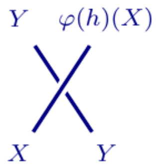
Orbifold data and G-crossed ribbon categories

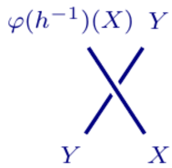
From Section 5 in CRS3 = 1809.01483, Carqueville, Runkel, Schaumann

We fix a finite group G, a ribbon fusion category $\mathcal{B}$, and a *ribbon crossed G-category* $\widehat{\mathcal{B}} = \bigoplus_{g\in G} \mathcal{B}_g$ such that $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{B}_g \neq 0$ for all $g \in G$. Roughly, this

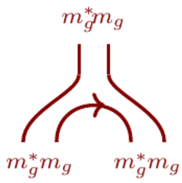
$\varphi(g)(\mathcal{B}_h) \subset \mathcal{B}_{g^{-1}hg}$ for all $g, h \in G$,

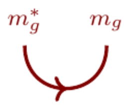
$c_{X,Y} \equiv$ 

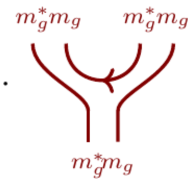
$: X \otimes Y \xrightarrow{\cong} Y \otimes \varphi(h)(X)$ if $Y \in \mathcal{B}_h$,

$\tilde{c}_{Y,X} \equiv$ 

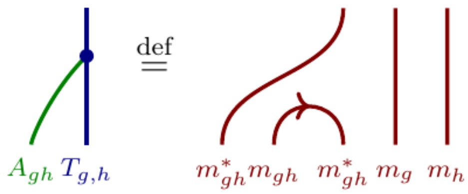
$: Y \otimes X \xrightarrow{\cong} \varphi(h^{-1})(X) \otimes Y$ if $Y \in \mathcal{B}_h$.

$A_g := m_g^* \otimes m_g$, 

, 

, $\dfrac{1}{d_{m_g}} \cdot$ 

, $d_{m_g} \cdot$ 

$T_{g,h} := m_{gh}^* \otimes m_g \otimes m_h$ with actions 

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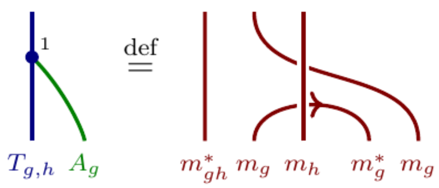
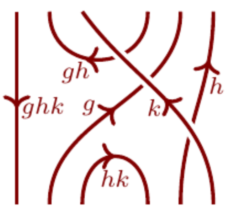
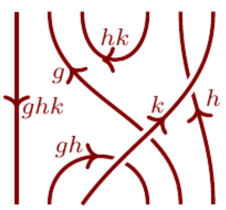


,

$\alpha_{g,h,k} \colon T_{g,hk} \otimes T_{h,k} \longrightarrow T_{gh,k} \otimes T_{g,h}$, $\quad \bar{\alpha}_{g,h,k} \colon T_{gh,k} \otimes T_{g,h} \longrightarrow T_{g,hk} \otimes T_{h,k}$

by

$\alpha_{g,h,k} \overset{\text{def}}{=}$ 

, $\bar{\alpha}_{g,h,k} \overset{\text{def}}{=}$ 

.

$A = \bigoplus_{g\in G} A_g$, $T = \bigoplus_{g,h\in G} {}_{gh}T_{g,h}$

1) Equivalence with A-modules

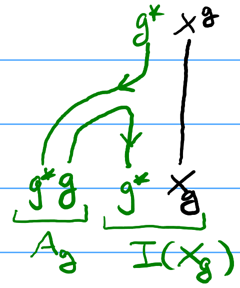
(This is just a warm-up, we do not actually need it)

Abbreviate $g := m_g \in \mathcal{B}_g$

Define functor $I : \widehat{\mathcal{B}} \longrightarrow A\text{-mod}_{\mathcal{B}}$

$$X_g \longmapsto g^* \otimes X_g \quad , \text{ action}$$

$$\left(X_g \xrightarrow{f} Y_g\right) \mapsto \mathrm{id}_{g^*} \otimes f$$



__Lem.:__ I is an equivalence of linear categories

__Pf__ $\mathcal{B} = \mathcal{B}_1 \curvearrowright \mathcal{B}_g$ is indec. $\mathcal{B}_1$-module, and A_g is inner hom $\rightsquigarrow$ Ostrik.

alternatively: Give explicit inverse: $\left(\bigoplus_{g \in G} g\right) \otimes_A (-)$ $\square$

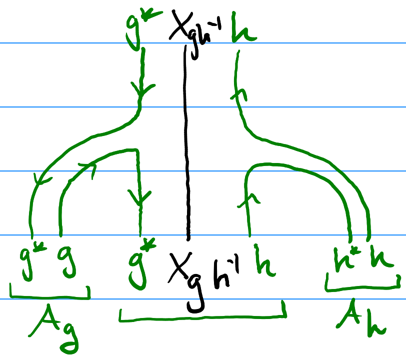
2) Fully faithful ⊗-functor to AA-bimodules

Define $\mathcal{J} : \hat{\mathcal{B}} \longrightarrow {}_A\mathcal{B}_A$ (A-A-bimodules in $\mathcal{B}$)

$$X \longmapsto \bigoplus_{g,h} g^* \otimes X_{g \cdot h^{-1}} \otimes h$$

$$(X \xrightarrow{f} Y) \longmapsto \bigoplus_{g,h} id_{g^*} \otimes f \otimes id_h$$

Actions on $\mathcal{J}(X)$:

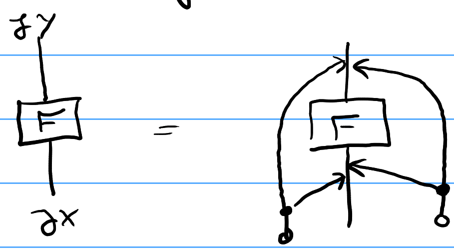


Lem.: $\mathcal{J}$ is fully faithful

Pf. faithful: ✓ (as id ⊗ – ⊗ id is injective on homs)

full: Let $\mathcal{J}(X) \xrightarrow{F} \mathcal{J}(Y)$ be a bimodule hom.

Then



and rhs is

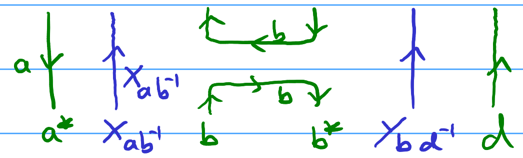
To be continued ...

Monoidal structure

$$(X \otimes Y)_a = \bigoplus_{b \in G} X_{a \cdot b} \otimes Y_{b^{-1}}$$

$$J(X \otimes Y) = \bigoplus_{g,h} g^* (XY)_{gh^{-1}} h$$

$$J(X) \underset{A}{\otimes} J(Y) = \bigoplus_{a,b,c,d} (a^* X_{ab^{-1}} b) \overset{\neq 0 \text{ only if } b = c}{\underset{A}{\otimes}} (c^* X_{cd^{-1}} d)$$

$$= \bigoplus_{a,b,d} \mathrm{im}\Big($$  $$\Big)$$

$$= \bigoplus_{a,b,d} a^* \otimes X_{ab^{-1}} \otimes Y_{bd^{-1}} \otimes d$$

$$= \bigoplus_{a,d} a^* \otimes \Big(\underbrace{\bigoplus_{b} X_{ab^{-1}} \otimes Y_{bd^{-1}}} \Big) \otimes d$$

$$= \bigoplus_{u} X_{ad^{-1}u} \otimes Y_{u^{-1}} \quad \text{for } u = d b^{-1}$$

$$= (XY)_{ad^{-1}}$$

Isomorphisms $J^2_{X,Y} : J(X) \underset{A}{\otimes} J(Y) \longrightarrow J(X \otimes Y)$

(which in above choice of $\otimes_A$ we can take to be $J^2_{X,Y} = \mathrm{id}$)

3) Tensor products with T

$$J(X) \underset{0}{\otimes} T$$

$$= \bigoplus_{abcd} \left[a^* X_{ab^{-1}}\, b\right] \underset{A}{\otimes} \left[(cd)^* \, c\, d\right]$$

