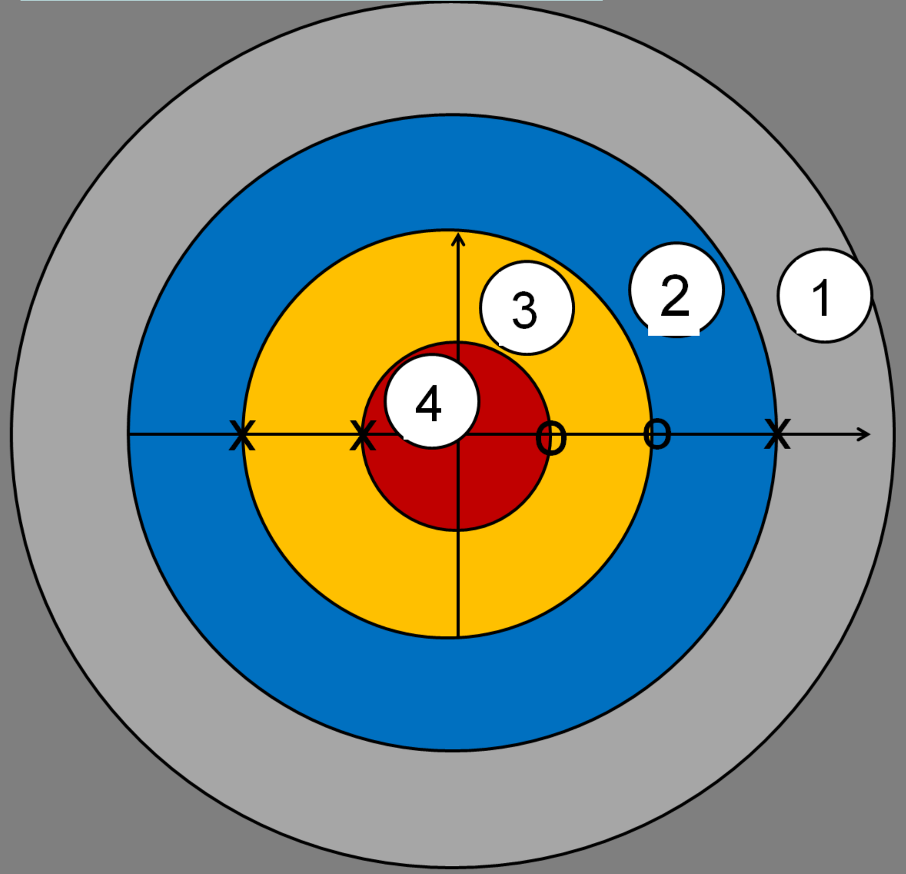
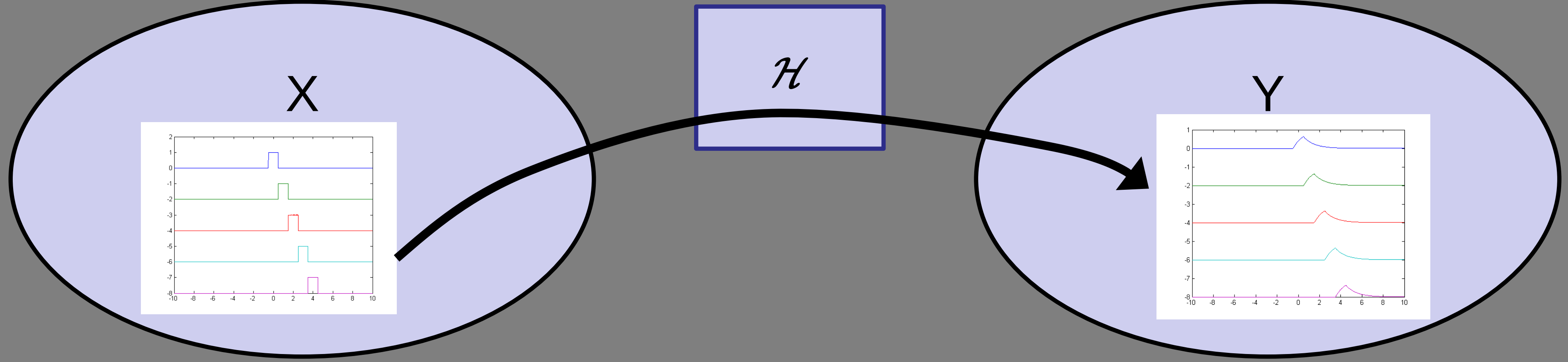
EEE304

Week 1: Review of Signals and Systems Fundamental Concepts

$$X(z)=\frac{(z-1)(z-2)}{(z-3)(z+1)(z+2)}$$





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Lecture 1.1a: Review of basic signals and their properties: Steps and Impulses

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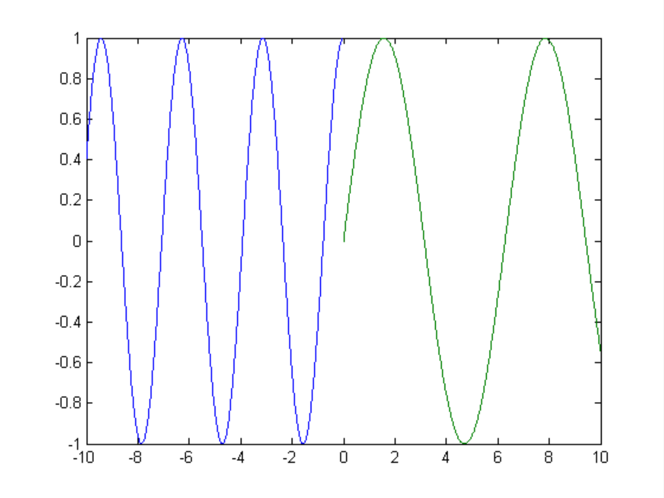
Unit Step

- Unit step: $u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$
  - The unit step serves as a set indicator, i.e., whether an argument belongs to a set or not. It is useful in writing compact expressions for “rule-based” functions
    - E.g..,$x(t) = \begin{cases} \sin t & \text{if } t \geq 0 \\ \cos 2t & \text{otherwise} \end{cases} = \sin t \, u(t) + \cos(2t)\,(1 - u(t))$

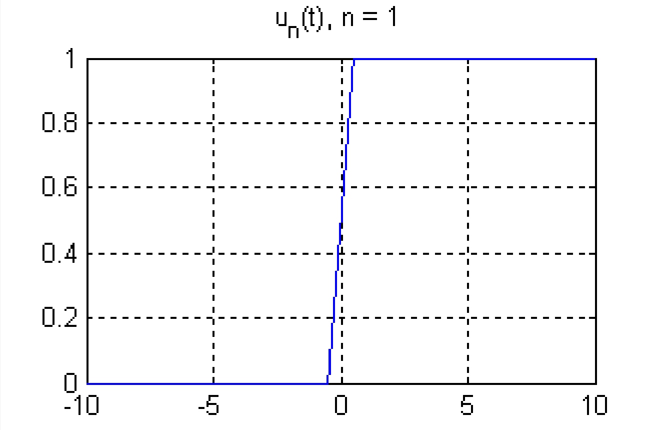


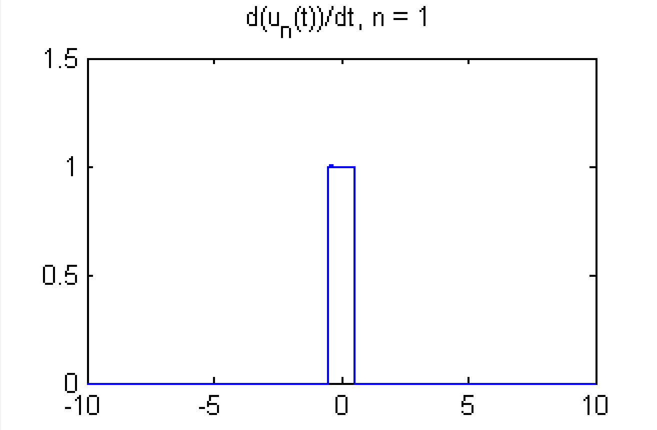
- A similar definition in Discrete-time signals (sequences)
- Unit step: $u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$
- Observation: In CT, $u(-t), \;\; 1 - u(t)$ are almost the same (they differ at single point). But in DT, $u(-n), \;\; 1 - u(n)$ are substantially different.

Unit Impulse

- Unit delta, or (Dirac) impulse: $\delta(t) = \dfrac{du(t)}{dt}$
  - The unit impulse can be viewed as the limit of the derivative of a continuous approximation of the unit step; one possible choice is $u_n(t) = \max[0, \min[1, nt + 1/2]]$

$u_1(t)$

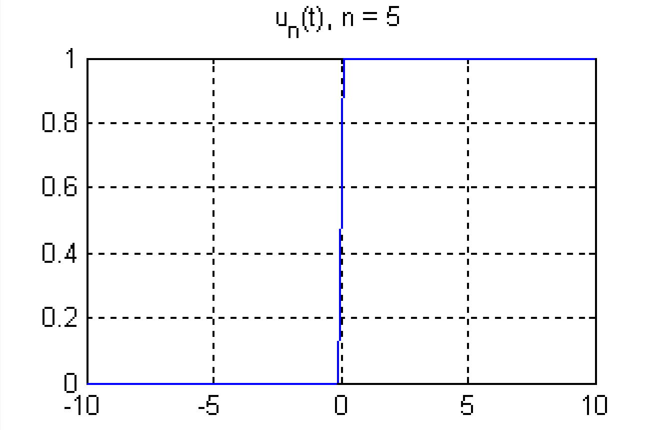


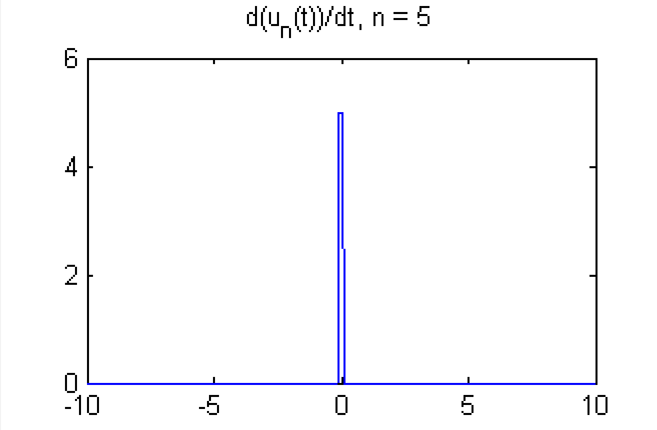


$$\delta_1(t) = \frac{du_1(t)}{dt}$$

⋮

$u_5(t)$





$$\delta_5(t) = \frac{du_5(t)}{dt}$$

⋮

$$u(t) = \lim_{n \to \infty} u_n(t)$$

$$\delta(t) \stackrel{dfn}{=} \lim_{n \to \infty} \delta_n(t)$$

Some common applications of the impulse

- Key properties of the impulse (sampling) $x(\tau)\delta(\tau) = x(0)\delta(\tau), \quad \int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$

Product with a continuous function: $\cos(\tau)\delta(\tau) = \cos(0)\delta(\tau) = \delta(\tau)$

Product and integration: $\int_{-\infty}^{\infty} \cos(\tau)\delta(\tau-1)d\tau = \int_{-\infty}^{\infty} \cos(1)\delta(\tau-1)d\tau = \cos(1)\int_{-\infty}^{\infty} \delta(\tau-1)d\tau = \cos(1)$

Product with a discontinuous function: $u(t)\delta(t) = \dfrac{u(0^+)+u(0^-)}{2}\delta(t) = \dfrac{1}{2}\delta(t), \quad (\delta \text{ is even})$

Product and integration: $\int_{-\infty}^{\infty} u(\tau)\delta(\tau-1)d\tau = \int_{-\infty}^{\infty} u(1)\delta(\tau-1)d\tau = 1\int_{-\infty}^{\infty} \delta(\tau-1)d\tau = 1$

Product and integration around a discontinuity: $\int_{-\infty}^{\infty} u(\tau-1)\delta(\tau-1)d\tau = \int_{-\infty}^{\infty} \dfrac{u(0^+)+u(0^-)}{2}\delta(\tau-1)d\tau = \dfrac{1}{2}$

Integration with the impulse on the boundary: $\int_{0}^{\infty} u(\tau)\delta(\tau)d\tau = \int_{0}^{\infty} u(0^+)\delta(\tau)d\tau = \dfrac{1}{2}$

Some common applications of the step

- Using steps to alter the interval of integration

Lower limit of integration and forward steps: $\int_{-\infty}^{\infty} x(\tau)u(\tau-1)d\tau = \int_{1}^{\infty} x(\tau)d\tau$

Upper limit of integration and backward steps: $\int_{-\infty}^{\infty} x(\tau)u(1-\tau)d\tau = \int_{-\infty}^{1} x(\tau)d\tau$

