Theoretical task 1

Solution should be short, mathematically precise and contain proof unless qualitative explanation/intuition is needed. Please underline your final answer for ease of checking. Solution should be handwritten, scanned and sent to mlaticl2019@yandex.ru due Feb 3, 23:59.

1. Consider real numbers $z_1, z_2, ...z_N$. Find such constant approximation μ of these numbers, so that

   (a) $\sum_{n=1}^{N} (z_n - \mu)^2$ is minimized.

   (b) $\sum_{n=1}^{N} |z_n - \mu|$ is minimized.

   Hint: if a function is convex, zero derivative is a sufficient condition of its global minimum. $\frac{d}{du}|u| = sign(u)$.

2. Suppose $x \in \mathbb{R}^D$ is a feature vector. Prove that whitening transformation $f = \Sigma^{-1/2}(x - \mu)$, where $\mu = \mathbb{E}x$, $\Sigma = cov[x, x]$, will give new feature vector f with:

   (a) $\mathbb{E}f = \mathbf{0}$ (all zeroes vector)

   (b) $cov[f, f] = I$ (identity matrix)

3. Under what selection of kernel $K(u)$ and bandwidth function $h(x)$ will Nadaraya-Watson regression reduce to K-NN regression?

4. Write stochastic gradient descent with minibatch size=1 for the following losses:

   (a) $\mathcal{L}(M) = [-M]_+$

   (b) $\mathcal{L}(M) = \ln(1 + e^{-M})$

5. Consider finding PCA components from a sequence of optimization tasks, applied to design matrix $X \in \mathbb{R}^{NxD}$. You know that

$$\begin{cases} \|Xa_1\|^2 \to \max_{a_k} \\ \|a_1\| = 1 \end{cases}$$

   gives eigenvector, corresponding to largest eigenvalue. Prove that

$$\begin{cases} \|Xa_k\|^2 \to \max_{a_k} \\ \|a_k\| = 1 \\ a_k^T a_1 = ... = a_k^T a_{k-1} = 0 \end{cases}$$

   will give eigenvector of $X^T X$ corresponding to k-th largest eigenvalue.

6. Derive analytical solution for weighted regression:

$$\sum_{n=1}^{N} w_n \left(x_n^T \beta - y_n\right)^2 \to \min_{\beta \in \mathbb{R}^D}$$

   in terms of matrix of weights diagonal matrix $W = diag\{w_1, ...w_N\} \in \mathbb{R}^{NxN}$, design matrix $X \in \mathbb{R}^{NxD}$ and outputs vector $Y \in \mathbb{R}^{Nx1}$, where D is the number of features.