STURM LIOUVILLE BOUNDARY VALUE PROBLEM

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by

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Sturm-Liouville problem

Abstract

Sturm-Liouville problem is a system consisting of a differential equation coupled with certain boundary condition in such a way that the differential operator is self-adjoint or Hermitian.Like any other eigenvalue problem,here we look for eigenfunctions that satisfy both differential equation and boundary conditions.

Introduction

Let [a,b] be a bounded interval in R .$C^2([a,b])$ denotes the space of functions with derivatives of second order continous upto the endpoints. $C_0^2([a,b])$ is the subspace of functions that vanish near the endpoints.
The differential equation

$$(p(x)y'(x))' - q(x)y(x) + \lambda r(x)y = 0, a \leq x \leq b \tag{1}$$

along with the boundary conditions

$$c_1 y(a) + c_2 y'(a) = 0, \qquad d_1 y(b) + d_2 y'(b) = 0 \tag{2}$$

is called a Sturm-Liouville equation (SLE). A value of the parameter λ for which a non- trivial solution ($y \neq 0$) exists for the boundary value problem (BVP)(1)&(2) is called an eigen-value of the problem and corresponding nontrivial solutions y(x) of BVP are called eigenfunctions which is associated with that eigenvalue.

REMARKS : Boundary condition:Let u$\in$ $C_c^0([a,b])$ .Boundary condition can be generalized as

$$Bu = \alpha u(a) + \beta u(b) + \gamma u'(a) + \delta u'(b)$$

where α ,β ,γ ,δ are real numbers.

1. Dirichlet Condition:$B_1 u = u(a), B_2 u = u(b)$
2. Neumann Condition:$B_1 u = u'(a), B_2 u = u'(b)$
3. Robin Condition:$B_1 u = u'(a) - \alpha u(a), B_2 u = u'(b) + \beta u(b)$

   Aforementioned conditions are said to be sepearated as each one is defined at a unique point.

4. Periodic Condition:$B_1 u = u(b) - u(a), B_2 u = u'(b) - u'(a)$

A system with a periodic boundary condition is not considered as a Sturm-Liouville problem because the boundary conditions are not separated.

The Sturm-Liouville Operator

Consider the Sturm-Liouville differential operator

$$L[y] = -(p(x)y'(x))' + q(x)y(x)$$

where $p > 0$,$r > 0$ and p',q and r are continuous on [a,b]. The differential equation takes the operational form

$$L[y] - r(x)y(x) = 0, \quad a \leq x \leq b$$

$$c_1 y(a) + c_2 y'(a) = 0$$
$$d_1 y(b) + d_2 y'(b) = 0$$

Lagrange's Identity

Let u and v be functions having continous second derivatives on the interval $a \leq x \leq b$.

$$\int_a^b L[u]v dx = \int_a^b [-(pu')'v + quv]dx$$

$$\int_a^b L[u]v dx = -p(x)u'(x)v(x)|_a^b + p(x)u(x)v'(x)|_a^b + \int_a^b [-u(pv')' + uqv]dx$$

$$= -p(x)[u'(x)v(x) - u(x)v'(x)]|_a^b + \int_a^b uL[v]dx$$

upon transposing the integral on the right side,we have

$$\int_a^b L[u]v - uL[v]dx = -p(x)[u'(x)v(x) - u(x)v'(x)]|_a^b,$$

which is **Lagrange's identity**.
Suppose that u and v satisfies the boundary conditions(2).

$$c_1 u(a) + c_2 u'(a) = 0$$
$$c_1 v(a) + c_2 v'(a) = 0$$

In matrix form this is equivalent to ,

$$\begin{pmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $(c_1, c_2) \neq (0,0)$, this equation can only hold if the matrix on the left is non- invertible. That is,

$$0 = \begin{vmatrix} u(a) & u'(a) \\ v(a) & v'(a) \end{vmatrix} = u(a)v'(a) - v(a)u'(a)$$

