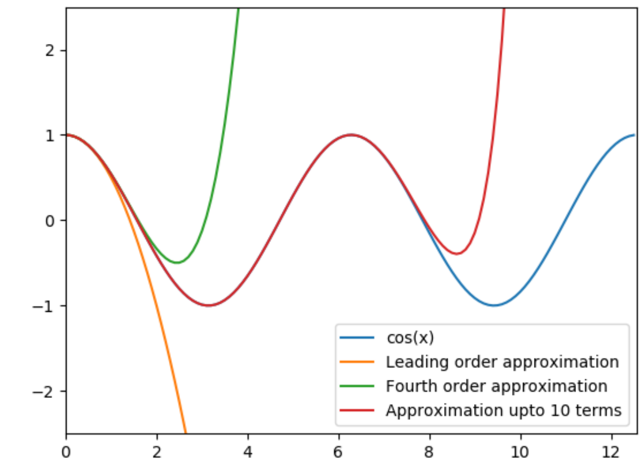
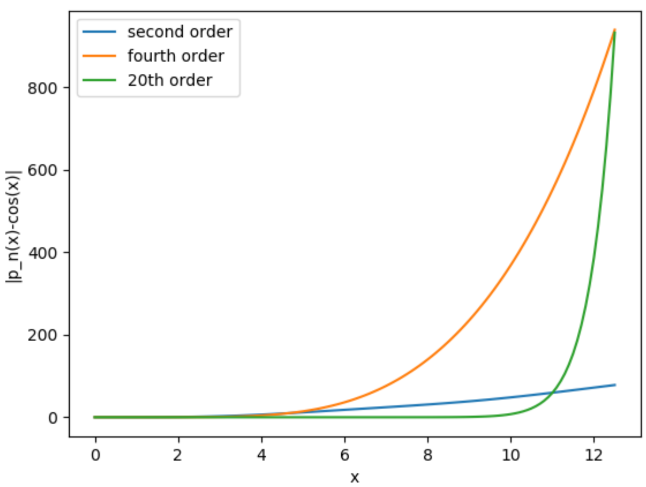
2. The taylor series approximation of $cos(x) = \sum_{i=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

The following is an image of the taylor series approximation up to second, fourth and twentieth order plotted against the original graph of $cos(x)$



These approximations continue to approximate $cos(x)$ to a better and better degree with higher and higher orders. In order to understand this, we can study $|p_n(x) - cos(x)|$ where $p_n(x)$ is the nth order approximation of $cos(x)$. Asymptotically, as $n \to \infty$ we expect our approximations to approach $g(x) = 0$. The following is the plot of $|p_2 - cos(x)|, |p_4 - cos(x)|$ and $|p_{20} - cos(x)|$, and it is evident that $p_n(x)$ is a better approximation as $n \uparrow$.



3. The classical formula for variance is $E[x^2] - E[x]^2 = \frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1} = \frac{\sum_{i=1}^{N}(x_i^2 - N\bar{x}^2)}{N-1}$. This is the formula that is computed in the first pseudocode.

The second pseudocode, we use a technique attributed to Welford. Where, we consider the following "trick". Let v_n be the variance of the first n terms. Then it can be shown that

$$(N-1)v_N - (N-2)v_{N-1} = (x_N - \bar{x}_N)(x_N - \bar{x}_{N-1})$$

.

Welford's technique uses this method to recursively compute
$(N-1)v_N = (N-2)v_{N-1} + (x_N - \bar{x}_N)(x_N - \bar{x}_{N-1})$, and the end of the computation, we divide by $(N-1)$ to obtain v_N as desired.
At each step, $\bar{x}_i$ is computed from $\bar{x}_{i-1}$ using the formula, $\bar{x}_i = \bar{x}_{i-1} + \frac{x_i - \bar{x}_{i-1}}{i}$

In the third pseudocode, we use a technique similar to Welford, which is attributed to Youngs and Cramer. Their technique is similar to that of Welford's technique - and this can be attained by just a few algebraic manipulation of the mean terms that appear in Welford's formula.

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4. A shooting method was implimented in python to solve the following boundary value problem

$$y'' = -\pi^2 y$$

with the boundary constraints $y(0) = 0$ and $y(\frac{1}{2}) = 1$.

