Can Conventional Measures Identify Geographically Varying Mixed Regression Relationships? A Simulation-based Analysis of Locally Weighted Regression

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$$Y = \beta_0 + \beta_1 * X_1 + \beta_2 * X_2 + \epsilon$$

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Locally Weighted Regression (LWR) to the Rescue?

OLS

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

LWR

$$\hat{\beta}(location_i) = (X'W(location_i)X)^{-1}(X'W(location_i)Y)$$

OLS

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

LWR

$$\hat{\beta}(location_i) = (X'W(location_i)X)^{-1}(X'W(location_i)Y)$$

$$w_{jj} = \left[1 - \left(\frac{d_{ij}}{d_{ik}}\right)^2\right]^2 \text{ if } d_{ij} < d_{ik}, \text{ otherwise} = 0,$$

$$\hat{\beta}(location_i) = (X'W(location_i)X)^{-1}(X'W(location_i)Y)$$

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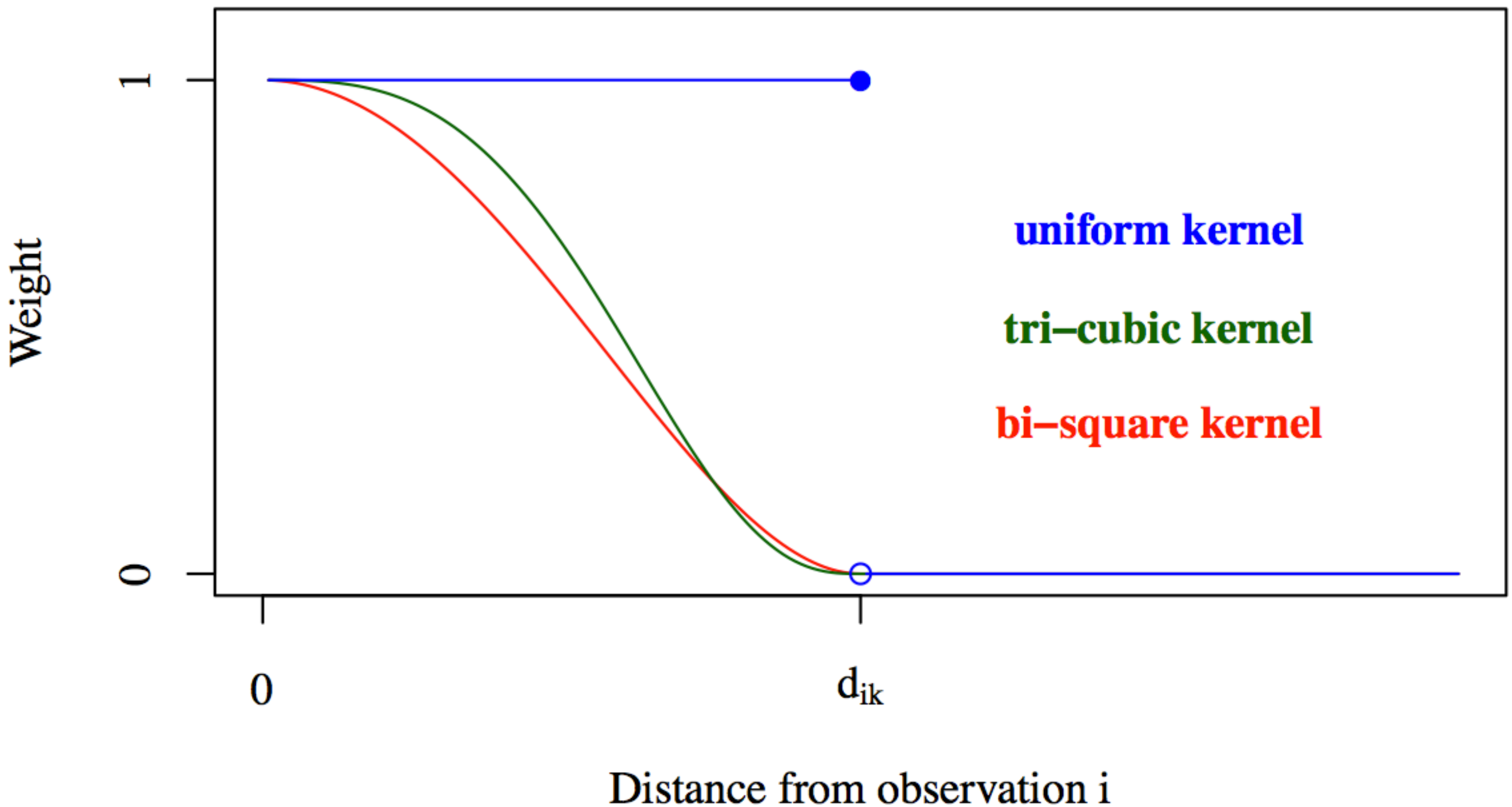

