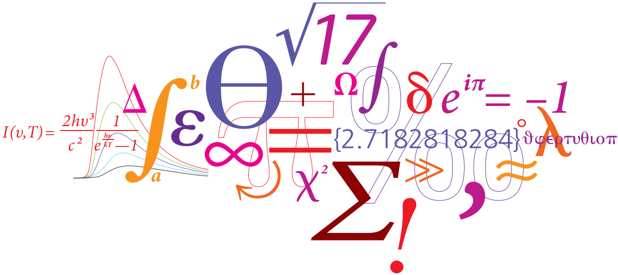
The Minimal Generating Set of the Presentation Ideal of Backelin Semigroup Ring

Arun Suresh

Georgia State University



Background

- **Numerical semigroups:** Subset S of $\mathbb{N}$ closed under $+$.
    - **Notation:** Let $a_1 < \cdots < a_r$. Denote by $\langle a_1, a_2, \ldots, a_r \rangle$ all the linear combinations of of $a_1, a_2, \ldots, a_r$.
        - A linear combination of $a_1 \ldots a_r$ looks like $a_1 n_1 + a_2 n_2 + \cdots + a_r n_r$ where $n_1, \ldots n_r$ are from $\mathbb{N}$
        - This is called the semigroup of $\mathbb{N}$ *generated by* $a_1, a_2, \ldots, a_r$
    - **Example:** Positive even numbers $\langle 2 \rangle$ form a semigroup.
- **Polynomial Ring:** $S = K[x_1, \ldots, x_r]$ is just a fancy name for the set of all polynomials in variables $x_1, \ldots, x_r$.
- **Canonical Homomorphism:** Consider a map ϕ such that $\phi : K[x_1, x_2, \ldots, x_r] \to K[t], \quad \phi(x_i) = t^{a_i}$.
    - The set of polynomials being mapped to 0 is called the *Presentation ideal I* of $K[x_1, x_2, \ldots, x_r]$

Background

- **Ideal generators** There is a finite list of polynomials in I such that every element of I can be written as linear combinations of elements of this list with coefficients from I.
  - The polynomials in this list are called **Ideal generators**
  - We study the size of this list as r varies.
- The presentation ideal is finitely generated and admits a unique minimal system of generators with cardinality is $\mu(H)$.

Example:

Let's say we have $r = 2$ and a semigroup $\langle 2, 3\rangle$.
Consider $f(x_1, x_2) = x_1^3 - x_2^2$, a polynomial in 2 variables.
Now, $\phi(f) = (t^2)^3 - (t^3)^2 = t^6 - t^6 = 0$. So f is in I
Also consider, $g(x_1, x_2) = x_1^9 - x_2^6$.
$\phi(g) = (t^2)^9 - (t^3)^6 = t^{18} - t^{18} = 0$. So g is in I as well.
But $g = (x_1^3 - x_2^2)(x_1^6 + x_1^3x_2^2 + x_2^4) = f(x_1^6 + x_1^3x_2^2 + x_2^4)$
In fact it is possible to show that **any** polynomial in I is a multiple of f.
So in this sense, f "generates" I and $\mu(\langle 2, 3\rangle) = 1$

Results from the past

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Known results for various values of r

- $r = 2 \implies \mu(H) = 1$ [Standard result]
  - **Note:** For any given $\langle a_1, a_2 \rangle$ there is always 1 generator
- $r = 3 \implies \mu(H) \leq 3$ [Herzog 1970]
  - if $\langle a_1, a_2, a_3 \rangle$ is symmetric, then $\mu(H) = 2$
    - **Example (Non-symmetric):** $H = \langle 5, 7, 9 \rangle$ with generating set

$$\{x_1^5 - x_2 x_3^2,\ x_2^2 - x_1 x_3,\ x_3^3 - x_1^4 x_2\}$$

    - **Example (Symmetric):** $H = \langle 6, 7, 8 \rangle$ with generating set

$$\{x_1^4 - x_3^3,\ x_2^2 - x_1 x_3\}$$

  - **Note:** For any given $\langle a_1, a_2, a_3 \rangle$ there is always 3 (or 2) generators.
- $r \geq 4 \implies \mu(H)$ can be arbitrarily large [Bresinsky 1975]

