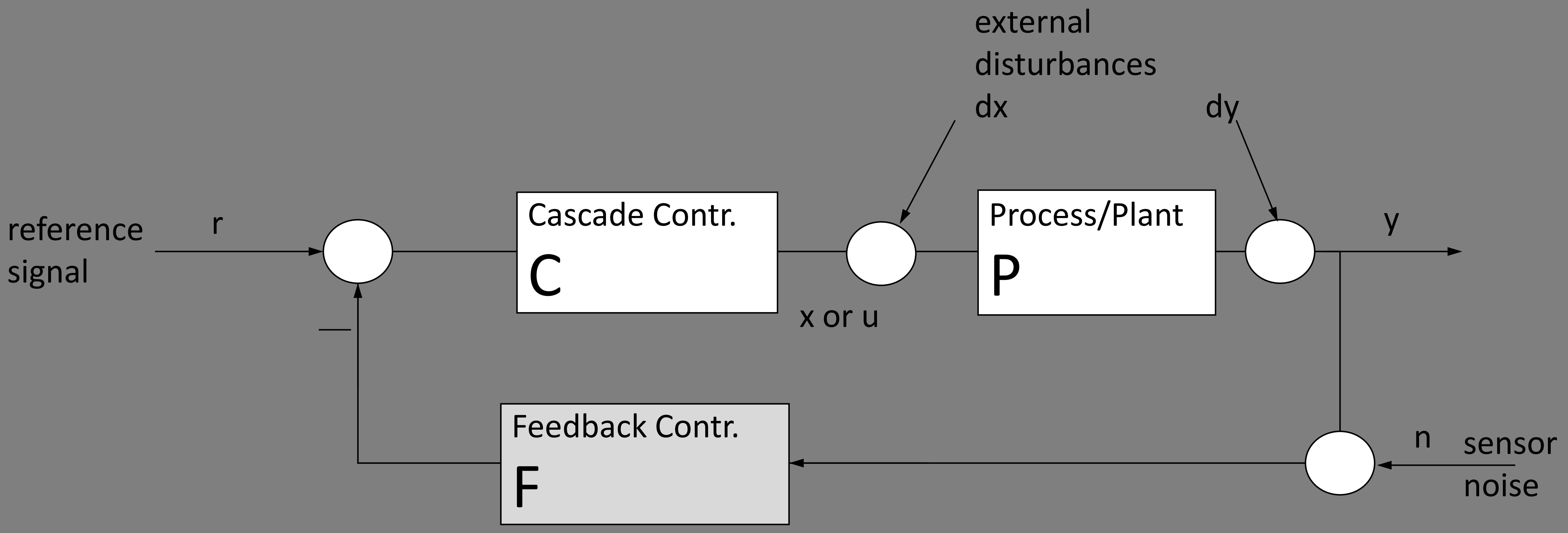
EEE304

Week 6: Feedback Systems II



Arizona State University

EEE304

Week 6: Feedback Systems II

Learning Objectives:

- Tune PID controllers for various cases of plants arising in practice
- DT controller implementation
    - Performance deterioration due to discretization
    - Adjustment of control specifications for DT implementation
    - All-discrete controller design
- Bandwidth Selection
    - Constraints from fundamental performance limitations
    - Constraints from model uncertainty

ARIZONA STATE UNIVERSITY

EEE304

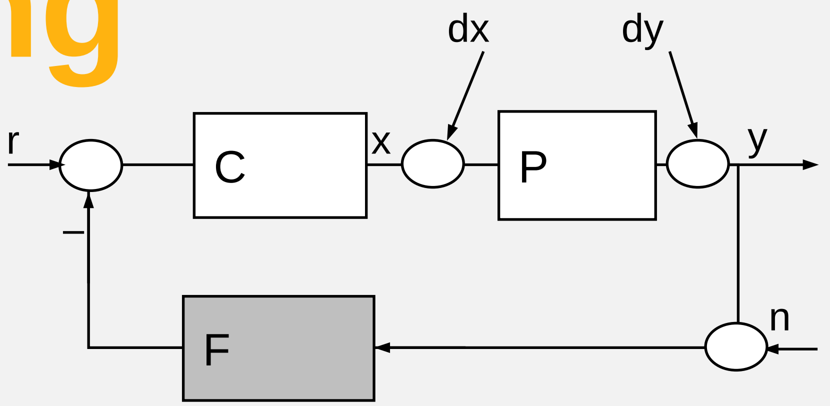
Lecture 6.1: Feedback Control Limitations and PID Controller Tuning

Arizona State University

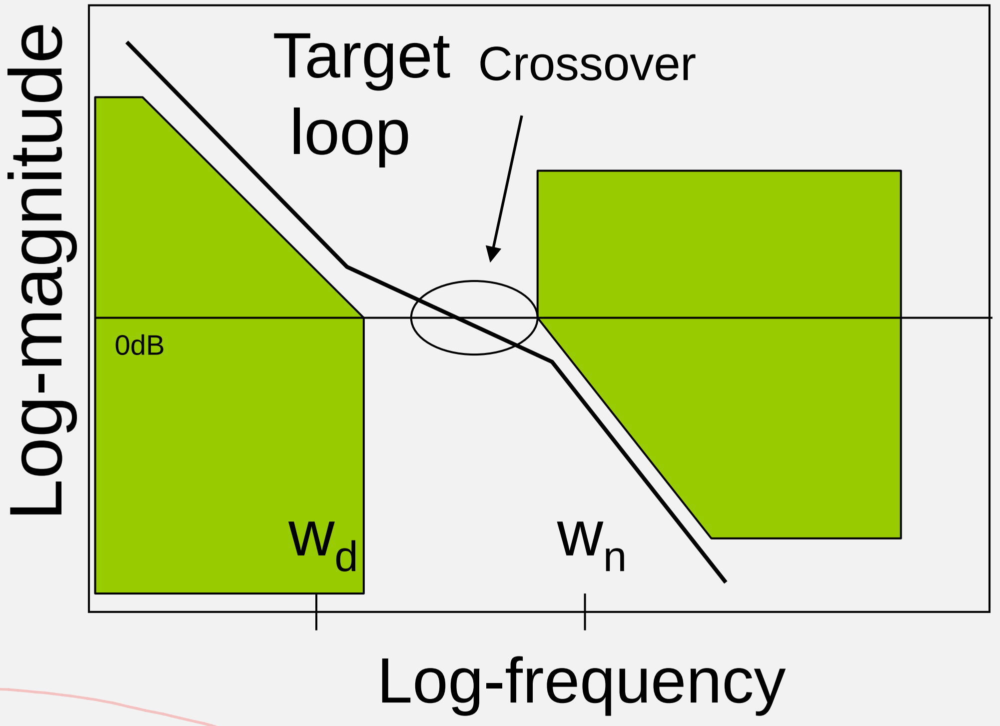
PID Tuning Overview

- Systematic algorithms for clear-cut cases.
- Experience and trial-and-error are required for the more complicated cases that are in the fringes of PID capabilities.
- Need to keep in mind the Fundamental Performance Limitations of feedback as well as the way PIDs can affect the closed-loop transfer function (poles, zeros) in order to pose a meaningful problem.
- A study of higher order case studies helps to establish the tuning procedure and reveal the potential adjustments to extend its domain.
- Also beneficial in addressing the controller discretization problem.

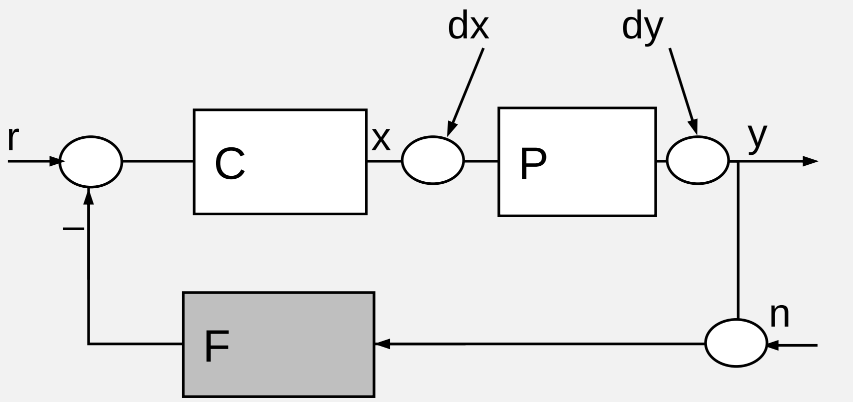
Essential Loop-Shaping for PID Tuning



1. Select the gain crossover frequency and Phase Margin specifications to satisfy the closed-loop control objectives.
    - Translate objectives to crossover/PM.
    - Iterate if necessary, observing limitations.
2. Compute the controller order and zero(s) from the PM inequality $\angle PC(j\omega_c) = \angle P(j\omega_c) + \angle C(j\omega_c) \geq -180^o + PM$
    - Do not use RHP zeros, avoid cancellations within the closed-loop bandwidth.
3. Adjust the controller gain from the crossover equation $|PC(j\omega_c)| = 1$ $\Rightarrow K = \dots$
4. Evaluate the various aspects of closed-loop performance, adjust crossover/PM and iterate.
    - Do not adjust gains directly.



Objectives and tuning parameters

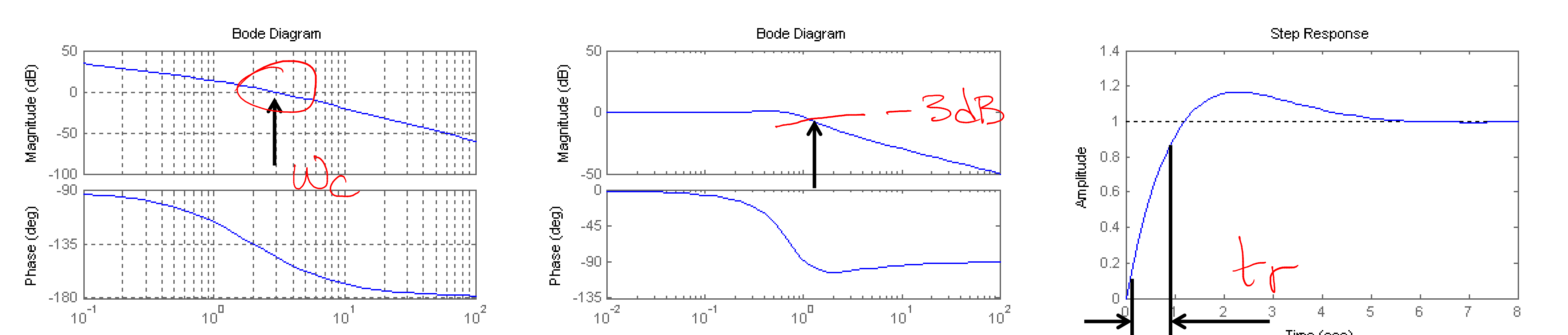


- Closed loop bandwidth (from practical constraints, rough estimate of sampling time) to crossover frequency:

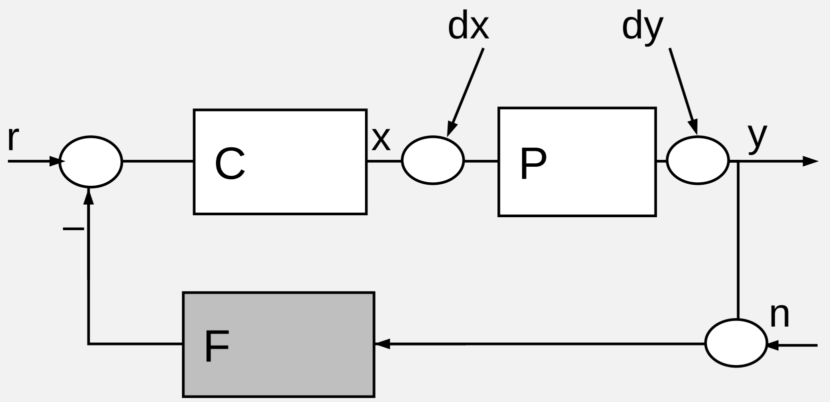
$$BW \approx 1.5\omega_c, \quad BW \approx \frac{2}{t_r}, \quad T_s \approx \frac{t_r}{6}$$

- Fundamental limitations (soft): $|RHP\ poles| < BW < |RHP\ zeros|$
- Performance objectives: $|Disturbance\ BW| < BW < |Noise\ BW|$
- Iterative adjustment: To improve disturbance attenuation, settling time, increase the crossover frequency. To reduce noise at the output, decrease crossover.

Gain Crossover and related specs.
Note:
Crossover/Bandwidth parameter first to define, last to compute.

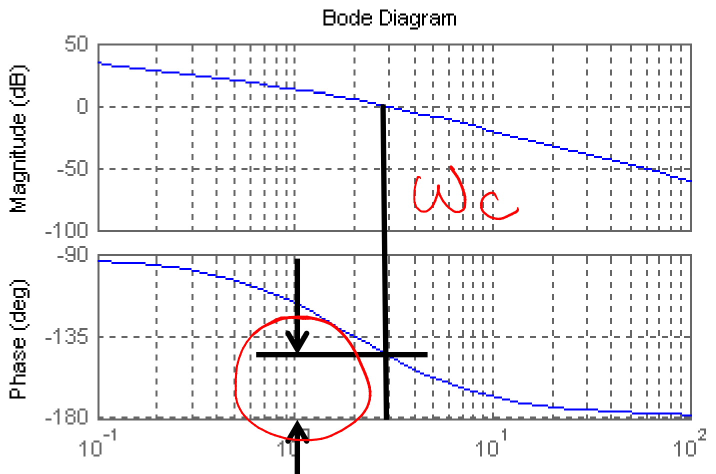


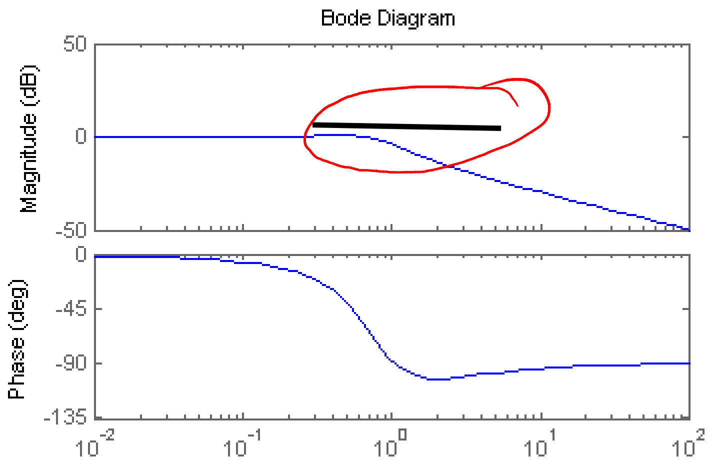
Objectives and tuning parameters

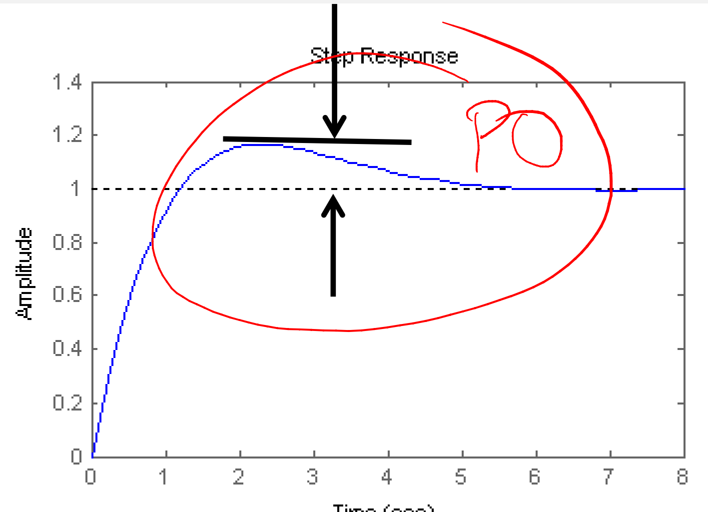


- Magitude Peak and Percent Overshoot have an inverse relationship to damping ratio and Phase Margin. PM = 60 deg. is a common first choice, motivated by optimal control results. Adjust iteratively by the same rule.
- Peak/PO are also affected by poles and zeros inside the closed loop bandwidth (if the controller requires a zero inside the BW, use a more conservative PM.)
- Large PM may be impossible to achieve for a given BW.
  - A compromise between PM and BW may be necessary in some cases. This is not entirely obvious by the PM/crossover equations which fix one point on the Nyquist plot, ignoring the rest.

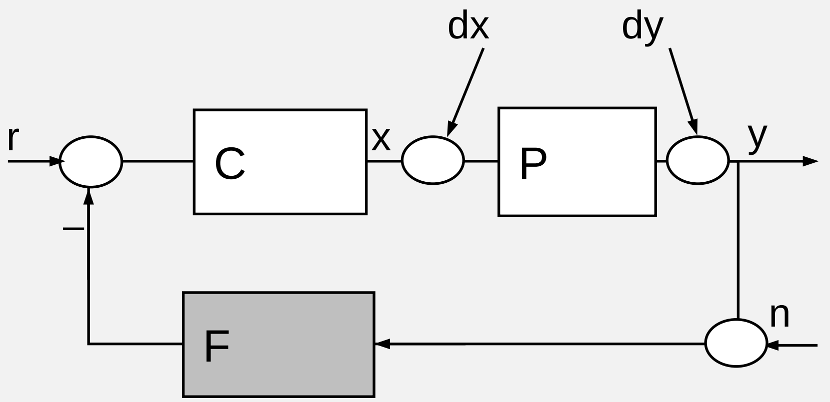
Phase Margin and related specs.
Note:
PM defines PID order and zero, and is the first to compute.





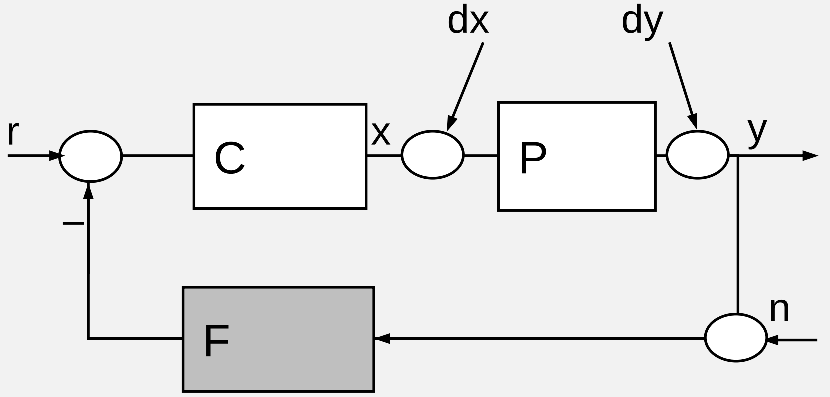


Objectives and tuning parameters



- Additional filters can be used to improve a loop shape but exercise caution with cancellations. E.g.,
  - Use a lowpass filter to attenuate resonant peaks (from flexible modes outside BW) or low bandwidth zeros (from inner-loop compensation inside BW).
  - Approximate cancellations of lightly damped modes may be possible (e.g., with a better damped term) but the design should account for inexact cancellations and nonzero residues.
  - Do not cancel poles more than an order of magnitude smaller than BW
  - Never cancel RHP modes!
- In the sequel, we analyze some case studies to illustrate these points.