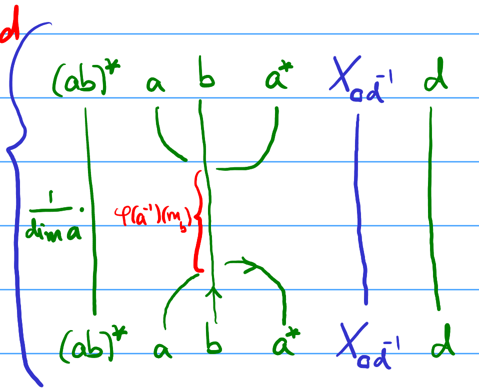
$$= \bigoplus_{acd} \underset{\substack{\uparrow \\ A_0}}{a^*} X_{a(cd)^{-1}} \underset{\substack{\uparrow \\ A_1}}{c} \underset{\substack{\uparrow \\ A_2}}{d}$$

must have $b = cd$

{ indices on which external $A_0 - A_1 A_2$-bimod. action is defined

$$T \otimes_1 J(X)$$

idempotent for $\otimes_A$ here is



$$= \bigoplus_{abcd} \left[(ab)^*\, a\, b\right] \underset{A}{\otimes} \left[c^* X_{cd^{-1}}\, d\right]$$

$c = a$

$$\cong \bigoplus_{abd} \underset{\substack{\uparrow \\ A_0}}{(ab)^*} \underbrace{\varphi(a^{-1})(m_b)}_{A_2} X_{ad^{-1}} \underset{\substack{\uparrow \\ A_1}}{d}$$

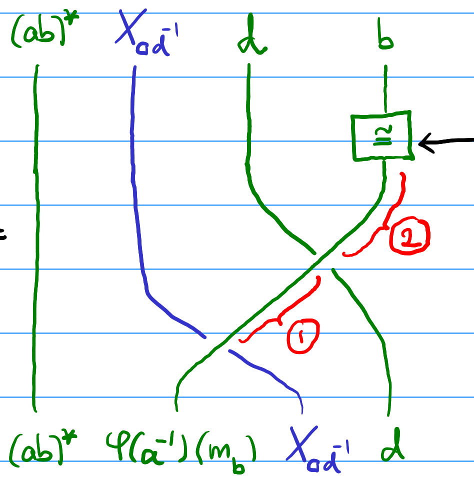
(note different order)

Would like to explicitly have order $A_0 - A_1 - A_2$, so use iso

$$\phi_1 : T \otimes_1 J(X) \xrightarrow{\sim} \bigoplus_{abd} \underset{\substack{\uparrow \\ A_0}}{(ab)^*} X_{ad^{-1}} \underset{\substack{\uparrow \\ A_1}}{d} \underset{\substack{\uparrow \\ A_2}}{b}$$

given by

$\phi_1 =$



① $\underbrace{\varphi(ad^{-1}) \circ \varphi(a^{-1})}_{\cong\, \varphi(a^{-1}ad) = \varphi(d)} (m_b)$

② $\underbrace{\varphi(d) \circ \varphi(ad^{-1}) \circ \varphi(a^{-1})}_{\cong\, \varphi(a^{-1}ad^{-1}d) = id} (m_b)$

iso from monoidal structure of $G \longrightarrow \mathrm{Auto}(\hat{\mathcal{B}})$
(note that we use a right action)

$$T \otimes_2 J(X)$$

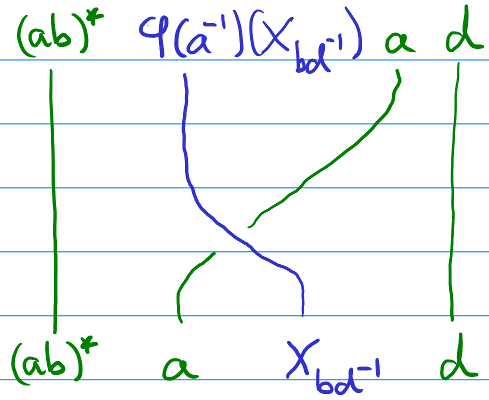
$$= \bigoplus_{abcd} \left[(ab)^* \; a \; \underbrace{b \,\big] \otimes_A \big[\, c^*}_{} \; X_{cd^{-1}} \; d \right]$$

$$\overset{b=c}{=} \bigoplus_{abd} (ab)^* \; a \; X_{bd^{-1}} \; d$$

Bring this to same order of tensor factors as in other cases

$$\phi_2 : T \otimes_2 J(X) \xrightarrow{\sim} \bigoplus_{abd} (ab)^* \underbrace{\varphi(a^{-1})(X_{bd^{-1}})}_{\in B_{abd^{-1}a^{-1}}} \; a \; d$$

$\phi_2 =$



$g = ab$

$a = h \quad d = k$

$g = hb \quad b = h^{-1}g$

Rewrite all with "external legs" (i.e. the ones giving the A-AA bimodule structure $\underset{A_0}{g^*} - \underset{A_1}{h} - \underset{A_2}{k}$:

$$J(X) \otimes_0 T = \bigoplus_{ghk} g^* \; X_{gk^{-1}h^{-1}} \; h \; k$$

$$T \otimes_1 J(X) \underset{\phi_1}{\cong} \bigoplus_{ghk} g^* \; X_{gk^{-1}h^{-1}} \; h \; k$$

$$T \otimes_2 J(X) \underset{\phi_2}{\cong} \bigoplus_{ghk} g^* \; \underbrace{\varphi(h^{-1})(X_{h^{-1}gk^{-1}})}_{\in B_{hh^{-1}gk^{-1}h^{-1}} = B_{gk^{-1}h^{-1}} \checkmark} \; h \; k$$

4) $\widehat{\mathcal{B}}^G$ and $\mathcal{B}_A$ as linear categories

from Etingof, Gelaki, Nikshych, Ostrik, Tensor Categories (AMS 2015)

Definition 2.7.2. A *G-equivariant object* in $\mathcal{C}$ is a pair (X, u) consisting of an object X of $\mathcal{C}$ and a family of isomorphisms $u = \{u_g : T_g(X) \xrightarrow{\sim} X \mid g \in G\}$, such that the diagram

$$\begin{tikzcd} T_g(T_h(X)) \arrow[r, "T_g(u_h)"] \arrow[d, "\gamma_{g,h}(X)"'] & T_g(X) \arrow[d, "u_g"] \\ T_{gh}(X) \arrow[r, "u_{gh}"] & X \end{tikzcd}$$

commutes for all $g, h \in G$. One defines morphisms of equivariant objects to be morphisms in $\mathcal{C}$ commuting with u_g, $g \in G$.

We need the version for right actions:

(X, η) where $\eta_g : \varphi(g)(X) \xrightarrow{\sim} X$ $(g \in G)$

such that

$$\begin{tikzcd} \varphi(g)\varphi(h)(X) \arrow[r, "\varphi(g)(\eta_h)"] \arrow[d, "\wr"'] & \varphi(g)(X) \arrow[d, "\eta_g"] \\ \varphi(hg)(X) \arrow[r, "\eta_{hg}"] & X \end{tikzcd}$$

morphisms : commute with η_g

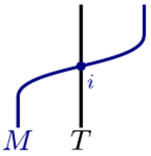
$$\begin{tikzcd} \varphi(g)(X) \arrow[r, "\eta_g"] \arrow[d, "\varphi(g)(f)"] & X \arrow[d, "f"] \\ \varphi(g)(Y) \arrow[r, "\eta_g"'] & X \end{tikzcd}$$

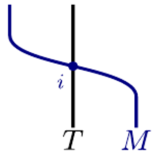
from 2002.00663, Mulevicius, Runkel :

Definition 3.1. Define the category $\mathcal{C}_{\mathbb{A}}$ to have:

- *Objects*: tuples $(M, \tau_1, \tau_2, \overline{\tau_1}, \overline{\tau_2})$, where
  - M is an A-A bimodule;
  - $\tau_1 : M \otimes_0 T \to T \otimes_1 M$, $\tau_2 : M \otimes_0 T \to T \otimes_2 M)$, $\overline{\tau_1} : T \otimes_1 M \to M \otimes_0 T$, $\overline{\tau_2} : T \otimes_2 M \to M \otimes_0 T$ are A-AAA-bimodule morphisms, denoted by

