EEE 480 HW # 1 PROBLEMS

Problem 1.3 A machine for making paper is shown in the Fig. 1.12. There are two main parameters under feedback control: the density of fibers as controlled by the consistency of the thick stock that flows from the headbox onto the wire, and the moisture content of the final product that comes out of the dryers. Stock from the machine chest is diluted by white water returning from under the wire as controlled by a control valve (CV). A meter supplies a reading of the consistency. At the"dry end" of the machine, there is a moisture sensor. Draw a component block diagram, and identify the seven components (i.e. process, actuator, sensor, reference input, controlled output, actuator output and sensor output.) for

a) control of consistency

b) control of moisture.

Problem 1.6 Draw the component block diagram for an elevator-position control. Indicate how you would measure the position of the elevator car. Consider a combined coarse and fine measurement system. What accuracies do you suggest for each sensor?

Problem 2.9 Use node analysis to write the dynamic equations for the circuits shown in Fig. 2.40 and listed below:

a) lead network

b) lag network

Problem 2.11a Use node analysis to write the dynamic equations for the op-amp circuits in Fig. 2.41a : first-order op-amp lead network. Assume ideal operational amplifiers in every case.

Problem 2.14 Find the differential equations for the circuit shown in Fig. 2.44, and put them in state-variable form.

Problem 2.30 Modify the equation of motion for the cruise control in Example 2.1, (Eq. 2.4: $\dot{v} + \frac{b}{m}v = \frac{u}{m}$) so that it has a control law; that is, let

$$u = K(v_r - v),$$

where
v_r = reference speed,
K = constant.

This is a "proportional" control law where the difference between v_r and the actual speed v is used as a signal to speed the engine up or slow it down. Put the equations in the standard variable form with v_r as the input and v as the state. Assume that $m = 1000$ kg and $b = 50$ N.sec/m, and find the response for a unit step in v_r using MATLAB or other CACSD tool. Using trial and error, find a value of K that you think would result in a control system in which the actual speed converges as quickly as possible to the reference speed with no objectionable behavior.

EEE 480 HW # 2 PROBLEMS

Problem 3.10 Write the dynamic equations describing the circuit in Fig. 3.34. Write the equations in both state-variable form and as a second-order differential equation in $y(t)$. Assuming a zero input, solve the differential equation for $y(t)$ using Laplace-transform methods for the parameter values and initial conditions shown in the figure. Verify your answer using the *initial* command in MATLAB.

Problem 3.11 Consider the standard second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Find the output of the system for the input shown in Fig. 3.35.

Problem 3.19 For a second-order system with transfer function

$$G(s) = \frac{3}{s^2 + 2s - 3}$$

determine the following:

a) DC gain;

b) the final value to a step input.

Problem 3.20

a) Compute the transfer function for the block diagram shown in Fig. 3.37. Note that a_i and b_i are constants.

b) Write the third-order differential equation that relates y and u. (*Hint*: Consider the transfer function.)

c) Write three simultaneous first-order (state-variable) differential equations using variables x_1, x_2 and x_3, as defined on the block diagram in Fig. 3.37 Notice how the same constant parameters enter the transfer function, the differential equations, and the matrices of the state-variable form. (This special structure is called the control canonical form).

Problem 3.21 Find the transfer functions for the block diagrams in Fig. 3.38. (Observe the obvious structural simplifications).

Problem 3.27 For the unity feedback system shown in Fig. 3.43, specify the gain K of the proportional controller so that the output $y(t)$ has an overshoot of no more than 10% in response to a unit step.

Problem 3.28 For the unity feedback system shown in Fig. 3.44, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 2% settling time of more than 0.1 sec. Verify your design using MATLAB.

EEE 480 HW # 3 PROBLEMS

Problem 4.12 A paper machine has a transfer function

$$G(s) = \frac{e^{-2s}}{3s+1}$$

where the input is stock flow onto the wire and the output is basis weight or thickness.

a) Find the PID-controller parameters using Zeigler-Nichols tuning rules.

b) The system becomes marginally stable for a proportional gain of $K_u = 3.044$ as shown by the impulse response in the Fig. 4.48. Find the optimal PID-controller parameters according to the Ziegler-Nichols tuning rules.

Problem 4.26 Consider the system shown in Fig. 4.55

a) Determine the transfer function from r to y.

b) Determine the transfer function from w to y.

c) Find the range of (K_1, K_2) for which the system is stable.

d) What is the system type with respect to r and w?

Problem Repeat examples 1.3.1 a,b,c from the notes on model approximations.

EEE 480 HW # 4 PROBLEMS

Problem 4.43 Suppose that unity feedback is to be applied around the following open loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

b) $KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

c) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

Problem 4.45 Find the range of K for which all the roots of the following polynomial are in the LHP.

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$$

Use MATLAB to verify your answer by plotting the roots of the polynomial in the s-plane for various values of K.

Problem 4.46 The transfer function of a typical tape-driven system is given by

$$G(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]}$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + G(s) = 0$.

EEE 480 HW # 5 PROBLEMS

Problem 5.2 Roughly sketch the root loci for the pole-zero maps shown in Fig. 5.57. Show asymptotes, centroids, a rough evaluation of arrival and departure angles, and the loci for positive values of the parameter K. Each pole-zero map is from a characteristic equation of the form

$$1 + K\frac{b(s)}{a(s)} = 0,$$

where the roots of the numerator b(s) are shown as small circles and the roots of the denominator are shown as $\times$'s on the s-plane. Note that in Fig. 5.57(c) there are two poles at the origin.

Problem 5.3 For the characteristic equation

$$1 + \frac{K}{s(s+1)(s+5)} = 0:$$

a) Draw the real-axis segment of the corresponding root locus.

b) Sketch the asymptotes of the locus for $K \to \infty$.

c) For what value of K are the roots on the imaginary axis?

EEE 480 HW # 6 PROBLEMS

Problem 5.21 Let

$$G(s) = \frac{1}{(s+2)(s+3)}$$

and

$$D(s) = K\frac{(s+a)}{(s+b)}.$$

Using root-locus techniques, find the values for the parameters a, b, and K of the compensation that will produce closed-loop poles at $s = -1 \pm j$ for the system shown in Fig. 5.61.

Problem 5.23 Suppose the unity unity feedback system of Fig. 5.61 has an open-loop plant given by $G(s) = 1/s^2$. Design a lead compensation to be added in series with the plant so that the dominant poles of the closed-loop system are located at $s = -2 \pm 2j$.

Problem 5.25 A numerically controlled machine tool has a transfer function given by

$$G(s) = \frac{1}{s(s+1)}.$$

Performance specifications require that, in the unity feedback configuration of Fig. 5.61, the closed-loop poles be located at $s = -1 \pm j\sqrt{3}$.

Problem 5.26 An armature-controlled DC motor with negligible armature resistance has a transfer function

$$G(s) = \frac{4}{s(s+0.5)}.$$

Design a series compensation D(s) in the unity feedback configuration of Fig. 5.61 to meet the following closed-loop specifications:

- The steady-state error to a unit ramp input must be less than 0.02.
- The damping ratio of the dominant closed-loop poles must be equal to 0.5.
- The undamped natural frequency of the dominant closed-loop poles must be 5 rad/sec.

Problem 5.49 Consider the third-order system shown in Fig. 5.77

a) Sketch the root locus for this system with respect to K, showing your calculations for the asymptote angles, departure angles and so on.

b) Using graphical techniques, locate carefully the point at which the locus crosses the imaginary axis. What is the value of K at that point?

c) Assume that, due to some unknown mechanism, the amplifier output is given by the following saturation nonlinearity (instead by a proportional gain K):

$$u = \begin{cases} e, & |e| \leq 1; \\ 1, & e > 1; \\ -1, & e < -1. \end{cases}$$

Qualitatively describe how you would expect the system to respond to a unit step input.

