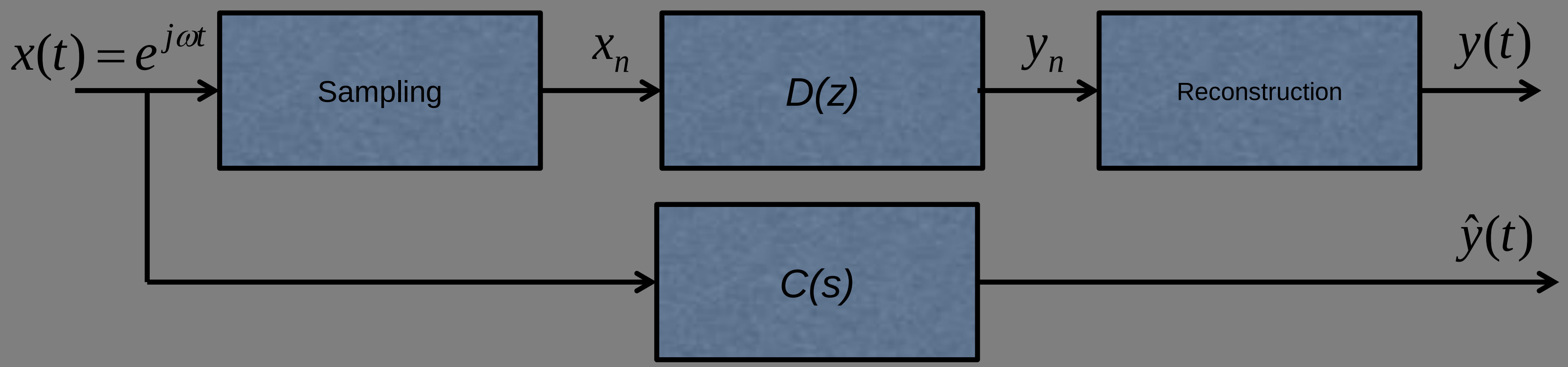
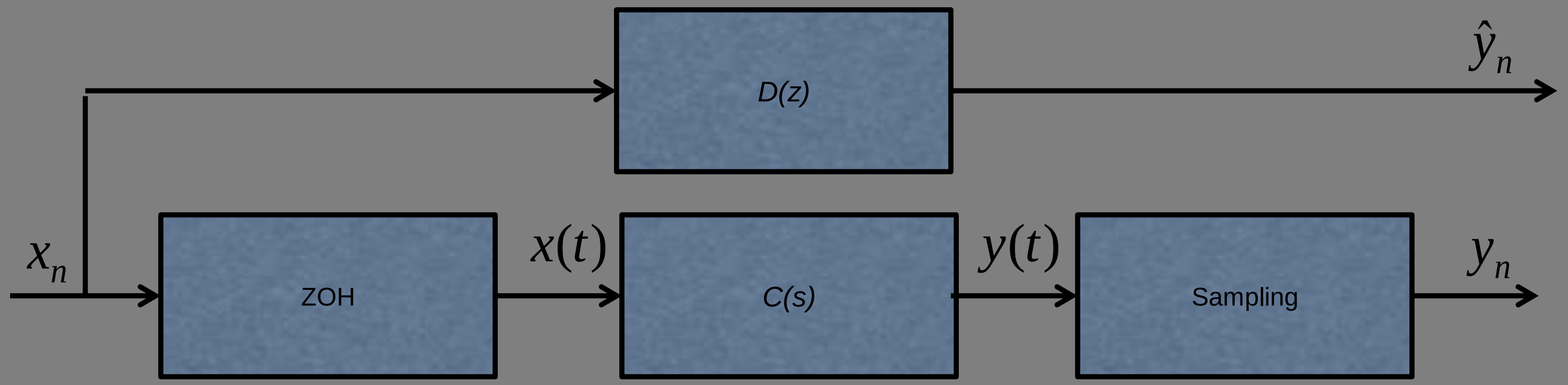
EEE304

Week 4: DT – CT filter equivalence





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Lecture 4.1: DT - CT Transfer Function Approximation

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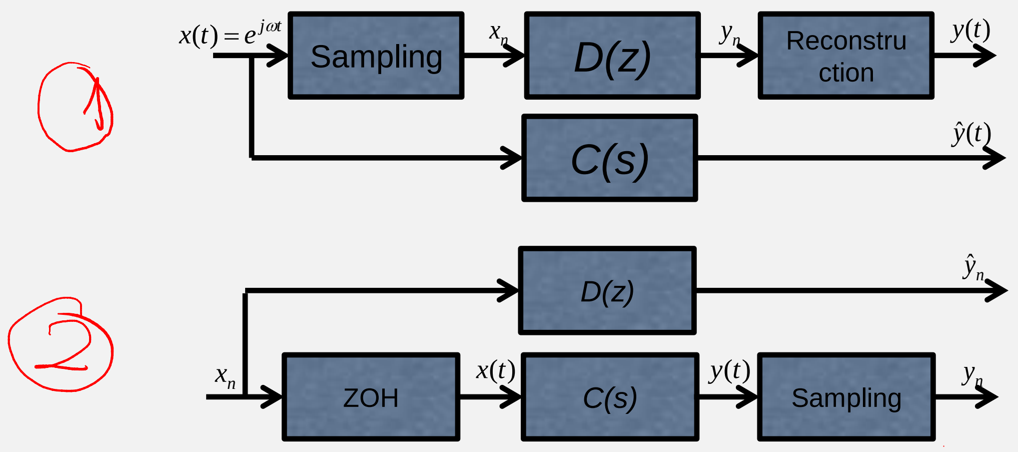
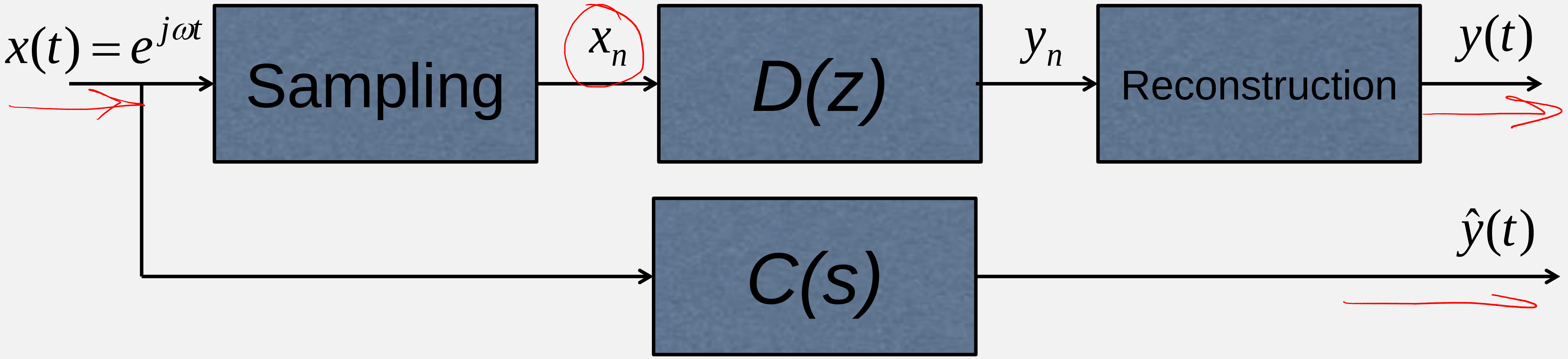
DT-CT filter "equivalence"



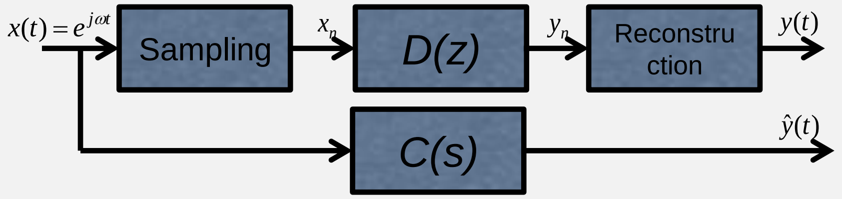
- DT digital filters replacing CT analog filters
  - Implementation, flexibility, consistency, power requirements, maintenance
  - Experience and specifications are available mostly in CT terms
- CT and DT (sampled-data) filters cannot be strictly equivalent, but we can define an approximation for two classes of problems:
  - 1 Given a bandlimited signal within the Nyquist frequency of the sampling system, determine H(z) or H(s) so that the CT output is identical to the ideal-filter reconstructed DT output.
  - 2 Given a sampled-data signal (pw constant) determine H(z) or H(s) so that the sampled CT output is identical to the DT output.

DT-CT filter "equivalence"

- Bandlimited signals: Matching the frequency response of the two filters



- Sampling x(t) we get $x_n = x(nT) = e^{j\omega nT} = \left(e^{j\omega T}\right)^n = \left(e^{j\Omega}\right)^n = z^n$ $\quad \Omega = \omega T$
- Then $y_n = D\left[e^{j\Omega}\right]\left(e^{j\Omega}\right)^n$
- After ideal reconstruction, $y(t) = REC\left\{D\left[e^{j\Omega}\right]e^{j\Omega n}\right\} = REC\left\{D\left[e^{j\Omega}\right]e^{j\omega Tn}\right\} = D\left[e^{j\Omega}\right]e^{j\omega t}$
- Matching the frequency responses $D\left(e^{j\omega T}\right) = C(j\omega)$



DT-CT filter "equivalence:" FE

- Frequency response matching filters $D(z)\big|_{z=e^{sT}} = C(s)$. The transformation $z = e^{sT}$ is not finite-dimensional and should be approximated for implementation.

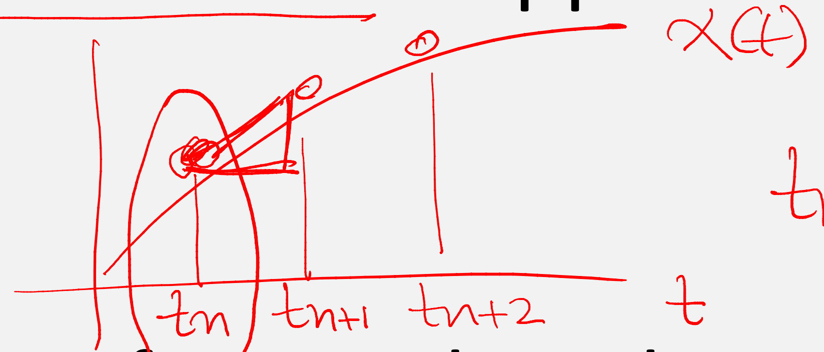
$|sT| << 1$

- One possibility is the so-called Forward Euler $z = 1 + sT$ (the first two terms of the Taylor series expansion). The FE approximates the frequency response of DT/CT filters for low frequencies such that |sT| << 1.

- The FE transformation can also be motivated by the forward Euler approximation of the derivative

$s = \frac{z-1}{T}$

$$sX(s) \leftrightarrow \frac{dx}{dt}(t_n) \approx \frac{x_{n+1}-x_n}{T} \leftrightarrow \frac{z-1}{T}X(z)$$



$t_n = nT$

- This approach yields a straightforward iterative solution of general nonlinear differential equations. (Higher order variants find extensive applications in numerical analysis.)

$$\frac{dx}{dt} = f(x) \Rightarrow \frac{x_{n+1}-x_n}{T} \approx f(x_n) \Rightarrow \boxed{x_{n+1} = x_n + Tf(x_n)}$$

$s = \frac{z-1}{T}$

DT-CT filter "equivalence:" Example FE1

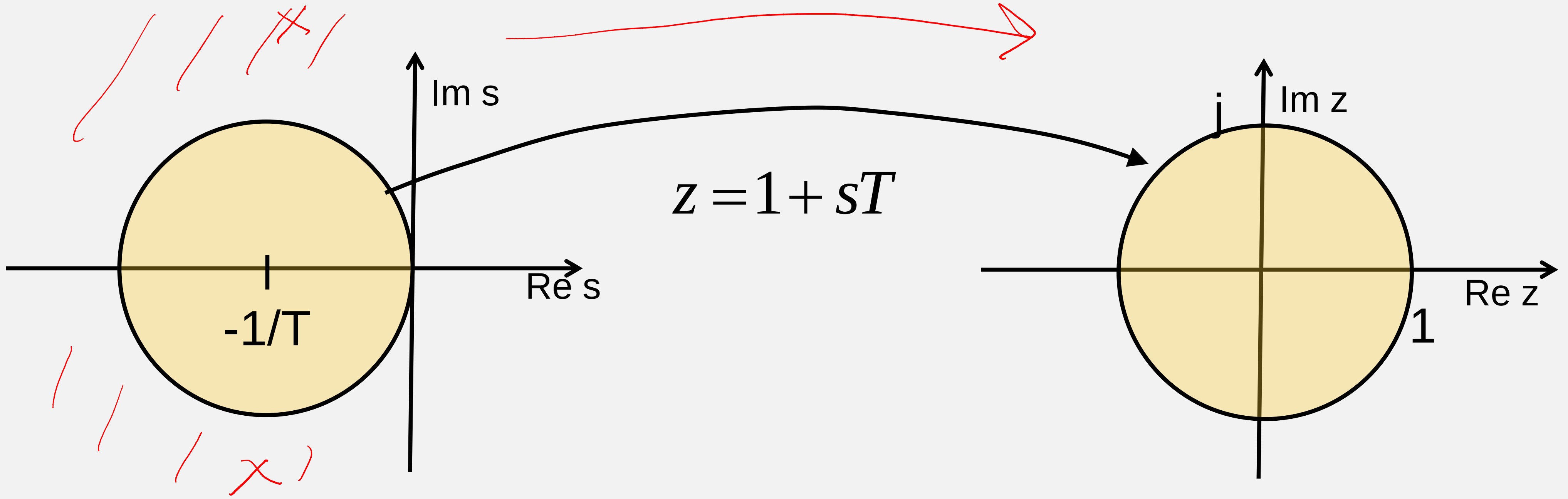
- The Forward Euler discretization of the CT transfer function $C(s) = \frac{1}{\tau s + 1}$

$$D(z) = C(s)\Big|_{s=\frac{z-1}{T}} = \frac{1}{\tau s + 1}\Big|_{s=\frac{z-1}{T}} = \frac{1}{\tau\frac{z-1}{T}+1} = \frac{\frac{T}{\tau}}{(z-1)+\frac{T}{\tau}} = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

At DC $(z=1)$

$$D(1) = \frac{T/\tau}{1-(1-T/\tau)} = 1 = C(0)$$

- As T increases, the approximation gets naturally worse and it even becomes unstable when $T/\tau > 2$
- It is often useful to visualize the map of characteristic domains with such transformations, e.g. stability.
  - Here we observe that the stable region in the z-domain (UC) is the map of a subset of the LHP, so there are other stable CT systems that do not map to stable DT systems.