Results from the past

- Closed forms for $\mu(H)$ when $r = 4$.
  - Bresinsky semigroup: Let $n \geq 4$, even
    $$H = \langle n^2+n,\ n^2-1,\ n^2+2n-1,\ n^2-n \rangle$$
    - $\mu(H) = 2n$
  - Arslan semigroup: Let $n \geq 2$, even
    $$H = \langle n^2+n,\ n^2+n+1,\ (n+1)^2,\ (n+1)^2+1 \rangle$$
    - $\mu(H) = 2n+2$
  - **Note:** So in this case, $\mu(H)$ depends on what $\langle a_1, a_2, a_3, a_4 \rangle$ is!
- When $r = 4$, all examples produce an **even** number for $\mu(H)$.

Project Motivation

- Backelin semigroup:
    - $\langle r(3n+2)+3, r(3n+2)+6, r(3n+2)+3n+4, r(3n+2)+3n+5 \rangle$
        - $n \geq 2$ and $r \geq 3n+2$
    - Computed using GAP to have $\mu(H) = 3n+4$ generators
    - First ever **odd** example!

The purpose of the project is to theoretically verify this result.

Crucial Theorem 1 - A barrage of binomials

Theorem (Herzog)

If $H = \langle a_1, a_2, a_3, a_4 \rangle$, then I is generated by

$$\{F_{\nu,\mu} | F_{\nu,\mu} = x^{\nu_1} y^{\nu_2} z^{\nu_3} w^{\nu_4} - x^{\mu_1} y^{\mu_2} z^{\mu_3} w^{\mu_4}, \quad \sum_i \nu_i a_i = \sum_i \mu_i a_i\}$$

Take Away: In our search for a minimal generating set, we only need to worry about binomials.

Crucial Theorem 2 - A Truly marvelous homomorphism

Theorem (Herzog)

Let $\pi : K[x_1, x_2, x_3, x_4] \to K[x_1, x_2, x_3]$ with $\pi(x_i) = x_i$ when $i \neq 4$ and $\pi(x_4) = 0$. Let $\overline{I} = \pi(I)$. If $g_1, \ldots g_m$, the standard basis of $\overline{I}$ are polynomials such that $\exists f_1 \ldots f_m$ with $\pi(f_i) = g_i$, then $f_1, \ldots f_m$ form the standard basis of $K[x_1, x_2, x_3, x_4]$.

Take Away: When we have a minimal generating set, we should be able to make a variable zero and the resulting monomials should not divide each other.

Minimal generating set for Backelin's semigroup

Consider the following four sets of polynomials

- $S_1 := \{x^{n-k}z^{3k-1} - y^{n-k+1}w^{3k-2} \qquad k \in \overline{1,n}\}$
- $S_2 := \{x^{r-k+3}y^{k-1} - z^{3(n-k)+2}w^{r-3(n-k)-1} \qquad k \in \overline{1,n}\}$
- $S_3 := \{x^{r-(n+k)+3}y^{n+k} - z^{3(n-k)+1}w^{r-3(n-k)+1} \qquad k \in \overline{1,n}\}$
- $S_4 := \{xw^3 - yz^3,\ x^n w^2 - y^{n+1}z,\ x^{r-n+2}y^n z - w^{r+2},\ x^{2n-1}zw - y^{2n+1}\}$

Claim: $S_1 \cup S_2 \cup S_3 \cup S_4$ is a list of minimal generators for the presentation ideal of Backelin's numerical semigroup ring.

Generating the Presentation ideal

An outline of the process.