Case Study 1: RHP Zeros/Delays

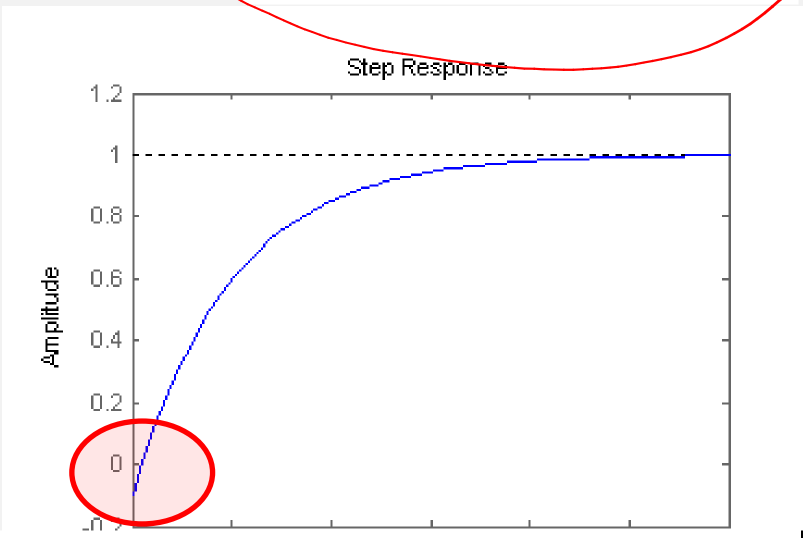


$$P(s) = \frac{-0.1s+1}{s+1}$$

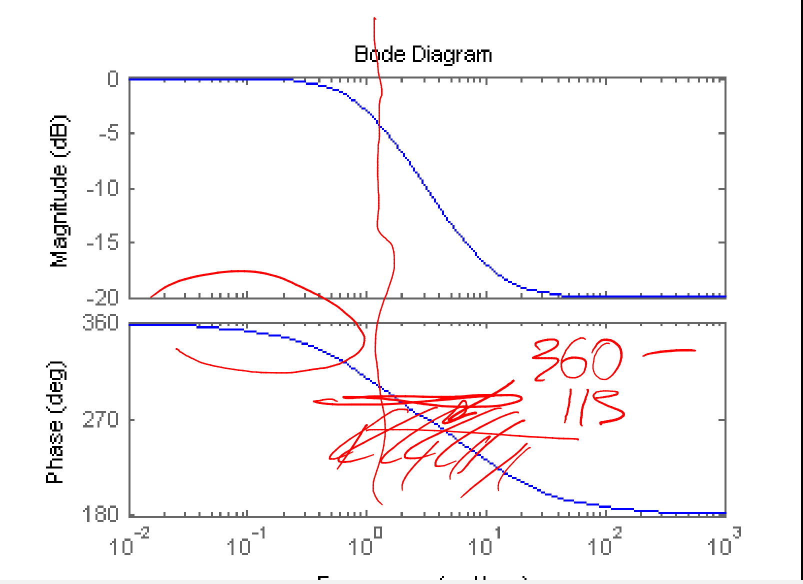
- Design a PID controller for a process with "inverse response"

$$P(s) = \frac{-\varepsilon s+1}{\tau s+1}$$

  - RHP zeros/delays appear in certain problems e.g., transportation delay, mixing, cart-and-pendulum, aircraft flight.



- From the Bode plot, the design is straightforward for low BW but becomes challenging for BW above 1



$$-180^o + PM \le \angle P + \angle C = \tan^{-1}(-\varepsilon\omega_c) - \tan^{-1}(\tau\omega_c) - 90^o + N\tan^{-1}(\tau_z\omega_c) - \tan^{-1}(\tau_p\omega_c)$$

INT

  - For PID, N=2 and $\tau_p > 0$. Then $\tan^{-1}(\tau_p\omega_c) \approx 6^o$, and expect at most N x 75 = 150 deg. from the zeros. Adding up the terms,

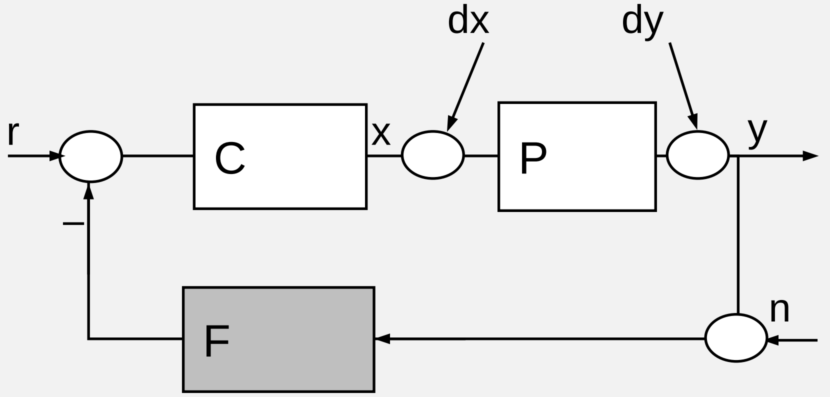
0.1 rad

$$-180^o + PM \le -90^o - 90^o - 90^o + 2\times 75^o - 6^o \Rightarrow \boxed{PM \le 54}$$ deg

  - A PI would limit the crossover $PM \le \angle P + 165^o : PM = 50 \Rightarrow \angle P > -115^o \Rightarrow \omega_c < 6.6 rad/s$

N = 1

Case Study 1: RHP Zeros/Delays



$$P(s) = \frac{-0.1s+1}{s+1}$$

$$C(s) = 1.4\frac{0.2s+1}{s}$$

- Let us start with a PM = 50deg. objective and BW = 1.5rad/s
  - In all cases, we will use the approximate BW-crossover relation as "exact".
- We compute zeros from the angle equation

$$-180^o + 50 \le \tan^{-1}(-0.1\times 1) - \tan^{-1}(1\times 1) - 90^o + N\tan^{-1}(\tau_z \times 1) - \tan^{-1}(\tau_p \times 1)$$

ω_c

- For PID, N=2 and $\tau_p = 0.1$. Then $2\tan^{-1}(\tau_z\omega_c) = 17^o$, so a PI would do.
- For PI, $\tan^{-1}(\tau_z\omega_c) = 11^o \Rightarrow \tau_z = 0.2$. Substituting into the crossover equation, we compute the PI gain:

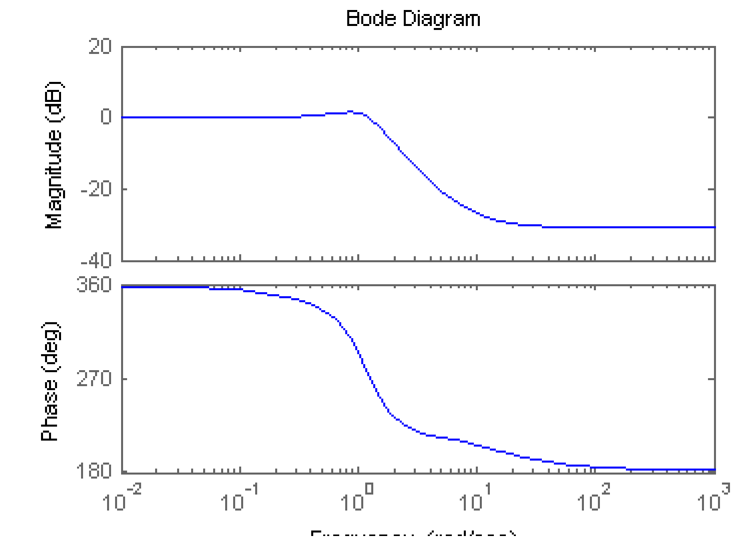
$\tau_p = 0$

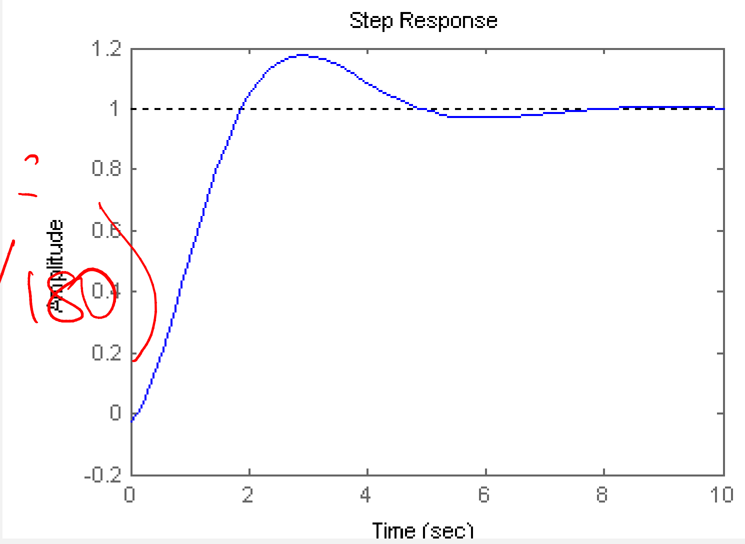
tau (11^o) ω_c

$$|P(j\omega_c)C(j\omega_c)| = \frac{|-0.1j+1|}{|j1+1|}|K|\frac{|0.2j+1|}{|j1|} = 1 \Rightarrow K = 1.4$$

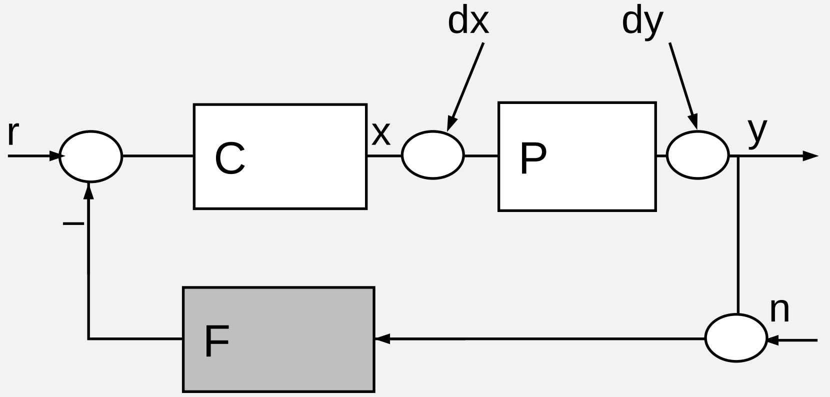
- Next, we form the controller and evaluate the closed-loop response.

in Matlab?
tau(11 × pi/180)





Case Study 1: RHP Zeros/Delays



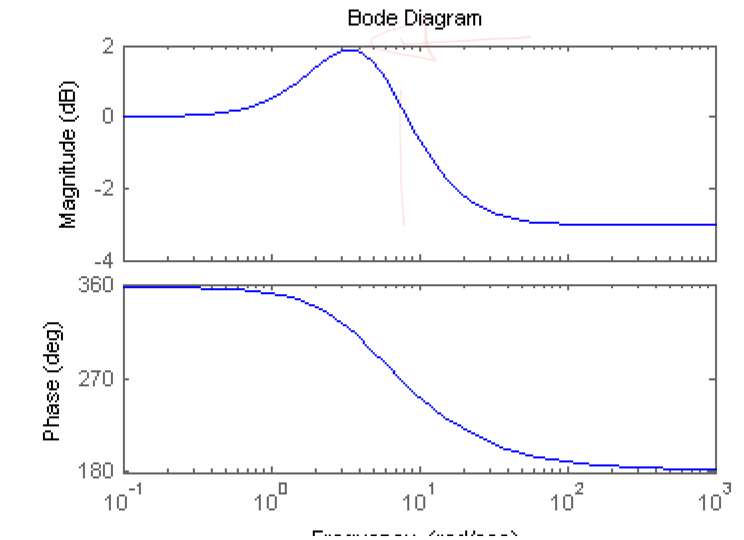
- Now, let PM = 50deg. and BW = 7.5rad/s.
- We compute zeros from the angle equation

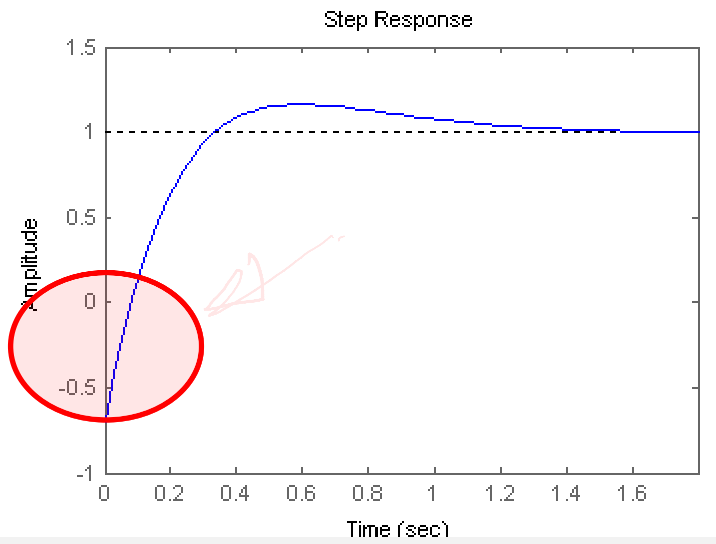
$$-180^o + 50 \leq \tan^{-1}(-0.1\times 5) - \tan^{-1}(1\times 5) - 90^o + N\tan^{-1}(\tau_z \times 5) - \tan^{-1}(\tau_p \times 5)$$

  - For PI, $\tan^{-1}(\tau_z \omega_c) = 65^o \Rightarrow \tau_z = 0.43$. Substituting into the crossover equation, we compute the PI gain, 9.6.
  - Next, we form the controller and evaluate the closed-loop response.
  - Notice the severe inverse response, an indication we are approaching the limitation from the RHP zero (at 10rad/s).

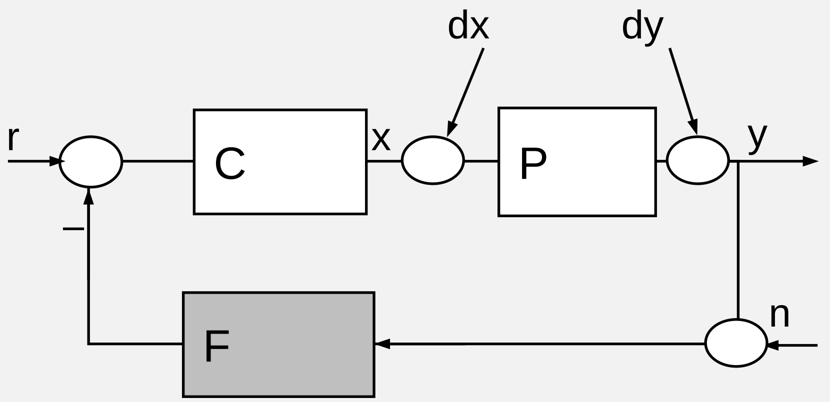
$$P(s) = \frac{-0.1s+1}{s+1}$$

$$C(s) = 9.6\frac{0.43s+1}{s}$$





Case Study 1: RHP Zeros/Delays



$$P(s) = \frac{-0.1s+1}{s+1}$$

$$C(s) = \frac{0.3882\,s^2 + 7.115\,s + 32.6}{0.01s^2 + s}$$

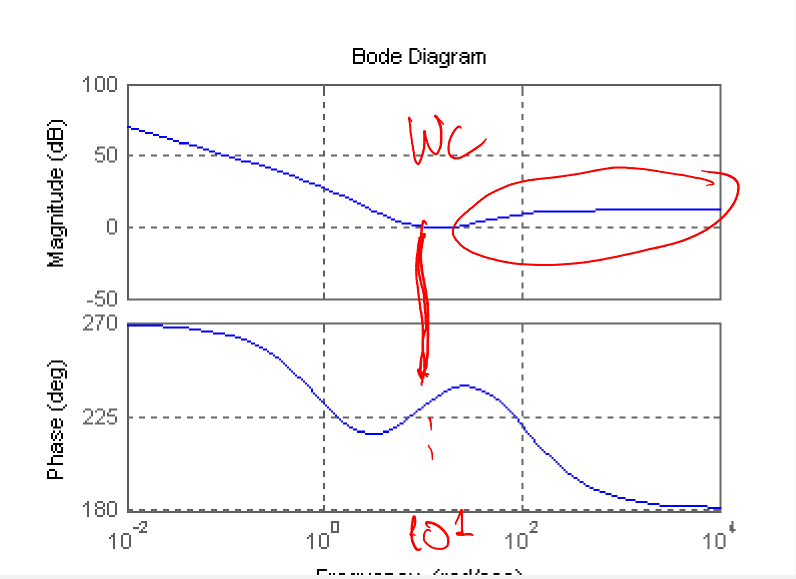
- What if PM = 50 and BW = 15? The PID phase says that it is feasible, but the fundamental limitation indicates a problem. We compute zeros from the angle equation

$$-180^o + 50 \le \tan^{-1}(-0.1\times 10) - \tan^{-1}(1\times 10) - 90^o + N\tan^{-1}(\tau_z \times 10) - \tan^{-1}(\tau_p \times 10)$$

ω_c

$$-180^o + 50 \le -130^o - 90^o + N\tan^{-1}(\tau_z \times 10) - 6^o$$

- For PID, $\tan^{-1}(\tau_z\omega_c) = 96^o/2 \Rightarrow \tau_z = 0.11$. Substituting into the crossover equation, we compute the PID gain, 32.6.
- Next, we form the controller and evaluate the closed-loop response. The loop is unstable and the problem can be seen in the Bode plot of PC. We have achieved -130deg phase at 10, but the magnitude does not roll-off, creating Nyquist encirclements.



Case Study 2: RHP Poles

$$P(s) = \frac{1}{s^2 - 10}$$

PID $\rightarrow$ $$C(s) = \frac{10.1s^2 + 62.3s + 96}{0.01s^2 + 1}$$

- Design a PID controller for an inverted pendulum. The inverted pendulum with a point mass satisfies

$$J\ddot{\theta} = T + mgL\sin\theta \approx T + mgL\theta \Rightarrow P(s) = \frac{1/mL^2}{s^2 - g/L}$$

External   Gravity $\sqrt{g/L}$

- From the Bode plot, the phase is constant at -180deg, so a PID is required. There is no indication where the crossover should be, other than the fundamental limitations.

