta z \to 0} \frac{j+1}{2j\,\Delta z} = \infty$

Limit does not exist

$\Rightarrow$ function discontinuous

2.4.8

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}$$

Let $x = 0$, $y = 1/n$ $n \to \infty$. Then $u(x,y) = \frac{0}{0 + 1/n^2} = 0$

If the limit exists, it must be 0. To show this, we must show that $\forall\, \varepsilon > 0\ \exists\, \delta(\epsilon)$: $\sqrt{x^2+y^2} < \delta \Rightarrow |u(x,y) - 0| < \epsilon$

Define $r = \sqrt{x^2+y^2}$. Then $|x| < r$, $x^2 < r^2$, $3y^2 < 3r^2$

Hence, $|u(x,y)| = \frac{|x|\,|x^2-3y^2|}{x^2+y^2} \leq \frac{|x|\left(|x^2| + 3|y^2|\right)}{x^2+y^2}$

$$\leq \frac{r\,(r^2+3r^2)}{r^2}$$

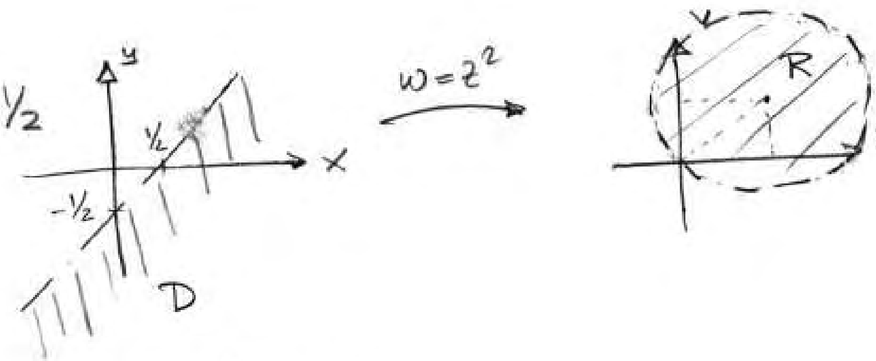
$$\leq 4r < 4\delta$$

Define $\delta(\epsilon) = \epsilon/4$ Then, $\sqrt{x^2+y^2} < \delta \Rightarrow |u(x,y)| < 4\epsilon/4 = \epsilon$.

$\Rightarrow$ $u(x,y)$ continuous at $(0,0)$.

2.6.14

$y < x - \frac{1}{2}$



$$w : D \to R \ (\text{onto})$$

First we need to show that any $z \in D$ maps to an $w \in R$

Then we must show that any $w \in R$ has a preimage $z \in D$ (onto)

1) $w = \frac{1}{z} = \frac{x - jy}{x^2+y^2} \Rightarrow u = \frac{x}{x^2+y^2}, \ v = \frac{-y}{x^2+y^2}$

$$\left|(u-1) + j(v-1)\right|^2 = \left(\frac{x}{x^2+y^2} - 1\right)^2 + \left(\frac{-y}{x^2+y^2} - 1\right)^2 = \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2} + 2$$

$$= \frac{1}{x^2+y^2} + \frac{2(y-x)}{x^2+y^2} + 2 = \underbrace{\frac{2(y-x+\frac{1}{2})}{x^2+y^2}}_{z\in D \Rightarrow (\,) < 0} + 2 < 2$$

$\therefore \ \underline{z \in D \Rightarrow w \in R}$

2) Let $w \in R$. Then $\left|(u-1) + j(v-1)\right|^2 < 2$. Then $z = \frac{1}{w} = \frac{u - jv}{u^2+v^2}$

or. $x = \frac{u}{u^2+v^2}, \ y = \frac{-v}{u^2+v^2}$

But $(u-1)^2 + (v-1)^2 < 2 \Rightarrow u^2 - 2u + 1 + v^2 - 2v + 1 < 2$

$$\Rightarrow u^2 + v^2 < 2(u+v)$$

$$\Rightarrow \frac{u+v}{u^2+v^2} > \frac{1}{2} \Rightarrow x - y > \frac{1}{2}$$

$$\Rightarrow y - x < -\frac{1}{2}$$

$\therefore \ \underline{w \in R \Rightarrow z \in D}$

$\therefore \ w = z^2$ maps D onto R.

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3.1.6 $P(z) = (z-z_1)(z-z_2)$, $z_1 \neq z_2$

$$P'(z) = z-z_2 + z-z_1 \Rightarrow \frac{P'(z)}{P(z)} = \frac{z-z_2+z-z_1}{(z-z_1)(z-z_2)}$$

$$= \underline{\frac{1}{z-z_1} + \frac{1}{z-z_2}}$$

3.2.14 $\omega = \cosh x \sin y - i \sinh x \cos y$

C-R: $u_x = v_y$, $u_y = -v_x$

$u_x = \frac{\partial}{\partial x}(\cosh x \sin y) = \sinh x \sin y$

$v_y = \frac{\partial}{\partial y}(-\sinh x \cos y) = -\sinh x(-\sin y)$
$= u_x$

$\cosh x = \frac{e^x+e^{-x}}{2}$

$\frac{d\cosh x}{dx} = \frac{e^x-e^{-x}}{2}$
$= \sinh x$

$u_y = \cosh x \cos y$

$v_x = -\cosh x \cos y = -u_y$

$\therefore$ {C-R hold $\forall$ x,y } & { continuity of partials } $\Rightarrow$

Analyticity $\forall$ x,y $\Rightarrow$ the function is <u>entire</u>.

Similarly for $w = \cosh x \cos y + i \sinh x \sin y$.

3.3.8 v harm. compl. of u $\Rightarrow$ v, u harmonic and CR(u,v) hold

u harmonic $\Rightarrow$ $-u$ harmonic $(u_{xx}+u_{yy}=0=-u_{xx}-u_{yy})$

$$\text{C-R}(v,-u): \left.\begin{aligned} v_x &= (-u)_y = -u_y \\ v_y &= -(-u)_x = u_x \end{aligned}\right\} \Leftrightarrow \text{C-R}(u,v)$$

$\Rightarrow$ <u>$-u$ harm. compl. of v</u>.