- In order to find the generators, we considered the different types of binomials we could have; and classified them as different types.
    - $x^{\nu_1}z^{\nu_3} - y^{\mu_2}w^{\mu_4}$
    - $x^{\nu_1}y^{\nu_2} - z^{\mu_3}w^{\mu_4}$
    - $x^{\nu_1}w^{\nu_4} - y^{\mu_2}z^{\mu_3}$
    - $\{x^{\nu_1} - y^{\mu_2}z^{\mu_3}w^{\mu_4},\ y^{\nu_2} - x^{\mu_1}z^{\mu_3}w^{\mu_4},\ z^{\nu_3} - x^{\mu_1}y^{\mu_2}w^{\mu_4},\ w^{\nu_4} - x^{\mu_1}y^{\mu_2}z^{\mu_3}\}$
- Considered each type, and found a set of generators for that type.
- The generators of the presentation ideal is then simply the union of the generators of each type.

It is clear that the proof boils down into studying multiple cases independently, each with sub-cases to consider. So in the following slide, we provide an example that illustrates the spirit of the proof.

Generating the Presentation ideal

Example

Notice that $f = x^{r-1}y^5 - z^{9n-12}w^{r-9n+15}$ is in I and is not in our list. We want to show that we can write f as a linear combination of things from our list. We do so as follows.

Notice that $x^{r-1}y^3 - z^{3n-10}w^{r-3n+11}$ is in our list $(S_2, k = 4)$.

We can now do,

$f - y^2(x^{r-1}y^3 - z^{3n-10}w^{r-3n+11})$
$= (x^{r-1}y^5 - z^{9n-12}w^{r-9n+15}) - (x^{r-1}y^5 - z^{3n-10}w^{r-3n+11}y^2)$
$= z^{3n-10}w^{r-3n+11}y^2 - z^{9n-12}w^{r-9n+15}$
$= w^{r-9n+15}z^{3n-10}(w^{6n-4}y^2 - z^{6n-2})$
$= w^{r-9n+15}z^{3n-10}(w^{3n-2}y + z^{3n-1})(w^{3n-2}y - z^{3n-1}) = g(w^{3n-2}y - z^{3n-1})$

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But $(w^{3n-2}y - z^{3n-1})$ is in our list! $(S_1, k = n)$.

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So, we have

$f - y^2(x^{r-1}y^3 - z^{3n-10}w^{r-3n+11}) = g(w^{3n-2}y - z^{3n-1})$
$\implies f = y^2(x^{r-1}y^3 - z^{3n-10}w^{r-3n+11}) + g(w^{3n-2}y - z^{3n-1})$

Meaning, f is indeed a linear combination of polynomials from our list.

Minimality

Claim

The list presented above is a minimal set.

Proof

Consider the π map discussed in Theorem 2 (that is, set $w = 0$). The list presented above under this map, yields a string of monomials
$\{z^{3n-1},\ xz^{3n-4}, \dots,\ x^{n-1}z^2\} \cup \{x^{r-n+3}y^{n-1},\ x^{r-n+4}y^{n-2}, \dots,\ x^{r+1}y^{n-1}\} \cup$
$\{x^{r-2n+3}y^{2n},\ x^{r-2n+4}y^{2n-1}, \dots,\ x^{r-n+2}y^{n+1}\} \cup$
$\{yz^3,\ x^{r+2},\ y^{n+1}z, x^{r-n+2}y^nz,\ y^{2n+1}\}$
where none of the monomials divide one another.
By the aforementioned theorem, the pre-images of these monomials must form a standard basis in $K[x, y, z, w]$ thereby ensuring minimality.

Conclusions and Acknowledgements

Conclusion

In this project we were able to:

- Theoretically verify that the minimal generating set for Backelin's semigroup ring is $3n + 4$ and thus provide a proof for the first odd example.
- Create grounds for more study and investigation by increasing the amount of examples available in the literature.
- Bring in new techniques of reduction the table, which could potentially be useful for solving problems of similar type

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Future Directions

- The minimal number of generators that we derived in this project are just a part of a more general set up. This number is called "The second betti number". There are similar computational results that indicate a closed form for the third and fourth betti number. Can we provide a theoretical proof for them as well?
- Generalizing Backelin's semigroup. Why is $r(3n+2)$ crucial? Can we produce a family of examples with similar parameters? If so, can this process produce infinitely many odd examples?

These are the questions we hope to answer in the upcoming year.

Thank you!

Arun Suresh

Georgia State University