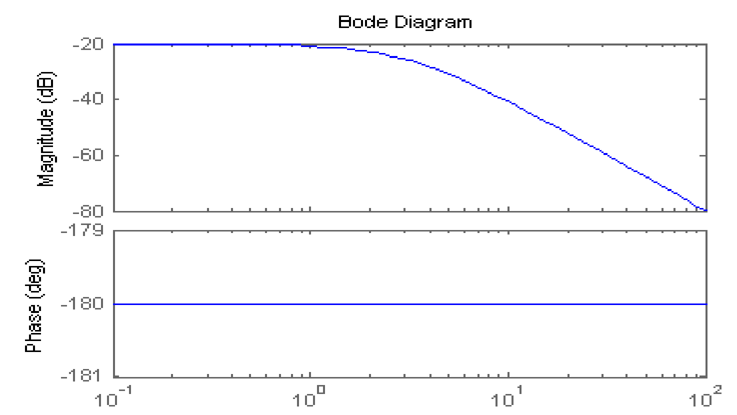
- Writing the PID phase equation for PM = 60deg., BW=15rad/s:

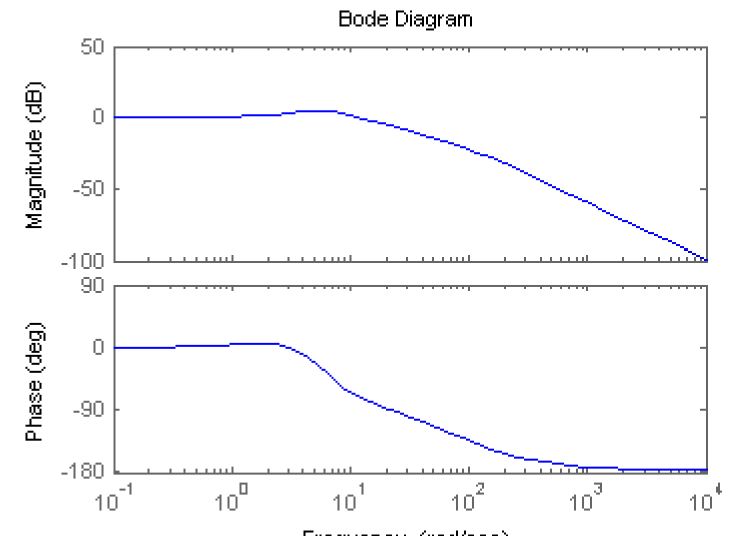
$$-180^o + PM \le -180^o - 90^o + 2 \times \tan^{-1}(\tau_z \omega_c) - 6^o \Rightarrow \tan^{-1}(\tau_z \omega_c) = 146/2$$

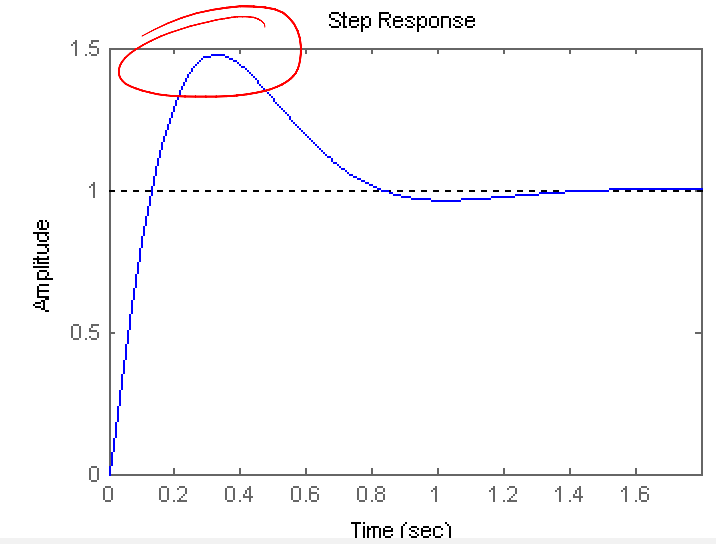
$$\Rightarrow \tau_z = 0.33, \quad |PC(j\omega_c)| = 1 \Rightarrow K = 96$$

$\omega_c = 10$

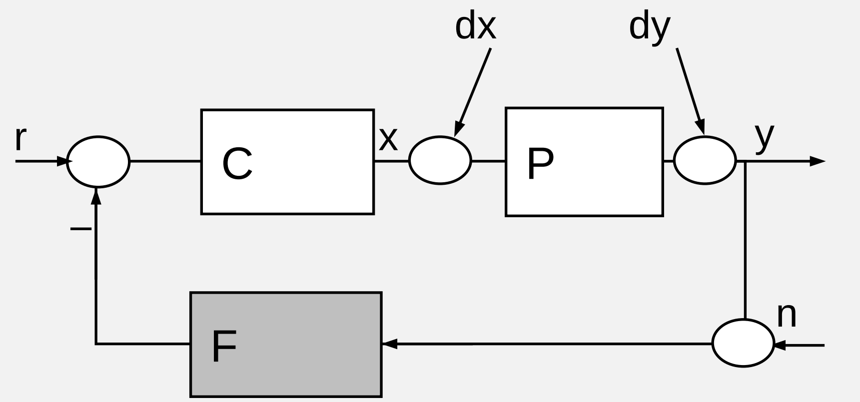
- We evaluate the closed loop response and we find an increased overshoot due to the zero inside the BW (at -3rad/s).







Case Study 2: RHP Poles



$$P(s) = \frac{1}{s^2 - 10}$$

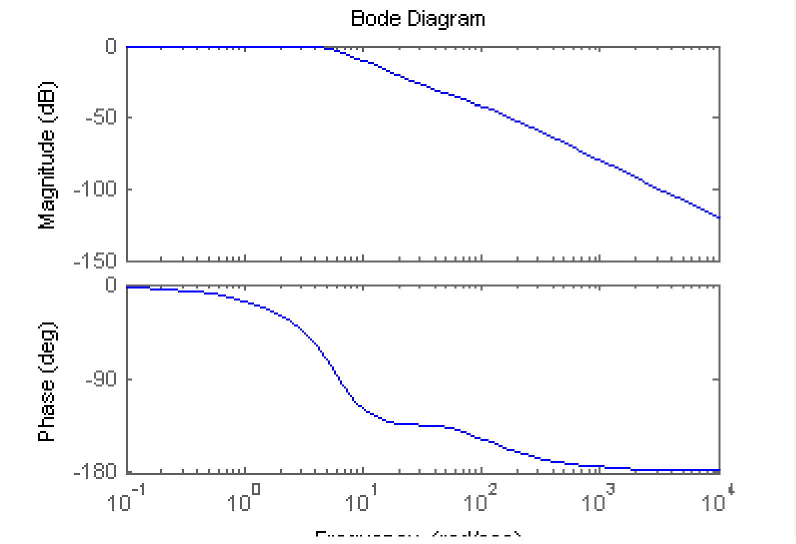
$$C(s) = \frac{10.1s^2 + 62.3s + 96}{0.01s^2 + 1}$$

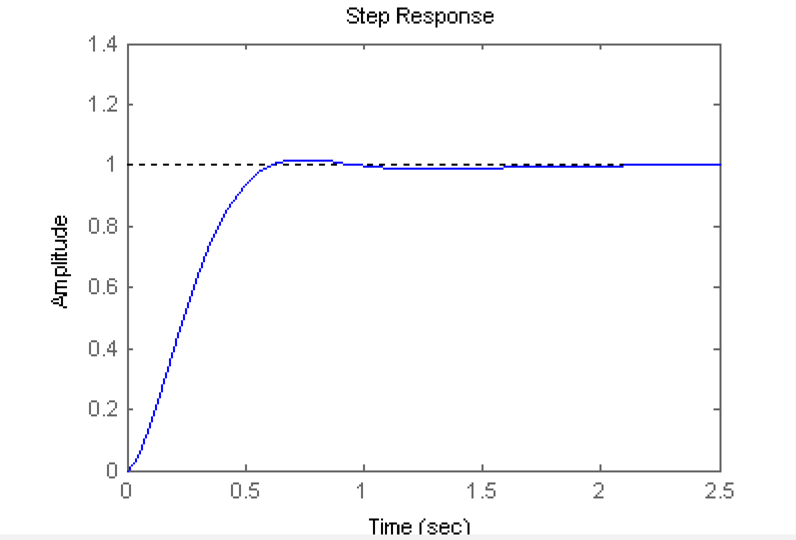
Comments:

$\sqrt{20}$

- The RHP pole is at the famous $\sqrt{\frac{g}{L}}$, showing why a human with BW around 1Hz can stabilize a 1m stick (pole at 4.5rad/s), but not a ¼ m pencil. The stabilization task becomes easier with friction that shifts the unstable pole left by about $-\varepsilon/2$.
- The resulting BW is 16, a good approximation of the desired 15rad/s. A second iteration could start with BW 15x15/16.
- A good exercise here is to design a command prefilter to avoid the 50% overshoot. We choose a pole and a zero around the peak magnitude frequency and after some trial-and-error we arrive at a good compromise for which overshoot is virtually none.

$$F_i(s) = \frac{0.04s+1}{0.4s+1}, \quad C \leftarrow C(s)F_i\ (s), \quad F(s) = \frac{0.4s+1}{0.04s+1},$$

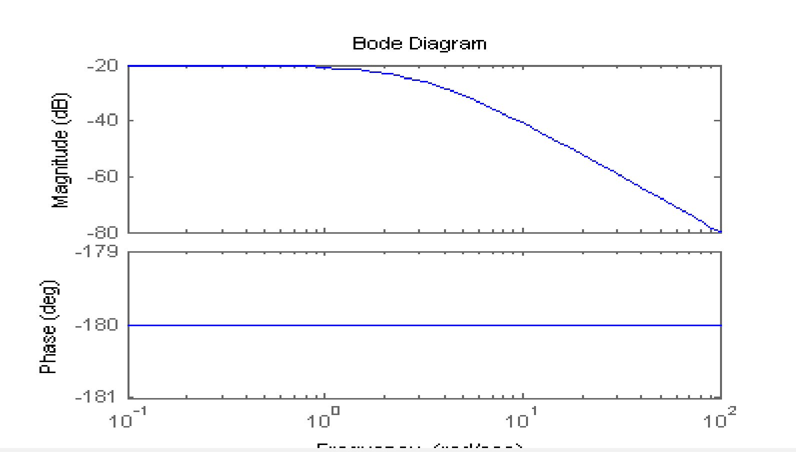


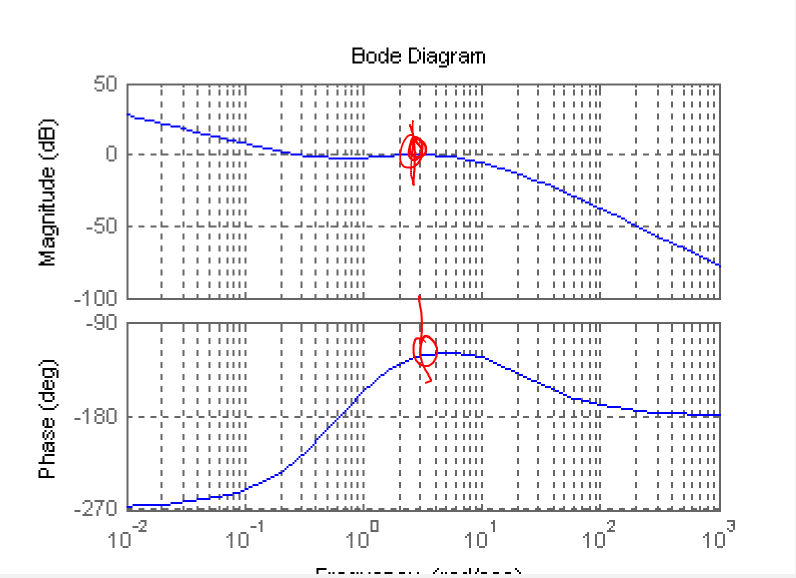


Case Study 2: RHP Poles

$$P(s) = \frac{1}{s^2 - 10}$$

$$C(s) = \frac{6.4s^2 + 7.9s + 2.4}{0.05s^2 + 1}$$





- Let us now attempt a controller design with BW at 3.
- The PM equation is the same, except now the crossover is at 2rad/s.

$$-180^o + PM \le -180^o - 90^o + 2\times\tan^{-1}(\tau_z\omega_c) - 6^o \Rightarrow \tan^{-1}(\tau_z\omega_c) = 146/2$$

$\omega_c = 2$

$$\Rightarrow \tau_z = 1.6,\ K = 2.4$$

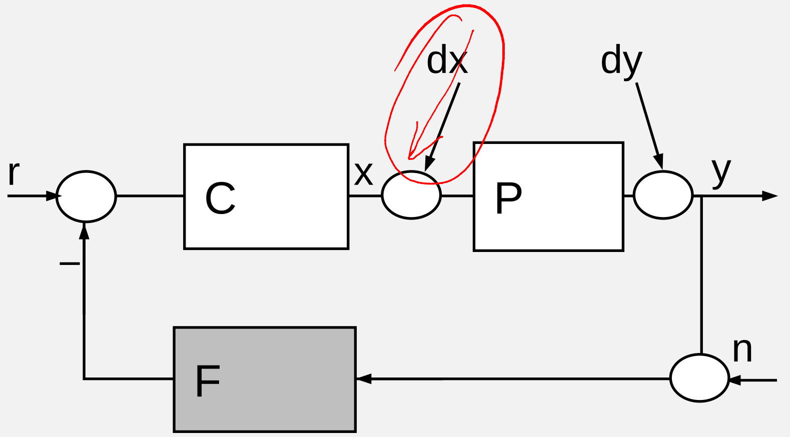
- We evaluate the closed loop response and, again, it is unstable. We have achieved the desired phase at the desired crossover but this controller has created another crossover frequency that changed the Nyquist encirclements.
- An obviously difficult problem: Design a PID to keep the BW low in the presence of significant delay.

EEE304

Lecture 6.2: Common extensions for PID Controller Tuning

ARIZONA STATE UNIVERSITY

Case Study 3: Cancellations



$$P(s) = \frac{1}{(s+0.1)(0.1s+1)}$$

$$C_n(s) = 9.6\frac{0.43s+1}{s}$$

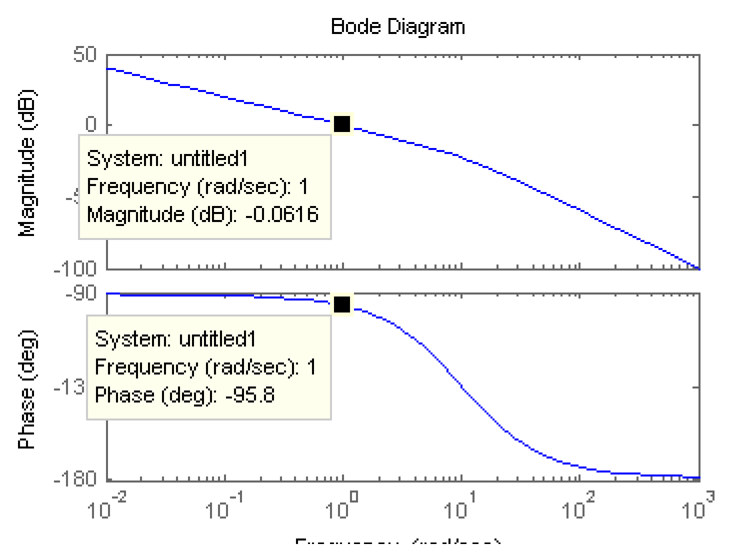
- Let BW = 1.5. Suppose that the plant has a slow mode, relative to the crossover.

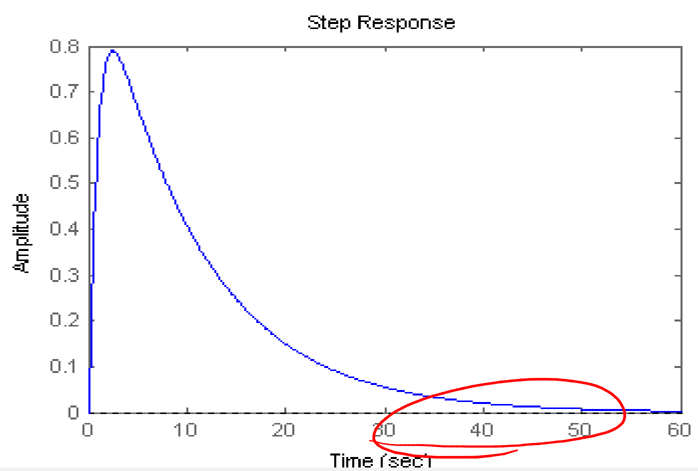
$$P(s) = \frac{1}{(s+0.1)(0.1s+1)}$$

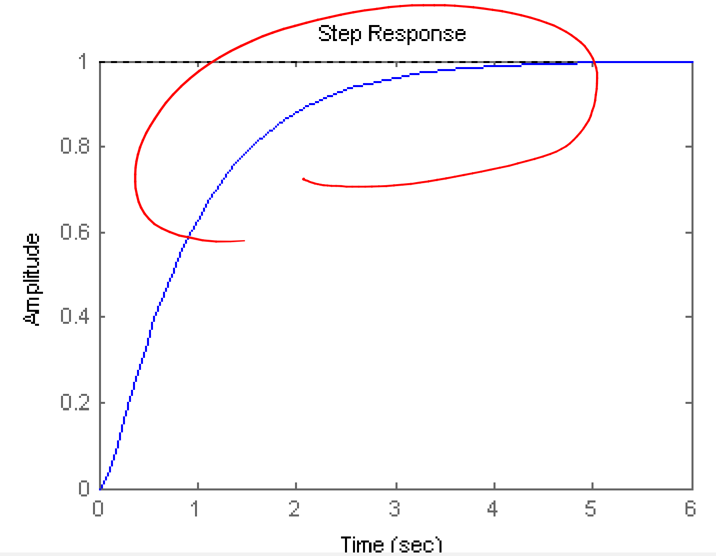
- It is tempting to try a "naïve" PID of the form

$$C_n(s) = K\frac{s+0.1}{s}$$

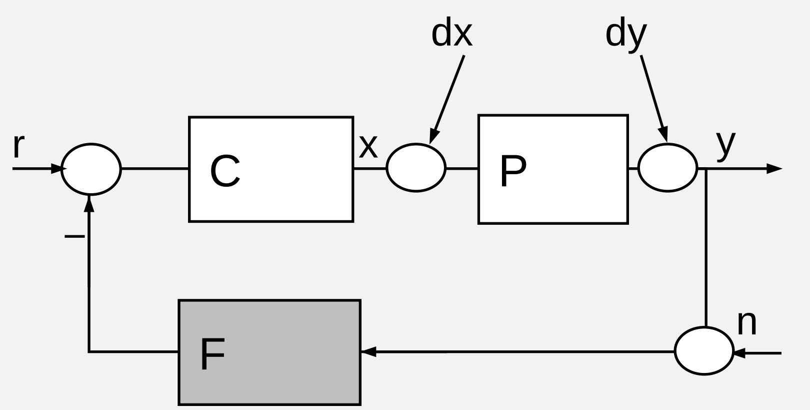
- The Bode plot of PCn shows that the controller has plenty of PM and the gain K = 1 is roughly correct and has a good step response.
- So, what's wrong? We can get an indication by looking at the input disturbance response that has a very long time constant.







Case Study 3: Cancellations



$$P(s)=\frac{1}{(s+0.1)(0.1s+1)}$$

$$C(s)=\frac{0.87s+0.5}{s}$$

- Let BW = 1.5 and perform a formal PI design. We solve the angle (or PM) equation for a 60 deg. PM.

