EEE 582 TEST # 2 SOLUTIONS

Problem 1

1. $P > 0$ since $P_{11} = 1 > 0$ and $\det P = 1/4 > 0$.
2. $\dot{V} = x^\top(A^\top P + PA + \dot{P})x = -x^\top Q x$, where

$$Q = \begin{pmatrix} -a(t) & -a(t)/2 \\ -a(t)/2 & 1 \end{pmatrix}$$

To guarantee stability, we must ask for $Q \geq 0$. This holds if $-a(t) \geq 0$ and $-a(t) - a^2(t)/4 \geq 0$. These two conditions are equivalent to $-4 \leq a(t) \leq 0$, for all t.

Problem 2

(A, C) c.o. is equivalent to the existence of L_o placing the eigenvalues of $A + L_o C$ to arbitrary locations. This is equivalent to the existence of $L_i = L_o - L$ placing the eigenvalues of $(A + LC) + L_i C$ to arbitrary locations. Hence, $(A + LC, C)$ is equivalently c.o.

Alternatively, one could compute the observability matrix after output injection

$$\begin{pmatrix} C \\ CA + (CL)C \\ CA^2(C2AL)C + (CLCL)C \\ \vdots \end{pmatrix}$$

and use the coefficients indicated by the parentheses to perform elementary row operations and recover the original observability matrix.

Problem 3

We must have $rank(Q_c) = rank([B, AB, A^2B] = 3$, i.e.,

$$rank \begin{pmatrix} b_1 & a_1 b_1 & a_1^2 b_1 \\ b_2 & a_1 b_2 & a_1^2 b_2 \\ b_3 & a_3 b_3 & a_3^2 b_3 \end{pmatrix} = 3$$

Now all $b_i \neq 0$, otherwise an entire row is zero. Then, left-multiplication with $\mathrm{diag}(1/b_1, 1/b_2, 1/b_3)$, which is invertible, does not change the rank of the matrix. Hence, controllability is equivalent to

$$rank \begin{pmatrix} 1 & a_1 & a_1^2 \\ 1 & a_1 & a_1^2 \\ 1 & a_3 & a_3^2 \end{pmatrix} = 3$$

This is a Vandermonde matrix and is nonsingular iff the a_i are distinct. So, (A, B) is c.c. iff $b_1 b_2 b_3 \neq 0$ and $a_1 \neq a_2,\ a_1 \neq a_3,\ a_2 \neq a_3$.

NOTE: If the b_i are functions of time then the rows of $\Phi_A(t, \tau)B(\tau)$ must be L.I. functions of τ. That is, $e^{a_i \tau} b_i(\tau)$, $i = 1, 2, 3$, must be L.I. functions. (The controllability matrix does not result in any simpler conditions.) In particular, this allows for $a_i = a_j$ as long as $b_i(\tau), b_j(\tau)$ are L.I. functions. On the other hand, even if $a_i \neq a_j$, controllability may be lost, e.g., when $b_i(\tau) = e^{-a_i \tau}$, $b_j(\tau) = e^{-a_j \tau}$. Of course, if any $b_i(\tau) \equiv 0$ (the zero function), then (A, B) is not controllable.

Problem 4

1. It is not true. For time varying ODE's, eigenvalues alone provide neither necessary nor sufficient conditions for stability. In fact, one can construct relatively simple matrices that have constant eigenvalues in the left half plane and are unstable, or constant eigenvalues in the right half plane and are stable. Only for slowly time varying ODE's a sufficient condition for stability is that the origin is uniformly exponentially stable (with a margin) and the rate of variation is sufficiently small (either $\|\dot{A}\|$, or $\int \|\dot{A}\|$, or $\int \|\dot{A}\|^2$.)

2. $\dot{x} = A(t)x$ US $\Rightarrow \|\Phi_A(t,\tau)\| \leq c$ for $t \geq \tau$. Consider now the system $\dot{x} = (A(t) - \delta I)x$. Then $\Phi_{A-\delta I}(t,\tau) = e^{-\delta(t-\tau)}\Phi_A(t,\tau)$. (Proof: $\Phi_{A-\delta I}(t,t) = I$ and $\frac{d}{dt}\Phi_{A-\delta I}(t,\tau) = \ldots = (A(t) - \delta I)\Phi_{A-\delta I}(t,\tau)$.) Hence, $\|\Phi_{A-\delta I}(t,\tau)\| \leq e^{-\delta(t-\tau)}\|\Phi_A(t,\tau)\| \leq ce^{-\delta(t-\tau)}$ for $t \geq \tau$. Hence, for $\delta > 0$, $A(t) - \delta I$ is ES.

Alternatively, there exists $P > 0$ such that $A^\top P + PA + \dot{P} \leq 0$. Then for $V = x^\top Px$ we have

$$\dot{V} = x^\top(A^\top P + PA + \dot{P})x - 2\delta x^\top Px \leq -2\delta V$$

Hence $V \leq V(0)e^{-2\delta t}$ and the equilibrium is ES.

3. Here we would like $A+\Delta$ to be stable. Rewrite $A+\Delta = (A+\delta I)+(-\delta I+\Delta)$ and suppose $(A+\delta I)$ is stable (A is stable with margin δ). Then, with $V = x^\top Px$ corresponding to the Lyapunov matrix of $A + \delta I$ as before, $\dot{V} \leq 2x^\top(-\delta + \|\sqrt{P^{-1}}\Delta\sqrt{P}\|)V$. For stability, we can state the condition $\|\sqrt{P^{-1}}\Delta\sqrt{P}\| < \delta$. This can always be satisfied with Δ small enough (relative to the exponential stability margin of A) but no simpler condition can be given. Even in the TI case, since Δ is a matrix perturbation that may not share the eigenvectors of A, the perturbation in the eigenvalues of the matrix A will be amplified by the condition number of $\sqrt{P}$.