EEE 480 HW # 7 PROBLEMS

Problem 6.11 Determine the range of K for which each of the following system is stable by making Bode plot for $K = 1$ and imagining the magnitude plot sliding up or down until instability results. Verify your answers using a root-locus plot (generated by hand or computer).

a) $KG(s) = \frac{K(s+2)}{(s+10)}$

b) $KG(s) = \frac{K}{(s+10)(s+2)^2}$

c) $KG(s) = \frac{K(s+10)(s+1)}{(s+100)(s+2)^3}$

Problem 6.14 Draw a Nyquist diagram for each of the following systems, and compare your result with that obtained using the MATLAB command *nyquist*:

a) $KG(s) = \frac{K(s+2)}{(s+10)}$

b) $KG(s) = \frac{K}{(s+10)(s+2)^2}$

c) $KG(s) = \frac{K(s+10)(s+1)}{(s+100)(s+2)^3}$

Using your plots, estimate the range of K for which each system is stable, and qualitatively verify your result using a root-locus plot (generated by hand or using MATLAB).

Problem 6.15 Draw a Nyquist diagram for each of the following systems in parts (a) through (c), choosing the contour to be to the right of any singularities on the $j\omega$-axis.

a) $KG(s) = \frac{K(s+1)}{s^2(s+10)}$

b) $KG(s) = \frac{K(s+1)}{s(s+2)}$

c) $KG(s) = \frac{K}{(s+2)(s^2+9)}$

d) Redo the Nyquist plots in parts (b) and (c), this time choosing the contour to be to the left of all singularities on the imaginary axis.

e) Use the Nyquist stability criterion to determine to determine the range K for which each system is stable, and verify your answer using a root-locus plot (generated by hand or computer).

Problem 6.22 For each of the open-loop frequency-response plots shown in Fig. 6.98, determine the following:

- the minimum number of open-loop poles n,
- the minimum number of zeros m,
- the pole-zero excess $n - m$,
- a pole-zero pattern in the s-plane that could correspond to the frequency response.

Problem 6.23 Associate the Nyquist plots, step response sand frequency responses, and frequency responses shown in Fig. 6.99. Justify your conclusions.

EEE 480 HW # 8 PROBLEMS

Problem 6.51

a) Skecth the locus of the closed-loop roots for the system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1.33)^3}.$$

b) Based on your root-locus sketch from part (a), find the value of the gain K at the stability boundary.

c) Is there a compensator that would stabilize this system and yield a closed-loop bandwidth of 10 rad/sec or greater? If so, find such a compensator.

Problem 6.52 Consider a type I unity feedback system with

$$G(s) = \frac{K}{s(s+1)}$$

Design a lead compensator so that $K_v = 12$ sec^{-1} and PM $> 40°$. Use MATLAB to verify that your design meets the specifications.

Problem 6.72 Assume that the system

$$G(s) = \frac{e^{-0.2s}}{s+10}$$

has a 0.2-sec time delay ($T = 0.2$ sec). While maintaining a phase margin of about 40°, find the maximum possible bandwidth using the following:

a) One lead-compensator section

$$D(s) = K\frac{s+a}{s+b},$$

where $b/a = 100$;

b) Two lead-compensator sections

$$D(s) = K(\frac{s+a}{s+b})^2,$$

where $b/a = 10$.

Problem 6.74 The pure time delay $x_{out}(t) = x_{in}(t-\tau)$ is represented by the transfer function $H(s) = e^{-s\tau}$.

a) Graph $|H(\omega)|$ and $\angle H(\omega)$ versus ω.

b) $H(s)$ cannot be represented exactly as a ratio of polynomials in s. Therefore, $H(s)$ cannot be described by a collection of poles and zeros. However, $H(s)$ may be approximated by a rational transfer function. Consider the first order Padé

$$H_1(s) = \frac{1-\tau s/2}{1+\tau s/2}.$$

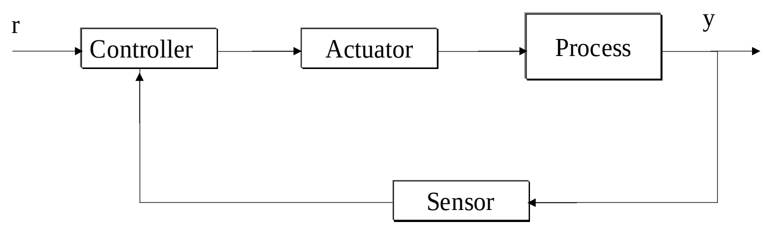
Graph $|H_1(\omega)|$ and $\angle H_1(\omega)$, and compare your results with those from part (a).

Problem 6.77 You are given the experimentally determined Bode plot shown in Fig 6.121. Design a compensation that will yield a crossover frequency of $\omega_c = 10$ rad/sec with PM $> 75°$.

EEE 480 HW # 1 SOLUTIONS

Problem 1.3

A generic block diagram of a feedback loop is shown in the figure below, including sensor and actuator components. Variations of this diagram can reflect a more detailed model of the process or the actuator as combinations of other blocks, and the possibility of measuring a variable which is different from (but related to) the controlled output. On the other hand, the actuator and process blocks are often combined so that a single model is used to describe the effect of the control input to the measured process output. The choice of the process representation depends on the availability of partial models for the various components as well as the existence of intermediate variables (external inputs or disturbances) whose effect on the process output needs to be investigated.



Generic Block Diagram of a Feedback Control Loop

For the first problem, the actuator is a valve controlling the white water flow. The typical inputs to such valves are a voltage or a pressure signal that is generated by the controller (control input). The output of the valve is the valve position. Alternatively, with some additional computations and modeling, the model of the actuator can include the flow characteristics of the line so that the actuator output is the white water flow. Regardless of the case the Process model should be consistent, describing the relationship between the actuator output and the process output. For the latter, the consistency can be used as a controlled variable. An alternative to that is to include wire speed and dispenser parameters so that the model output is the fiber density (or paper weight) that is one of the important characteristics of the final product. The described sensor measures consistency. Such a sensor could provide an analog signal to the controller. It may also include a microcontroller to convert the consistency measurement directly to fiber density. (Modern sensors scan the paper sheet and process the measurements to generate estimates of both paper weight and moisture at given points along the paper production line.) The reference signal is a consistency set-point (or fiber density, depending on the sensor).

For the second case of moisture control the actuator is again a valve, receiving a voltage or pressure control input signal. This time the valve controls the flow of steam to the dryer drums. The process model should now describe the effect of the actuator output (either valve position or steam flow) on the reel moisture. Notice that such a model would require the knowledge of the paper water content (consistency) at the beginning of the dryer drum sequence. In other words, the moisture loop is affected by the consistency loop and a full multivariable controller design may be necessary in order to achieve good performance. The sensor measures the moisture of the paper for which the controller has a reference set-point.

As an alternative control system for the moisture, the steam flow is adjusted by a local (inner-loop) controller that controls the temperature of the dryer drums. This is desirable since constraints on the drum temperatures can be imposed to ensure paper integrity at all times. (In contrast, using the steam flow to control moisture directly may result in high drum temperatures during transients that would in turn cause the paper to become too dry and possibly break.) For this control system, the control input is the reference (set-point) drum temperature sent to the local temperature controller. The process model then relates drum temperature to reel moisture with the starting consistency as an external parameter.

Problem 1.6

The actuator for the elevator position system is a power control circuitry driving a motor that supplies the necessary torque to raise or lower the elevator cabin. The motor then turns a wheel that moves the elevator cables and the cabin.

A coarse measurement of the elevator position can be obtained by connecting a potentiometer to the motor shaft through a set of gears that reduce the total number of revolutions of the shaft down to one. The potentiometer resistance can be read as an analog or a digital signal. For the second case, assuming an 8-bit A/D converter (cheap) and a 30 m tall building, the resolution of the position measurement is approximately 10 cm. For a more accurate measurement, we could increase the A/D resolution and the quality of the potentiometer, a possibly expensive approach. Or, we could add a second, free turning potentiometer that makes a full revolution for a fraction of the overall height. For example, suppose that we keep only the 6 MSB's from the first potentiometer (to avoid noise problems), resulting in a 0.5m resolution. Then, we could set the second potentiometer gears to yield a full revolution every 0.5m. Keeping 6 MSB's from this measurement too, we obtain a resolution for the combined measurement of about 1cm. The stopping position of the elevator at each floor can be monitored and fine-tuned using a proximity sensor (optical or magnetic). Finally, notice that the controller should maintain smooth and consistent operation of the elevator, regardless of the weight, and without excessive speed or acceleration. For this reason, the reference input could be generated internally as a ramp of the desired maximum speed with tapered (smooth) initial and final segments.