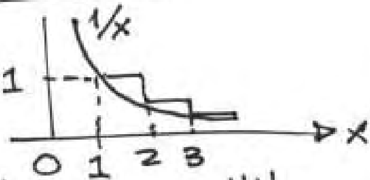
4.1.6

$\sum_{1}^{\infty} \frac{1}{n} + \frac{j}{2^n}$ converges $\Leftrightarrow$ $Re\{\ \}$ converges
$Im\{\ \}$ converges

$Re\{\ \} = \sum_{1}^{\infty} \frac{1}{n}$ which is known to diverge.

$\Rightarrow \sum_{1}^{\infty} \frac{1}{n} + \frac{j}{2^n}$ diverges.

Note:



$\frac{1}{x} < \left[\frac{1}{x}\right]$ for $x \geq 1$

where $[\cdot]$ = integer part.

$\Rightarrow \int_{1}^{N+1} \frac{1}{x} dx < \int_{1}^{N+1} \left[\frac{1}{x}\right] dx = \sum_{1}^{N} \frac{1}{n}$

$\Rightarrow \ln(N+1) < \sum_{1}^{N} 1/n \quad$ 5 $\quad \ln(N+1)$ diverges as $N \to \infty$

$\Rightarrow \sum_{1}^{\infty} 1/n$ diverges

4.1.10

$$\sum_{1}^{\infty} \frac{(i)^n}{n} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Both series for the real and imaginary parts are alternating and decreasing in magnitude. Therefore, (Leibnitz) they converge. Hence, $\sum_{1}^{\infty} \frac{(i)^n}{n}$ converges.

EEE 550 HW #3 SOLUTIONS

4.3.6 $|z|<1$; $z = re^{j\theta}$ (polar form with $r=\sqrt{x^2+y^2}$)

$\Leftrightarrow r<1.$

$\Rightarrow z^n = r^n e^{jn\theta} \Rightarrow |z^n| = r^n |e^{jn\theta}| = r^n.$

Now, $\forall \varepsilon>0 \; \exists N \geq \frac{\ln \epsilon}{\ln r}$ (for $\epsilon<1$, 0 otherwise)

such that $n>N \Rightarrow |z^n - 0| = r^n < r^N$

But $\ln r^N = N \ln r \underset{(\ln r<0)}{\leq} \frac{\ln \epsilon}{\ln r} \ln r = \ln \epsilon$

From the monotonicity of $\ln(.)$ $\Rightarrow r^N < \epsilon \Rightarrow |z^n - 0| < \epsilon$

$\Rightarrow \lim_{n\to\infty} z^n = 0$

4.3.12 $\sum_0^\infty r^n e^{jn\theta} \underset{z=re^{j\theta}}{=} \sum_0^\infty z^n \underset{|z|<1}{=} \frac{1}{1-z} = \frac{1}{1-re^{j\theta}}$

$= \frac{1}{(1-r\cos\theta) - j(r\sin\theta)} = \frac{(1-r\cos\theta)+j(r\sin\theta)}{(1-r\cos\theta)^2+(r\sin\theta)^2}$

$= \frac{(1-r\cos\theta)+j(r\sin\theta)}{1+r^2-2r\cos\theta}.$ $(r<1)$

Now, $\sum_0^\infty r^n e^{jn\theta} = \underbrace{\sum_0^\infty r^n \cos n\theta}_{\text{Real}} + j \underbrace{r^n \sin n\theta}_{\text{Imag}} = \underbrace{\frac{1-r\cos\theta}{1+r^2-2r\cos\theta}}_{\text{Real}} + j \underbrace{\frac{r\sin\theta}{1+r^2-2r\cos\theta}}_{\text{Imag}}$

Equating real and imaginary parts on both sides, we obtain the required relationships.

Notice that each series is absolutely convergent for $r<1$ (e.g. from the comparison principle)

4.4.4

$$\sum_0^\infty (n+1)^2 z^n = \frac{1+z}{(1-z)^3} \;;\quad \text{D'Alembert: } \lim \frac{(n+2)^2}{(n+1)^2} = 1$$

$$\Rightarrow \text{roc} = \{ |z| < 1 \}$$

For $|z|<1$, $\sum_0^\infty z^n = \frac{1}{1-z}$. Furthermore in roc, the power series is analytic and can be differentiated term-by-term.

$$\therefore \frac{\partial}{\partial z} \sum_0^\infty z^n = \sum_{n=1}^\infty n z^{n-1} = \frac{1}{(1-z)^2}. \quad \text{Re-indexing } (k=n-1)$$

we have $\sum_{k=0}^\infty (k+1) z^k = \frac{1}{(1-z)^2}$. $(*)$

Differentiating again: $\frac{\partial}{\partial z} \sum_{k=0}^\infty (k+1) z^k = \sum_{k=1}^\infty (k+1) k z^{k-1} =$

(re-indexing $n=k-1$) $= \sum_{n=0}^\infty (n+2)(n+1) z^n = \frac{2}{(1-z)^3}$

Expanding the series terms $\sum_{n=0}^\infty (n+1)^2 z^n + (n+1) z^n = \frac{2}{(1-z)^3}$

from $(*)$ $\Rightarrow$ & convergence of each series

$$\sum_{n=0}^\infty (n+1)^2 z^n = \frac{2}{(1-z)^3} - \sum_{n=0}^\infty (n+1) z^n = \frac{2}{(1-z)^3} - \frac{1}{(1-z)^2}$$

$$= \frac{2-(1-z)}{(1-z)^3} = \underline{\underline{\frac{1+z}{(1-z)^3}}}$$

4.4.6 $\sum c_n z^n$ converges for $z_1 = 4-i$ & diverges for $z_2 = 2+3i$

From Thm 4.14 $(\alpha=0)$ power series converge in a disc $|z|<\rho$.

Since $z_1 \in D \Rightarrow |4-i| < \rho \Rightarrow \sqrt{17} < \rho$.

But $|z_2| = \sqrt{4+9} = \sqrt{13} < \sqrt{17} \Rightarrow z_2 \in D \Rightarrow \sum c_n z^n$ must converge at z_2. Hence, there is no such power series.

