The Generators, Relations and Type of the Backelin Semigroup

Arun Suresh, Florian Enescu

Georgia State University

Background

- $H = \langle a_1, \ldots, a_h \rangle$ is a *numerical semigroup*
  - $H \subseteq \mathbb{N}$
  - $\mathbb{N} \setminus H$ is finite
  - H is closed under $+$.
- h is called the *embedding dimension* of H.
  - For this talk, $h = 4$.
- $K[H] = K[t^a | a \in H]$ is the numerical semigroup ring associated to H
- $\phi : K[x_1, \ldots, x_h] \to K[t]$ with $\phi(x_i) = t^{a_i}$
- $\ker(\phi) = I_H$ is the presentation ideal.
- $\mu(I_H)$ is the cardinality of the minimal generating set of I_H

Background

- It is known that I_H is binomial
  - $I_H = (x^u - x^v : u, v \in \mathbb{N}^n, \sum_{i=1}^{h} u_i a_i = \sum_{i=1}^{h} v_i a_i)$

    where for $u = (u_1, \ldots, u_h)$, we let $x^u = x_1^{u_1} \cdots x_h^{u_h}$.
- Projdim$(K[H]) = 3$.
  - Therefore, it admits 4 betti numbers $(\beta_0, \beta_1, \beta_2, \beta_3)$
  - The n^{th} betti number is the cardinality of the n^{th} syzygy of $k[H]$.
    - $\beta_0(K[H]) = 1$
    - $\beta_1(K[H])$ is the cardinality of any minimal generating set of I_H .
    - $\beta_{h-1}(K[H]) = \beta_3(K[H])$ is called the Cohen-Macaulay type of $K[H]$.

Background

- (Fröberg, et. al.) The Cohen-Macaulay type of H is the cardinality of the set

$$PF(H) = \{x \in \mathbb{Z} \setminus H : x + h \in H \text{ for all } h \in H\}$$

  - This, in fact, equals the (Cohen-Macaulay) type of $K[H]$. [Stamate]
- Examples of semigroups with unbounded type and fixed embedding dimension have been always of interest and are scarce in the literature.

Background

- Let $S = K[x_1, \ldots, x_h]$ and $I \leq S$.
- Consider the standard grading on S
- For an $f \in I$, initial form of f is its nonzero homogeneous part of least degree.
    - With respect to the standard grading
- Let $I^* = (f^* : f \in I)$
- We say, $f_1, \ldots, f_k \in I$ form a standard basis for I if $I^* = (f_1^*, \ldots, f_k^*)$

Background

- The associated graded ring with respect to $\mathfrak{m} = (t^h : h \in H \setminus \{0\})$ is given by

$$\mathrm{gr}_{\mathfrak{m}} k[H] = \bigoplus_{i \geq 0} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$$

- This object is also called the tangent cone of $k[H]$.
- $\mathrm{gr}_{\mathfrak{m}} k[H] \cong S/I_H^*$
- Its betti numbers form an upper bound for the betti numbers of $K[H]$.
- When equality holds, the $K[H]$ is said to be of homogeneous type

The First Betti Number

Results from the past

Known results for various values of h

- $h = 2 \implies \mu(I_H) = 1$ [Standard result]
- $h = 3 \implies \mu(I_H) \leq 3$ [Herzog 1970]
- $h \geq 4 \implies \mu(I_H)$ can be arbitrarily large [Bresinsky 1975]

4-generated semigroups

- Bresinsky semigroup: Let $n \geq 4$, even

$$H = \langle n^2 + n,\ n^2 - 1,\ n^2 + 2n - 1,\ n^2 - n \rangle$$

  - $\mu(I_H) = 2n$

- Arslan's semigroup: Let $n \geq 2$

$$H = \langle n(n+1), n(n+1) + 1, (n+1)^2, (n+1)^2 + 1 \rangle$$

  - $\mu(I_H) = 2n + 2$

- $\mu(I_H)$ is even in both examples
- **Natural Question:** Is there an example with $\mu(I_H)$ odd while arbitrarily large?