Problem 2.9

a. Writing the voltage current relationships we have

$$\begin{aligned} i_2 &= \frac{y}{R_2} = i_1 + i_c \\ C\dot{v}_c &= i_c \\ v_c &= u - y \\ i_1 &= \frac{u-y}{R_1} \end{aligned}$$

Eliminating all the variables other than u, y we obtain the differential equation

$$C\dot{y} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right) y = C\dot{u} + \frac{1}{R_1} u$$

Taking the Laplace transform of both sides we find that the circuit transfer function is

$$\frac{Cs + \frac{1}{R_1}}{Cs + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

Its DC gain is $1/(1 + R_1/R_2) < 1$ and at high frequencies its magnitude approaches 1. Its phase is positive between the zero and the pole and hence its name of a Lead network.

b. As before, we write the voltage-current relationships to obtain the circuit differential equation

$$C(R_1 + R_2)\dot{y} + y = CR_2\dot{u} + u$$

Taking the Laplace transform of both sides we find that the circuit transfer function is

$$\frac{CR_2 s + 1}{C(R_1 + R_2)s + 1}$$

Its DC gain is 1 and at high frequencies its magnitude approaches $R_2/(R_1 + R_2) < 1$. Its phase is negative between the pole and the zero and hence its name of a Lag network.

Notice that in both of these examples, the equations were particularly simple and the derivation of the final equation presented no difficulty, even using a "brute force" approach. A more systematic version of this is taken in Problem 2.14.

Problem 2.9

From the ideal op-amp equations ($v_+ = v_-, i_+ = i_- = 0$) we have the well-known input output relationship

$$v_o = -\frac{R_3}{Z_{in}} v_{in}$$

where

$$Z_{in} = R_1 + \frac{R_2}{\frac{1}{R_2} + Cs}$$

is the input impedance. Hence,

$$\frac{v_o}{v_{in}} = -\left(\frac{R_3}{R_1 + R_2}\right) \frac{CR_2 s + 1}{C\frac{R_2 R_1}{R_1 + R_2} s + 1}$$

Since the pole is larger than the zero, this is a Lead circuit.

Here, instead of using the differential equations to describe the voltage-current relationships, we use the respective Laplace transforms to obtain the final transfer function. If necessary, a differential equation description can then be derived by writing the ODE that corresponds to that transfer function.

Problem 2.14

For this problem we write the current voltage equations for the three dynamic elements and the nodal equations using the same currents and voltages.

$$\begin{aligned}
&C_1 \dot{v}_1 = i_1 \ ; \quad L\dot{i}_L = v_L \ ; \quad C_2 \dot{v}_2 = i_2 \\
&\frac{v_{in} - v_1}{R_1} + \frac{v_L - v_1}{R_2} = i_1 \\
&\frac{v_1 - v_L}{R_2} + \frac{v_2 - v_L}{R_3} = i_L \\
&\frac{v_L - v_2}{R_3} = i_2 \\
&v_o = v_2
\end{aligned}$$

The "natural" definition of the states for circuit problems is capacitor voltages and inductor currents, so $x = [v_1, i_L, v_2]^\top$. In addition to that, let $z = [i_1, v_L, i_2]^\top$ a vector with auxiliary variables. With these definitions, a state-space version of the above set of algebraic-differential equations is

$$\underbrace{\begin{pmatrix} C_1 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & C_2 \end{pmatrix}}_{E} \frac{dx}{dt} = z$$

$$\left(\underbrace{\begin{matrix} \frac{1}{R_1} + \frac{1}{R_2} & 0 & 0 \\ \frac{1}{R_2} & -1 & \frac{1}{R_3} \\ 0 & 0 & \frac{1}{R_3} \end{matrix}}_{A_1} \middle| \underbrace{\begin{matrix} 1 & -\frac{1}{R_2} & 0 \\ 0 & -\frac{1}{R_2} - \frac{1}{R_3} & 0 \\ 0 & -\frac{1}{R_3} & 1 \end{matrix}}_{A_2}\right) \begin{pmatrix} x \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{R_1} \\ 0 \\ 0 \end{pmatrix}}_{B_1} v_{in}$$

$$v_o = \underbrace{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}_{C} x + 0 v_{in}$$

Since A_2 is invertible, we can solve for z and substitute back into the differetial equation:

$$\begin{aligned}
z &= -A_2^{-1} A_1 x + A_2^{-1} B_1 v_{in} \\
E\dot{x} &= -A_2^{-1} A_1 x + A_2^{-1} B_1 v_{in} \\
\dot{x} &= Ax + Bv_{in} \\
v_o &= Cx
\end{aligned}$$

where $A = -E^{-1} A_2^{-1} A_1$, $B = E^{-1} A_2^{-1} B_1$. The last two equations are in the standard state-space form (with zero direct throughput). Even though the state representation matrices are defined in terms of other matrices, their computation in a MATLAB environment is quite straightforward.

Problem 2.30

The model for the cruise control system is $m\dot{v} = -bv + u$. Letting $u = K(v_r - v)$ we obtain the closed-loop model $m\dot{v} = -(b + K)v + Kv_r$, or

$$\dot{v} = -\frac{b+K}{m}v + \frac{K}{m}v_r$$

The speed of convergence of the response to steady-state depends on the system pole (1/time-constant) $-(b + K)/m$. So, for fast convergence K should be as large as possible. Moreover, the final steady-state to a constant input v_r is given by

$$0 = -\frac{b+K}{m}v + \frac{K}{m}v_r \;\Rightarrow\; v = \frac{K}{b+K}v_r$$

This implies that larger K result in a smaller steady-state error.

On the other hand, using large controller gains may cause "objectionable behavior" in the form of large acceleration due to noisy measurements. (The fact that the maximum acceleration (i.e., u) is finite can be handled through control input saturation instead of decreasing the controller gain.) To capture this effect, let us suppose that the speed measurement contains a noise component n, so that $u = K(v_r - v - n)$. Re-writing the closed-loop equation,

$$\dot{v} = -\frac{b+K}{m}v + \frac{K}{m}v_r - \frac{K}{m}n$$

We would like the noise-dependent acceleration to be smaller than a prescribed constant that defines the tolerable acceleration limit. That is $\frac{K}{m}n_{max} \leq a_{max}$.

Using trial and error in MATLAB, we can define the transfer functions from v_r to v and from n to v by the following commands.

```
>> b=50;m=1000;nm=.5*1.6*1000/3600;
>> k=400;
>> g=tf({k/m; [k/m 0]},{[1 (b+k)/m];[1 (b+k)/m]})
>> step(g)
```

Here, the maximum noise (0.5 mph, converted to SI units) is used to normalize the acceleration transfer function. Generating the step response of this model produces two plots: The first one is the response to a step change in the commanded speed. We would like for that to converge quickly to a value near unity. (The difference-times-100 is the percent error.) The second plot is the acceleration response to a step change in the noise. The initial peak of this response is the maximum acceleration produced by noise variations within the prescribed n_{max} level. We would like that peak to be below a_{max}, say 0.01g = 0.1 m/s/s. We may now iterate k to investigate the trade-off between these objectives. The optimum value for the controller gain is around 450 resulting in a 10% steady-state error and about 6 s settling time.

An improvement of this performance requires a dynamic compensator and some (reasonable) assumptions on the noise spectral properties. Also notice that, because of the simplicity of the model, the same result can be obtained analytically, without trial-and-error.

EEE 480 HW # 2 SOLUTIONS

Problem 3.10

The voltage-current relationships are

$$\begin{aligned} C\frac{dv_C}{dt} &= i_C = i_R = i_L \\ L\frac{di_L}{dt} &= v_L \\ v_L + v_R + v_C &= u\ ; \quad v_R = i_R R \end{aligned}$$

and $y = v_C$. Define the usual two states $x = [v_C, i_L]^\top$ and eliminate the rest of the variables to get the state-space equations

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1/C \\ -1/L & -R/L \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/L \end{pmatrix} u \\ y &= [1,\ 0]x + [0]u \end{aligned}$$

From this, $\dot{y} = x_2/C$; differentiate y once more and eliminate x_2 to get the differential equation

$$\ddot{y} + \frac{R}{L}\dot{y} + \frac{1}{LC}y = \frac{1}{LC}u$$

For zero input and the given R, L, C values, the homogeneous response satisfies

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + Y(s) = 0$$

Hence,

$$Y(s) = \frac{s+2}{s^2+2s+1} = \frac{s+1+1}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2} \text{ (easy PFE!)}$$

for which the inverse Laplace transform is

$$y(t) = e^{-t}\mathcal{U}(t) + te^{-t}\mathcal{U}(t)$$

A numerical computation of the response using MATLAB's lsim yields the same result.