Upper limit of integration and backward steps: $\int_{-\infty}^{\infty} x(\tau)u(t-\tau)u(1-\tau)d\tau = \int_{-\infty}^{\min(1,t)} x(\tau)d\tau$

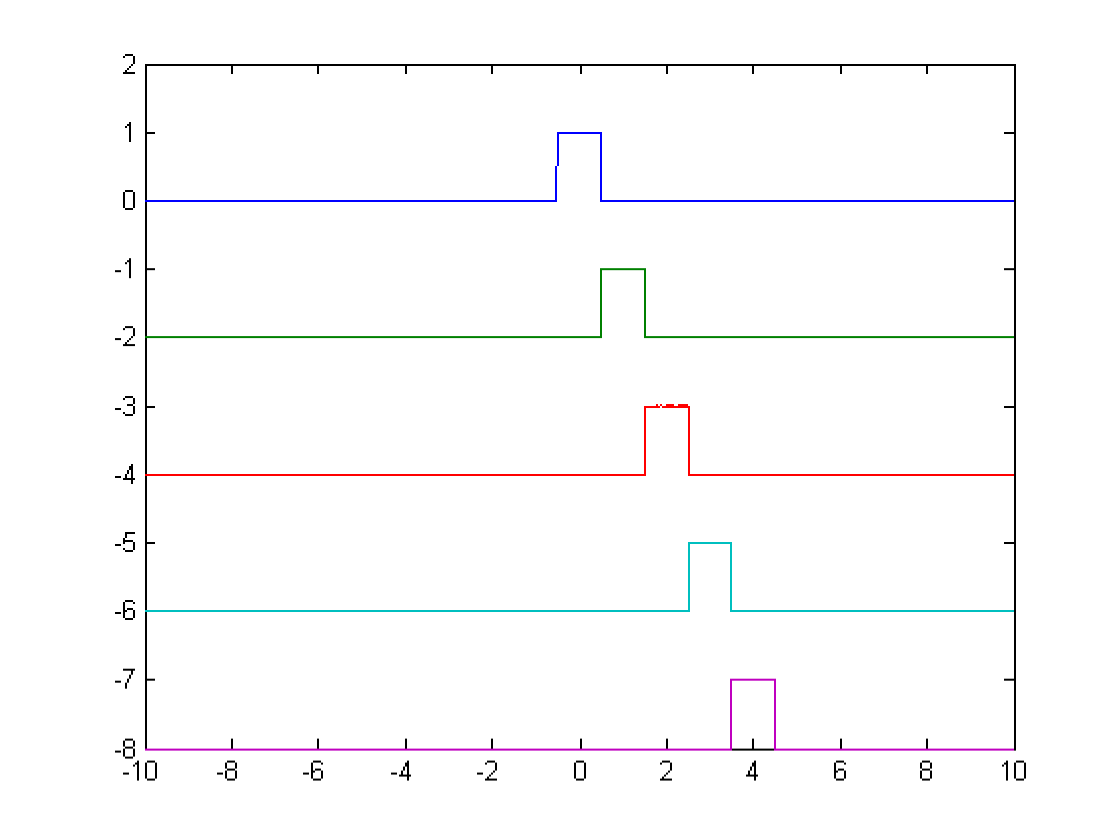
Upper and lower limit of integration and steps: $\int_{-\infty}^{\infty} x(\tau)u(t-\tau)u(\tau-1)d\tau = \left[\int_{1}^{t} x(\tau)d\tau\right]\mathrm{u(t-1)}$

Signal parametrization

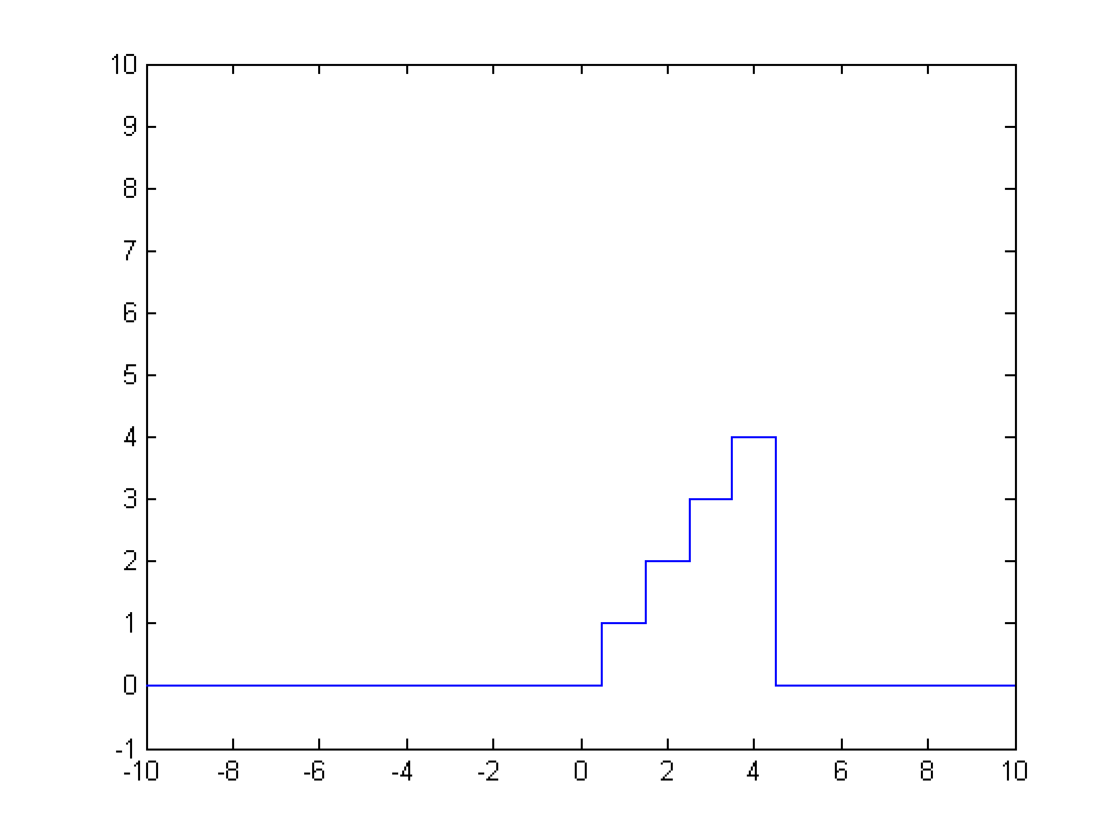
- The response of an LTI system to an impulse characterizes its response to arbitrary signals. At the heart of this result is the so-called “signal parametrization” in terms of shifted impulses

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t)\cdot 1 = x(t)$$

Shifted impulses (with offset) d0, d1, d2, …



Approximate ramp d0*0+d1*1+ d2*2+d3*3+ d4*4+…



Discrete-time development

- Unit delta, or (Kronecker) impulse: $\delta(n) = u(n) - u(n-1) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

  (an ordinary sequence)

- Signal parametrization in terms of shifted impulses

$$\sum_k x(k)\delta(n-k) = \sum_k x(n)\delta(n-k) = x(n)\sum_k \delta(n-k) = x(n)\cdot 1 = x(n)$$

- These CT/DT identities lead to the convolution integral and convolution sum, describing the output of LTI systems in terms of the input and a characteristic of the system, the “impulse response.”

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Lecture 1.1b: Review of basic systems and their properties: Impulse response

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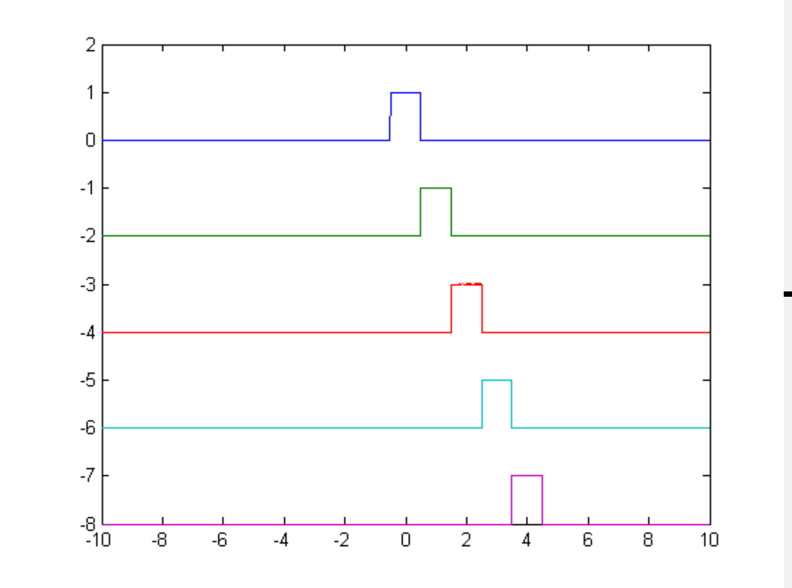
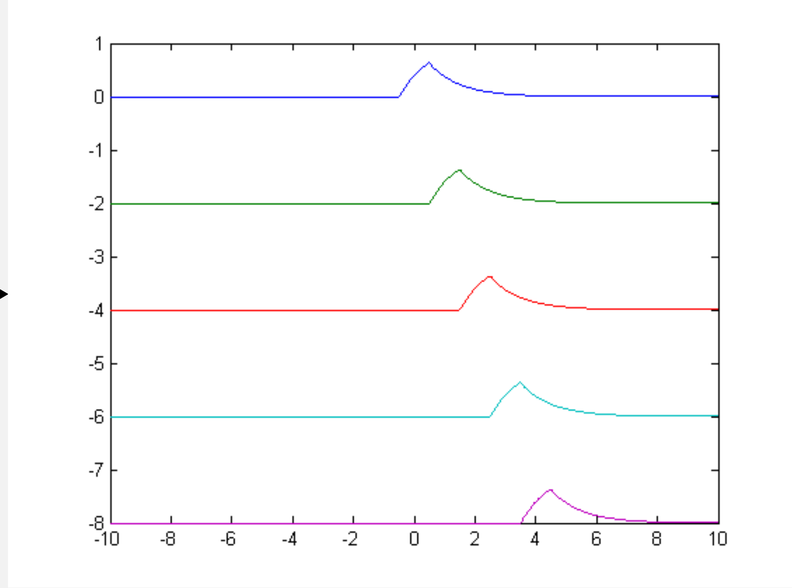
Impulses and LTI systems (CT)

- For the parametrization of the output of an LTI system in terms of shifted impulse responses (convolution integral) we operate with the system on x:

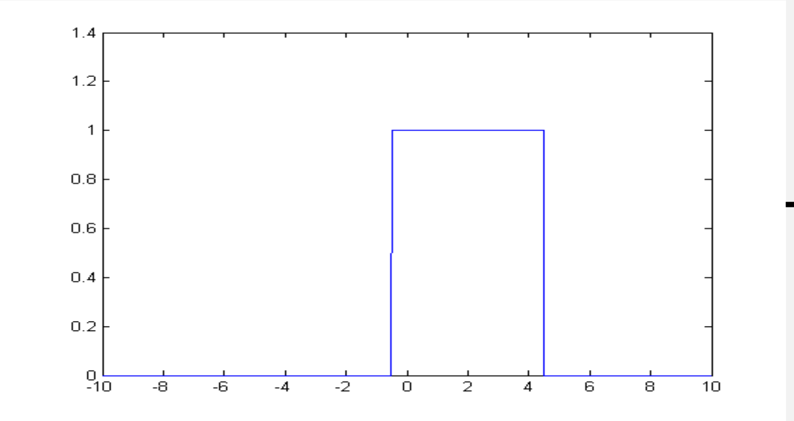
$$\mathcal{H}[x(t)]=\mathcal{H}\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]=\int_{-\infty}^{\infty}\mathcal{H}[x(\tau)\delta(t-\tau)]d\tau=\int_{-\infty}^{\infty}x(\tau)\mathcal{H}[\delta(t-\tau)]d\tau=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=y(t)$$



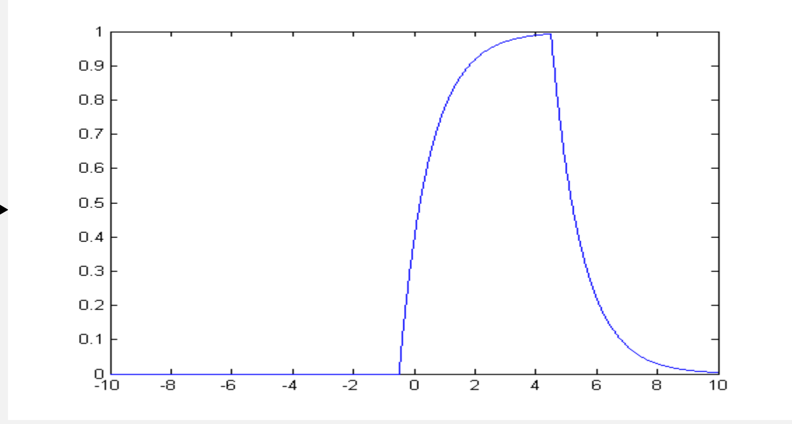
$$\longrightarrow \boxed{H(s)=\frac{1}{s+1}} \longrightarrow$$



Adding the signals to produce a “pulse” and its response



$$\longrightarrow \boxed{H(s)=\frac{1}{s+1}} \longrightarrow$$



Computation details (CT)

It is a good practice to derive here the formulae of the various responses mentioned in the last example:

$$\mathcal{H}[u(t)] = \int_{-\infty}^{\infty} u(\tau) e^{-(t-\tau)} u(t-\tau) d\tau = \int_{0}^{t} e^{-(t-\tau)} d\tau u(t) = e^{-(t)} e^{\tau} \Big|_{0}^{t} u(t-0) = (1-e^{-t}) u(t)$$

$$\mathcal{H}[pulse(t)] = \mathcal{H}[u(t) - u(t-1)] = (1-e^{-t}) u(t) - (1-e^{-(t-1)}) u(t-1)$$

Now, adding the five one-second pulse responses to obtain the five-second pulse response is a tedious but straightforward exercise (the intermediate terms cancel out).