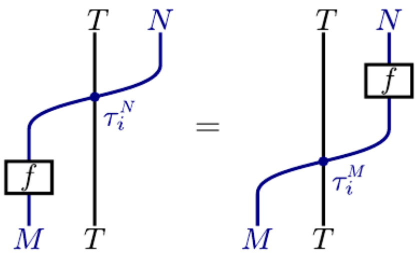
$$\tau_i = \quad , \qquad \overline{\tau_i} := \quad , \qquad i = 1, 2 \,, \tag{3.1}$$



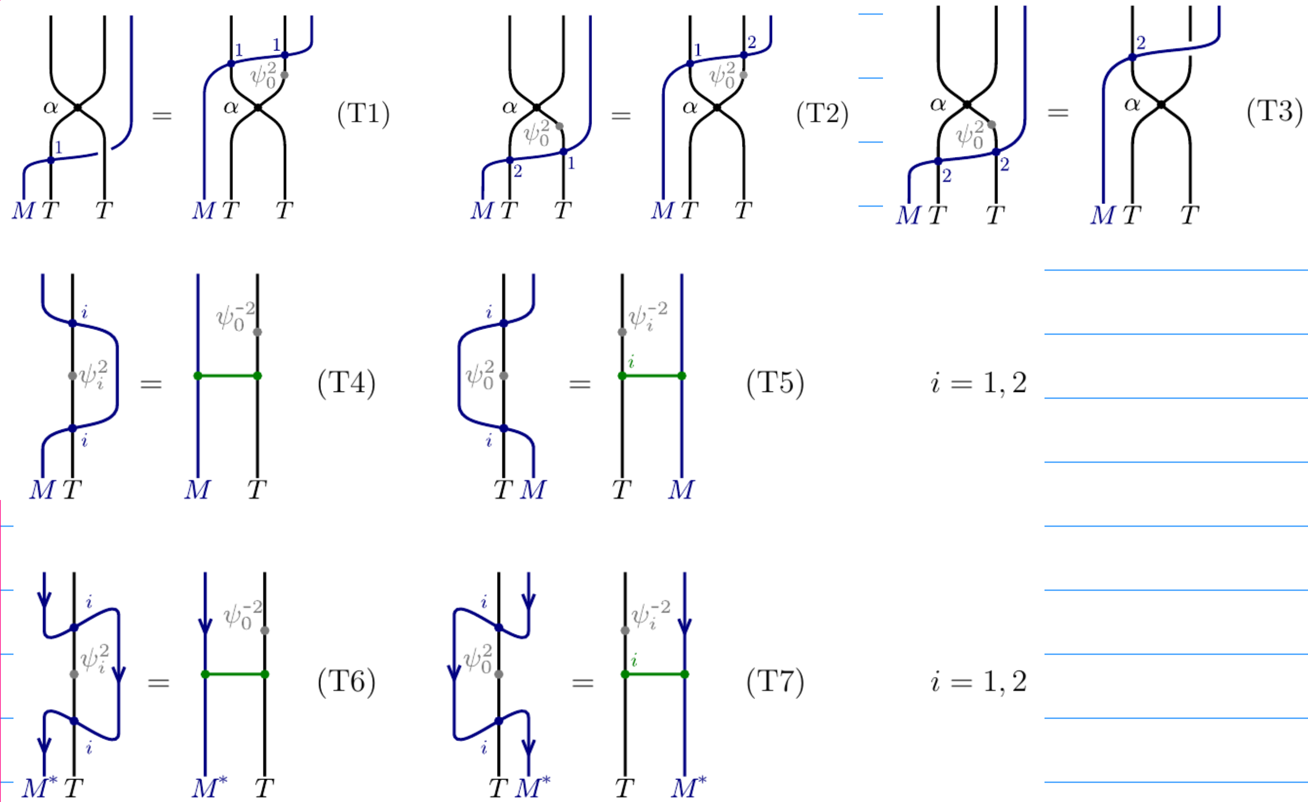


  such that the identities in Figure 3.1 are satisfied. (Recall from the end of Section 2.2 that the notation $\otimes_0$ refers to the left-A-action on T.)

- *Morphisms*: A morphism $f : (M, \tau_1^M, \tau_2^M, \overline{\tau_1^M}, \overline{\tau_2^M}) \to (N, \tau_1^N, \tau_2^N, \overline{\tau_1^N}, \overline{\tau_2^N})$ is an A-A-bimodule morphism $f : M \to N$, such that $\tau_i^N \circ (f \otimes_0 \mathrm{id}_T) = (\mathrm{id}_T \otimes_i f) \circ \tau_i^M$, $i = 1, 2$, or graphically



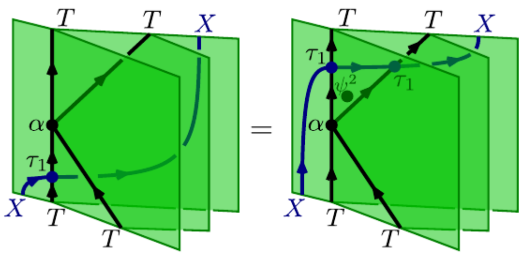
$$, \quad i = 1, 2 \,. \tag{M}$$



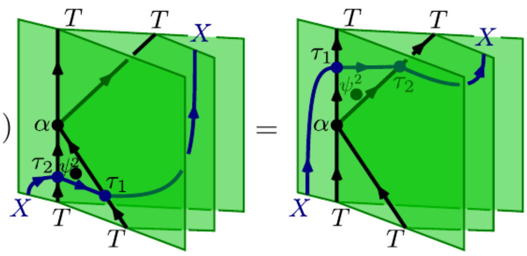
Note: If τ_1, τ_2 are invertible (w.r.t $\otimes_A$) this fixes $\overline{\tau_1}, \overline{\tau_2}$ uniquely, so we will only give τ_1, τ_2.

(Aside (not needed): Conditions as defect pictures)

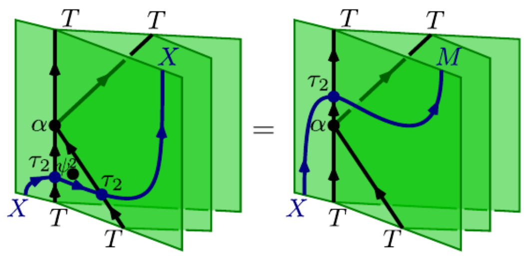
from 2109.04754, Carqueville, Mulevicius, Runkel, Schaumann, Scherl :



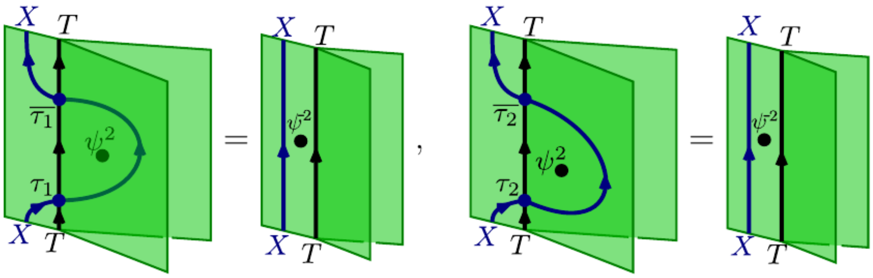
(T1)



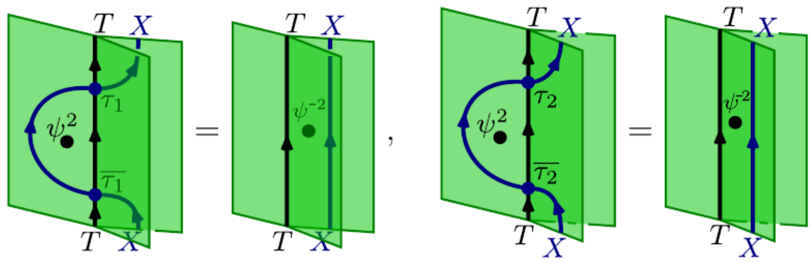
(T2)