EEE 550 HW#4 SOLUTIONS

5.3.1 a) $4^i = \left(e^{\ln 4}\right)^i = e^{i\ln 4} = \boxed{\cos(\ln 4) + i\sin(\ln 4)}$

b) $\mathrm{Log}(1+i)$: $1+i = \sqrt{2}\, e^{i\pi/4} = e^{\ln\sqrt{2} + i\pi/4}$

$$\Rightarrow \mathrm{Log}(1+i) = \ln\sqrt{2} + i\pi/4$$

$$\Rightarrow (1+i)^{\pi i} = \exp\left[\left(\ln\sqrt{2} + i\pi/4\right)\pi i\right] = \exp\left(-\frac{\pi^2}{4} + \ln\sqrt{2}\,\pi i\right)$$

$$= \boxed{e^{-\pi^2/4}\left[\cos\left(\pi\ln\sqrt{2}\right) + i\sin\left(\pi\ln\sqrt{2}\right)\right]}$$

c) $\mathrm{Log}(-1) = \mathrm{Log}\left(e^{i\pi}\right) = i\pi$

$$\Rightarrow (-1)^{1/\pi} = \exp\left[(i\pi)\frac{1}{\pi}\right] = \exp(i) = \boxed{\cos 1 + i\sin 1}$$

d) $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2e^{i\pi/3} = e^{\ln 2 + i\pi/3}$

$$\Rightarrow \left(1 + i\sqrt{3}\right)^{i/2} = \exp\left[\left(\ln 2 + i\pi/3\right)\cdot\frac{i}{2}\right] = \exp\left[i\frac{\ln 2}{2} - \frac{\pi}{6}\right]$$

$$= \boxed{\frac{\cos\frac{\ln 2}{2} + i\sin\frac{\ln 2}{2}}{e^{\pi/6}}}$$

5.4.7 a) $\dfrac{d\sin(1/z)}{dz} = \dfrac{d\sin s}{ds}\cdot\dfrac{ds}{dz}\Big|_{s=1/z} = \left(\cos\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)$; $z \neq 0$

b) $\dfrac{d\, z\tan z}{dz} = \tan z + z\dfrac{d\tan z}{dz} = \tan z + \dfrac{1}{\cos^2 z}$; $\cos z \neq 0$

c) $\dfrac{d\sec z^2}{dz} = 2z\dfrac{d\sec s}{ds}\Big|_{s=z^2} = 2z\sec z^2\tan z^2 = 2z\dfrac{\sin z^2}{\cos^2 z^2}$, $\cos z^2 \neq 0$

Note: $\cos z = 0$ when $x = \frac{\pi}{2} \pm k\pi$, $y = 0$

5.4.10

b) $\cos z = 2 = \cos x\left(\frac{e^y+e^{-y}}{2}\right) + i\sin x\left(\frac{e^y-e^{-y}}{2}\right)$

Equating real & imaginary parts,

$$\sin x\,\frac{e^y-e^{-y}}{2} = 0 \Leftrightarrow \begin{cases} \sin x = 0 & \text{or} \\ y = 0 \end{cases}$$

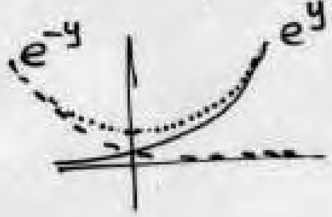
If $y=0$, then $\frac{e^y+e^{-y}}{2} = 1$ and $\cos x$ must be 2

for which there is no solution.

So it must be that $\sin x = 0$. Then $\cos x = \pm 1$

but $\frac{e^y+e^{-y}}{2} > 0$ so $\cos x$ must be positive $\Rightarrow x = 2k\pi$

Then, y must be such that $\frac{e^y+e^{-y}}{2} = 2$



$e^y+e^{-y} = 4$ has two solutions $\pm y_*$ and y_* is a numerically computed root between 1 and 4. (e.g. bisection)

So, $z \in \{2k\pi \pm i y_*\}_{k\in\mathbb{Z}}$

e) $\cosh z = 1 = \cosh x \cos y + i \sinh x \sin y$

$$\sinh x \sin y = 0 \Leftrightarrow \begin{cases} \sinh x = 0 & \text{or} \\ \sin y = 0 \end{cases}$$

Since $\cosh x > 0$, $\cos y$ must be positive, So if $\sin y = 0$

then $y = 2k\pi$. Then $\cosh x = 1 \Rightarrow x = 0$.

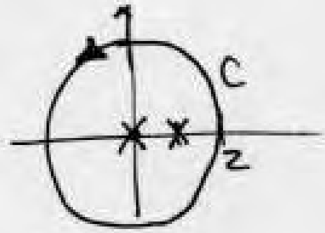
If $\sinh x = 0$ then $x = 0$ and $\cosh x = 1$. So $\cos y = 1$

$\Rightarrow y = 2k\pi$.

Hence, $z = \{2k\pi i\}_{k\in\mathbb{Z}}$

6.3.5

a). $\displaystyle\int_C \frac{2z-1}{z(z-1)}\,dz$



Two singularities @ 0 and 1.

Perform a PFE (since Cauchy's Integral formula is in section 6.5)

$$\frac{2z-1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{2z-1}{z(z-1)}\,dz = \int_C \frac{1}{z}\,dz + \int_C \frac{1}{z-1}\,dz = \int_{C_r(0)} \frac{1}{z}\,dz + \int_{C_r(1)} \frac{1}{z-1}\,dz$$

$(r \to 0)$

$$= 2\pi i + 2\pi i = 4\pi i.$$

Note: $\displaystyle\int_{C_r(1)} \frac{1}{z}\,dz = 0$ since $1/z$ is analytic in $D_r(1)$. Similarly for $\displaystyle\int_{C_r(0)} \frac{1}{z-1}\,dz$.

b) $\displaystyle\int_C \frac{2z-1}{z(z-1)}\,dz$

$$= \int_C \frac{1}{z}\,dz + \int_C \frac{1}{z-1}\,dz = 2\pi i + 0 = 2\pi i.$$

Cauchy-Goursat

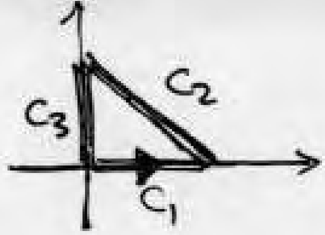
NOTE: Using Thm 6.10, for (a) without PFE

$$\int_C \frac{2z-1}{z(z-1)}\,dz = \int_{C_r(0)} \frac{\frac{2z-1}{z-1}}{z}\,dz + \int_{C_r(1)} \frac{\frac{2z-1}{z}}{z-1}\,dz$$

Then, $\frac{2z-1}{z-1}$ analytic in $D_r(0) \Rightarrow$.

$$\int_{C_r(0)} \frac{\frac{2z-1}{z-1}}{z}\,dz = 2\pi i \left.\frac{2z-1}{z-1}\right|_{z=0} = 2\pi i.$$

similarly for the rest...