Impulses and LTI systems (DT)

- The discrete-time analog also starts with the input parametrization:

$$x(n) = \sum_k x(k)\delta(n-k)$$

Then, operating with the system on both expressions of the input signal

$$\mathcal{H}[x(n)] = \mathcal{H}\left[\sum_k x(k)\delta(n-k)\right] = \sum_k x(k)\mathcal{H}[\delta(n-k)] = \sum_k x(k)h(n-k) = y(n)$$

- The last expression is the parametrization of the output of linear systems in terms of shifted impulse responses (convolution sum)

Computation Details (DT)

- Performing the computations for the analogous pulse example in DT:

$$\mathcal{H}[u(n)] = \sum_{k=-\infty}^{\infty} u(k)\lambda^{n-k}u(n-k) = \sum_{k=0}^{n} \lambda^{n-k}u(n-0) = \lambda^n \frac{1-\lambda^{-(n+1)}}{1-\lambda^{-1}}u(n-0) = \frac{1-\lambda^{(n+1)}}{1-\lambda}u(n)$$

- Here, we recall the well-known computation for geometric series:

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} \lambda^k = \lambda^0 + \lambda^1 + \lambda^2 + \ldots + \lambda^n \\
\lambda S_n &= \lambda \sum_{k=0}^{n} \lambda^k = \lambda^1 + \lambda^2 + \lambda^3 + \ldots + \lambda^{n+1} \\
(1-\lambda)S_n &= \lambda^0 - \lambda^{n+1} \\
S_n &= \frac{1-\lambda^{n+1}}{(1-\lambda)}
\end{aligned}$$

Computational Details (DT)

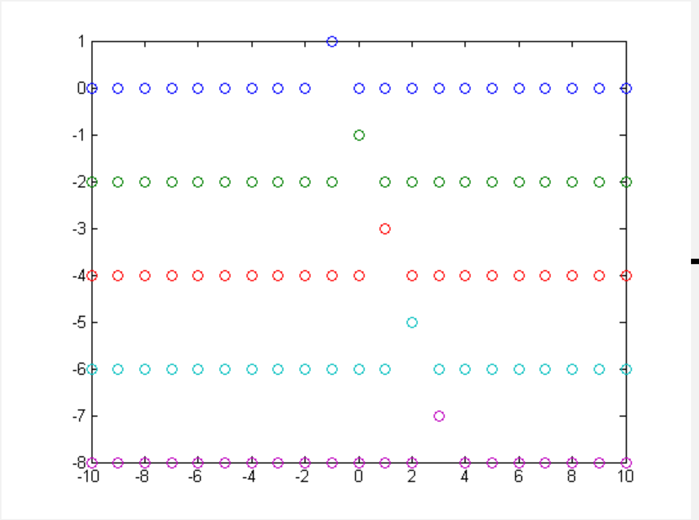
- Thus,

$$\mathcal{H}[pulse(n)]=\mathcal{H}[u(n)]-\mathcal{H}[u(n-1)]=\frac{1-\lambda^{(n+1)}}{1-\lambda}u(n)-\frac{1-\lambda^{(n)}}{1-\lambda}u(n-1)=\frac{1}{1-\lambda}\delta(n)-\frac{\lambda^{(n)}}{1-\lambda}\left[\lambda u(n)-u(n-1)\right]=$$

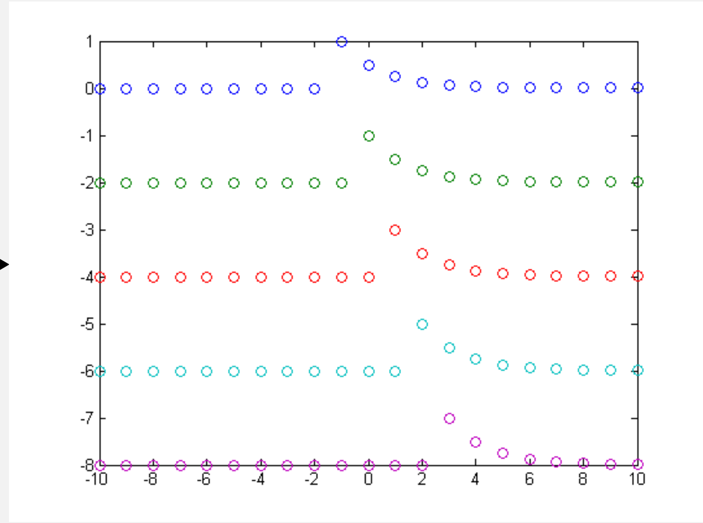
$$=\frac{1}{1-\lambda}\delta(n)-\frac{\lambda^{(n)}}{1-\lambda}\left[\lambda u(n)-u(n-1)\right]=\frac{1}{1-\lambda}\delta(n)-\frac{\lambda^{(n)}}{1-\lambda}\left[(\lambda-1)u(n)+\delta(n)\right]=\lambda^{n}u(n)$$

- This is simply the impulse response since u(n) – u(n-1) = δ(n)

$$\mathcal{H}[pulse_2(n)]=\mathcal{H}[u(n)]-\mathcal{H}[u(n-2)]=\ldots=\lambda^{n}u(n)+\lambda^{n-1}u(n-1)=\delta(n)+\left(\lambda^{n}+\lambda^{n-1}\right)u(n-1)$$



$$H(z)=\frac{z}{z-0.5}$$



MATLAB sample code for the examples

```
t=[-10:.01:10]; % define a time vector
n=1;   % pick n for the step approximation
u0=max(0,min(1,n*t+1/2));   % define shifted steps
u1=max(0,min(1,n*(t-1)+1/2));
u2=max(0,min(1,n*(t-2)+1/2));
u3=max(0,min(1,n*(t-3)+1/2));
u4=max(0,min(1,n*(t-4)+1/2));
d1=([diff(u1) 0]*100);   % compute derivatives (approx)
d0=([diff(u0) 0]*100);
d2=([diff(u2) 0]*100);
d3=([diff(u3) 0]*100);
d4=([diff(u4) 0]*100);
plot(t,[d0 ;d1-2 ;d2-4 ;d3-6; d4-8]); pause  % plot results

H=tf(1,[1 1])   % define a system by its transfer function
y0=lsim(H,d0,t);  % compute the system response to input d0
y1=lsim(H,d1,t);
y2=lsim(H,d2,t);
y3=lsim(H,d3,t);
y4=lsim(H,d4,t);
plot(t,[y0 y1-2 y2-4 y3-6 y4-8]); pause  % plot results
```

MATLAB sample code for the examples

```
N=[-10:10]; % define a time vector
n=10;   % pick n for the step approximation
u0=max(0,min(1,n*N+1));   % define shifted steps
u1=max(0,min(1,n*(N-1)+1));
u2=max(0,min(1,n*(N-2)+1));
u3=max(0,min(1,n*(N-3)+1));
u4=max(0,min(1,n*(N-4)+1));
d1=([diff(u1) 0]);   % compute differences(approx)
d0=([diff(u0) 0]);
d2=([diff(u2) 0]);
d3=([diff(u3) 0]);
d4=([diff(u4) 0]);
plot(N,[d0 ;d1-2 ;d2-4 ;d3-6; d4-8],'o'); pause  % plot results

H=tf([1 0],[1 -.5],1)   % define a system by its transfer function
y0=lsim(H,d0,N);  % compute the system response to input d0
y1=lsim(H,d1,N);
y2=lsim(H,d2,N);
y3=lsim(H,d3,N);
y4=lsim(H,d4,N);
plot(N,[y0 y1-2 y2-4 y3-6 y4-8],'o'); pause  % plot results
```

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Lecture 1.2: Review of basic signals and their properties: Exponentials

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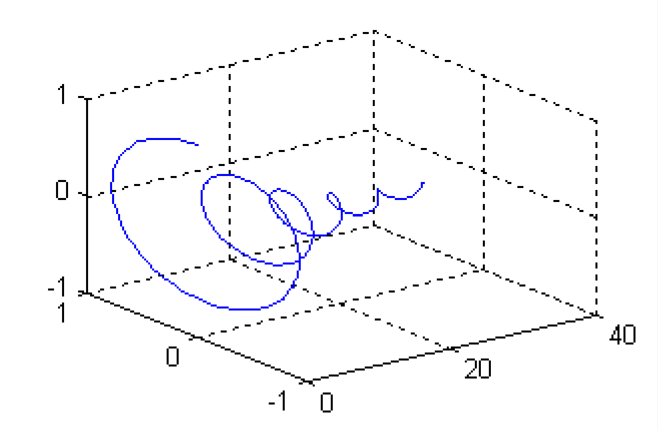
Exponentials

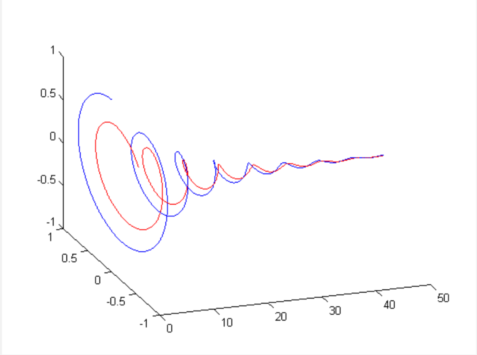


- CT Exponential: $e^{st} = e^{\sigma t + j\omega t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$
- When σ = 0, the exponential is periodic with period 2π/ω
- When σ = 0, the exponential has magnitude 1:

$$\left\| e^{j\omega t} \right\| = \sqrt{\cos^2 \omega t + \sin^2 \omega t} = 1$$

- DT Exponential: $e^{sTn} = z^n = \rho^n e^{j\Omega n} = \rho^n(\cos\Omega n + j\sin\Omega n)$
- When ρ = 1, the exponential is periodic with period N if and only if $\Omega = \dfrac{2\pi}{N}$
- When ρ = 1, the exponential has magnitude 1.
- Note: ω in rad/sec, $\Omega = \omega T$ in rad/sec × sec/sample = rad/sample

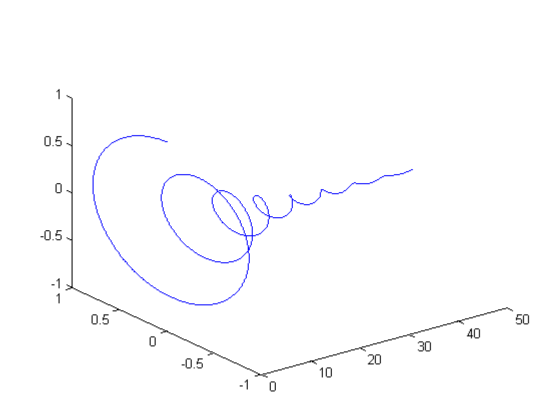
Exponentials and LTI systems (CT)



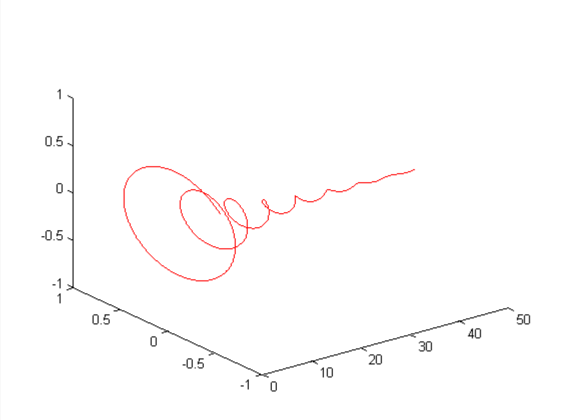
- Parametrization of the output of LTI systems in terms of exponentials (transfer function)

$$\mathcal{H}[x(t)=e^{st}]=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}x(t-\tau)h(\tau)d\tau=\int_{-\infty}^{\infty}e^{s(t-\tau)}h(\tau)d\tau=e^{st}\int_{-\infty}^{\infty}e^{-s\tau}h(\tau)d\tau=H(s)e^{st}=y(t)$$

$e^{(-0.1+j)t}$



$$\xrightarrow{e^{st}} \boxed{H(s)=\frac{1}{s+1}} \xrightarrow{H(s)e^{st}}$$



$$\frac{e^{(-0.1+j)t}}{(-0.1+j)+1}$$

Here, the Laplace transform of the impulse response (h(t)), also known as the transfer function, arises naturally as the coefficient of the output exponential.