Bandwidths are commonly selected with…

Leave One Out Cross Validation

Akaike Information Criterion

Generalized Cross Validation

Standardized Cross Validation

$$LOOCV = \frac{1}{N}\sqrt{\sum_{i=1}^{N}(y - \hat{y}_{\neq i})^2},$$

$$SCV_i(k) = \frac{(y_i - \hat{y}_{\neq i}(k))^2}{\sum_k (y_i - \hat{y}_{\neq i})^2}$$

$$SCV(k) = \sum_i SCV_i(k)$$

S Farber and A Páez. A systematic investigation of cross-validation in GWR model estimation: empirical analysis and Monte Carlo simulations. *Journal of Geographical Systems*, 9(4):371–396, 2007.

$$GCV = n * \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{(n - v_1)^2},$$

$v_1 =$ "effective number of parameters"

$$GCV = n * \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{(n - v_1)^2},$$

$$v_1 = \text{“effective number of parameters”}$$

$v_1 = \mathrm{tr}(\mathbf{S})$, where the matrix $\mathbf{S}$ is the “hat matrix” which maps y onto $\hat{y}$,

$$\hat{y} = \mathbf{S}y,$$

and each row of $\mathbf{S}$, r_i is given by:

$$r_i = X_i(X'W_iX)^{-1}X'W_i.$$

$$AIC = 2 * n * ln(\hat{\sigma}) + n * ln(2 * \pi) + n * \frac{n + v_1}{n - 2 - v_1}$$

"Global" OLS GGG

"Local"Regression LLL

“Global” OLS	GGG
	LGG
	GLG
	GGL
Mixed Models	LLG
	LGL
	GLL
“Local”Regression	LLL

The Researcher's Problem

Choose a bandwidth <u>and</u> a model

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Choose a bandwidth <u>and</u> a model

Can we do both with conventional metrics?

Experiment Data Generation Process

$$Y_i = \beta_0(East_i, North_i) + \beta_1(East_i, North_i) * X_{1i} + \beta_2(East_i, North_i) * X_{2i} + \epsilon_i$$