6.3.6 C: 

parametrization of C: $x+jy = \begin{cases} t+j0\ ,\ t\in[0,1],\ dz=dt \\ 1-t+jt\ ,\ t\in[0,1],\ dz=-dt+jdt \\ 0+j(1-t)\ \ t\in[0,1],\ dz=-jdt \end{cases}$

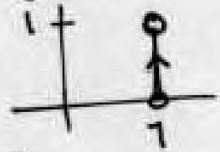
a). $\displaystyle\int_C 1\,dz = \underbrace{\int_0^1 dt}_{C_1} + \underbrace{\int_0^1 (-1+j)\,dt}_{C_2} + \underbrace{\int_0^1 -j\,dt}_{C_3}$

$= 1 + (-1+j) - j = 0.$

b) $\displaystyle\int_C z\,dz = \int_0^1 t\,dt + \int_0^1 (-1+j)(1-t+jt)\,dt + \int_0^1 -j\,(j(1-t))\,dt$

$= \cancel{\tfrac{1}{2}t^2\Big|_0^1} + (-1+j)\left[t-\tfrac{1}{2}t^2+\tfrac{1}{2}jt^2\right]\Big|_0^1 + \left[t-\cancel{\tfrac{1}{2}t^2}\right]_0^1$

$= (-1+j)\left(1-\tfrac{1}{2}+\tfrac{1}{2}j\right)+1 = -\dfrac{(1-j)(1+j)}{2}+1 = 0$

6.4.11 $\displaystyle\int_C \operatorname{Log} z\,dz$ C: 

Consider the path  $C_1 = t\ ,\ t\in[1,\sqrt{2}]$

$C_2 = \sqrt{2}e^{j\theta}\ \ \theta\in[0,\frac{\pi}{4}]$

Log z is analytic in a domain containing C, C_1, C_2

so its integral is path-independent.

$\Rightarrow \displaystyle\int_C \operatorname{Log} z\,dz = \int_{C_1} \operatorname{Log} z\,dz + \int_{C_2} \operatorname{Log} z\,dz.$

Evaluating each integral separately,

$$\int_{C_2} \mathrm{Log}\, z\, dz = \int_0^{\pi/4} (\ln\sqrt{2} + i\theta)\, i\sqrt{2} e^{i\theta} d\theta$$

$$= i\sqrt{2}\ln\sqrt{2} \int_0^{\pi/4} e^{i\theta} d\theta + i\, i\sqrt{2} \int_0^{\pi/4} \theta e^{i\theta} d\theta$$

$$= \sqrt{2}\ln\sqrt{2}\left[e^{i\pi/4} - 1\right] - \sqrt{2}\left[e^{i\pi/4}(1 - i\pi/4) - 1\right]$$

$$\int_{C_1} \mathrm{Log}\, z\, dz = \int_0^{\ln\sqrt{2}} t e^t dt = \left[te^t - e^t\right]\Big|_0^{\ln\sqrt{2}}$$

let $z = e^t$
$t \in [\ln 1, \ln\sqrt{2}]$

$$= \sqrt{2}\ln\sqrt{2} - \sqrt{2} + 1$$

Then, $$\int_C \mathrm{Log}\, z\, dz = \sqrt{2}\ln\sqrt{2} - \sqrt{2} + 1 + \sqrt{2}\ln\sqrt{2}\, e^{i\pi/4}$$

$$- \sqrt{2}\ln\sqrt{2} - \sqrt{2}e^{i\pi/4} + \sqrt{2}\, i\frac{\pi}{4} e^{i\pi/4} + \sqrt{2}$$

Use $e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$$= 1 + (1+i)\left(\log\sqrt{2} - 1 + i\pi/4\right)$$

$$= \boxed{\log\sqrt{2} - \frac{\pi}{4} + i\left(\log\sqrt{2} - 1 + \pi/4\right)}$$

6.5.13

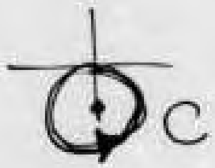
a) $\displaystyle\int_C \frac{\sin z}{z^2+1}\,dz$

$$= \int_C \frac{\sin z/(z+i)}{z-i}\,dz \qquad \frac{\sin z}{z+1} \text{ analytic in } D_1(i)$$

$$= 2\pi i \left.\frac{\sin z}{z+i}\right|_{z=i} = 2\pi i\,\frac{\sin i}{2i} = \pi \sin i$$

$$= \pi\left(\frac{e^{ii}-e^{-ii}}{2i}\right) = \pi i\left(\frac{e^{1}-e^{-1}}{2}\right) = \boxed{\pi i \sinh 1}$$

b)



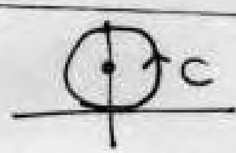
$\sin z/_{z-i}$ analytic in $D_1(-i)$

$$\int_C \frac{\sin z/_{z-i}}{z+i}\,dz = 2\pi i \left.\frac{\sin z}{z-i}\right|_{z=-i} = 2\pi i\,\frac{\sin -i}{-2i}$$

$$= \pi \sin i = \boxed{\pi i \sinh 1}$$

6.5.14

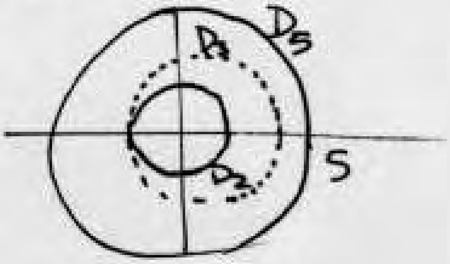
$\displaystyle\int_C \frac{1}{(z^2+1)^2}\,dz$



$$= \int_C \frac{1/(z+i)^2}{(z-i)^2}\,dz = 2\pi i\,\frac{d}{dz}\left.\frac{1}{(z+i)^2}\right|_{z=i}$$

$$= 2\pi i \left.\frac{-2}{(z+i)^3}\right|_{z=i} = 2\pi i\,\frac{-2}{(2i)^3} = \boxed{\frac{\pi}{2}}$$

6.6.5



f analytic in D_5, $|f(z)| \leq 10$ on $C_3(1)$

a) From Cauchy's inequality $|f(z)| \leq 10$ on $C_3(1)$ which is in D_5 where f is analytic.