Problem 3.11

The input can be written as

$$u(t) = \int_{-\infty}^{t} [\mathcal{U}(\tau) - \mathcal{U}(\tau-1) - \mathcal{U}(\tau-2) + \mathcal{U}(\tau-3)]d\tau$$

Since the integrator is also an LTI system, it commutes with $G(s)$. Hence the output $y(t)$ can be written as

$$y(t) = \int_{-\infty}^{t} [y_s(\tau) - y_s(\tau-1) - y_s(\tau-2) + y_s(\tau-3)]d\tau$$

where $y_s(t)$ is the step response of $G(s)$, for which analytical expressions are available.

For a further simplification, recall that the integrator is TI, hence it commutes with shifts. So,

$$y(t) = y_i(t) - y_i(t-1) - y_i(t-2) + y_i(t-3)$$

where

$$y_i(t) = \int_{-\infty}^{t} y_s(\tau)d\tau$$

The analytical computation of this integral is tedious but straightforward.

Problem 3.19

The DC-gain is $G(0) = -1$.

The final value to a step input is not well defined since G is not stable. However, if $y(0) = -1, \dot{y}(0) = 0$, then $y(t) = -1$ for $t > 0$.

Problem 3.20

We have that

$$\begin{aligned}
Y(s) &= b_1 X_1(s) + b_2 X_2(s) + b_3 X_3(s) \\
sX_1(s) &= -a_1 X_1(s) - a_2 X_2(s) - a_3 X_3(s) + U(s) \\
& \quad X_1(s) = sX_2(s) = s^2 X_3(s)
\end{aligned}$$

hence,

$$\begin{aligned}
& [s^3 + a_1 s^2 + a_2 s + a_3] X_3(s) = U(s) \\
Y(s) &= [b_1 s^2 + b_2 s + b_3] X_3(s) \\
&= \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} U(s)
\end{aligned}$$

For the differential equation for y, use the Laplace transform property for the differentiation to write

$$\begin{aligned}
& [s^3 + a_1 s^2 + a_2 s + a_3] Y(s) = [b_1 s^2 + b_2 s + b_3] U(s) \\
\Rightarrow \quad & \frac{d^3}{dt^3} y(t) + a_1 \frac{d^2}{dt^2} y(t) + a_2 \frac{d}{dt} y(t) + a_3 y(t) = b_1 \frac{d^2}{dt^2} u(t) + b_2 \frac{d}{dt} u(t) + b_3 u(t)
\end{aligned}$$

Finally, with $x = [x_1, x_2, x_3]^\top$, the state-space equations are

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \\
y &= [b_1,\ b_2,\ b_3] x + [0] u
\end{aligned}$$

Problem 3.21

Taking advantage of the nested structure of these block diagrams, the three transfer functions can be written virtually by inspection:

$$\begin{aligned}
G_a(s) &= \frac{G_1}{1 + G_1} + G_2 \\
G_b(s) &= \frac{G_1}{1 + G_1 G_2} G_3 \frac{G_4}{1 + G_4 G_5} G_6 + G_7 \\
G_c(s) &= \left[G_1 \frac{G_2}{1 + G_2} G_3 + G_6 \right] \frac{G_4}{1 + G_4} G_5 + G_7
\end{aligned}$$

Notice that summation nodes with three or more inputs can be split into two or more nodes; moreover, consecutive summation nodes commute.

Problem 3.27

The closed-loop transfer function is

$$G_c = \frac{K}{s^2 + 2s + K}$$

This is the standard second order system for which (as shown in the textbook) a 10% or less overshoot is obtained for $\zeta \geq 0.6$. Here,

$$\begin{aligned}
& 2\zeta w_n = 2, \quad w_n^2 = K \\
\Rightarrow \quad & \zeta = 1/\sqrt{K} \\
\Rightarrow \quad & K \leq \frac{1}{0.6^2} = 2.78
\end{aligned}$$

Problem 3.28

The closed-loop transfer function is

$$G_c = \frac{100K}{s^2 + (a+25)s + 100K + 25a}$$

Again, this is the standard second order system for which a 25% or less overshoot is obtained approximately for $\zeta \geq 0.4$. Also, the settling time is approximately $4/\zeta w_n$, so $\zeta w_n \geq 40$. Here,

$$\begin{aligned}
& 2\zeta w_n = a + 25, \quad w_n^2 = 100K + 25a \\
\Rightarrow \quad & a \geq 80 - 25 = 55 \\
& \frac{a+25}{2\sqrt{100K + 25a}} \geq 0.4 \\
\Rightarrow \quad & K \leq \frac{a^2 + 34a + 25^2}{64}
\end{aligned}$$

One possible combination is $a = 55, \; K = 86.25$. A MATLAB simulation shows that the actual response has a slightly higher overshoot and a shorter settling time, but within the accuracy of the approximations.

EEE 480 HW # 3 SOLUTIONS

Problem 4.12

1. Using the analytical version of ZN-tuning, $R = K/\tau = 1/3$, $L = 2$, so $K_p = 1.8$, $T_I = 4$, $T_D = 1$.

2. Using the experimental version of ZN, the ultimate gain is 3.04 so $K_p = 1.82$. Then, from the graph the ultimate period is between 5 and 10 sec, say 7.5 sec. From this, $T_I = 3.75$, $T_D = 0.94$.

(The agreement between the two methods is reasonably good, and that is expected since the data were produced by the model.)

Problem 4.26

1.

$$G_{ry}(s) = \frac{10(k_1 + k_2 s)}{(s^2 + s + 20)s + 10(k_1 + k_2 s)}$$

2.

$$G_{wy}(s) = \frac{10s}{(s^2 + s + 20)s + 10(k_1 + k_2 s)}$$

3. Characteristic polynomial: $s^3 + s^2 + (20 + 10k_2)s + 10k_1$. Performing the Routh test:

$$\begin{array}{c|ll} s^3 & 1 & 20 + 10k_2 \\ s^2 & 1 & 10k_1 \\ s^1 & \frac{-(10k_1 - 20 - 10k_2)}{1} & \\ s^0 & 10k_1 & \end{array}$$

Thus, we obtain the stability conditions: $k_1 > 0$, $k_2 - k_1 > 2$ (intersection of two half-spaces).

4. The output error depends on r as $r - y = (1 - G_{ry})r$. This transfer function is

$$1 - G_{ry} = \frac{s^3 + s^2 + 20s}{(s^2 + s + 20)s + 10(k_1 + k_2 s)}$$

and has a simple zero at 0. Hence, the system type with respect to r is I.

The dependence of the output error on w is given by G_{wy} itself, which has a simple zero at 0. Hence, the system type with respect to w is I.

EEE 480 HW # 3 SOLUTIONS

Problem 4.43

1. Form the Routh array of $s^4 + 2s^3 + 3s^2 + 8s + 8$:

$$
\begin{array}{c|ccc}
s^4 & 1 & 3 & 8 \\
s^3 & 2 & 8 & 0 \\
s^2 & -1 & 8 & \\
s^1 & 24 & & \\
s^0 & 8 & &
\end{array}
$$

There are two sign changes in the first column and therefore the corresponding closed loop system is unstable (two RHP poles).

2. Form the Routh array of $s^3 + s^2 + 2s + 8$:

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 1 & 8 \\
s^1 & -6 & \\
s^0 & 8 &
\end{array}
$$

There are two sign changes in the first column and therefore the corresponding closed loop system is unstable.

3. Form the Routh array of $s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 1$:

$$
\begin{array}{c|ccc}
s^5 & 1 & 3 & 4 \\
s^4 & 2 & 7 & 1 \\
s^3 & -1/2 & 7/2 & \\
s^2 & 0(\epsilon) & 1 & \\
s^1 & 1/2\epsilon & & \\
s^0 & 1 & &
\end{array}
$$

There are two sign changes in the first column and therefore the corresponding closed loop system is unstable.