This shows that, $[u'(x)v(x) - u(x)v'(x)]|_a^b = 0$ in accordance with the above argument.Then Lagrange's identity reduces to

$$\int_a^b (L[u]v - uL[v])dx = 0$$

The inner product (.,.) of two real valued functions u and v on an interval $a \leq x \leq b$ is defined as

$$(u, v) = \int_a^b u(x)v(x)r(x)dx$$

,where r(x) is said to be the weight function and for the eigenspace is a hilbert space, we need $r(x) > 0$. Using the notion of inner product,the reduced Lagrange's identity can be written as

$$(\frac{1}{r}L[u], v) - (u, \frac{1}{r}L[v]) = 0$$

i.e,

$$(\frac{1}{r}L[u], v) = (u, \frac{1}{r}L[v])$$

then L is said to be a **Hermitian operator**.
L being a hermitian operator has analogous properties as that of a hermitian matrix.
Let X and Y be two complex vectors such that $X = (x_1, x_2, x_3, ..., x_n)$ and $Y = (y_1, y_2, y_3, ..., y_n)$.An inner product can be defined as

$$(X, Y) = \sum_{i=1}^{n} x_i \bar{y}_i$$

Let A be a n*n matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The inner product $(AX, Y) = \bar{y_1}(a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n) + \bar{y_2}(a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n) + ... + \bar{y_n}(a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n)$. Adjoint of the matrix A is

$$A^{\dagger} = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{nn} \end{bmatrix}$$

The inner product, $(X, A^{\dagger}Y) = \bar{y_1}(a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n) + \bar{y_2}(a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n) + ... + \bar{y_n}(a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n)$ Hence,

$$(AX, Y) = (X, A^{\dagger}Y)$$

If $A^{\dagger} = A$ then A is said to be self-adjoint or Hermitian ,which implies (AX,Y)=(X,AY).

Theorem 1. *All the eigenvalues of the Sturm Liouville problem are real.*

Proof. Suppose that λ is a (possibly complex) eigenvalue of the Sturm-Liouville problem and that ϕ is a corresponding eigenfunction, also possibly complex-valued. Let us write $\lambda = \mu + i\nu$ and $\phi(x) = U(x) + iV(x)$, where μ, ν, U(x), and V(x) are real. Then, if we let u = ϕ and also v = ϕ, we have

$$(L[\phi], \phi) = (\phi, L[\phi])$$

We have $L[\phi] = \lambda r\phi$, so $(\lambda r\phi, \phi) = (\phi, \lambda r\phi)$.

$$\int_a^b \lambda r(x)\phi(x)\bar{\phi}(x)dx = \int_a^b \phi(x)\bar{\lambda}\bar{r}(x)\bar{\phi}(x)$$

Since r(x) is real, $\bar{r}(x) = r(x)$, $(\lambda - \bar{\lambda}) \int_a^b r(x)\phi(x)\bar{\phi}(x)dx = 0$ The integrand in the above equation is non-negative and not-identically zero. Since the integrand is also continuous, it follows that the integral is positive. Therefore, the factor $(\lambda - \bar{\lambda}) = 2i\nu$ must be zero. Hence λ is real, so the theorem is proved.
Remark : This result is analogous to the case of a Hermitian matrix,where all eigenvalues are real.

□

Theorem 2. *If ϕ_m and ϕ_n are two independent eigenfunctions of the Sturm Liouville problem corresponding to eigenvalues λ_m and λ_n, respectively, and if,then*

$$\int_a^b r(x)\phi_m(x)\phi_n(x) = 0$$

Proof. $L[\phi_m] = \lambda_m r\phi_m, L[\phi_n] = \lambda_n r\phi_n$
By Lagrange's identity for Sturm-Liouville case,

$$(L[\phi_m], \phi_n) = (\phi_m, L[\phi_n])$$

i.e, $\int_a^b \lambda_m r\phi_m\bar{\phi}_n dx - \int_a^b \phi_m\bar{\lambda}_n\bar{r}\bar{\phi}_n = 0$
As $\phi_n(x), \lambda_n, r(x)$ are real ,

$$(\lambda_m - \lambda_n)\int_a^b r(x)\phi_m(x)\phi_n(x) = 0$$

As $(\lambda_m - \lambda_n) \neq 0$ implies,

$$(\phi_m, \phi_n) = \int_a^b r(x)\phi_m(x)\phi_n(x) = 0$$

$\square$

The eigenfunctions of a regular Sturm-Liouville boundary value problem corresponding to distinct eigenvalues are othogonal w.r.t. weight function r(x) on [a,b] .
We state without proof

Theorem 3. *The Sturm-Liouville problem has an infinite number of eigenvalues, which can be written in increasing order as* $\lambda_1 < \lambda_2 < ... < \lambda_n < ...$ *such that* $\lim_{n\to\infty} \lambda_n = \infty$.