Project Motivation

- Backelin semigroup: Let $n \geq 2$ and $r \geq 3n+2$
    - $\langle r(3n+2)+3, r(3n+2)+6, r(3n+2)+3n+4, r(3n+2)+3n+5\rangle$
- Betti Sequence: $(1, 3n+4, 6n+4, 3n+2)$ [Stamate 2017]
    - Obtained through computations in Singular and GAP
- First potential example with $\mu(I_H)$ odd while arbitrarily large
- First example with unbounded type and fixed embedding dimension.

Goals of the project

- Produce an explicit minimal generating set for I_H
    - Therefore, show $\mu(I_H) = 3n + 4$
- We verify that the type of this semigroup is $3n + 2$.
- We show that the backelin semigroup ring is of homogeneous type.

Set up

- Let $n \geq 2$, $r \geq 3n+2$

$$H_{n,r} = \langle r(3n+2)+3, r(3n+2)+6, r(3n+2)+3n+4, r(3n+2)+3n+5 \rangle$$

- Let $S = K[x,y,z,w]$ and $\phi : S \to K[H]$ defined by

$$\phi(x) = t^{r(3n+2)+3},\ \phi(y) = t^{r(3n+2)+6},$$

$$\phi(z) = t^{r(3n+2)+3n+4},\ \phi(w) = t^{r(3n+2)+3n+5}$$

- $I_H = \ker(\phi)$
  - graded under this nonstandard $\mathbb{N}$ grading.
- $\mu(I_H)$ is the cardinality of any minimal set of graded generators of I_H.

Minimal generating set for Backelin's semigroup

Consider the following four sets of polynomials

- $S_1 := \{x^{n-k}z^{3k-1} - y^{n-k+1}w^{3k-2} \qquad k \in \overline{1,n}\}$
- $S_2 := \{x^{r-k+3}y^{k-1} - z^{3(n-k)+2}w^{r-3(n-k)-1} \qquad k \in \overline{1,n}\}$
- $S_3 := \{x^{r-(n+k)+3}y^{n+k} - z^{3(n-k)+1}w^{r-3(n-k)+1} \qquad k \in \overline{1,n}\}$
- $E := \{xw^3 - yz^3,\ x^n w^2 - y^{n+1}z,\ x^{r-n+2}y^n z - w^{r+2},\ x^{2n-1}zw - y^{2n+1}\}$

Theorem (1)

The set $\Lambda = S_1 \cup S_2 \cup S_3 \cup E$ *generates the defining ideal of* $K[H]$

Theorem (2)

1. *The minimal generating set of the* I_H *is given by* Λ
2. *The type of* $K[H]$ *is* $3n+2$ *and the sequence of Betti numbers of* $K[H]$ *is* $(1, 3n+4, 6n+5, 3n+2)$.

Herzog's theorem: Setup

- Let $S = K[x_1, \ldots, x_h]$ and $I \leq S$.
- Consider the standard grading on S
- For an $f \in I$, initial form of f is its nonzero homogeneous part of least degree.
  - With respect to the standard grading
- Let $I^* = (f^* : f \in I)$
- $f_1, \ldots, f_k \in I$ form a standard basis for I if $I^* = (f_1^*, \ldots, f_k^*)$

Herzog's theorem:

Theorem (Herzog)

Let $I \subseteq \mathfrak{n} = (x_1, \ldots, x_n)$ *be an ideal in* $S = K[x_1, \ldots, x_h]$*. Let* $\hat{S} = K[[x_1, \ldots, x_h]]$ *and assume that* x_1 *is a nonzerodivisor on* $\hat{S}/I\hat{S}$*. Let* $\pi : S \to K[x_2, \ldots, x_h]$ *defined by* $\pi(x_1) = 0, \pi(x_i) = x_i$ *for all* $i = 2, \ldots, h$ *and denote* $\overline{I} = \pi(I)$*. Assume that* $g_1, \ldots, g_r$ *form a standard basis for* $\overline{I}$ *in* $K[x_2, \ldots, x_h]$ *and let* $f_i \in I$ *such that* $\pi(f_i) = g_i$ *and* $deg(f_i^*) = deg(g_i^*)$ *for all* $i = 1, \ldots, r$*. Let* $\overline{S} = \pi(S) = K[x_2, \ldots, x_n]$ *and* $\overline{\mathfrak{n}} = \pi(\mathfrak{n}) = (x_2, \ldots, x_n)$*.*

