EEE HW #3 SOLUTIONS

#10. $$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{pmatrix} s & -1 \\ 1 & s+2 \end{pmatrix}^{-1}\right\} =$$

$$\mathcal{L}^{-1}\frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix} = \boxed{\begin{pmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{pmatrix}}.$$

C.H. $$e^{At} = \beta_0(t)I + \beta_1(t)A : \quad \begin{cases} e^{-t} = \beta_0 + \beta_1(-1) \\ te^{-t} = \beta_1 \end{cases}$$

$$\Rightarrow \beta_0 = e^{-t}+te^{-t}$$

$$= \begin{bmatrix} e^{-t}+te^{-t} & \\ & e^{-t}+te^{-t} \end{bmatrix} + te^{-t}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \ldots \text{ as before}$$

J.F. $$E = \begin{bmatrix} 1 & 1/2 \\ -1 & 1/2 \end{bmatrix}, \quad J = \begin{bmatrix} -1 & 1 \\ & -1 \end{bmatrix} \Rightarrow e^{Jt} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

$$\Rightarrow e^{At} = E e^{Jt} E^{-1} = \begin{bmatrix} e^{-t} & te^{-t}+e^{-t}/2 \\ -e^{-t} & -te^{-t}+e^{-t}/2 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1 & 1 \end{bmatrix}$$

$$= \ldots \text{ as before}$$

#11. T.F: $$C(sI-A)^{-1}B+D = (1\ \ 1)\frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s+1}$$

STEP RESP. $$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-1}{s+1}\right\} = 1-e^{-t}, \quad t \geq 0$$

St.sp: $$y(t) = Ce^{+At}\underbrace{x(0)}_{=0} + \int_0^t Ce^{A(t-\tau)}B\underbrace{u(\tau)}_{=1}d\tau$$

$$= \int_0^t [1\ \ 1]\left.\begin{pmatrix} e^{-t}+te^{-t} & te^{-t} \\ -te^{-t} & e^{-t}-te^{-t} \end{pmatrix}\right|_{t-\tau}\begin{pmatrix} 0 \\ 1 \end{pmatrix}d\tau$$

$$= \int_0^t e^{-(t-\tau)}d\tau = 1-e^{-t}, \quad t \geq 0$$

#12 Controllable canonical form:

$$A=\begin{pmatrix}-2 & -1\\ 1 & 0\end{pmatrix},\quad B=\begin{pmatrix}1\\0\end{pmatrix},\quad C=(0,\ 1),\quad D=0$$

Other zero-state equivalent (I/O equivalent) realizations of higher dimension can be obtained by adding a state that receives no contribution from the input or other states and/or has no contri to the output or other states. To show that it is not Topologically Equivalent to another, it is sufficient to choose different eigenvalues for the additional state.
(Since $\det(\lambda I - P^{-1}AP) = \det P^{-1}(\lambda PIP^{-1} - A)P =$
$= \det P^{-1} \det(\lambda I - A) \det P = \det(\lambda I - A)$
$\Rightarrow$ eigenvalues are preserved after similarity transformation)

Thus: SS1. $A=\left(\begin{array}{cc|c}-2 & -1 & \\ 1 & 0 & \\ \hline & & 0\end{array}\right)$, $B=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $C=(0,1,0)$
$D=0$

SS2. $A=\left(\begin{array}{cc|c}-2 & -1 & \\ 1 & 0 & \\ \hline & & 1\end{array}\right)$, $B=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $C=(0,1,0)$
$D=0.$

SS3: Similarity transf on SS1 with a "random" matrix $P=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

$\Rightarrow \bar{A}=P^{-1}AP=\begin{bmatrix}-3&-4&-4\\1&1&1\\0&0&0\end{bmatrix}$ $\bar{B}=P^{-1}B=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\bar{C}=CP=\begin{bmatrix}0&1&1\end{bmatrix}$, $\bar{D}=D=0$

#13. Relative degree $r = \partial d - \partial n$ for $H(s) = \frac{n(s)}{d(s)}$

$\Rightarrow s^{n} H(s) = \frac{s^{n} n(s)}{d(s)}$ is biproper, so there is a direct -throughput (u appears in y directly, or its state space realization has $D \neq 0$.)