I.e., in terms of exponentials, LTI systems are described by multipliers, while in terms of shifted impulses they are described by convolution integrals

Frequency-domain response of LTI systems

The corresponding parametrization of the output of LTI systems in the frequency domain (complex exponentials) uses the inverse Laplace transform

$$x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds \Rightarrow \text{Bromwich integral, contour in ROC}$$

$$\mathcal{H}[x(t)] = \mathcal{H}\left[\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds\right] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)\mathcal{H}[e^{st}]ds =$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s)H(s)e^{st}ds = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} Y(s)e^{st}ds \Leftrightarrow \quad Y(s) = H(s)X(s)$$

Note: $\mathcal{H}[x(t)]$ is bad notation. Writing it, we simply mean that this is the response of the system $\mathcal{H}$ with input x, a time function. The response, or output, is itself a time function. A more precise and unambiguous notation would be y = $\mathcal{H}[x]$ and its value at time t would be y(t) = $\mathcal{H}[x](t)$.

Discrete-time LTI systems in the frequency domain

- Parametrization of the output of linear systems in terms of exponentials (transfer function)

$$\mathcal{H}[x(n)=z^n]=\sum_k x(k)h(n-k)=\sum_k x(n-k)h(k)=\sum_k z^{n-k}h(k)=z^n\sum_k z^{-k}h(k)=z^nH(z)=y(n)$$

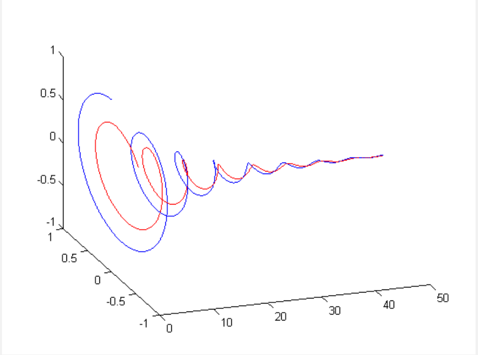
Similarly, the output in the discrete-time frequency domain is

$$Y(z)=H(z)X(z)$$

Again, the $\mathcal{Z}$-transform of the impulse response (h(n)), also known as the transfer function, arises naturally as the coefficient of the output exponential.

- Thus, in terms of exponentials, LTI systems are described by multipliers, both in CT and DT. When the exponentials are sinusoids, the description is related to the Fourier “frequency response” of the system.

Exponentials and LTI systems



- A more familiar look:
  - In continuous-time, the cosine is expressed in terms of complex exponentials and the response to a cosine can be found as a simple linear combination of responses to exponentials

$$\mathcal{H}[x(t)=\cos\omega t]=\mathcal{H}[\tfrac{e^{j\omega t}+e^{-j\omega t}}{2}]=\frac{1}{2}\left[H(j\omega)e^{j\omega t}+H(-j\omega)e^{-j\omega t}\right]=\ldots=|H(j\omega)|\cos\left[\omega t+\angle H(j\omega)\right]$$

  - Here, the derivation makes use of properties of the transfer function of real systems to express the output as a cosine of different amplitude and phase.
  - Analogous statements can be made for discrete-time systems

$$\mathcal{H}[x(n)=\cos\Omega n]=\mathcal{H}[\tfrac{e^{j\Omega n}+e^{-j\Omega n}}{2}]=\frac{1}{2}\left[H(e^{j\Omega n})e^{j\Omega n}+H(e^{-j\Omega n})e^{-j\Omega n}\right]=\ldots=|H(e^{j\Omega n})|\cos\left[\Omega n+\angle H(e^{j\Omega n})\right]$$

MATLAB sample code for the examples

```
X=exp(-0.1*t).*exp(j*t);  % define a complex exponential
plot3(t,imag(X),real(X)); pause  % plot it in 3D (Re, Im, time)
s=-0.1+j;      % define the complex frequency
Hs=1/(s+1);    % evaluate the transfer function
plot3(t,imag(Hs*X),real(Hs*X),'r');pause  % plot the 3D output
```

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Lecture 1.3: Review of Basic Systems and their properties

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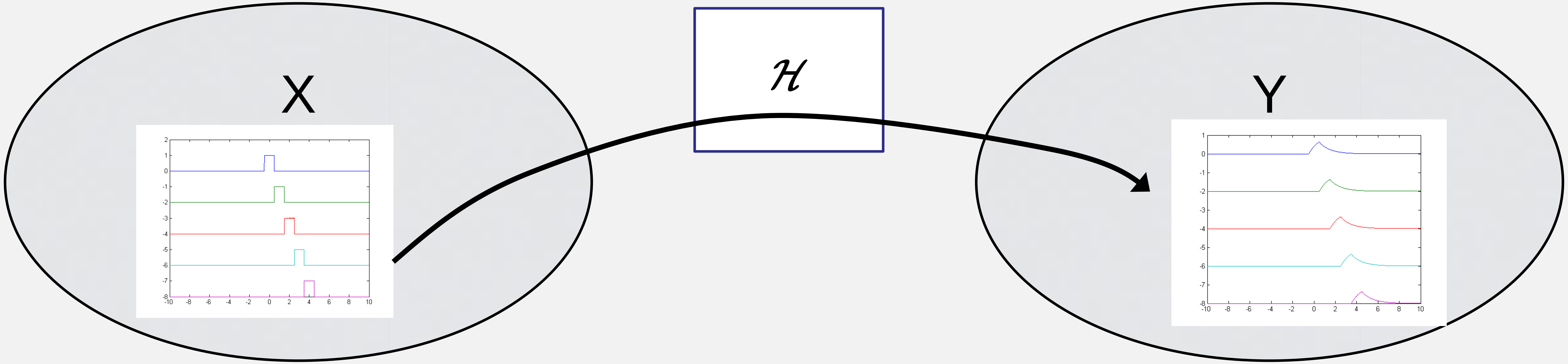
Systems: Basic definitions

- Systems are maps from a space of functions to another space of functions



- In continuous time, typical examples of the spaces X, Y are the spaces of piece-wise continuous functions, energy functions, bounded functions. In discrete time, typical spaces are bounded sequences, etc.
- We write $\mathcal{H}: X \mapsto Y, \quad y = \mathcal{H}[x], \quad y(t) = \mathcal{H}[x](t), \quad y(n) = \mathcal{H}[x](n)$
- For LTI, we also have $y(t) = (h * x)(t), \quad Y(s) = H(s)X(s), \quad Y(z) = H(z)X(z),$

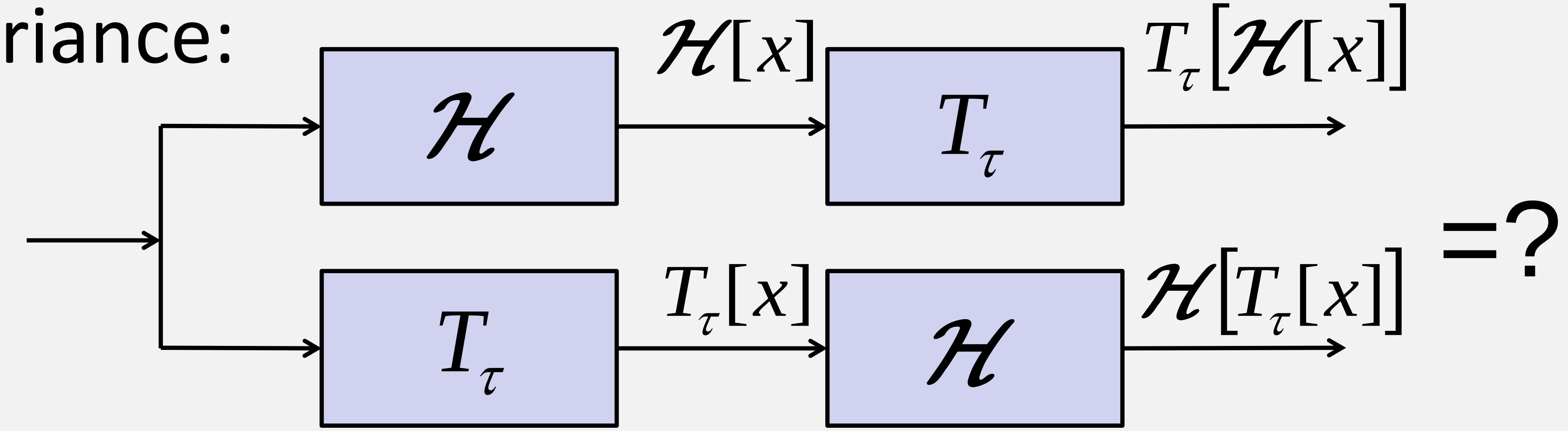
System properties

- We say that a system is
  - Linear, if $\mathcal{H}[ax_1 + bx_2] = a\mathcal{H}[x_1] + b\mathcal{H}[x_2], \quad \forall x_1, x_2, \quad \forall a, b$
  - Time-Invariant, if $\mathcal{H}T_\tau = T_\tau \mathcal{H}, \quad \forall \tau$, where $T_\tau : x(t) \mapsto T_\tau[x](t) = x(t-\tau)$ is the "shift operator" or "time-delay system"
  - Memoryless, if the value of $y(t)$ depends only on the value of $x(t)$
  - Causal, if the value of $y(t)$ depends only on the values of $x(\tau), \tau \leq t$
  - Stable, if bounded inputs produce bounded outputs.

$$x : \{\exists B > 0 : |x(t)| < B, \forall t\} \Rightarrow y = \mathcal{H}[x] : \{\exists C > 0 : |y(t)| < C, \forall t\}$$

System properties

- Analyzing system properties from the definition:
  - Linearity: $\mathcal{H}[x_1 + x_2] = \mathcal{H}[x_1] + \mathcal{H}[x_2]; \quad \mathcal{H}[ax] = a\mathcal{H}[x]$
  - Time-Invariance:



  - Stability: Bounding of system output with a sequence of inequalities, e.g.,

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \right| \leq \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|d\tau \leq \int_{-\infty}^{\infty} B_x |h(t-\tau)|d\tau \leq B_x \int_{-\infty}^{\infty} |h(t-\tau)|d\tau \leq B_x \gamma(\mathcal{H})$$

System properties: Examples I

- Some examples of systems and the properties they satisfy are:
  - $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ Linear, Time-Invariant (LTI)
  - $y(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$ LTI, Causal, Memory, Unstable
  - $y(t) = \int_{-\infty}^{\infty} x(\tau)u(-t-\tau)d\tau$ L, TV, Not-Causal, Memory, Unstable
  - $y(t) = \int_{-\infty}^{\infty} x(\tau)e^{t-\tau}u(-t+\tau)d\tau$ LTI, Not-Causal (Anti-Causal), Memory, Stable

System properties: Examples II

- Some examples of systems and the properties they satisfy are:
  - $y(t) = dx(t)/dt$ LTI, Causal, Memory, Unstable
  - $y(t) = x(at), \quad a \neq 1$ LTV, Not-Causal, Memory, Stable
  - $y(t) = \sin(5t)x(t)$ LTV, Causal, Memoryless, Stable
  - $y(t) = x(t - t_0), \quad t_0 > 0$ LTI, Causal, Memory, Stable
  - $y(t) = x(t + t_0), \quad t_0 > 0$ LTI, Not-Causal (Anti-Causal), Memory, Stable

System properties: Examples III

- More examples of systems and the properties they satisfy are:
  - $y(t)=\sin(5x(t))$ Non-Linear, TI, Causal, Memoryless, Stable
  - $y(t)=\sin(5x(t+1))$ Nonlinear, TI, Not-Causal, Memory, Stable
  - $y(t)=\sin(5t+1)x(t)$ LTV, Causal, Memoryless, Stable
  - $y(t)=e^{t-1}x(t)$ LTV, Causal, Memoryless, Unstable
  - $y(t)=e^{x(t-1)}$ NL, TI, Causal, Memory, Stable

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Lecture 1.3 Addendum: Derivations of the system properties

System properties: Examples I

#1. Linearity: $\mathcal{H}$: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

$$x(t) = ax_1(t) + bx_2(t) => \ y(t) = \int_{-\infty}^{\infty} [ax_1(\tau) + bx_2(\tau)]h(t-\tau)d\tau$$