ω_c

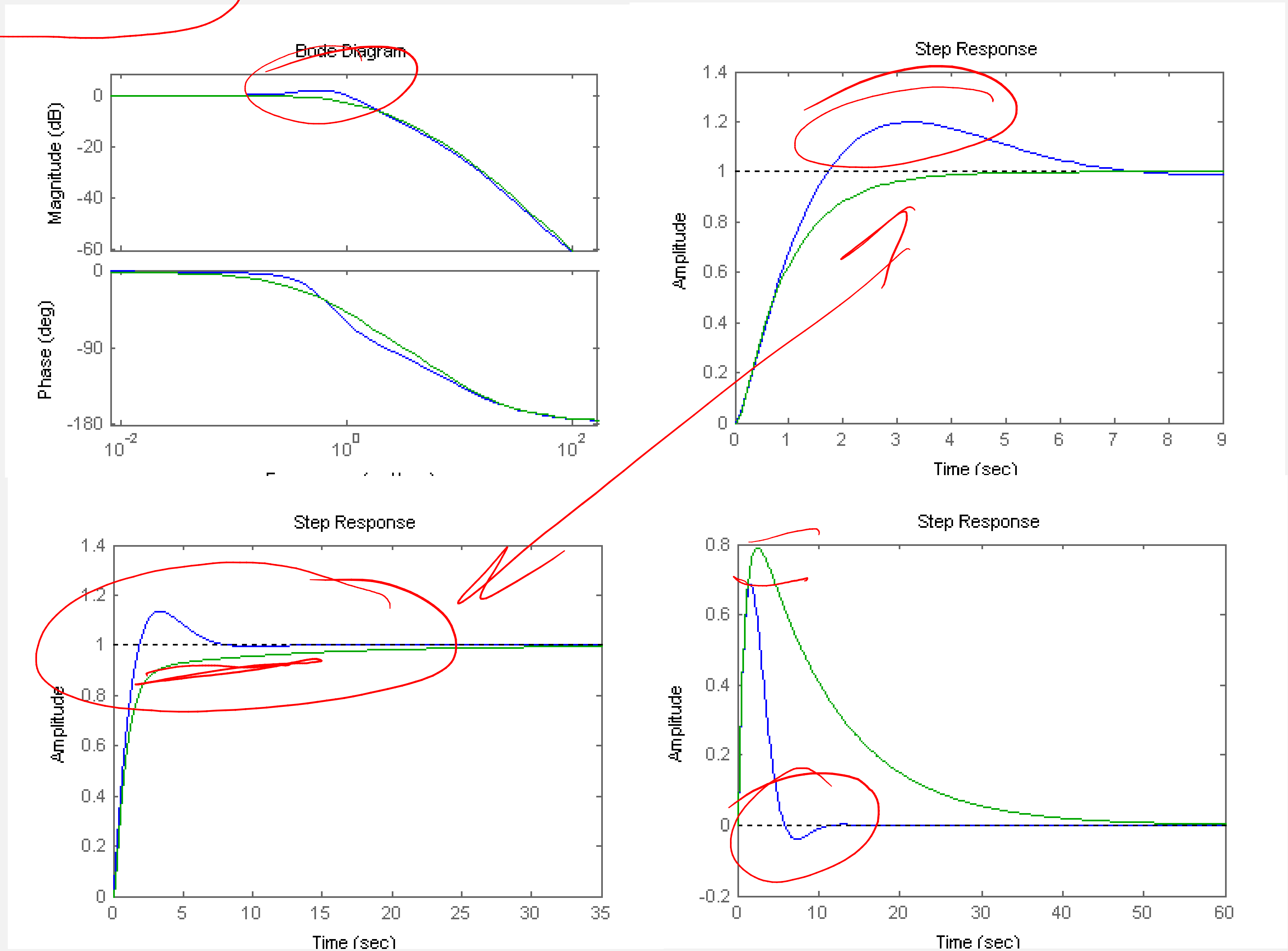
$$-180^o+60^o\le-\tan^{-1}(0.1\times1)-\tan^{-1}(1/0.1)-90^o+\tan^{-1}(\tau_z\times1)$$

- We find $\tau_z=1.7$, and from the crossover equation $K=0.5$.

$$C(s)=\frac{0.87s+0.5}{s}$$

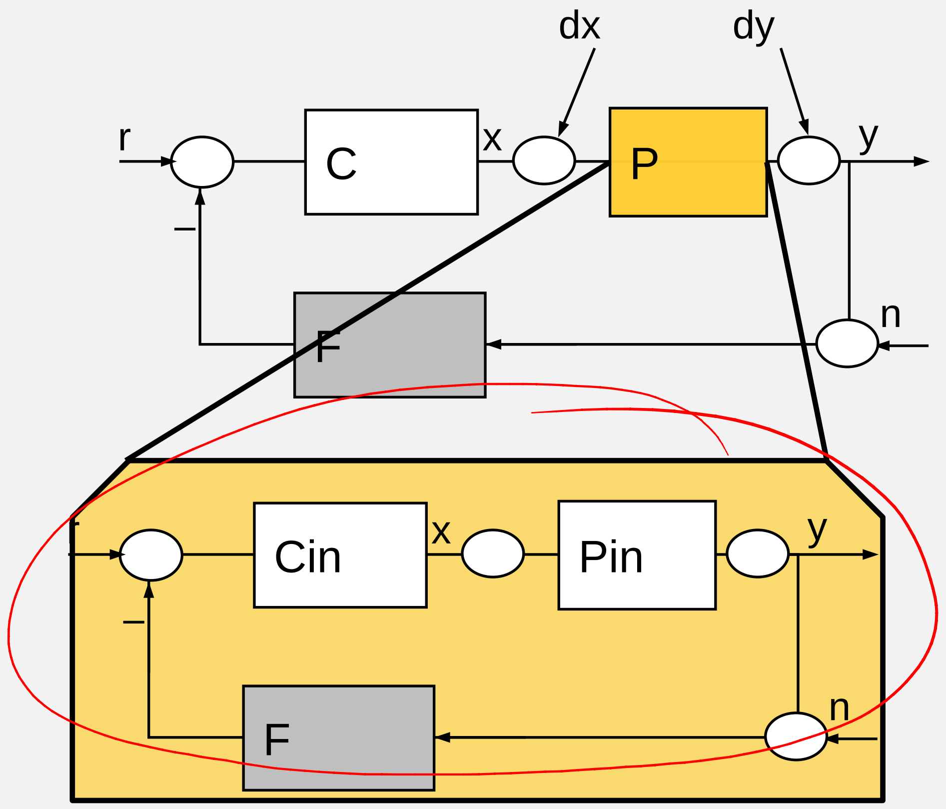
- Comparing with the Cn controller, we find that the closed loop BW are almost the same and the step response of Cn has no overshoot. But the difference in disturbance rejection is striking in favor of the formal design.

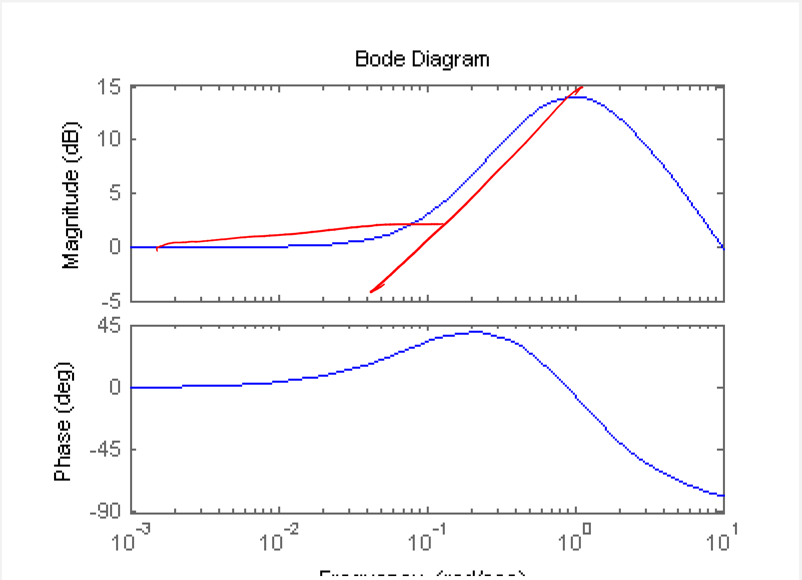
- There is also a considerable difference in the step response if the plant slow pole changes to -0.2. Then the imperfect cancellation produces a lingering response.



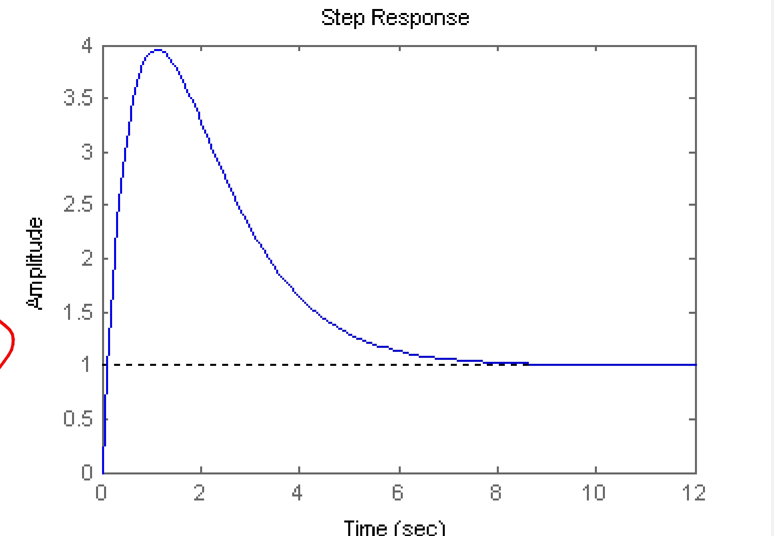
Case Study 4: Inner loop zero

- In some cases, there are plants exhibiting a lead behavior, that is a slow zero followed by faster poles. Such a behavior may arise in an inner-outer loop design where a local controller controls a fast variable and then a slower outer loop controller controls the global variable. In this case, P is itself a closed loop system of a controller Cin and a plant Pin.
- There is a wealth of inner-outer loop designs in the industry and an interesting one appears in paper machines. An inner loop PID controlling the temperature of a drum following a slow heat transfer equation and requiring significant phase lead. Then, an outer loop PID controls the final output which is the paper moisture, by adjusting the set-point of the temperature controller. Here the moisture system may be faster but the lead zero from the inner loop still appears in the effective plant transfer function.
- As an exaggerated example, here we take P(s) with a zero one decade below the poles. Clearly, PID's only have adjustable zeros and they are not suitable for such problems.





$$P(s) = \frac{10s+1}{(s+1)^2}$$



$$P(s) = \frac{10s+1}{(s+1)^2}$$

$$C(s) = \frac{0.2}{s}$$

$$C_o(s) = \frac{1.6s+1.16}{s} \times \frac{0.1s+1}{10s+1}$$

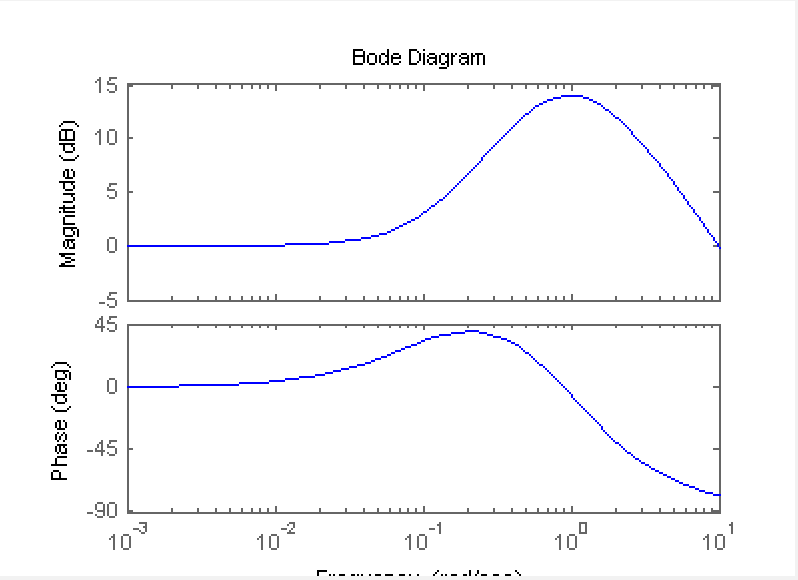
Case Study 4: Inner loop zero

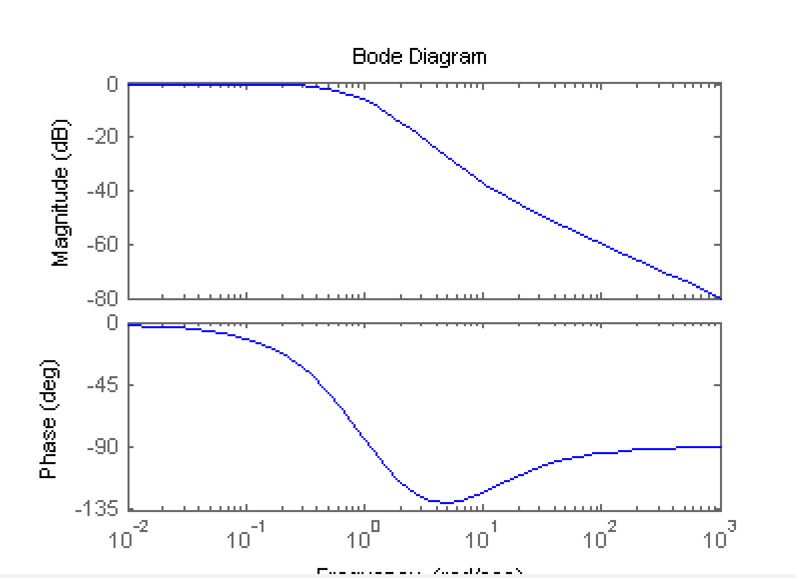
- To design a controller for a plant with slow lead zero, we augment the PID with a lowpass filter, canceling the zero. Such an action will reduce the loop gain at high frequencies, but this is a lesser problem.
- For comparison, we perform one design with the plant as is, and one with a filter $C_F(s) = \frac{0.1s+1}{10s+1}$
- The first case with a target BW 1.5rad/s requires no PI zero and the controller is simply an integrator 0.2/s (gain computation from the crossover equation).
- For the second case, the plant is $P(s)C_F(s)$, the zero is canceled and replaced by the much faster 0.1s+1, so effectively only the double pole at 1 remains. Its contribution is 2 x 45 = 90deg. so a PI controller is sufficient.

P(s) Int → PI-zero

$$-180^o + 60^o \le \tan^{-1}(0.1\times 1) - 2\tan^{-1}(1\times 1) - 90^o + \tan^{-1}(\tau_z \times 1)$$

$$\tan^{-1}(\tau_z \omega_c) = 54^o \Rightarrow \tau_z = 1.4. \quad |PC(j\omega_c)| = 1 \Rightarrow K = 1.16$$





Case Study 4: Inner loop zero

$$P(s) = \frac{10s+1}{(s+1)^2}$$

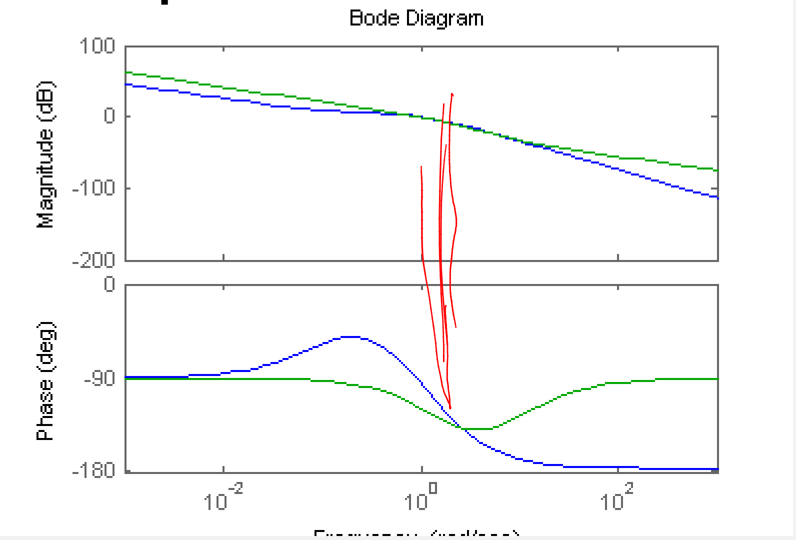
$$C(s) = \frac{0.2}{s}$$

$$C_o(s) = \frac{1.6s+1.16}{s} \times \frac{0.1s+1}{10s+1}$$

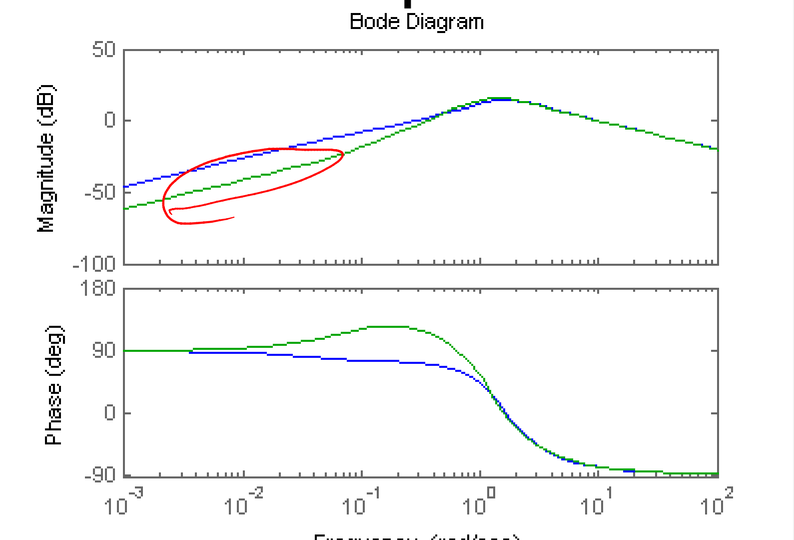
Finally, we compare the responses of the two controllers:

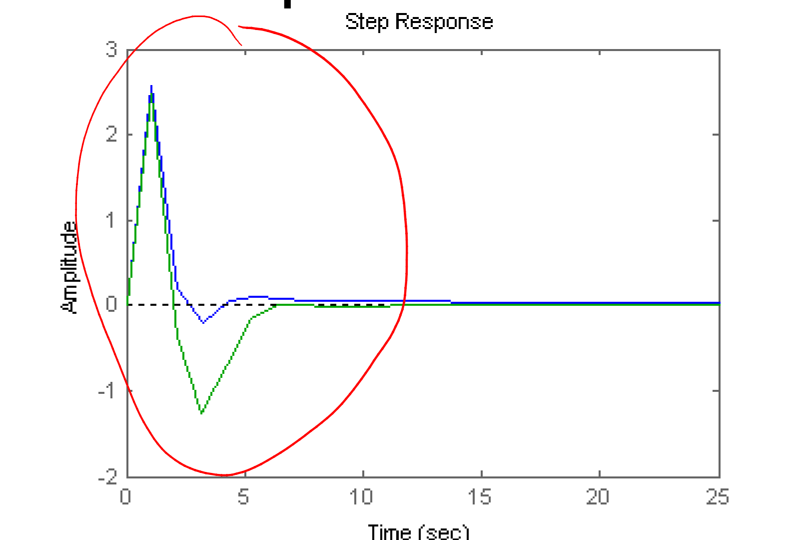
- The standard loop (blue trace) has a better PM but its gain at low frequencies is lower. (It attenuates high frequencies better because it is a strictly proper integrator.
- The filtered loop has better low frequency disturbance rejection but also much faster settling of the step response because it lacks the slow mode attracted by the zero. This mode will show up again if the plant zero moves, but will have a lower residue nevertheless.