$$n \in \{50, 100, 200, 400, 800\}$$

$$\sigma^2 \in \{0.25, .5, 1, 2, 3\}$$

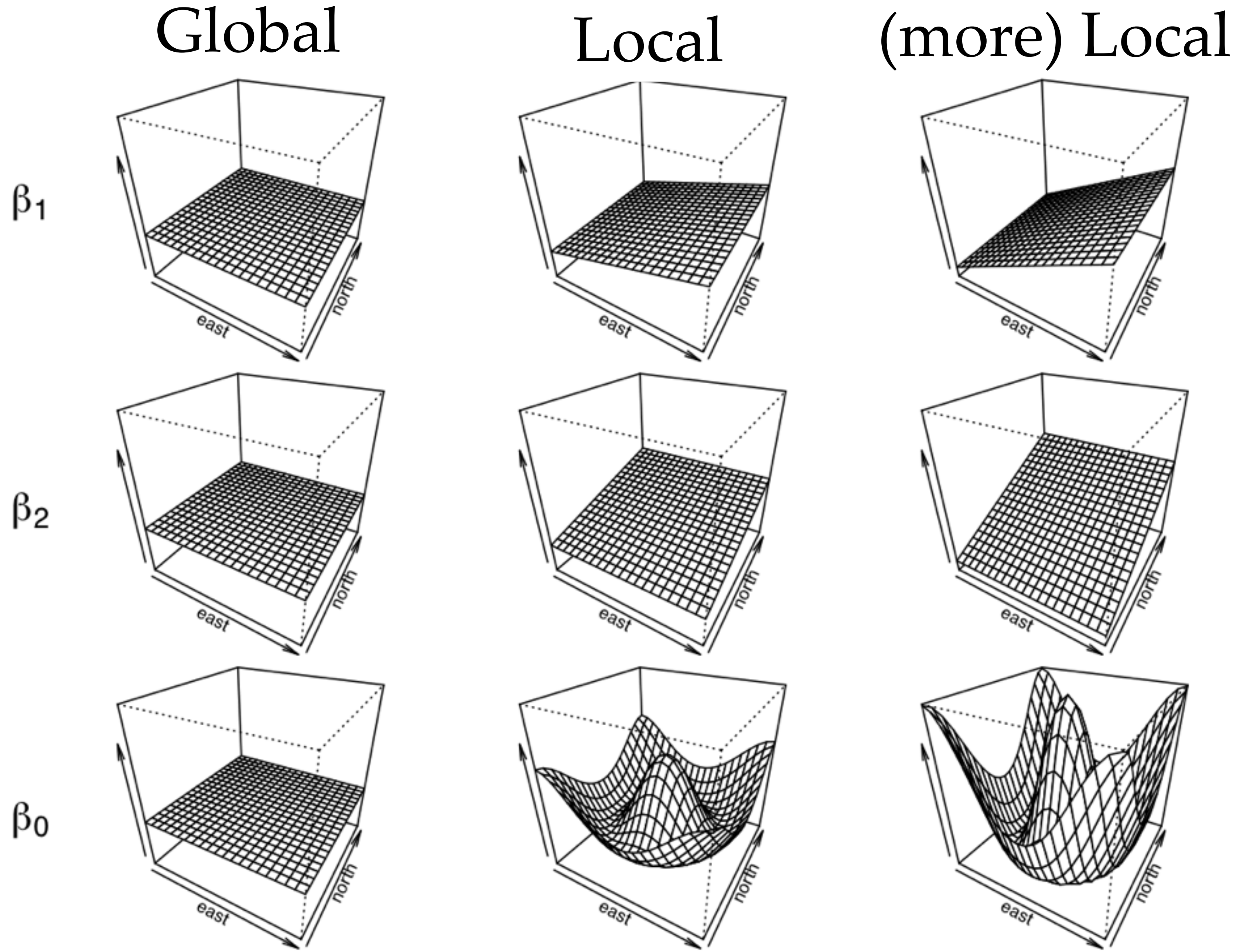
$$X_1 \sim u[0, 1]$$

$$X_2 \sim u[0, 1]$$

$$East \sim u[0, 1]$$

$$North \sim u[0, 1]$$

Coefficient Spatial Variation



With Our Data…

$$[Y, X_1, X_2, East, North]$$

Estimate all models

GGG

$$\left.\begin{array}{l} \text{LGG} \\ \text{GLG} \\ \text{GGL} \\ \text{LLG} \\ \text{LGL} \\ \text{GLL} \\ \text{LLL} \end{array}\right\}$$

7 bandwidths each

50 combinations total

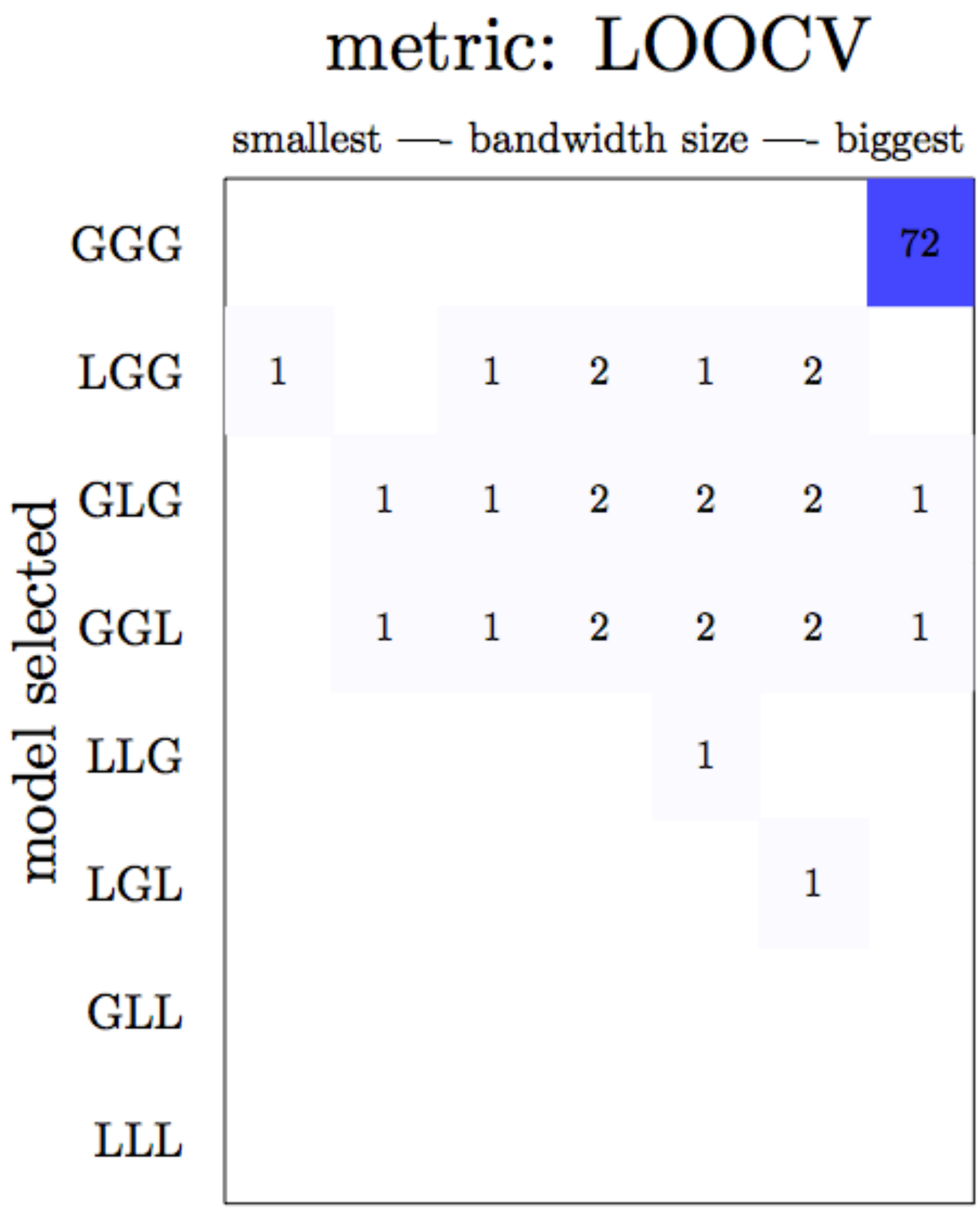
Calculate the values of the four metrics (LOOCV, GCV, SCV, AIC)