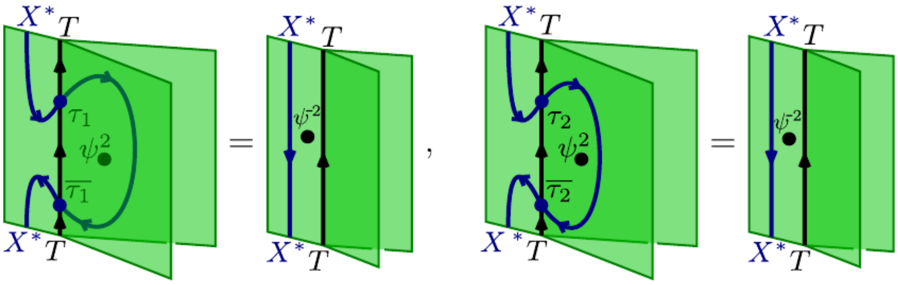
(T3)



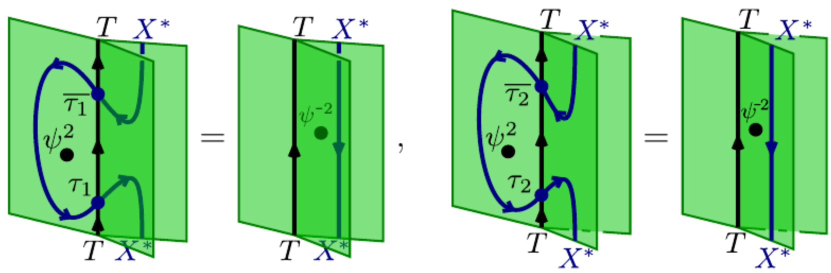
(T4)



(T5)



(T6)



(T7)

5) Linear functor $\hat{\mathcal{B}}^G \longrightarrow \mathcal{B}_A$.

Want to define

$$E : \hat{\mathcal{B}}^G \longrightarrow \mathcal{B}_A$$

On objects: Let $(X, \eta) \in \hat{\mathcal{B}}^G$ be given.

Write $E(X,\eta) = (M, \tau_1, \tau_2)$

- $M := J(X)$ from Sec. 2.
- $\tau_1 : J(X) \otimes_0 T \xrightarrow{\phi_1^{-1}} T \otimes_1 J(X)$ (cf. Sec 3 for ϕ_1)

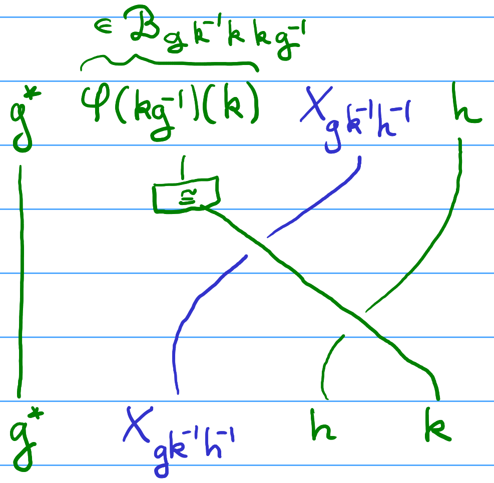
$$J(X)\otimes_0 T = \bigoplus_{ghk} g^* X_{gk^{-1}h^{-1}}\, h\, k$$

- $\tau_2 : J(X)\otimes_0 T \dashrightarrow T \otimes_2 J(X)$

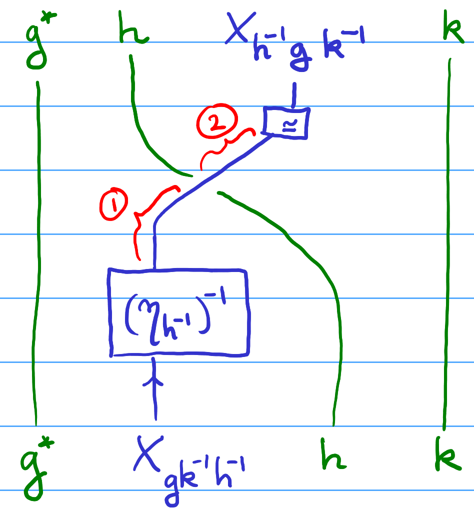
$$\begin{tikzcd}
J(X)\otimes_0 T \arrow[r, dashed, "\sim"] \arrow[d, equal] & T\otimes_2 J(X) \arrow[d, "\Phi_2", "\wr"'] \\
\bigoplus_{ghk} g^* X_{gk^{-1}h^{-1}}\, h\, k & \bigoplus_{ghk} g^* \varphi(h^{-1})(X_{h^{-1}gk^{-1}})\, h\, k \arrow[l, "id \otimes \eta_{h^{-1}} \otimes id"']
\end{tikzcd}$$

this is the place where equiv. structure enters.

$\tau_1 =$



$\tau_2 =$

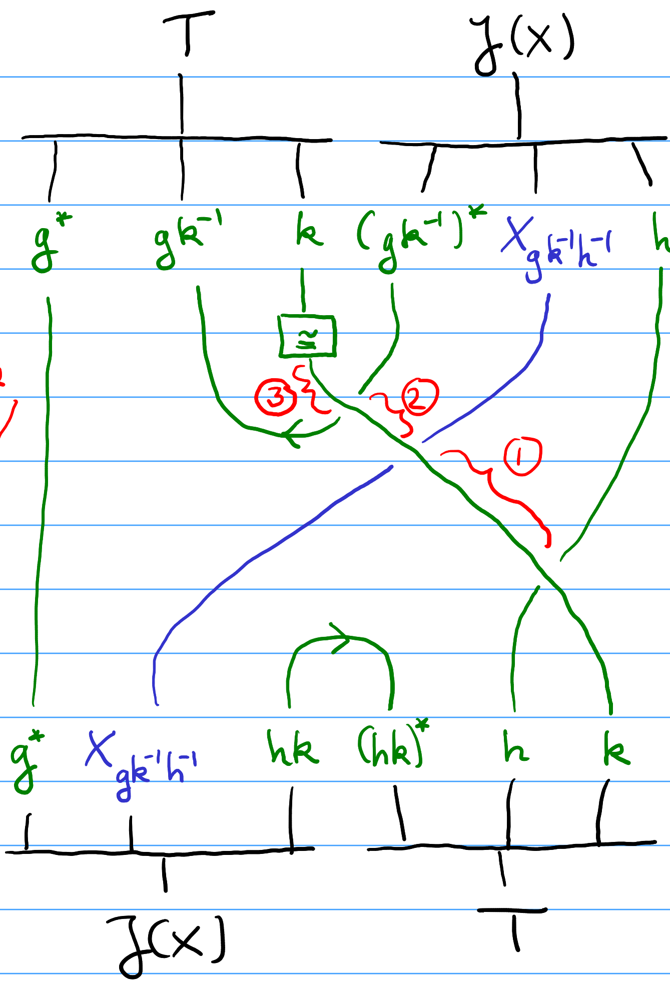


① $\varphi(h^{-1})(X_{h^{-1}gk^{-1}})$ ② $\varphi(h)\varphi(h^{-1})(X_{h^{-1}gk^{-1}})$