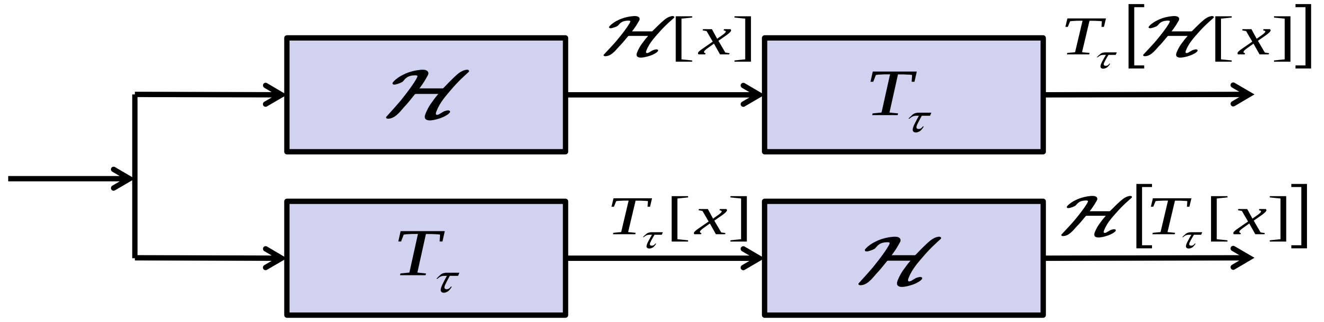
$$= \int_{-\infty}^{\infty} ax_1(\tau)h(t-\tau)d\tau + \int_{-\infty}^{\infty} bx_2(\tau)h(t-\tau)d\tau = ay_1(\tau) + by_2(\tau)$$

#1. Time Invariance:

$H[x] = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau => \ T_d\big[H[x]\big] = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\,|_{t=t-d} \ = \int_{-\infty}^{\infty} x(\tau)h(t-d-\tau)d\tau$

$T_d[x] = x(t-d) => \ H\big[T_d[x]\big] = \int_{-\infty}^{\infty} x(\tau-d)h(t-\tau)d\tau \xrightarrow{\tau'=\tau-d} \int_{-\infty}^{\infty} x(\tau')h(t-(\tau'+d))d\tau' = \int_{-\infty}^{\infty} x(\tau)h(t-d-\tau)d\tau$

$=> \ T_d\big[H[x]\big] = H\big[T_d[x]\big]$ (TI)



System properties: Examples I

#2. Causality: $\mathcal{H}$: $y(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$ => causal

#2. Stability: $\mathcal{H}$: $x(t) = u(t) => y(t) = \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \int_{0}^{t} 1 d\tau\, u(t) = tu(t)$

⇨ Unstable (unbounded output from bounded input)

#3. Time Invariance: $\mathcal{H}$: $y(t) = \int_{-\infty}^{\infty} x(\tau)u(-t-\tau)d\tau$

$H[x] = \int_{-\infty}^{\infty} x(\tau)u(-t-\tau)d\tau => T_d\big[H[x]\big] = \int_{-\infty}^{\infty} x(\tau)u(-t-\tau)d\tau\,|_{t=t-d} = \int_{-\infty}^{\infty} x(\tau)u(-t+d-\tau)d\tau$

$T_d[x] = x(t-d) => H\big[T_d[x]\big] = \int_{-\infty}^{\infty} x(\tau-d)u(-t-\tau)d\tau \xrightarrow{\tau'=\tau-d} \int_{-\infty}^{\infty} x(\tau')u(-t-(\tau'+d))d\tau' = \int_{-\infty}^{\infty} x(\tau)u(-t-d-\tau)d\tau => T_d\big[H[x]\big] \neq H\big[T_d[x]\big]$ (TV)

System properties: Examples I

#3. Causality: $\mathcal{H}$: $y(t) = \int_{-\infty}^{\infty} x(\tau)u(-t-\tau)d\tau = \int_{-\infty}^{-t} x(\tau)d\tau$ => Non causal, $y(-1)$ requires inputs $x(t), t \in (-\infty, 1]$.

#4. Causality: $\mathcal{H}$: $y(t) = \int_{-\infty}^{\infty} x(\tau)e^{t-\tau}u(-t+\tau)d\tau = \int_{t}^{\infty} x(\tau)e^{t-\tau}d\tau$ => Anticausal, $y(t)$ requires inputs $x(\tau), \tau \in [t, \infty,)$.

#4. Stability: let $|x(t)| < B$. $|y(t)| \leq \int_{t}^{\infty} |x(\tau)|e^{t-\tau}d\tau \leq Be^{t}\int_{t}^{\infty} e^{-\tau}d\tau \leq$ $-Be^{t}[e^{-\infty} - e^{-t}] \leq B$ => BIBO Stable.

System properties: Examples II

#2. Time Invariance: $\mathcal{H}$: $y(t) = x(at)$

$$H[x] = x(at) \Rightarrow T_d\big[H[x]\big] = x(at - ad)$$
$$T_d[x] = x(t-d) \Rightarrow H\big[T_d[x]\big] = x(at - d)$$
$$\Rightarrow T_d\big[H[x]\big] \neq H\big[T_d[x]\big]\ for\ a \neq 1, d \neq 0,$$ => TV

#2. Causality: We can always find t to satisfy $at > t, or, (a-1)t > 0$ => Not causal

#3. Time Invariance: $\mathcal{H}$: $y(t) = \sin(5t)\, x(t)$

$$H[x] = \sin(5t)\, x(t) \Rightarrow T_d\big[H[x]\big] = \sin(5t - 5d)\, x(t-d)$$
$$T_d[x] = x(t-d) \Rightarrow H\big[T_d[x]\big] = \sin(5t)\, x(t-d)$$
$$\Rightarrow T_d\big[H[x]\big] \neq H\big[T_d[x]\big],$$ => TV

System properties: Examples III

#3. Causality: $\mathcal{H}$: $y(t) = \sin{(5t + 1)}x(t)$

$y(t)\ only\ requires\ x(t)$

$\sin(5t + 1)$ is computed by the system (System has a "clock"!)

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Lecture 1.4: Properties of LTI systems in the Time-Domain

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Properties of LTI systems (CT)

$$y = \mathcal{H}[x] = h * x$$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau; \quad Y(s) = H(s)X(s)$$

$$y(n) = \sum_{k} h(n-k)x(k); \quad Y(z) = H(z)X(z)$$

- Here, we focus on LTI systems. Using the general parametrization in terms of their impulse response, we can obtain equivalent conditions for the remaining properties. For the continuous-time (CT) case,

$$y = \mathcal{H}[x] = h * x \Leftrightarrow y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

- The above LTI system is:
  - Causal, if and only if $h(t) = 0, \text{ for } t < 0$. Alternatively, $h(t) = h(t)u(t)$.
  - Memoryless, if and only if $h(t) = k\delta(t)$, for some constant *k*. Here, $\delta(t)$ is the Dirac delta.
  - Stable, if and only if $\int_{-\infty}^{\infty} |h(t)| d\tau < \infty$. In this case, we say that *h* is “absolutely integrable.

Properties of LTI systems (DT)

$$y = \mathcal{H}[x] = h * x$$
$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau; \quad Y(s) = H(s)X(s)$$
$$y(n) = \sum_k h(n-k)x(k); \quad Y(z) = H(z)X(z)$$

- Analogous statements are valid for discrete-time (DT) systems

$$y = \mathcal{H}[x] = h * x \Leftrightarrow y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k) \quad \left(alt. \sum_k h(n-k)x(k) \right)$$

- The above LTI system is:
  - Causal, if and only if $h(n) = 0, \text{ for } n < 0.$ Alternatively, $h(n) = h(n)u(n)$
  - Memoryless, if and only if $h(n) = k\delta(n)$, for some constant *k*. Here, $\delta(n)$ is the Kronecker delta.
  - Stable, if and only if $\sum_n |h(n)| < \infty$. In this case, we say that *h* is “absolutely summable.”

Examples of LTI system properties I

- Determine the properties of an LTI system from its impulse response:
  - $h(t) = (e^{-2t})u(t)$ Memory, Causal, Stable
  - $h(t) = (e^{-2t})u(-t)$ Memory, Anti-Causal, Unstable
  - $h(t) = (e^{2t})u(-t)$ Memory, Anti-Causal, Stable
  - $h(t) = (e^{2t})u(t)$ Memory, Causal, Unstable
  - $h(t) = (17)\delta(t)$ Memoryless, Causal, Stable

Examples of LTI system properties II

- Determine the properties of an LTI system from its impulse response:
  - $h(n) = (-3)^n u(n)$ Memory, Causal, Unstable
  - $h(n) = 0.3^n u(-n)$ Memory, Anti-Causal, Unstable
  - $h(n) = (-0.3)^n u(-n)$ Memory, Anti-Causal, Unstable
  - $h(n) = (-0.3)^n u(n)$ Memory, Causal, Stable
  - $h(n) = (17)\delta(n)$ Memoryless, Causal, Stable

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Lecture 1.5a: Properties of CT LTI systems in the Frequency Domain: ROC

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System properties from the transfer function: Region of Convergence

- The development of equivalent conditions for the properties of LTI systems from its transfer function (TF) hinges on the concept of the region of convergence (ROC) of the corresponding transform ($\mathcal{L}$,$\mathcal{Z}$)

- The ROC of the transform is the region where the corresponding integral or summation converge.

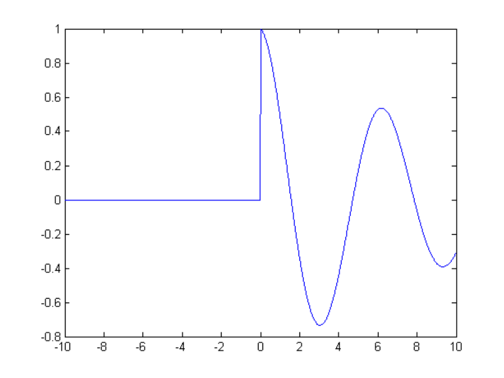
$$X(s) = \int_{-\infty}^{\infty} e^{-st} x(t) dt \qquad X(z) = \sum_{k} z^{-k} x(k)$$

By convergence we mean the formal existence of the limits

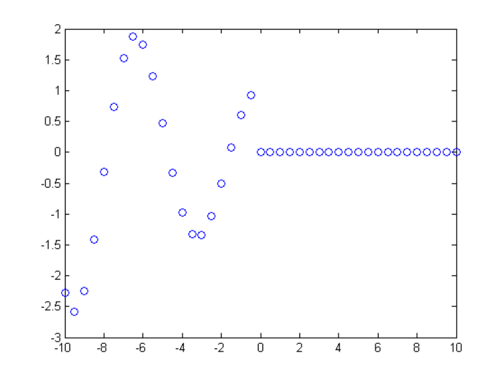
$$X(s) = \lim_{N,M \to \infty} \int_{-M}^{N} e^{-st} x(t) dt \qquad X(z) = \lim_{N,M \to \infty} \sum_{k=-M}^{N} z^{-k} x(k)$$

Region of Convergence of transforms

- The analysis of convergence also makes use of the concept of left- and right- sided functions and sequences:
  - x(t) is right sided if x(t) = x(t)u(t), e.g., a right sided CT signal is



  - x(k) is left sided if x(n) = x(n)u(-n), e.g., a left-sided DT signal is



etc.

- Fact: Any signal can be written as a sum of a right-sided and a left-sided one.

$$x(t) = x_+(t) + x\_(t) \qquad x(n) = x_+(n) + x\_(n)$$

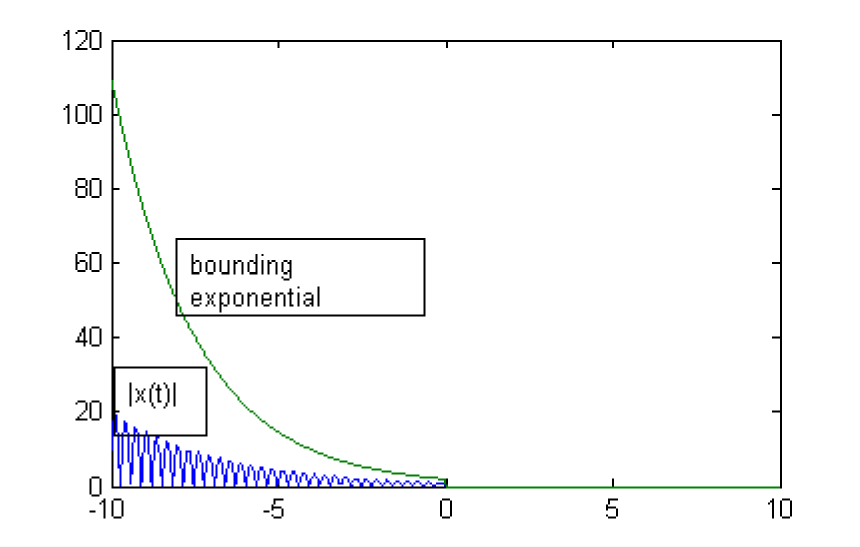
The analysis of the ROC of the transforms uses this decomposition