$s = \frac{z-1}{T}$

DT-CT filter "equivalence:" Example FE1

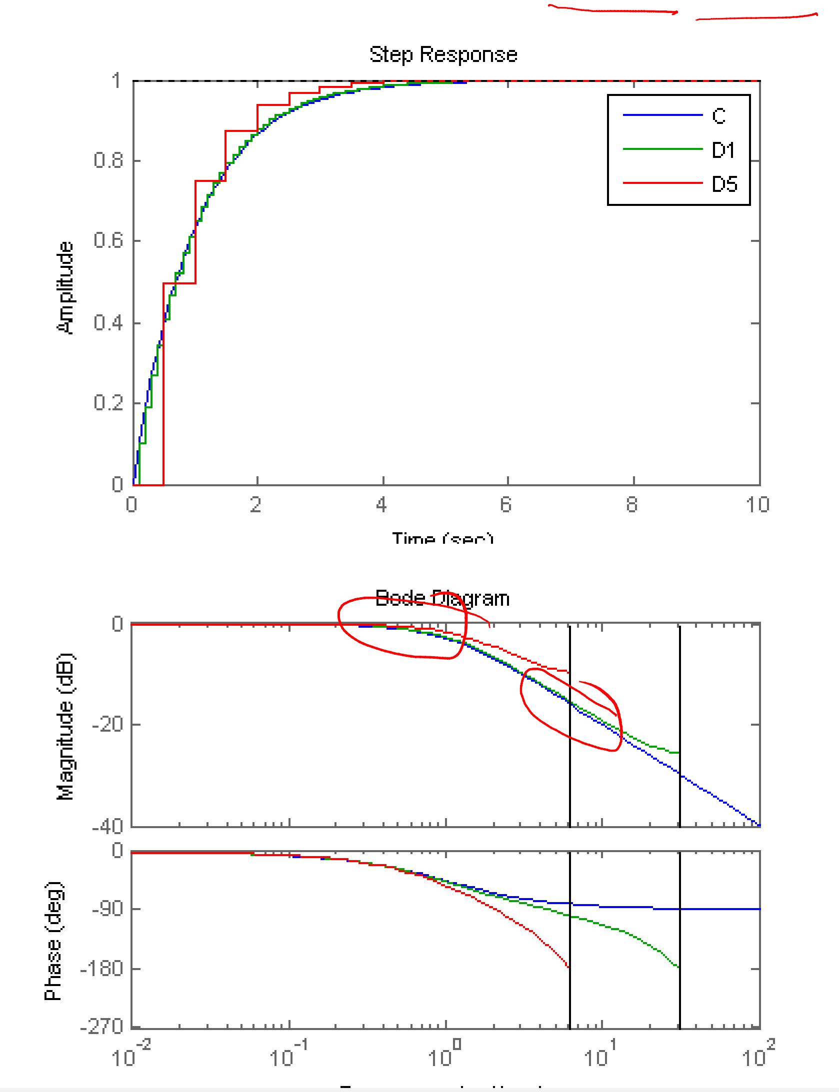
- The Forward Euler discretization

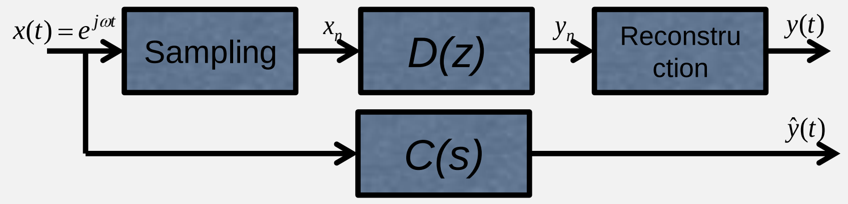
$$C(s) = \frac{1}{\tau s + 1} \xleftrightarrow{FE} D(s) = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

In the examples :

$\tau = 1, \quad T = 0.1, 0.5$

- The approximation is well-behaved for $T/\tau < 1$
- FE yields a quick conversion and preserves the relative degree of the transfer function.
- Its main limitation is that the sampling rate must be faster than the fastest system pole.

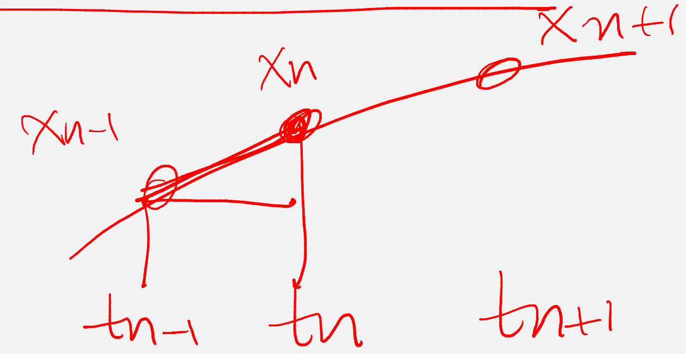




DT-CT filter “equivalence:” BE

- Frequency response matching $D(z)\big|_{z=e^{sT}} = C(s)$, approximation of $z = e^{sT}$.
- Another possibility is the so-called Backward Euler $z^{-1} = 1 - sT \approx e^{-sT}$.
- The BE approximation is also valid only for low frequencies: $|sT| << 1$.
- The BE transformation can also be motivated by the backward Euler approximation of the derivative

$$sX(s) \leftrightarrow \left.\frac{dx}{dt}\right|_{t_n} \approx \frac{x_n - x_{n-1}}{T} \leftrightarrow \frac{1 - z^{-1}}{T} X(z) = \frac{z-1}{Tz} X(z)$$



- This approach yields an implicit iterative solution of general nonlinear differential equations:

$$\frac{dx}{dt} = f(x) \Rightarrow \frac{x_n - x_{n-1}}{T} \approx f(x_n) \Rightarrow \boxed{x_n - Tf(x_n) = x_{n-1}}$$

$$s = \frac{z-1}{Tz}$$

$s = \frac{z-1}{Tz}$

DT-CT filter "equivalence:" Example BE1

- The Backward Euler discretization of the CT transfer function $C(s) = \frac{1}{\tau s + 1}$

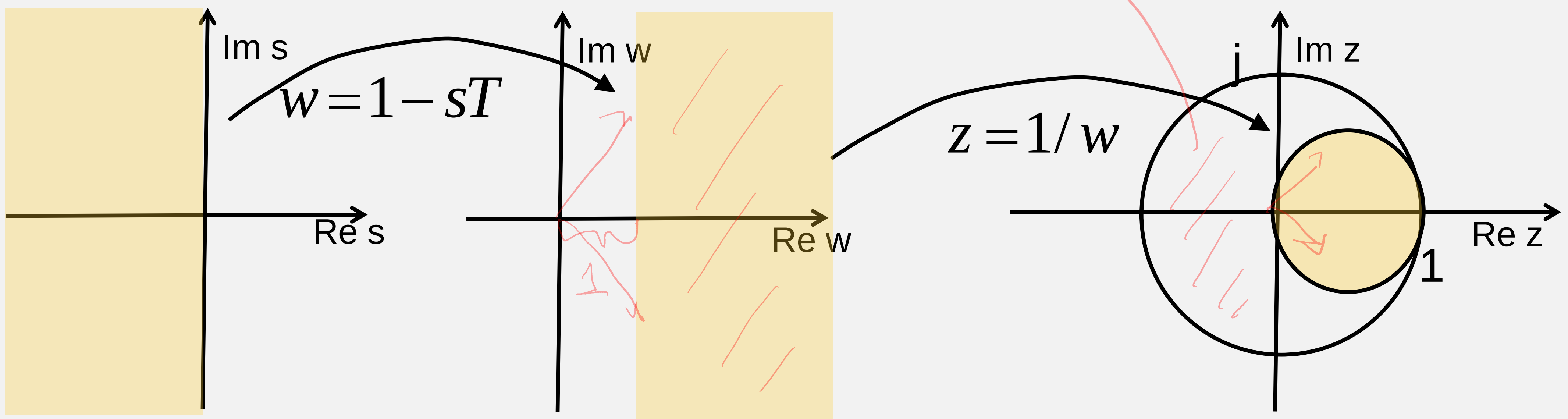
$$D(z) = C(s)\Big|_{s=\frac{z-1}{Tz}} = \frac{1}{\tau s+1}\Bigg|_{s=\frac{z-1}{Tz}} = \frac{1}{\tau\frac{z-1}{Tz}+1} = \frac{\frac{T}{\tau}z}{(z-1)+\frac{T}{\tau}z} = \frac{\frac{T}{\tau}z}{\left(1+\frac{T}{\tau}\right)z-1}$$

pole $\frac{1}{1+\frac{T}{\tau}}$

- As T increases, the approximation gets naturally worse but never becomes unstable for $\tau > 0$
- Visualizing the map of the CT stability domain:

map to unstable CT systems

- Here we observe that the stable region in the s-domain (LHP) maps into a subset of the UC, so there are other stable DT systems that are not maps of stable CT systems.



$$s = \frac{z-1}{Tz}$$

DT-CT filter "equivalence:" Example BE1

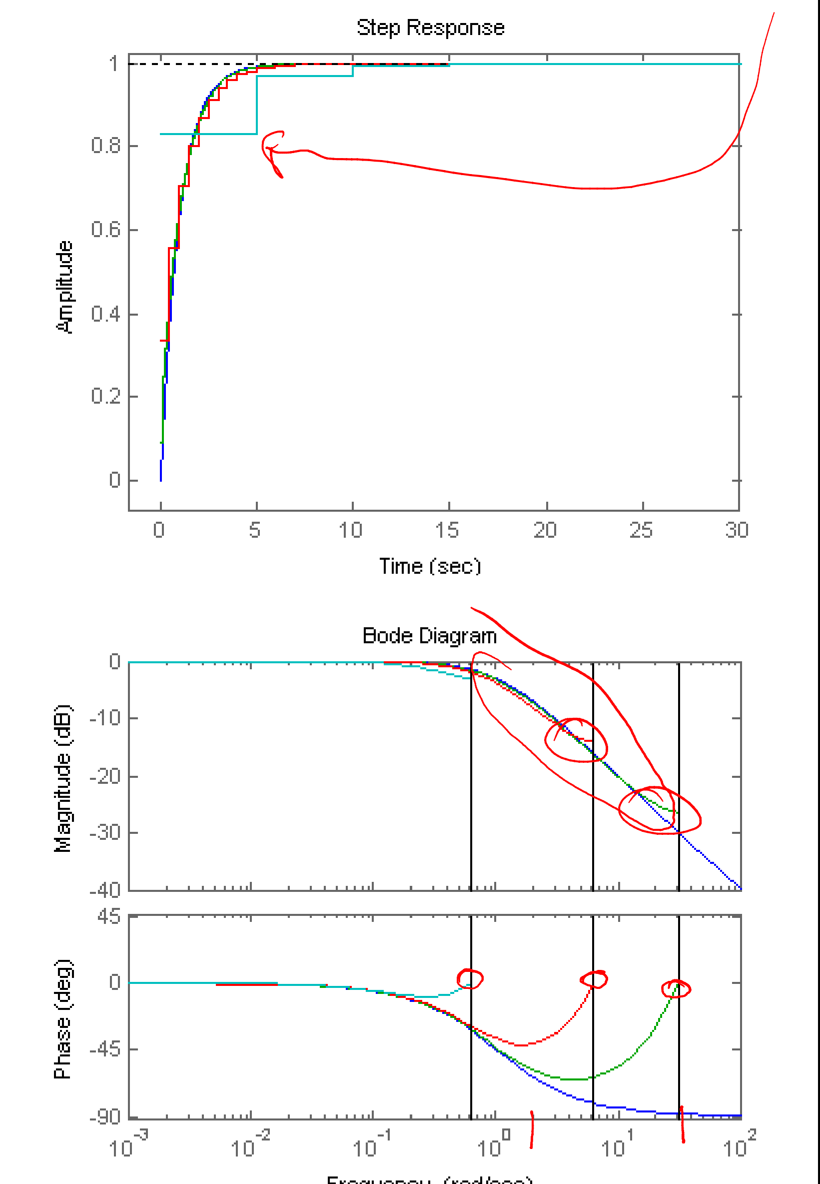
- The Backward Euler discretization

$$C(s) = \frac{1}{\tau s + 1} \xleftrightarrow{BE} D(z) = \frac{\frac{T}{\tau} z}{\left(1 + \frac{T}{\tau}\right) z - 1}$$

DC: $D(1) = 1$

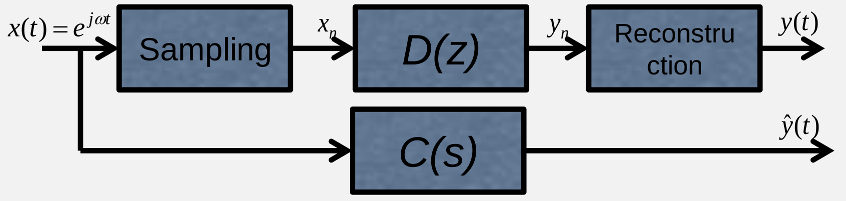
In the examples: $\tau = 1, \quad T = 0.1, 0.5, 5$

- "Justifiable" response for any *T*. In the limit as T = inf, D(z) = 1.
- BE yields biproper transfer functions and that can be an issue in applications where strict causality is required.
- BE yields an implicit conversion but it is relatively simple.



@ Nyquist Freq: $z = e^{j\pi} = -1 \Rightarrow D(z) = \frac{-T/\tau}{-1 - T/\tau - 1} > 0$

$\Rightarrow \angle D(e^{j\pi}) = 0$



DT-CT filter "equivalence:" Tustin

- A better approximation to achieve frequency response matching is the so-called Tustin transformation (BILINEAR)

$$s = \frac{az+b}{cz+d}$$

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} = \frac{1+sT/2}{1-sT/2} \Leftrightarrow s = \frac{2}{T}\cdot\frac{z-1}{z+1}$$

This transformation provides a good approximation up to frequencies close to the Nyquist frequency (π/T), e.g., up to 1/3-Nyquist.

