Can Conventional Measures Identify Geographically Varying Mixed Regression Relationships? A Simulation-based Analysis of Locally Weighted Regression

Aaron Swoboda[*]

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Regional scientists are increasingly utilizing data analysis techniques that allow for spatial heterogeneity of various forms. Regression specifications like Geographically Weighted Regression and Locally Weighted Regression allow the regression parameters to vary over space thereby reflecting spatially non-stationary relationships. Such spatial heterogeneity is appealling in a regional science context in which location is assumed to matter in various ways. Recent work has begun to estimate "mixed" relationships in which some variables exert non-stationary effects on the dependent variable, while others exhibit a constant effect over space. With so many possibilities available, researchers face a daunting task determining which variables to estimate in a (non-)stationary fashion. In this paper we simulate "mixed" regression relationships to determine whether commonly used metrics (for example Leave One Out Cross Validation and the Akaike Information Criterion) can adequately be used to distinguish between the different possibilities.

1 Background

Imagine a simple linear model,

$$Y = \beta_0 + \beta_1 * X_1 + \beta_2 * X_2 + \epsilon. \tag{1}$$

In addition to the three variables listed above (Y, X_1, and X_2), assume we know the geographical location for each of our N observations. Thus, our data consists of an $N \times 5$ matrix, where Y may be house prices, X_1 and X_2 could be the living space and lot size associated with each house, and the final two columns determine the location of the observations (for instance, latitude and longitude, or distances north and east from a prescribed point).

The simple model in (1) exemplifies spatial stationarity in the parameters: the β coefficients are constant over space. Alternatively, the coefficients could exhibit

[*] Carleton College, email: aswoboda@carleton.edu

spatial non-stationarity, in which case one, two, or all three of the β coefficients vary with location. This has a natural interpretation in the current real estate example: location matters. However, location can matter in different ways. For instance, if the value of land varies over space, then we would expect the coefficient on lot size to vary over space, while it is also possible that the intercept varies over space to reflect variation in prices of similar houses in different locations.

It is possible to parameterize the variation in coefficients, for instance by including a variable measuring the distance from an observation to an important amenity such as the Central Business District and then this distance variable could be interacted with variables whose value are predicted to vary over space. However, it is plausible that the variation in coefficients might not be easily parameterized (for instance, if land values are a non-monotonic function of distance). Researchers may instead interact variables with fixed effects for cities or census tracts. However, such strategies require the analyst to make assumptions that severely limit the type and degree of variation in the parameters. For instance, interaction terms with geographic boundaries assume discrete differences in the value of parameters across the boundaries, while instead the parameters may instead be a continuous function of location. Additionally, numerous interaction terms may unduly reduce the degrees of freedom.

1.1 Weighted Regression to the Rescue?

Locally Weighted Regression (LWR) techniques are one set of potential solutions to the challenge described. It is a weighted least squares methodology in which regression coefficients are estimated as a function of the local data. Cleveland and Devlin (1988) provides one of the first descriptions of the method, while more recently Brunsdon et al. (1998b) and Fotheringham et al. (2002) have explicitly incorporated geography/location into the methodology. Specifically, the coefficients are estimated,

$$\hat{\beta}(location_i) = (X'W(location_i)X)^{-1}(X'W(location_i)Y), \tag{2}$$

where X is a $n \times m$ matrix of independent variables, W_i is the $n \times n$ weights matrix, and Y is the $n \times 1$ vector of dependent variable values. The weights matrix, W_i is a diagonal matrix where element w_{jj} denotes the weight that the j^{th} data point will receive in the regression coefficients estimated at location i in the dataset.[1] We employ a bi-square weights function and a k-nearest neighbor bandwidth approach as described in equation (3),

$$w_{jj} = \left[1 - \left(\frac{d_{ij}}{d_{ik}}\right)^2\right]^2 \text{ if } d_{ij} < d_{ik}, \text{ otherwise} = 0, \tag{3}$$

where d_{ij} denotes the distance between observations i and j, and d_{ik} is the distance from observation i to the k^{th} nearest observation. This function assigns weights close to 1 for data points near observation i, weights positive but closer to zero for

[1] If all diagonal elements of W are one, then (2) simplifies to the standard OLS estimation.

observations farther away, and zero for all $n-k$ observations farther away than the k^{th} nearest observation.

A key decision in estimating LWR models is choosing the number of observations to include in the bandwidth. Bandwidths that are too large in the presence of spatial non-stationarity create bias in the regression estimates (the large bandwidth creates weights matrices that are similar over space and therefore the regression coefficients are forced to be similar when they should vary over space). Bandwidths that are too small add unneccessary error in our estimates by excluding informative observations. Often, researchers choose a bandwidth my minimizing a cross validation metric.

1.2 Mixed Geographic Models

Thus far two models have been discussed - the standard OLS model in which all of the coefficients are stationary and a local model in which all coefficients are estimated locally using some bandwidth. The two cases are identical in the extreme case of an infinite bandwidth. Other "mixed" models are also possible in which some coefficients are assumed to be stationary while others are estimated locally. In the example described in Equation (1) with three coefficients, there are are eight ($2\times2\times2$) total models, ranging from the LWR model in which all three coefficients are assumed to be "Local", three models in which two coefficients are "Local" and one is assumed to be "Global", three models in which one coefficient is "Local" and the remaining two coefficients are "Global" and the OLS model in which all three coefficients are "Global".

In the presence of mixed models, the researcher is tasked with simultaneously choosing which variables to treat as spatially non-stationary and the bandwidth at which to perform the analysis. Little is known about model performance when models are selected across multiple mixed models and among multiple different potenial bandwidth sizes. This paper uses simulated data generated under multiple conditions to begin to answer some of the outstanding questions in the area of geographically mixed models.

2 Experimental Design

2.1 Data Generation Process

For a given sample size, n, we generate an $n\times4$ matrix of uniformly independently distributed values between 0 and 1.[2] Let the first two of these columns be called $East$ and $North$ and denote the location of each of our observations within a two-dimensional unit grid (East, North), while the remaining two columns be X_1 and X_2 and serve as the independent variables in our regression equation.

Our goal is to generate data according to the following process:

$$Y_i = \beta_0(East_i, North_i)+\beta_1(East_i, North_i)*X_{1i}+\beta_2(East_i, North_i)*X_{2i}+\epsilon_i. \quad (4)$$

where ϵ_i is normally distributed with mean zero and variance σ^2. Equation 4 states that the value of the dependent variable for a given observation depends on X_1,

[2] For our simulation we set $n \in \{50, 100, 200, 800\}$.

X_2, a normally distributed error term, and three β coefficients that are a function of geographic location.

For our simulations, the relationship between β_1 and location is given by,

$$\beta_1(East, North, \eta_1) = 2 + \eta_1 \times (East - .5), \tag{5}$$

where $\eta_1 \in \{0, 2, 4\}$ represents a "degree of spatial variation" parameter. When $\eta_1 = 0$, equation (5) simplifies to $\beta_1 = 2$ and there is no spatial variation in the coefficient while β_1 exhibits spatial non-stationarity when $\eta_1 \in \{2, 4\}$. Due to the functional form of equation (5) and the uniform distribution of the *East* variable between zero and one, the expected value of β_1 is two for all η_1. Figure 1 shows a visual representation of the possible values for β_1 as a function of location and η_1.

Similar to β_1, the relationship between location and β_2 is given by,

$$\beta_2(East, North, \eta_2) = 2 + \eta_2 \times (North - .5) \tag{6}$$

and is displayed in the second row of Figure 1.

The main difference between equations (5) and (6) is that β_2 is a function of *North*, while β_1 is a function of *East*, thus the β_1 and β_2 coefficients are orthogonal to one another. In order to introduce spatial non-stationarity in the β_0 coefficient without also creating high and/or perfect collinearity among the coefficients, we generated the β_0 coefficient by the following,

$$\beta_0(East, North, \eta_0) = 2 + \eta_0 \times \left[C + cos\left(2\pi \times \sqrt{\frac{(East - .5)^2 + (North - .5)^2}{.5}} \right) \right], \tag{7}$$

where C is a constant selected to yield $E(\beta_0) = 2$. Equation (7) is presented visually in the third row of Figure 1.

Figure 1 shows the three different formulations possible for the three model coefficients. The leftmost column shows each coefficient constant over space, the middle column shows "some" spatial variation and the rightmost column shows more spatial variation. In other words, the left column shows the stationary, or "Global", specification for each coefficient, while the middle and right columns show the non-stationary, or "Local", specifications for the coefficients.