Is the model with the optimized metric value the correct model?

Start Simple… True Model: GGG

Model Selected	Metric: LOOCV	GCV	SCV	AIC	
GGG	72	0	8	0	3/3 Correct
LGG	7	28	29	28	
GLG	8	36	22	37	2/3 Correct
GGL	8	33	22	34	
LLG	1	1	5	0	
LGL	2	1	5	1	1/3 Correct
GLL	1	1	8	0	
LLL	0	0	1	0	0/3 Correct
	100	100	100	100	

True Model: GGG



True Model: GGG

metric: GCV

smallest —- bandwidth size —- biggest

model selected							
GGG							
LGG	1	1	2	2	2	4	17
GLG		1	2	2	3	5	23
GGL		1	1	2	3	4	21
LLG							
LGL							
GLL							
LLL							

metric: SCV

smallest —- bandwidth size —- biggest

model selected							
GGG							8
LGG			2	12	10	5	1
GLG			2	6	7	5	2
GGL			1	7	8	4	1
LLG					2	2	1
LGL					2	2	1
GLL				1	3	3	1
LLL						1	

metric: AIC

smallest —- bandwidth size —- biggest

model selected							
GGG							
LGG			1	2	2	4	19
GLG			1	2	3	6	25
GGL			1	2	3	4	23
LLG							
LGL							
GLL							
LLL							

True Model: GGG

Model Selected	Coefficient RMSE $\widehat{\beta_0}$	$\widehat{\beta_1}$	$\widehat{\beta_2}$
GGG	6	6	7
LGG	3	23	24
GLG	22	4	24
GGL	23	23	3
LLG	5	3	33
LGL	5	36	2
GLL	32	4	4
LLL	4	2	3
	100	100	100

True Model: GGG

metric: B0RMSE

smallest —- bandwidth size —- biggest

model selected							
GGG							6
LGG						1	3
GLG	9	3	2	2	2	2	1
GGL	9	3	3	3	2	1	1
LLG							4
LGL							4
GLL	14	5	4	4	3	2	1
LLL							4

metric: B1RMSE

smallest —- bandwidth size —- biggest

model selected							
GGG							6
LGG	8	4	3	3	2	2	1
GLG							3
GGL	10	3	3	2	2	2	1
LLG							3
LGL	15	5	4	4	3	2	3
GLL							4
LLL							2

metric: B2RMSE

smallest —- bandwidth size —- biggest

model selected							
GGG							7
LGG	9	4	3	2	2	2	2
GLG	10	4	2	1	2	2	2
GGL							3
LLG	12	5	4	4	3	3	2
LGL							2
GLL							4
LLL							3

