# Can Conventional Measures Identify Geographically Varying Mixed Regression Relationships? A Simulation-based Analysis of Locally Weighted Regression

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$$Y = \beta_0 + \beta_1 * X_1 + \beta_2 * X_2 + \epsilon$$

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Locally Weighted Regression (LWR)
to the Rescue?

# OLS

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

# LWR

$$\hat{\beta}(location_i) = (X'W(location_i)X)^{-1}(X'W(location_i)Y)$$

# OLS

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

# LWR

$$\hat{\beta}(location_i) = (X'W(location_i)X)^{-1}(X'W(location_i)Y)$$

$$w_{jj} = \left[1 - \left(\frac{d_{ij}}{d_{ik}}\right)^2\right]^2 \text{ if } d_{ij} < d_{ik}, \text{ otherwise} = 0,$$

Bandwidths are commonly selected with…

Leave One Out Cross Validation

Akaike Information Criterion

Generalized Cross Validation

Standardized Cross Validation

$$LOOCV = \frac{1}{N}\sqrt{\sum_{i=1}^{N}(y - \hat{y}_{\neq i})^2},$$

$$SCV_i(k) = \frac{(y_i - \hat{y}_{\neq i}(k))^2}{\sum_k (y_i - \hat{y}_{\neq i})^2}$$

$$SCV(k) = \sum_i SCV_i(k)$$

S Farber and A Páez. A systematic investigation of cross-validation in GWR model estimation: empirical analysis and Monte Carlo simulations. *Journal of Geographical Systems*, 9(4):371–396, 2007.

$$GCV = n * \sum_{i=1}^{n} \frac{(y_i - \hat{y}_i)^2}{(n - v_1)^2},$$

$$v_1 = \text{“effective number of parameters”}$$

$$AIC = 2 * n * ln(\hat{\sigma}) + n * ln(2 * \pi) + n * \frac{n + v_1}{n - 2 - v_1}$$

“Global” OLS GGG

“Local”Regression LLL

| | |
|---|---|
| "Global" OLS | GGG |
| | LGG |
| | GLG |
| | GGL |
| Mixed Models | LLG |
| | LGL |
| | GLL |
| "Local" Regression | LLL |

# The Researcher's Problem

Choose a bandwidth <u>and</u> a model

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Choose a bandwidth <u>and</u> a model

Can we do both with conventional metrics?

# Experiment Data Generation Process

$$Y_i = \beta_0(East_i, North_i) + \beta_1(East_i, North_i) * X_{1i} + \beta_2(East_i, North_i) * X_{2i} + \epsilon_i$$