Version with $\otimes_A \rightsquigarrow \otimes$ to make composition easier

$\underbrace{\cdot\,(\dim m_{kg^{-1}} \cdot \dim m_{hk})^{-1/2}}$

$\tau_1 =$



① $\varphi(h^{-1})(k)$

② $\varphi(hkg^{-1})\varphi(h^{-1})(k)$

③ $\underbrace{\varphi(gk^{-1})\varphi(hkg^{-1})\varphi(h^{-1})(k)}$

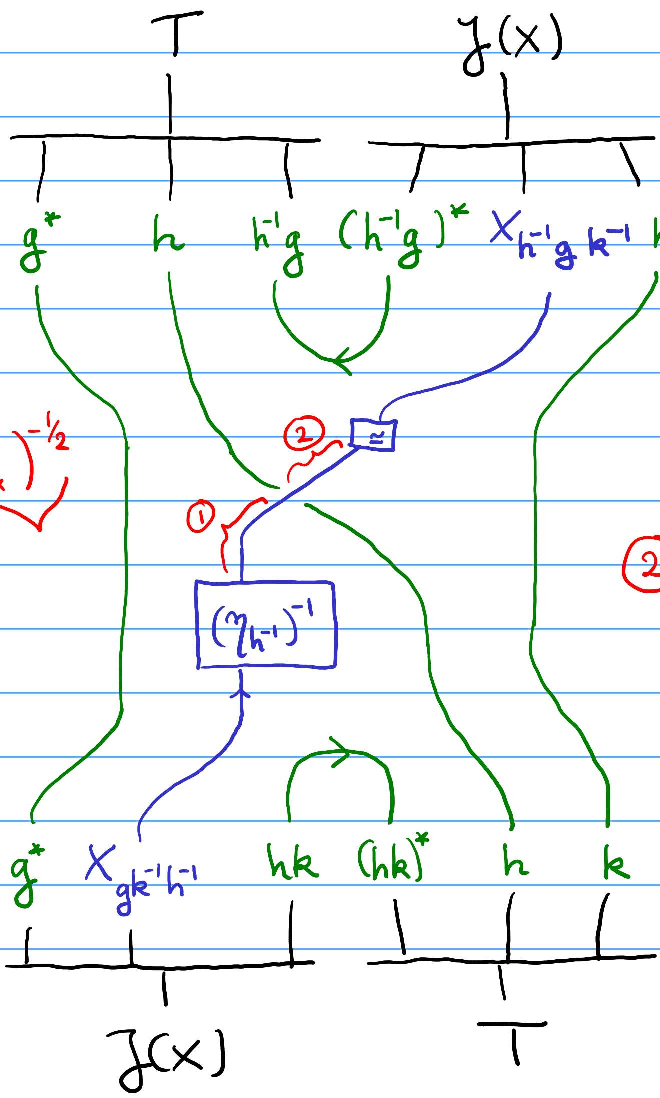
$\simeq \varphi(h^{-1}hkg^{-1}gk^{-1})$

$= id$ ✓

maybe need some factors, maybe not, omit for now and see what happens

$\underbrace{(\dim m_{h^{-1}g} \cdot \dim m_{hk})^{-1/2}}$

$\tau_2 =$



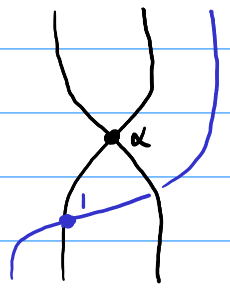
① $\varphi(h^{-1})(X_{h^{-1}gk^{-1}})$

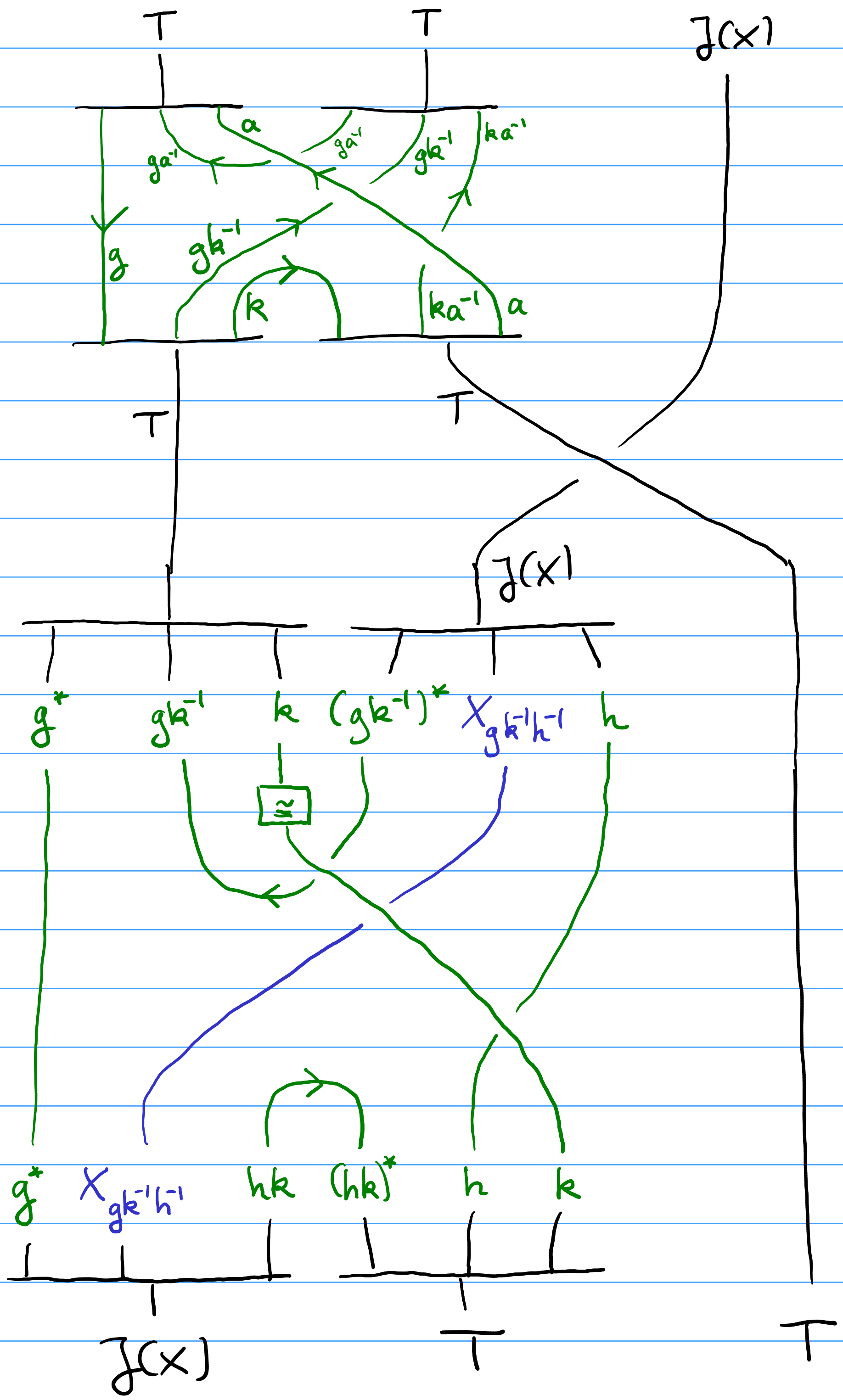
② $\varphi(h)\varphi(h^{-1})(X_{h^{-1}gk^{-1}})$