Then $|f^{(4)}(1)| \leq \frac{4!\,(10)}{3^4} = \boxed{\frac{80}{27}}$

b) For a bound on $f^{(4)}(0)$, consider D_2 which is the largest disk with center 0 and contained in $D_3(1)$

Now, f is analytic in D_5 and therefore in D_3 and by the maximum modulus theorem,

$$\max_{z \in D_3} |f(z)| \leq \max_{z \in C_3(1)} |f(z)| \leq 10$$

$$\overset{D_2 \subset D_3}{\Rightarrow} \max_{z \in D_2} |f(z)| \leq 10$$

$$\overset{\text{Cauchy}}{\Rightarrow} |f^{(4)}(0)| \leq \frac{4!\,(10)}{2^4} = \boxed{15}$$

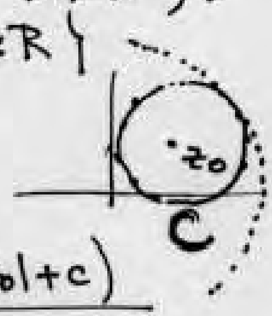
6.6.6

Let f be entire and $|f(z)| \leq M|z|+c$, $\forall z$ (need a c, otherwise $f(0)=0$).

a). Consider a point z_0 and the circle $C: \{|z-z_0| = R\}$

Then f is analytic on the disk and

$$|f(z)| \leq M|z|+c \leq M(|z_0|+R)+c, \quad \forall z \in C$$



By Cauchy's inequality, $\left|f^{(n)}(z_0)\right| \leq \dfrac{n!\,(MR+M|z_0|+c)}{R^n}$

Letting $R \to \infty$ we get $|f'(z_0)| \leq M$,

$$\left|f^{(2)}(z_0)\right| \leq \frac{K}{R} \text{ and as } R\to\infty \quad |f^2(z_0)| = 0.$$

Similarly for $n > 2$.

Now the right handsides are independent of z_0 so taking $\sup\limits_{z_0}$ we get

$$|f'(z)| \leq M \ \forall z, \qquad f^{(n)}(z) = 0 \ \forall z, \ \forall n \geq 2$$

b) f' is entire (since f is) and bounded $\overset{\text{Liouville}}{\Longrightarrow} f'(z) = \alpha$ (a constant). Let F be its antiderivative, e.g.

$F(z) = \int_0^z f'(\xi)\,d\xi + \underbrace{F(0)}_{\text{const}}$. Then, it must be that

$F(z) = \alpha(z-0) + F(0) = \alpha z + F(0)$. ($F$ is welldefined)

But $\frac{d}{dz}(F-f) = F'(z) - f'(z) = 0 \ \forall z \Rightarrow F-f$ is a constant. $\Rightarrow$ $\boxed{f(z) = \alpha z + \beta.}$

EEE 550 HW 5 SOLUTIONS

7.1.3 a) $\sum_{1}^{\infty} \frac{1}{k^2} z^k$ Radius of convergence $= \frac{1}{\limsup \sqrt[k]{1/k^2}} = 1$

$\therefore$ the issue is convergence on the boundary.

From M-test, $\sum \left|\frac{1}{k^2} z^k\right| \le \sum 1/k^2 < \infty$

$\Rightarrow$ uniform convergence in $\bar{D}_1(0)$

b). $\sum_{0}^{\infty} \frac{1}{(z^2-1)^k}$ For the given set $D_2^c(0)$, the M test yields $\frac{1}{(|z^2-1|)^k} \le \frac{1}{(|z|^2-1)^k} \le \frac{1}{(2^2-1)^k} = \frac{1}{3^k} = M_k$

Since $\sum_{0}^{\infty} M_k = \sum_{0}^{\infty} 1/3^k$ converges, $\sum_{0}^{\infty} 1/(z^2-1)^k$ converges uniformly in $D_2^c(0)$.

Notice that the series converges in $\{z : |z^2-1| > 1\}$.

But $|z^2-1| > |z|^2 - 1 \Rightarrow \{ |z|^2 - 1 > 1 \Rightarrow |z^2-1| > 1 \}$

Hence $z \in D_{\sqrt{2}}^{o}(0) \Rightarrow |z^2-1| > 1$, or $D_{\sqrt{2}}^{c}(0) \subset \{z : |z^2-1| > 1\}$

$\therefore$ Uniform convergence in $D_{\sqrt{2+\epsilon}}^{o}(0)$, for $\epsilon > 0$.

The given set $D_2^c(0)$ is $D_{\sqrt{2+\epsilon}}^{c}(0)$ for $\epsilon = 2$.

c) $\sum_{0}^{\infty} \frac{z^k}{z^{2k}+1}$ cannot converge in a domain containing the roots of $z^{2k}+1$, i.e. the unit circle.

For $z \in \bar{D}_r(0)$, $r < 1$, $\left|\frac{z^k}{z^{2k}+1}\right| \le \frac{r^k}{1-r^{2k}} \le \frac{r^k}{1-r^2} = M_k$ $\quad k \ge 1$

$\therefore$ by the M-test $\sum_{1}^{\infty} M_k$ converges and $\sum_{1}^{\infty} \frac{z^k}{z^{2k}+1}$ converges uniformly in $\bar{D}_r(0)$, $r < 1$.

Hence, $\sum_{0}^{\infty} \frac{z^k}{z^{2k}+1} = \frac{1}{2} + \sum_{1}^{\infty} \frac{z^k}{z^{2k}+1}$ also converges uniformly in $\overline{D}_r(0)$, $r<1$.

The series also converges uniformly in $D_r^c(0)$, $r>1$. Take ρ: $r>\rho>1$. Then $\left|\frac{z^k}{z^{2k}+1}\right| = \left|\frac{1}{z^k+z^{-k}}\right| \leq \frac{1}{|z|^k-|z|^{-k}}$

$$< \frac{1}{r^k-r^{-k}}$$

But $\frac{1}{r^k-r^{-k}} < \frac{1}{\rho^k}$ for sufficiently large k, say $k>N$

$\Rightarrow \sum_{N}^{\infty} \frac{z^k}{z^{2k}+1}$ converges uniformly in $D_r^c(0)$, $r>1$, and so does $\sum_{0}^{\infty} \frac{z^k}{z^{2k}+1}$.