True Model: GGG

metric: B0RMSE

model selected	smallest —-			bandwidth size			—- biggest
GGG							6
LGG						1	3
GLG	9	3	2	2	2	2	1
GGL	9	3	3	3	2	1	1
LLG							4
LGL							4
GLL	14	5	4	4	3	2	1
LLL							4

metric: B1RMSE

model selected	smallest —-			bandwidth size			—- biggest
GGG							6
LGG	8	4	3	3	2	2	1
GLG							3
GGL	10	3	3	2	2	2	1
LLG							3
LGL	15	5	4	4	3	2	3
GLL							4
LLL							2

metric: B2RMSE

model selected	smallest —-			bandwidth size			—- biggest
GGG							7
LGG	9	4	3	2	2	2	2
GLG	10	4	2	1	2	2	2
GGL							3
LLG	12	5	4	4	3	3	2
LGL							2
GLL							4
LLL							3

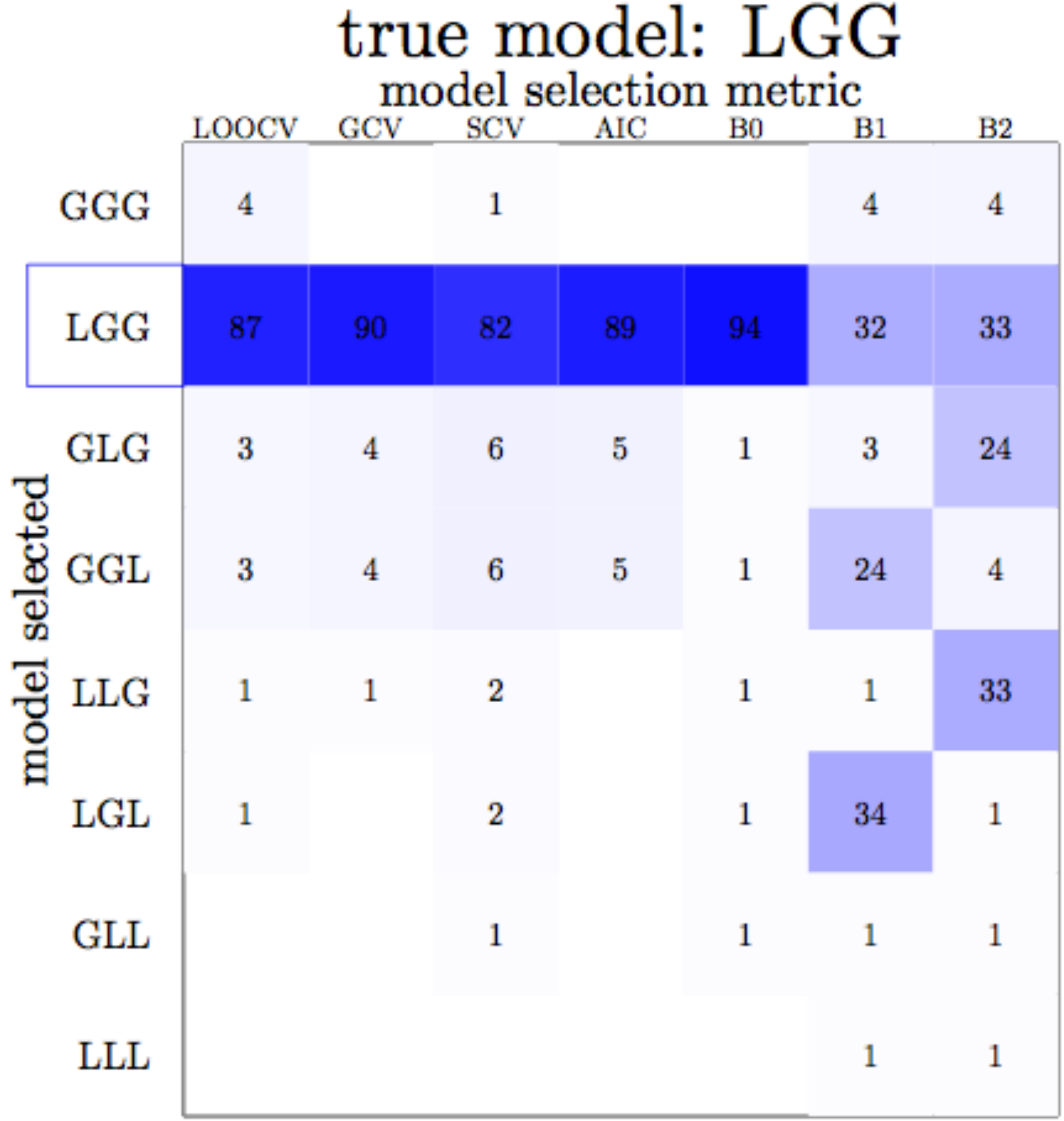
Most accurate coefficient estimates:

- Tend not to be the correct model
- Allow other coefficients to vary

true model: GGG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	72		8		6	6	7
LGG	7	28	29	28	3	23	24
GLG	8	36	22	37	22	4	24
GGL	8	33	22	34	23	23	3
LLG	1	1	5		5	3	33
LGL	2	1	5	1	5	36	2
GLL	1	1	8		32	4	4
LLL			1		4	2	3


true model: LGG
model selection metric
LOOCV
GCV
SCV
AIC
B0
B1
B2
model selected
GGG
LGG
GLG
GGL
LLG
LGL
GLL
LLL
4
1
4
4
87
90
82
89
94
32
33
3
4
6
5
1
3
24
3
4
6
5
1
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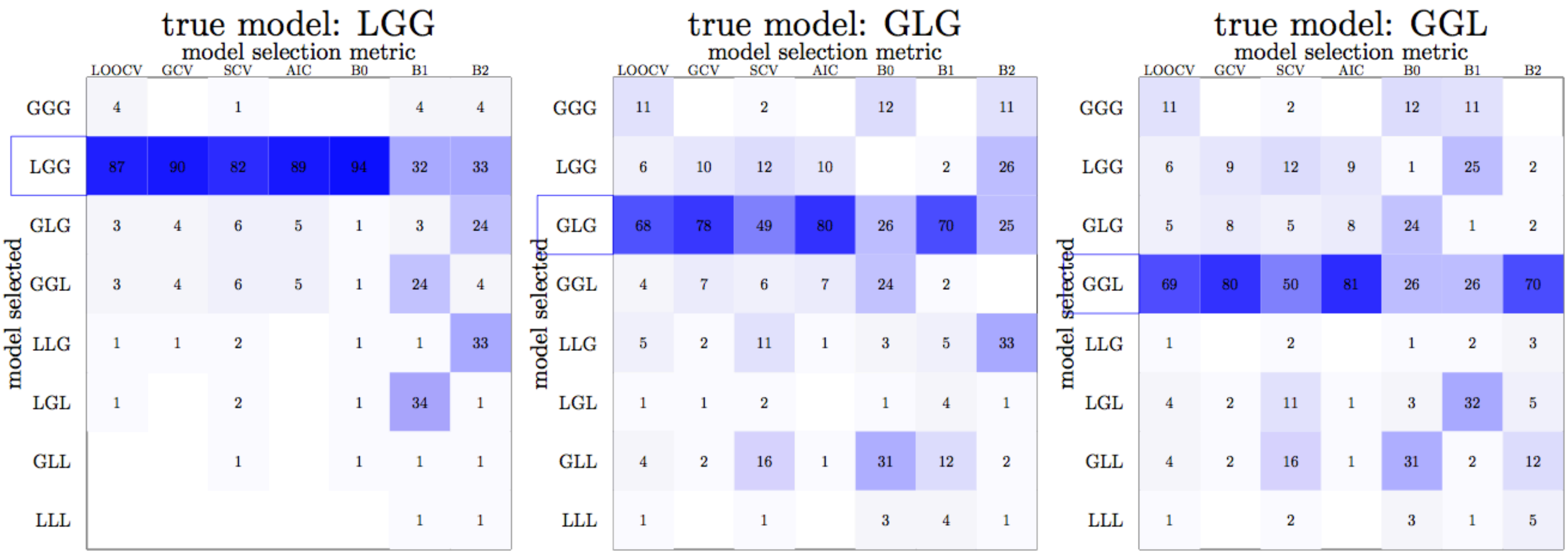
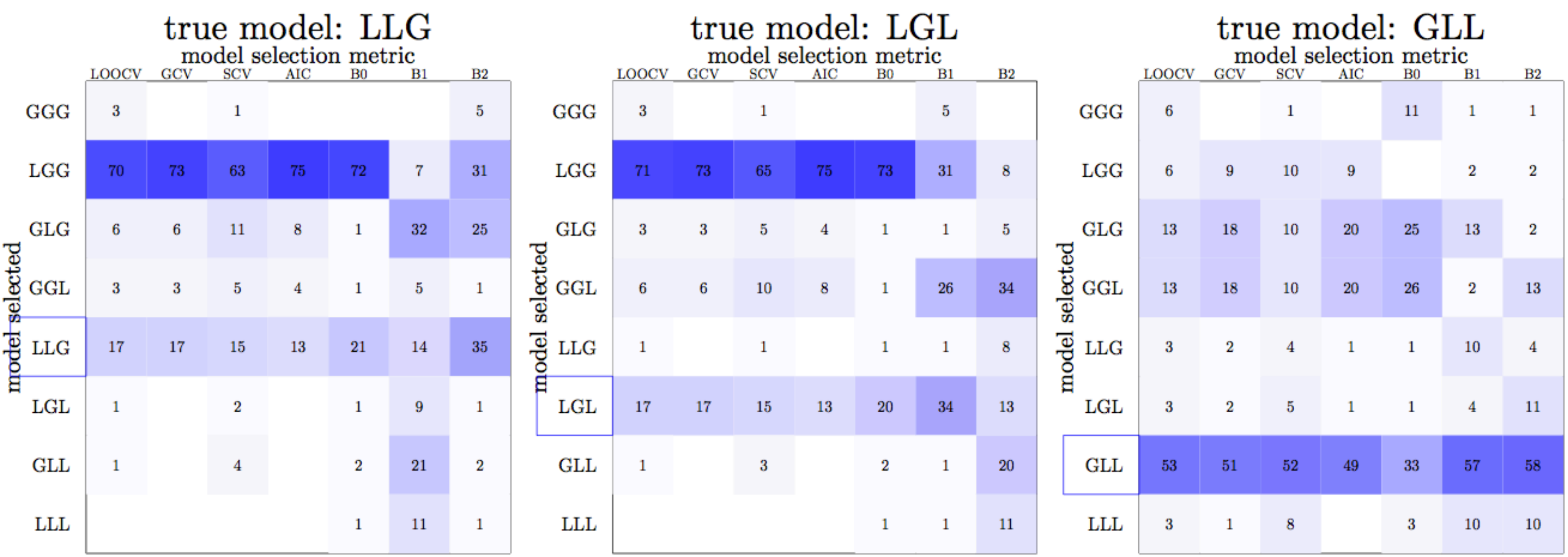
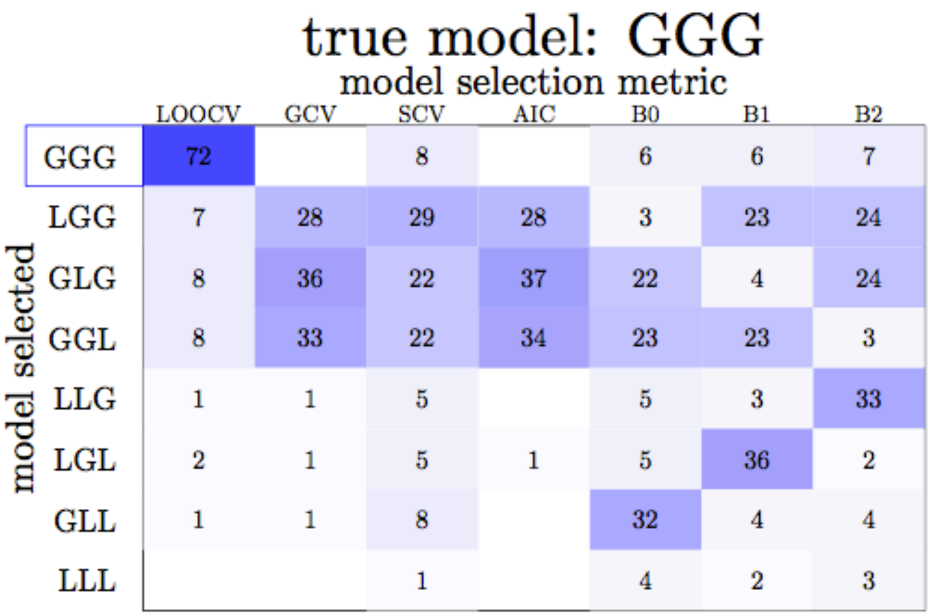
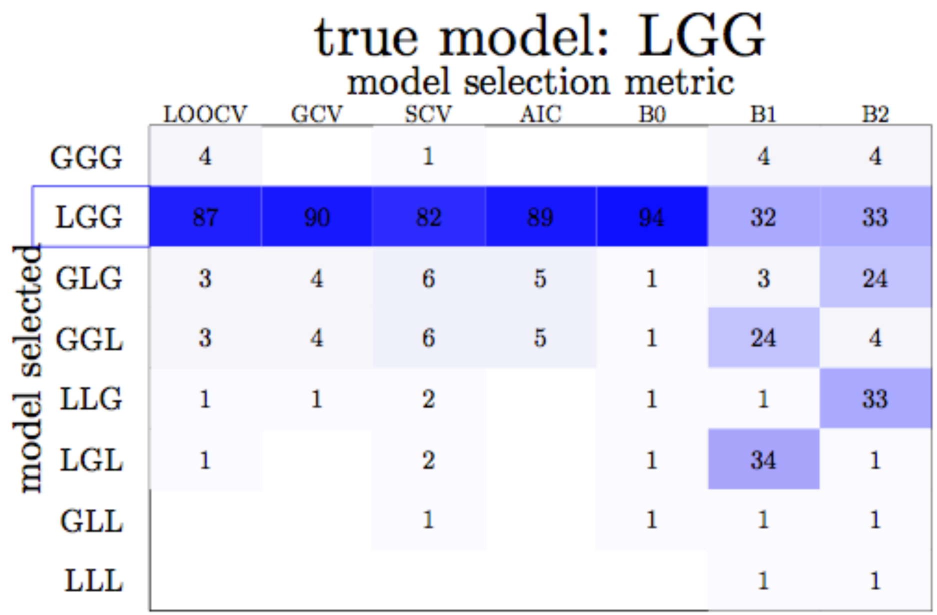
1
1
1
1
1
1