In each instance of our simulation we select a degree of spatial variation (none, some, more) for each of the three coefficients. These 27 different combinations can be categorized into one of eight possible models depending on whether each of the three coefficients is "Global" or "Local"[3]

In addition to changing the degree of spatial variation in the three regression coefficients, we also implement different error term variances, σ^2, to explore the impact of "noisier" data. All told, each simulation can be described as a quintuple

[3] Specifically, the eight models are GGG, GGL, GLG, LGG, GLL, LGL, LLG, and LLL.

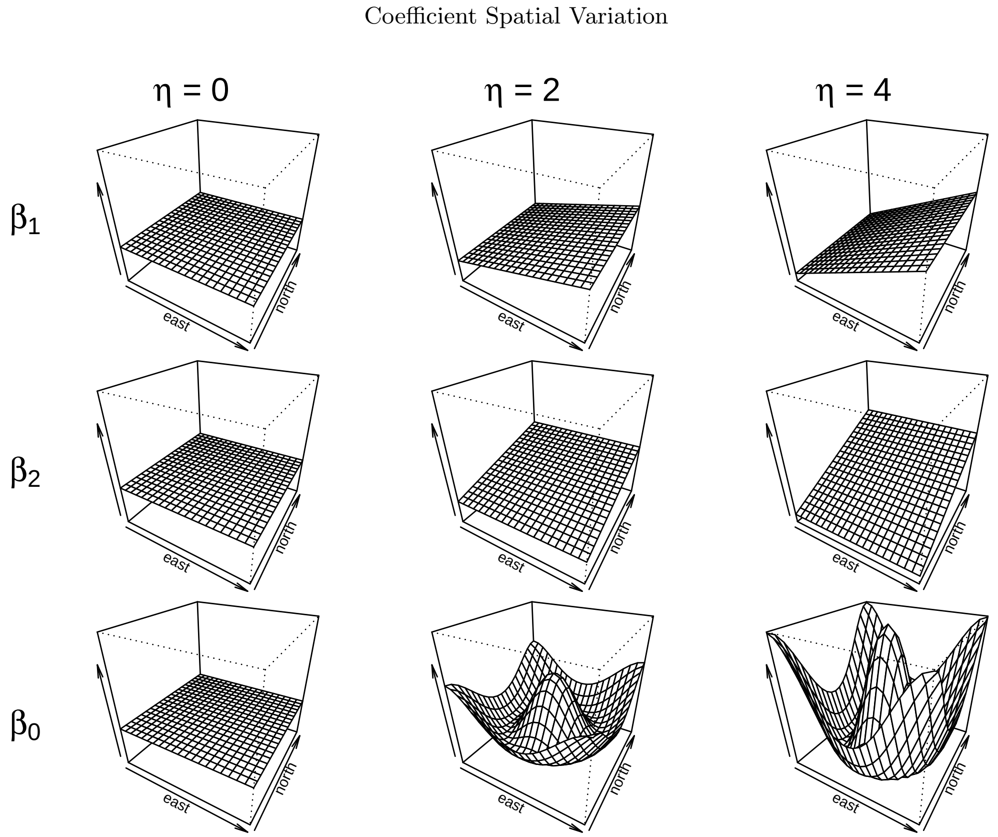


Figure 1: A Visual Representation of the Spatial Variation for the Three Coefficients as Described in Equations (5) - (7)

choosing from the following values:

$$\begin{aligned} \eta_0 &\in \{0, 2, 4\} \\ \eta_1 &\in \{0, 2, 4\} \\ \eta_2 &\in \{0, 2, 4\} \\ n\ &\in \{50, 100, 200, 400, 800\} \\ \sigma^2 &\in \{0.25, .5, 1, 2, 3\} \end{aligned} \tag{8}$$

yielding a total of 675 unique combinations. We replicate each combination 100 times for a total of 67,500 simulation iterations.

2.2 Model Selection

After generating a dataset according the parameters described above, we approach the data as a researcher might - trying to understand the underlying DGP with only an $n \times 5$ matrix of data representing the dependent variable, X_1, X_2, and the $East$ and $North$ values for each observation. We estimate 50 different model-bandwidth combinations for each dataset - seven different bandwidths[4] for each of the seven mixed models with at least one "Local" variable plus the standard OLS model, which we denote as "GGG."

For each model-bandwidth combination we calculate several model selection metrics. Researchers may use metrics like the Leave One Out Cross Validation (LOOCV), Generalized Cross Validation (GCV), Standardized Cross Validation (SCV) and Akaike Information Criterion (AIC) to choose bandwidth sizes and models. In addition to comparing these metric values across the model-bandwidth combinations, we also calculate the Root Mean Square Error (RMSE) for each of the three model coefficients, which allows us to compare the accuracy of the coefficient estimates across model-bandwidth combinations.

2.2.1 Leave One Out Cross Validation

Perhaps the most common cross validation metric used in the literature is the Leave One Out Cross Validation score (LOOCV), which is calculated as follows,

$$LOOCV = \frac{1}{N}\sqrt{\sum_{i=1}^{N}(y - \hat{y}_{\neq i})^2}, \tag{9}$$

where $\hat{y}_{\neq i}$ represents the dependent variable estimate for observation i while excluding observation i from the regression. This prevents the observation from having undue influence in the regression with small bandwidths and overfitting the model. Such a model, while intuitively appealing, can be computationally expensive, as regressions must be estimated first while excluding individual observations to calculate the LOOCV and then again while including the observation to obtain the

[4] Specifically, we use the following bandwidths: n, $n\left(\frac{2}{3}\right)$, $n\left(\frac{2}{3}\right)^2$, $n\left(\frac{2}{3}\right)^3$, $n\left(\frac{2}{3}\right)^4$, $n\left(\frac{2}{3}\right)^5$, and $n\left(\frac{2}{3}\right)^6$ rounded up to the nearest whole number. For the smallest sample size, 50, the smallest bandwidth is five observations.

regression coefficients. Perhaps because of the intuitive appeal and despite the computational intensity, the LOOCV remains one of the most popular model selection metrics. A sampling of recent articles that use LOOCV in the context of LWR model selection include: Brunsdon et al. (1998a), Huang and Leung (2002), Lloyd and Shuttleworth (2005), Cho et al. (2006), Yu (2006), Cho et al. (2009a), Cho et al. (2009b), and Huang et al. (2010).

2.2.2 Generalized Cross Validation

An alternative cross validation metric is known as the Generalized Cross Validation (GCV) score. The GCV score is a convenient model selection metric that rewards models that provide a good fit to the data, while penalizing models with a greater number of model parameters (Loader, 1999; McMillen and Redfearn, 2010). One advantage over the LOOCV is that it requires calculating the regressions only once per location and explicitly calculates the leverage each observation has over the regression coefficients. The GCV score calculation is detailed in equation (10),

$$GCV = n * \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{(n - v_1)^2}, \tag{10}$$

where $\hat{y}_i$ is the predicted dependent variable value for observation i, and v_1 can be interpreted as the "effective number of model parameters," and calculated as $v_1 = \text{tr}(\mathbf{S})$, where the matrix $\mathbf{S}$ is the "hat matrix" which maps y onto $\hat{y}$,

$$\hat{y} = \mathbf{S}y, \tag{11}$$

and each row of $\mathbf{S}$, r_i is given by:

$$r_i = X_i(X'W_iX)^{-1}X'W_i. \tag{12}$$

A few examples of works using the GCV include: McMillen and Redfearn (2010), Sunding and Swoboda (2010), Paez et al. (2011), Geniaux et al. (2011), and McMillen (2012).

2.2.3 Standardized Cross Validation

The (Row-)Standardized Cross Validation Score was suggested by (Farber and Páez, 2007) and elaborated on in (Paez et al., 2011) as an alternative to conventional metrics like LOOCV and GCV. This metric is designed to limit the influence of outliers which may disproportionately impact the choice of bandwidth. The Standardized Cross Validation score for a given observation i and bandwidth k is,

$$SCV_i(k) = \frac{(y_i - \hat{y}_{\neq i}(k))^2}{\sum_k (y_i - \hat{y}_{\neq i})^2}, \tag{13}$$

and the total score for bandwidth k is then,

$$SCV(k) = \sum_i SCV_i(k). \tag{14}$$

Equation (13) calculates a partial score for each observation as a proportion of the total squared deviance at that observation across the different bandwidths, while (14) then calculates the sum across all observations for a given bandwidth. Note that, contrary to the other metrics described here, the SCV score has to be calculated after all possible bandwidths have been implemented.