7.2.4 $f(z) = \begin{cases} 1 & \text{if } z=0 \\ \frac{\sin z}{z} & \text{otherwise.} \end{cases}$

a) At $z=0$, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{\sin \Delta z}{\Delta z} - 1}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\frac{\Delta z - \Delta z^3/3! + \dots}{\Delta z} - 1}{\Delta z} = \lim_{\Delta z \to 0} \underbrace{\frac{\Delta z}{3!} + \dots}_{\text{conv. abs}} = 0$$

In a nbhd of 0 $f'(z) = \frac{\cos z}{z} - \frac{\sin z}{z^2}$

$\Rightarrow f$ is analytic at 0.

b). For the Maclaurin series, invoke uniqueness and expand $\sin z$ as $z - z^3/3! + \dots \Rightarrow \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \dots$

c) $g(z) = \int_C f(\zeta)\, d\zeta$

$f(\zeta)$ is $1 - \frac{z^2}{3!} + \dots$ so it is the limit of the sequence of partial sums, which is uniformly convergent in a domain containing C. Hence, exchanging limits and integration, $g(z) = \int_0^z 1 - \frac{z^2}{3!} + \dots$

$$= z - \frac{1}{3}\frac{z^3}{3!} + \dots$$

7.2.7

a) $f(z) = \sum_{n=0}^{\infty} \left[3 + (-1)^n\right]^n z^n$

Convergence: (must check!) $\frac{1}{\limsup \sqrt[n]{|(3+(-1)^n)^n|}} = \frac{1}{\limsup |3+(-1)^n|} = \frac{1}{4}$

Unif. convergence in $\overline{D}_{\frac{1}{4}-\epsilon}(0)$ $\frac{1}{4} > \epsilon > 0$

Derivative is evaluated at 0 which is inside $\overline{D}_{\frac{1}{4}-\epsilon}(0)$

Then $\frac{d^3f}{dz^3}(0) = 3!\, a_3 = 3!\ [3 + (-1)^3]^3 = 3!\, 2^3 = 8 \cdot 3 \cdot 2$

$\Rightarrow \underline{\frac{d^3f}{dz^3} = 48}$

b) $g(z) = \sum_{1}^{\infty} \frac{(1+i)^n}{n} z^n$; $\limsup \left|\frac{1+i}{\sqrt[n]{n}}\right| = 2 \Rightarrow r = 1/2$

$\frac{d^3g}{dz^3}(0) = 3!\, a_3 = \frac{(1+i)^3}{3} \cdot 6 = \underline{2(1+i)^3}.$

c). $h(z) = \sum_0^\infty \frac{1}{(\sqrt{3}+i)^n} z^n$ $\qquad \limsup \frac{1}{\sqrt[n]{|\sqrt{3}+i|^n}} = \limsup \frac{1}{|\sqrt{3}+i|} = \frac{1}{\sqrt{10}}$

$r = \sqrt{10}$.

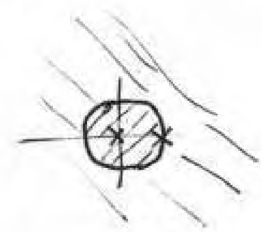
$$\frac{d^3h}{dz}(0) = 3!\, a_3 = 6/(\sqrt{3}+i)^3$$

7.3.4 $\quad \sum_0^\infty \frac{1}{(z-1)^n} = \sum_0^\infty w^n \Big|_{w = 1/z-1} = \frac{1}{1-w}$ for $|w|<1$

$$\Rightarrow \sum_0^\infty \frac{1}{(z-1)^n} = \frac{1}{1-\left(\frac{1}{z-1}\right)} \quad \text{for } \left|\frac{1}{z-1}\right| < 1$$

$$= \frac{z-1}{z-2} \quad \text{for } |z-1| > 1.$$

7.3.9 $\quad f(z) = \frac{1}{z(4-z)^2}$: poles at $0, 4$,
Laurent ROC
$D_4(0)$, $\overline{D}_4^{\,c}(0)$



1. Inside $D_4(0)$.: Observe that $\frac{1}{(4-z)^2} = \frac{d}{dz}\left(\frac{1}{4-z}\right)$

Also, $\frac{1}{4-z} = \frac{1/4}{1-z/4} = 1/4 \sum_0^\infty \frac{z^k}{4^k} = \sum_0^\infty \frac{z^k}{4^{k+1}}$

$$\Rightarrow \frac{1}{(4-z)^2} = \sum_0^\infty \frac{k\, z^{k-1}}{4^{k+1}}$$

$$\Rightarrow \frac{1}{z(4-z)^2} = \sum_{k=0}^{\infty} \frac{k z^{k-2}}{4^{k+1}} = 0 + \frac{z^{-1}}{4^2} + \sum_0^{\infty} \frac{(k+2) z^k}{4^{k+3}}$$

Outside $D_4(0)$: Similarly,

$$\frac{1}{4-z} = \frac{-1/z}{1-4/z} = -(1/z) \sum_0^{\infty} \left(\frac{4}{z}\right)^k$$

$$\Rightarrow \frac{1}{(4-z)^2} = \sum_1^{\infty} \frac{4^{k-1} k}{z^{k+1}}$$

$$\Rightarrow \frac{1}{z(4-z)^2} = \sum_1^{\infty} \frac{k 4^{k-1}}{z^{k+2}}$$

7.4.3

a) $z^2(z-\sin z)$ is an entire function $\Rightarrow$ No finite singularities

but $z^{-2}(z-\sin z) = z^{-2}(z - z + z^3/3! + \dots)$

$= z/3! + \dots$ (Taylor), Entire

$\Rightarrow$ A removable singularity at 0.

b) $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^3 3!} + \dots$: Essential singularity at 0

(infinite number of $\frac{1}{z}$-powers)

c). $z \exp(1/z) = z\left(1 + \frac{1}{z} + \frac{1}{z^2 2!} + \dots\right) = z + 1 + \frac{1}{2! z} + \dots$

Essential singularity at 0.

d) $\tan z = \frac{\sin z}{\cos z}$: singular when $\cos z = 0$

Examine the type of zeros of the inverse :

$\frac{\cos z}{\sin z} = 0$ at $\frac{\pi}{2} + n\pi$; (Notice that $\sin z \neq 0$)

Then, $\frac{d}{dz}\left(\frac{\cos z}{\sin z}\right) = \frac{-\sin z}{\sin z} + \frac{\cos^2 z}{-\sin^2 z} = -1 \neq 0$

$\Rightarrow$ the zeros are simple.