Example:Consider the differential equation,

$$y\prime\prime + \lambda y = 0, y(0) = y(1) = 0$$

This is a Sturm-Liouville equation with p(x)=1,q(x)=0 and r(x)=1.
If $\lambda = 0$,the general solution is

$$y = ax + b$$

The two boundary conditions require that $a = b = 0$ .So the boundary problem has no non-trivial solutions when $\lambda = 0$ .Hence $\lambda = 0$ is not an eigenvalue.
If $\lambda > 0$,we have the general solution as

$$y = a\sin(\sqrt{\lambda}x) + bcos(\sqrt{\lambda}x)$$

.On computation,the boundary condition require that b=0.For having a non-trivial solution $a \neq 0$ and $a\sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = n\pi$.So the eigenfunction and eigenvalue are respectively

$$\lambda_n = n^2\pi^2, y_n = a\sin(n\pi x)$$

The eigenvalues here, form an infinite sequence,as said in the previous theorem.

Example: Consider the differential equation,

$$y'' + \lambda y = 0, y(0) = 0, y'(1) + y(1) = 0$$

This is a Sturm-Liouville equation with p(x)=1,q(x)=0 and r(x)=1. If $\lambda = 0$,the general solution is

$$y = ax + b$$

The two boundary conditions require that $a = b = 0$ .So the boundary problem has no non-trivial solutions when $\lambda = 0$ .Hence $\lambda = 0$ is not an eigenvalue.
If $\lambda > 0$, a plausible option for y is e^{rx} ,then we have the characteristic equation as $r^2 + \lambda = 0$ i.e,$r = \pm i\sqrt{\lambda}$, which implies the general solution is a linear combination of $e^{i\sqrt{\lambda}x}$and $e^{-i\sqrt{\lambda}x}$ .Using Euler's identity the general solution can be expressed as

$$y = a\sin(\sqrt{\lambda}x) + b\cos(\sqrt{\lambda}x)$$

The boundary condition at x=0 require that, b=0 and at x=1 require

$$a(\sin(\sqrt{\lambda}) + \sqrt{\lambda}\cos(\sqrt{\lambda}) = 0.$$

For a non-trivial solution, we need $a \neq 0$ and the eigenvalue λ must satisfy the condition

$$\sin(\sqrt{\lambda}) + \sqrt{\lambda}\cos(\sqrt{\lambda} = 0$$

.If in any case $\cos(\sqrt{\lambda}) = 0$,then $\sin(\sqrt{\lambda}) = 1$ and then the above condition doesn't hold.So assuming $\cos(\sqrt{\lambda}) \neq 0$,we have $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$,whose solution can be determined graphically by plotting $\sqrt{\lambda}$ and $\tan(\sqrt{\lambda})$ on a common axes and identifying the points of intersection.

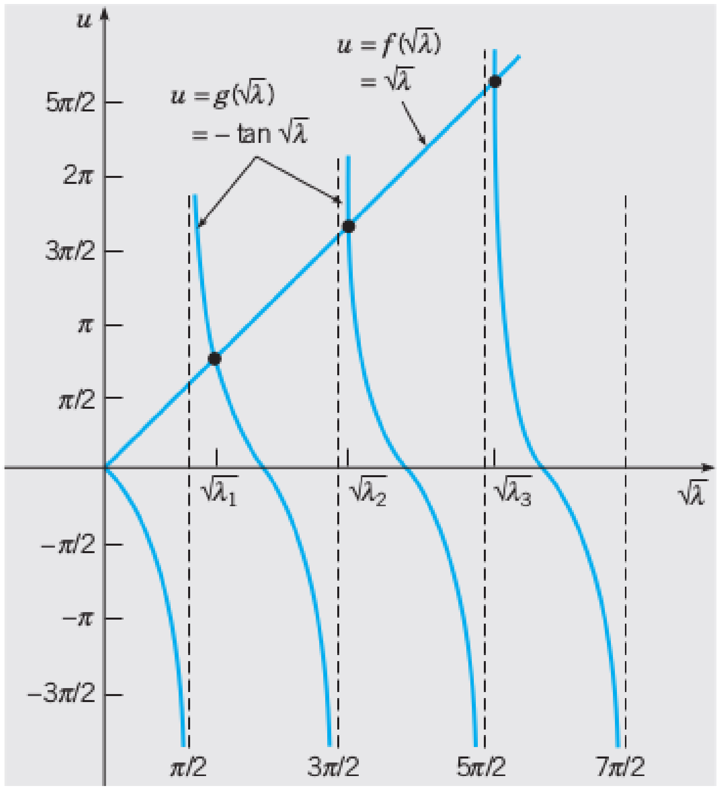

Figure 1: graphical solution of $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$