2.2.4 Akaike Information Criterion

As noted in (Fotheringham et al., 2002), the well-known Akaike Information Criterion is calculated in the geographically weighted regression framework as follows,

$$AIC = 2 * n * ln(\hat{\sigma}) + n * ln(2 * \pi) + n * \frac{n + v_1}{n - 2 - v_1} \tag{15}$$

where $\hat{\sigma}$ is the estimated standard error of the regression, n is the sample size, and v_1 remains the "effective number of parameters" estimated by the model as described above. The AIC remains another popular metric in the literature, being used recently by: Foody (2003), Kestens et al. (2005), Yu (2006), Yu et al. (2007), Yu (2007), Borst and Mccluskey (2007), Partridge and Rickman (2007), Cahill and Mulligan (2007), Eckey et al. (2007), Partridge et al. (2008), Tu and Xia (2008), Helbich and Leitner (2009), Hanink et al. (2010), Sá et al. (2010), Pineda Jaimes et al. (2010), Axhausen and Löchl (2010), and Haynes (2011).

2.2.5 Coefficient Root Mean Square Error

The goal of regression analyses tends to be the efficient, consistent, and unbiased estimation of a particular variable coefficient rather than necessarily correctly identifying the exact model specification. We therefore also calculated the Root Mean Square Error for each regression coefficient across all of our model specifications as,

$$\text{RMSE } \widehat{\beta}_m = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\widehat{\beta}_{mi} - \beta_{mi})^2}, \tag{16}$$

where i denotes the observation, n is the sample size, and $m \in \{0, 1, 2\}$ specifies the model coefficient in question. Unlike the metrics described previously, these model performance metrics are not available to researchers with observational data. These measures of estimated coefficient accuracy can only be calculated because we know the true underlying data generating process.

3 Simulation Results

We compare four important cross-validation/information criteria: Leave One Out Cross Validation (LOOCV), Generalized Cross Validation (GCV), Standardized Cross Validation (SCV), and the Akaike Information Criterion (AIC). How frequently can researchers utilizing these metrics identify the correct model among the various possible combinations? Are certain metrics more/less prone to false positive/negatives? Do they suggest no spatial variation when in fact it exists? Do they suggest spatial variation when in fact there is not?

3.1 Starting Simple: All Coefficients are Spatially Stationary

We begin by examining the simulation resuts for the spatially stationary data generation process. With no spatial variation for any of the coeffients, these data are consistent with standard OLS regression. We label this model ‘GGG’ to denote that all three coefficients are ‘Global’ rather than ‘Local’.[5] Table 1 displays the percentage of simulation iterations that each of the eight different mixed GWR models was ‘selected’ by each of the four different metrics: LOOCV, GCV, SCV, and AIC. Correspondingly, each column sums to 100 (subject to rounding error).

		Metric				
		LOOCV	GCV	SCV	AIC	
Model Selected	GGG	72	0	8	0	3/3 Correct
	LGG	7	28	29	28	2/3 Correct
	GLG	8	36	22	37	
	GGL	8	33	22	34	
	LLG	1	1	5	0	1/3 Correct
	LGL	2	1	5	1	
	GLL	1	1	8	0	
	LLL	0	0	1	0	0/3 Correct
		100	100	100	100	

Table 1: Distribution of Model Selected by Metric when True Model = GGG (All Coefficients are Non-Stationary).
Cell values denote the percentage of simulations in which each model yielded the best metric value. For instance, the GGG model had the smallest LOOCV value for 72 percent of our simulations. Each column sums to 100 subject to rounding error. Cell shading denotes the number of coefficients that are correctly identified as stationary or not.

Table 1 shows a distinct difference between LOOCV and the three other metrics. Almost three-fourths of the time the LOOCV was minimized using the model that was “correct” across all three coefficients. Conversely, the SCV metric selected the correct (‘GGG’) model in less than 10 percent of the simulations and both the GCV and AIC metrics selected the correct model less than 1 percent of the time. Interestingly, while the GCV, SCV, and AIC metrics did not choose the correct model nearly as frequently as the LOOCV metric, they tend to make a correctly identify the spatial (non-)stationarity for two out of three coefficients. The GCV and AIC metric almost exclusively selected one of the ‘LGG’, ‘GLG’, and ‘GGL’ models. Almost 20 percent of the time the SCV metric selected a model that was incorrect about two (‘LLG’, ‘LGL’, ‘GLL’) or all (‘LLL’) of the coefficients stationarity.

At first glance, the high frequency of type one error displayed in Table 1 is frustrating. Table 2 shows the distribution of models selected by having the smallest RMSE for each coefficient.

An interesting pattern emerges upon examination of Table 2. The largest value

[5] The eight GWR models representing the unique mixture/combinations of (non-)stationarity across the three coefficients are labeled: GGG, LGG, GLG, GGL, LLG, LGL, GLL, and LLL.

		Coefficient RMSE		
		$\widehat{\beta_0}$	$\widehat{\beta_1}$	$\widehat{\beta_2}$
Model Selected	GGG	6	6	7
	LGG	3	23	24
	GLG	22	4	24
	GGL	23	23	3
	LLG	5	3	33
	LGL	5	36	2
	GLL	32	4	4
	LLL	4	2	3
		100	100	100

Table 2: Distribution of Model Selected by RMSE when True Model = GGG (All Coefficients are Non-Stationary).
Cell values denote the percentage of simulations in which each model yielded the lowest RMSE value for each coefficient. For instance, the GGG model had the smallest RMSE $\widehat{B_0}$ value for 6 percent of our simulations. Each column sums to 100 subject to rounding error. Shaded cells denote model for which the spatial non-stationarity of the column coefficient is correct.

in each column represents approximately one-third of the simulations, but is not the perfect model. The 'GGG' row of Table 2 shows that the correct model yielded the smallest RMSE for a given $\widehat{B}$ in only 6 to 7 percent of simulations. Instead, the model most frequently minimizing the RMSE for $\widehat{B}$ correctly identifies the spatial stationarity of the coefficient in question, but incorrectly treats both of the other coefficients as non-stationary. For instance, in 32 percent of these simulations with a true 'GGG' model, the smallest RMSE for $\widehat{B_0}$ was obtained using the 'GLL' model. In each column the four respective models that correctly identify the spatial non-stationarity of the respective coefficient yield the most accurate estiamtes of the coefficient in question for approximately 85 percent of our simulations. It is relatively rare for the most accurate stationary regression coefficient estimates to be obtained from a model that incorrectly identifies it as spatially non-stationary. Approximately half of our simulations yielded minimum RMSEs using models that were correct about the coefficient in question, but were incorrect about one of the two remaining coefficients.

3.1.1 Model and Bandwidth

The previous section explored the model selected by different metrics in the presence of an underlying globally stationary data generation process. The results showed that the correct model, 'GGG', was only selected relatively frequently by the LOOCV metric. However, we have also seen that some of the most accurate estimates of the individual regression coefficients come from incorrect models. We have not yet quantified how wrong these incorrect model are. For instance, a model may incorrectly dentify a coefficient as spatially non-stationary, but might estimate a very small degree of variation in the coefficient by using a large bandwidth relative to our sample size. That is, it is potentially very different to select the 'GGL' model instead of the 'GGG' with a large vs. small bandwidth. A small bandwidth can yield more variation in coefficient estimates across our sample, while a large bandwidth will restrict the non-stationary coefficients to be more similar.

In each of our simulations we estimated the seven models allowing non-stationarity in at least coefficient for seven different bandwidths, ranging from using all of the data to just under 10 percent of the observations in the mixed models with the smallest bandwidth. Figure 2 shows the distribution of model selected and bandwidth size for each of the seven metrics we've discussed.

3.2 All Local Coefficients

The previous section investigated the simulation results when the true model was stationary across all three coefficients. In this section we examine the opposite extreme: all non-stationary coefficients. To begin, we construct a figure similar to Figure 2 but with a true model of 'LLL.'

