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1): Given function,

$$f = 2x_1^2 - 4x_1x_2 + 1 - 5x_2^2 + x_2.$$

Gradient, $g = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$

$$= \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$$

Hessian, $H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

Now we have to find the Eigen values of the matrix.

$$[A - \lambda I] = \begin{bmatrix} 4-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$$

$$(4-\lambda)(3-\lambda) - (-4)(-4)$$

$$\Rightarrow 4(3-\lambda) - \lambda(3-\lambda) - 16$$

$$\Rightarrow 12 - 4\lambda - 3\lambda + \lambda^2 - 16$$

$$\Rightarrow \lambda^2 - 7\lambda - 4$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+7 \pm \sqrt{49 + 16}}{2}$$

$$= \frac{7 + \sqrt{65}}{2}, \quad \frac{7 - \sqrt{65}}{2}$$

Here, $\lambda_1 = \frac{7+\sqrt{65}}{2} > 0$

$\lambda_2 = \frac{7-\sqrt{65}}{2} < 0$

Which means it is indefinite.

Thus, f is a saddle point.

We know that at the saddle point, gradient is 0.

Thus,

$$4x_1 - 4x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

$$-4x_1 + 3x_2 + 1 = 0$$

$$\Rightarrow -4x_1 + 3x_1 + 1 = 0$$

$$\Rightarrow -x_1 = -1$$

$$\Rightarrow \boxed{x_1 = 1}$$

$$\therefore \boxed{x_2 = 1}$$

Thus, the saddle point is $(1, 1)$.

$$f(x) = f(x_0) + \overbrace{\cancel{g^T_{x_0}(x - x_0)}}^{=0} + \frac{1}{2}(x - x_0)^T H_{x_0}(x - x_0)$$

$$= f(x_0) + \frac{1}{2}(x - x_0) H_{x_0}(x - x_0)$$

$$\Rightarrow f(x) - f(x_0) = \frac{1}{2}(x - x_0) H_{x_0}(x - x_0)$$

Here if $g < 0$

$$4x_1 - 4x_2 < 0$$

$$-4x_1 + 3x_2 + 1 < 0$$

Downslope area:

$$\begin{cases} x_1 < x_2 \\ x_1 > \frac{3}{4}x_2 + 1/4 \end{cases}$$

2): (i): $f = (x_1+1)^2 + x_2^2 + (x_3-1)^2 =$

$\Rightarrow x_1 = 1 - 2x_2 - 3x_3$

$\Rightarrow f = (-2x_2 - 3x_3 + 2)^2 + x_2^2 + (x_3-1)^2$

$= 4x_2^2 + 9x_3^2 + 4 + 12x_2x_3 - 8x_2 - 12x_3 + x_2^2 + x_3^2 - 2x_3 + 1$

$= 5x_2^2 + 10x_3^2 - 8x_2 - 14x_3 + 12x_2x_3 + 5$

Now we have to find the gradient to find the point,

$$g = \begin{bmatrix} \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$= 10x_2 + 0 - 8 + 0 + 12x_3 = 0$

$\Rightarrow \boxed{10x_2 + 12x_3 - 8 = 0} \Rightarrow \boxed{5x_2 + 6x_3 - 4 = 0}$

$20x_3 + 12x_2 - 14 = 0 \Rightarrow \boxed{6x_2 + 10x_3 - 7 = 0}$

$\begin{bmatrix} 10 & 12 \end{bmatrix}$ $\begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

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$x_3 = 11/14$

$x_2 = -1/7$

$$x_1 = \frac{1}{1} - 2\left(\frac{-1}{7}\right) - 3\left(\frac{11}{14}\right)$$

$$= \frac{1}{1} + \frac{2}{7} - \frac{33}{14}$$

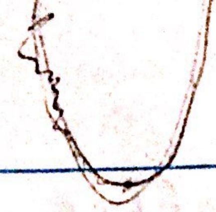
$$\frac{14 + 4 - 33}{14}$$

$$= \frac{-15}{14}$$

$$\boxed{\text{So the point is } \left(\frac{-15}{14}, -1/7, \frac{11}{14}\right)}$$

$$\text{Hessian, } H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 x}{\partial x_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix}$$



Let $A = \begin{bmatrix} 5 & 6 \\ 6 & 10 \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} 5-\lambda & 6 \\ 6 & 10-\lambda \end{bmatrix}$