The first three positive solutions of Eq. $\sqrt{\lambda} = -\tan(\sqrt{\lambda})$ are $\lambda_1 \cong 2.029$, λ_2 , $\cong 4.913$, and $\lambda \cong 7.979$.

From the figure,we can approximate other roots as

$\sqrt{\lambda} \cong (2n-1)\pi/2$, for n=4,5,6,...
The solutions of the differential equation i.e, the eigenfunction corresponding to eigenvalue λ_n is

$$\phi_n(x, \lambda_n) = k_n \sin(\sqrt{\lambda_n} x); n = 1, 2, ...,$$

where $k_n$is an arbitrary constant.

Considering the case $\lambda < 0$.Let $\lambda = -\mu$,so that $\mu > 0$. Then

$$y'' - \mu y = 0,$$

and it's general solution is ,

$$y = a\sinh(\sqrt{\mu}x) + b\cosh(\sqrt{\mu}x)$$

where $\mu > 0$. Proceeding as in the previous case, we find that must satisfy the equation

$$\sqrt{\mu} = -\tanh(\sqrt{\mu})$$

The solutiions of this equation can be graphically determined by identifying the points of intersection.

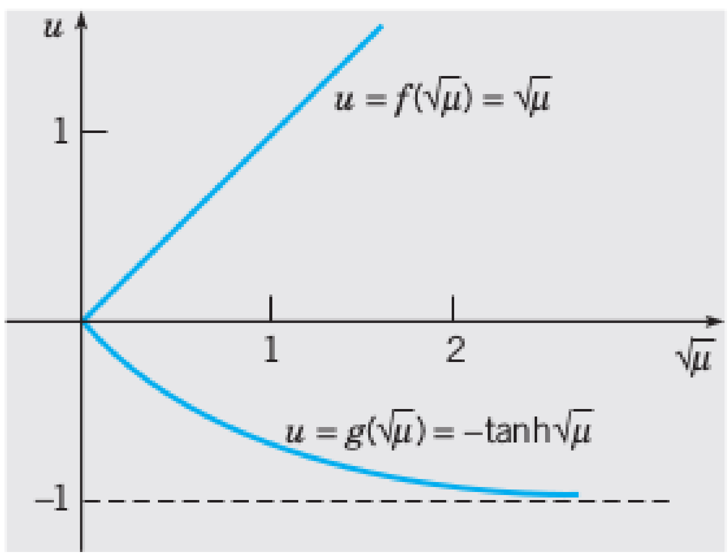

Figure 2: graphical solution of $\sqrt{\mu} = -\tanh(\sqrt{\mu})$

it is clear that the graphs of $f(\sqrt{\mu}) = \sqrt{\mu}$ and $g(\sqrt{\mu}) = -\tanh(\sqrt{\mu})$ intersect only at the origin. Hence there are no positive values of that satisfy the equation,$\sqrt{\mu} = -\tanh(\sqrt{\mu})$ and hence the boundary value problem has no negative eigenvalues.

Nonhomogenous Boundary Value Problems

Consider the differential equation ,

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x)$$

where μ is a constant and f is a given function on $a \leq x \leq b$, along with the boundary conditions

$$c_1 y(0) + c_2 y'(a) = 0, d_1 y(b) + d_2 y'(b) = 0$$

.Assume that p,p',q and r are continous on $a \leq x \leq b$ and that $p(x) > 0$ and $r(x) > 0$.
One way to solve non-homogenous problem is using the eigenfunction expansion for the corresponding homogeneous Sturm-Liouville problem.
Consider the non-homogeneous Sturm- Liouville problem

$$(p(x)y')' + q(x)y + \mu r(x)y = f(x)$$

with boundary conditions

$$c_1 y(0) + c_2 y'(a) = 0, d_1 y(b) + d_2 y'(b) = 0$$

Suppose λ_n and y_n are the eigenvalues and eigenfunctions of the homogeneous problem