Realization of $\frac{dy}{dt}$: $\left.\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}\right\} \Rightarrow \frac{dy}{dt} = C\dot{x} + D\dot{u}$.

If $D \neq 0$ then $\frac{dy}{dt}$ is not realizable and requires $\frac{du}{dt}$ (i.e., $sH(s)$ is improper)

$\Rightarrow \frac{dy}{dt} = CAx + CBu$.

To realize $\frac{d^2y}{dt^2}$, CB must be zero, in which case

$\frac{d^2y}{dt^2} = CA\dot{x} = CA^2x + CABu$

$\Rightarrow$ for relative degree r, $D = 0$, $CA^{i}B$, $i = 0, 1 \ldots r-2$ must be zero and $CA^{r-1}B \neq 0$.

#14. New states $\tilde{x} = e^{-At}x$: $\dot{\tilde{x}} = -Ae^{-At}x + e^{-At}(Ax+Bu) = e^{-At}Bu$

$y = Cx = Ce^{At}\tilde{x} \Rightarrow (\tilde{A}, \tilde{B}, \tilde{C}) = (0, e^{-At}B, Ce^{At})$

e^{-At} is Lyapunov only if all eigenvalues of A are on the $j\omega$-axis and its maximum Jordan block has size 1.

#15. $\dot{X} = Ae^{At}Ce^{Bt} + e^{At}CBe^{Bt}$

$= AX + XB$ (since $Be^{Bt} = e^{Bt}B$)

$X(0) = e^{A0}Ce^{B0} = C \Rightarrow X(t)$ is the unique sol'n

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#16. $\Phi^{-1}(t,t_0) = \Phi(t_0,t)$.

Then, $\frac{d}{dt}\Phi(t_0,t) = \frac{d}{dt}\Phi^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)\frac{d\Phi(t,t_0)}{dt}\Phi^{-1}(t,t_0)$

$= -\Phi(t_0,t) \cdot A\Phi(t,t_0)\Phi^{-1}(t,t_0)$

$= -\Phi(t_0,t)A$.

#17. We solve $A^TP + PA = -I$ for the CT systems

and $A^TPA - P = -I$ for the DT systems

We will have asymptotic stability iff P exists and $P > 0$.

①. $P = \frac{1}{8}\begin{pmatrix} 10 & 2 \\ 2 & 3 \end{pmatrix} > 0. \Rightarrow$ A.S.

② $P = \frac{1}{8}\begin{pmatrix} -6 & 2 \\ 2 & 1 \end{pmatrix} \not> 0 \Rightarrow$ not A.S.

③ $P = \frac{1}{10}\begin{pmatrix} -38 & -16 \\ -16 & -12 \end{pmatrix} \not> 0 \Rightarrow$ not A.S

④. $P = \begin{pmatrix} 6.878, & 5.344 \\ 5.344, & 5.878 \end{pmatrix} > 0 \Rightarrow$ A.S.

#18. ① $A^TM + MA + 2\mu M = (A+\mu I)^T M + M(A+\mu I) = -I$

$\Rightarrow \operatorname{Re}\operatorname{Eig}(A+\mu I) < 0 \Rightarrow \operatorname{Re}\operatorname{Eig}(A) < -\mu$

② $\Leftrightarrow A^TMA - \mu^2 M = -\mu^2 I$

$\Rightarrow \left(\frac{A}{\mu}\right)^T M \left(\frac{A}{\mu}\right) - M = -I$

$\Rightarrow \left|\operatorname{Eig}\left(\frac{A}{\mu}\right)\right| < 1 \Rightarrow |\operatorname{Eig}(A)| < \mu$