1. *Then* $f_1, \ldots, f_r$ *form a standard basis for* I*.*
2. x_1 *is a nonzerodivisor on* $gr_{\mathfrak{n}}(S/I)$*.*
3. *We have a graded* K*-algebra isomorphism*

$$\frac{gr_{\mathfrak{n}}(S/I)}{x_1 \cdot gr_{\mathfrak{n}}(S/I)} \simeq gr_{\overline{\mathfrak{n}}}(\overline{S}/\overline{I}).$$

Using Herzog's Theorem

Proof of theorem 2 assuming theorem 1.

1

- From Herzog's theorem, let $\pi(x) = x$, $\pi(y) = y$, $\pi(z) = z$ and $\pi(w) = 0$.
- Let $I = (\Lambda)$ and $\pi(I) = \overline{I}$
- Betti numbers are invariant under modding out by a nonzero divisor in S/I.
  - Suffices to find the betti numbers of $\overline{I}$.
- From thm(1) the defining ideal is I. Letting $w = 0$, we get $\overline{I}$ is generated by

$$x^{n-k}z^{3k-1},\ x^{r-k+3}y^{k-1},\ x^{r-(n+k)+3}y^{n+k},\quad k = 1, \ldots, n$$

  and

$$yz^3, y^{n+1}z, x^{r-n+2}y^n z, y^{2n+1}$$

- Clearly this forms a minimal set of generators for $\overline{I}$.
- $\mu(\overline{I}) = 3n + 4$
- Cardinality of $\Lambda = 3n + 4$ and Λ generates I.
- So Λ is a minimal generating set for I.

$\square$

Proof of theorem 2 assuming theorem 1.

2

- We begin by computing the type of $k[H]$ = the type of $\overline{I}$.
- We will find the set $\mathcal{B}$ of monomials whose images form a basis for
$$\frac{\overline{I}:(x,y,z)}{\overline{I}}.$$
- It can be seen that
$$x^{n-k}z^{3k-2} \text{ for } k=2,\dots,n$$
$$x^{r-(n+k)+3}y^{n+k-1},\ x^{r-k+2}y^{k-1}z \text{ for } k=1,\dots,n$$
$$x^{r-n+1}y^n z,\ x^{n-2}y^n z^2,\ x^{r-2n+2}y^{2n}$$
have non-zero images in $\frac{\overline{I}:(x,y,z)}{\overline{I}}$
- Take a monomial $x^a y^b z^c \in \mathcal{B}$.
- We will show it belongs to the above list by examining possible values for c.
- Clearly $c \leq 3n-2$
- If $c=0$, $b \leq 2n$.
  - $b=2n$, then $a=r-2n+2$.
  - $n \leq b < 2n$, then $b=n+k-1$ for $1 \leq k \leq n$. $a=r-(n+k)+3$.
  - $b<n$ is not possible.
- $c=1,\dots$ can be checked similarly. This gives $\beta_3(K[H])=3n+2$
- $\beta_0=1$, $\beta_1=3n+4$ and $\beta_3=3n+2 \implies \beta_2=6n+5$.

$\square$

Corollary

1. *The set* $S_1 \cup S_2 \cup S_3 \cup E$ *forms a standard basis for the defining ideal* I *of* $K[H]$.
2. *Let* $\mathfrak{n} = (x, y, z, w)$ *in* $K[x, y, z, w]$ *which maps onto the maximal graded ideal of* $K[H] = K[x, y, z, w]/I$. *Then* $K[H]$ *and* $\mathrm{gr}_{\mathfrak{n}}(K[H])$ *have the same Betti numbers.*

Proof.