$\Rightarrow$ $\tan z$ has poles of order 1 at $\frac{\pi}{2} + n\pi$

e) $\frac{\sin z}{z(z+1)} = \frac{z - z^3/3! + \dots}{z(z+1)} = \frac{1 - z^2/3! + \dots}{z+1}$

Note $\sin z \neq 0$ at $z = -1$ $\Rightarrow$

$\begin{cases} \text{A removable singularity at } 0 \\ \text{A pole of order 1 at } -1. \end{cases}$

f) $\frac{z}{\sin z}$: Since $\frac{\sin z}{z} \neq 0$ the singularity at 0 is removable

The singularities at $n\pi$ $(n \neq 0)$ are poles of order 1 since $\frac{d}{dz}\left(\frac{\sin z}{z}\right) = \frac{\cos z}{z} - \frac{\sin z}{z^2} \neq 0$

g) $\dfrac{e^z - 1}{z} = \dfrac{1 + z + \frac{z^2}{2!} + \dots - 1}{z} = 1 + z/2! + \dots$ (entire)

$\Rightarrow$ Removable singularity at 0

h). $\dfrac{\cos z - \cos 2z}{z^4} = \dfrac{1 - z^2/2! + z^4/4! + \dots - 1 + 4z^2/2! - 16z^4/4! + \dots}{z^4}$

$$= \frac{3}{2} z^{-2} + 15/4! + \dots$$

A pole of order 2 at 0.

Notice that the functions may also have singularities at ∞. To determine the type of these singularities we examine $f(1/z)$ as $z \to 0$.

E.g. $f(z) = z^{-2}(z - \sin z)$

$$f(1/z) = z^2\left(\frac{1}{z} - \sin\frac{1}{z}\right) = z^2\left(\frac{1}{z} - \frac{1}{z} + \frac{1}{z^3 3!} + \dots\right)$$

$$= \frac{1}{z\,3!} + \dots \Rightarrow \text{Essential singularity at } 0$$

$\Rightarrow f(z)$ has an essential singularity at ∞.

E.g $f(z) = \sin\left(\frac{1}{z}\right)$; $f(1/z) = \sin(z)$ no singularity at 0

$\Rightarrow$ not singular (analytic) at ∞.

Eq. $f(z) = z\exp(1/z)$; $f(1/z) = \frac{1}{z}\exp z$; pole of order 1 at 0

$\Rightarrow$ pole of order 1 at ∞.

EEE 550 HW#6 SOLUTIONS

8.2.4 f/g has a simple pole at z_0. Then

$$\mathrm{Res}\left(f/g, z_0\right) = (z-z_0)\frac{f(z)}{g(z)}\Big|_{z_0} = \frac{f(z_0)}{g_1(z_0)}$$

where g_1: $g(z) = (z-z_0)\,g_1(z)$

But $\frac{dg}{dz}(z) = (z-z_0)\frac{dg_1}{dz} + g_1(z)$

And $\frac{dg}{dz}(z_0) = g_1(z_0)$. Hence, $\mathrm{Res}\left(f/g, z_0\right) = \frac{f(z_0)}{g'(z_0)}$.

8.5.1 1) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+9}\,dx$; $P=1$, $Q=x^2+9$, $\alpha=1$ | $f = \frac{e^{iz}}{x^2+9}$, $\mathrm{Res}(f, i3) = \frac{e^{-3}}{2i3}$

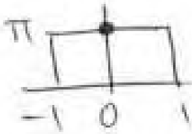
$$PV\int_{-\infty}^{\infty} \frac{\cos x}{x^2+9}\,dx = -2\pi \sum \mathrm{Im}\,\mathrm{Res}\left[f, i3\right] = \boxed{\frac{\pi}{3e^3}}$$

2) $$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+9}\,dx = 2\pi \sum \mathrm{Re}\,\mathrm{Res}\left[f, i3\right] = \boxed{0}$$

8.7.9 $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$; $P=1$, $Q=z$, $\alpha=1$ | $f = \frac{e^{iz}}{z}$, 1 pole on real axis

$$\int = \pi \sum \mathrm{Re}\,\mathrm{Res}(f, t_i) = \pi\,\mathrm{Re}\,e^{i0} = \boxed{\pi}$$

Fourier version: $\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \int_{-\infty}^{\infty} \frac{\sin t}{t} e^{j\omega t}\, dt \Big|_{\omega=0}$

$= \pi \mathcal{F}\left\{ \frac{\sin t}{\pi t} \right\}\Big|_{\omega=0} =$  $= \pi.$

8.7.10 $\quad 1 - e^{i2z} = e^{iz}\left(e^{-iz} - e^{iz}\right)\left(\frac{2i}{2i}\right)$

$= -2i\, e^{iz} \sin z$

$= -2i\,(\cos z + i \sin z) \sin z$

$= 2\sin^2 z - i \cos z \sin z$

$\Rightarrow \int_{\mathbb{R}} \frac{\sin^2 z}{z^2} = \frac{1}{2}\,\mathrm{Re} \int_{\mathbb{R}} \frac{1-e^{i2z}}{z^2}$; $\frac{1-e^{i2z}}{z^2}$: simple pole at 0

$\stackrel{\text{Jordan}}{=} \frac{1}{2}\,\mathrm{Re}\left[2\pi i \sum \mathrm{Res}\left[\cancel{f, z_j}\right] + \pi i \sum \mathrm{Res}\left[f, t_j\right] \right]$

$= \frac{1}{2}\,\mathrm{Re}\left[\pi i\, \mathrm{Res}\left[\frac{1-e^{i2z}}{z^2}, 0 \right] \right]$

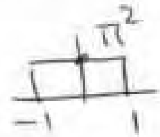
$= \frac{1}{2}\,\mathrm{Re}\; \pi i \; \frac{1-e^{i2z}}{z} \Big|_{z=0}$

$\stackrel{\text{L'Hopital}}{=} \frac{1}{2}\,\mathrm{Re}\; \pi i \; \frac{-2i\, e^{i2z}}{1} \Big|_{z=0} =$

$= \frac{1}{2}\,\mathrm{Re}\; \pi i\,(2i) = \boxed{\pi}$

Fourier Version: $\displaystyle \int \frac{\sin^2 t}{t^2}\,dt = \int \pi^2 \left|\frac{\sin t}{\pi t}\right|^2 dt$

$$\overset{\text{Parseval}}{=} \frac{1}{2\pi}\int \left|\mathcal{F}\left\{\pi \cdot \frac{\sin t}{\pi t}\right\}\right|^2 d\omega$$

$$= \frac{1}{2\pi}\int$$  $$= \frac{2\pi^2}{2\pi} = \boxed{\pi}$$

8.8.9 z analytic and $|h(z)| < 1$ in $\overline{D_1(0)}$ (The closure of the disk). Define $g(z) = h(z) - z^n$ and $f(z) = z^n$. Both are meromorphic in $D_1(0)$ and $|g(z) + f(z)| = |h(z)| < 1 = |f(z)|$ on $C_1(0)$

Hence, by Rouché, $\underbrace{Z_f}_{n \text{ at the origin}} - \underbrace{P_f}_{0} = Z_g - \underbrace{P_g}_{0} \Rightarrow Z_g = n$

So $g(z)$ has n zeros inside $C_1(0)$.