Loop transfer function

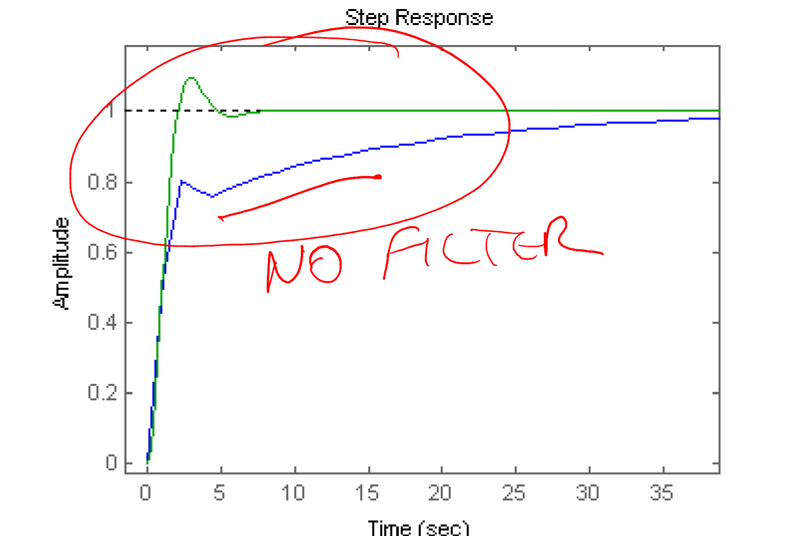


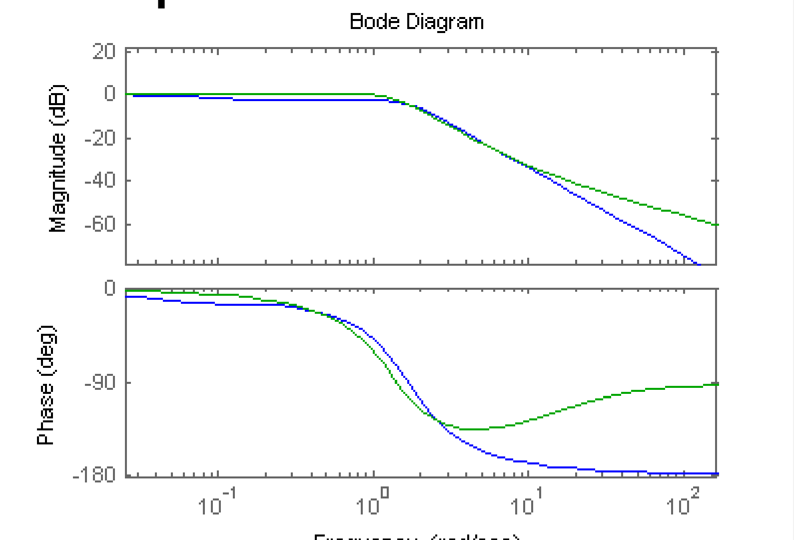
Input Disturbance-2-Output



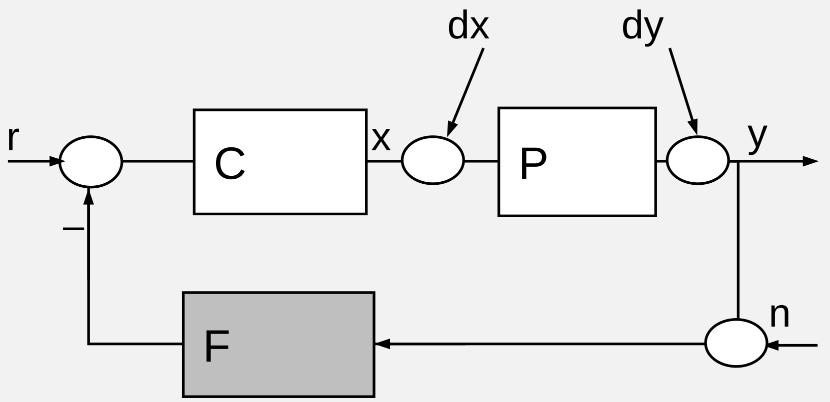


Reference-2-Output



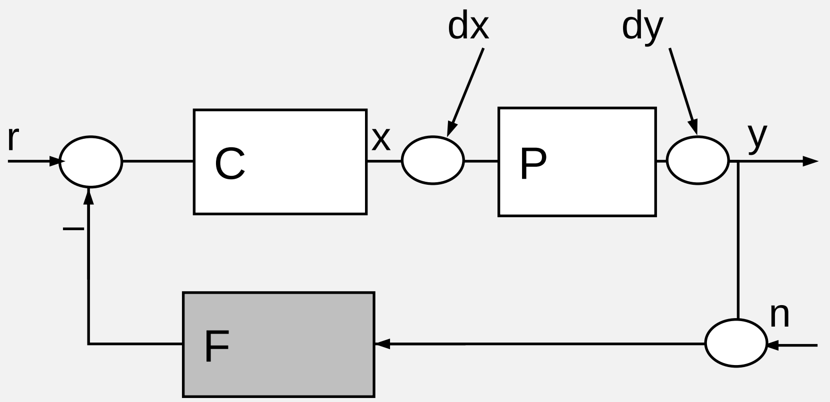


Case Study 5: Flexible mode filtering

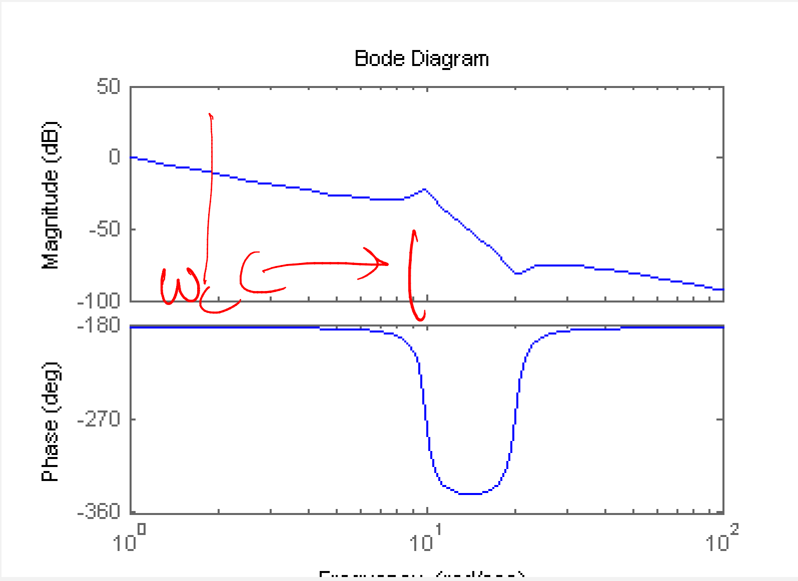


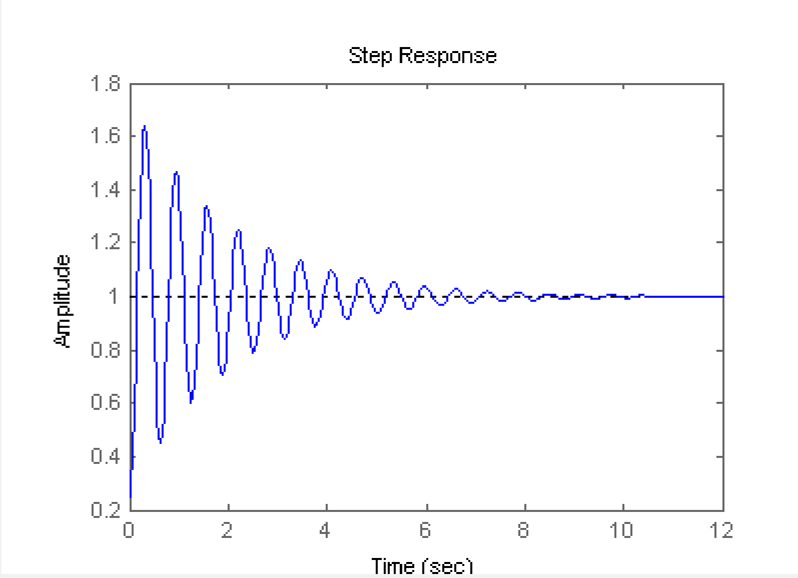
- A flexible mode or resonance in the plant is another type of behavior that cannot be compensated by the single or double zero of the PID, let alone that, even if an exact cancellation is possible, it is not recommended. The resonance means that the system has a high gain at that frequency. Its cancellation helps the response to commands but ignores input disturbances that produce a very oscillatory response.
- To the extent that PID compensation is feasible, there are two possible designs. One where the flexible mode is left at the frequencies below BW and one where it is above BW. In the latter more common case, it makes sense to filter and attenuate it. Cancellation is always “dangerous” since any shift of the plant resonance will immediately show up.

Case Study 5: Flexible mode filtering

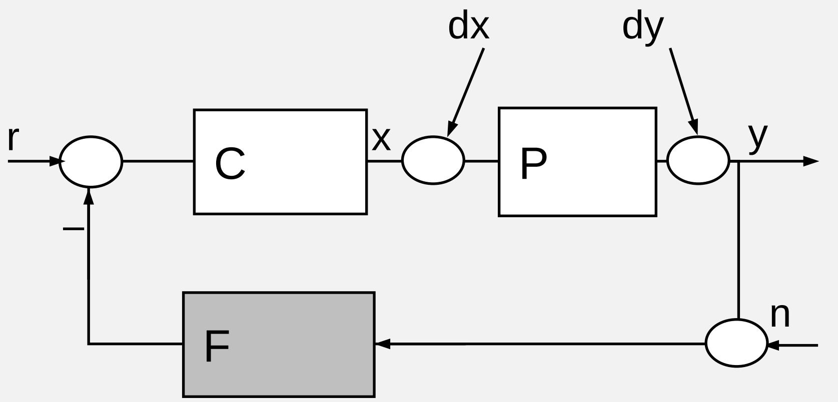


- Flexible modes typically show up as lightly damped pole-zero pairs. Examples are the flexible pendulum in the lab (with the pendulum dynamics for the rigid body) or a disk drive arm position controller (double integrator dynamics for the rigid body). Resonances also show up in power systems control.
- Here we consider the plant $P(s) = \dfrac{0.25s^2 + 0.5s + 100}{s^2 + s + 100} \times \dfrac{1}{s^2}$
- The step response of the flexible mode alone is also shown in the plots for reference.
- The PID control is uneventful up to the frequency range where the resonance becomes significant.





Case Study 5: Flexible mode filtering



$$P(s) = \frac{0.25s^2 + 0.5s + 100}{(s^2 + s + 100)s^2}$$

- We design a PID for BW 1.5rad/s and PM = 40 deg. (Here the designs become very conservative for larger PM or even infeasible.) The controller is

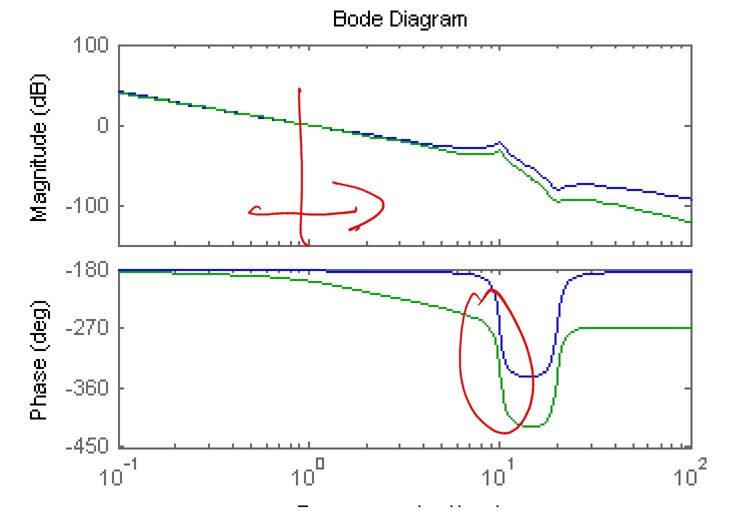
$$C_o(s) = \frac{0.14(2.48s+1)^2}{0.1s^2 + s}$$

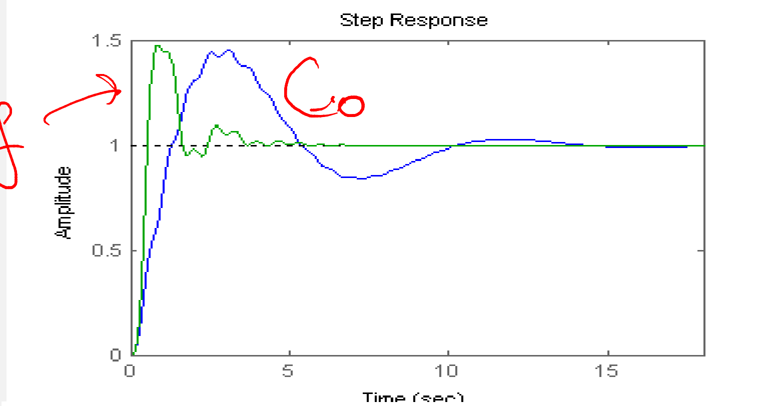
- For higher BW the controller solution using the PM equation is not stabilizing. The resonance creates too much phase lag that is not seen at the intended crossover.

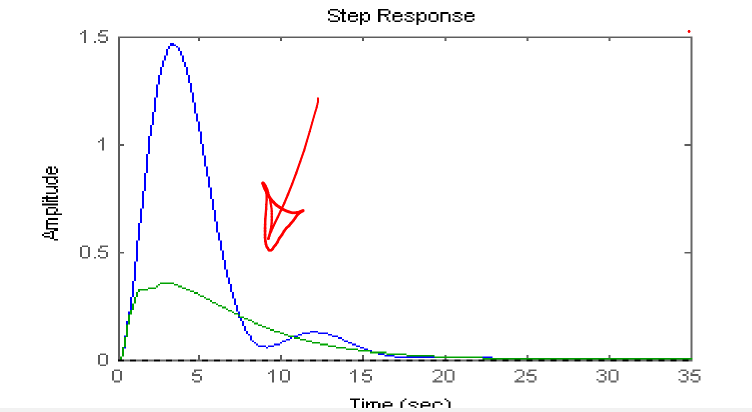
- Instead, we use a lowpass filter $F_o(s) = \frac{1}{0.3s+1}$ to attenuate the resonance and design a PID for a BW of 4.5rad/s with PM=30:

$$C_f(s) = \frac{0.33(2.37s+1)^2}{(0.033s^2 + s)(0.3s+1)}$$

- The responses are improved (green traces), and so is disturbance rejection. The benefits, however, depend on the precise characteristics of the modes.





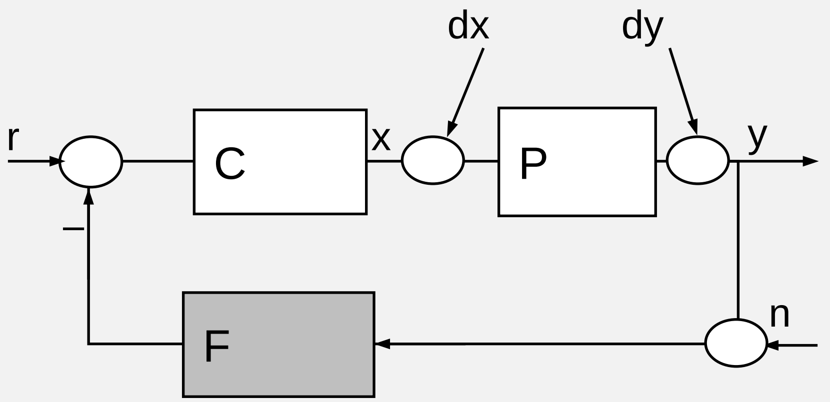


EEE304

Lecture 6.3: DT-PID and Controller Discretization

Arizona State University

Controller Discretization



- PID Discrete Implementation

$$x_k = K_p e_k + K_i s_k + K_d (e_k - e_{k-1})$$

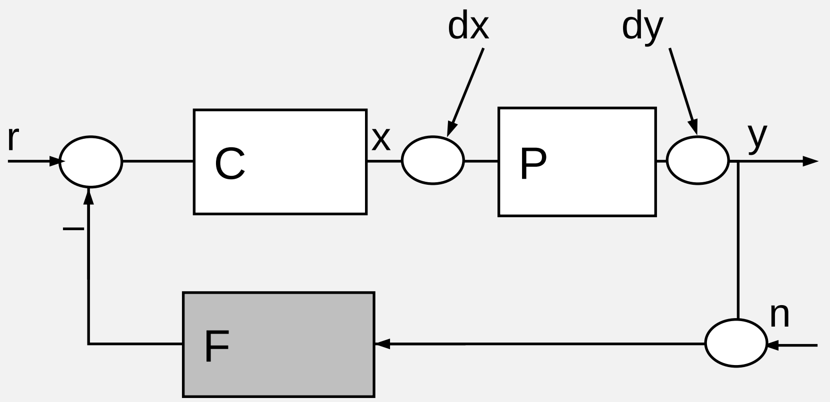
$$s_{k+1} = s_k + e_k$$

- several different but equivalent implementation equations,
- e.g., $\Delta x_k = x_{k+1} - x_k = K_p (e_{k+1} - e_k) + K_i (e_k) + K_d (e_{k+1} - 2e_k + e_{k-1})$
- Integrator windup: Nonlinear behavior when the control input saturates (can lead to instability)
- Remedy: Anti-windup modification (limited integrators)

$$s_{k+1} = \min[\max\{ s_k + e_k , \tfrac{x_{\min}}{K_i}\}, \tfrac{x_{\max}}{K_i}]$$

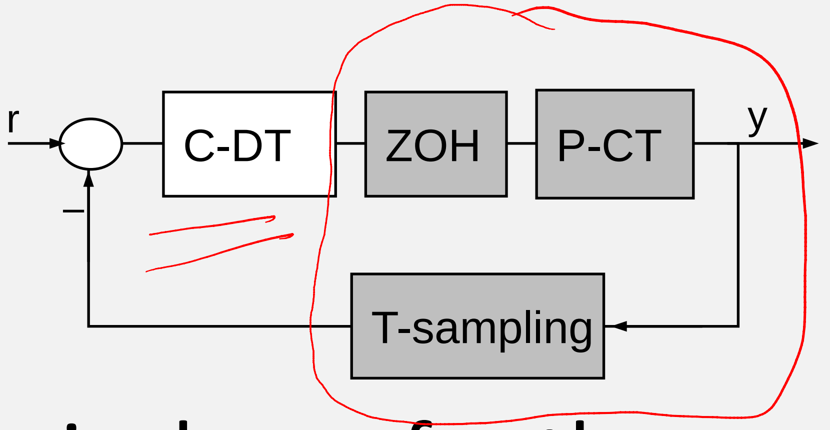
but notice that the saturation of the incremental form is much easier!