- Tustin belongs to the general class of "bilinear" transformations. It is also a special case of the Padé rational approximations of the delay.
- "Pre-warped" versions of Tustin allow the exact matching at some specific frequency of interest.
- MATLAB: c2d(C,T,'Tustin'), d2c(D,'Tustin')

$$s = \frac{2}{T}\cdot\frac{z-1}{z+1}$$

DT-CT filter "equivalence:" Example T1

- The Tustin discretization of the CT transfer function $C(s) = \dfrac{1}{\tau s + 1}$

$$D(z) = C(s)\Big|_{s=\frac{2(z-1)}{T(z+1)}} = \frac{1}{\tau s + 1}\Bigg|_{s=\frac{2(z-1)}{T(z+1)}} = \frac{1}{\tau\dfrac{2(z-1)}{T(z+1)}+1} = \frac{T(z+1)}{2\tau(z-1)+T(z+1)} = \frac{T(z+1)}{(2\tau+T)z-(2\tau-T)}$$

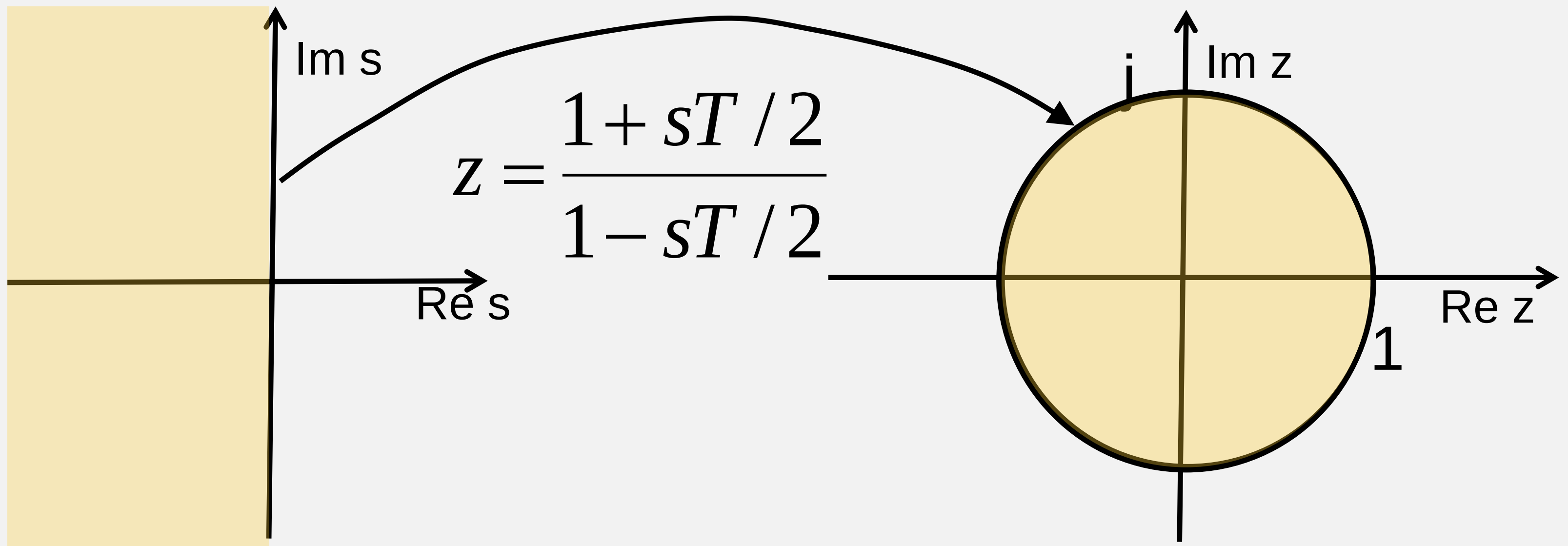
- As T increases, the approximation gets naturally worse but never becomes unstable for $\tau > 0$

pole $\left|\dfrac{2\tau - T}{2\tau + T}\right| < 1$

- Visualizing the map of the CT stability domain:

  - Here we observe that the entire stable region in the s-domain (LHP) maps onto the UC.
  - For *sT/2 = a+jb*, $|z| = \dfrac{\sqrt{(1+a)^2+b^2}}{\sqrt{(1-a)^2+b^2}} < 1$ for $a < 0$

    in LHP



$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter "equivalence:" Example T1

In the examples:

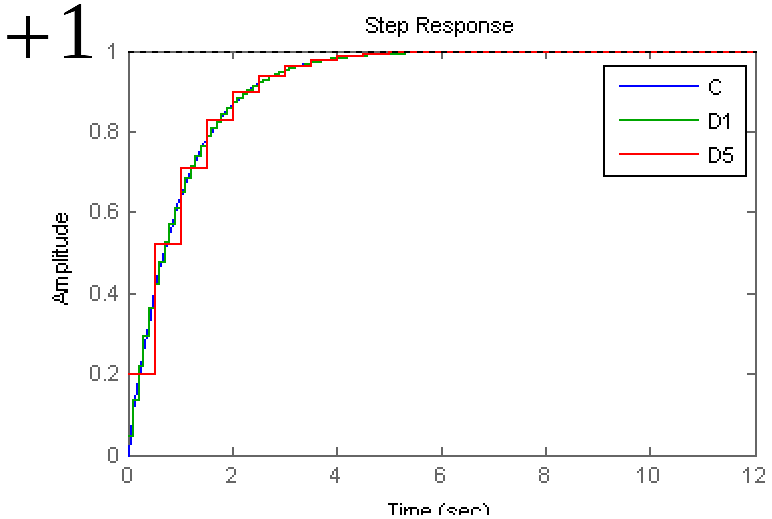
$\tau = 1, \quad T = 0.1, 0.5$

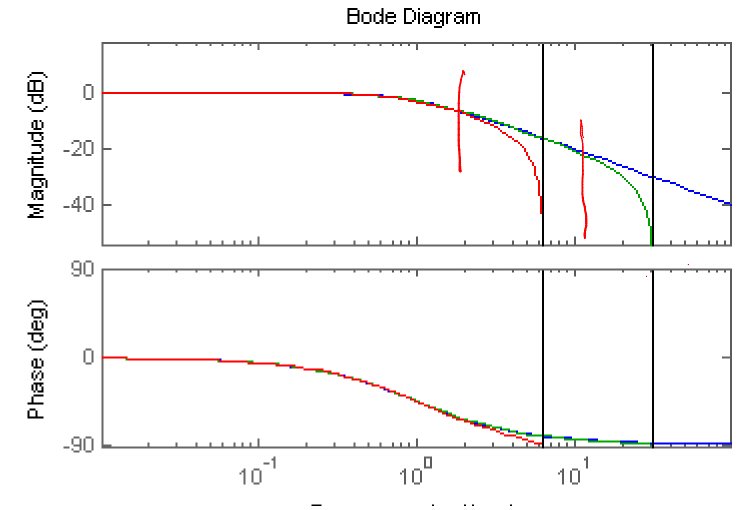
- The Tustin discretization of the CT transfer function $C(s) = \dfrac{1}{\tau s + 1}$

$$D1(z) = \frac{\dfrac{T}{(2\tau + T)}(z+1)}{z - \dfrac{(2\tau - T)}{(2\tau + T)}} = \frac{0.04762\,\mathrm{z} + 0.04762}{\mathrm{z} - 0.9048}; T = 0.1,$$

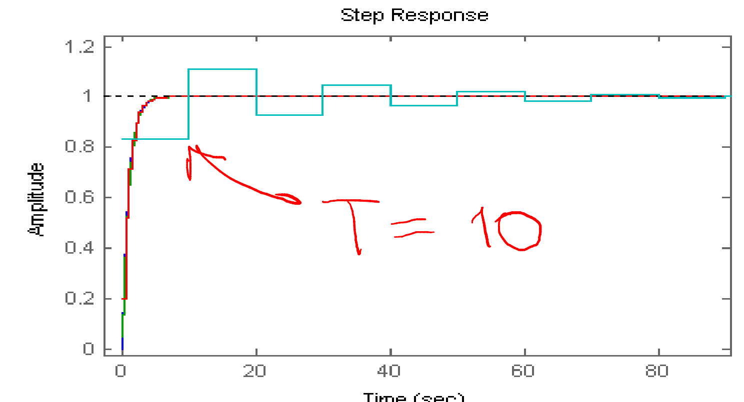
$$D5(z) = \frac{0.2\,\mathrm{z} + 0.2}{\mathrm{z} - 0.6}; T = 0.5$$

$k(z+1)$ $z = -1$ when $\omega = \pi$





- The approximation always interpolates the step response samples and even for T = 10, the response is "justifiable".
- Tustin equivalents always contain direct throughput (not strictly causal). That may present a problem in certain applications.



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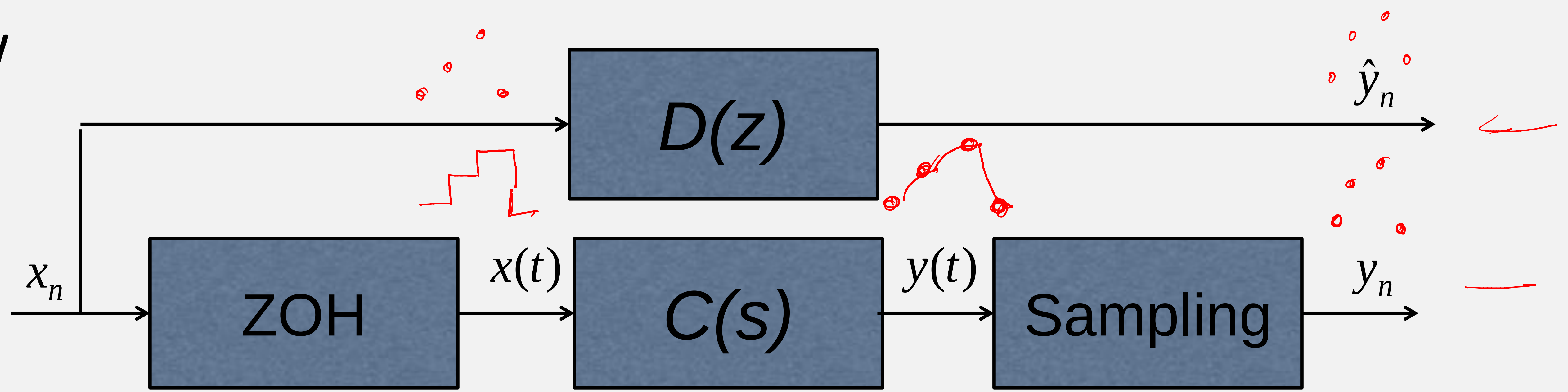
Lecture 4.2: DT - CT Sampled-Response Equivalence

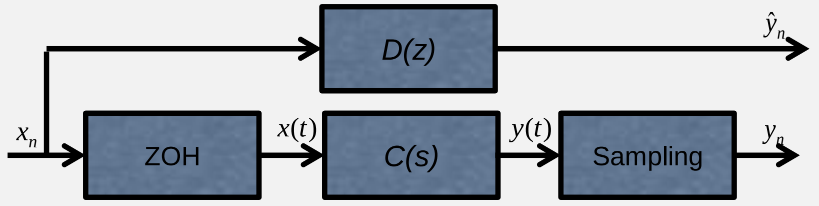
DT-CT filter "equivalence": ZOH

- Piece-wise constant signals: Matching the sampled step response of the two filters is a problem arising frequently in sampled-data systems, computer controlled systems etc. The setting for this problem is shown below



- Here, the input is assumed to be a sampled signal, that is reconstructed using a ZOH. On the other end, we only try to match the sampled responses. Such a matching can be exact and we refer to D(z) as the ZOH-equivalent of C(s).

DT-CT filter "equivalence": ZOH



$$D(z) = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\Big|_{t=nT}\right\}$$

- Computing the ZOH-equivalent discretization is fairly tedious. Consider the example $C(s) = \dfrac{1}{s+1}$, to illustrate the steps.

C(S) PFE

1. Compute the step response of C: $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{-1}{s+1}\right\} = (1-e^{-t})u(t)$
2. Sample the step response: $y_n = y(nT) = (1-e^{-nT})u(nT) = (1-\lambda^n)u_n; \quad \lambda = e^{-T}$
3. Find the $\mathcal{Z}$-transform of y_n: $Y(z) = \mathcal{Z}\{y_n\} = \frac{z}{z-1} - \frac{z}{z-\lambda}; \quad \lambda = e^{-T}$
4. Compute the corresponding t.f.: $D(z) = \frac{z-1}{z}Y(z) = 1 - \frac{z-1}{z-\lambda} = \frac{1-\lambda}{z-\lambda}; \quad \lambda = e^{-T}$

$Y_{STEP}(z) = D(z) \cdot \frac{z}{z-1}$

$$D(z) = \frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\bigg|_{t=nT}\right\}$$

DT-CT filter “equivalence:” Example Z1

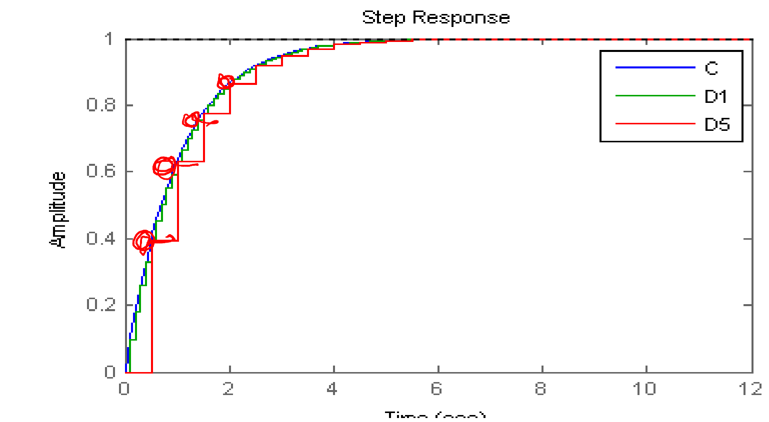
- The ZOH discretization of the CT transfer function $C(s) = \dfrac{1}{\tau s + 1}$

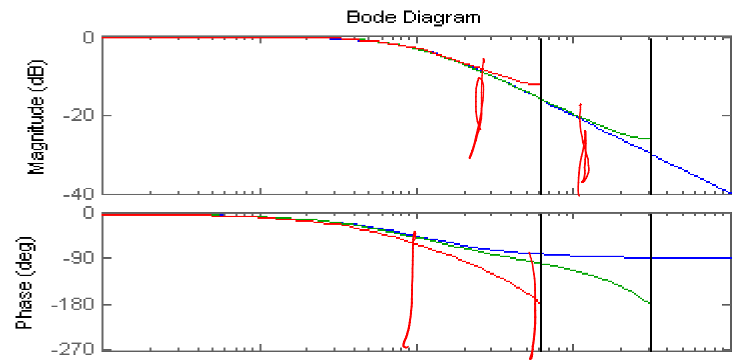
D1(1) = 1 = C(0)

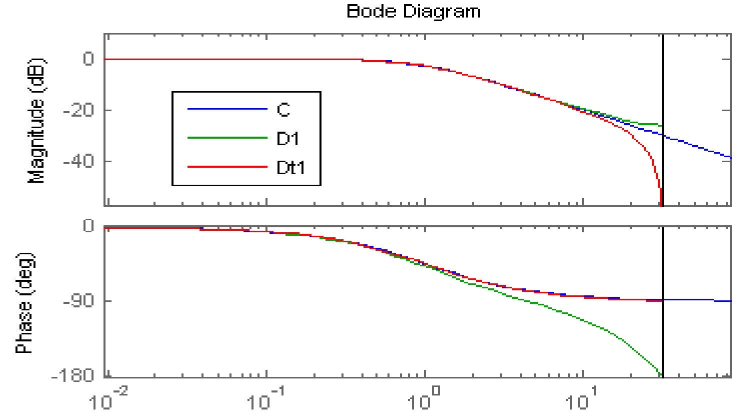
$$D1(z) = \frac{1-e^{-0.1}}{z-e^{-0.1}};T = 0.1,\quad D5(z) = \frac{1-e^{-0.5}}{z-e^{-0.5}};T = 0.5$$

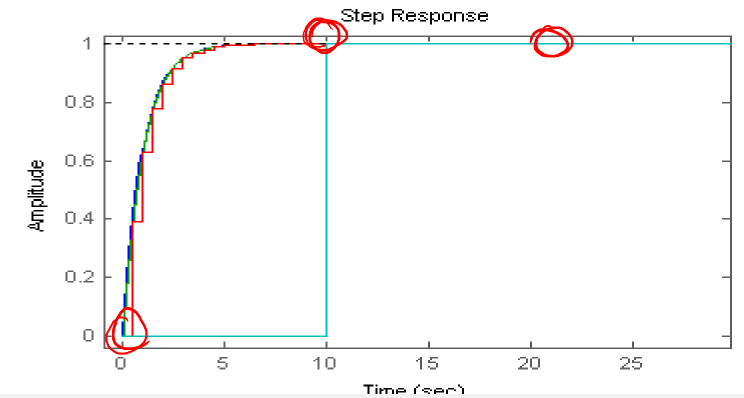
$\tau = 1$

- The approximation is always well-behaved and lags behind the CT response by ½ sample time on the average.
- Magnitude approximation is good, even past ½ -Nyquist but phase deviates rapidly even at 1/10-Nyquist (see comparison with Tustin-equivalent).
- As T increases, the approximation gets worse but even for T = 10, the response is “reasonable” (one-step, “dead-beat”).
- In MATLAB, c2d(C,T,’zoh’), d2c(C,’zoh’). (Default)