Problem 4.45

Form the Routh array of $s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K$:

$$
\begin{array}{c|lll}
s^5 & 1 & 10 & 5 \\
s^4 & 5 & 10 & K \\
s^3 & 8 & 5 - K/5 & \\
s^2 & K/8 + 55/8 & K & \\
s^1 & \frac{-K^2 - 350K + 1375}{5(K+55)} & & \\
s^0 & K & &
\end{array}
$$

Thus, we obtain the stability conditions: $K > -55$, $K > 0$, $-K^2 - 350K + 1375 > 0$. Their intersection is $0 < K < 3.88$, which agrees with the results from `rlocus(1,[1 5 10 10 5 0]);`.

Problem 4.46

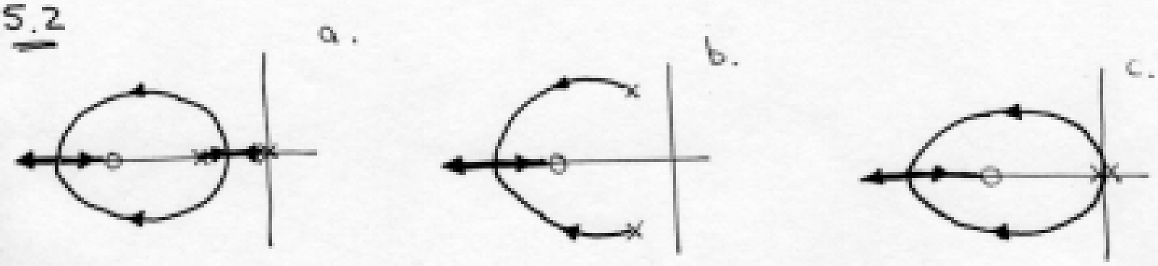
Form the Routh array of $s^5 + 1.9s^4 + 5.1s^3 + 6.2s^2 + (2 + K)s + 4K$: (here, the `conv` function is useful in computing the various terms as polynomials of K)

$$
\begin{array}{c|lll}
s^5 & 1 & 5.1 & K + 2 \\
s^4 & 1.9 & 6.2 & 4K \\
s^3 & 1.8368 & -1.1053K + 2 & \\
s^2 & 1.1433K + 4.1312 & 4K & \\
s^1 & \frac{-1.2636K^2 - 9.6267K + 8.2624}{1.1433K + 4.1312} & & \\
s^0 & 4K & &
\end{array}
$$

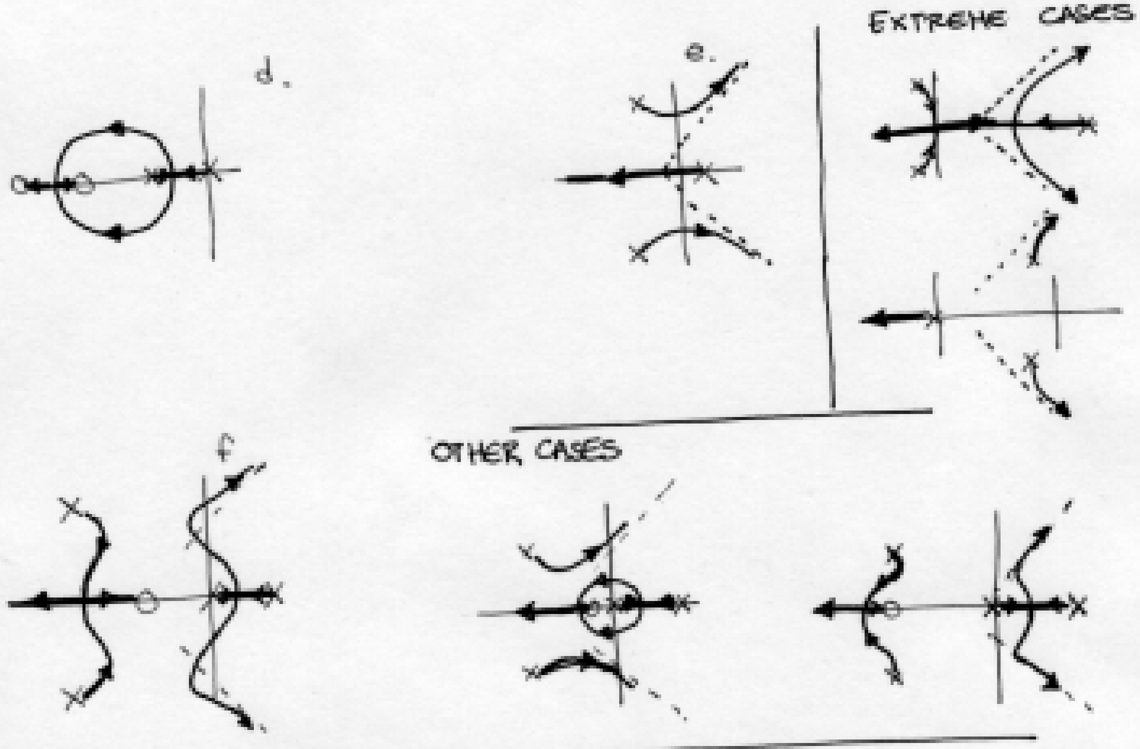
Thus, we obtain the stability conditions: $K > -3.613$, $K > 0$, $-8.397 < K < 0.7787$. Their intersection is $0 < K < 0.7787$, which agrees with the results from `rlocus`.

EEE 480 HW #5 SOLUTIONS

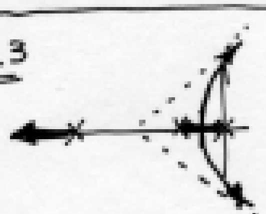
5.2



Note: b and c are portions of a.



5.3



Char. poly. $s^3 + 6s^2 + 5s + K$

Routh:

s^3	1	5
s^2	6	K
s^1	$\frac{1}{6}(30-K)$	
s^0	K	

$s^1 = 0$ when $\boxed{K = 30}$. $\Rightarrow$ $j\omega$ axis crossing at the roots of $6s^2 + 30$

$\rightarrow \boxed{\pm j\sqrt{5}}$

EEE 480 HW # 6 SOLUTIONS

Problem 5.21

The root-locus asymptotes for pure-gain compensation are at -2.5. Since the required closed-loop pole locations are $-1 \pm j$, we need to introduce a lag compensator. Observe that asymptotes at -1 are obtained for poles at $0, -2$. Hence, a possible controller would cancel the system pole at -3 and put a pole at the origin. That is,

$$C(s) = K\frac{s+3}{s}$$

To compute K, we can form the closed-loop characteristic equation (simple enough), $(s^2+2s+K)(s+3)$. One of its roots is at -3 due to the cancellation; the other two should be at $-1 \pm j$ and have magnitude $\sqrt{2}$, hence $K = 2$.

Problem 5.23

Here, the pure-gain asymptotes are at 0 and we need a lead compensator to achieve the desired specification. At $-2 \pm j2$, the system double integrator contributes $2 \times 135^o = 270^o$, so the compensator should contribute 90^o phase lead.

Choosing the pole angle as 10^o, the zero should contribute 100^o. Computing the corresponding locations:

- pole: $\tan 10^o = 0.176 \Rightarrow p = \frac{2}{0.176} + 2 = 13.3$.
- zero: $z = 2 - (2)(0.176) = 1.65$.
- gain: $K = \left|\frac{s^2(s+p)}{s+z}\right|\Big|_{s=-2+j2} = \frac{(8)(\sqrt{11.3^2+2^2})}{\sqrt{0.35^2+2^2}} = 45.21$.

The final compensator is

$$C(s) = 45.21\frac{s+1.65}{s+13.3}$$

Problem 5.25

Here, the pure-gain asymptotes are at 0.5 and we need a lead compensator to achieve the desired specification. While the procedure used in Problem 5.23 is applicable, let us use a different approach that can yield quick results for simple problems.

Guided by the expected root-locus of the compensated system, we select the desired closed-loop characteristic polynomial as

$$(s^2+2s+4)(s+10) = s^3+12s^2+24s+40$$

The first factor is from the specified dominant pair and the second factor is a reasonable choice for the third closed-loop pole. Next, we write the closed-loop characteristic polynomial in terms of the lead compensator parameters

$$(s^2+s)(s+p) + K(s+z) = s^3 + (1+p)s^2 + (K+p)s + Kz$$

Equating the coefficients of the two polynomials we find:

- $p = 12 - 1 = 11$
- $K = 24 - p = 13$
- $z = 40/K = 3.08$.