Controller Discretization



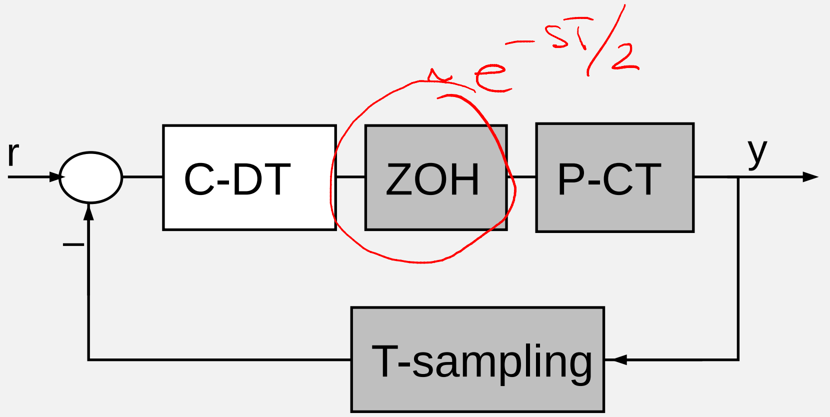
- For fast sampling rates, satisfying the Forward Euler constraint, any “C2D” method results in similar controllers. The gains in the previous equation can be simply taken as the CT gains.
- But with more general PID definitions (with lowpass filters, pseudo-differentiators, cancellations, feedforward filters) one needs a more formal discretization approach.
- The situation is exacerbated when the sampling rates are not very fast. Such problems appear with the use of cheap processors/microcontrollers, in demanding applications (embedded local controllers have better disturbance rejection characteristics over centralized slower systems because of fast sampling). Another reason is the use of sensors that can measure a quantity of interest directly but at a lower rate.

Controller Discretization



- To address this problem, we notice that the suitable form of equivalence for the plant is that of a ZOH.
- The ZOH-equivalent plant is controlled by the DT controller with piecewise constant signals and DT stability is necessary and sufficient for CT stability. Unfortunately, the DT controller design equations are considerably more "messy" so alternative procedures have been developed:
  - Convert the DT loop to the so-called "w-plane" through the D2C Tustin transformation. The "w-plane" offers an equivalent CT system with necessary and sufficient stability conditions (recall the mapping of the LHP onto the UC by Tustin), where all the CT insight and design methods apply. Then convert back to the "z-plane" using the C2D Tustin to obtain the DT controller. This method offers equivalence and non-conservative results. However, approaching the Nyquist frequency, the intra-sample behavior of the CT system may no longer be desirable. Furthermore, the conversions can be cumbersome, if done manually.

Controller Discretization



- A simpler alternative that sacrifices some performance near the Nyquist frequency can be defined by noting that ZOH is roughly equivalent to ½-sample delay.
  - This is accurate in terms of phase but the magnitude starts deviating near Nyquist frequency.
- Taking that into account, the PM equation is adjusted to obtain the desired PM with the ZOH.
  - Effectively, we can solve the same equation adding an extra margin of half-sample-time. (A magnitude correction does not offer a consistent advantage)

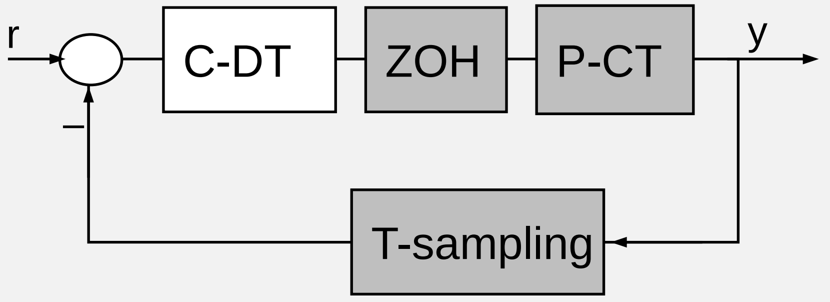
$$-180^o + PM \le \angle P + \angle C + \angle ZOH = \angle P - 90^o + N\tan^{-1}(\tau_z \omega_c) - \tan^{-1}(\tau_p \omega_c) - T_s \omega_c / 2$$

$$|PC(j\omega_c)| = 1$$

- The resulting CT controller is then converted to DT using Tustin that preserves phase very well. It is always a biproper controller so in case a strictly-proper (and strictly causal) controller is required, an additional delay should be added to the plant.

$-T_s \omega_c$

DT PID Example 1



$$P(s) = \frac{1}{(s+0.1)}$$

$$C_T(z) = \frac{0.87\,z - 0.82}{z-1}$$

- Consider the plant $P(s) = \frac{1}{(s+0.1)}$. Design a DT-PID controller for BW = 1.5rad/s, PM = 60deg, using Ts = 0.1s.
- We substitute in the PM equation

ZOH ~3°

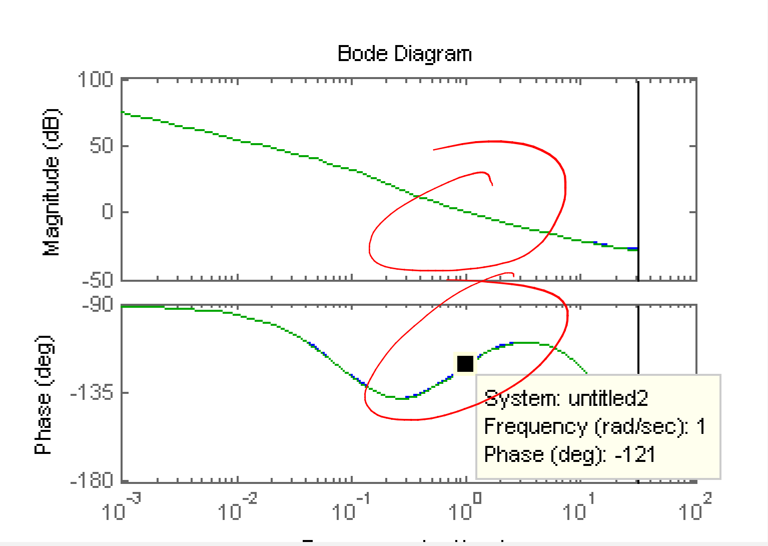
$$-180^o + 60^o \le \angle P + \angle C + \angle ZOH = -84 - 90^o + N\tan^{-1}(\tau_z 1) - \tan^{-1}(0.1) - 0.1/2$$

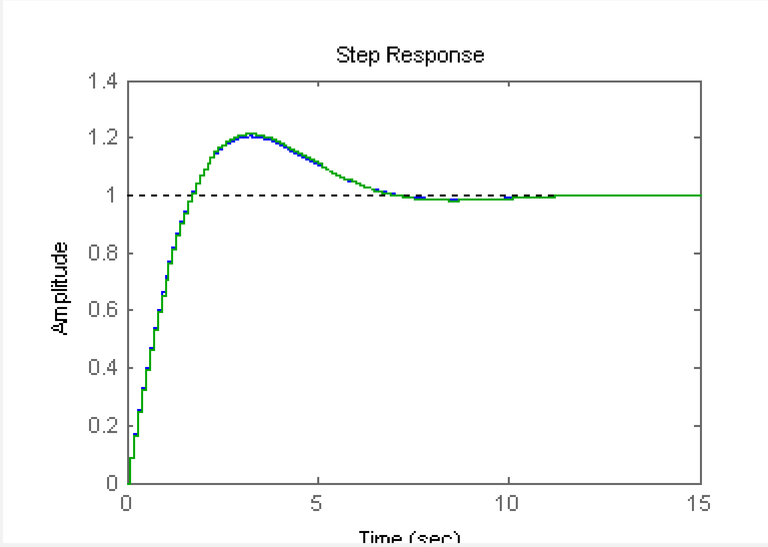
$$\Rightarrow N = 1, \quad \tau_z = \tan(54)/1 = 1.4$$

$$|PC(j\omega_c)| = 1 \Rightarrow K = 0.59$$

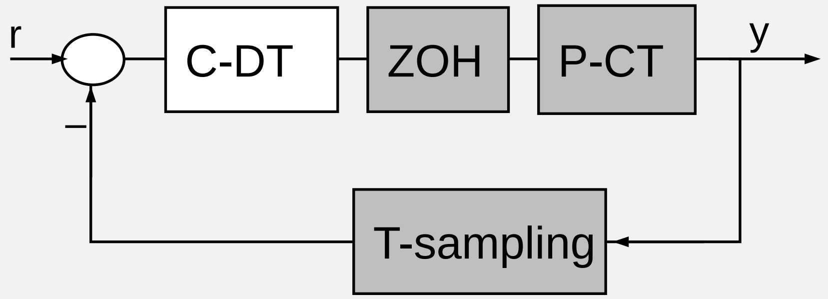
$\frac{1}{T_p s+1}$ if $N=2$

- Then the CT-controller is $C(s) = \frac{0.85\,s + 0.54}{s}$ and its Tustin equivalent is $C_T(z) = \frac{0.87\,z - 0.82}{z-1}$. For comparison, its FE equivalent would be $C_{FE}(z) = \frac{0.85\,z - 0.79}{z-1}$
- We compare the responses of the two controllers and they are virtually identical.
- Note: For PI, there is no active constraint for FE discretization.





DT PID Example 2



$$P(s) = \frac{1}{(s+0.1)}$$

$$C_T(z) = \frac{1.11z - 0.813}{z-1}$$

- Consider the plant $P(s) = \frac{1}{(s+0.1)}$. Design a DT-PID controller for BW = 1.5rad/s, PM = 50deg, using Ts = 1s.
- We substitute in the PM equation

ZOH

$$-180^o + 50^o \le \angle P + \angle C + \angle ZOH = -84 - 90^o + N\tan^{-1}(\tau_z 1) - \tan^{-1}(0.1) - T_s/2$$

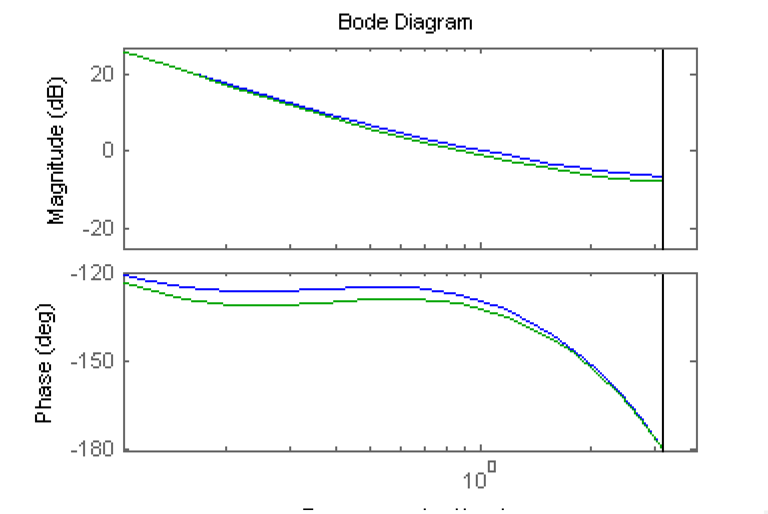
$$\Rightarrow N = 1, \quad \tau_z = \tan(73)/1 = 3.26$$

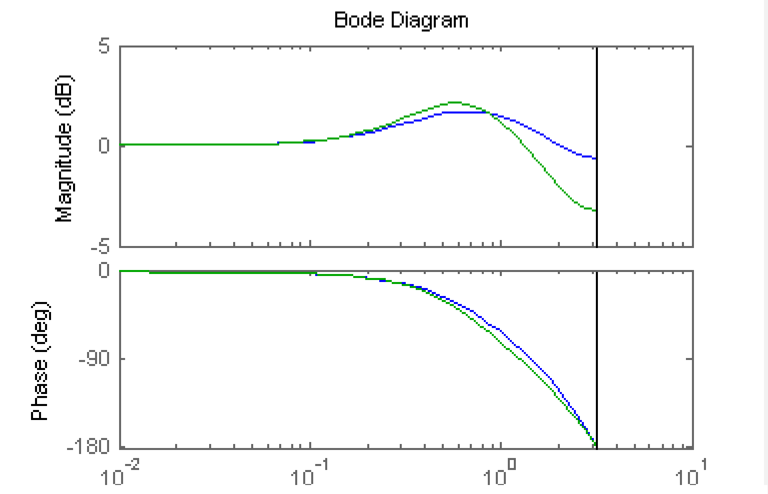
$$|PC(j\omega_c)| = 1 \Rightarrow K = 0.295$$

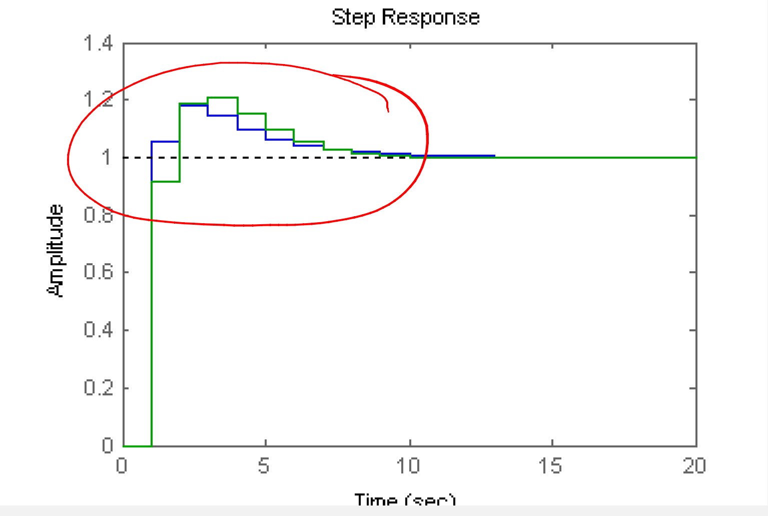
- Then the CT-controller its Tustin and FE equivalents are

$$C(s) = \frac{0.961s + 0.295}{s}; \quad C_T(z) = \frac{1.11z - 0.813}{z-1}; \quad C_{FE}(z) = \frac{0.961z - 0.666}{z-1}$$

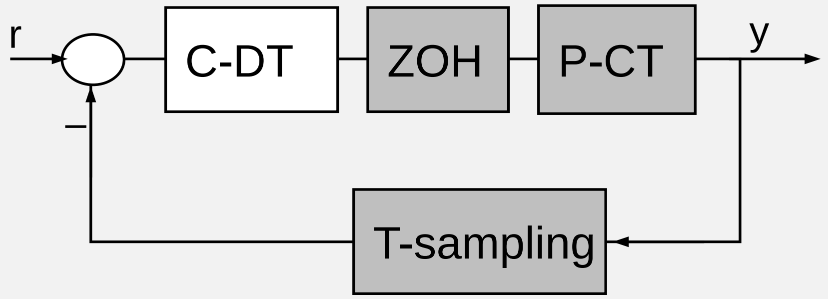
- We compare the Loop transfer function, closed-loop frequency and step responses of the two controllers. They are not as close any more and they resemble less a CT response. When the crossover approaches the Nyquist frequency, several of the approximations break down and this simple design approach is no longer reliable.







DT PID Example 3



$$P(s) = \frac{1}{(s+0.1)}$$

$$C_D(z) = \frac{0.937z - 0.889}{z-1}$$

- Consider the plant $P(s) = \dfrac{1}{(s+0.1)}$. Use the DT equations to design a DT-PI controller for BW = 1.5rad/s, PM = 50deg, using Ts = 1.5s.
- The DT-PI has the form $C_D(z) = \dfrac{K(z-a)}{(z-1)}$ Direct DT design
- The ZOH plant is $P_D(z) = \dfrac{1.393}{z-0.861}$ ← MATLAB Computations
- We substitute in the PM equation and evaluate at $\boxed{\Omega_c = \omega_c T_s} = 1.5 rad/smpl$

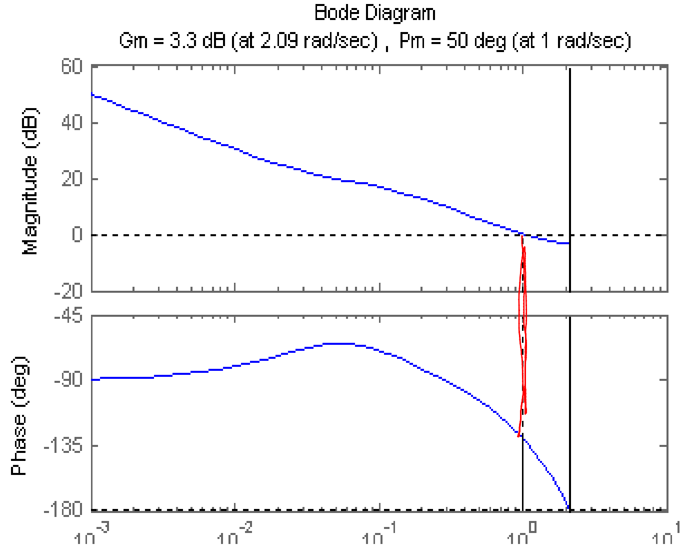
$$-180^o + 50^o \le \angle P_D + \angle C_D = -\tan^{-1}\frac{\sin\Omega_c}{\cos\Omega_c - 0.861} - \tan^{-1}\frac{\sin\Omega_c}{\cos\Omega_c - 1} + \tan^{-1}\frac{\sin\Omega_c}{\cos\Omega_c - a}\Bigg|_{\Omega_c = 1.5}$$

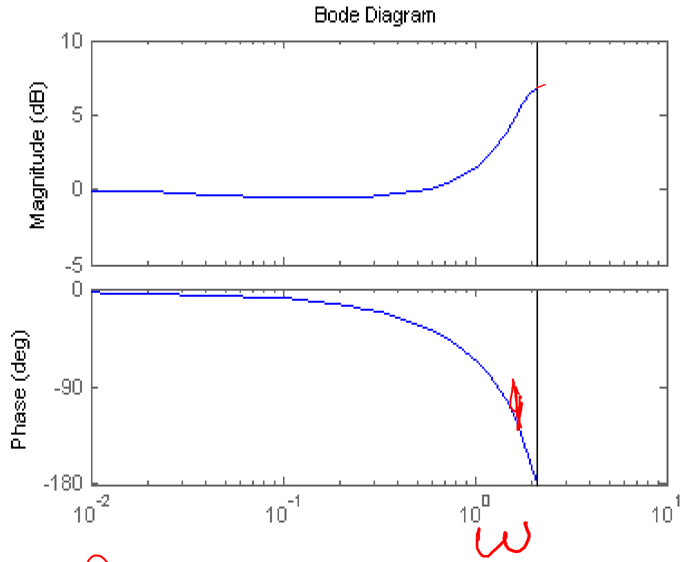
$z = e^{j\Omega_c} = \cos\Omega_c + j\sin\Omega_c$ Int