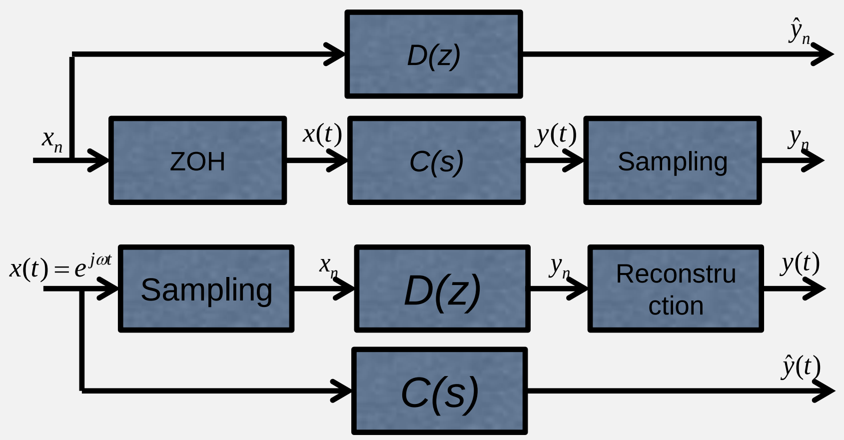








DT-CT filter "equivalence": Comparison of methods



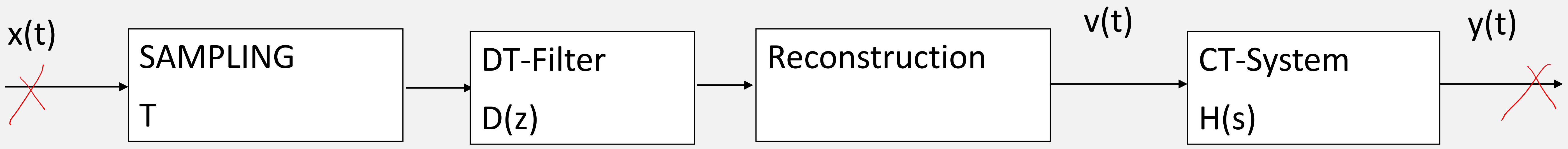
- Which is the "best" method? (Application dependent)
- Forward Euler is simple, explicit, but has stability constraints.
- Tustin is the prevalent method for frequency response approximation. It also has certain system properties. It is multiplicative, i.e., it can be applied to the entire system or its components.
- The ZOH-equivalent is an exact method to match the CT-DT step responses at the sampling points. It is not multiplicative, i.e., cannot be performed as a combination of factors:

$$\mathcal{ZOH}\{C_1(s)C_2(s)\} \neq \mathcal{ZOH}\{C_1(s)\}\mathcal{ZOH}\{C_2(s)\}$$

- There are other approximation methods (impulse invariance, first-order hold) but with limited applicability

Sampling and DT processing example

- Suppose that we want to implement an equalization prefilter for a system with transfer function $H(s) = \frac{1}{s^2 + 0.5s + 1}$. We want to eliminate its peak magnitude and extend its bandwidth from 1.5 to 3 rad/s



- We design a cascade equalizer G(s) such that the equalization system matches the frequency response of a Butterworth filter with bandwidth 3 rad/s: $W_B(s) = \frac{9}{s^2 + 4.243s + 9}$ $= \frac{Y(z)}{X(z)}$

$$G(s) = \{H(s)\}^{-1} W_B(s) = \frac{9(s^2 + 0.5s + 1)}{s^2 + 4.243s + 9}$$

- We “arbitrarily” select a sampling time of 0.1 (Nyquist frequency 31.4 rad/s) to implement the filter. We assume the existence of adequate AntiAliasing and Reconstruction filters and select

$$D(z) = c2d\{G(s), 0.1, 'tustin'\} = \frac{8.833z^2 - 17.62z + 8.789}{z^2 - 1.958z + 0.9585}$$

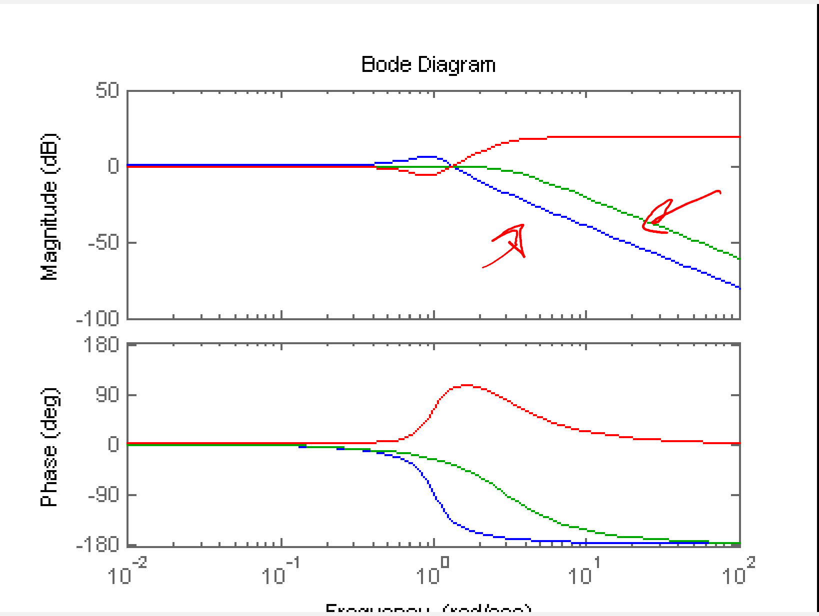
Sampling and DT processing example (cont.)

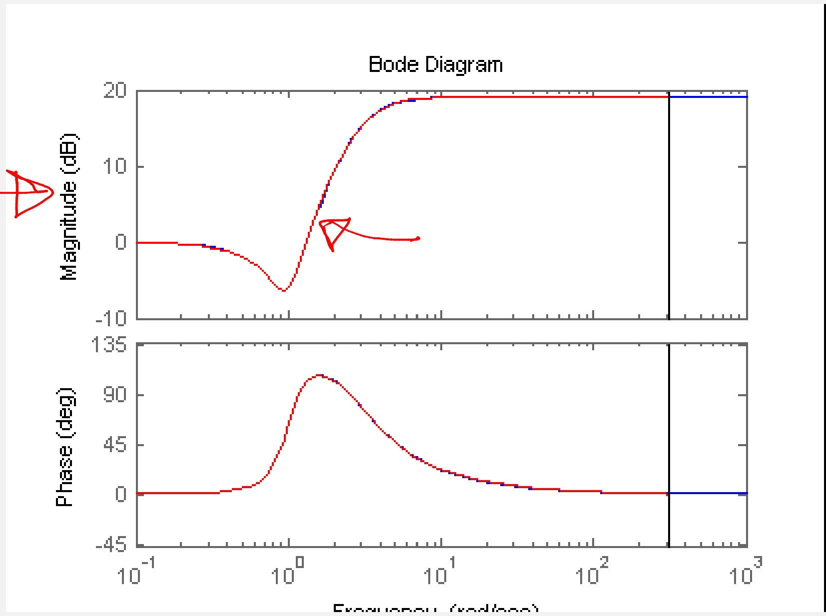
- Comparison with the ideal CT equalization filter
- >> bode(H,W,W/H)
- Note: Here H is minimum phase (no RHP zeros) so W/H is a stable filter.

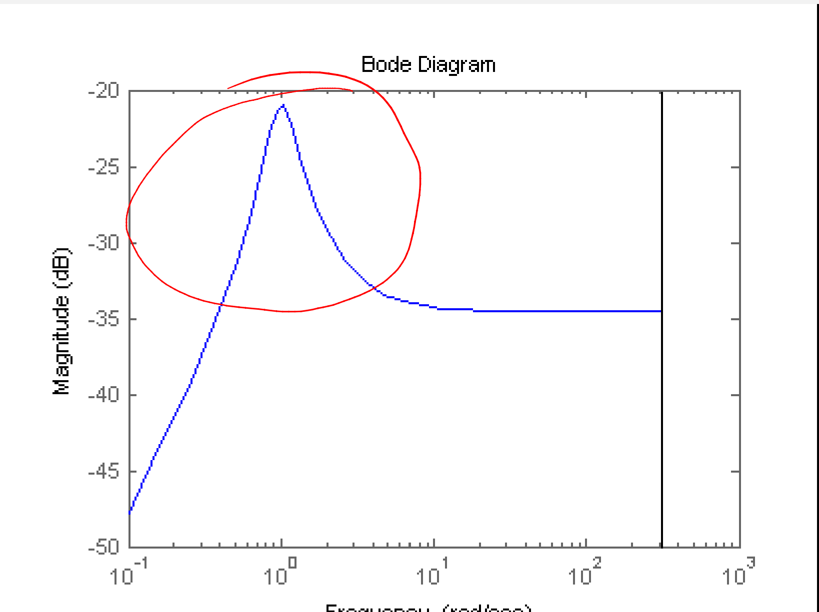
  In general, such an approximation requires more elaborate tools.
- Now, the matching between the CT (blue) and the DT (red) equalizers is nearly perfect. (It helps that they are biproper and that the sampling rate is an order of magnitude above bandwidth
- What if we used a ZOH approximation?

  There is a difference but it is small, -20 dB as the peak of the "relative error" $\dfrac{D(z)-D_{zoh}(z)}{D(z)}$. Again, we are sampling relatively fast so that differences in approximations are small.

$$c2d\{G(s),0.1,'tustin'\}=\frac{8.833z^2-17.62z+8.789}{z^2-1.958z+0.9585};\quad c2d\{G(s),0.1,'zoh'\}=\frac{9z^2-17.95z+8.952}{z^2-1.958z+0.9585}$$

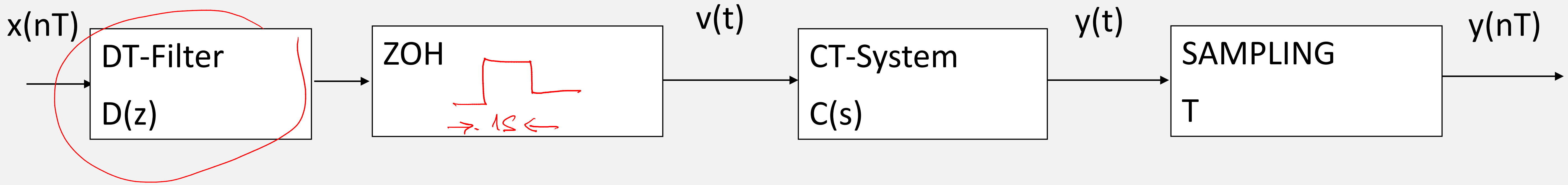






Feedback DT compensation example

- In a feedback system the system is a cascade of two transfer functions $C(s)=\frac{1}{(s+1)}\cdot\frac{1}{(s+1)}$ and control signals are applied by a DAC every 1sec. In a portion of a feedback analysis problem we want to describe the output of the system for an arbitrary input.

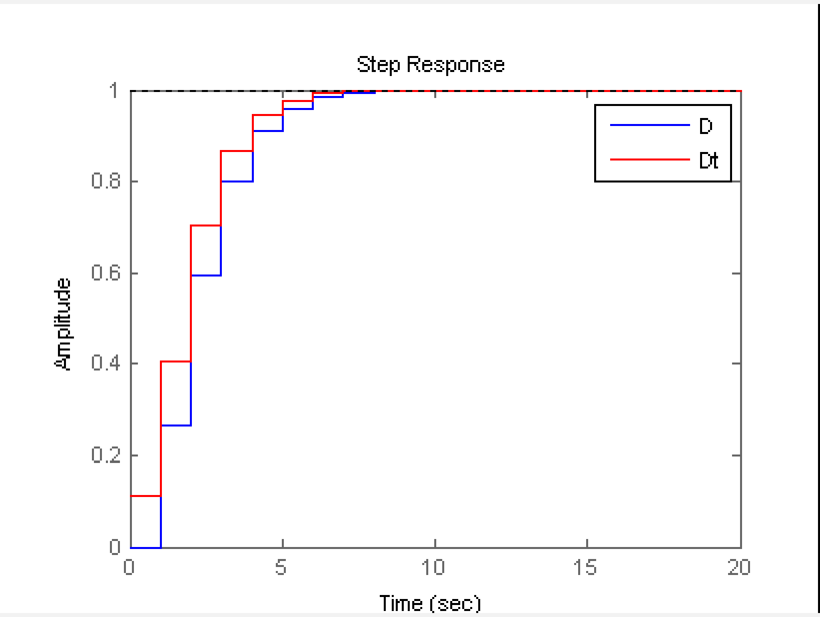


- The ZOH equivalent of C(s) is $D(z)=\frac{0.2642\,z+0.1353}{z^2-0.7358\,z+0.1353}$
- >> D=c2d(C,1,'zoh')
- This transfer function describes exactly the output to any input x(nT) at the sampling instants. If the intermediate output values are needed, they could be obtained by integrating the CT differential equation with a constant input in the last interval.

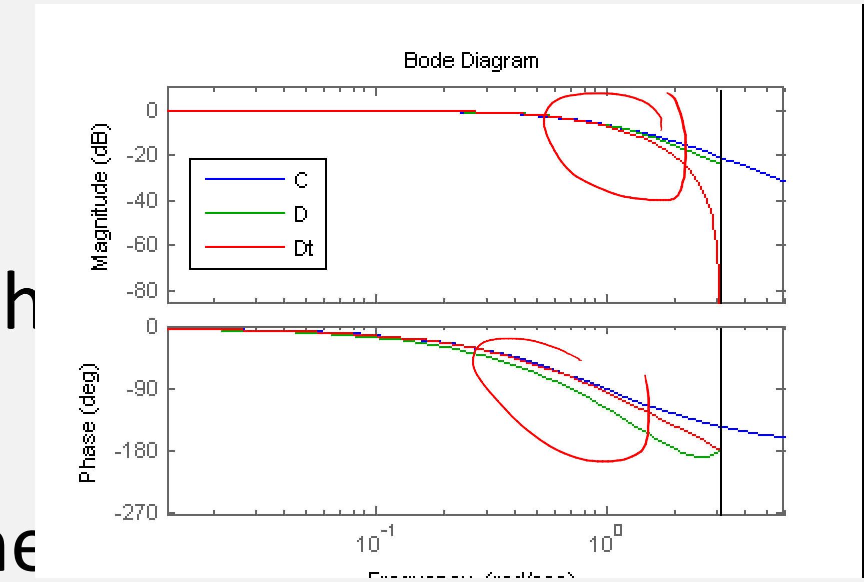
Feedback DT compensation example

- The ZOH equivalent is computed for the entire system, from the ZOH to the sampler.
- In contrast, the Tustin equivalent can be computed in a modular fashion, as well.

$$D(z) = \frac{0.2642\,z + 0.1353}{z^2 - 0.7358\,z + 0.1353} \qquad D_{Tustin}(z) = \frac{0.1111 z^2 + 0.2222\,z + 0.1111}{z^2 - 0.6667\,z + 0.1111}$$



- There is an obvious and appreciable difference in the step response samples. This is more pronounced because the sampling frequency is relatively small (for the system bandwidth), as it is often the case for such problems.
- Note that the Tustin equivalent may still be a better representation of the frequency response, but does not describe the sampled response as well.