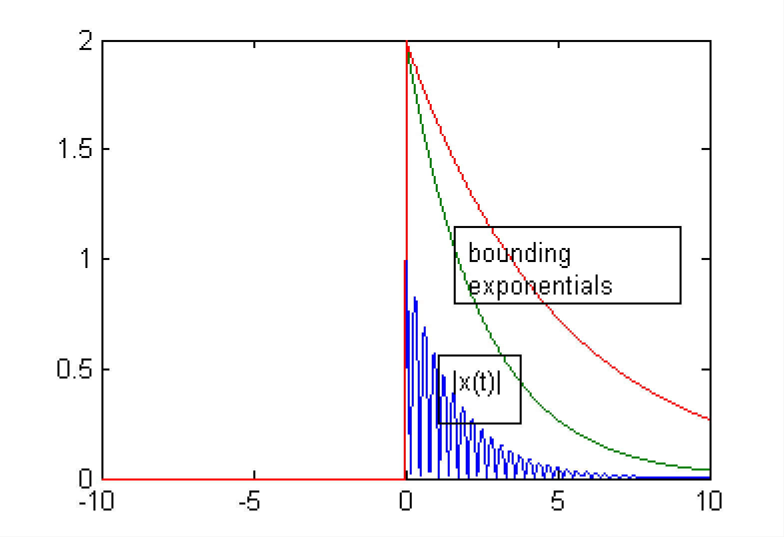
Region of Convergence of transforms (CT)

- It is now straightforward to see that if the Laplace transform of a right-sided function converges for $s = s_0$, then it converges for all s with $\operatorname{Re} s = \operatorname{Re} s_0$

  Furthermore, it also converges for all s with $\operatorname{Re} s \geq \operatorname{Re} s_0$

- For left-sided signals, the Laplace transform converges for s with $\operatorname{Re} s \leq \operatorname{Re} s_0$





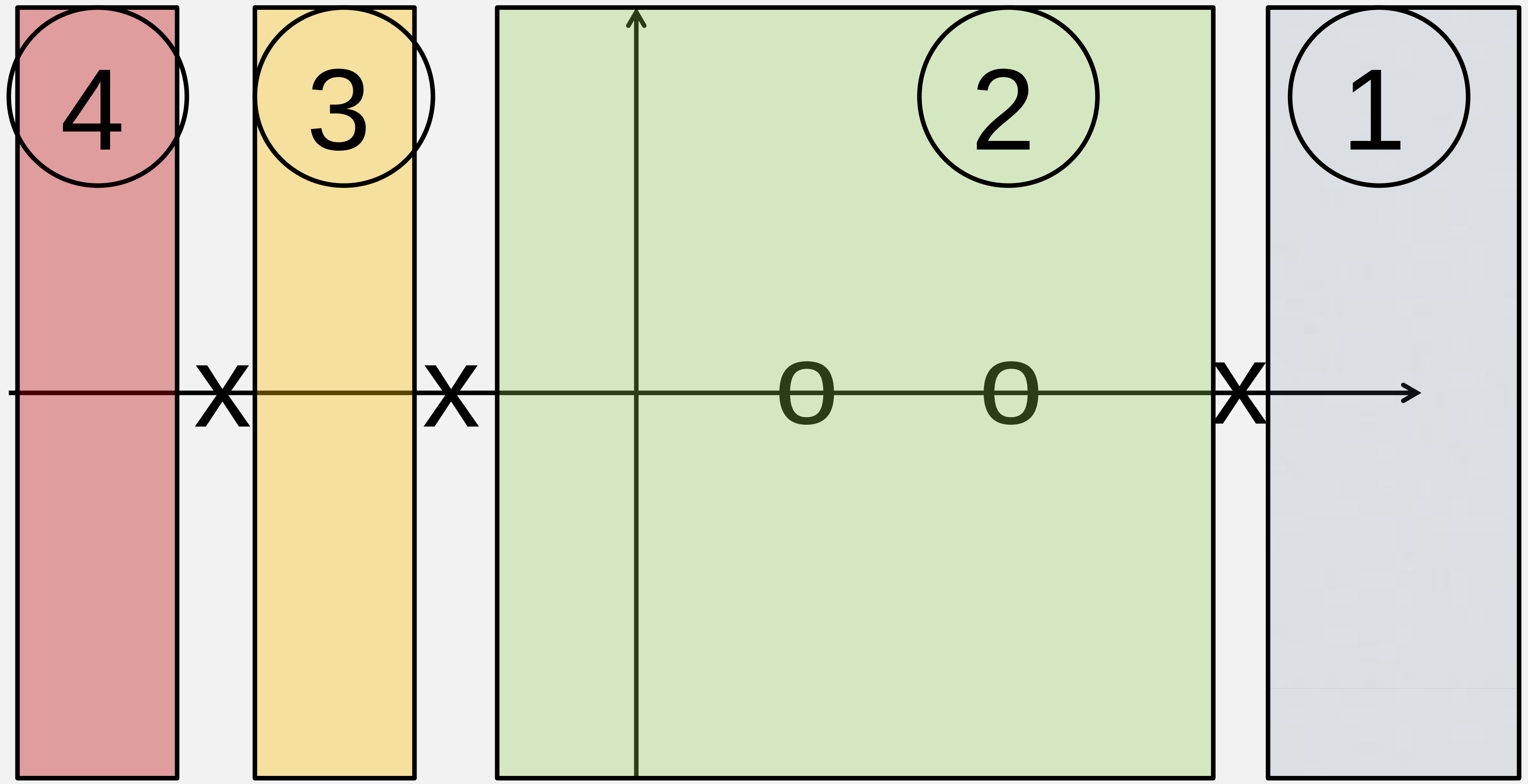
Properties of ROC of transforms (CT)

- It follows that right-sided signals have ROC that extends to $+\infty$ and left-sided signals have ROC that extends to $-\infty$.
- Since the ROC cannot (by definition) contain singularities, it is limited by the singularities (poles) of the transform function.
- Combining the left- and right-sided parts of the function, the ROC of the Laplace transform must have the form of vertical strips, with singularities on its boundary.
- The singularities to the left of the ROC correspond to right-sided time functions, and singularities to the right of the ROC correspond to left-sided time functions.

Examples of ROC of transforms (CT)

- As an example, consider the Laplace transform $X(s) = \dfrac{(s-1)(s-2)}{(s-3)(s+1)(s+2)}$
- We sketch its pole-zero plot (poles = ‘x’, zeros = ‘o’)

The possible ROC’s are highlighted with different colors. For example, ROC 1 corresponds to a right-sided function, ROC 4 to a left-sided, etc.

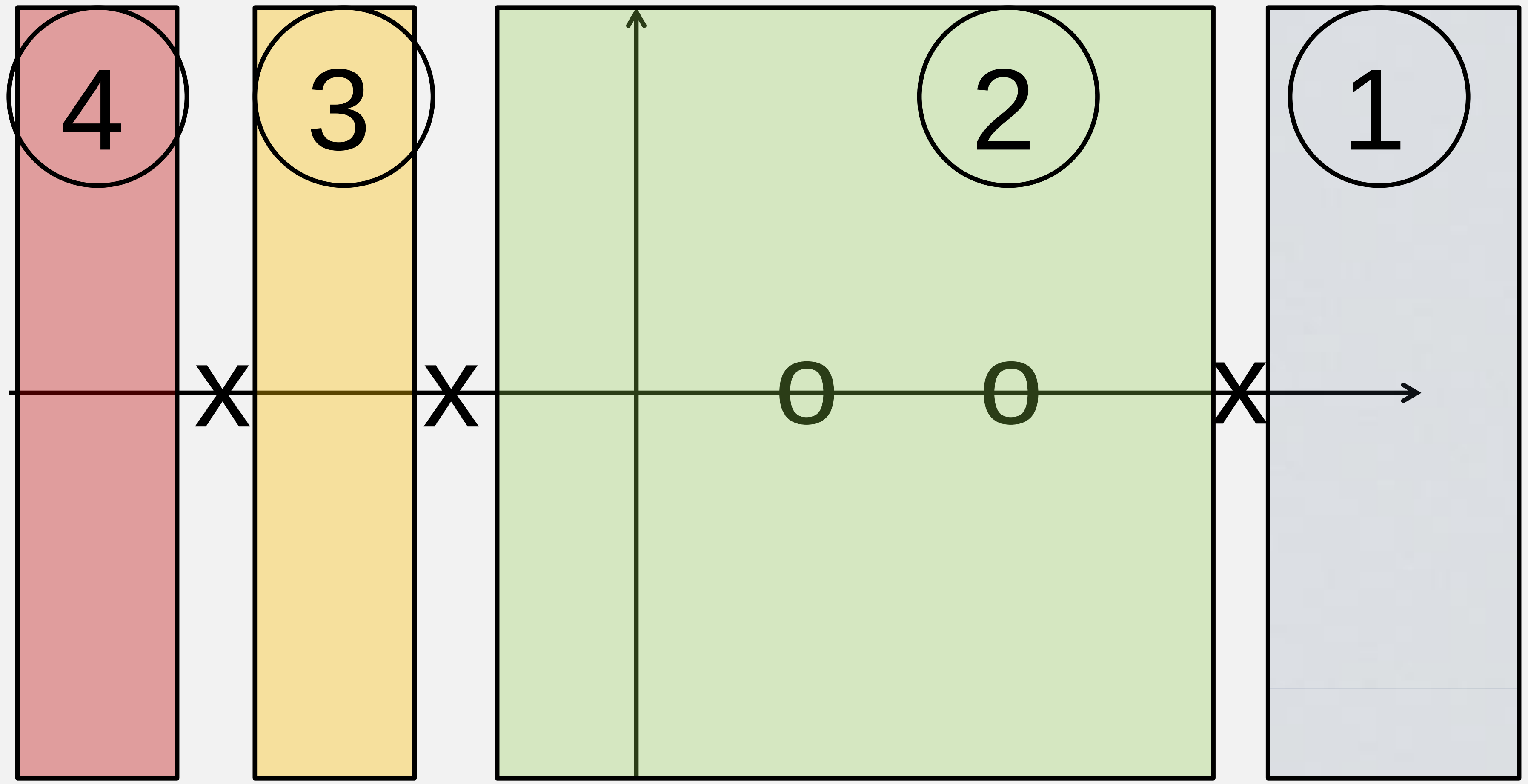


For the ROC 2, say, the pole at 3 should be inverted as left-sided and the poles at -1, -2 as right-sided. Thus, writing the PFE of X(s), we have

$$X(s) = \frac{(s-1)(s-2)}{(s-3)(s+1)(s+2)} = \frac{0.1}{(s-3)} + \frac{2.4}{(s+2)} + \frac{-1.5}{(s+1)} \Rightarrow x(t) = -0.1e^{3t}u(-t) + 2.4e^{-2t}u(t) - 1.5e^{-t}u(t)$$

Examples of ROC of transforms (CT)

- As an example, consider the Laplace transform $X(s) = \dfrac{(s-1)(s-2)}{(s-3)(s+1)(s+2)}$
- We sketch its pole-zero plot (poles = 'x', zeros = 'o')



Another example:

For the ROC 1, all poles should be inverted as right-sided. Thus, writing the PFE of X(s), we have (same PFE):

$$X(s) = \frac{(s-1)(s-2)}{(s-3)(s+1)(s+2)} = \frac{0.1}{(s-3)} + \frac{2.4}{(s+2)} + \frac{-1.5}{(s+1)} \Rightarrow x(t) = 0.1e^{3t}u(t) + 2.4e^{-2t}u(t) - 1.5e^{-t}u(t)$$

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Lecture 1.5b: Properties of DT LTI systems in the Frequency Domain: ROC

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Region of Convergence of transforms (DT)

- Analogous statements hold in DT and the Z-transform

$$X(z) = \sum_k z^{-k} x(k)$$

- Here, the quantity that determines the convergence of the Z transform is the magnitude of the discrete complex frequency z.

- Right-sided sequences have Z-transforms that converge for $|z| > |z_0|$. Their ROC extends to infinity.

- Left-sided sequences have Z-transforms that converge for $|z| < |z_0|$. Their ROC extends to zero.