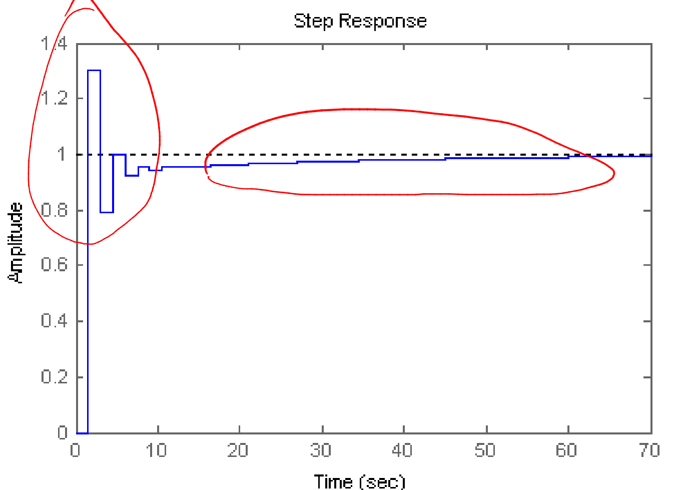
$$-130^o \le 0.90 + 0.82 + \tan^{-1}\frac{\sin\Omega_c}{\cos\Omega_c - a} \Rightarrow -228.6 = 131.4^o \le \tan^{-1}\frac{\sin\Omega_c}{\cos\Omega_c - a} \Rightarrow a = 0.949$$

$$\Rightarrow C_D(z) = \frac{K(z-0.949)}{z-1}. \quad |PC(j\omega_c)| = 1 \Rightarrow K = 0.937$$

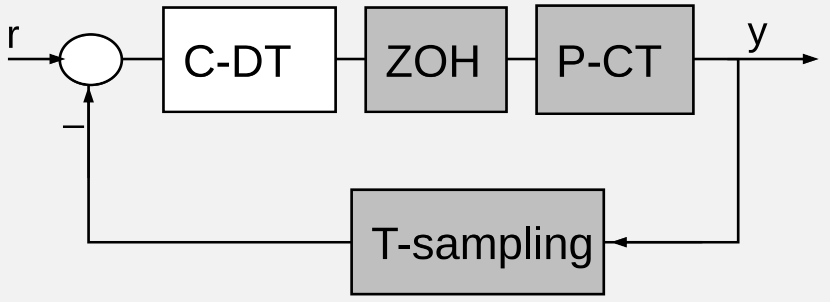
- We evaluate the Loop transfer function and closed-loop responses. The closed-loop is still stable, but it is quite different from our CT expectations due to the proximity of the crossover to the Nyquist frequency.







DT PID Example 3: w-plane



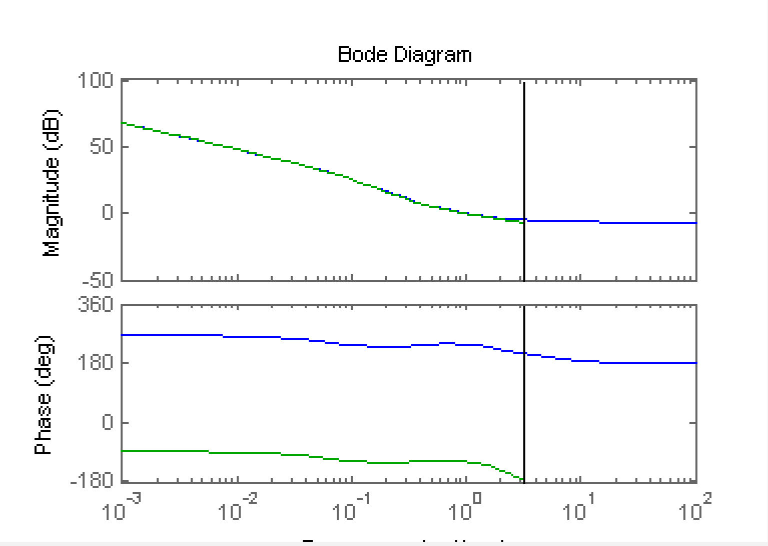
- Consider the plant $P(s)=\dfrac{1}{(s+0.1)(s+1)}$. Design a DT-PID controller for BW = 1.5rad/s, PM = 60deg, using Ts = 1. (This is near half the Nyquist rate corresponding to the required BW.) 3rad/s – 0.5Hz – 2s TUSTIN D2C
- We find the ZOH equivalent and its w-plane equivalent

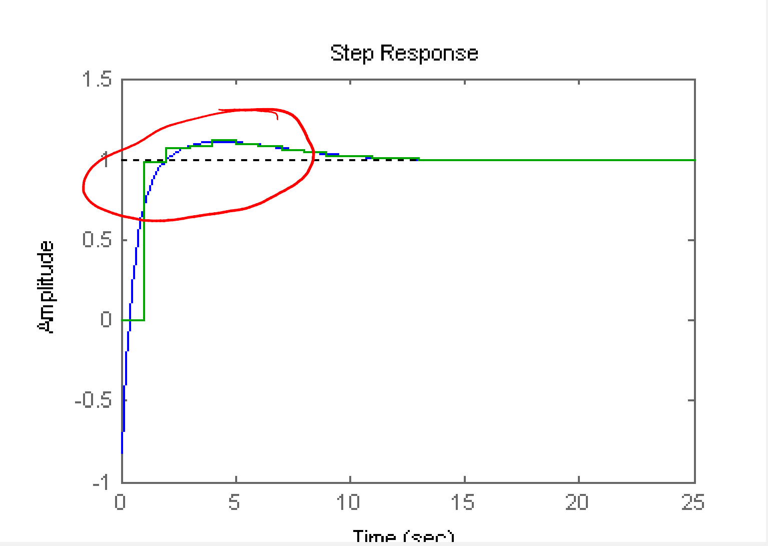
$$P_d(z)=\frac{0.355\,z+0.247}{z^2-1.27\,z+0.333},\quad P_w(w)=\frac{-0.0416\,w^2-0.379\,w+0.924}{w^2+1.024\,w+0.0924}$$

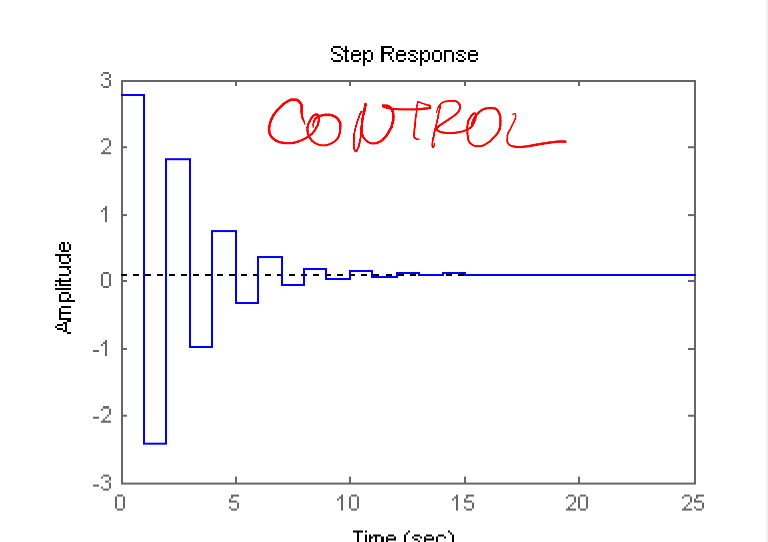
- We now design the PID controller for Pw and then convert to DT using Tustin. TUSTIN C2D

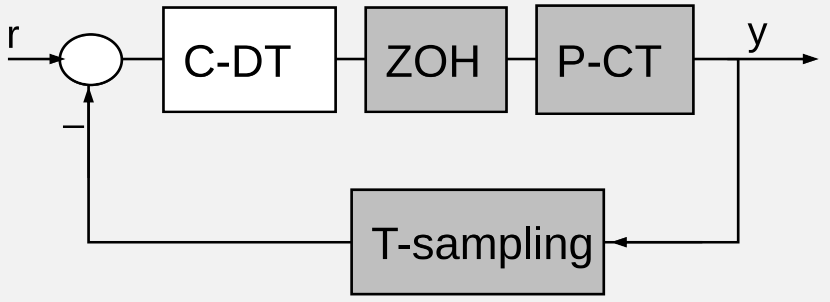
$$C(w)=\frac{1.08\,w^2+1.04\,w+0.249}{0.1w^2+w},\quad C_T(z)=\frac{2.76\,z^2-3.39\,z+1.04}{z^2-0.333\,z-0.667}$$

- Despite the coarse sampling, the DT system is a fairly close interpolation of the desired continuous system in specs and response.
  - The price for this result is the “violent” control action (last figure). Normally such extremes are to be avoided because of sensitivity to noise.









DT PID Discussion

- Sampling time selection (for faithful interpolation):

  Phase Margin : $\frac{T_s \omega_c}{2} \sim 0.1 = 6^o (\times \frac{\pi}{180})$ other choices in $(3^o - 12^o)$ range

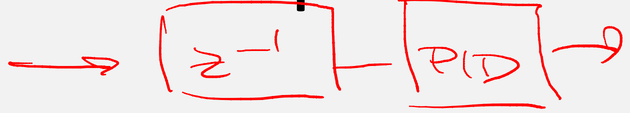
  6 samples per rise time : $T_s = \frac{t_r}{6} = \frac{2}{6BW} = \frac{1}{3BW} = \frac{T_{Nyquist}}{3\pi} \sim \frac{T_{Nyquist}}{10}$

- DT-Controller design can suffer from insufficient roll-off for low sampling rates.
  - Using PM as a stability metric relies on the shape of the Nyquist plot. If sampling very slowly, the crossover comes near the Nyquist frequency with little range left for the magnitude to decay. Many approximations break down at that stage.
- Add a delay for strictly proper and strictly causal controllers.
  - Can be relaxed by imposing some timing assumptions.

$x_k = f(y_k)$

$x_k = f(y_{k-1})$

- Notice the dependence of the procedure on sampling time: difficult to perform a parametric study or optimization.



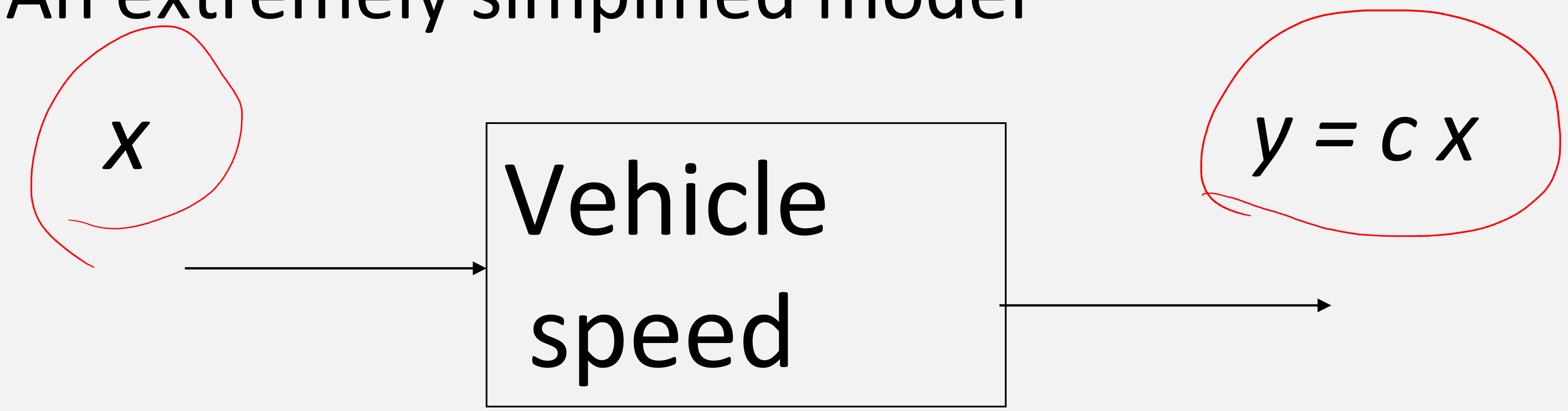
EEE304

Lecture 6.4: PID Control: Revisiting the Control Objectives

Arizona State University

Two basic concepts of feedback systems: Performance, Robustness

- An extremely simplified model

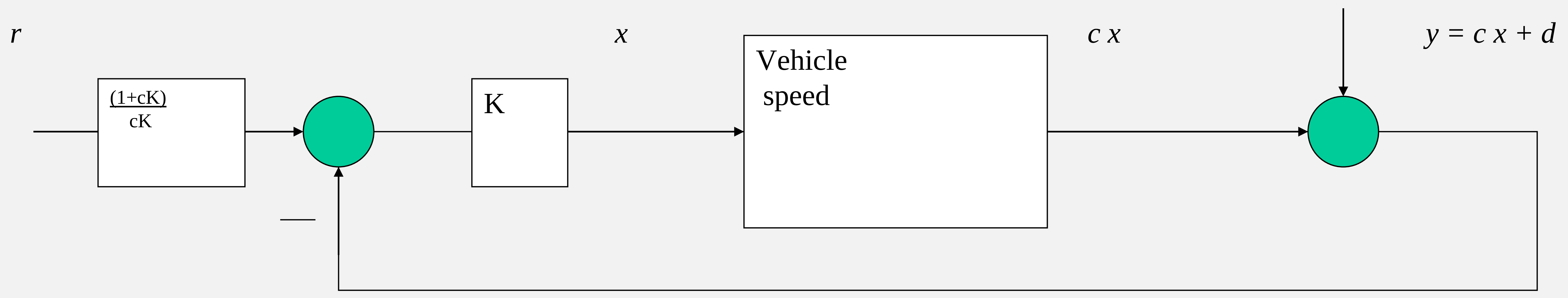


(All models are wrong, some are useful[G.E. Box]*)*

- Objective $y = r$ (e.g., c=100, r=50)
- Easy choice: $x = r/c$

Performance...

- Add a disturbance: $y = cx+d$ (say $d = -20$)
- Then $y = c(r/c)+d$ = 50-20 = 30 :(
- Measure y and close the loop:



- $x = K(r_f\text{-}y)$, $r_f = (1+cK)/cK$ (adjusted ref.)
- Then $y = cK(r_f - y) + d = \ldots = r + d/(1+cK)$

Performance...

- Conclusion: As the controller gain (K) increases, the effect of the disturbance is reduced.

- ***Rule#1: High gain => Performance***

- Similar conclusion for variations in c (a bit more messy)
- So, take K = 1000000…
- Too good, too easy? …

Performance...

- Suppose there is an unmodeled small delay between application of input and output: $y(t) = cx(t-\tau)$
- Without the disturbance, $y(t) = cK[r_f(t-\tau) - y(t-\tau)]$
- Look at the sample times $\tau, 2\tau, 3\tau, \ldots$: $y[n\tau] = -cK\, y[(n-1)\tau] + cKr_f$
- If $cK > 1$, $|y(n\tau)|$ grows exponentially! :(
- For a continuous time version of this argument, use a Pade approximation of the delay:

$$e^{-s\tau} \cong \frac{1-s\,\tau/2}{1+s\,\tau/2} : y = c\frac{1-s\,\tau/2}{1+s\,\tau/2}K(y-r_f) \Rightarrow y = \frac{cK(1-s\,\tau/2)}{(1-cK)\,\tau/2\, s+(1+cK)}r_f$$

$$poles: \frac{1+cK}{1-cK}\frac{2}{\tau}$$

- Hence, the closed loop is unstable for $cK>1$

Performance...

- Conclusion: High controller gains can cause instability due to unmodeled dynamics.
- Alternative interpretation: Our controller did not respect the limitations of the model

  (Models have limitations, stupidity does not![Michael Athans])

- ***Rule#2: Respect the uncertain[Gunter Stein]***

- NOTE: For unstable systems, low controller gains can lead to instability as well, but this is predicted by the model.

Robust controller design

- A nominal model by itself is not very useful. We also need an uncertainty estimate.

- Multiplicative uncertainty (relative error)

$$P(s) = [I + \Delta_m(s)]P_0(s)$$

$$|\Delta_m(j\omega)| = \frac{|P(j\omega) - P_0(j\omega)|}{|P_0(j\omega)|} \approx \underbrace{\frac{|y_0(j\omega) - y(j\omega)|}{|y_0(j\omega)|}}_{\text{estimation from data}}$$

- Estimate the bound analytically or from data.

- *Small Gain Theorem:* the controller should satisfy

$$|T(jw)| < 1/ |D_m(jw)|$$

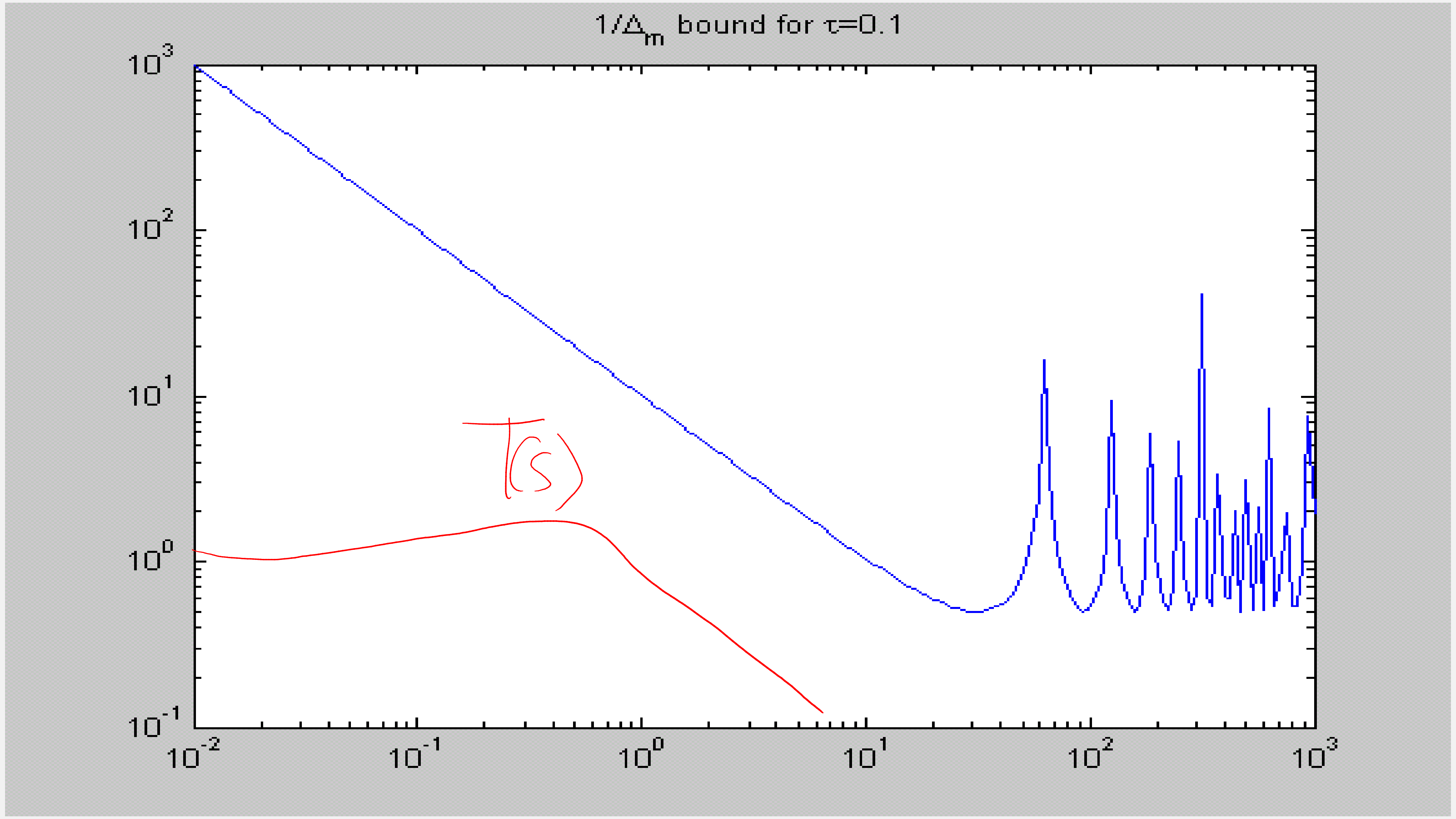
where $T = P_0C/(1+P_0C)$ (complementary sensitivity)

Robust controller design

- In our case, P_0=c, C=K, so T = cK/(1+cK)

T is a constant, so we must have T< 1/2 for all w.