EEE304

Lecture 4.3: Examples on DT - CT Equivalence

ARIZONA STATE UNIVERSITY

$$s=\frac{z-1}{T}$$

DT-CT filter "equivalence:" Example FE2

- To illustrate the implication of the FE sampling rate constraint, let us consider the system

$$C(s)=\frac{1}{(s+1)(0.01s+1)}$$

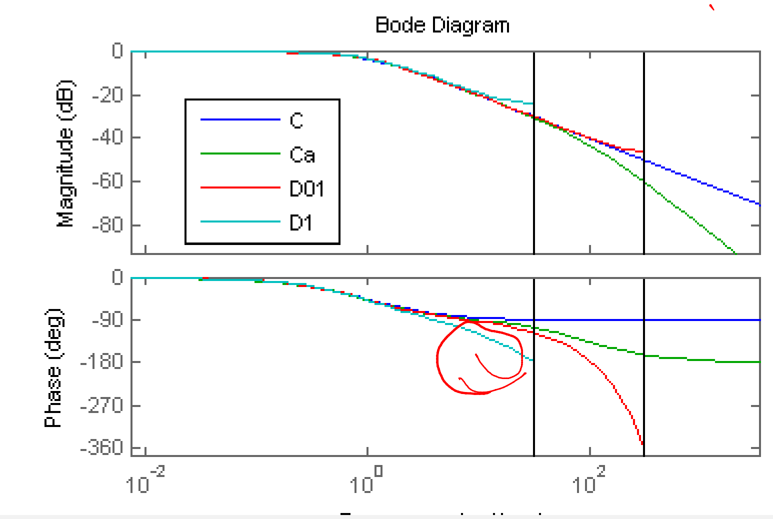
Such characteristics arise in systems that are composed of slow and fast subsystems, e.g., mechanical and electrical components.

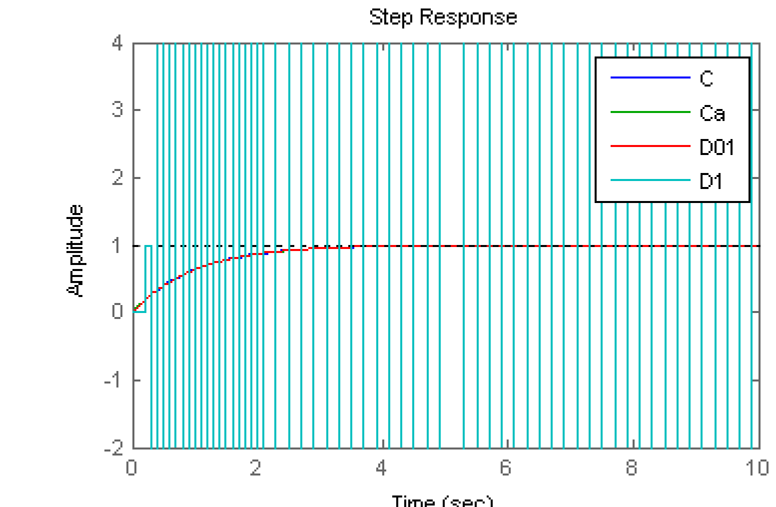
- PFE yields $C(s)=\frac{1.01}{(s+1)}+\frac{-0.0101}{(0.01s+1)}$, indicating that $C(s)\approx\frac{1}{(s+1)}$, something that is also verified by the step response.

- The stability constraint for FE discretization of this transfer function is $T/0.01<2$. For T = 0.1 the discretization is unstable, even though the dominant part of the system is well approximated.

In the examples :

$\tau=1, 100, \quad T=0.01, 0.1$





$$s = \frac{z-1}{T}$$

DT-CT filter "equivalence:" Example FE3

$2\zeta\omega_0 = 0.2$

$\omega_0 = 1$

$\zeta = 0.1$

- Consider the discretization of the CT transfer function $C(s) = \boxed{\dfrac{1}{s^2 + 0.2s + 1}}$ using FE transformation $s = (z-1)/T$ for T = 0.1, 0.5

$$D1(z) = \frac{0.01}{z^2 - 1.98z + 0.99};\ T = 0.1, \quad D5(z) = \frac{0.25}{z^2 - 1.9z + 1.15};\ T = 0.5$$

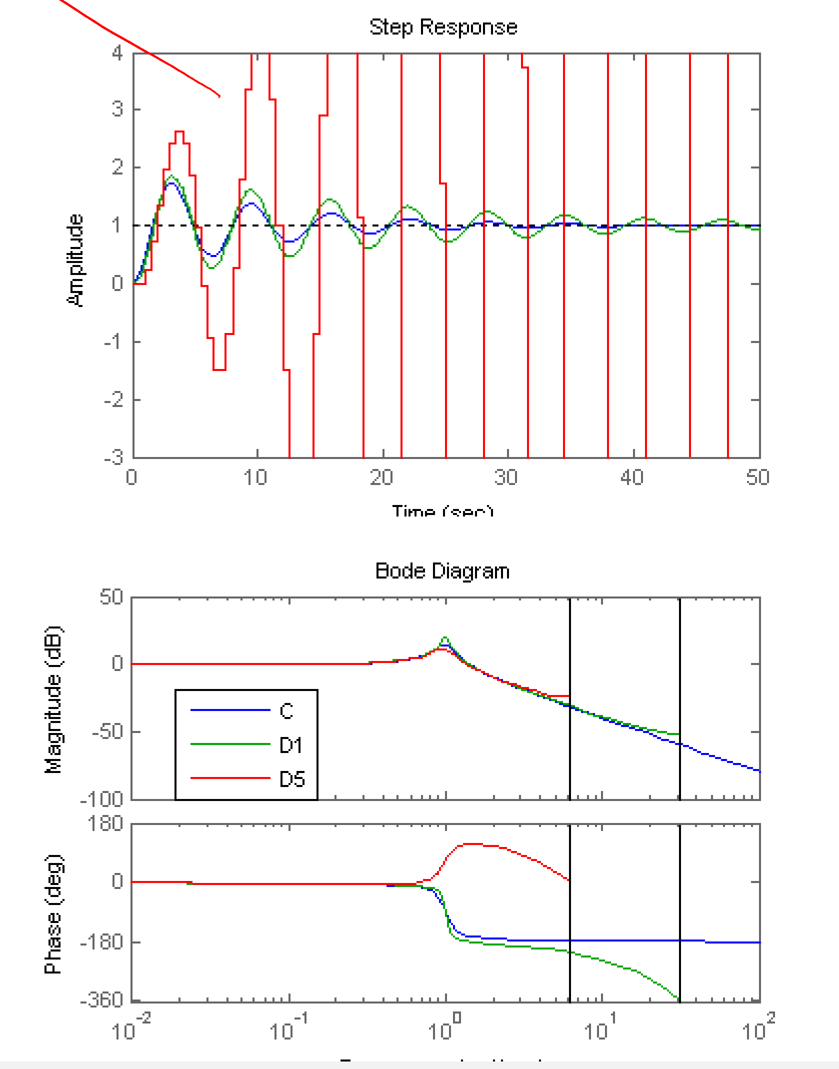
In the examples : $T = 0.1, 0.5$

- We observe that the resonance peak is over-estimated for T = 0.1 and under-estimated for T = 0.5.

- For T = 0.5 the DT "equivalent" is unstable. Here, the sampling time does not satisfy the general condition for complex poles

$$\left|1 + poles[C(s)] \times T\right| < 1, \quad \text{or,} \quad \left|\left(1 + \mathrm{Re}[p_i]T\right) + \left(\mathrm{Im}[p_i]T\right)\right| < 1$$

- This condition eventually can be written as $|p_i|T < 2\cos\left(\angle p_i\right)$



$$s = \frac{z-1}{Tz}$$

DT-CT filter “equivalence:” Example BE2

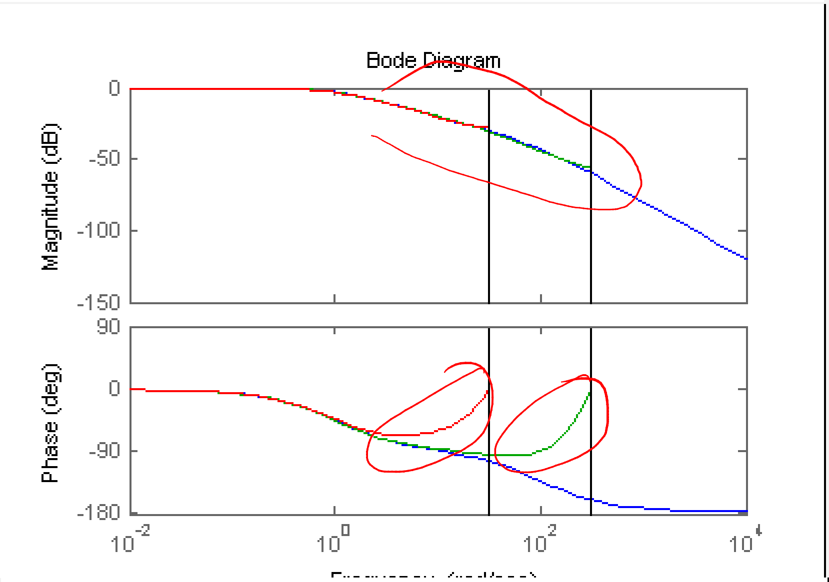
- To illustrate the BE equivalent properties, we consider the system $C(s) = \dfrac{1}{(s+1)(0.01s+1)}$

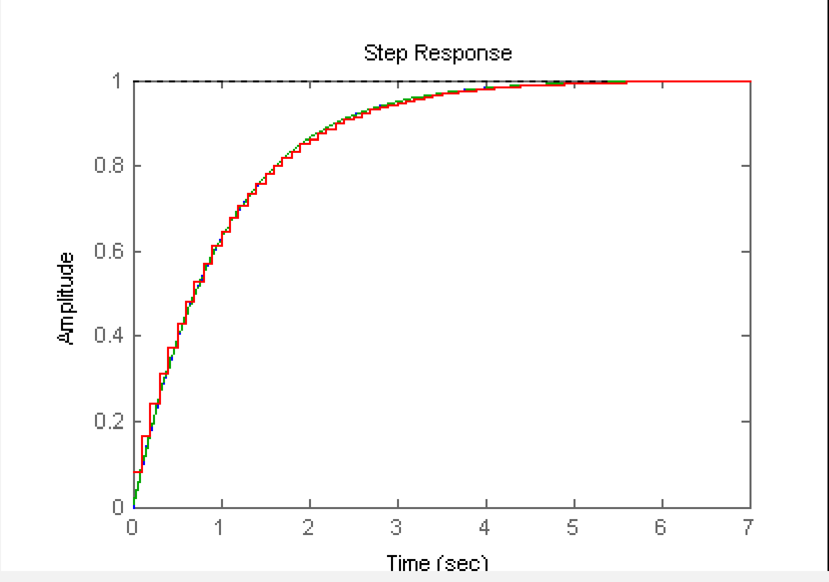
$$D01(z) = \frac{0.00495\,z^2}{z^2 - 1.49\,z + 0.495};\; T = 0.01, \quad D1(z) = \frac{0.08264\,z^2}{z^2 - z + 0.08264};\; T = 0.1$$

- We note that the poles of D1 are 0.9, 0.09, as expected from the slow-fast pole structure of C(s) (similarly with D01).
- The frequency response approximation is very good in terms of magnitude but the phase starts deviating one decade below Nyquist. Step response approximation is also very good.

In the examples :

$\tau = 1, 100, \quad T = 0.01, 0.1$





$$s = \frac{z-1}{Tz}$$

DT-CT filter "equivalence:" Example BE3

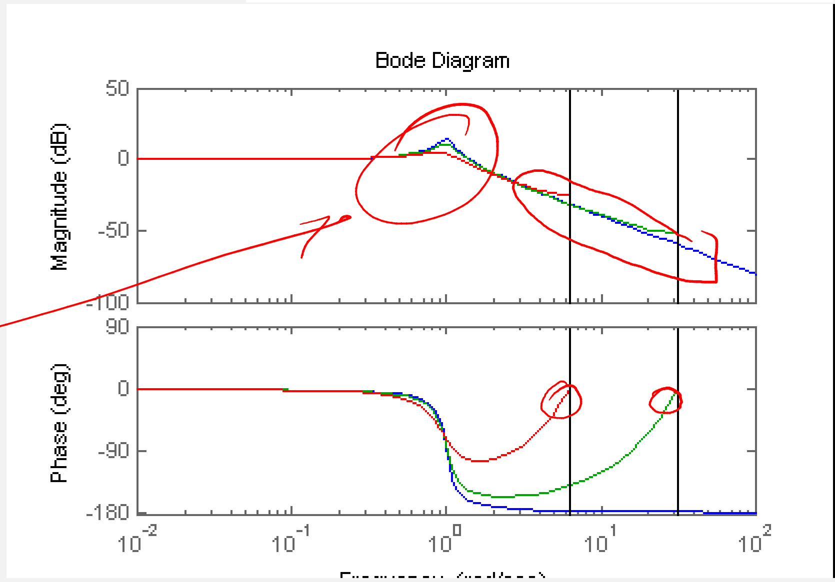
- Consider the discretization of the CT transfer function $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$ using BE transformation $s = (z-1)/Tz$ for T = 0.1, 0.5

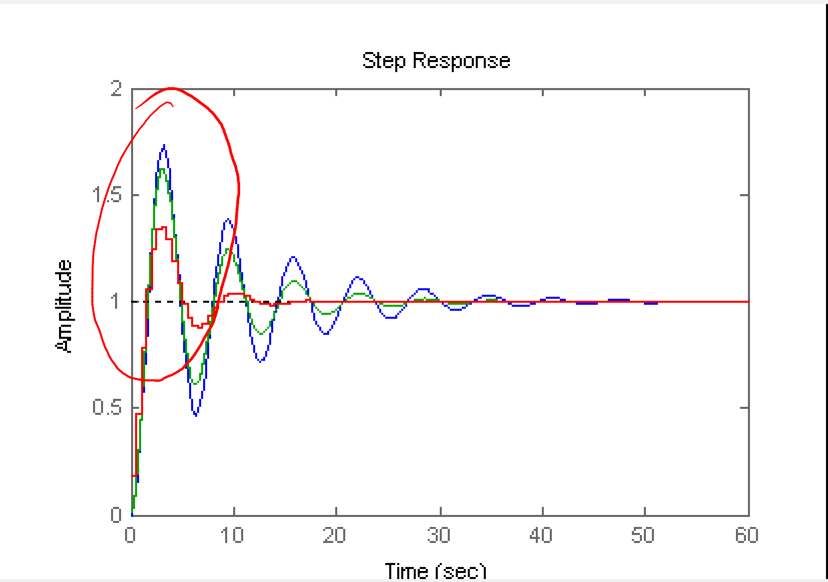
$$D1(z) = \frac{0.009709\, z^2}{z^2 - 1.961\, z + 0.9709}; T = 0.1, \quad D5(z) = \frac{0.1852\, z^2}{z^2 - 1.556\, z + 0.7407}; T = 0.5$$

In the examples:

$T = 0.1, 0.5$

- We observe that the resonance peak is under-estimated for T = 0.1 and even more for T = 0.5.
- The DT "equivalent" is always stable even if the approximation is not as good.





