Computation as Platonic Shadow

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Abstract

We propose a philosophy of mathematics that synthesizes formalism and platonism by proposing "semantics" as dual to computation. We draw an exact parallel between physics and mathematics by proposing that *computation* (including proof) takes the role of experiments in physics, while the definitions and theory we build in mathematics playing the analogous role to theory in physics.

Given recent accomplishments in machine learning and artificial intelligence, it is now possible to imagine a near-future scenario where a computer oracle can output proofs of arbitrary mathematical statements at a superhuman level. In such a scenario, it is not clear any more what role mathematicians could play in society or even what being a mathematician would entail.

In parallel fields like chess that have already undergone this AI revolution, human chess players have changed their activities from exploring and searching for new ideas towards interpreting and understanding the outputs of their computer systems. Will mathematicians undergo such a transformation too? And if so, what implications does it have for mathematics as a discipline? To explore these questions, it presumably helps to first try and understand what mathematicians currently do, and what mathematics as a discipline is. This short essay addresses this question. We suggest that mathematics in practice is much closer to an experimental science like physics in very precise ways. In exploring this similarity, we will have reason to demarcate a fundamental divide between syntactics and semantics in (human) thought, and a novel suggestion for where certainty in mathematics stems from.

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The Cyclic Nature of Mathematical Practice

To a first approximation, there are two dominant, countervailing attitudes towards what mathematics is: *Platonism* and *Formalism*.

The first view is intentionally vague and centers mathematics within the broader domain of the sciences - mathematical objects are "real", and mathematicians study them in similar ways to how physicists might study electrons. Almost every working mathematician assumes this attitude while working on mathematics and has a strong sense that the mathematical objects they study are external to them, and have their own independent existence and structure. However, just as universally, mathematicians often have trouble justifying this attitude, especially when pressed about the correctness of mathematics as a whole.

The second view grew in strength as a reaction to the "foundational crisis" at the turn of the twentieth century and is a more mechanical view of mathematics. It is the attitude that we present to students, especially when we ask how mathematics is to be certified correct. However, very few working mathematicians take this view seriously due to the common objection that, divorced of intuition, the foundational axioms are meaningless and not obviously interesting. Nevertheless, this view has gained prominence recently with the rise of automated theorem provers.

I propose that these views can be reconciled by understanding mathematics as a cyclic process that moves between computation and theory-building. This cycle proceeds as follows:

1. We perform computations - physical processes that generate observations and constitute mathematical experiments.

2. Based on these observations, we build theories (definitions, axioms, rules) to explain what we've observed.

3. Within these theories, we make new computations that both test our theories and generate new observations.

4. These observations lead to theory refinement, beginning the cycle anew.

Crucially, we see computation as a *physical process* and as serving a very similar role to experiments in the other sciences. Conversely, we see the act of coming up with new definitions, theories, axiomatic systems and frameworks also as complete analogs to theory building in the other sciences.

And the role of such theory building is the same in either case - to *explain* the results of the experiments/computations and to recursively allow us to push the boundaries of our knowledge.

Computation and Mathematical Experiments

To ground this cycle, let us first turn to the notion of computation. The theory of computability grew out of the "foundational crisis" and in particular Gödel's analysis of the incompleteness of axiomatic methods. At its core, a computation is a class of functions that convert an input string to an output string through a finite sequence of steps, with each step performing a simple operation (for example, a "find and replace"). To be clear, a computation for us is to be considered as a protocol for a class of physical processes, similar to how a physics experiment describes in principle a physical process that can be carried out under a variety of circumstances.

The notion of computation makes precise the formalist attitude. Under this view, mathematics consists of finding a computational reduction from "conjectures" (output strings) to "axioms" (input strings) using the "rules of deduction" (valid steps).

Definition 1. A mathematical experiment is a derivation of a statement from an axiomatic framework following the corresponding rules of deduction. A negative result is the derivation of a contradiction.

We will see later how the notion of computation as an interactive experimental protocol clarifies the Platonist viewpoint.

As mathematicians carry out experiments through computation and proof construction, they simultaneously build and refine theories to explain their observations. To take a concrete example, the entire edifice of scheme theory and étale cohomology was pursued in trying to explain Weil's observations and conjectures about the connection between point counts over a finite field and the complex topology of a set of equations. This example is far from unique - as just another of many possible examples, modern analysis similarly stems from an attempt to put on consistent foundations the successes of the earlier differential calculus. This process mirrors almost exactly the way our theories in physics (say general relativity) are constructed, with the only difference being what counts as an "experimental result" in the two domains.

Truth and validity in mathematics

In the above view, a valid proof is an insufficient criterion for correctness in mathematics - just as how a valid derivation from Newton's laws of gravity might still not be physically valid. Indeed, Gödel's theorems already inform us that in any sufficiently rich axiomatic framework, one can never be perfectly confident in its consistency and hence, a derivation of a proof is not a sufficient reason to believe that the negation of the statement is unprovable. Therefore, in our view, an axiomatic framework is always to be taken as provisional, a current "best" attempt at explaining the observations, but one that is always subject to revision. Where then do we derive confidence in our mathematical results?

In this respect too, mathematics is no different from the other sciences. In physics, valid experiments constrain the possible theories that could be true and increase our confidence in the theories that predict and/or explain the experiments. Moreover, our confidence in any one theory increases as we conduct further experiments that test the theory by testing its conclusions. Similarly in mathematics, our belief in an axiomatic framework increases as we use it to derive proofs and perform computations in without running into a contradiction. At a more granular level, our confidence in the truth of a statement is partially increased by a valid proof, but in practice, our confidence rests far more on a statement being used often in deriving new results (without running into a contradiction). Indeed, an isolated statement with a complicated proof is often considered with suspicion. On the other hand, an often used statement is almost certainly "true" even if we find that the original proof has a flaw in it. This explains the initially surprising observation that a vast majority of mathematics papers contain flaws in them and any individual proof is extremely fragile to any mistake; nevertheless, mathematics as a profession is remarkably resilient, and commonly believed statements are almost never found to be wrong. For a remarkable recent case study in this, see here.