Figure 3 reveals a stark pattern. Although the true underlying model is non-stationary, none of the four metrics choose the 'LLL' model in more than 10 percent of our simulations. Instead, the 'LGG' model is selected over 50 percent of the time by each metric. The LOOCV, GCV, and AIC metrics all select the 'LGG' model with the smallest bandwidth much more frequently than any other model/bandwidth combination. For instance, LOOCV and GCV select the 'LGG' model with the smallest bandwidth over 50 percent of the time.

The second row of results in Figure 3 displays the model and bandwidth combinations that yield the smallest RMSE for the coefficients in our model. The pattern for the smallest RMSE associated with the intercept term, β_0 closely resembles the pattern of models and bandwidths for LOOCV, GCC, and AIC. Roughly 10 percent of simulations have the smallest RMSE using the 'LLL' model, while roughly half of the simulations selected the 'LGG' model with the smallest bandwidth. However, we see a different pattern for β_1 and β_2. In both instances, no single model and bandwidth combination yielded the smallest RMSE more than 20 percent of the time (the 'GLL' model with the second largest bandwidth). That is, the intercept was fixed while the other two coefficients were treated as non-stationary and the bandwidth was comprised of approximately two-thirds of the observations.

The simulations contained in Figure 3 actually contain a lot of variation in the amount of coefficient non-stationarity. Recalling that we implemented two amounts of spatial variation in each coefficient (think of it as 'some' and 'more') there are actually eight combinations of 'LLL' models ranging from "some, some, some," to "more, more, more." We therefore present a comparison of the following three models, the 'GGG' model we've already discussed, the 'LLL' model containing the smallest amount of overall spatial non-stationarity in all coefficients (the "some, some, some" model), and the largest amount of overall spatial non-stationarity (the "more, more, more" model).

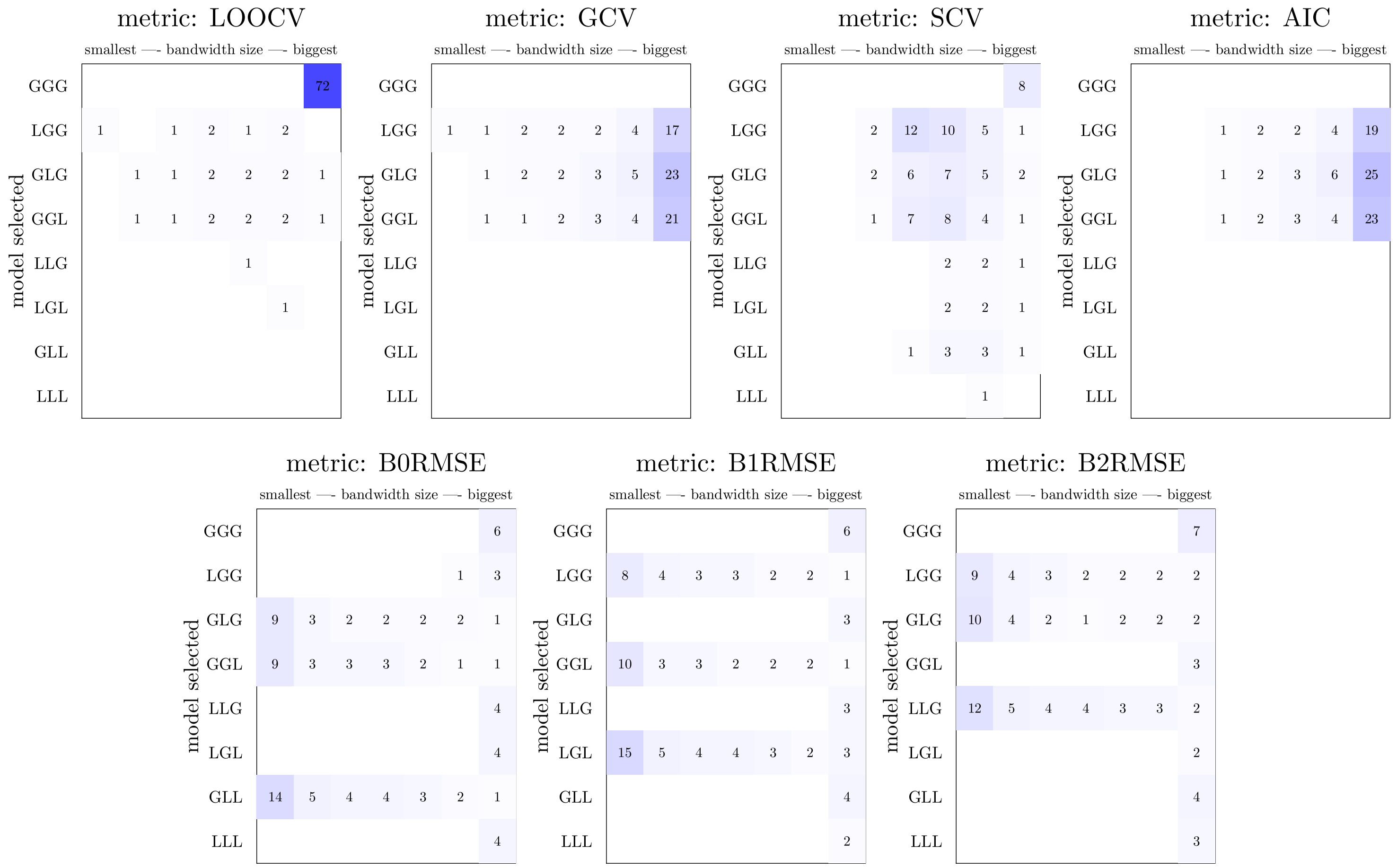


Figure 2: Model and Bandwidth Selected by Metric when the True Model = ‘GGG’ (All coefficients are stationary) Each subfigure shows the percentage of simulations a given model and bandwidth combination was selected among the 50 possible combinations (7 bandwidths for each of the seven mixed models plus the GGG model) for a given metric. The sum of all values in a given subfigure sum to 100 subject to rounding error. For convenience, cell values less than 0.5 percent are omitted. The color saturation of the cells helps denote the magnitude of the values.