$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter “equivalence:” Example T2

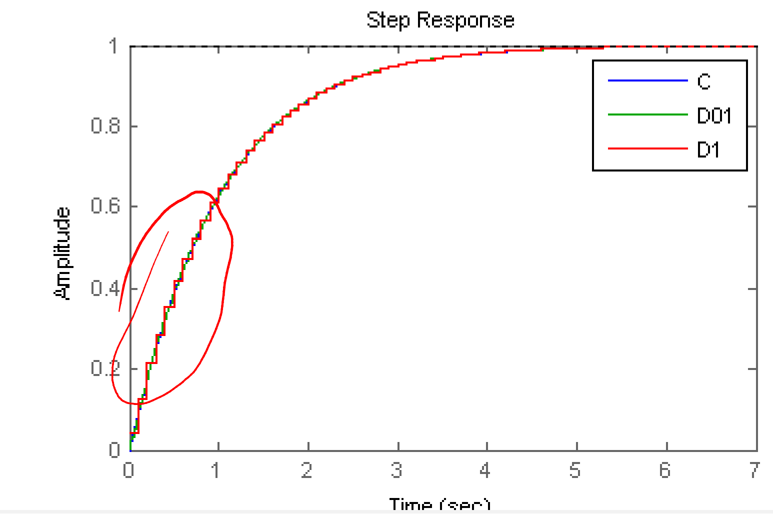
- Returning to the example with fast and slow dynamics, $C(s) = \dfrac{1}{(s+1)(0.01s+1)}$
- We compute the Tustin discretization for T = 0.01 and 0.1

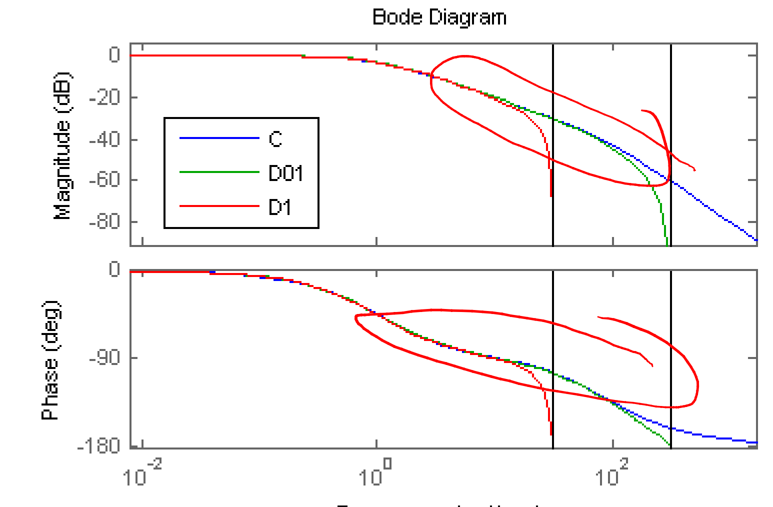
$$D01(z) = \frac{0.001658\,\mathrm{z}^2 + 0.003317\,\mathrm{z} + 0.001658}{\mathrm{z}^2 - 1.323\,\mathrm{z} + 0.33}; T = 0.01,$$

$$D1(z) = \frac{0.03968\,\mathrm{z}^2 + 0.07937\,\mathrm{z} + 0.03968}{\mathrm{z}^2 - 0.2381\,\mathrm{z} - 0.6032}; T = 0.1$$

In the examples :

$\tau = 1, 100, \quad T = 0.01, 0.1$





- We observe that the discretization approximates the CT system very well up to the usual 1/3-Nyquist frequency.
- The fast pole, appearing at 100 rad/s has a small effect on the Tustin-discretized transfer functions, as it is intuitively expected. (But a formal comparison with the 1st order transfer function is not straightforward)

$$s = \frac{2}{T} \cdot \frac{z-1}{z+1}$$

DT-CT filter "equivalence:" Example T3

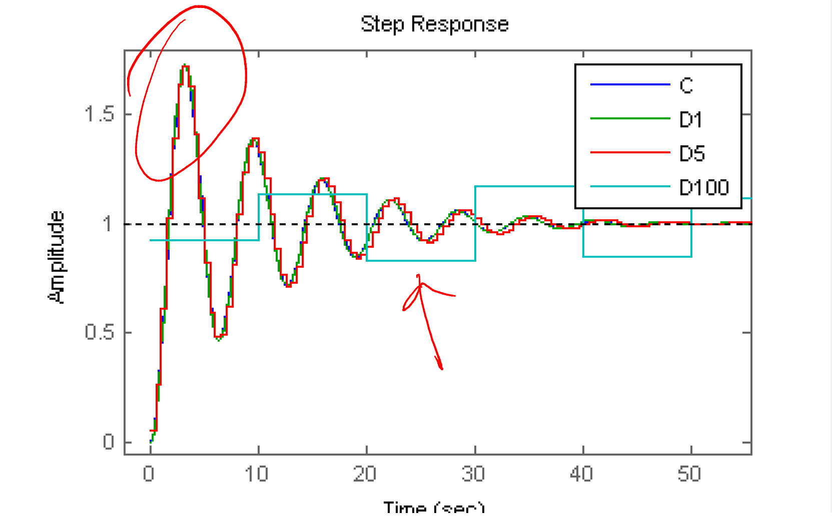
- The Tustin discretization of the CT transfer function $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$

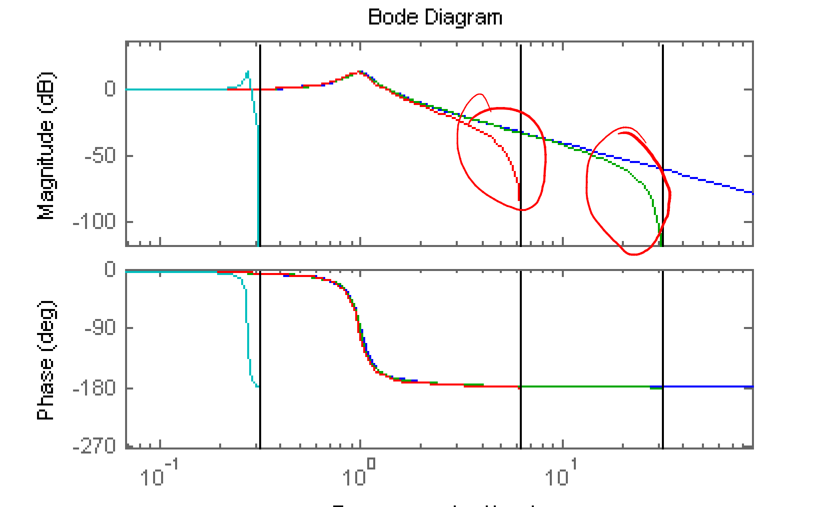
$$D1(z) = \frac{0.002469\, z^2 + 0.004938\, z + 0.002469}{z^2 - 1.97\, z + 0.9802}; T = 0.1,$$

$$D5(z) = \frac{0.05618\, z^2 + 0.1124\, z + 0.05618}{z^2 - 1.685\, z + 0.9101}; T = 0.5$$

In the examples : $T = 0.1, 0.5, 10$

- The approximation is always well-behaved and interpolates closely the step response samples.
- As T increases, the approximation gets worse but only since the Nyquist frequency decreases. Even for T = 10, the response is stable, although far from the CT response.





$$D(z)=\frac{z-1}{z}\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot C(s)\right\}\Big|_{t=nT}\right\}$$

DT-CT filter "equivalence:" Example Z2

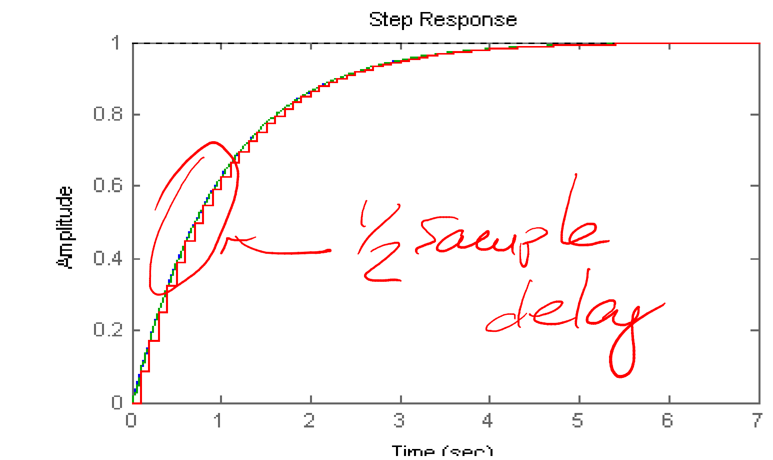
In the examples :

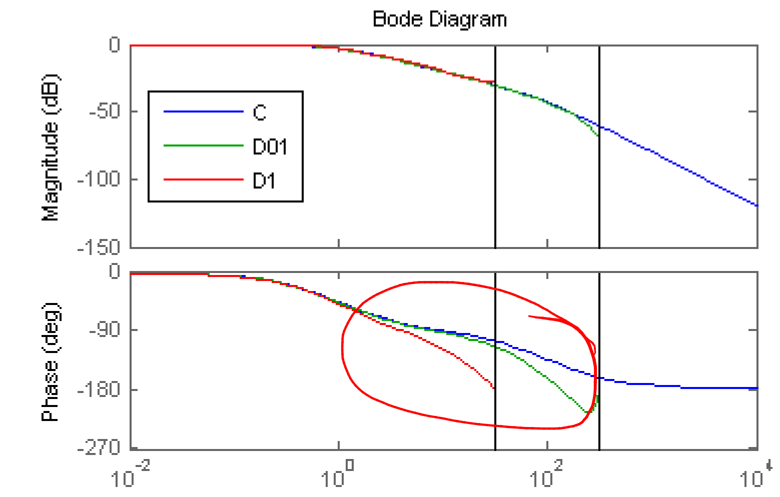
$\tau = 1, 100, \quad T = 0.01, 0.1$

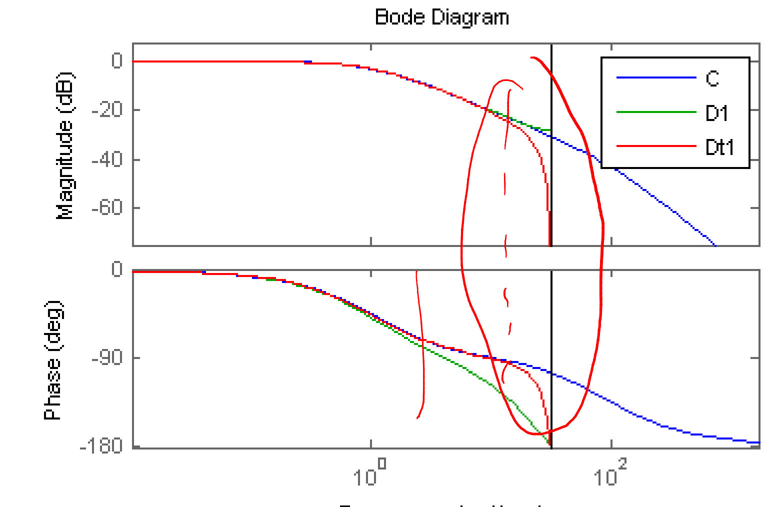
- For the transfer function $C(s)=\dfrac{1}{(s+1)(0.01s+1)}$
- We compute the ZOH discretization for T = 0.01 and 0.1

$$D01(z)=\frac{0.003666\,z+0.002624}{z^2-1.358\,z+0.3642};\, T=0.01,$$
$$D1(z)=\frac{0.08602\,z+0.009135}{z^2-0.9049\,z+4.108e-5};\, T=0.1$$

- We observe that the discretization approximates the CT system very well up to the 1/10-Nyquist frequency (phase deviations).
- Comparison with the Tustin-discretized system illustrates the phase lag of the ZOH-discretization.







$$D(z) = \frac{z-1}{z}\mathcal{Z}\left\{ \mathcal{L}^{-1}\left\{ \frac{1}{s}\cdot C(s) \right\} \Big|_{t=nT} \right\}$$

DT-CT filter "equivalence:" Example Z3

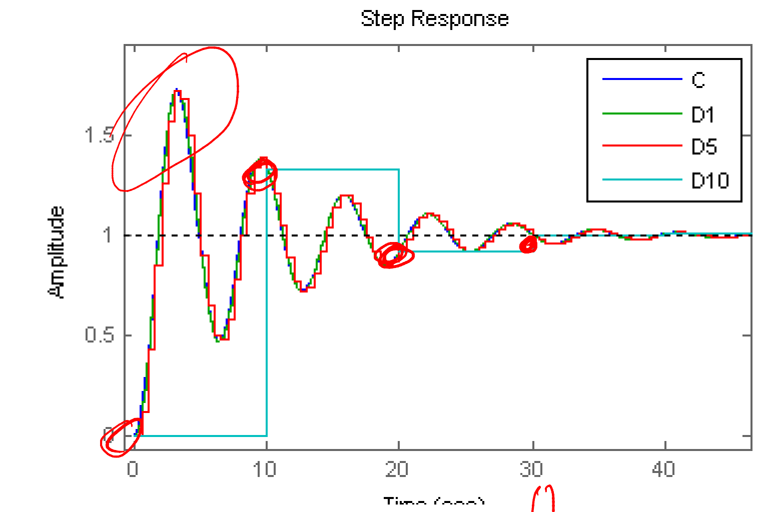
$\text{In the examples}: \; T = 0.1, 0.5, 10$

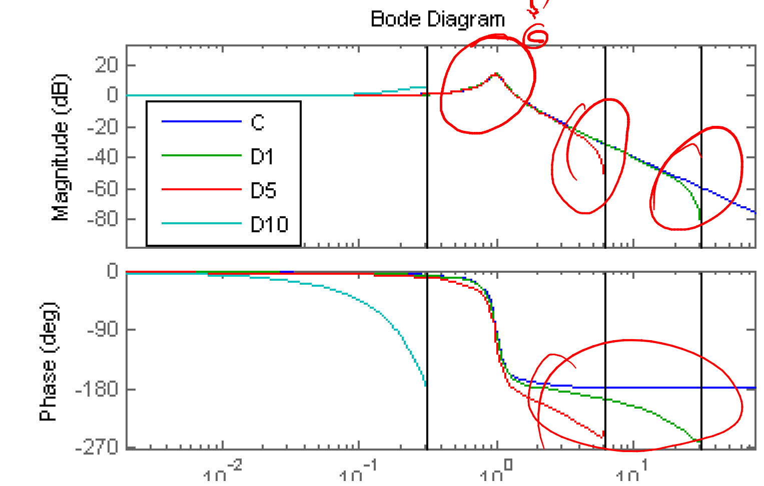
- The ZOH discretization of $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$

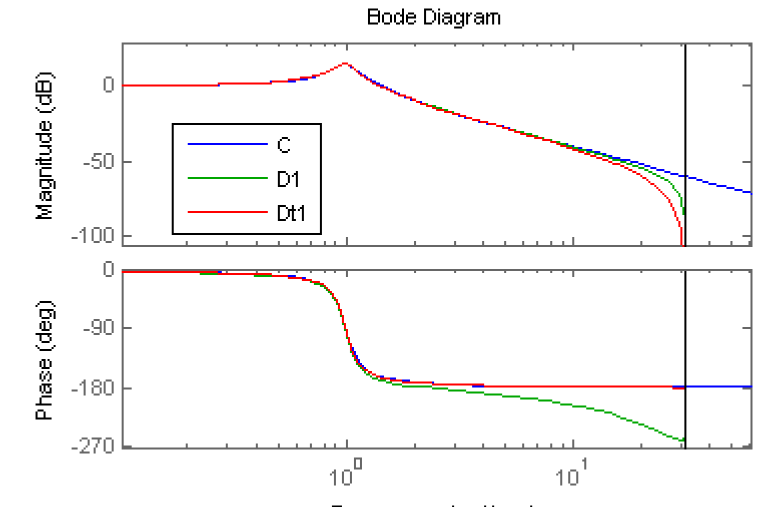
$$D1(z) = \frac{0.004963\,z + 0.00493}{z^2 - 1.97\,z + 0.9802}; \; T = 0.1,$$

$$D5(z) = \frac{0.1185\,z + 0.1145}{z^2 - 1.672\,z + 0.9048}; \; T = 0.5$$

- The approximation is always well-behaved and matches the CT step response samples at the sampling instants.
- As T increases, the approximation gets worse but remains reasonable and stable. Even for T = 10, the response correctly interpolates the sampled output although far from the complete CT response.
- Comparison with Tustin-discretization shows similar (or better) magnitude matching but much worse phase approximation.