Region of Convergence of transforms (DT)

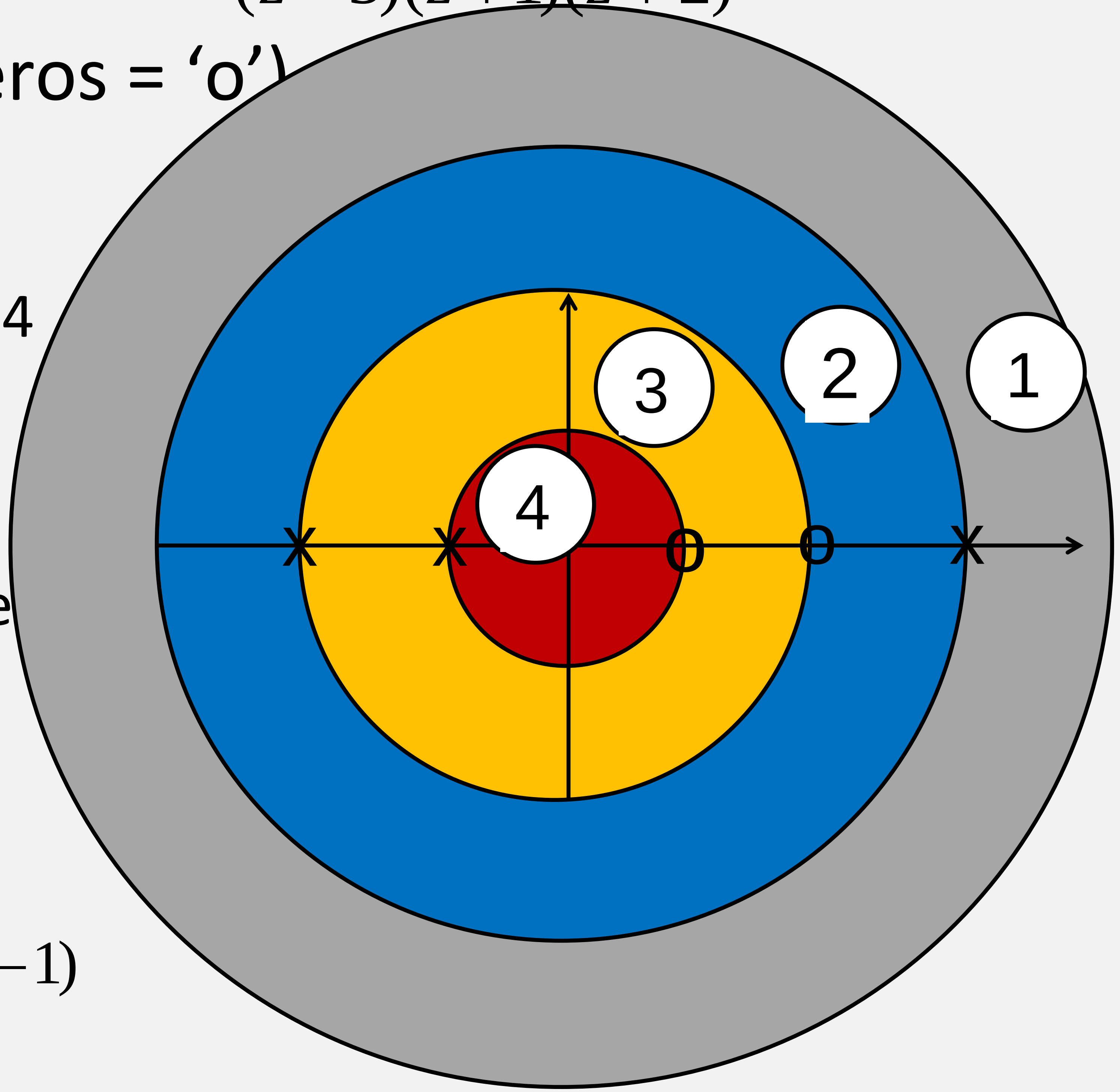
- For general sequences, the ROC of the Z transform is a so-called annular region, that is limited by the singularities of the function.
- Poles smaller than the inner radius correspond to right-sided sequences,
- Poles larger than the outer radius correspond to left-sided sequences.
- Notice the difference in DT between right- and left-sided sequences:

$$\mathcal{Z}^{-1}\{z/(z-a)\big|_{RS}\} = a^n u(n)$$

$$\text{But, } \mathcal{Z}^{-1}\{z/(z-a)\big|_{LS}\} = -a^n u(-n-1)$$

Examples of ROC of transforms (DT)

- As an example, consider the Z transform $$X(z)=\frac{(z-1)(z-2)}{(z-3)(z+1)(z+2)}$$
- We sketch its pole-zero plot (poles = ‘x’, zeros = ‘o’)



The possible ROC’s are indicated by the numbers. For example, ROC 1 corresponds to a right-sided function, ROC 4 to a left-sided, etc.

For the ROC 2, say, the pole at 3 should be inverted as left-sided and the poles at -1, -2 as right-sided. Thus, writing the PFE of X(z), we have

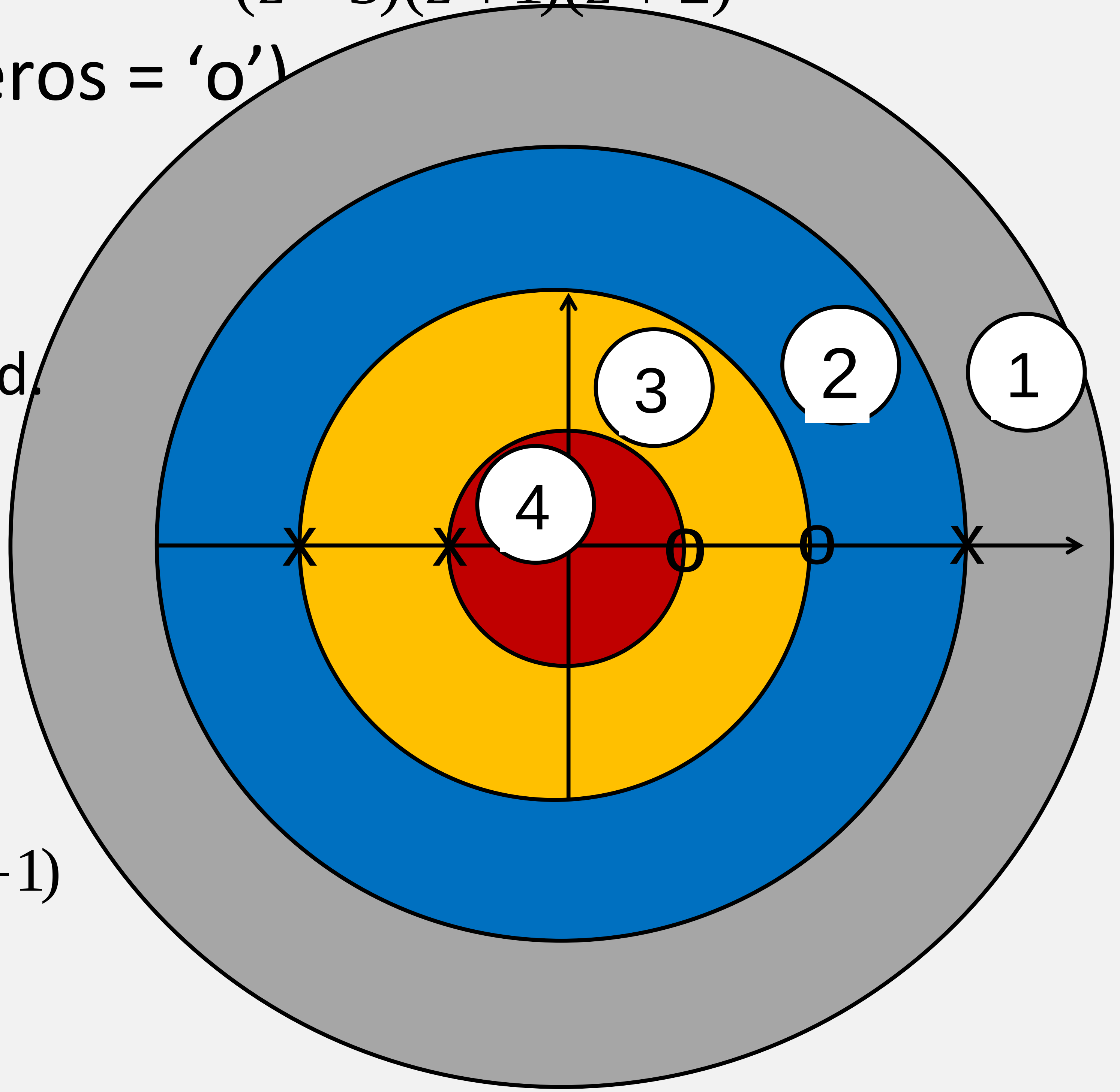
$$X(z)=\frac{(z-1)(z-2)}{(z-3)(z+1)(z+2)}=\frac{0.1}{(z-3)}+\frac{2.4}{(z+2)}+\frac{-1.5}{(z+1)}$$

$$\Rightarrow x(n)=-0.1(3)^{n-1}u(-n)+2.4(-2)^{n-1}u(n-1)-1.5(-1)^{n-1}u(n-1)$$

$$\text{Note}:\ \mathcal{Z}^{-1}\{z/(z-a)\big|_{LS}\}=-a^n u(-n-1)$$

Examples of ROC of transforms (DT)

- As an example, consider the Z transform $X(z)=\dfrac{(z-1)(z-2)}{(z-3)(z+1)(z+2)}$
- We sketch its pole-zero plot (poles = 'x', zeros = 'o')



As another example

For the ROC 1, all the poles should be inverted as right-sided. Thus, writing the same PFE of X(z), we have

$$X(z)=\frac{(z-1)(z-2)}{(z-3)(z+1)(z+2)}=\frac{0.1}{(z-3)}+\frac{2.4}{(z+2)}+\frac{-1.5}{(z+1)}$$

$$\Rightarrow x(n)=0.1(3)^{n-1}u(n)+2.4(-2)^{n-1}u(n-1)-1.5(-1)^{n-1}u(n-1)$$

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Lecture 1.6: Properties of LTI systems in the Frequency Domain: Examples

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Properties of LTI systems (CT)

$$y = \mathcal{H}[x] = h * x$$

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau; \quad Y(s) = H(s)X(s)$$

$$y(n) = \sum_{k} h(n-k)x(k); \quad Y(z) = H(z)X(z)$$

- In our framework, LTI systems are associated with an impulse response or a transfer function:

$$y = \mathcal{H}[x] = h * x \Leftrightarrow y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau, \quad Y(s) = H(s)X(s); \quad ROC_{H(s)} = \Re\{s\} \in \mathcal{M}$$

- Consequently, we can ascertain all the system properties from its Transfer Function (TF), as well as, its Impulse Response (IR)

- Some common problem statements:
    - Given an IR, determine the ROC of the corresponding TF
    - Given a TF and ROC, determine the corresponding IR
    - Given a TF and ROC determine the system properties (without explicitly finding the IR)
    - Given a TF determine the ROC that corresponds to certain system properties.

Properties of LTI systems (CT)

$$y=\mathcal{H}[x]=h*x$$
$$y(t)=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau;\quad Y(s)=H(s)X(s)$$
$$y(n)=\sum_{k}h(n-k)x(k);\quad Y(z)=H(z)X(z)$$

- Thus, the system

$$y=\mathcal{H}[x]=h*x \Leftrightarrow y(t)=\int_{-\infty}^{\infty}h(t-\tau)x(\tau)d\tau,\quad Y(s)=H(s)X(s);\quad ROC_{H(s)}=\Re\{s\}\in\mathcal{M}$$

with transfer function H(s) is:

- Causal, iff* its ROC extends to $+\infty$. Anti-Causal, iff* its ROC extends to $-\infty$.
- Memoryless, if and only if $H(s)=k$, for some constant *k*.
- Stable, if and only if its ROC contains the $j\omega$–axis. In this case, poles with negative real parts correspond to right-sided functions and poles with positive real parts correspond to left-sided functions.

*: iff for rational transfer functions, otherwise $+/-\infty$ in ROC only implies right/left-sided impulse response.

Examples of LTI system properties (Frequency Domain, CT)

Determine the transfer function ROC for an LTI system to be stable/causal:

- $H(s) = \dfrac{1}{(s+1)(s+2)}$ Causal: ROC= $\{s: -1 < \operatorname{Re} s\}$ Stable: ROC= $\{s: -1 < \operatorname{Re} s\}$
- $H(s) = \dfrac{1}{(s-1)(s-2)}$ Causal: ROC= $\{s: 2 < \operatorname{Re} s\}$ Stable: ROC= $\{s: \operatorname{Re} s < 1\}$
- $H(s) = \dfrac{1}{(s)(s-2)}$ Causal: ROC= $\{s: 2 < \operatorname{Re} s\}$ Stable: ROC= ϕ
- $H(s) = \dfrac{1}{(s+1)(s-2)}$ Causal: ROC= $\{s: 2 < \operatorname{Re} s\}$ Stable: ROC= $\{s: -1 < \operatorname{Re} s < 2\}$

Properties of LTI systems (DT)

$$y = \mathcal{H}[x] = h * x$$
$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau; \quad Y(s) = H(s)X(s)$$
$$y(n) = \sum_{k} h(n-k)x(k); \quad Y(z) = H(z)X(z)$$

- Analogous statements are valid for discrete-time (DT) systems

$$y = \mathcal{H}[x] = h * x \Leftrightarrow y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k), \quad Y(z) = H(z)X(z), \quad ROC_{H(z)} = |z| \in \mathcal{M}$$

- The above LTI system with transfer function H(z) is:
  - Causal, iff its ROC extends to ∞. Anti-Causal, iff its ROC extends to 0.
  - Memoryless, if and only if $H(z) = k$, for some constant *k*.
  - Stable, if and only if its ROC contains the unit circle. In this case, poles with modulus (magnitude) less than one correspond to right-sided functions and poles with modulus greater than one correspond to left-sided functions.