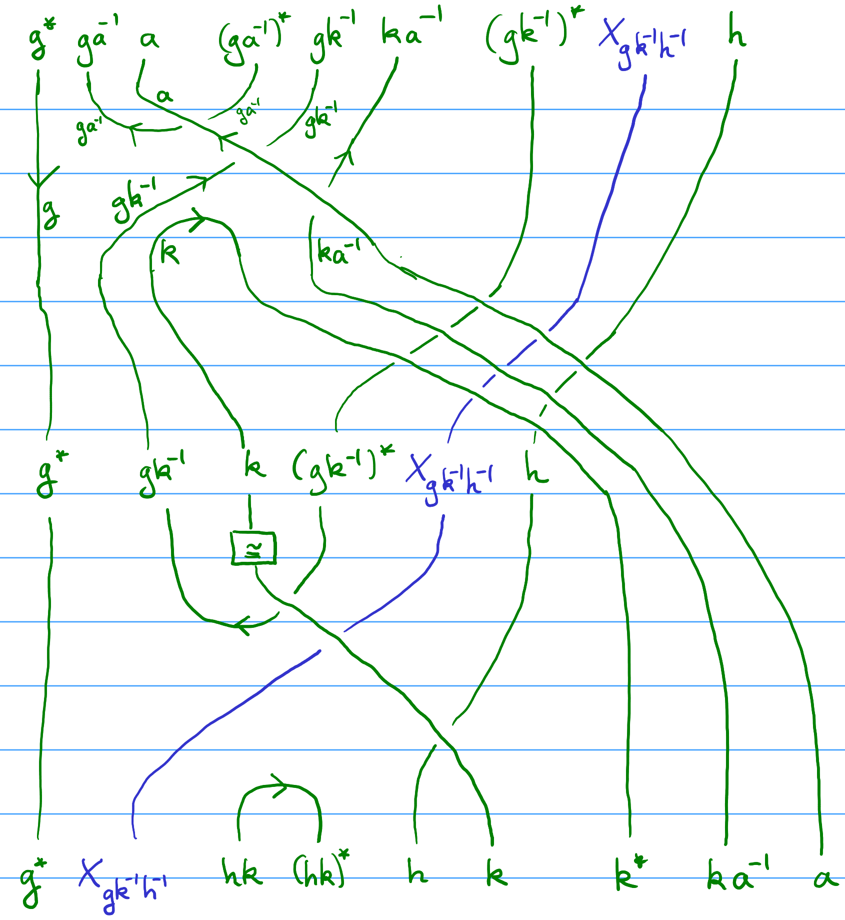
6) Check T1 – T7 for $E(X,\eta)$

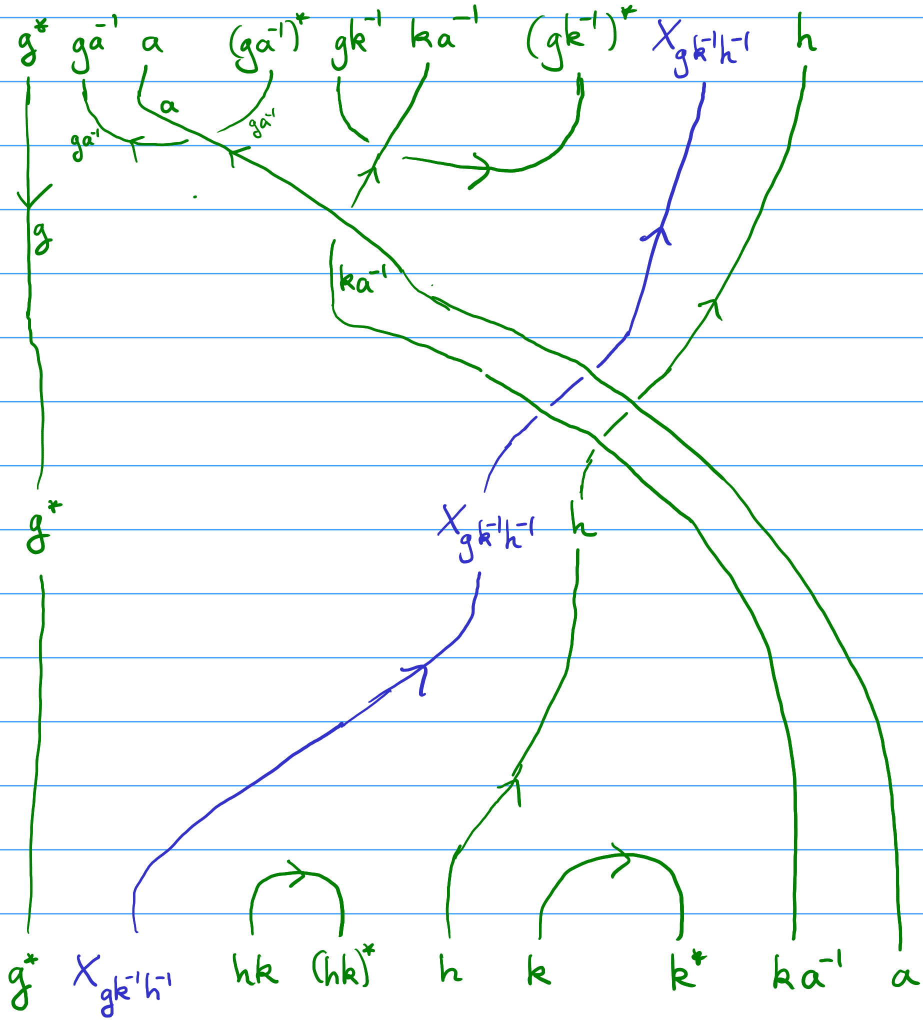
T1

lhs

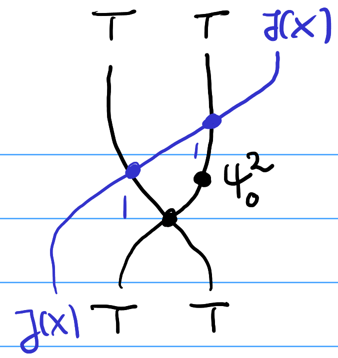


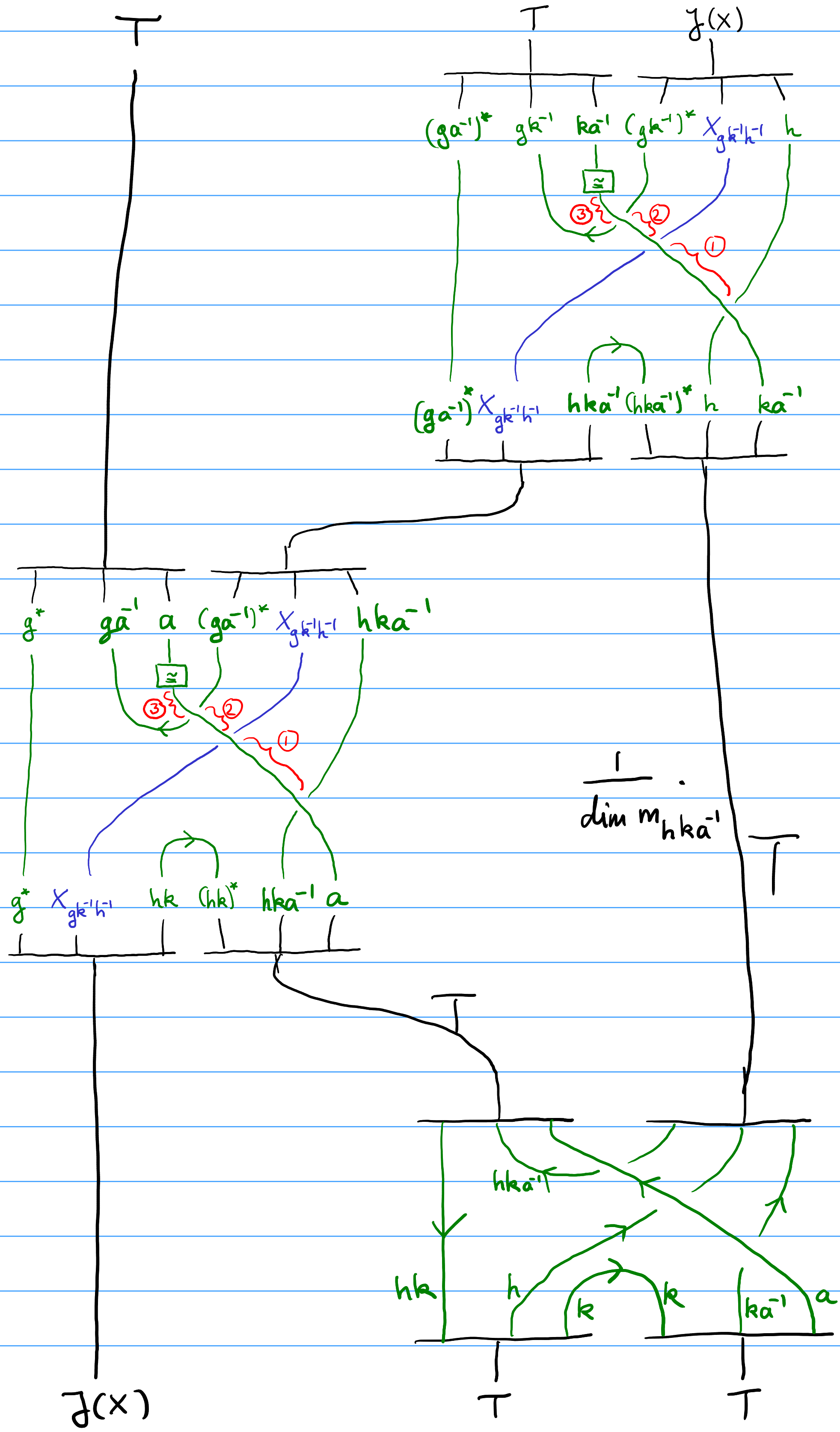



g* ga⁻¹ a (ga⁻¹)* gk⁻¹ ka⁻¹ (gk⁻¹)* X_{gk⁻¹h⁻¹} h
g* gk⁻¹ k (gk⁻¹)* X_{gk⁻¹h⁻¹} h
≅
g* X_{gk⁻¹h⁻¹} hk (hk)* h k k* ka⁻¹ a