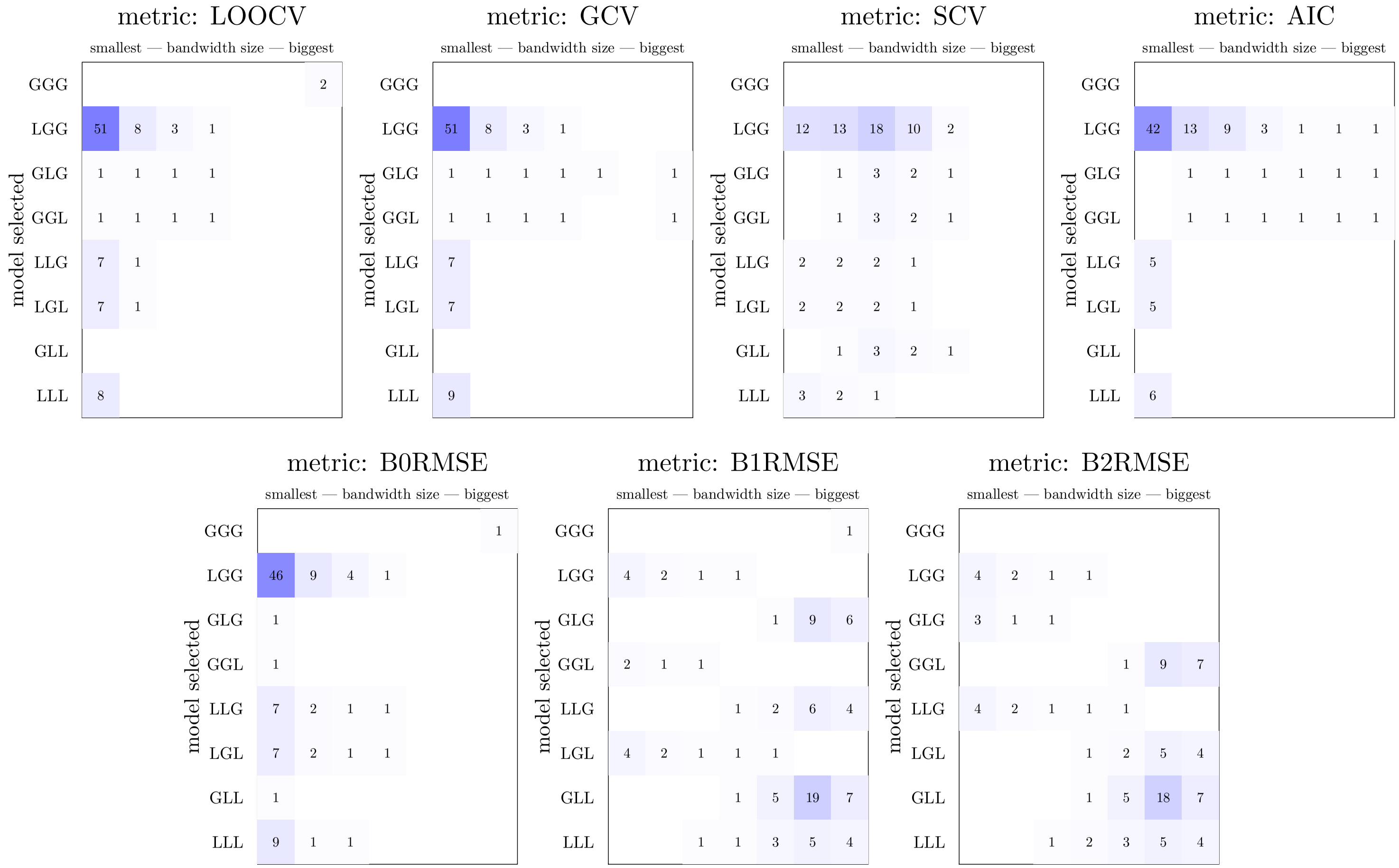
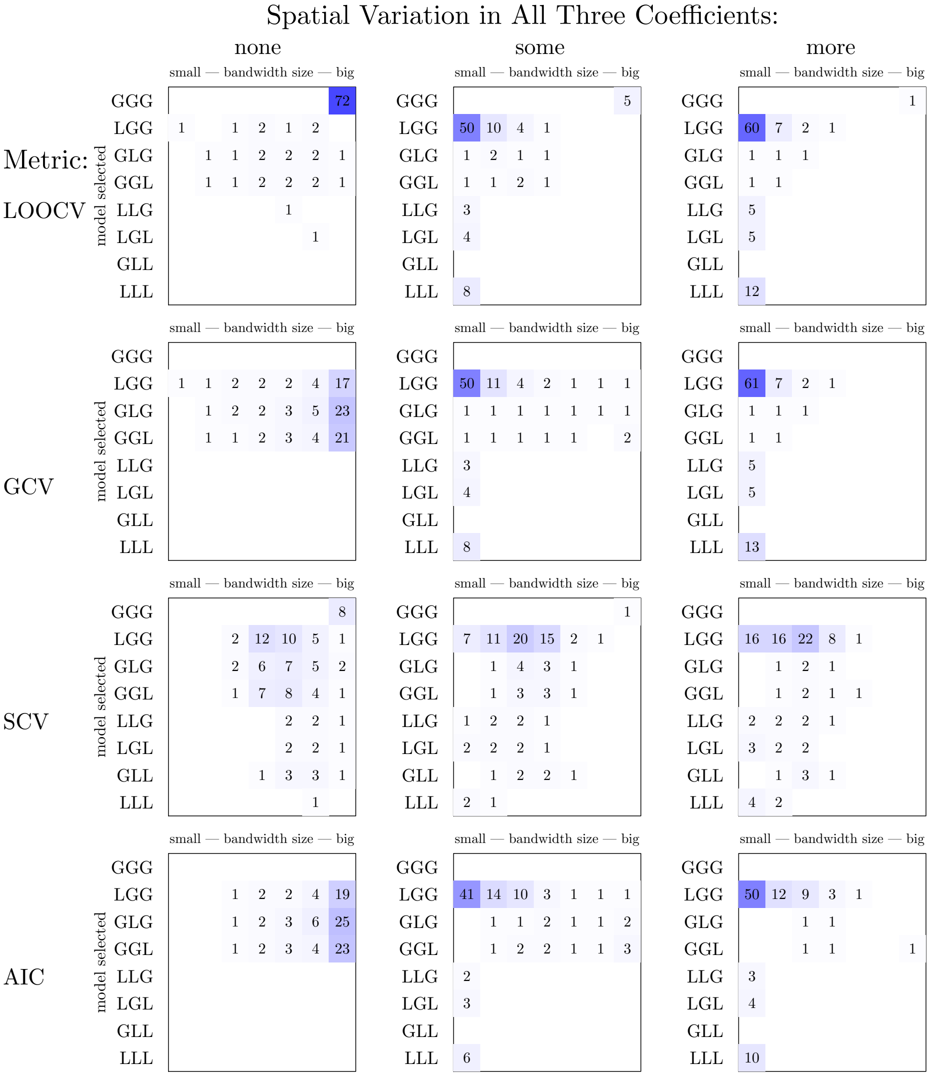


Figure 3: Model and Bandwidth Selected by Metric when the True Model = 'LLL' (All coefficients are non-stationary)
Each subfigure shows the percentage of simulations a given model and bandwidth combination was selected among the 50 possible combinations (7 bandwidths for each of the seven mixed models plus the GGG model) for a given metric. The sum of all values in a given subfigure sum to 100 subject to rounding error. For convenience, cell values less than 0.5 percent are omitted. The color saturation of the cells helps denote the magnitude of the values.



Spatial Variation in All Three Coefficients:

Metric: LOOCV

none — small — bandwidth size — big

model selected							
GGG							72
LGG	1		1	2	1	2	
GLG		1	1	2	2	2	1
GGL		1	1	2	2	2	1
LLG					1		
LGL						1	
GLL							
LLL							

some — small — bandwidth size — big

model selected							
GGG							5
LGG	50	10	4	1			
GLG	1	2	1	1			
GGL	1	1	2	1			
LLG	3						
LGL	4						
GLL							
LLL	8						

more — small — bandwidth size — big

model selected							
GGG							1
LGG	60	7	2	1			
GLG	1	1	1				
GGL	1	1					
LLG	5						
LGL	5						
GLL							
LLL	12						

GCV

none — small — bandwidth size — big

model selected							
GGG							
LGG	1	1	2	2	2	4	17
GLG		1	2	2	3	5	23
GGL		1	1	2	3	4	21
LLG							
LGL							
GLL							
LLL							

some — small — bandwidth size — big

model selected							
GGG							
LGG	50	11	4	2	1	1	1
GLG	1	1	1	1	1	1	1
GGL	1	1	1	1	1		2
LLG	3						
LGL	4						
GLL							
LLL	8						

more — small — bandwidth size — big

model selected							
GGG							
LGG	61	7	2	1			
GLG	1	1	1				
GGL	1	1					
LLG	5						
LGL	5						
GLL							
LLL	13						

SCV

none — small — bandwidth size — big

model selected							
GGG							8
LGG			2	12	10	5	1
GLG			2	6	7	5	2
GGL			1	7	8	4	1
LLG					2	2	1
LGL					2	2	1
GLL				1	3	3	1
LLL						1	

some — small — bandwidth size — big

model selected							
GGG							1
LGG	7	11	20	15	2	1	
GLG		1	4	3	1		
GGL		1	3	3	1		
LLG	1	2	2	1			
LGL	2	2	2	1			
GLL		1	2	2	1		
LLL	2	1					

more — small — bandwidth size — big

model selected							
GGG							
LGG	16	16	22	8	1		
GLG		1	2	1			
GGL		1	2	1	1		
LLG	2	2	2	1			
LGL	3	2	2				
GLL		1	3	1			
LLL	4	2					

AIC

none — small — bandwidth size — big

model selected							
GGG							
LGG			1	2	2	4	19
GLG			1	2	3	6	25
GGL			1	2	3	4	23
LLG							
LGL							
GLL							
LLL							

some — small — bandwidth size — big

model selected							
GGG							
LGG	41	14	10	3	1	1	1
GLG		1	1	2	1	1	2
GGL		1	2	2	1	1	3
LLG	2						
LGL	3						
GLL							
LLL	6						

more — small — bandwidth size — big

model selected							
GGG							
LGG	50	12	9	3	1		
GLG			1	1			
GGL			1	1			1
LLG	3						
LGL	4						
GLL							
LLL	10						

Figure 4: Model and Bandwidth Selected by Metric for ‘None’, ‘Some’, and ‘More’ Spatial Variation in the Three Regression Coefficients
Each subfigure shows the percentage of simulations a given model and bandwidth combination was selected among the 50 possible combinations (7 bandwidths for each of the seven mixed models plus the GGG model) for a given metric. The sum of all values in a given subfigure sum to 100 subject to rounding error. For convenience, cell values less than 0.5 percent are omitted. The color saturation of the cells helps denote the magnitude of the values.