Analytically, we can easily deduce (using the characteristic polynomial of this DE, or simply by observation) that the solution is $y = sin(\pi x)$. However, in order to solve this boundary value problem numerically, we turn the BV problem into a Initial value problem by assuming an arbitrary value a for $y'(0)$. Let $z = y'(x)$ Now, we are equipped with the following Initial value problems in z, namely

$$\begin{aligned} z = y'(x) &= f_1(x, y, z), \qquad z(0) = a \\ z' = -\pi^2 * y(x) &= f_2(x, y, z), \qquad y(0) = 0 \end{aligned}$$

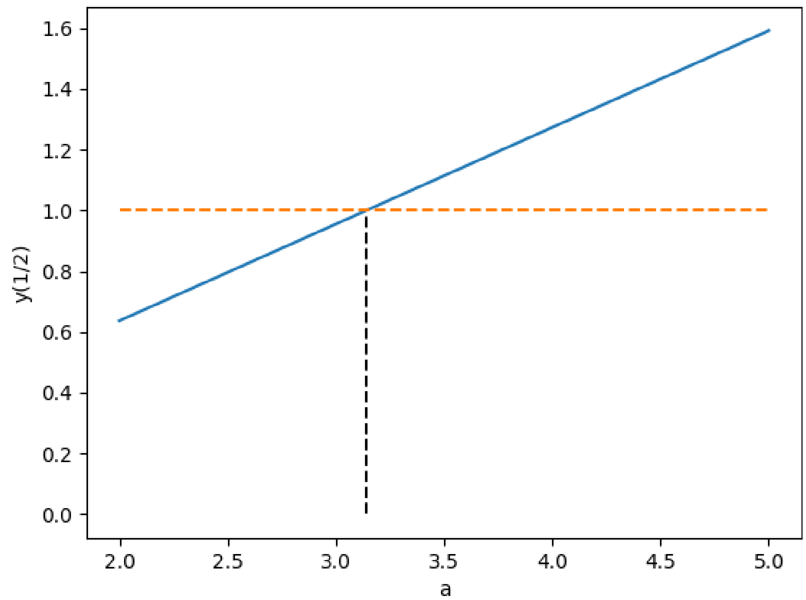
Initially, we have $x = 0, y = 0, z = a$, therefore, we can numerically solve for the next data point (taking a stepsize h) for x, y and z using the discrete equations

$$\begin{aligned} x_n &= x_{n-1} + h \\ y_n &= y_{n-1} + f_1(x_{n-1}, y_{n-1}, z_{n-1})h \\ z_n &= z_{n-1} + f_2(x_{n-1}, y_{n-1}, z_{n-1})h \end{aligned}$$

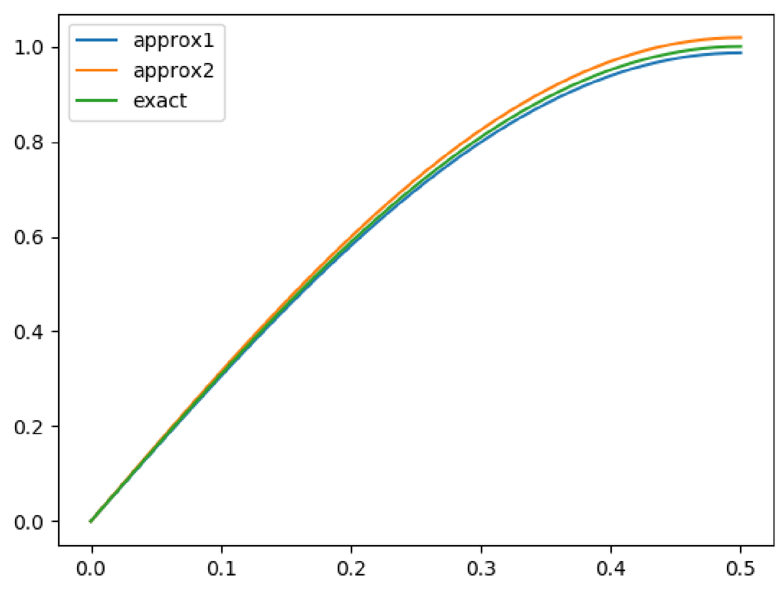
After iterating until $x = \frac{1}{2}$, we can check the value of $y(1/2)$ against $sin(\frac{\pi}{2}) = 1$.

We can now, iterate through other values of a, and plot $y(\frac{1}{2})$ against its respective a-value. The a value at which this plot interescts the function $f(a) = 1 = sin(\frac{\pi}{2})$, will be the critical a_0 value for which our BVP is equal to the IVP with the IC being $y(0) = 0,\ y'(0) = a_0$.

The plot of $y(\frac{1}{2})$ vs. a is presented below. It is evident that it intersects the function $f(a) = 1$ at π which we can analytically confirm is correct.



The actual plot of $\sin(x)$ is plotted below alongside the two closest approximations we obtained using the shooting method



It can be seen that the obtained functions closely approximate $sin(x)$, any deviation from the actual function can be attributed to a large stepsize of 0.1 for the a value.