1. This is a direct consequence of Herzog's Theorem (1)
2. This is similar to the proof of Theorem 2 (1), except here we send $x \to 0$.
   - This satisfies the hypothesis needed for Herzog's Theorem (3).
   - It is straightforward to show that the betti numbers are preserved.

□

Proving Theorem 1

- I_H is the presentation ideal of $K[H]$.
- As mentioned before I_H is generated by binomials

$$x^{\nu_1}y^{\nu_2}z^{\nu_3}w^{\nu_4} - x^{\mu_1}y^{\mu_2}z^{\mu_3}w^{\mu_4}$$

  with $\sum_{i=1}^{4} a_i\nu_i = \sum_{i=1}^{4} a_i\mu_i = d$
- d is the total degree of the binomial under this induced grading.
- Since I_H is prime, each binomial is the difference of non-overlapping monomials
- Let $I = (\Lambda)$
- It is clear that $I \subseteq I_H$. We only need to show $I_H \subseteq I$.
- The proof goes by induction on d

Proving Theorem 1

- For any $d \geq 1$, $b = x^{\nu_1} y^{\nu_2} z^{\nu_3} w^{\nu_4} - x^{\mu_1} y^{\mu_2} z^{\mu_3} w^{\mu_4}$ of degree d is either
  - in I (or)
  - in the ideal generated by binomials of I_H of degree strictly less than d.
- If b satisfies the second condition, we will say that it "reduces" to a lower degree.
- Our analysis will consider all possible types of binomials in I
  - assumed non-overlapping.

Proving Theorem 1

- We categorize the binomials in I_H into types
- For example,
  - **Type** $x^{\nu_1}y^{\nu_2} - z^{\mu_3}w^{\mu_4}$
  - **Type** $x^{\nu_1}z^{\nu_3} - y^{\mu_2}w^{\mu_4}$ and so on...
- Studying each type just means studying expressions like $\sum_{i=1}^{4} a_i\nu_i = \sum_{i=1}^{4} a_i\mu_i$
- Which turned out to be expressions like (for instance in the fist type)

$$\nu_1[r(3n+2)+3]+\nu_2[r(3n+2)+6] = \mu_3[r(3n+2)+3n+4]+\mu_4[r(3n+2)+3n+5]$$

- For each type we find a lower bound for total degree of the monomials
  - Under the standard grading
  - In the first type, say, we show $\nu_1 + \nu_2 \geq r+2$ and $\mu_3 + \mu_4 \geq r+1$
- Using these bounds, we can parameterize the ν's and μ's
  - Following that we use modular arguments to "reduce" the homogeneous degree of the binomials.

Example

Take $xyw^{6n-1} - z^{6n+1} \in I_H$.
Notice $z^{3n-1} - yw^{3n-2}$ and $xw^3 - yz^3 \in \Lambda$.
So, we write

$$(xyw^{6n-1} - z^{6n+1}) - z^{3n+2}(z^{3n-1} - yw^{3n-2}) = yw^{3n-2}(xw^{3n+1} - z^{3n+2})$$

Now consider $xw^{3n+1} - z^{3n+2} \in I_H$, and write

$$xw^{3n+1} - z^{3n+2} + z^3(z^{3n-1} - yw^{3n-2}) = xw^3 - yz^3 \in \Lambda$$

Notice that at every step of the equality, the homogeneous degree of the binomial dropped.
Working backwards, we conclude that $xyw^{6n-1} - z^{6n+1} \in (\Lambda)$.

Future Directions

- There is a need for a better theoretical framework that helps produce other examples.
- Finding a minimal generating set that also serves as a Gröbner basis under some order, and producing a minimal free resolution of the semigroup ring.
    - This has been done for a few other famous examples in the literature.

Thank you!

Arun Suresh, Florian Enescu

Georgia State University