3.3 One Local Coefficient

We now examine our results when exactly one model coefficient is non-stationary. Figure 5 shows that most metrics can correctly identify the proper mixed model when only one coefficient is non-stationary. Interestingly, the most accurate estimates of the coefficients again are not necessarily associated with the correct overall model.

3.4 Two Local Coefficients

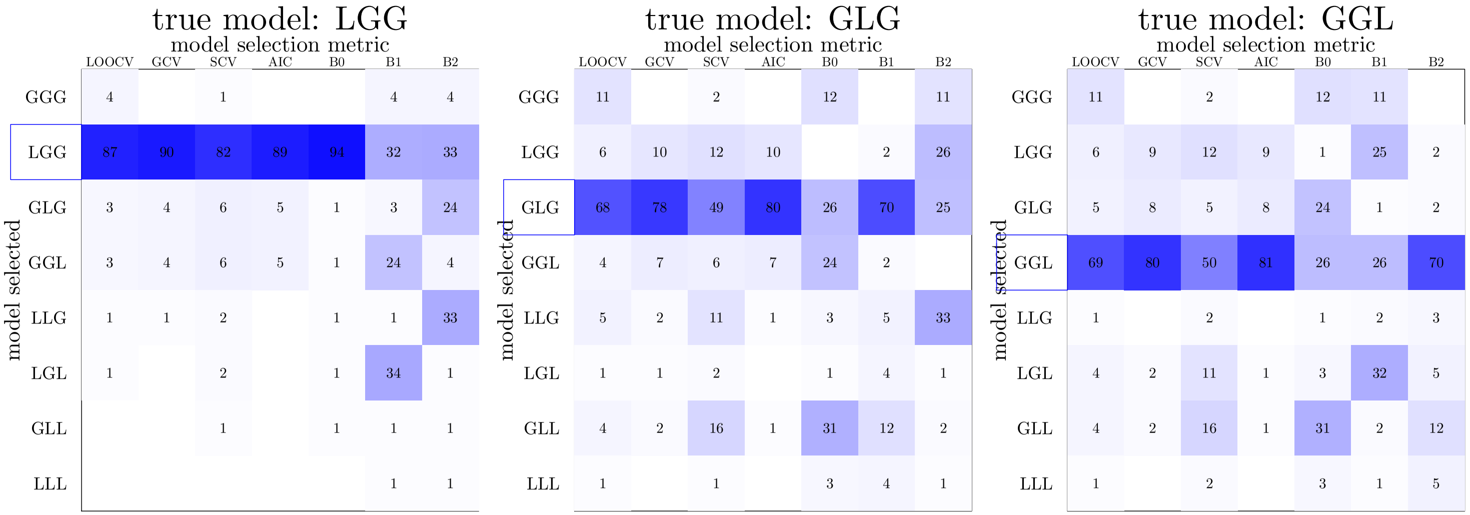
We now examine our results when exactly one model coefficient is non-stationary. Figure 6 shows that these conventional metrics are less successful at identifying the correct model under these conditions. Notably, only the "GLL" model is easily identified with regularity. Figure 6 shows that the "LLG" and "LGL" models are not misidentified uniformly across the other seven models. Instead, the "LGG" model is the most commonly selected.

4 Discussion and Future Work

Figure 7 summarizes the results of model selection across our eight different models. The models are arranged such that the left and right columns of Figure 7 differ only by whether or not the model constant is stationary (left) or not (right). This arrangement shows an interesting pattern. In all four cases when the model intercept is considered "Local" (non-stationary), the most commonly selected model by most metrics is "LGG" model. In other words, if the model's constant is varying over space, these conventional metrics select a model that correctly identifies the spatially varying intercept, but seems indifferent to whether or not the other coefficients are stationary or not. What accounts for this behavior? Could it be that the specification and parameters selected in this work make proper identification of the spatial variation (or not) in the intercept term more important?

A close examination of the rightmost columns in each table of Figure 7 reveals other patterns, too. The models associated with the smallest Root Mean Square Error for each coefficient are not necessarily with the correct model. For instance, when the true model is "LGG" (in the upper right hand corner). We see that this is an example of when the conventional cross validation metrics routinely identify "LGG" as the correct model. The most accurate estimates of the interecpt term are obtained by the true model over 90 percent of the time. However, the most accurate estimates of the other two coefficients (and perhaps the coefficients researchers are most interested in the value of) are only obtained by the correct model roughly one third of the time. Other model forumations have the smallest RMSE for these two coefficients just as frequently (LGL for β_1 and LLG for β_2). Are there meanginful patterns to be found in these results?

Thus far, this work has only examined the performance of different metrics in identifying the correct model and which models result in the smallest RMSE for each coefficient in our model. We have not yet compared the coefficient RMSE across model selection metrics. That is, does the model selected by the LOOCV



true model: LGG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	4		1			4	4
LGG	87	90	82	89	94	32	33
GLG	3	4	6	5	1	3	24
GGL	3	4	6	5	1	24	4
LLG	1	1	2		1	1	33
LGL	1		2		1	34	1
GLL			1		1	1	1
LLL						1	1

true model: GLG

model selection metric

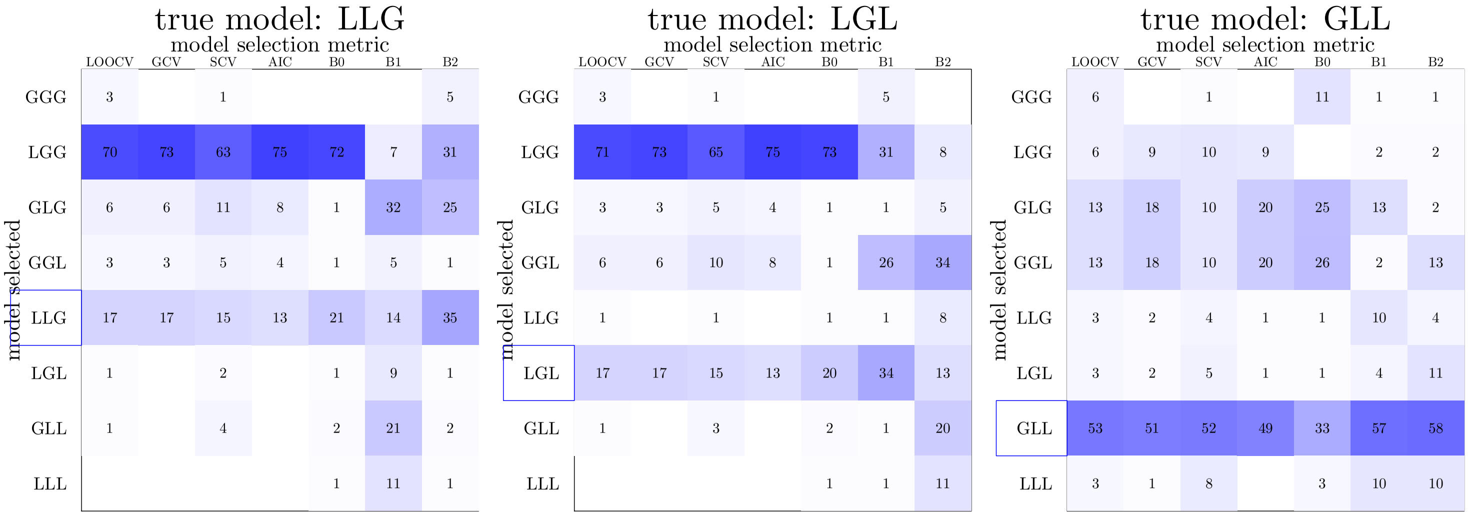
model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	11		2		12		11
LGG	6	10	12	10		2	26
GLG	68	78	49	80	26	70	25
GGL	4	7	6	7	24	2	
LLG	5	2	11	1	3	5	33
LGL	1	1	2		1	4	1
GLL	4	2	16	1	31	12	2
LLL	1		1		3	4	1

true model: GGL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	11		2		12	11	
LGG	6	9	12	9	1	25	2
GLG	5	8	5	8	24	1	2
GGL	69	80	50	81	26	26	70
LLG	1		2		1	2	3
LGL	4	2	11	1	3	32	5
GLL	4	2	16	1	31	2	12
LLL	1		2		3	1	5

