

# An Introduction to Mathematical Reasoning with Applications to Induction and Contradiction Proofs

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Show & Tell Karumi

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[https://github.com/asuncionjc/  
Show\\_and\\_Tell\\_Mathematical\\_Proofs](https://github.com/asuncionjc/Show_and_Tell_Mathematical_Proofs)



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# Outline

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- 2 Proof by contradiction
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**Bolzano's theorem:** If  $f$  is a continuous function defined on a closed interval  $[a, b]$  such that  $\text{sign}(f(a)) \neq \text{sign}(f(b))$ .

Then there exists a point  $c$  in the open interval  $(a, b)$  satisfying  $f(c) = 0$ .



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### Mistakes in proofs

**Conjecture:**  $1 = 2$ .

*Assume  $a, b$  are two equal positive integers:*

- ①  $a = b$
- ②  $a^2 = ab$
- ③  $a^2 - b^2 = ab - b^2$
- ④  $(a - b)(a + b) = b(a - b)$
- ⑤  $a + b = b$
- ⑥  $2b = b$
- ⑦  $2 = 1$

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**Where is the error?**

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### Previous concepts

- **Definition:** A real number  $r$  is *rational* if there exists integers  $p$  and  $q$ , with no common factors and  $q \neq 0$  such that  $r = p/q$ . A real number that is not rational is called *irrational*.
- **Theorem:** Let  $a$  be an integer number such that  $a^2$  is even. Then  $a$  is also even.

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## Contradiction!

On the one hand, we assumed  $p$  and  $q$  have no common factors. On the other hand, if  $p$  and  $q$  are even, 2 is a common factor. Therefore,  $\sqrt{2}$  is irrational.

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**Theorem:** Let  $a$  be an integer number such that  $a^2$  is even. Then  $a$  is also even.

- ❶ Assume  $a$  is odd.
- ❷  $a = 2k + 1$  for some integer  $k$ .
- ❸  $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ .
- ❹  $a^2$  is odd.

## Contradiction!

Since, by hypothesis,  $a^2$  is even.

Therefore,  $a$  is even.

# Another example?

**Theorem:** For every natural number  $a > 2$  and  $a$  prime, then it holds that  $a$  is odd.



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**Theorem:** For every natural number  $a > 2$  and  $a$  prime, then it holds that  $a$  is odd.

- ① Assume  $a$  is even.
- ②  $a = 2k$ , for some  $k$ . Since  $a > 2$ , then  $k \neq 1$ .
- ③  $a$  is composite.

**Contradiction!**

Since, by hypothesis,  $a$  is prime.

Therefore,  $a$  is odd.

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- ① Base case ( $n = 1$ ).
- ② Assume the result for all  $k < n$  (Induction hypothesis).
- ③ Prove for  $n$ .

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**Theorem:** For all  $n \in \mathbb{N}$ :

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

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1  $(n = 1) \sum_{i=1}^1 i = 1 = \frac{1(1+1)}{2}.$

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- ❷ (Induction hypothesis) The result is true for all  $k < n$ .

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$$\begin{aligned} \sum_{i=1}^n i &= \left( \sum_{i=1}^{n-1} i \right) + n = \frac{(n-1)(n-1+1)}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \\ &= \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}. \end{aligned}$$

# Your turn!

**Theorem:** For all  $n \in \mathbb{N}$  and  $r \neq 1$ :

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}$$



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- ①  $(n = 0) : \sum_{i=0}^0 r^i = r^0 = 1 = \frac{1-r}{1-r}.$
- ② (Induction hypothesis) The result is true for all  $k < n$ .
- ③ 
$$\sum_{i=0}^n r^i = \sum_{i=0}^{n-1} r^i + r^n = \frac{1-r^n}{1-r} + r^n = \frac{1-r^n+r^n(1-r)}{1-r} = \frac{1-r^n+r^n-r^{n+1}}{1-r}.$$

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- ❶  $(n = 1) : 11^1 - 6 = 5$  which is divisible by 5.
- ❷ (Induction hypothesis) The result is true for all  $k < n$ , and therefore  $11^k = 5m + 6$ , for some integer  $m$ .
- ❸  $11^n - 6 = (11 \cdot 11^{n-1}) - 6 = 11 \cdot (5m + 6) - 6 = 11 \cdot 5m + 66 - 6 = 11 \cdot 5m + 60 = 5 \cdot (11m + 12)$ , which is divisible by 5.

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- Some basic concepts about the mathematical results and how to prove them.
- Proof by contradiction.
- Proof by induction.

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