EEE304

Lecture 4.4: DT - CT Equivalence: Comparisons

Arizona State University

DT-CT filter “equivalence:” Resonant poles and notch zeros

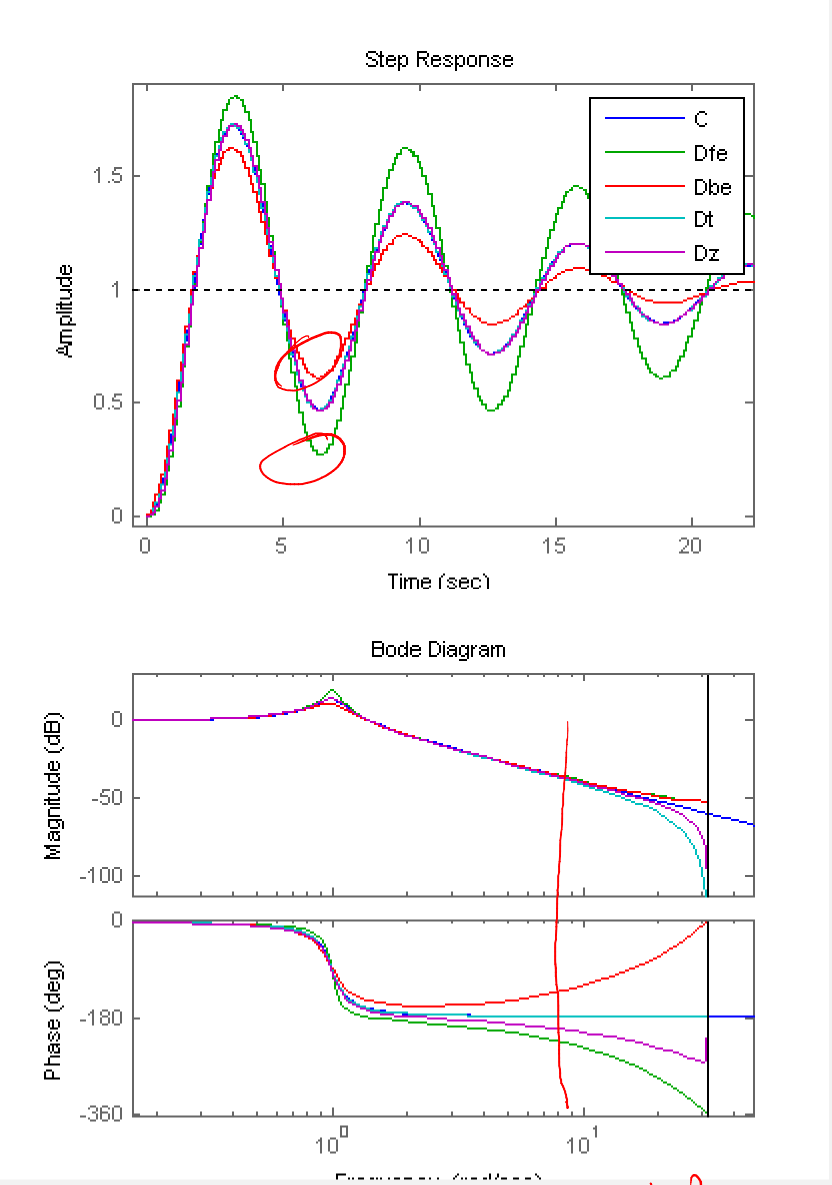
In the examples:

$$T = 0.1$$

- To compare the results from the different methods we consider the resonant system again $C(s) = \dfrac{1}{s^2 + 0.2s + 1}$

  Its bandwidth is 1.55rad/s so 3rad/s (0.5Hz) sampling rate would be the “limit” for its discretization.

- The FE method fails starting at T=0.2 (cmp. Mapping of the stability region) so we compare the responses for T=0.1s. The two Euler methods, being simple approximations, over or under estimate the resonance. But Tustin and ZOH provide a more faithful representation of the system closer to Nyquist frequency



ω_{Nyq}

DT-CT filter "equivalence:" Resonant poles and notch zeros

$$C(s)=\frac{1}{s^2+0.2s+1}$$

In the examples :

$T=1$

- To observe the behavior of more extreme cases of conversion we examine the case T=1s. Here we eliminate the FE method from the list.
- We observe that the ZOH, even delayed, provides a more faithful representation of the resonance while BE almost misses it completely.
- It is worthwhile to note that the zeros of the Tustin equivalent at z = -1 force its magnitude to 0 at pi which worsens the magnitude approximation.
- Matlab commands use the older function "bilin"

```
Dfe=tf(bilin(ss(C),1,'fwdrec',T))
Dbe=tf(bilin(ss(C),1,'bwdrec',T))
```

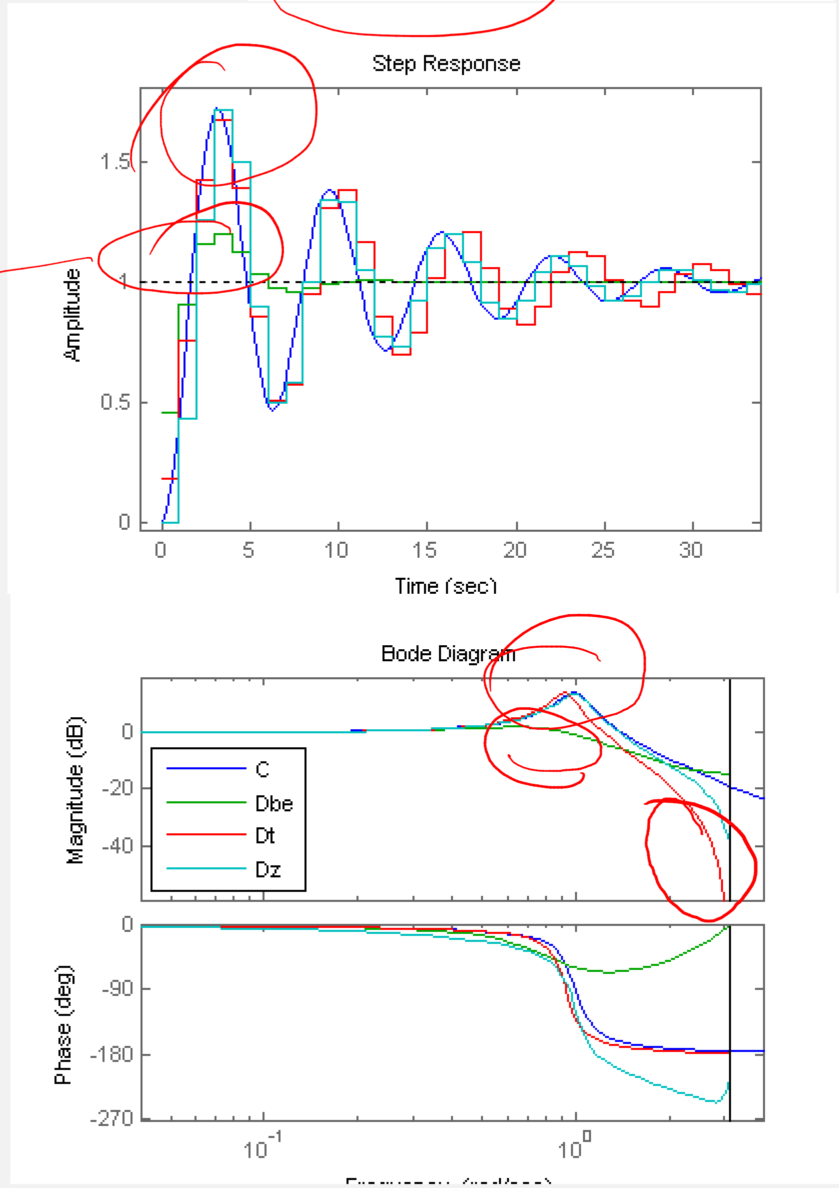
FE

BE

1 for C2D

-1 for D2C

BE

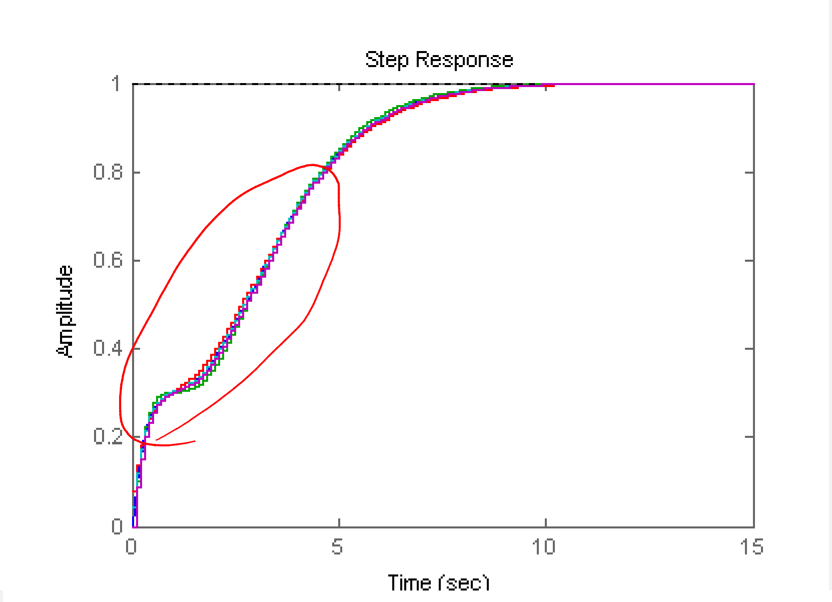


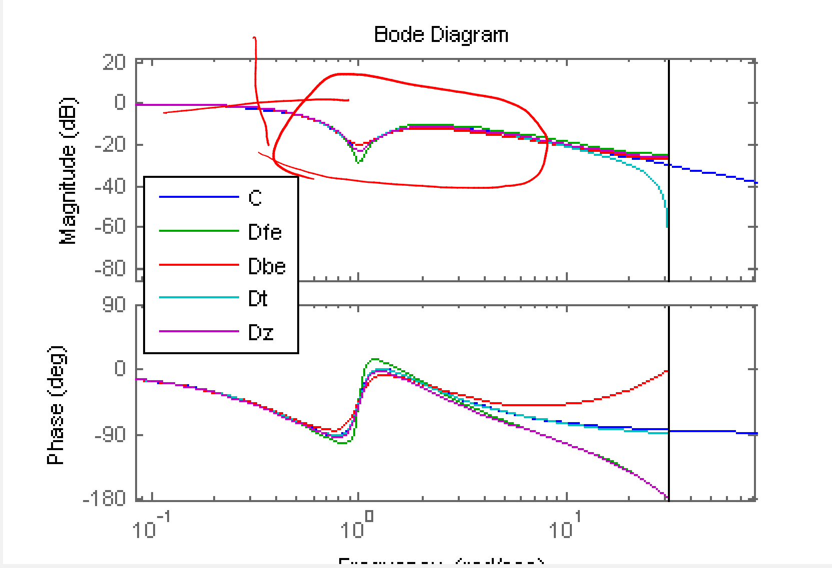
DT-CT filter "equivalence:" Resonant poles and notch zeros

In the examples :

$T = 0.1$

- The approximation differences are more subtle when we consider notch zeros in the transfer functions. Here, stability is no longer an issue and approximation failure is signified by increased errors (hence, becoming more difficult to detect).
- We consider the transfer function $C(s) = \dfrac{s^2 + 0.2s + 1}{(s+1)^3}$, whose bandwidth is 0.37rad/s. We first try T = 0.1, more than a decade above Nyquist rate.
- All methods provide an accurate frequency response and step response, with the Euler methods having more error.

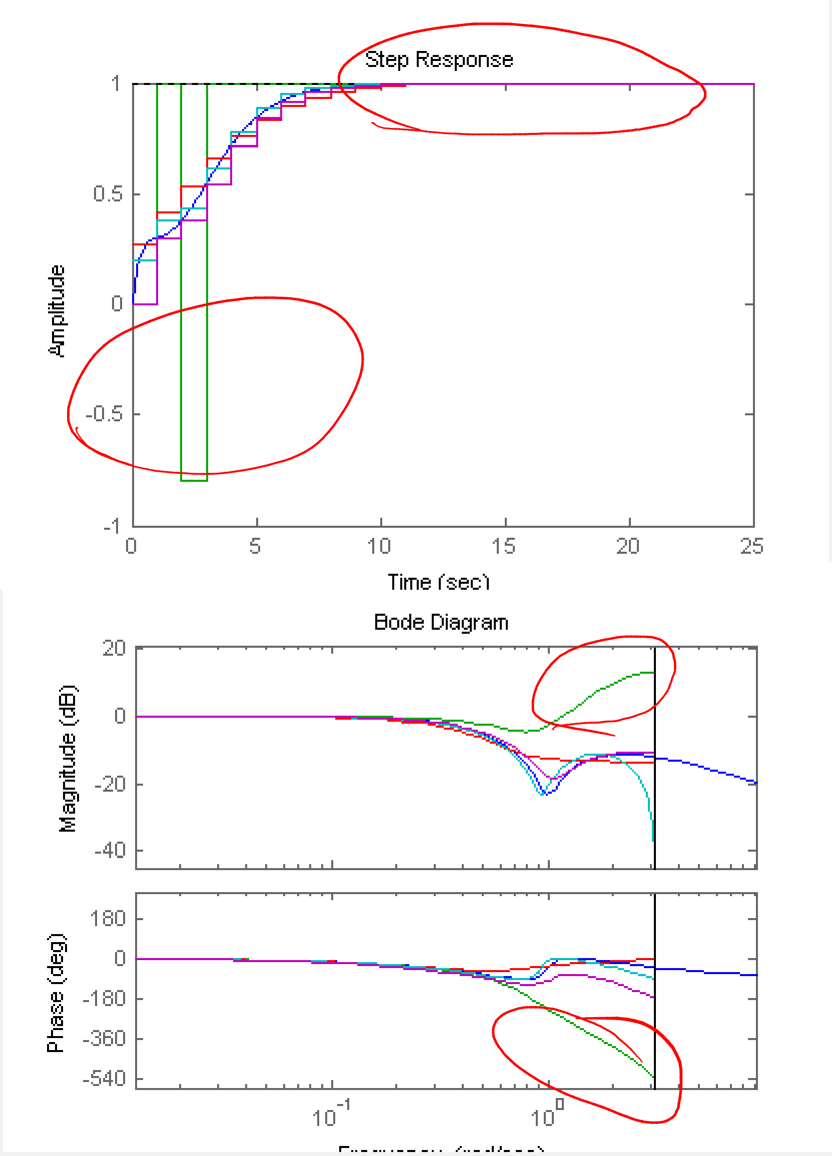




DT-CT filter "equivalence:" Resonant poles and notch zeros

In the examples :

$T = 1$



- With T = 1, the FE generates very large transient errors , although it still converges to the correct final value.
- From the rest, Tustin appears to be closer to the notch zero.
- One should keep some perspective, however, since such comparisons can quickly lose their practical meaning. (Sampling at 1s to detect a zero at 1rad/s).