The final compensator is

$$C(s) = 13\frac{s+3.08}{s+11}$$

This approach is based on the so-called "pole-placement" design which computes the parameters of a general (high-order) compensator to assign the closed-loop poles to desired locations. The advantage of pole placement is that all the poles of the closed-loop system are assigned to prescribed locations and, thus, it can be used to stabilize arbitrary systems. For this, the compensator should have enough parameters (compensator order $\geq$

system order-1) and, in general, the design requires the solution of a linear system of equations. However, for simple systems the solution can be obtained by inspection. (More details in EEE 482.)

Problem 5.26

From the given specifications the dominant closed-loop poles should be at $-2.5 \pm j4.33$. Furthermore, for the steady-state error to ramps, we use the Laplace final value theorem to compute the minimum compensator DC-gain as follows:

$$e_{ss,r} = \lim_{s\to 0} s\frac{1}{1+C(s)G(s)}\frac{1}{s^2} = \lim_{s\to 0}\frac{1}{sC(s)G(s)} = \lim_{s\to 0}\frac{0.5}{4C(s)}$$

For this to be less than 0.02, $C(0)$ should be greater than 6.25.

The largest $C(0)$ is obtained by placing the compensator zero as deep as possible in the left-half plane. Computing the required lead angle we find

$$\theta_{lead} = 120^o + 114.8^o - 180^o = 54.8^o$$

Hence, the largest possible zero would be at $-(4.33/\tan 54.8^o + 2.5) = -5.55$. The corresponding compensator would have a pole at $-\infty$ ($C(s) = K(s+5.55)$) and the value for K would be 1.125. This results in $C(0) = 6.25$. In other words, the specifications are achievable with a PD compensator. Such a compensator, however, is not implementable in a strict sense. For this reason, we must perform a lead-lag design to achieve the required specifications.

First, we design a lead compensator to place the dominant closed-loop poles at the desired locations. Selecting a zero angle contribution of 70^o the pole angle should be 15.2^o, resulting in the locations $z = 4.076, p = 18.43$. The corresponding gain is

$$K = \frac{5|2+j4.33||15.93+j4.33|}{4|1.576+j4.33|} = 21.36$$

Hence,

$$C_{Lead}(s) = 21.36\frac{s+4.076}{s+18.43}$$

Next, we compute the required compensator DC-gain to achieve the ramp steady-state specification. With the lead compensator, $sC_{Lead}(s)G(s)|_{s=0} = 37.8$, so the additional required DC-gain is $50/37.8 = 1.323$. We now design a lag compensator of the form $\frac{s+z}{s+p}$ with $p < z \ll 2.5$ The angle contribution of the lag should be small at the dominant poles so that their location will change only slightly. Notice that there is no gain adjustment of the lag compensator (in a lead-lag design) since at the dominant poles, $\frac{s+z}{s+p} \simeq 1$. Selecting $z = 0.5$, we compute $p = 0.5/1.323 = 0.378$. (The corresponding angle contribution is $63.9 - 65.2 = 1.3^o$ which is indeed very small.)

Thus, the final compensator is

$$C(s) = 21.4\frac{(s+4.08)(s+0.5)}{(s+18.4)(s+0.378)}$$

It should be emphasized that this "step-by-step" design of a lead-lag compensator is quite simple but leaves much room for improvement. The closed-loop system will have a pole-zero near cancellation at approximately -0.5. This implies that there will be a residual part in the response with time constant 8s, when the dominant part of the response has a time constant $1.8/w_n \sim 0.36$s. Depending on the application, such a residual slow response may or may not be important. Any improvement in this aspect of the design requires the increase of the lag zero which cannot be done independent of the lead design.

As an alternative to the previous "classical" lead-lag design, let us consider the design of a PID compensator which will automatically achieve the ramp steady-state specification. The PID is a special case of lead-lag compensators, having a pole at the origin, a fast pole due to the approximate differentiator and two zeros to be selected. Let us first select the fast pole to be at $s = -20$ and the two zeros to be identical. (It is fairly obvious that this is a feasible choice, contributing a small angle at the dominant pole location.) Computing the angle contributions at $-2.5 + j4.33$ we have

$$2\theta_z = 2\times 120 + 114.8 + \tan^{-1}\frac{4.33}{17.5} - 180 \Rightarrow \theta_z = 94.35 \Rightarrow z = 2.5 - \frac{4.33}{13.15} = 2.17$$

The corresponding gain K is

$$K = \frac{5^2|2+j4.33||17.5+j4.33|}{4|0.33+j4.33|^2} = 28.5$$

The final compensator is

$$C(s) = 28.5\frac{(s+2.17)^2}{s(s+20)}$$

(This can be converted to a standard PID form after a few straightforward manipulations.)

Problem 5.49

The root locus for this transfer function has one pole moving towards the zeros on the real axis. The other two poles branch out towards the RHP and then circle around the zeros; then one branches towards the second zero and the other towards $-\infty$. The shape of the root-locus shows that for small feedback gains, the closed-loop system is unstable.

The jw-axis crossover can be found by a standard Routh argument. Alternatively, because of the special structure of this system, we may observe that when there is a pole on the jw-axis, the three poles contribute an angle -3×90^o. Hence the angle from the zeros must be 2×45^o and the jw-axis pole must be at $j1$. The value of the gain for which this pole is obtained is $K = 1/2$.

The saturation nonlinearity decreases effective compensator gain for large error signals. From the root locus plot, this would eventually lead to instability if the signal magnitude grows large enough. On the other hand, starting with initial conditions near zero and having a small reference signal, the error signal is small and the saturation operates in the linear region. [1]

In our case, a unit step input produces an initial error equal to 1 and the output moves in the direction of the reference (Laplace initial value theorem). From a quick computation, we find that the linear system ($K = 1$) exhibits a 60% overshoot. Hence, in its first cycle, the error stays in the linear region of the saturation so the response should be identical to the linear system response. The same argument applies to the rest of the response so we expect to see no difference between the linear and the nonlinear system responses.

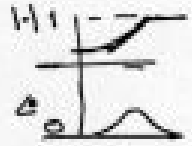
To analyze the large-reference case, observe that the critical value of the error is 2. Beyond this value the loop is unstable at least in a local sense. The error response to a step reference will reach the critical value at the end of the first cycle when the step amplitude is 2/0.3=3.3. For such a reference we expect to see significant differences between the linear and the nonlinear system responses.

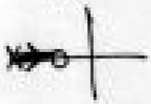
A Simulink simulation of the system is consistent with the above arguments. In fact, the nonlinear system becomes unstable for reference amplitudes beyond 3.1. This is because for errors larger than 1, the saturation decreases the apparent damping ratio and increases the overshoot.

[1] The small signal stability argument can be generalized to arbitrary systems and an at least conservative estimate of the stability region can be computed systematically. Instability conditions are much harder to obtain and the corresponding analysis often relies on the specific system structure.

EEE 480 HW # 7 SOLUTIONS

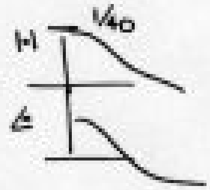
6.11

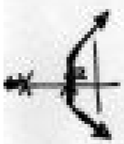
a) $K \frac{s+2}{s+10}$, $K > 0$. Bode 

R/L 

No phase crossover frequency, stable open loop

$\Rightarrow$ Gain margin $= \infty$ $\Rightarrow$ Closed loop stable for $K \in [0, \infty)$

b) $\frac{K}{(s+10)(s+2)^2}$, $K>0$, Bode 



Phase crossover at $\omega = 6.62$ where $|G| = -55.1$ dB

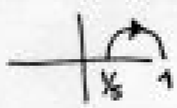
Open loop stable sys. $\rightarrow$ Gain Margin $-$ 55.1 dB or

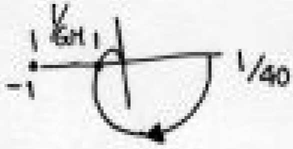
closed loop stable for $K \in [0, 568.8)$.

