EEE 582, Test 1 **NAME: __SOLUTIONS__**

Oct. 12, 2011, 4 Problems, Equal Credit, Closed-book, Closed-notes, 1 sheet of formulae allowed

Problem 1. (Short questions)

1. Let ||.|| be a matrix norm; is it always true that $\|AB\| \le \|A\|\,\|B\|$?

   Not unless the norm is consistent. Induced norms are always consistent. Also the Frobenius norm (sum of entries squared) is consistent. The maximum absolute element norm is not consistent.

2. When is $e^{A+B} = e^A e^B$. How does it help in computing the matrix exponential of a Jordan block?

   When A and B commute. A Jordan block can be written as $\lambda I + N$, where N is a nilpotent matrix. Since $\lambda I, N$ commute, $e^{\lambda I+N} = e^{\lambda I}e^N = e^{\lambda}Ie^N$, i.e., a scalar-times-identity times a finite sum.

3. Let $\Phi(t,\tau)$ be the State Transition Matrix of $\dot{x} = A(t)x$. Compute the derivative $\frac{\mathrm{d}\Phi(t,\tau)}{\mathrm{d}\tau}$

   $\frac{d}{d\tau}\Phi(t,\tau) = \frac{d}{d\tau}\Phi^{-1}(\tau,t) = -\Phi^{-1}(\tau,t)\frac{d}{d\tau}\Phi(\tau,t)\Phi^{-1}(\tau,t) = -\Phi(t,\tau)A(\tau)\Phi(\tau,t)\Phi^{-1}(\tau,t) = -\Phi(t,\tau)A(\tau)$

4. Find the least squares solution of $\begin{bmatrix}1\\0\\1\end{bmatrix}x = \begin{bmatrix}1\\1\\1\end{bmatrix}$, where x in $\mathbf{R}^1$

$$x_{LS} = (A^TA)^{-1}A^Tb = \left(\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{2}2 = 1$$

Problem 2. Realize in state space the transfer function $G(s) = \frac{2s+1}{(s+1)(s+2)(s+3)}$

Approach 1: Write the PFE of G(s) and realize in a diagonal/Jordan form.

$$G(s) = \frac{\frac{-1}{2}}{(s+1)} + \frac{\frac{-3}{-1}}{(s+2)} + \frac{\frac{-5}{2}}{(s+3)} \Rightarrow A = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}, B = \begin{bmatrix}-\frac{1}{2}\\ 3\\ -\frac{5}{2}\end{bmatrix}, C = \begin{bmatrix}1 & 1 & 1\end{bmatrix}, D = 0$$

Approach 2: Expand the numerator and denominator of G(s) and realize in a companion form.

$$G(s) = \frac{2s+1}{s^3+6s^2+11s+6} \Rightarrow A = \begin{bmatrix}-6 & -11 & -6\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, B = \begin{bmatrix}1\\0\\0\end{bmatrix}, C = \begin{bmatrix}0 & 2 & 1\end{bmatrix}, D = 0$$

Problem 3. Find the state transition matrix for

$$A = \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

Using Problem 1.2,

$$e^{At} = e^{-t}\exp\left(\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = e^{-t}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + t\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = \begin{bmatrix}e^{-t} & te^{-t} & 0\\ 0 & e^{-t} & 0\\ 0 & 0 & e^{-t}\end{bmatrix}$$

Problem 4. A 4x4 matrix has eigenvalues {1,1,1,1}. Write all the possible Jordan canonical forms in the following cases:

1. Null(I-A) has dimension 3
2. $(I-A)^2 = 0$

1: 3 Jordan blocks => 2 blocks of size 1 and 1 of size 2: $\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & 1\\ & & & 1\end{bmatrix}$

2: Max. size of Jordan block = 2 => 2 blocks of size 1 and 1 of size 2; 2 Jordan blocks of size 2:
$\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & 1\\ & & & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1 & & \\ & 1 & & \\ & & 1 & 1\\ & & & 1\end{bmatrix}$. Also $\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}$ since it is not stated that $I - A \neq 0$.