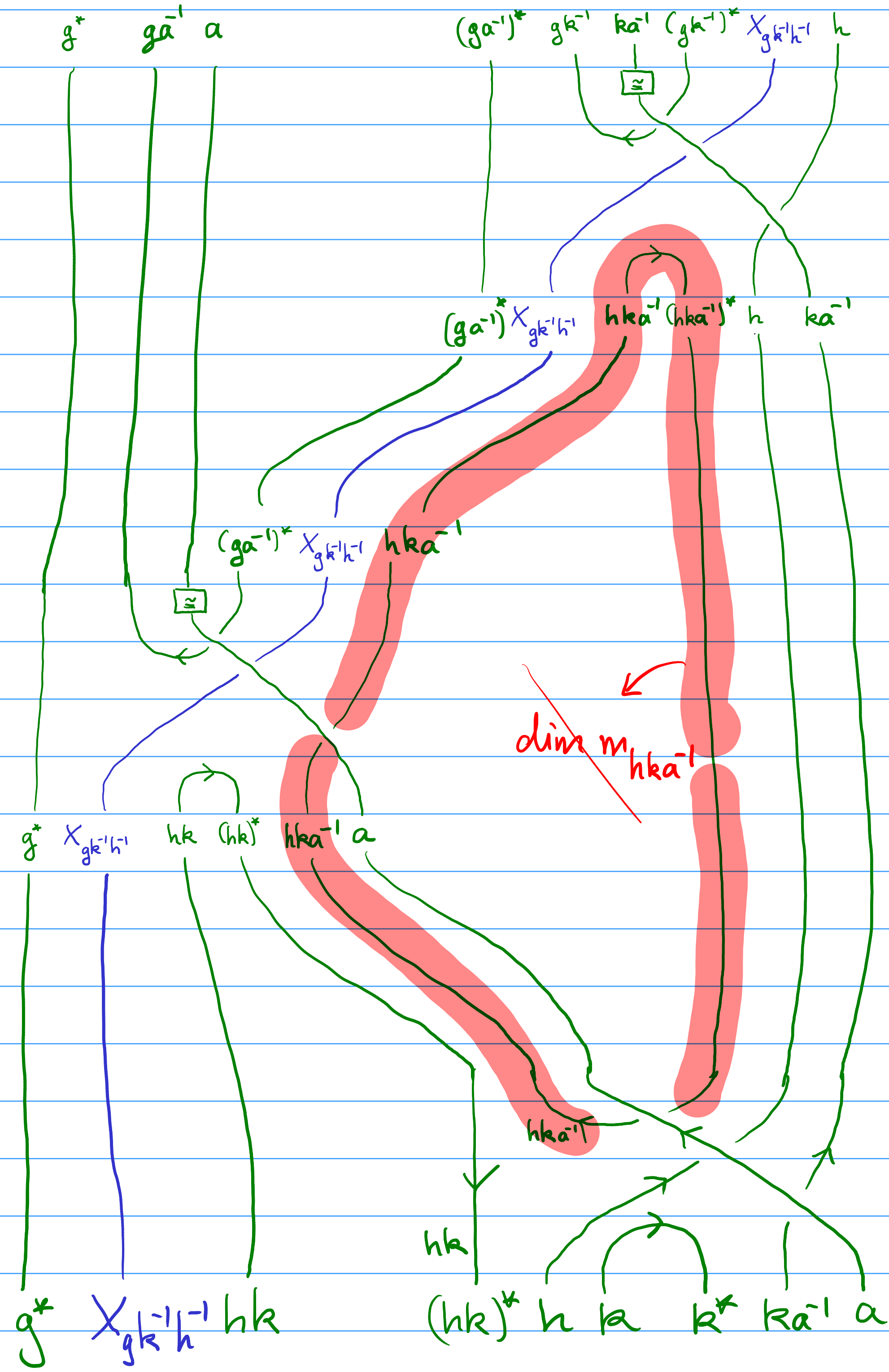

g* ga⁻¹ a (ga⁻¹)* gk⁻¹ ka⁻¹ (gk⁻¹)* X_{gk⁻¹h⁻¹} h
g* X_{gk⁻¹h⁻¹} h
g* X_{gk⁻¹h⁻¹} hk (hk)* h k k* ka⁻¹ a

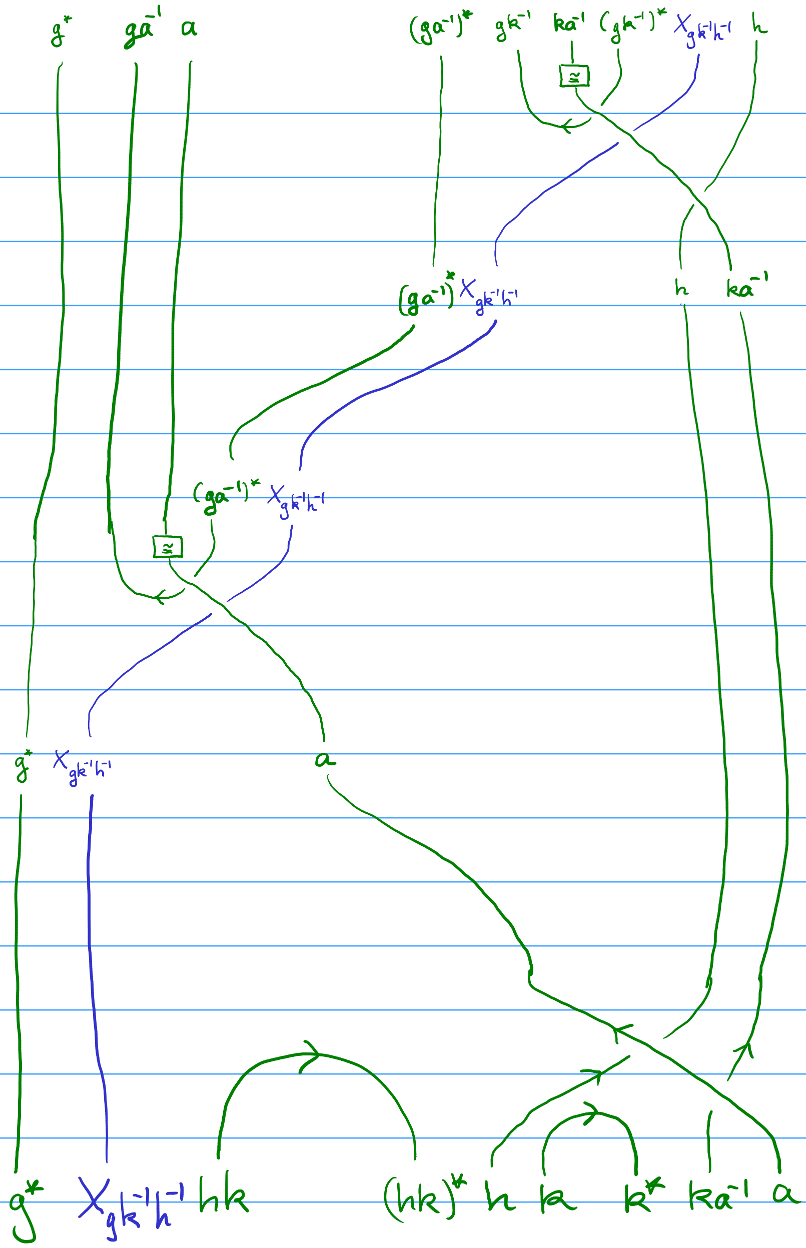
rhs



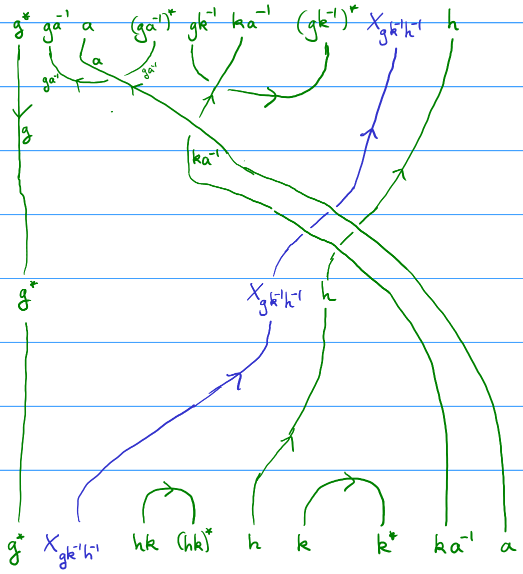


$\cancel{\dfrac{1}{\dim m_{hka^{-1}}}}$.