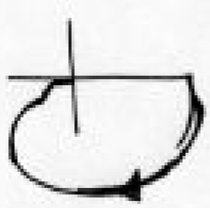
c). $K \frac{(s+10)(s+1)}{(s+100)(s+2)^3}$ No phase crossover $\Rightarrow$ Gain margin $= \infty$

+ Open loop stable $\rightarrow$ Closed loop stability for $K \in [0, \infty)$

6.14

a)  $(\omega > 0)$ stable for $K > 0$

b)  $(\omega > 0)$ stable for $K < GM \simeq 568$

c)  $(\omega > 0)$

Note: $|G|$ increases initially

but $\angle G \simeq \tan^{-1}\left(\frac{\omega}{1}\right) - 3\tan^{-1}\left(\frac{\omega}{2}\right)$

$\simeq \omega - \frac{3}{2}\omega \downarrow$

for small ω.

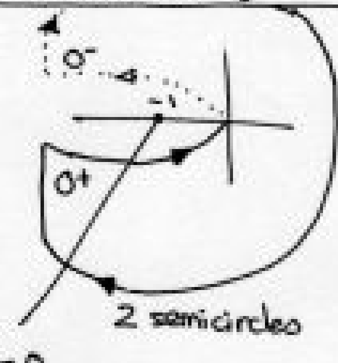
Stable for $K>0$

$\boxed{6.15}$

a) $\frac{s+1}{s^2(s+10)}$

RL

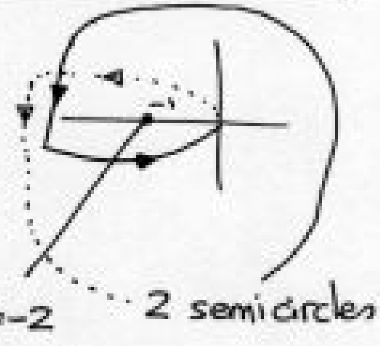
R-turn at 0^-



N=0

$\Rightarrow Z = 0+0.$

L-turn at 0^-

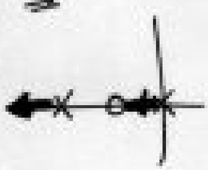


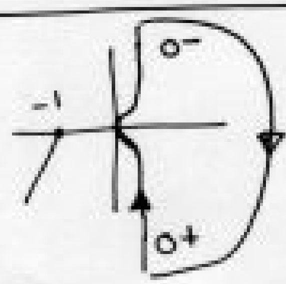
N=-2

$Z = -2+2 = 0$

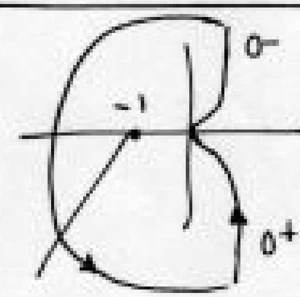
b) $\frac{s+1}{s(s+2)}$

RL





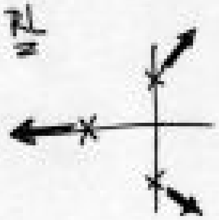
$N=0,\ Z=0+0$



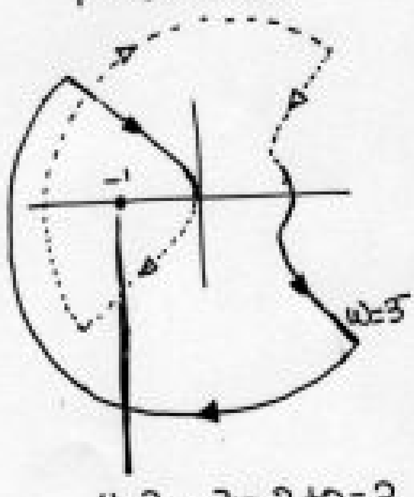
$N=-1,\ Z=-1+1=0$

c) $\frac{1}{(s+2)(s^2+9)}$

RL

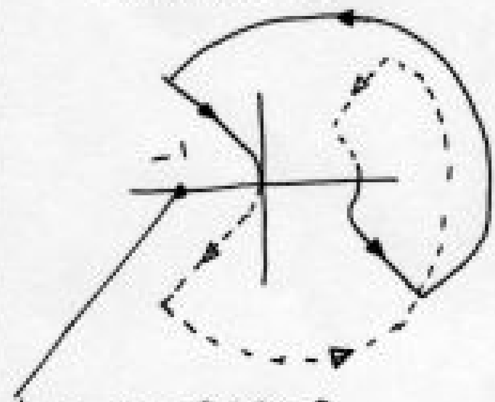


R-turn at 3^-



$N=2,\ Z=2+0=2$

L-turn at 3^-



$N=0,\ Z=0+2=2.$

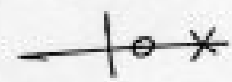
$\boxed{6.22}$

a) $n=1$, $m=1$, $n-m=0$ 

Lag, loss of phase

$G(s) \not\to 0$ as $s \to \infty$

b) same as a but magnitude increases while phase drops $\Rightarrow$ zero in RHP; pole must also be in RHP to return to the same phase at ∞.

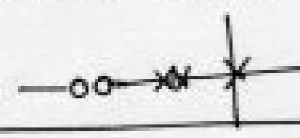


c) $n=1$, $m=1$, $n-m=0$ phase increase + mag. increase

$\to$ Lead 

d). 1 integrator (mag $\to \infty$ at $\omega \to 0$), two poles (angle drops 2 quadrants), one zero (angle comes to $-180°$), one zero (change of quadrant)

final angle $\to -90°$ $\Rightarrow$ 1 more pole than zeros.



$\boxed{6.23}$

I - b - 3 distance from -1 point; "well damped" closed-loop

II - c - 4 lightly damped, but the phase margin

III - a - 2 undamped response: Nyquist passes thru -1, Mag - closed loop $\to \infty$ at some ω.

IV - d - 1 Encircles -1 $\Rightarrow$ unstable closed-loop. d - because PM is $-$ve.

Note: We assume that the open-loop system has no poles in the RHP.

EEE 480 HW # 8 SOLUTIONS

NOTE: Ballpark computations relating bandwidth (w_{BW}) and crossover frequency (w_{GC}).

In classical compensator designs, it is often useful to convert closed-loop bandwidth specifications to crossover specifications. The closed-loop bandwidth is the frequency where the complementary sensitivity T drops below $\sqrt{0.5}$ of its initial (DC) magnitude. Recall that

$$T(s) = \frac{L(s)}{1 + L(s)}$$

where $L(s)$ is the loop transfer function (typically $L = GC$, G being the plant and C being the compensator transfer function).

In a first approximation, $w_{BW} \approx w_{GC}$. This is justified by the observation that the loop transfer function magnitude decreases (usually rapidly) past the crossover frequency. While this is a good ballpark estimate, the two frequencies can differ by approximately a factor of 2.

For a more refined approximation, we can use some assumptions on the rate of decay and the phase of the loop transfer function to develop a relationship between w_{BW} and w_{GC}. Since such a relationship is used before the compensator is designed, it is important to express our assumptions in terms of quantities that appear in the typical design specifications.

First, observe that at the crossover frequency

$$|T(jw_{GC})| = \left| \frac{L(jw_{GC})}{1 + L(jw_{GC})} \right| = \frac{1}{|1 + \cos\phi + j\sin\phi|}$$

where $\phi = 180 - PM$, in degrees. Furthermore,

$$\frac{|T(jw_{BW})|}{|T(jw_{GC})|} = \frac{|L(jw_{BW})|}{|L(jw_{GC})|} \frac{|1 + L(jw_{GC})|}{|1 + L(jw_{BW})|}$$

For simplicity, we take the last factor to be approximately 1. Also, for typical designs, $L(jw)$ decreases with a rate around 20db/dec around the crossover frequency. This implies that

$$\frac{|L(jw_{BW})|}{|L(jw_{GC})|} \approx \frac{w_{GC}}{w_{BW}}$$

(Depending on prior knowledge, one can use the expression $(w_{GC}/w_{BW})^m$ where $m \approx 0.5 - 1.5$ for a 10-30db/dec, respectively.) Substituting these expressions in the definition of the bandwidth $|T(jw_{BW})| = \sqrt{0.5}|T(0)|$, we get

$$w_{BW} \approx w_{GC} \frac{\sqrt{2}}{\sqrt{(1 + \cos\phi)^2 + \sin^2\phi}} \frac{|1 + L(0)|}{|L(0)|}$$

When the plant and/or the controller contain an integrator, this expression simplifies to

$$w_{BW} \approx w_{GC} \frac{\sqrt{2}}{\sqrt{(1 + \cos\phi)^2 + \sin^2\phi}}$$

For example, for a 40^o phase margin the last expression yields $w_{BW} \approx 2w_{GC}$, while for a 60^o phase margin $w_{BW} \approx \sqrt{2}w_{GC}$.

