HW 3 - q3

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1 $$\phi_x a_x + \phi_y a_y + \phi_t = 0 \quad \text{——— (1)}$$

$$\begin{bmatrix} a_x \\ a_y \\ 1 \end{bmatrix} = \frac{\alpha}{|\nabla_2 \phi(x,t)|} \begin{bmatrix} \phi_x(\vec{x},t) \\ \phi_y(\vec{x},t) \end{bmatrix} \quad \text{——— (2)}$$

$$a_x = \frac{\alpha}{|\nabla_2 \phi(x,t)|} \cdot \phi_x(\vec{x},t) \qquad a_y = \frac{\alpha \phi_y(\vec{x},t)}{|\nabla_2 \phi(x,t)|}$$

$$\frac{\alpha \phi_x^2(\vec{x},t)}{|\nabla_2 \phi(\vec{x},t)|} + \frac{\alpha \phi_y^2(\vec{x},t)}{|\nabla_2 \phi(\vec{x},t)|} + \phi_t(\vec{x},t) = 0$$

$$\alpha = \frac{-\phi_t(\vec{x},t) \cdot |\nabla_2 \phi(x,t)|}{\phi_x^2 + \phi_y^2} = \frac{-\phi_t(\vec{x},t)}{|\nabla_2 \phi(\vec{x},t)|}$$

$$\vec{a} = \frac{\alpha}{|\nabla_2 \phi(\vec{x},t)|} \cdot \nabla_2 \phi(\vec{x},t)$$

$$= \frac{-\phi_t(\vec{x},t)}{|\nabla_2 \phi(\vec{x},t)|^2} \cdot \nabla_2 \phi(\vec{x},t)$$

is the best estimate for $\vec{a}$.

HW3 - q3b

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(2) S.T. $$\nabla_3 \phi(\vec{x},t) = \frac{Im\left[s^*(\vec{x},t)\ \nabla_3 s(\vec{x},t)\right]}{|s(\vec{x},t)|^2}$$

and $$|s(\vec{x},t)| = \rho(\vec{x},t)$$

and $$s(\vec{x},t) = \rho(\vec{x},t)\ exp\left(j\phi(\vec{x},t)\right)$$

$$\nabla_3 s(\vec{x},t) = \begin{bmatrix} \frac{\partial}{\partial x} s(x,y,t) \\ \frac{\partial}{\partial y} s(x,y,t) \\ \frac{\partial}{\partial t} s(x,y,t) \end{bmatrix}$$

$$= \begin{bmatrix} \rho(\bar{x},t)\ exp(j\phi(\bar{x},t))\ \phi_x j + exp(j\phi(x,t))\ \rho_x \\ \rho(\bar{x},t)\ exp(j\phi(\bar{x},t))\ \phi_y j + exp(j\phi(x,t))\ \rho_y \\ \rho(\bar{x},t)\ exp(j\phi(\bar{x},t))\ \phi_t j + exp(j\phi(\bar{x},t))\ \rho_t \end{bmatrix}$$

$$= s(\bar{x},t)\ \nabla_3\phi(\bar{x},t)\, j + \nabla_3\rho(\bar{x},t)\ exp(j\phi(\bar{x},t))$$

Substituting $\nabla_3 s(\bar{x},t)$, we get:

$$\frac{Im\left[s^*(\bar{x},t)\cdot \nabla_3 s(\bar{x},t)\right]}{|s(\bar{x},t)|^2}$$

$$= \frac{Im\left[s^*(\bar{x},t)\ s(\bar{x},t)\ \nabla_3\phi(\bar{x},t) j + s^*(\bar{x},t)\ \nabla_3\rho(\bar{x},t)\ exp(j\phi(\bar{x},t))\right]}{|s(\bar{x},t)|^2}$$

$$= \frac{Im\left[|s(\vec{x},t)|^2\ \nabla_3\phi(\bar{x},t) j\right] + Im\left[s^*(\bar{x},t)\ \nabla_3\rho(\bar{x},t)\ \frac{s(\bar{x},t)}{\rho(\bar{x},t)}\right]}{|s(\bar{x},t)|^2}$$

$$= \frac{|s(\bar{x},t)|^2\ \nabla_3\phi(\bar{x},t)}{|s(\bar{x},t)|^2} + \frac{Im\left[\frac{\nabla_3\rho(\bar{x},t)}{\rho(\bar{x},t)}\ |s(\bar{x},t)|^2\right]}{|s(\bar{x},t)|^2}$$

$$= \nabla_3\phi(\bar{x},t) + 0 = \nabla_3\phi(\bar{x},t)$$

Hence, shown.