Compare rhs to lhs ✓



7) E is fully faithful

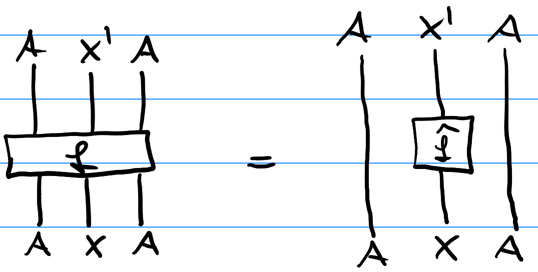
E is faithful: clear as $\mathcal{J}$ is already faithful and morphisms in $\mathcal{B}_A$ are just subspaces of bimodule maps

E is full: Let $f : E(X,\eta) \longrightarrow E(X',\eta')$ be a morphism in $\mathcal{B}_A$. This means:

1) $f : \mathcal{J}(X) \longrightarrow \mathcal{J}(X')$ is an A-A-bimodule map

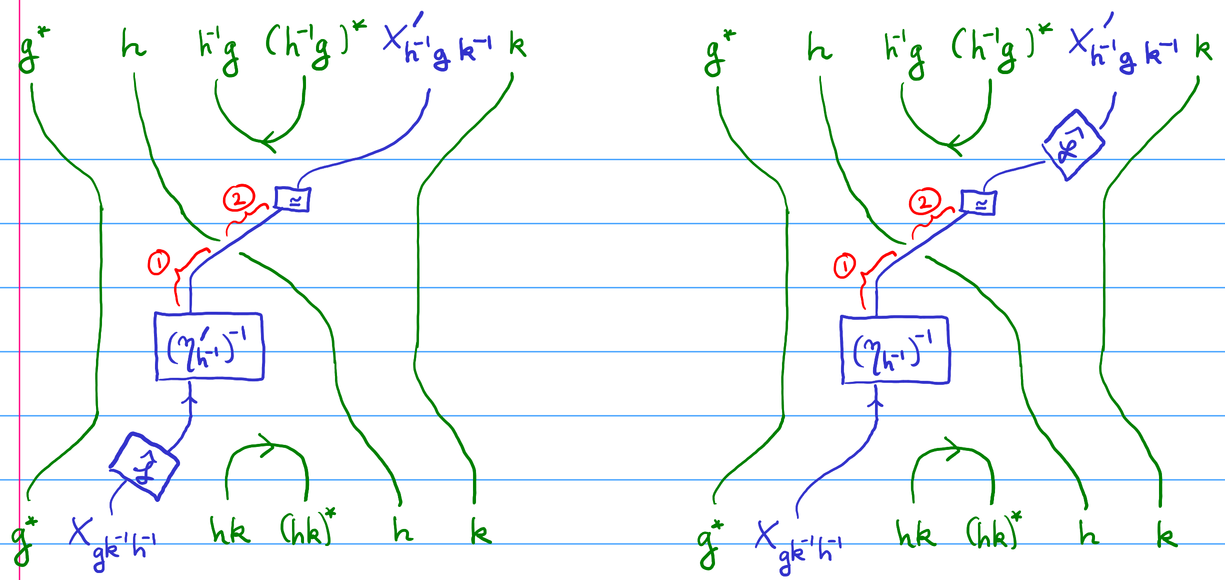
2)
$$\begin{tikzcd} \mathcal{J}(X) \otimes_0 T \arrow[r, "f \otimes id"] \arrow[d, "\tau_i"] & \mathcal{J}(X') \otimes_0 T \arrow[d, "\tau_i"] \\ T \otimes_i \mathcal{J}(X) \arrow[r, "id \otimes_i f"] & T \otimes_i \mathcal{J}(X') \end{tikzcd}$$
commutes for $i = 1,2$

By the lemma in part 2 there is a (unique) $\hat{f} : X \to X'$ in $\hat{\mathcal{B}}$ s.th. $\mathcal{J}(\hat{f}) = f$. In diagrams:

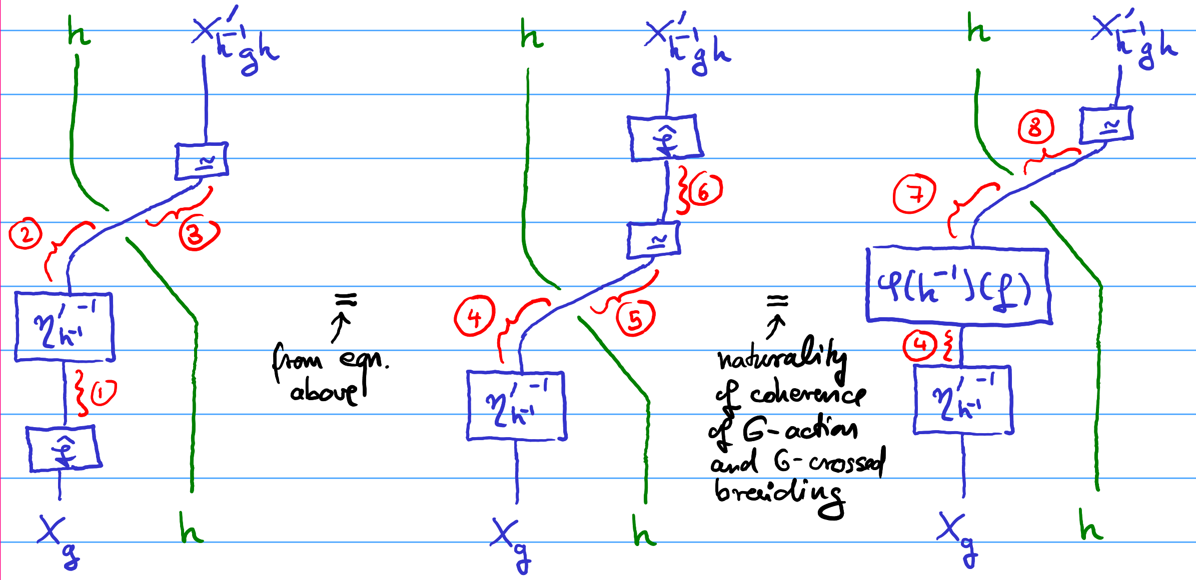


Condition 2 for $i=1$ gives nothing new, but for $i=2$ the calc. on next page togethere with invertibility of σ-crossed braiding gives

$$\left[X_g \xrightarrow{\hat{f}} X'_g \xrightarrow{(\eta'_{h^{-1}})^{-1}} \varphi(h^{-1})(X'_{h^{-1}gh}) \right]$$
$$= \left[X_g \xrightarrow{(\eta_{h^{-1}})^{-1}} \varphi(h^{-1})(X_{h^{-1}gh}) \xrightarrow{\varphi(h^{-1})(\hat{f})} \varphi(h^{-1})(X'_{h^{-1}gh}) \right] \quad (*)$$



Set $k = h^{-1}$ and trace out the m_a-strands not braided with X to get



① X'_g

② $\varphi(h^{-1})(X'_{h^{-1}gh}) \in \mathcal{B}_{hh^{-1}ghh^{-1}}$ ✓

③ $\varphi(h)\varphi(h^{-1})(X_{h^{-1}gh})$

④ $\varphi(h^{-1})(X_{h^{-1}gh})$

⑤ $\varphi(h)\varphi(h^{-1})(X_{h^{-1}gh})$

⑥ $X_{h^{-1}gh}$

⑦ $\varphi(h^{-1})(X'_{h^{-1}gh})$

⑧ $\varphi(h)\varphi(h^{-1})(X'_{h^{-1}gh})$

Conjugating (*) with η, doing $\bigoplus_{g\in G}$, and renaming $h^{-1} \rightsquigarrow a$ gives

$$\left[\varphi(a)(X) \xrightarrow{\eta_a} X \xrightarrow{\hat{f}} X'\right] = \left[\varphi(a)(X) \xrightarrow{\varphi(a)(\hat{f})} \varphi(a)(X') \xrightarrow{\eta'_a} X'\right]$$

Thus $\hat{f} : X \to X'$ is also a morph. $(X,\eta) \to (X',\eta')$ in $\hat{\mathcal{B}}^G$. ✓

Next steps:

* show that image of functor indeed satisfies all conditions T1-T7

* check further properties of functor
    - monoidal
    - braided and ribbon
    - essentially surjective