Examples of LTI system properties (Frequency Domain, DT)

Determine the transfer function ROC for an LTI system to be stable/causal:

- $H(z)=\dfrac{1}{(z+0.1)(z-0.2)}$ Causal: ROC= $\{z:0.2<|z|\}$ Stable: ROC= $\{z:0.2<|z|\}$
- $H(z)=\dfrac{1}{(z+1.1)(z-2)}$ Causal: ROC= $\{z:2<|z|\}$ Stable: ROC= $\{z:|z|<1.1\}$
- $H(z)=\dfrac{1}{(z+1)(z-2)}$ Causal: ROC= $\{z:2<|z|\}$ Stable: ROC= ϕ
- $H(z)=\dfrac{1}{(z-0.1)(z-2)}$ Causal: ROC= $\{z:2<|z|\}$ Stable: ROC= $\{z:0.1<|z|<2\}$

Notes on system properties and response computations

- Fourier transform = Laplace evaluated at s = jw, if ROC includes the jw-axis. E.g., impulse responses of stable systems, energy signals
- Power signals have Fourier transforms containing impulses and their Laplace transforms are different (at s = jw).
- Increasing exponentials or polynomials do not have a Fourier transform
- Analysis of transient responses is easier with Laplace
  - After PFE, poles should be inverted according to their location relative to the ROC
  - Periodic signals, if necessary, should be handled after projection to right sided and left sided components.

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Lecture 1.7a: Examples of system response computations

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Notes on system properties and response computations

- Analysis of periodic responses is easier with Fourier
    - FT of periodic signals is composed of impulses (through Fourier Series expansion). Then, system output is also impulses weighted by the T.F. (sampling property of impulses)
- Analysis of transient signals, would involve the convolution of the function transforms with, e.g., steps, and as such, it can be easier with Laplace
- Similarly in Discrete-time with Z- and DTFT
- Stability: ROC (of IR) must include the jw axis (CT) or the unit circle (DT). For causal systems (ROC includes +inf) this implies that all the poles must be in the left half-plane (CT) or inside the unit circle (DT).

Computation of system responses

Problem 1: Consider the filter with impulse response $h(t) = e^{-t}u(t-1) - e^{-2t}u(t)$.

1. Find the transfer function

$$H(s) = \frac{1}{e}L\{e^{-(t-1)}u(t-1)\} - L\{e^{-2t}u(t)\} = \frac{1}{e}\frac{e^{-s}}{(s+1)} - \frac{1}{(s+2)}$$

2.1 Find the Laplace transform of the output when $x(t) = \sin(t)u(t)$

$$y(s) = \left(\frac{1}{e}\frac{e^{-s}}{(s+1)} - \frac{1}{(s+2)}\right)\frac{1}{s^2+1}$$

2.2 Find the output by taking the inverse Laplace transform

$$y(t) = L^{-1}\left\{\frac{1}{e}\frac{e^{-s}}{(s+1)}\frac{1}{(s^2+1)}\right\} - L^{-1}\left\{\frac{1}{(s+2)}\frac{1}{(s^2+1)}\right\}$$

$$= \frac{1}{e}L^{-1}\left\{\frac{A}{s+1} + \frac{B}{s-j} + \frac{B^*}{s+j}\right\}_{\{t=t-1\}} - L^{-1}\left\{\frac{C}{s+2} + \frac{D}{s-j} + \frac{D^*}{s+j}\right\}$$

Computation of system responses

$$= \frac{1}{e} L^{-1} \left\{ \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2j(j+1)}}{s-j} + \frac{B^*}{s+j} \right\}_{\{t=t-1\}} - L^{-1} \left\{ \frac{1/5}{s+2} + \frac{1/2j(2+j)}{s-j} + \frac{D^*}{s+j} \right\}$$

$$= \frac{1}{e} \left\{ \frac{1}{2} e^{-t} u(t) + 2Re \left[\frac{1}{2j(j+1)} e^{jt} u(t) \right] \right\}_{\{t=t-1\}} - \left\{ \frac{1}{5} e^{-2t} u(t) + 2Re \left[\frac{1}{2j(2+j)} e^{jt} u(t) \right] \right\}$$

$$= \frac{1}{e} \left\{ \frac{1}{2} e^{-t} u(t) + Re \left[\left| \frac{1}{j(j+1)} \right| e^{j\angle \frac{1}{j(j+1)}} e^{jt} u(t) \right] \right\}_{\{t=t-1\}} - \left\{ \frac{1}{5} e^{-2t} u(t) + Re \left[\left| \frac{1}{j(2+j)} \right| e^{\angle \frac{1}{j(2+j)}} e^{jt} u(t) \right] \right\}$$

$$= \frac{1}{2e} e^{-(t-1)} u(t-1) + \frac{1}{\sqrt{2}e} \cos\left(t - 1 - \frac{\pi}{2} - \tan^{-1} 1\right) u(t-1) - \frac{1}{5} e^{-2t} u(t)$$
$$- \frac{1}{\sqrt{5}} \cos\left(t - \frac{\pi}{2} - \tan^{-1} \frac{1}{2}\right) u(t)$$

$$y(t) = \frac{1}{2e} e^{-(t-1)} u(t-1) + \frac{1}{\sqrt{2}e} \sin(t - 1 - \tan^{-1} 1) u(t-1) - \frac{1}{5} e^{-2t} u(t) - \frac{1}{\sqrt{5}} \sin\left(t - \tan^{-1} \frac{1}{2}\right) u(t)$$

Computation of system responses

2.3. Can you obtain the same result using Fourier Transforms?

The system is stable, hence the Fourier transform of its transfer function exists. The Fourier transform of the input also exists (in the sense of distributions) and so does the Fourier transform of the output. Hence, it is possible to obtain the same result. However, the appearance of impulses in the transform (due to the unit step) make the computation considerably more involved.

3.1. Find the Fourier transform of the output when $x(t) = \cos(t)$

The system is stable, so the frequency response satisfies $H(j\omega) = H(s)|_{s=j\omega}$

$$H(j\omega) = \frac{1}{e}\frac{e^{-j\omega}}{(j\omega+1)} - \frac{1}{(j\omega+2)}$$

Furthermore, $X(j\omega) = \pi\{\delta(\omega+1) + \delta(\omega-1)\}$. Hence,

Computation of system responses

$$Y(j\omega) = \left\{\frac{1}{e}\frac{e^{-j\omega}}{(j\omega+1)} - \frac{1}{(j\omega+2)}\right\}\pi\{\delta(\omega+1)+\delta(\omega-1)\}$$

$$= \pi\left\{\frac{1}{e}\frac{e^{j1}}{(-j+1)} - \frac{1}{(-j+2)}\right\}\delta(\omega+1) + \pi\left\{\frac{1}{e}\frac{e^{-j1}}{(j+1)} - \frac{1}{(j+2)}\right\}\delta(\omega-1)$$

3.2. Find the output by taking the inverse Fourier transform

$$y(t) = \frac{1}{2}\left\{\frac{1}{e}\frac{e^{j1}}{(-j+1)} - \frac{1}{(-j+2)}\right\}e^{-jt} + \frac{1}{2}\left\{\frac{1}{e}\frac{e^{-j1}}{(j+1)} - \frac{1}{(j+2)}\right\}e^{jt}$$

$$y(t) = 2Re\left[\frac{1}{2}\left\{\frac{1}{e}\frac{e^{-j1}}{(j+1)} - \frac{1}{(j+2)}\right\}e^{jt}\right] = \rho\cos(t+\phi)$$

Where

$$\rho = \left|\left\{\frac{1}{e}\frac{e^{-j1}}{(j+1)} - \frac{1}{(j+2)}\right\}\right| = 0.458, \quad \phi = \angle\left\{\frac{1}{e}\frac{e^{-j1}}{(j+1)} - \frac{1}{(j+2)}\right\} = -173.2^{o}$$

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Lecture 1.7b: Examples of system response computations

ARIZONA STATE UNIVERSITY

Computation of system responses

Problem 2:

Consider the continuous time causal filter with transfer function

$$H(s) = \frac{s}{(s-1)(s-2)}$$

1. Compute the response of the filter to $x(t) = u(t)$.

$$y(s) = \frac{s}{(s-1)(s-2)}\frac{1}{s}; ROC = \{Re\ s > 2\} \cap \{Re\ s > 0\} \text{ (right-sided poles)}$$

$$y(s) = \frac{-1}{(s-1)} + \frac{1}{(s-2)} => y(t) = -e^{t}u(t) + e^{2t}u(t)$$

2. Compute the response of the filter to $x(t) = u(-t)$.

$$ROC = \{Re\ s > 2\} \cap \{Re\ s < 0\} = \phi \text{ (no intersection, response is not well defined)}$$

Computation of system responses

3. Repeat parts 1 and 2 for a stable system with the same transfer function.

3.1 $y(s) = \frac{s}{(s-1)(s-2)}\frac{1}{s}; ROC = \{Re\ s < 1\} \cap \{Re\ s > 0\}$ (left-sided poles for H, ROC includes jw-axis)

$$y(s) = \frac{-1}{(s-1)} + \frac{1}{(s-2)} \Rightarrow y(t) = +e^{t}u(-t) - e^{2t}u(-t)$$

3.2 $y(s) = \frac{s}{(s-1)(s-2)}\frac{-1}{s}; ROC = \{Re\ s < 1\} \cap \{Re\ s < 0\}$ (left-sided poles for H, ROC includes jw-axis)

$$y(s) = \frac{+1}{(s-1)} + \frac{-1}{(s-2)} \Rightarrow y(t) = -e^{t}u(-t) + e^{2t}u(-t)$$

Computation of system responses

Problem 3: Consider the discrete time stable filter with transfer function

$$H(z) = \frac{z}{(z-0.1)(z-0.2)}$$

1. Compute the response of the filter to $x[n] = u[n]$.

$y(z) = \frac{z}{(z-0.1)(z-0.2)} \frac{z}{z-1}; ROC = \{|z| > 0.2\} \cap \{|z| > 1\}$ (right-sided poles for H, ROC includes unit circle)

$y(z) = z\left\{\frac{-2.5}{(z-0.2)} + \frac{1.1}{(z-0.1)} + \frac{1.3889}{z-1}\right\} \Rightarrow$

$y(n) = -2.5(0.2)^n u(n) + 1.1(0.1)^n u(n) + 1.3889u(n)$

(other equivalent expressions are also possible)

2. Repeat part 1 for a causal filter with the same transfer function.

Ans: $ROC = \{|z| > 0.2\}$ also corresponds to the causal filter ROC, so the response is the same as in Part 1.

Computation of system responses

Problem 4: Consider the discrete time causal filter with transfer function

$$H(z) = \frac{z}{(z-0.1)(z-0.2)}$$

1. Compute the response of the filter to $x(n) = e^{j\frac{2\pi}{12}\mathrm{n}}$

$$y(n) = H\left(e^{j\frac{2\pi}{12}}\right)e^{j\frac{2\pi}{12}\mathrm{n}} = \frac{e^{j\frac{2\pi}{12}}}{\left(e^{j\frac{2\pi}{12}}-0.1\right)\left(e^{j\frac{2\pi}{12}}-0.2\right)}e^{j\frac{2\pi}{12}\mathrm{n}}$$

$$y(n) = \frac{\cos\frac{2\pi}{12}+j\sin\frac{2\pi}{12}}{\left(\cos\frac{2\pi}{12}+j\sin\frac{2\pi}{12}-0.1\right)\left(\cos\frac{2\pi}{12}+j\sin\frac{2\pi}{12}-0.2\right)}e^{j\frac{2\pi}{12}\mathrm{n}}$$

$$y(n) = \frac{0.866+j0.5}{(0.866+j0.5-0.1)(0.866+j0.5-0.2)}e^{j\frac{2\pi}{12}\mathrm{n}} = (1.005+j0.844)e^{j\frac{2\pi}{12}\mathrm{n}} = 1.313e^{j\left(\frac{2\pi}{12}\mathrm{n}-0.699\right)}$$

MATLAB code for the examples

```
display('Problem 1')
e=exp(1);
H=1/e*tf(1,[1 1]);
H.iodelay=1
HH=ss(H)-ss(tf(1,[1 2]));  % convert to state space for MATLAB to handle delays
[m,p]=bode(HH,1)
x=1/e*(e^-j)/(j+1)-1/(j+2)  % alternative computation
abs(x)
angle(x)*180/pi

display('Problem 2')
num=[1 0],den=conv([1 0],conv([1 -1],[1 -2]))
[r,p,k]=residue(num,den)

display('Problem 3')
num=[1 0],den=conv([1 -1],conv([1 -0.1],[1 -0.2])) % factor out one z in num
[r,p,k]=residue(num,den)

display('Problem 4')
H=tf([1 0],[1 -.2],1)*tf(1,[1 -.1],1)
[m,p]=bode(H,2*pi/12)
z=exp(j*2*pi/12)          % alternative computation
x=z/(z-.1)/(z-.2)
angle(x)
abs(x)
```