EEE 582 **HW # SVD**

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% SVD application in the modeling of a noisy oscillatory signal
%     as the output of an autoregressive model:
%     y(n)=[y(n-1),y(n-2),...]*q

% Define the signals
t=(0:.01:10)';          % time
n=rand(size(t))-.5;     % noise
y=sin(2*t+1)+sin(10*t+1)+.02*n/2;    % measurement
N0=10:length(t);        % fitting window
F=tf(.01,[1 -.99],.01);y=lsim(F,y); % Optional Filtering for frequency-weighted fit

% Form the regressor by taking lags of the output
W=[y(N0-1),y(N0-2),y(N0-3),y(N0-4),y(N0-5),y(N0-6),y(N0-7),y(N0-8),y(N0-9)];
NN=0.02*[n(N0-1),n(N0-2),n(N0-3),n(N0-4),n(N0-5),n(N0-6),n(N0-7),n(N0-8),n(N0-9)];
q=W\(y(N0)), % least squares fit
plot(t,y,t(N0),W*q,t(N0),W*q-y(N0)); pause    % check the fit

% Autoregressive transfer function: resonance at the oscillation frequency
g=tf(1,[1 -q'],.01), bode(g)                  % check the t.f.
% Model order: How many columns of W do you need?
s=svd(W)             % svd of regressor
s2=svd(W'*W)          % svd of gramian

% Questions:
% 1. What is the relationship between s and s2? How many lags do you need
% in the model?
% 2. The singular values of W appear to reach a floor related to the noise.
%    Derive this value analytically and verify with an example.
% 3. What is ithe effect of the noise amplitude?
% 4. What happens when the signal is composed of two frequencies, say 10
% and 2?
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Answers

1. $s = \sqrt{s2}$; we need at least two lags (2nd order difference equation) to describe a sinusoidal solution.

2. Letting W_0 be the deterministic component and n be the noise, $W = W_0 + n$, $W^T W = (W_0+n)^T (W_0+n) = W_0^T W_0 + n^T n$ since $n \perp W$ (noise uncorrelated with signal). Furthermore, since each sample of the noise is independent, $n^T n = N\rho I$, where ρ is the noise variance and N is the number of points. For the uniform distribution in the interval $[a, b]$, with mean $\bar{n} = (a + b)/2$, $var(n) = \frac{1}{b-a}\int_a^b (n - \bar{n})^2 dn = (b - a)^2/12$. In our case the distribution is symmetric, zero mean. Let r denote the maximum amplitude. Then, $\rho = r^2/3$. In the program, $r = 0.01$, $N = 1001$, so $rho = 0.033$. The small eigenvalues of $s2$ range in 0.0355 to 0.0316, which agrees with the theoretical value.

3. Increasing the noise amplitude increases the singular values of W and introduces a bias. Denoting by '$\sharp$' the left inverse (LS solution), we have

$$q = W^{\sharp} y = (W_0 + n)^{\sharp} (y_0 + n_{k+1}) = (W_0^T W_0 + \rho I)^{-1} W_0^T y_0$$

The higher the ρ, the more the solution deviates from the nominal one $(W_0^T W_0)^{-1} W_0^T y_0$ For example, we find by trial and error that when the noise amplitude is 0.45, the resonance is smeared and is bearly recognizable.

4. When two frequencies are present, the identification is more difficult. With noise amplitude 0.01, the frequency 2 is not recognized (because it does not possess enough energy-cycles in the data interval). Reducing the noise amplitude by more than a factor of 50 allows for the second peak to appear in the fitted model. Alternatively, introducing a low-pass filter to pre-process the data has a similar effect since it attenuates the noise at high frequencies and effectively reduces the variance entering the regressor matrix.