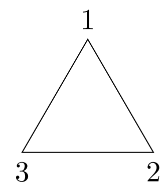
Phys 7325 Homework 3

Professor Halverson

Due: November 15, 2019

1. Non-abelian Discrete Groups. 10 pts.

In class we saw that $\phi \to -\phi$ is a symmetry of the Lagrangian when every term in the Lagrangian is even in ϕ. But there can also be non-abelian discrete actions on fields, i.e. actions that don't commute. As a warmup to get used to non-abelian discrete groups, consider actions on the triangle



one called R, which rotates clockwise by 120 degrees, and another called F, which flips the triangle along the axis connecting 1 to the midpoint of 2 and 3. To be clear, the numbers label the vertices so when you act on the triangle the numbers move with the vertices. Call this initial configuration $(1, 2, 3)$, where the ordering in this list is the clockwise ordering of the triangle.

1) 3 pts. Acting with R and F (and compositions of them) in all possible ways, compute the unique triangle configurations.

2) 3 pts. The action of this group is non-abelian if $[F, R] \neq 0$. Show this in an example via subsequent actions on the above triangle.

3) 3 pts. Suppose that F and R act instead on fields (ϕ_1, ϕ_2, ϕ_3) where R acts by shifting to the right, in which case the third field becomes the first, and F acts by swapping the second two entries in the vector. I.e., the fields are transforming under F and R exactly like the triangle vertices. Show that $\mathcal{L} = \sum_i \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - g\phi_1\phi_2\phi_3 - M^2(\phi_1^2 + \phi_2^2 + \phi_3^2) - K^2(\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1)$ is invariant. We have a field theory with a non-abelian discrete symmetry!

4) 1 pt. Does this symmetry have a Noether's current? If so, what is it? If not, why?

2. Higher Order Perturbation Theory. 10 pts.

Let's do a higher order perturbation in a familiar theory. Compute $G^{(4)}(x_1, x_2, x_3, x_4)$ (as defined in class) in ϕ^4 theory at $O(\lambda^2)$, expressing the result both in terms of Feynman diagrams and not in terms of Feynman diagrams. Since the analytical expression is the result, make sure it is clear how your diagrams correlate with the analytical expression; i.e. it is fine to deviate slightly from the Feynman rule conventions in the book, as long as your rules are clear.

Hint: We computed $G^{(2)}(x_1, x_2)$ at $O(\lambda^2)$ in class. There were seven diagrams, and the sum of their coefficients was $(2(5) - 1)!!$, which was not an accident. A similar result holds here, which may be used to check your results. If you would rather write a code to do the combinatorics, feel free, as long as you produce the source and output.

Encouragement: If you're doing this and it feels too tedious to be correct, keep on going. This is a very tedious calculation designed to get you used to the combinatorics once and for all, and also to help convince you of Wick's theorem.

3. One Motivation for Supersymmetry. 5 pts.

There are a number of motivations for supersymmetry (SUSY), which is a symmetry between bosons and fermions. In condensed matter systems they are sometimes studied because they can give rise to interesting properties, such as a large ground state degeneracy. In high energy physics, one of the primary reasons is that SUSY provides a solution to the weak hierarchy problem, which in a broader context (as relevant for condensed matter) can be thought of as softenining the divergence properties of scalars.

Our goal in this problem is to understand how the introduction of new particles can soften the severity of scalar divergences, as measured by the severity of the divergence of certain amplitudes (or, more precisely, the degree of its cutoff dependence).

a) Consider the Yukawa interaction

$$\mathcal{L}_y = -y\, h\bar{\psi}\psi \tag{1}$$

where h is a real scalar field, ψ is a Dirac spinor of mass m_f, and y is the Yukawa coupling. The point is to have h be our Higgs boson in this toy model. The interaction gives a one-loop contribution at order y^2 to the 2-point amplitude for h, which is intimately tied to quantum corrections to the Higgs mass. Compute that amplitude contribution, but don't evaluate the momentum integral.

b) Now suppose we have some complex scalars ϕ_L and ϕ_R of mass m_L and m_R that interact with h as

$$\mathcal{L}_s = -\lambda\, h^2(\phi_L^*\phi_L + \phi_R^*\phi_R), \tag{2}$$

where λ is a coupling constant. These interactions also gives a loop contribution to the 2-point amplitude for h. Compute it, but don't evaluate the momentum integral.

c) Sum the results from the first two parts and study the most divergent pieces of the sum, which are quadratically divergent as discussed in class. Determine a relationship between y and λ under which this quadratically divergent piece is zero, if the masses of all particles in the loops (i.e. ϕ_L, ϕ_R, ψ) are the same. Note: this may require using some fermion identities to simplify the result from a).

d) So adding scalars in this special way gets rid of the quadratically divergent piece. Of course, in high energy physics this is ludicrous: we can't just add a new particle of the same mass for every particle that's already been discovered at colliders, since they would have been discovered long ago. But what if the new particles were heavier? Give an argument as to why the divergence problem is still softened even if the new scalars are heavier than the fermions, i.e. there is some mass splitting.

There is much more to the story than this, for example the systematic construction of supersymmetric Lagrangians. But you have now seen the basic idea, and in fact SUSY Lagrangians give nice relationships between y and λ of precisely the required type.