It would be extremely interesting to develop alternative foundations for mathematics where the validity of a mathematical statement or framework is not absolute (i.e., derivable from axioms) but tentative and proportional to the "probability" that the framework in question can be used to derive a contradiction. The paper [GBTC$^+$16] goes a long way towards such an alternative foundation - they propose a way of assigning probabilities to as-yet-unproven statements that faithfully reflect their "truth" status.

The Reality of Mathematical Objects

Mathematical objects *feel* real to working mathematicians in a deep, visceral sense. What might explain this? I will argue that indeed mathematical objects feel real for the same reason that a table feels real. In doing so, I hope to broaden and identify more precisely when objects "feel real", but at the same time I will also stay completely silent on what the "true nature of reality is".

So when does something feel real? I believe that this happens precisely when our mental model (or hypothesis, theory, expectation, etc.) of the object is repeatedly tested by interacting with it and the perceived results match our expectations. In the case of a table, it feels more real to us as we interact with it by seeing it, touching it, and interacting with it in a variety of other physical ways. In all these interactions, we verify our intuitive physical model of the world, and hence reinforce our belief in the "realness" of the table. But at the same time, it is clear that the "table" has no real place in the fundamental laws of physics as currently written down, so the *only* sense in which the table is real is that it is a useful concept *for us* as we interact with the world around us.

Similarly, a mathematical object "feels real" to us as we interact with it in a variety of ways that do not lead to contradictions. In this case, interacting with a mathematical object is a purely computational activity but no less "real" for that. And, as with a table, our intuitive mental model of a mathematical object is often divorced from the axiomatic framework in which it receives a precise definition. Indeed, that this can happen at all is the only reason axiomatic frameworks are more malleable than the fundamental objects of mathematics (such as the natural numbers or the topological surfaces).

And this is exactly what leads to our strong intuition that "truth" is a valid concept. As we discussed before, our experimental results in mathematics can only indicate "contradictions" but never rule them out in sufficiently rich theories. And in particular, they never discuss "Truth" directly and our intuition that propositions can be "true" is directly analogous to our intuition that certain objects and processes are "real".

And this phenomenon leads us to the major, hidden theme of this essay: a distinction between syntax and semantics in thought.

Semantics and syntax

Mathematical logic too tries to capture this dichotomy in its distinct subfields of proof theory and model theory. Proof theory is quite literally a study of the computational traces - proofs, without reference to the “meaning” of any such syntactic proof. Conversely, model theory attempts to ground meaning within set theory.

But this process of *grounding meaning* within a theory finds much broader use in mathematics and does not depend on set theory (which is after all just another example of an axiomatic system with no ”physical” existence). For example, one could define any specific group by specifying a specific set of generators and presentations and prove various properties of such a specific group through manipulating these generators and presentations. Indeed, in practice this is how one works with a group and almost never by literally iterating through its elements (for groups that are reasonably big). Or one could work at a higher level of abstraction and prove statements about groups en mass. In this case, one could very reasonably interpret abstract group theoretic proofs as ”syntactic” and their interpretation with respect to a fixed group defined by a fixed set of generators and relations as ”semantic”, all the while completely avoiding any talk of sets.

Is it possible to ground all semantics as simply translations from a more abstract to a less abstract set of syntactic theories? My intuition is that there is indeed a crucial difference between semantics and syntax that is not captured in the above examples. While I do not know how to make this precise yet, I would like to outline some of my attendant intuitions about this division.

Syntactic processes are characterized by being explicit, local and procedural. They is moreover fragile and errors can be catastrophic. Semantic understanding on the other hand is characterized by being implicit, global and emphasizes coherence. It is relatively resistant to loss of information and the entire model can be recovered from partial information. Moreover, the same semantic content can often correspond to many different syntactic expressions of it.

In fact, the dichotomy here seems reflective of a deep dichotomy inherent to cognition itself, especially with the rise of artificial intelligences. The time seems ripe for a more formalized understanding of this distinction between semantic and syntactic understanding. The theory of computation seems very well adapted to how syntactic thought works, and the theory that has been developed so far certainly seems adequate to at least express our intuitions about it. On the other hand, the formalization of semantic understanding is in a much more nascent stage. At least in rough outline, such a theory should have the following properties:

1. The "meaning" of a concept should be purely determined by its relation to other concepts. Here, "relation" is used in a very flexible way - quite possibly, concepts should be allowed to act on other concepts in order to generate new concepts for example.

2. This space of concepts should allow us to "interpret" syntactic traces that we observe in the world - and perhaps even let us carve up a continuous outer world into discrete syntactic traces.

3. It should be possible to learn and refine this space of concepts in response to computational traces, i.e., experimental data of various forms including internally generated ones. This process should be inherently resistant to noise.

4. And conversely, it should be possible to produce computational traces given this space of concepts - for example, generate an attempt at a proof from mathematical understanding.

Such a theory would take as its ultimate goal a unified understanding of learning across all cognitive agents and at all levels, from humans, animals, unicellular organisms all the way to artificial machine intelligence, and from learning about basic distinctions in the world as in the earliest moments of an organism all the way to abstract concepts like mathematics.

References

[GBTC+16] Scott Garrabrant, Tsvi Benson-Tilsen, Andrew Critch, Nate Soares, and Jessica Taylor. Logical induction. *arXiv preprint arXiv:1609.03543*, 2016.