

EEF 582 HW #2 SOLUTIONS

#5) All solutions are of the form $x_0 + N(A)$, where x_0 is a particular sol'n and $N(A)$ is the null space of A .

A particular sol'n can be derived in more or less systematic ways, e.g. arbitrarily define $x_3 = x_4 = 0$ & solve

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 = 1, x_1 = -1 \text{ (but we must have initially observed that } \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ is nonsingular).}$$

$N(A)$ can be derived in terms of vectors whose span is the space, i.e., two (in this case) independent solutions of $Ax = 0$. [Here, $\dim(R(A)) + \dim(N(A)) = \dim(\text{Domain}) = 4$]

$$\Rightarrow N(A) = \text{span} \left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right]. \text{ (Mathlab produces an orthonormal basis)}$$

\Rightarrow All solutions are of the form $x =$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{matrix} \xi \\ \eta \end{matrix}$$

The min norm solution can be found by solving $\min \|N\xi + x_0\|$, or simply by using the formula

$$x_{\min} = A^T(AA^T)^{-1}b = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -4 & 10 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

#6) A has rank 2 \Rightarrow its null space is trivial \Rightarrow the minimum norm solution is unique. It can be computed by $x_{LS} = (A^T A)^{-1} A^T b \Rightarrow$

$$x_{LS} = \frac{1}{21} \begin{pmatrix} 2 & -5 \\ -5 & 23 \end{pmatrix} \begin{pmatrix} 9 \\ 2 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 8 \\ 1 \end{pmatrix}.$$

#7)

We have $\text{tr}(A) = 1, 1, 2$ so for $\lambda = 1$

$$N(A - \lambda I) = N\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \dim N(A - \lambda I) = 2$$

\Rightarrow 2 Jordan blocks correspond to $\lambda = 1 \Rightarrow$

$$J = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}.$$

For the rest of the computation we need to find

the similarity matrix: $E = \begin{pmatrix} 1 & 0 & 0.577 \\ 0 & 1 & -0.577 \\ 0 & 0 & 0.577 \end{pmatrix}$ so $A = E J E^{-1}$

$$A^{-1} = E J^{-1} E^{-1} = \begin{bmatrix} 1 & 0 & 0.23 \\ 0 & 1 & -0.23 \\ 0 & 0 & 0.24 \end{bmatrix}$$

$$e^{At} = E e^{Jt} E^{-1} = \begin{bmatrix} 1 & 0 & 0.577 \\ 0 & 1 & -0.577 \\ 0 & 0 & 0.577 \end{bmatrix} \begin{bmatrix} e^t & & \\ & e^t & \\ & & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1.732 \end{bmatrix}$$

$$= \begin{pmatrix} e^t & 0 & e^{3t}/3 \\ 0 & e^t & -e^{2t}/\sqrt{3} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1.732 \end{pmatrix} = \begin{bmatrix} e^t & 0 & e^{3t} - e^t \\ 0 & e^t & e^{2t} - e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

$$\#8) \sigma^2 = \text{eig}(A^T A) = \text{eig}(A^2) = \text{eig}(A) \Rightarrow \sigma = |a|.$$

#9) Verify that

$$\begin{pmatrix} 1 & 0 \\ b^T & 1 \end{pmatrix} \begin{pmatrix} 1 + ab^T & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 + ba^T \end{pmatrix}$$

Taking determinants of both sides

$$(1) \det(I + ab^T)(1) = \det(1 + a^T b)$$

So for a scalar $a^T b \Rightarrow \det(1 + ab^T) = 1 + a^T b.$