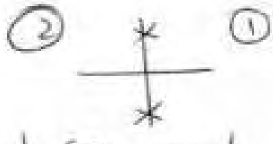
Implication: Say $h(z) = \sum_0^{n-1} a_k z^k$. Then

$$|h(z)| \le \sum |a_k||z^k| \le \sum |a_k|$$

So, if $\sum |a_k| < 1$ the polynomial

$P(z) = z^n + \sum_0^{n-1} a_k z^k$ has roots inside the unit circle (stable in D-T)

11.9.7 $\quad Y(s) = \dfrac{1}{s^2+4}$



Two possible ROC: causal (1) and anti-causal (2)

Applying the analysis presented in-class

$$y(t) = \begin{cases} -\sum \text{Res}\left[Y(s)\,e^{st}, s_R\right] & t<0 \\ \sum \text{Res}\left[Y(s)\,e^{st}, s_L\right] & t>0 \end{cases}$$

where s_R, s_L are the poles to the Right or left of the Bromwich contour, respectively. Hence,

ROC 1 :

$$y(t) = \begin{cases} 0 & t<0 \\ \text{Res}\left[\dfrac{e^{st}}{(s+2i)(s-2i)}, 2i\right] + \text{Res}\left[\dfrac{e^{st}}{(s+2i)(s-2i)}, -2i\right] \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ \dfrac{e^{i2t}}{4i} + \dfrac{e^{-i2t}}{-4i} = \dfrac{1}{2}\sin 2t & t>0 \end{cases}$$

ROC (2) :

$$y(t) = \begin{cases} -\text{Res}\left[\dfrac{e^{st}}{(s+2i)(s-2i)}, 2i\right] - \text{Res}\left[\dfrac{e^{st}}{(s+2i)(s-2i)}, -2i\right] & t<0 \\ 0 & t>0 \end{cases}$$

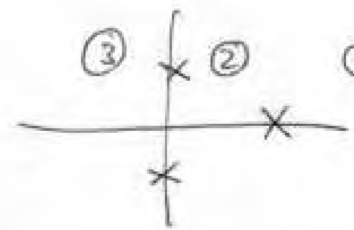
$$= \begin{cases} -\frac{1}{2}\sin 2t & \text{for } t<0 \\ 0 & \text{for } t>0. \end{cases}$$

11.9.8 $\quad Y(s) = \dfrac{s+3}{(s-2)(s+j)(s-j)}$

Let's try the PFE for this one:

$$Y(s) = \frac{5/5}{s-2} + \frac{Bs+C}{s^2+1}$$

$$= \frac{1}{s-2} - \frac{s+1}{s^2+1}$$



Now we have three possibilities of ROC.

ROC ①, causal: $\mathcal{L}_2^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{-\frac{s+1}{s^2+1}\right\}$$

$$= e^{2t}u(t) - \cos t\, u(t) - \sin t\, u(t).$$

ROC ②: Causal $\pm j$, Anticausal 2

$$\mathcal{L}_2^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{s+1}{s^2+1}\right\} + R\mathcal{L}^{-1}R\left\{\frac{1}{s-2}\right\}$$

$$= -\cos t\, u(t) - \sin t\, u(t) + R\mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\}$$

$$= -\cos t\, u(t) - \sin t\, u(t) + R\left[-e^{-2t}u(t)\right]$$

$$= -\cos t\, u(t) - \sin t\, u(t) - e^{2t}u(-t).$$

$\mathcal{L}$: unilateral, causal Laplace, $\mathcal{L}_2$ bilateral Laplace.

ROC ③ Anticausal:

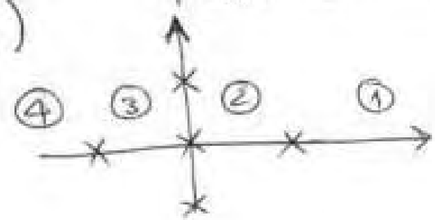
$$\mathcal{L}_2^{-1}\{Y(s)\} = R\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+1}\right\} + R\mathcal{L}^{-1}\left\{\frac{-1}{s+2}\right\}$$

$$= R\left[\cos t\, u(t) - \sin t\, u(t)\right] + \left(-e^{2t} u(-t)\right)$$

$$= \cos t\, u(-t) + \sin t\, u(-t) - e^{2t} u(-t).$$

11.9.9. $Y(s) = \dfrac{s^3+s^2-s+3}{s(s^4-1)}$ poles at $0, \pm 1, \pm i$

Four possible ROC:



$$Y(s) = \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1} + \frac{E}{s}$$

where $A = \left.\dfrac{s^3+s^2-s+3}{s(s-1)(s^2+1)}\right|_{s=-1} = \cdots$

$B = \cdots$

$E = \cdots$

C, D: Write two eqn & solve them

The inversion of each term is then tabulated as follows

Term	Causal $\mathcal{L}_2^{-1}/_{ROC}$	Anticausal $\mathcal{L}_2^{-1}/_{ROC}$
$\frac{A}{s+1}$	$Ae^{-t}u(t)$ ROC: 1, 2, 3	$-Ae^{-t}u(-t)$ ROC: 4
$\frac{B}{s-1}$	$Be^{t}u(t)$ ROC: 1	$-Be^{t}u(-t)$ ROC: 2, 3, 4
$\frac{Cs+D}{s^2+1}$	$(C\cos t + D\sin t)u(t)$ ROC 1, 2	$-(C\cos t + D\sin t)u(-t)$ ROC: 3, 4 (also see 11.9.8)
$\frac{E}{s}$	$E\,u(t)$ ROC 1, 2	$-E\,u(-t)$ ROC: 3, 4

Then mix and match according to the ROC of interest.