An Introduction to Mathematical Reasoning with Applications to Induction and Contradiction Proofs

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Show & Tell Karumi

XX XXXX 2019

Outline

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Bolzano's theorem: If f is a continuous function defined on a closed interval $[a,b]$ such that $sign(f(a)) \neq sign(f(b))$.
Then there exists a point c in the open interval (a,b) satisfying $f(c)=0$.

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- Thesis:
  - $\exists c \in (a, b) : f(c) = 0$

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Mistakes in proofs

Conjecture: $1 = 2$.

Assume a, b are two equal positive integers:

1. $a = b$
2. $a^2 = ab$
3. $a^2 - b^2 = ab - b^2$
4. $(a - b)(a + b) = b(a - b)$
5. $a + b = b$
6. $2b = b$
7. $2 = 1$

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Where is the error?

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1. Assume the thesis is false.
2. Show that this assumption leads to a contradiction.
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Theorem: $\sqrt{2}$ is irrational.

Previous concepts

- **Definition:** A real number r is *rational* if there exists integers p and q, with no common factors and $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.
- **Theorem:** Let a be an integer number such that a^2 is even. Then a is also even.

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Contradiction!

On the one hand, we assumed p and q have no common factors.
On the other hand, if p and q are even, 2 is a common factor.
Therefore, $\boldsymbol{\sqrt{2}}$ **is irrational**.

It's your turn!

Theorem: Let a be an integer number such that a^2 is even. Then a is also even.

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Theorem: Let a be an integer number such that a^2 is even. Then a is also even.

1. Assume a is odd.
2. $a = 2k + 1$ for some integer k.
3. $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
4. a^2 is odd.

Contradiction!

Since, by hypothesis, a^2 is even.

Therefore, a is even.

Another example?

Theorem: For every natural number $a > 2$ and a prime, then it holds that a is odd.

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Theorem: For every natural number $a > 2$ and a prime, then it holds that a is odd.

1. Assume a is even.
2. $a = 2k$, for some k. Since $a > 2$, then $k \neq 1$.
3. a is composite.

Contradiction!
Since, by hypothesis, a is prime.
Therefore, a is odd.

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