Figure 5: Model Selection with One Local Coefficient
This figure shows the model selected by each of the seven different metrics for the three underlying models containing one Local (and two Global) model coefficients. Values in each cell refer to the percentage that each model is selected by the metric. As such, each column sums to 100. The color saturation for each cell helps denote the relative frequency.



true model: LLG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	3		1				5
LGG	70	73	63	75	72	7	31
GLG	6	6	11	8	1	32	25
GGL	3	3	5	4	1	5	1
LLG	17	17	15	13	21	14	35
LGL	1		2		1	9	1
GLL	1		4		2	21	2
LLL					1	11	1

true model: LGL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	3		1			5	
LGG	71	73	65	75	73	31	8
GLG	3	3	5	4	1	1	5
GGL	6	6	10	8	1	26	34
LLG	1		1		1	1	8
LGL	17	17	15	13	20	34	13
GLL	1		3		2	1	20
LLL					1	1	11

true model: GLL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	6		1		11	1	1
LGG	6	9	10	9		2	2
GLG	13	18	10	20	25	13	2
GGL	13	18	10	20	26	2	13
LLG	3	2	4	1	1	10	4
LGL	3	2	5	1	1	4	11
GLL	53	51	52	49	33	57	58
LLL	3	1	8		3	10	10

Figure 6: Model Selection with Two Local Coefficients
This figure shows the model selected by each of the seven different metrics for the three underlying models containing two Local (and one Global) model coefficients. Values in each cell refer to the percentage that each model is selected by the metric. As such, each column sums to 100. The color saturation for each cell helps denote the relative frequency.

true model: GGG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	72		8		6	6	7
LGG	7	28	29	28	3	23	24
GLG	8	36	22	37	22	4	24
GGL	8	33	22	34	23	23	3
LLG	1	1	5		5	3	33
LGL	2	1	5	1	5	36	2
GLL	1	1	8		32	4	4
LLL			1		4	2	3

true model: LGG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	4		1			4	4
LGG	87	90	82	89	94	32	33
GLG	3	4	6	5	1	3	24
GGL	3	4	6	5	1	24	4
LLG	1	1	2		1	1	33
LGL	1		2		1	34	1
GLL			1		1	1	1
LLL						1	1

true model: GLG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	11		2		12		11
LGG	6	10	12	10		2	26
GLG	68	78	49	80	26	70	25
GGL	4	7	6	7	24	2	
LLG	5	2	11	1	3	5	33
LGL	1	1	2		1	4	1
GLL	4	2	16	1	31	12	2
LLL	1		1		3	4	1

true model: LLG

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	3		1				5
LGG	70	73	63	75	72	7	31
GLG	6	6	11	8	1	32	25
GGL	3	3	5	4	1	5	1
LLG	17	17	15	13	21	14	35
LGL	1		2		1	9	1
GLL	1		4		2	21	2
LLL					1	11	1

true model: GGL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	11		2		12	11	
LGG	6	9	12	9	1	25	2
GLG	5	8	5	8	24	1	2
GGL	69	80	50	81	26	26	70
LLG	1		2		1	2	3
LGL	4	2	11	1	3	32	5
GLL	4	2	16	1	31	2	12
LLL	1		2		3	1	5

true model: LGL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	3		1			5	
LGG	71	73	65	75	73	31	8
GLG	3	3	5	4	1	1	5
GGL	6	6	10	8	1	26	34
LLG	1		1		1	1	8
LGL	17	17	15	13	20	34	13
GLL	1		3		2	1	20
LLL					1	1	11

true model: GLL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	6		1		11	1	1
LGG	6	9	10	9		2	2
GLG	13	18	10	20	25	13	2
GGL	13	18	10	20	26	2	13
LLG	3	2	4	1	1	10	4
LGL	3	2	5	1	1	4	11
GLL	53	51	52	49	33	57	58
LLL	3	1	8		3	10	10

true model: LLL

model selection metric

model selected	LOOCV	GCV	SCV	AIC	B0	B1	B2
GGG	2				1	1	
LGG	63	65	55	69	61	8	8
GLG	5	5	7	7	2	16	6
GGL	4	5	7	6	2	5	17
LLG	8	8	8	5	11	14	10
LGL	8	8	8	6	11	9	14
GLL	1	1	8		2	31	30
LLL	9	9	6	6	12	15	15

Figure 7: This figure shows the model selected by each of the seven different metrics across all eight different true underlying data generation processes. Values in each cell refer to the percentage that each model is selected by the metric. As such, each column (of each table) sums to 100. The color saturation for each cell helps denote the relative frequency. The color scale is consistent across all tables.

systematically yield different RMSE values for the coefficients than any of the other metrics? Additionally, how might other selection strategies fare? For instance, future work should examine the hypothesis testing procedure put forth by Leung et al. (2000) and the Monte Carlo simulation procedure described in Fotheringham et al. (2002).

References

Kay W Axhausen and Michael Löchl. Modeling hedonic residential rents for land use and transport simulation while considering spatial effects. *Journal of Transport and Land Use*, 3(2):39–63, 2010. doi: 10.1598/jtlu.v3i2.117.

B Y Richard A Borst and William J Mccluskey. Using Geographically Weighted Regression to Detect Housing Submarkets: Modeling Large-Scale Spatial Variations in Value. *Journal of Property Tax Assessment Administration*, 5(1):21–54, 2007. ISSN 15483614. URL http://www.redi-bw.de/db/ebsco.php/search.ebscohost.com/login.aspx?direct=true&db=buh&AN=31846286&site=ehost-live.

Chris Brunsdon, Stewart Fotheringham, and Martin Charlton. Geographically weighted regression-modelling spatial non-stationarity. *Journal of the Royal Statistical Society Series D The Statistician*, 47(3):431–443, 1998a. ISSN 00390526. doi: 10.1111/1467-9884.00145. URL http://www.jstor.org/stable/2988625.

Chris Brunsdon, Stewart Fotheringham, and Martin Charlton. Geographically weighted regression-modelling spatial non-stationarity. *Journal of the Royal Statistical Society Series D The Statistician*, 47(3):431–443, 1998b. ISSN 00390526. doi: 10.1111/1467-9884.00145. URL http://www.jstor.org/stable/2988625.

M Cahill and G Mulligan. Using Geographically Weighted Regression to Explore Local Crime Patterns. *Social Science Computer Review*, 25(2):174–193, 2007. ISSN 08944393. doi: 10.1177/0894439307298925. URL http://ssc.sagepub.com/cgi/doi/10.1177/0894439307298925.

Seong-hoon Cho, J. M Bowker, and William M. Park. Measuring the Contribution of Water and Green Space Amenities to Housing Values: An Application and Comparison of Spatially Weighted Hedonic Models. *Journal of Agricultural and Resource Economics*, 31(3):485–507, 2006.

Seong-hoon Cho, D Christopher, M William, and Seung Gyu Kim. Spatial and Temporal Variation in the Housing Market Values of Lot Size and Open Space. *Land Economics*, 85(1):51–73, 2009a.

Seong-Hoon Cho, Dayton M Lambert, Roland K Roberts, and Seung Gyu Kim. Moderating urban sprawl: is there a balance between shared open space and housing parcel size? *Journal of Economic Geography*, 10(5):763–783, 2009b. ISSN 14682702. doi: 10.1093/jeg/lbp048. URL http://joeg.oxfordjournals.org/cgi/doi/10.1093/jeg/lbp048.

William S Cleveland and Susan J Devlin. Locally Weighted Regression: An Approach to Regression Analysis by Local Fitting. *Journal of the American Statistical Association*, 83(403):596–610, June 1988. doi: 10.1080/01621459.1959.10501996.

Hans-Friedrich Eckey, Reinhold Kosfeld, and Matthias Türck. Regional Convergence in Germany: a Geographically Weighted Regression Approach. *Spatial Economic Analysis*, 2(1):45–64, 2007. ISSN 17421772. doi: 10.1080/17421770701251905. URL `http://www.tandfonline.com/doi/abs/10.1080/17421770701251905`.

S Farber and A Páez. A systematic investigation of cross-validation in GWR model estimation: empirical analysis and Monte Carlo simulations. *Journal of Geographical Systems*, 9(4):371–396, 2007.

