

An Introduction to Mathematical Reasoning with Applications to Induction and Contradiction Proofs

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Show & Tell Karumi

February 1st 2019

Outline

- 1 What is a proof?
- 2 Proof by contradiction
- 3 Proof by induction
- 4 Conclusions

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Bolzano's theorem: If f is a continuous function defined on a closed interval $[a, b]$ such that $\text{sign}(f(a)) \neq \text{sign}(f(b))$.

Then there exists a point c in the open interval (a, b) satisfying $f(c) = 0$.

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 - $\exists c \in (a, b) : f(c) = 0$

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Mistakes in proofs

Conjecture: $1 = 2$.

Assume a, b are two equal positive integers:

- ① $a = b$
- ② $a^2 = ab$
- ③ $a^2 - b^2 = ab - b^2$
- ④ $(a - b)(a + b) = b(a - b)$
- ⑤ $a + b = b$
- ⑥ $2b = b$
- ⑦ $2 = 1$

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Where is the error?

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Previous concepts

- **Definition:** A real number r is *rational* if there exists integers p and q , with no common factors and $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.
- **Theorem:** Let a be an integer number such that a^2 is even. Then a is also even.

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Contradiction!

On the one hand, we assumed p and q have no common factors. On the other hand, if p and q are even, 2 is a common factor. Therefore, $\sqrt{2}$ is irrational.

It's your turn!

Theorem: Let a be an integer number such that a^2 is even. Then a is also even.

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Theorem: Let a be an integer number such that a^2 is even. Then a is also even.

- ① Assume a is odd.
- ② $a = 2k + 1$ for some integer k .
- ③ $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
- ④ a^2 is odd.

Contradiction!

Since, by hypothesis, a^2 is even.

Therefore, a is even.

Another example?

Theorem: For every natural number $a > 2$ and a prime, then it holds that a is odd.

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Theorem: For every natural number $a > 2$ and a prime, then it holds that a is odd.

- ① Assume a is even.
- ② $a = 2k$, for some k . Since $a > 2$, then $k \neq 1$.
- ③ a is composite.

Contradiction!

Since, by hypothesis, a is prime.

Therefore, a is odd.

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- ② Assume the result for all $k < n$ (Induction hypothesis).
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- ❸ $\sum_{i=1}^n i = \left(\sum_{i=1}^{n-1} i \right) + n = \frac{(n-1)(n-1+1)}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$.

Your turn!

Theorem: For all $n \in \mathbb{N}$ and $r \neq 1$:

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}$$

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Theorem: For all $n \in \mathbb{N}$ and $r \neq 1$:

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}$$

- ① $(n = 0) : \sum_{i=0}^0 r^i = r^0 = 1 = \frac{1-r}{1-r}.$
- ② (Induction hypothesis) The result is true for all $k < n$.
- ③
$$\sum_{i=0}^n r^i = \sum_{i=0}^{n-1} r^i + r^n = \frac{1-r^n}{1-r} + r^n = \frac{1-r^n+r^n(1-r)}{1-r} = \frac{1-r^n+r^n-r^{n+1}}{1-r}.$$

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- ① $(n = 1) : 11^1 - 6 = 5$ which is divisible by 5.
- ② (Induction hypothesis) The result is true for all $k < n$, and therefore $11^k = 5m + 6$, for some integer m .
- ③ $11^n - 6 = (11 \cdot 11^{n-1}) - 6 = 11 \cdot (5m + 6) - 6 = 11 \cdot 5m + 66 - 6 = 11 \cdot 5m + 60 = 5 \cdot (11m + 12)$, which is divisible by 5.

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- Some basic concepts about the mathematical results and how to prove them.
- Proof by contradiction.
- Proof by induction.

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