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0$$

with the same boundary conditions.
Suppose we can write $y(x) = \sum_n b_n y_n(x)$,here b_n is unknown. we have $(p(x)y_n')' + q(x)y_n + \lambda_n r(x)y_n = 0$ as y_n is an eigenfunction corresponding to eigenvlue λ_n of the homogenous problem. We can write ,

$$(p(x)y_n')' + q(x)y_n + \mu r(x)y_n = (\mu - \lambda_n)r(x)y_n$$

$$(p(x)y')' + q(x)y + \mu r(x)y = \sum_n b_n((p(x)y_n')' + q(x)y_n + \mu r(x)y_n)$$

i.e,

$$f(x) = \sum_n b_n(\mu - \lambda_n)r(x)y_n$$

if , $\frac{f(x)}{r(x)} = \sum_n c_n y_n$, such that $c_n = b_n\ (\mu - \lambda_n)$
The coeficient c_n can be found out by using the property of orthogonality.

$$\int_a^b \frac{f(x)}{r(x)} y_n r(x) dx = c_n \int_a^b y_n^2(x) r(x)$$

which implies

$$c_n = \frac{\int_a^b f(x) y_n dx}{\int_a^b y_n^2(x) r(x)}$$

Now we can get the expression for b_n from the above equation for c_n.The solution for the non-homogenous problem can be expressed as

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{\mu - \lambda_n} y_n(x)$$

There is a possibility that μ can be equal to one of the eigenvalue of corresponding homogenous problem, $\mu = \lambda_n$.Then the equation relating b_n and c_n takes the form as $(0 * b_n - c_n = 0)$.Then there arises two cases,when $\mu = \lambda_n$ and $c_n \neq 0$,then there is no value for b_n that satisfies $(0 * b_n - c_n = 0)$(*) so there is no solution for the non-homogenous problem.The second case arises when $\mu = \lambda_n$ and $c_n = 0$,then the eqn (*) is satisfied regardless of b_n and there

exist solution for the non-homogenous problem.

The expression for c_n is given by

$$c_n = \int_a^b f(x)y_n(x)dx$$

as c_n=0 ,

$$\int_a^b f(x)y_n(x)dx = 0$$

which implies,when $\mu = \lambda_n$ the non-homogenous boundary value problem is solvable only if f is orthogonal to the eigenfunction corresponding to λ_n.

Example:Consider the non-homogenous boundary value problem

$$y'' + 2y = x, y(0) = 0, y(1) = 0$$

the corresponding homogenous problem is

$$y'' + 2y = 0, y(0) = 0, y(1) = 0$$

Let y_n denote the solution of the homogenous problem.Thus we can write the solution of non-homogenous problem as $y(x) = \sum_n b_n y_n(x)$. We have the eigenfunctions of homogenous problem $y_n = \sin(n\pi x)$ and eigenvalues $\lambda_n = (n\pi)^2$.
We have

$$c_n = \frac{\int_a^b f(x)y_n dx}{\int_a^b y_n^2(x)r(x)}$$

here in this problem $f(x) = x$ and $r(x) = 1$.On computing we get

$$c_n = 2\int_0^1 x\sin(n\pi x)dx = \frac{2}{n\pi}(-1)^{n+1}$$

From c_n we can find b_n

$$b_n = \frac{2}{n\pi(2-(n\pi)^2)}(-1)^{n+1}$$

. The solution of the non-homogenous problem can be expressed as

$$y(x) = \sum_{n}^{\infty} \frac{2}{n\pi(2-(n\pi)^2)}(-1)^{n+1}\sin(n\pi x)$$

In this case μ and λ_n will never be equal.

Remark : The nonhomogeneous boundary value problem has a unique solution for each continuous f whenever μ is different from all the eigenvalues of the corresponding homogeneous problem; the solution can be expressed as

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{\mu - \lambda_n} y_n(x)$$

and this series converges for each x in $a \leq x \leq b$. If μ is equal to an eigenvalue λ_m of the corresponding homogeneous problem, then the nonhomogeneous boundary value problem has no solution unless f is orthogonal to eigenfunction corresponding to λ_m.

Reference

1. Elementary Differential Equations and Boundary Value Problems,Boyce & DiPrima