G.M M Foody. Geographical weighting as a further refinement to regression modelling: An example focused on the NDVIrainfall relationship. *Remote Sensing of Environment*, 88(3):283–293, December 2003. ISSN 00344257. doi: 10.1016/j.rse.2003.08.004. URL `http://eprints.soton.ac.uk/14528/http://linkinghub.elsevier.com/retrieve/pii/S0034425703001949`.

A. Stewart Fotheringham, Chris Brunsdon, and Martin Charlton. *Geographically Weighted Regression: the analysis of spatially varying relationships.* John Wiley & Sons, West Sussex, England, 2002.

Ghislain Geniaux, Jean-Sauveur Ay, and Claude Napoléone. A SPATIAL HEDONIC APPROACH ON LAND USE CHANGE ANTICIPATIONS. *Journal of Regional Science*, 51(5):967–986, 2011. ISSN 00224146. doi: 10.1111/j.1467-9787.2011.00721.x. URL `http://doi.wiley.com/10.1111/j.1467-9787.2011.00721.x`.

Dean M Hanink, Robert G Cromley, and Avraham Y Ebenstein. Spatial Variation in the Determinants of House Prices and Apartment Rents in China. *The Journal of Real Estate Finance and Economics*, 2010. ISSN 08955638. doi: 10.1007/s11146-010-9262-3. URL `http://www.springerlink.com/index/10.1007/s11146-010-9262-3`.

K E Haynes. The Employment Effects of New Business Formation: A Regional Perspective. *Economic Development Quarterly*, 25(3):282–292, 2011. ISSN 08912424. doi: 10.1177/0891242411407310. URL `http://edq.sagepub.com/cgi/doi/10.1177/0891242411407310`.

Marco Helbich and Michael Leitner. Spatial Analysis of the Urban-to-Rural Migration Determinants in the Viennese Metropolitan Area. A Transition from Suburbia to Postsuburbia? *Applied Spatial Analysis and Policy*, 2(3):237–260, 2009. ISSN 1874463X. doi: 10.1007/s12061-009-9026-8. URL `http://www.springerlink.com/content/pkq04616t6577843`.

Bo Huang, Bo Wu, and Michael Barry. Geographically and temporally weighted regression for modeling spatio-temporal variation in house prices. *International Journal of Geographical Information Science*, 24 (3):383–401, 2010. ISSN 13658816. doi: 10.1080/13658810802672469. URL http://www.informaworld.com/openurl?genre=article&doi=10.1080/13658810802672469&magic=crossref.

Yefang Huang and Yee Leung. Analysing regional industrialisation in Jiangsu province using geographically weighted regression. *Journal of Geographical Systems*, 4(2):233–249, 2002. ISSN 14355930. doi: 10.1007/s101090200081. URL http://www.springerlink.com/openurl.asp?genre=article&id=doi:10.1007/s101090200081.

Yan Kestens, Marius Thériault, and François Des Rosiers. Heterogeneity in hedonic modelling of house prices: looking at buyers household profiles. *Journal of Geographical Systems*, 8(1):61–96, 2005. ISSN 14355930. doi: 10.1007/s10109-005-0011-8. URL http://www.springerlink.com/index/10.1007/s10109-005-0011-8.

Yee Leung, Chang-Lin Mei, and Wen-Xiu Zhang. Testing for spatial autocorrelation among the residuals of the geographically weighted regression. *Environment and Planning - Part A*, 32(5):871–890, 2000. ISSN 0308518X. doi: 10.1068/a32117. URL http://www.envplan.com/abstract.cgi?id=a32117.

Chris Lloyd and Ian Shuttleworth. Analysing commuting using local regression techniques: scale, sensitivity, and geographical patterning. *Environment and Planning - Part A*, 37(1):81–103, 2005. ISSN 0308518X. doi: 10.1068/a36116. URL http://www.envplan.com/abstract.cgi?id=a36116.

Clive Loader. *Local Regression and Likelihood*. Springer-Verlag, New York, NY, 1999.

Daniel P. McMillen. Perspectives on Spatial Econometrics: Linear Smoothing With Structured Models. *Journal of Regional Science*, 52(2):192–209, May 2012. ISSN 00224146. doi: 10.1111/j.1467-9787.2011.00746.x. URL http://doi.wiley.com/10.1111/j.1467-9787.2011.00746.x.

Daniel P. McMillen and Christian L. Redfearn. Estimation and Hypothesis Testing for Nonparametric Hedonic House Price Functions. *Journal of Regional Science*, 50(3):712–733, April 2010. ISSN 00224146. doi: 10.1111/j.1467-9787.2010.00664.x. URL http://doi.wiley.com/10.1111/j.1467-9787.2010.00664.x.

Antonio Paez, Steven Farber, and David Wheeler. A simulation-based study of geographically weighted regression as a method for investigating spatially varying relationships. *Environment and Planning A*, 43(12):2992–3010, 2011.

M D Partridge, D S Rickman, K Ali, and M R Olfert. The Geographic Diversity of U . S . Nonmetropolitan Growth Dynamics : A Geographically Weighted Regression Approach. *Land Economics*, 84(2):241–266, 2008. ISSN 00237639. URL http://www.scopus.com/inward/record.url?eid=2-s2.0-44649124821&partnerID=40&md5=1385582b8ae5c778c531d9481977e14c.

Mark D Partridge and Dan S Rickman. Persistent pockets of extreme American poverty and job growth: Is there a place-based policy role? *Journal of Agricultural and Resource Economics*, 32(1):201–224, 2007. ISSN 01621912. URL https://libserv7.princeton.edu:82/pul/nph-pul2.cgi/000000A/http/www.jstor.org/stable/info/40987359=3f&Search=3dyes&searchText=3dpersistent&searchUri=3d=252Faction=252FdoBasicSearch=253FQuery=253Dpersistent=252Brural=252Bpoverty=2526acc=253Don=2526wc=253Don.

Noel Bonfilio Pineda Jaimes, Joaquín Bosque Sendra, Montserrat Gómez Delgado, and Roberto Franco Plata. Exploring the driving forces behind deforestation in the state of Mexico (Mexico) using geographically weighted regression. *Applied Geography*, 30(4):576–591, 2010. ISSN 01436228. doi: 10.1016/j.apgeog.2010.05.004. URL http://linkinghub.elsevier.com/retrieve/pii/S0143622810000561.

Ana C L Sá, José M C Pereira, Martin E Charlton, Bernardo Mota, Paulo M Barbosa, and A Stewart Fotheringham. The pyrogeography of sub-Saharan Africa: a study of the spatial non-stationarity of fireenvironment relationships using GWR. *Journal of Geographical Systems*, 13(3):227–248, 2010. ISSN 14355930. doi: 10.1007/s10109-010-0123-7. URL http://www.springerlink.com/index/10.1007/s10109-010-0123-7.

David L. Sunding and Aaron M. Swoboda. Hedonic analysis with locally weighted regression: An application to the shadow cost of housing regulation in Southern California. *Regional Science and Urban Economics*, 40(6):550–573, November 2010. ISSN 01660462. doi: 10.1016/j.regsciurbeco.2010.07.002. URL http://linkinghub.elsevier.com/retrieve/pii/S0166046210000608.

Jun Tu and Zong-Guo Xia. Examining spatially varying relationships between land use and water quality using geographically weighted regression I: Model design and evaluation. *Science of the Total Environment*, 407(1):358–378, 2008. URL http://www.ncbi.nlm.nih.gov/pubmed/18976797.

Dan-Lin Yu. Spatially varying development mechanisms in the Greater Beijing Area: a geographically weighted regression investigation. *The Annals of Regional Science*, 40(1):173–190, January 2006. ISSN 0570-1864. doi: 10.1007/s00168-005-0038-2. URL http://www.springerlink.com/index/10.1007/s00168-005-0038-2.

Danlin Yu. Modeling owner-occupied single-family house values in the city of milwaukee: A geographically weighted regression approach. *GIScience Remote Sensing*, 44(3):267–282, 2007. ISSN 15481603. doi: 10.2747/1548-1603.44.3.267. URL http://bellwether.metapress.com/openurl.asp?genre=article&id=doi:10.2747/1548-1603.44.3.267.

Danlin Yu, Yehua Dennis Wei, and Changshan Wu. Modeling spatial dimensions of housing prices in Milwaukee, WI. *Environment and Planning BPlanning Design*, 34(6):1085–1102, 2007. ISSN 02658135. doi: 10.1068/b32119. URL http://www.envplan.com/abstract.cgi?id=b32119.