As usual, these expressions are not exact and have a limited range of validity. Their best use is often to adjust the crossover frequency selection in the second design iteration.

Problem 6.51

The root-locus shows 4 asymptotes at 90^o angles. The maximum gain for stability is 2.74 (from the MATLAB plot, or a Routh array computation). For the stabilizing gains, the dominant pole-pair bandwidth does not exceed 1, so for a closed-loop bandwidth of 10 we need to design a lead compensator.

Let us first select a phase margin of 50^o for the closed loop system. Using the previous approximation we select the crossover frequency as $w_{GC} = 10/1.67 \approx 6$. At that frequency, the plant transfer function has phase -322^o and, therefore, we need 192^o of phase lead. For this we need at least 3 lead elements and we select a compensator of the form

$$C(s) = K \left(\frac{s + z}{s + p} \right)^3$$

As usual $\sqrt{zp} = w_{GC} = 6$, for maximum efficiency, while each element should contribute $192/3 = 64^o$ of phase lead. From the latter, $\sqrt{z/p} = 0.231$ yielding

$$\begin{aligned} z &= 1.385 \\ p &= 26 \end{aligned}$$

From the Bode plot of CG with $K = 1$, we find that the compensated transfer function has amplitude -101db at $w = 6$, so $K = 112200$. Then, the final compensator is

$$C(s) = 112200 \left(\frac{s + 1.385}{s + 26} \right)^3$$

The closed-loop frequency response with this compensator has a bandwidth of 11.3 and its step response shows a 20% overshoot, 0.18 rise time and 0.9 settling time (all units SI).

Problem 6.52

The K_v specification requires that the loop transfer function (plant and compensator) satisfy $sL(s)|_{s=0} = 12$. For a pure gain compensator, this would require $K = 12$, resulting in an unacceptable phase margin of 16^o. We now need to design a lead compensator to increase the PM, without affecting the low-frequency portion of the plant (with $K = 12$). Using the corresponding lead design procedure we find that the actual PM (before compensation) is 16^o. Adding a 6^o safety (since the crossover frequency will move slightly after compensation) we need a phase lead of $40 - 16 + 6 = 30^o$. From this angle we determine that $\sqrt{z/p} = 0.577 = -4.77$db. We now find the new (compensated) crossover frequency as the frequency where $|KG(jw)| = -4.77$db. From the Bode plot of KG, this frequency is 4.5. Hence, $\sqrt{z/p} = 4.5$ from which,

$$\begin{aligned} z &= 2.6 \\ p &= 7.8 \end{aligned}$$

To leave the low frequencies unaffected, the compensator is implemented in the time constant form, i.e.,

$$C(s) = 12 \frac{0.385s + 1}{0.128s + 1}$$

The compensated system has $PM = 43^o$ and $K_v = 12$, as specified ($G(s) = 1/(s^2 + s)$).

Notice that these specifications are not complete as they allow for possibly undesirable designs. For example, the compensator $C(s) = \frac{s+0.01}{s}$ provides 50^o phase margin and trivially satisfies the K_v specification ($K_v = \infty$). However, its closed-loop response has much larger rise-time and exhibits a very slow final convergence to the steady-state.

For an alternative approach, we can attempt to convert the K_v specification to a bandwidth (and crossover) specification. This is with the understanding that such a design will be by nature iterative, especially if the K_v specification must be met exactly. To get a ballpark figure for the bandwidth, we can use the K_v interpretation as a steady-state error to ramp commands and look at a first order closed-loop approximation $BW/(s + BW)$. This suggests that $BW \approx 12$ and, from the previous bandwidth-crossover relations, $w_{GC} \approx 6$. We now perform a lead design to provide $PM = 40^o$ at $w = 6$. The required phase lead is 30^o, for which $\sqrt{z/p} = 0.577$. After computing the pole, zero, and gain, the final compensator is

$$C(s) = 63.7 \frac{s + 3.46}{s + 10.4}$$

This compensator results in $K_v = \frac{63.7 \times 3.46}{10.4} = 21.2$. This is considerably higher than requested. In this problem, it is easy to adjust the compensator in order to meet the K_v specification exactly. This is because lower gains correspond to larger phase margins. For example, we may now reduce the compensator gain by $12/21.7$ to get

$$C(s) = 35.2 \frac{s + 3.46}{s + 10.4}$$

This compensator results in $K_v = 12$ and $PM = 42^o$, as requested.

Problem 6.72

This problem shows how poorly stated objectives in feedback systems can easily produce ill-posed problems. Here, the delay contributes a lot of phase lag without the corresponding amplitude attenuation. Then, adding phase lead can produce the desired phase at a given frequency but the magnitude will be constant or increasing. From the Nyquist plot, it is obvious that such a controller will destabilize the system. Since the pole/zero ratio in the controller is fixed, there is only limited freedom on the application of the phase lead.

In addition to that, the phase margin no longer represents a good estimate of the Nyquist distance from the -1 point. This allows large peaks in the sensitivity and complementary sensitivity for the same PM specification. In turn, the usual bandwidth and crossover frequency interpretations are also lost.

In order to analyze the properties of the various designs more easily, it is convenient to work with the Pade approximation of the delay. This allows the use of standard Matlab commands to form the closed-loop system and check step and frequency responses (otherwise, the closed-loop must be simulated in Simulink). The results obtained with Pade approximation are qualitatively similar to the pure delay case but differences of 20% are not uncommon.

Let us discuss the case of the single lead compensator first. Let $C(s) = \frac{s/z+1}{s/(100z)+1}$. Observe that when the compensator zero z is greater than 10, then the loop gain $|CG|$ is monotonically decreasing. There is a single crossover frequency with the usual PM interpretation. Keeping the PM the same (40^o), the compensator gain can be determined from the Bode plot of $CG(s)$ for each value of z. Notice that the crossover frequency is computed only numerically. The maximum crossover frequency is obtained for $z \to 10$ and is near 26 with a compensator gain slightly above 10. This produces an extremely oscillatory closed-loop system since the distance of the Nyquist plot from -1 becomes arbitrarily small.

On the other hand, $z < 10$ produces a loop gain that peaks at some frequency past z. From the Nyquist plot, the only possibility to stabilize this system is to choose a small-enough gain so that the entire Nyquist plot is inside the unit circle. There is no crossover frequency for this design. Also, the closed-loop system is quite different from the typical second-order transfer function so any bandwidth considerations are irrelevant.

The situation with second order lead is even worse. Essentially the only possibilities for stabilization are: 1) small gain, or 2) very large z rendering the lead compensator virtually ineffective.

With the small modification this problem can become meaningful. Suppose that the transfer function is $G(s) = \frac{e^{-0.2s}}{s(s+10)}$. Now, the lead compensator is indeed needed to increase the closed-loop bandwidth with adequate phase margin. For the Pade approximation, the maximum phase from the compensator (78.6^o) can be used to provide 40^o PM up to a crossover frequency of 9.2. The corresponding gain is 12.6. Even though this design is far more reasonable than the previous ones, it is still quite oscillatory due to the proximity of the -1 point to the Nyquist plot (the PM is again a poor indicator of stability margin). For the system with the pure delay the same computations result in a slightly smaller crossover bound (7.8).

Problem 6.74

The magnitude of the pure delay transfer function is 1 at all frequencies. Its phase is linear ($-w\tau$ in rad/s). Its Pade approximation has the same magnitude (1 at all frequencies, an "all-pass" transfer function). The phase of the Pade approximation is $-2\tan^{-1} w\tau/2$. At $1/\tau$ their difference is reasonably small (about 5^o) but at $2/\tau$ there is a significant deviation of about 20^o.

Pade approximations should be used very carefully so that their limits of validity are not exceeded.

Problem 6.77

From the frequency response of the system, the PM specification requires a lead compensator providing 75^o phase lead at 10 rad/s. This implies that $\sqrt{zp} = 10$ and $\sqrt{z/p} = 0.132$. At that frequency the plant magnitude is approx. 1.5 so the compensator gain is $\frac{1}{(1.5)(.132)} = 5$. The final compensator is

$$C(s) = 5\frac{s+1.32}{s+76}$$