true model: LGG
true model: GLG
true model: GGL
model selection metric
LOOCV
GCV
SCV
AIC
B0
B1
B2
model selected
GGG
LGG
GLG
GGL
LLG
LGL
GLL
LLL


true model: LLG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 3 1 5
LGG 70 73 63 75 72 7 31
GLG 6 6 11 8 1 32 25
GGL 3 3 5 4 1 5 1
LLG 17 17 15 13 21 14 35
LGL 1 2 1 9 1
GLL 1 4 2 21 2
LLL 1 11 1
true model: LGL
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 3 1 5
LGG 71 73 65 75 73 31 8
GLG 3 3 5 4 1 1 5
GGL 6 6 10 8 1 26 34
LLG 1 1 1 1 8
LGL 17 17 15 13 20 34 13
GLL 1 3 2 1 20
LLL 1 1 11
true model: GLL
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 6 1 11 1 1
LGG 6 9 10 9 2 2
GLG 13 18 10 20 25 13 2
GGL 13 18 10 20 26 2 13
LLG 3 2 4 1 1 10 4
LGL 3 2 5 1 1 4 11
GLL 53 51 52 49 33 57 58
LLL 3 1 8 3 10 10


true model: GGG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 72 8 6 6 7
LGG 7 28 29 28 3 23 24
GLG 8 36 22 37 22 4 24
GGL 8 33 22 34 23 23 3
LLG 1 1 5 5 3 33
LGL 2 1 5 1 5 36 2
GLL 1 1 8 32 4 4
LLL 1 4 2 3


true model: LGG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 4 1 4 4
LGG 87 90 82 89 94 32 33
GLG 3 4 6 5 1 3 24
GGL 3 4 6 5 1 24 4
LLG 1 1 2 1 1 33
LGL 1 2 1 34 1
GLL 1 1 1 1
LLL 1 1

true model: GLG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	11		2		12		11
LGG	6	10	12	10		2	26
GLG	68	78	49	80	26	70	25
GGL	4	7	6	7	24	2	
LLG	5	2	11	1	3	5	33
LGL	1	1	2		1	4	1
GLL	4	2	16	1	31	12	2
LLL	1		1		3	4	1

true model: LLG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	3		1				5
LGG	70	73	63	75	72	7	31
GLG	6	6	11	8	1	32	25
GGL	3	3	5	4	1	5	1
LLG	17	17	15	13	21	14	35
LGL	1		2		1	9	1
GLL	1		4		2	21	2
LLL					1	11	1

true model: GGL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	11		2		12	11	
LGG	6	9	12	9	1	25	2
GLG	5	8	5	8	24	1	2
GGL	69	80	50	81	26	26	70
LLG	1		2		1	2	3
LGL	4	2	11	1	3	32	5
GLL	4	2	16	1	31	2	12
LLL	1		2		3	1	5

true model: LGL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	3		1			5	
LGG	71	73	65	75	73	31	8
GLG	3	3	5	4	1	1	5
GGL	6	6	10	8	1	26	34
LLG	1		1		1	1	8
LGL	17	17	15	13	20	34	13
GLL	1		3		2	1	20
LLL					1	1	11

true model: GLL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	6		1		11	1	1
LGG	6	9	10	9		2	2
GLG	13	18	10	20	25	13	2
GGL	13	18	10	20	26	2	13
LLG	3	2	4	1	1	10	4
LGL	3	2	5	1	1	4	11
GLL	53	51	52	49	33	57	58
LLL	3	1	8		3	10	10

true model: LLL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	2				1	1	
LGG	63	65	55	69	61	8	8
GLG	5	5	7	7	2	16	6
GGL	4	5	7	6	2	5	17
LLG	8	8	8	5	11	14	10
LGL	8	8	8	6	11	9	14
GLL	1	1	8		2	31	30
LLL	9	9	6	6	12	15	15

Some Conclusions and Questions

LOOCV is pretty good!

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What happens, ceteris paribus, with greater error variance?