DT-CT filter "equivalence:" Butterworth filters, revisited

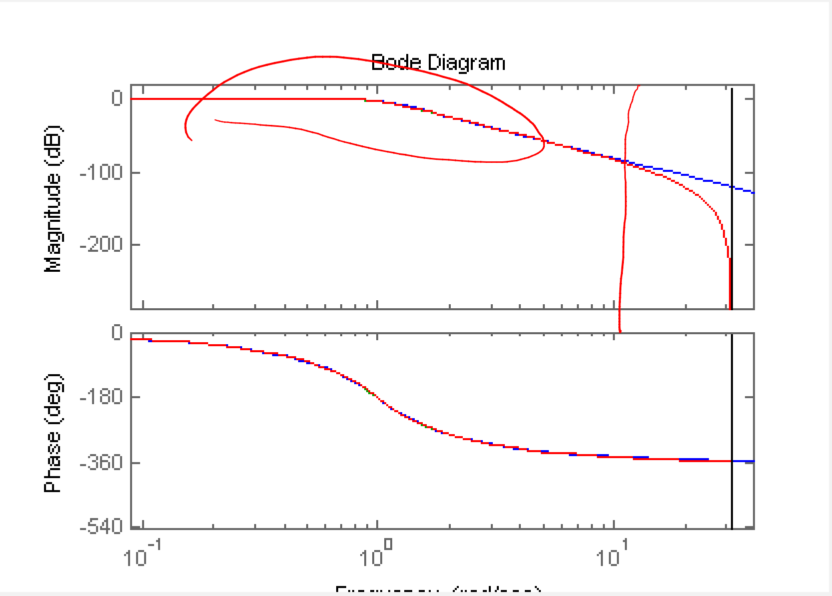
- Butterworth filters have formulae for CT and DT design.
- The DT Butterworth can be approximately obtained from its CT version with a Tustin transformation

```
>> [n,d]=butter(4,1,'s'),W=tf(n,d),Wd=c2d(W,.1,'tustin')
>> [n,d]=butter(4,1/31.4),D=tf(n,d,.1)
```

- The approximation is good even for frequencies very near Nyquist frequency. At T = 1, with DT cutoff frequency 0.3 out of 1, the relative error is below -10dB and that after frequency 1rad/s

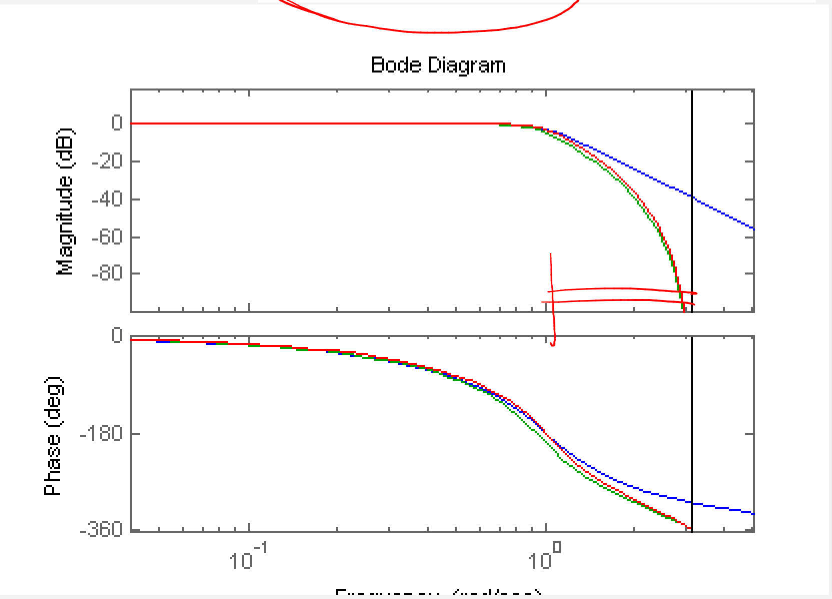
In the example:

$$T = 0.1$$



In the example:

$$T = 1$$

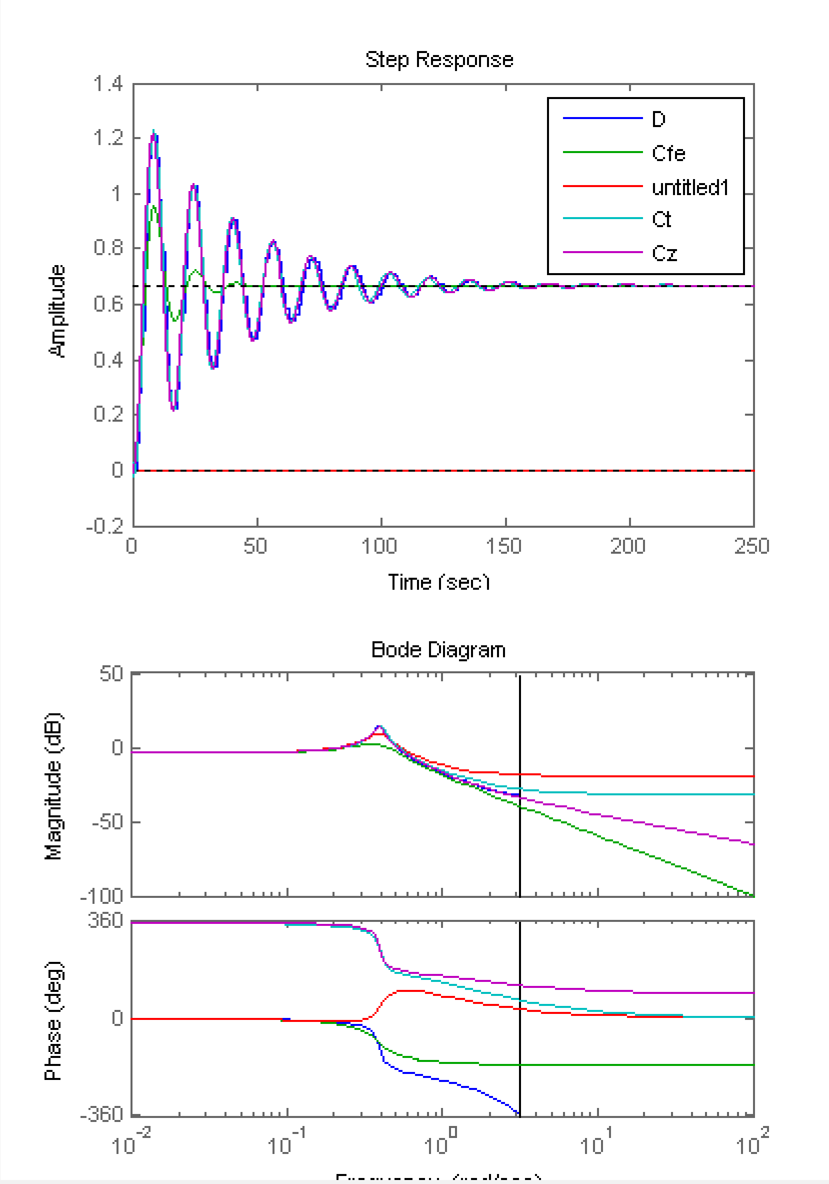


DT-CT filter “equivalence:” Converse problems

In the example :

$T = 1$

- In some cases, the converse approximation is needed. For example, let $D(z) = \dfrac{1}{z^2 - 1.8z + 0.95}$, for which we seek the CT equivalent.
  - This was the case when we wanted to generate the asymptotes of Bode plots for DT systems, or when we try to find the equivalent analog filter.
  - Here the approximations are straightforward as before and for T = 1 (if not specified) or any other value.
  - In this case, the BE can yield an unstable CT for a stable DT (the LHP was mapping into a subset of the UC).
  - The approximation is expected to be good for poles near 1.



DT-CT filter "equivalence:" Converse problems

In the example :

$$T = 1$$

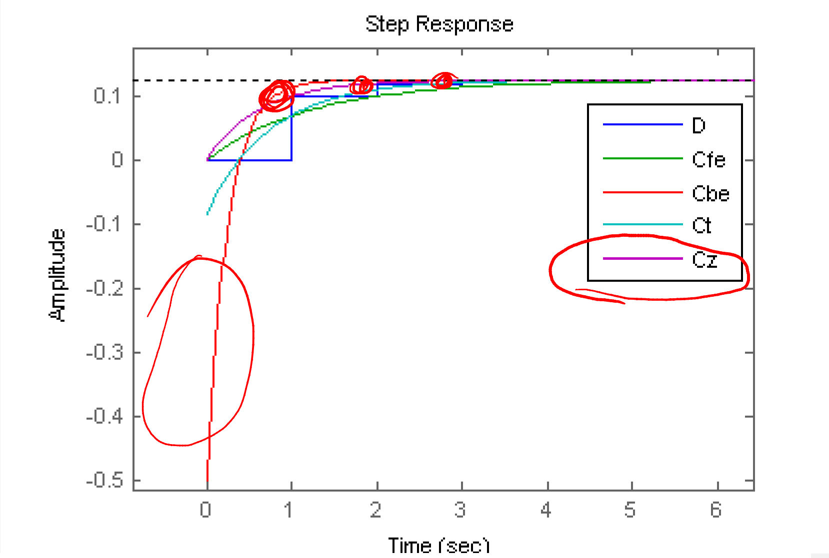
- Poles that are far from 1 do not correspond to CT analogues or their approximations are poor, especially the Euler equivalents.

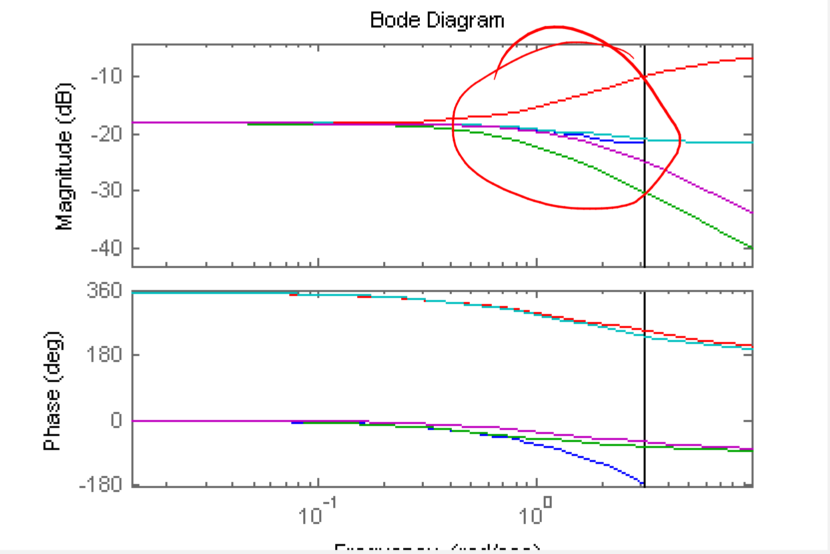
$$\boxed{D(z) = \frac{1}{z - 0.2}}$$

0.2^n

$$C_{fe}(s) = \frac{0.1}{s + 0.8}; \quad C_{be}(s) = \frac{-0.5s + 0.5}{s + 0.4}$$

$$C_{Tustin}(s) = \frac{-0.08333s + 0.1667}{s + 1.33}; \quad C_{zoh}(s) = \frac{0.2012}{s + 1.609}$$

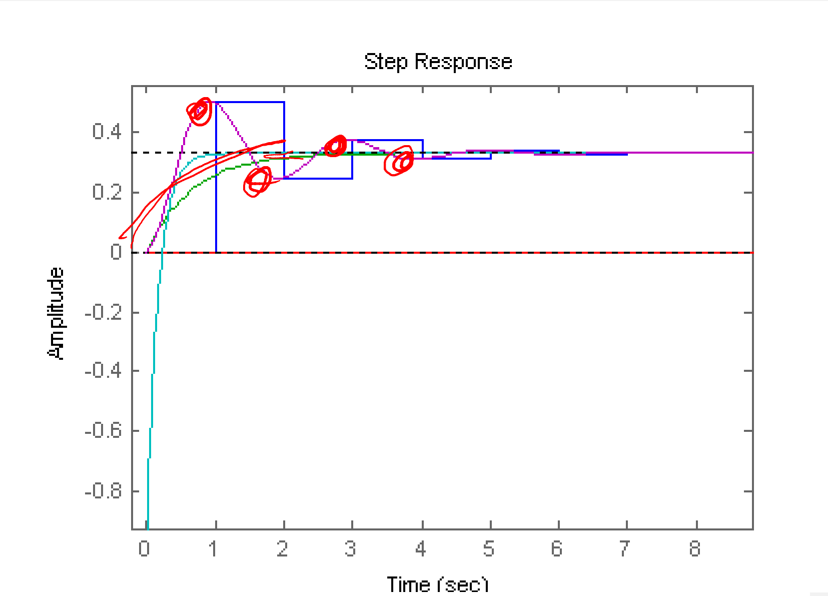


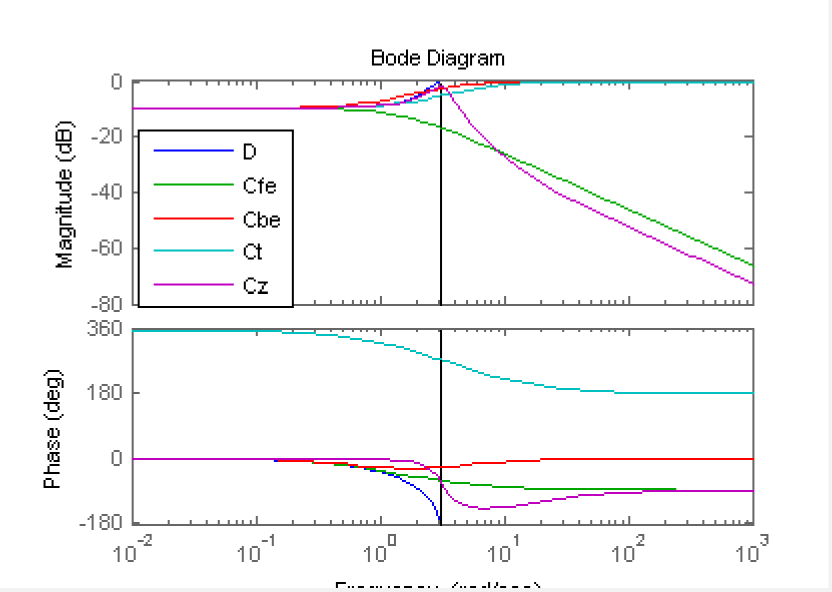


DT-CT filter "equivalence:" Converse problems

In the example: $T = 1$

- Poles with negative real parts indicate a ringing response and cannot be adequately represented in CT. The ZOH approximation may increase the transfer function order to achieve that (MATLAB) while the rest attempt an approximation of the average system.





pole in LHP → $(-0.5)^n$

$$D(z) = \frac{0.5}{z + 0.5}$$

$$C_{fe}(s) = \frac{0.5}{s + 1.5}; \quad C_{be}(s) = \frac{s - 1}{s - 3}$$

$$C_{Tustin}(s) = \frac{-s + 2}{s + 6}; \quad C_{zoh}(s) = \frac{0.231s + 3.45}{s^2 + 1.386s + 10.35}$$

Higher Order!

DT-CT filter “equivalence:” Concluding remarks

- There are several methods to obtain DT-CT equivalent systems. The dominant ones are:
  - Tustin (bilinear) for its matching properties of the frequency response
  - ZOH for interpolating the sampled output exactly (Feedback Systems)
  - Forward Euler for its simplicity, but paying attention to its domain of validity
- In all cases, reasonable approximations are obtained for frequencies well-below the Nyquist frequency of the sampling system, although Tustin can often provide a good approximation up to 1/3 Nyquist and ZOH interpolates the sampled data regardless of the rate.