$= (5-\lambda)(10-\lambda) - 36$

$= 5(10-\lambda) - \lambda(10-\lambda) - 36$

$\Rightarrow 50 - 5\lambda - 10\lambda + \lambda^2 - 36$

$= \lambda^2 - 15\lambda + 14$

$\Rightarrow \lambda^2 - \lambda - 14\lambda + 14$

$\Rightarrow \lambda(\lambda - 1) - 14(\lambda - 1)$

$\Rightarrow (\lambda - 14)(\lambda - 1)$

$\Rightarrow \boxed{\lambda = 14, 1}$

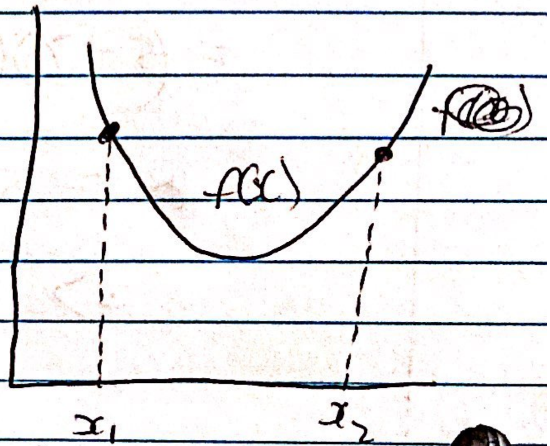
Both are +ve. Thus it is a convex fund.

3). a) $f(x)$ and $g(x)$ are convex functions.

We have to see if $(af(x)+bg(x))$ is also a convex function.

Let x_1 and x_2 be be any ponts on the function.

Here $\lambda x_1 + (1-\lambda)x_2$.



Here $f: X \to R$ is convex.

This means,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

Also,

$$g(\lambda x_1 + (1-\lambda)x_2) \le \lambda g(x_1) + (1-\lambda) g(x_2).$$

Which gives,

$$af(\lambda x_1 + (1-\lambda)x_2) + bg(\lambda x_1 + (1-\lambda)x_2) \le \lambda(af(x_1) + bg(x_1)) + (1-\lambda)(af(x_2) + bg(x_2))$$

Thus, $af(x) + bg(x)$ is convex.

3 b) Here $j(x) = f(g(x))$

Differentiating w.r.t x.

$$\frac{d\,j(x)}{dx} = f'(g(x)).g'(x).$$

Using chain rule,

$$\frac{d^2 j(x)}{dx^2} = f''(g(x))g'(x).g'(x) + f'(g(x))g''(x)$$

$$= f''(g(x))g'^2(x) + f'(g(x))g''(x)$$

Here the condition is,

$$\boxed{f''(g')^2 + f'g'' > 0}$$

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4): $f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0)$

For convex functions, $f(x): X \to R$ and for $x_0, x_1 \in X$

$$f(\lambda y + (1-\lambda)x) \leq \lambda f(y) + (1-\lambda) f(x)$$

Here $\lambda \in [0,1]$

$$f(x + \lambda(y-x)) \leq (1-\lambda) f(x) + y.$$

(i): f is convex.
Thus using first order Taylor's expansion,

$$f(y) \geq f(x) \nabla f(x)^T (y-x)$$

Also,

$$\nabla^2 f(x) \geq 0$$

$$f(x + \lambda(y-x)) \leq f(x) + \lambda (f(y) - f(x))$$

$$f(y) - f(x) \geq \frac{f(x + \lambda(y-x)) - f(x)}{\lambda}$$

$$f(y) \geq f(x) + \frac{f(x+\lambda(y-x)) - f(x)}{\lambda}$$

$$\lim_{t \to 0} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda} = f'(x)(y-x)$$

$$f(y) \geq f(x) + f'(x)(y-x)$$

$$w = \lambda x + (1-\lambda) y \quad \forall \lambda \in [0,1]$$

$$f(x) \geq f(w) + f'(w)(x-w)$$

$$f(y) \geq f(w) + f'(w)(y-w)$$

5): a) We shodd minimize the error between I and $a_k^T P$, we should adjusting p. The unconstrained optimisation problem is,

$$\min_p \sum_{k=1}^{m} a_k^T P_i - I)^2$$

Here, $0 \leq P_i \leq P_{max} \quad \forall\, i = 1, \ldots, n$

Thus, $f(P) = (a^T P - I)^2$.

$$g = 2(a^T P - I)a$$

$$H = 2aa^T \geq 0$$

b): By using lemma,

$$\text{if } d^T H d \geq 0$$

Here $d \neq 0$

This means that H is psd.

$$\Rightarrow d^T H d = 2d^T[a][a]^T d$$

$$= 2a_k^2 \geq 0$$

Hessian is psd

Thus, It is a convex problem.

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c): For unique solution,
it should be stricty convex.

Which means the Hessian should be positive definite,

$\Rightarrow \sum_k 2d^T a_k a_k^T d = \sum_k 2 u_k^2 > 0$

Here $d \neq 0$.

Here we cannot find d for all $u_k^2 = 0$.

Which means

The numbes of missors > Number of lamps.

which is necessary condition for 1 solution.

d): If the number of missors are more than ~~tea~~ ~~~~ the number of lamps, then there is a half unique solution.