=> cK<1



For the plot,

```
>> w=logspace(-2,3,200);
>> d=exp(j*w*.1);
>> loglog(w,1./abs(1-d))
```

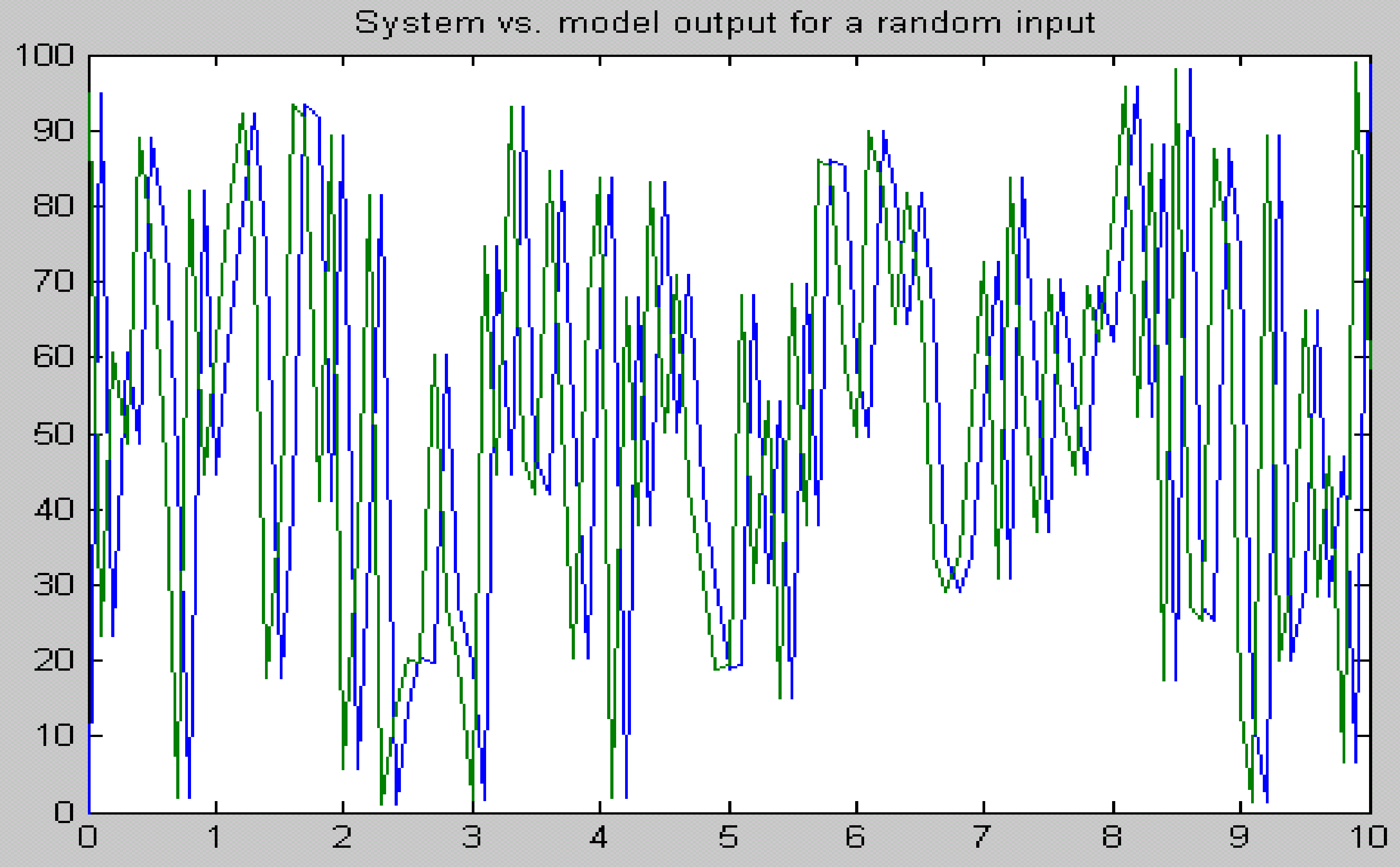
Robust controller design

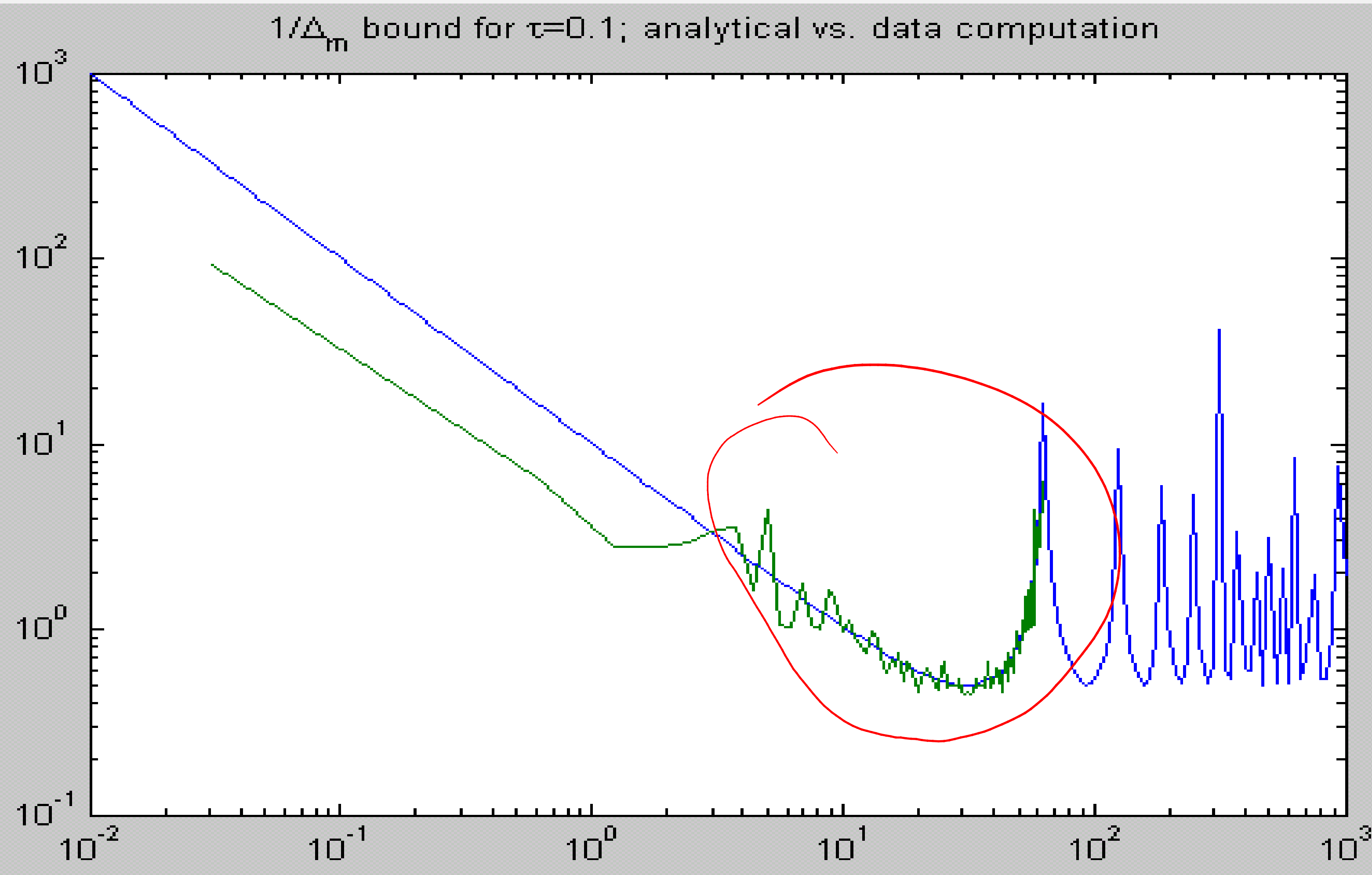
- Estimating the uncertainty from data:
  - Collect data with a random input

>> dd=abs(fft(y-y0)./fft(y0));

>> ww=([1:pts]'-1)*2*pi/pts/dt;ww(1)=ww(2)/20;

- %fft frequencies for pts number of points, sampled every dt.





Robust controller design

- Dynamic controllers: Getting better performance at some frequencies.
- The uncertainty constraint is not active at low frequencies => we should be able to increase the gain there.
- PI control: $x = (K_p s+K_i)/s[e]$, (e = error = r-y)
  - For simplicity, consider the I-part only; for our easy system we do not need the P- nor the D-part. We loose some performance at high frequencies but our typical d is a low frequency signal.
  - Controller gain = $|C(jw)| = K_i/w$; large at low frequencies. Also interpreted as an internal model of constant disturbances

Robust controller design

- Find the complementary sensitivity (nominal)

$$T(s) = \frac{cK_i/s}{1 + cK_i/s} = \boxed{\frac{cK_i}{s + cK_i}}$$

- Should have $|T(jw)| < 1/|D_m(jw)|$
- Roughly, closed-loop bandwidth = $cK_i < w_{max}$ (uncertainty crossover frequency), which is ~ 10.
- Leave some margin too, so $K_i = 2/c$
- Analysis: (takes a while)

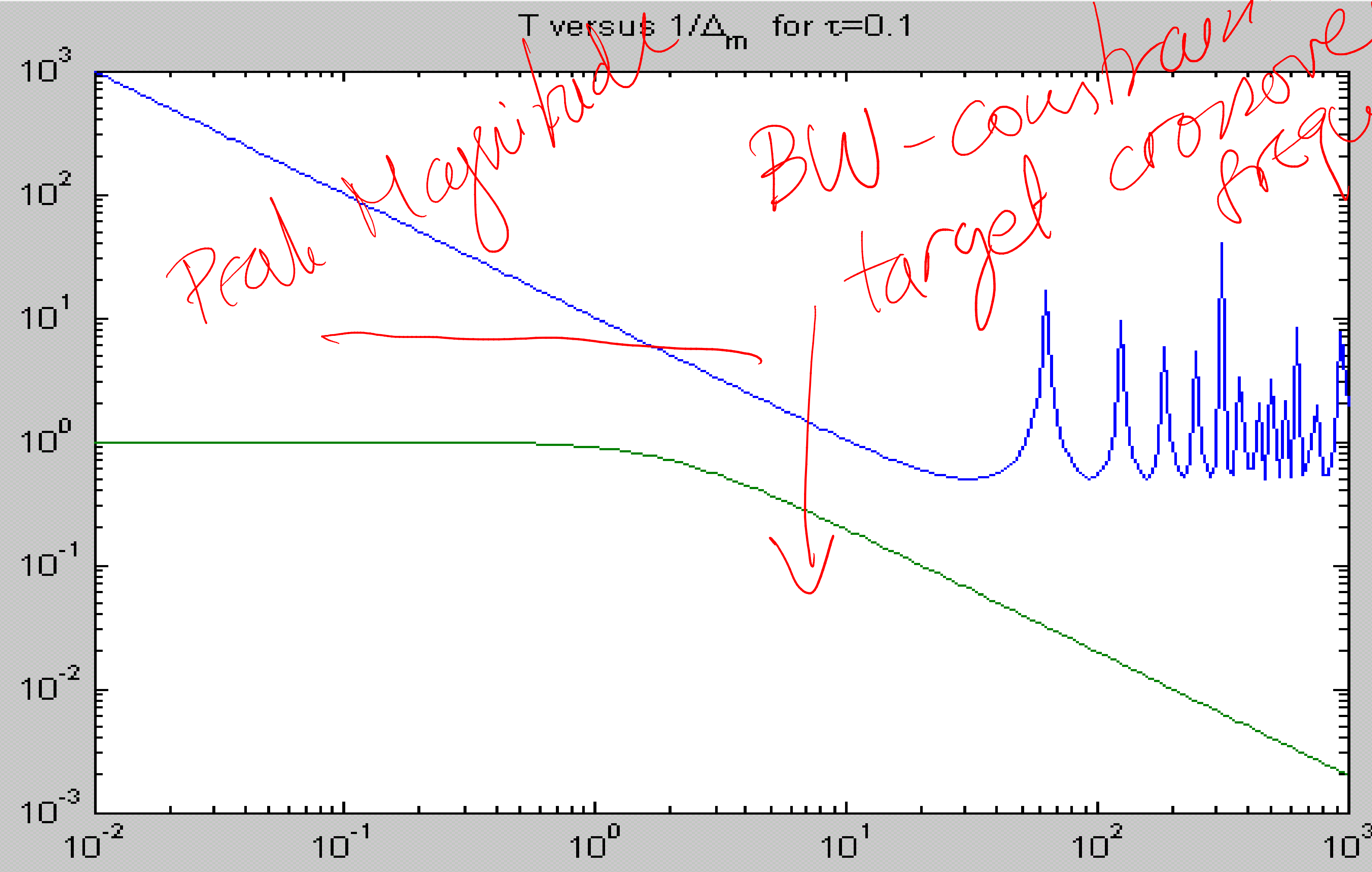
  Stable closed loop, *y* converges to *r* for any constant d, small errors for slowly varying d! :)

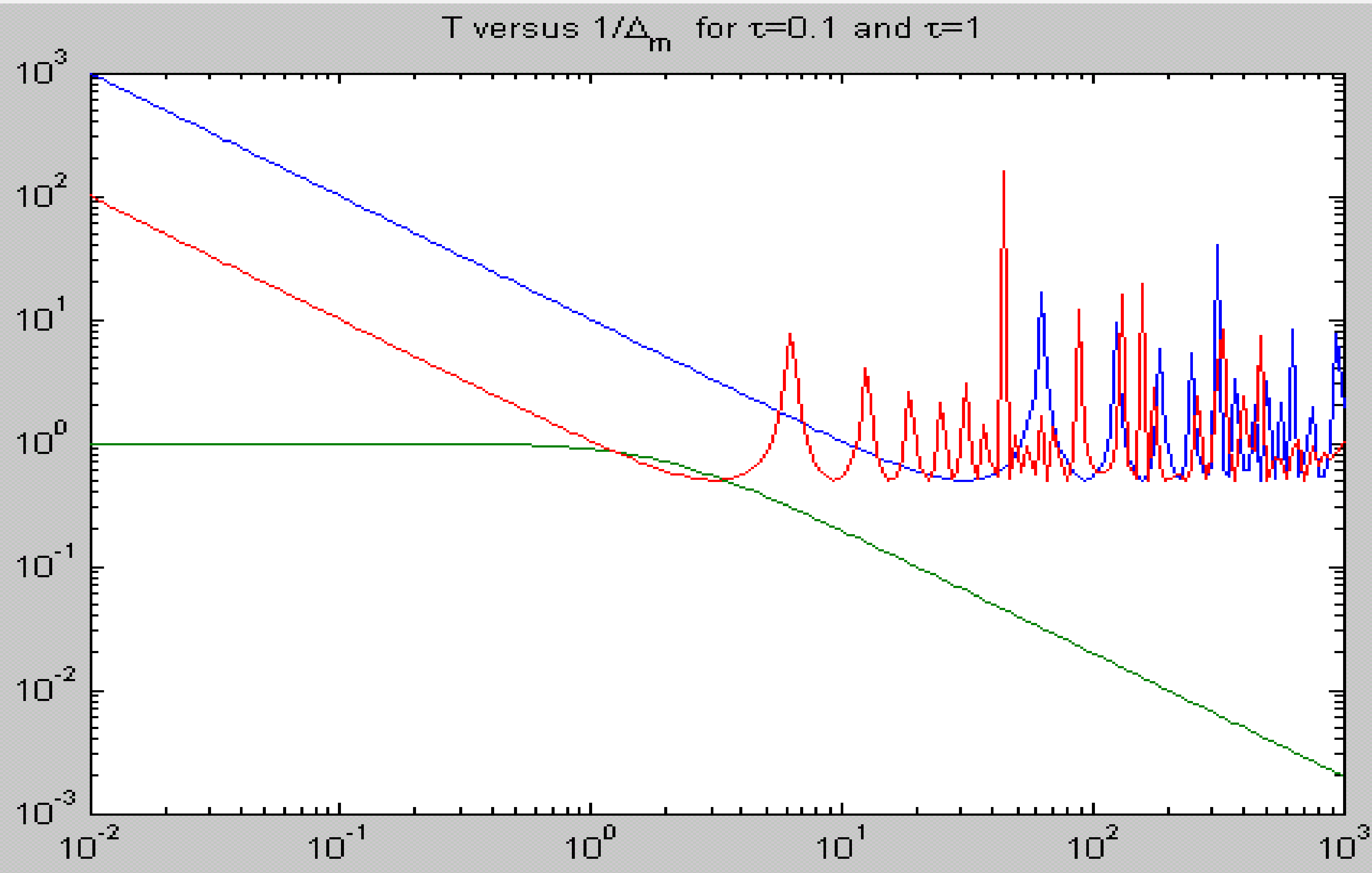
Robust controller design

The better performance has a price: This controller cannot tolerate arbitrary delays…

Left figure: closed-loop T vs. uncertainty bound;

Right figure: what happens if the delay increases…





Conclusions

- Concepts demonstrated by this example:
  - Performance-robustness tradeoff
    - Improvement with dynamic controllers
  - Our expectations from the model
  - Model uncertainty restricts our expectations from the controller
- Concepts carefully hidden in this example:
  - High order systems can impose further constraints on what is achievable
  - Nominal controller design can get quite complicated for high order systems