An Introduction to Mathematical Reasoning with Applications to Induction and Contradiction Proofs

Mª Asunción Jiménez Cordero
asuncionjc@us.es

Departamento de Estadística e Investigación Operativa
and Instituto de Matemáticas de la Universidad de Sevilla,
Sevilla, Spain

Show & Tell Karumi

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https://github.com/asuncionjc/Show_and_Tell_Mathematical_Proofs

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Outline

1. What is a proof?
2. Proof by contradiction
3. Proof by induction
4. Conclusions

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Then there exists a point c in the open interval (a, b) satisfying $f(c) = 0$.

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- **Hypothesis:**
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  - $f : [a, b] \rightarrow \mathbb{R}$ is continuous.
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  - $\exists c \in (a, b) : f(c) = 0$

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Mistakes in proofs

Conjecture: $1 = 2$.

Assume a, b *are two equal positive integers:*

1. $a = b$
2. $a^2 = ab$
3. $a^2 - b^2 = ab - b^2$
4. $(a - b)(a + b) = b(a - b)$
5. $a + b = b$
6. $2b = b$
7. $2 = 1$

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Where is the error?

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Previous concepts

- **Definition:** A real number r is *rational* if there exists integers p and q, with no common factors and $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.
- **Theorem:** Let a be an integer number such that a^2 is even. Then a is also even.

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Contradiction!

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2. $\sqrt{2} = \frac{p}{q}$, p and q without common factors.
3. $2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2$.
4. p^2 is even and, by the previous theorem, p is also even.
5. $p = 2s$, for some integer s.
6. $(2s)^2 = 2q^2 \Rightarrow 4s^2 = 2q^2 \Rightarrow 2s^2 = q^2$.
7. q^2 is even, and so it is q.

Contradiction!

On the one hand, we assumed p and q have no common factors.
On the other hand, if p and q are even, 2 is a common factor.
Therefore, $\boldsymbol{\sqrt{2}}$ **is irrational**.

It's your turn!

Theorem: Let a be an integer number such that a^2 is even. Then a is also even.

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Theorem: Let a be an integer number such that a^2 is even. Then a is also even.

1. Assume a is odd.
2. $a = 2k + 1$ for some integer k.
3. $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$
4. a^2 is odd.

Contradiction!

Since, by hypothesis, a^2 is even.

Therefore, $\boldsymbol{a}$ **is even**.

Another example?

Theorem: For every natural number $a > 2$ and a prime, then it holds that a is odd.

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Theorem: For every natural number $a > 2$ and a prime, then it holds that a is odd.

1. Assume a is even.
2. $a = 2k$, for some k. Since $a > 2$, then $k \neq 1$.
3. a is composite.

Contradiction!

Since, by hypothesis, a is prime.

Therefore, $\boldsymbol{a}$ **is odd**.

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2. Assume the result for all $k < n$ (Induction hypothesis).
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1. $(n = 1)$ $\sum\limits_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$.
2. (Induction hypothesis) The result is true for all $k < n$.
3. $\sum\limits_{i=1}^{n} i = \left(\sum\limits_{i=1}^{n-1} i\right) + n = \frac{(n-1)(n-1+1)}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$.

Your turn!

Theorem (Geometric series): For all $n \in \mathbb{N}$ and $r \neq 1$:

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

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1. $(n=0)$: $\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{1-r}{1-r}$.
2. (Induction hypothesis) The result is true for all $k < n$.
3. $\sum_{i=0}^{n} r^i = \sum_{i=0}^{n-1} r^i + r^n = \frac{1-r^n}{1-r} + r^n = \frac{1-r^n+r^n(1-r)}{1-r} = \frac{1-r^n+r^n-r^{n+1}}{1-r} = \frac{1-r^{n+1}}{1-r}$.

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Theorem: For all $n \in \mathbb{N}$:

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1. $(n = 1) : 11^1 - 6 = 5$ which is divisible by 5.
2. (Induction hypothesis) The result is true for all $k < n$, and therefore $11^k = 5m + 6$, for some integer m.
3. $11^n - 6 = \left(11 \cdot 11^{n-1}\right) - 6 = 11 \cdot (5m + 6) - 6 = 11 \cdot 5m + 66 - 6 = 11 \cdot 5m + 60 = 5 \cdot (11m + 12)$, which is divisible by 5.

Outline

- Some basic concepts about the mathematical results and how to prove them.
- Proof by contradiction.
- Proof by induction.

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Thank you for your attention!

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