$$n \in \{50, 100, 200, 400, 800\}$$

$$\sigma^2 \in \{0.25, .5, 1, 2, 3\}$$

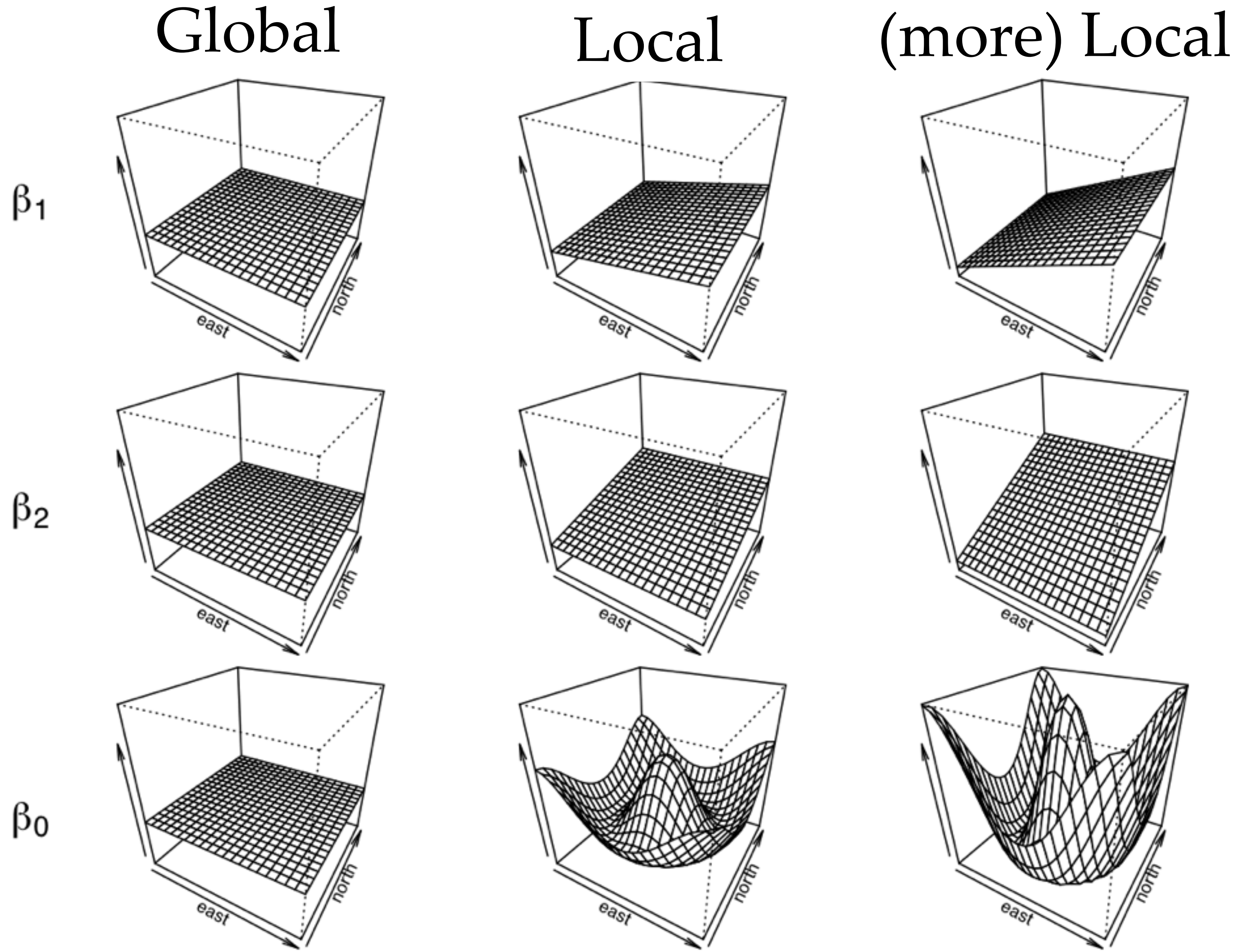
$$X_1 \sim u[0,1]$$

$$X_2 \sim u[0,1]$$

$$East \sim u[0,1]$$

$$North \sim u[0,1]$$

# Coefficient Spatial Variation



# With Our Data…

$$[Y, X_1, X_2, East, North]$$

Estimate all models

| |
|---|
| GGG |
| LGG |
| GLG |
| GGL |
| LLG |
| LGL |
| GLL |
| LLL |

7 bandwidths each

50 combinations total

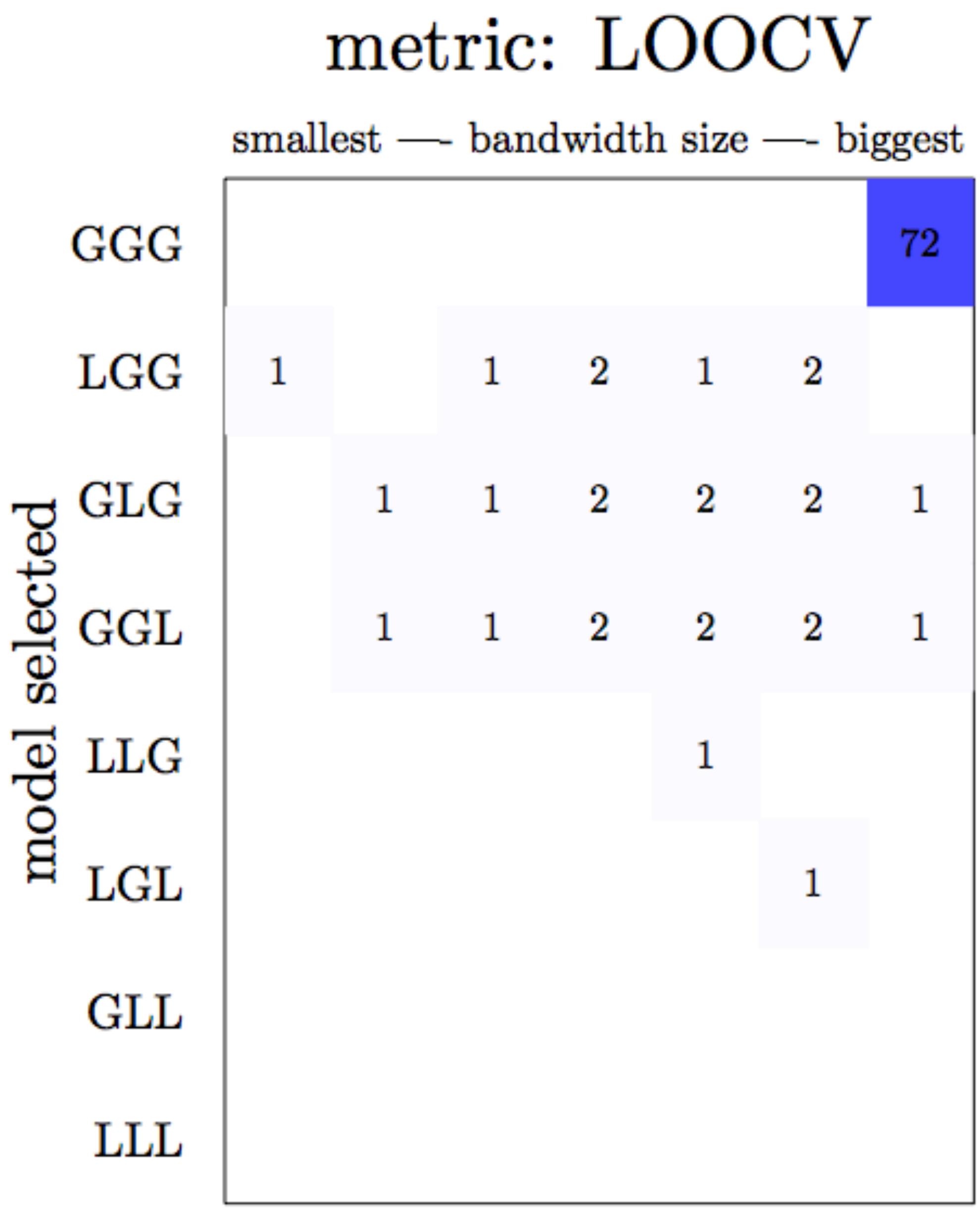
Calculate the values of the four metrics (LOOCV, GCV, SCV, AIC)

Is the model with the optimized metric value the correct model?

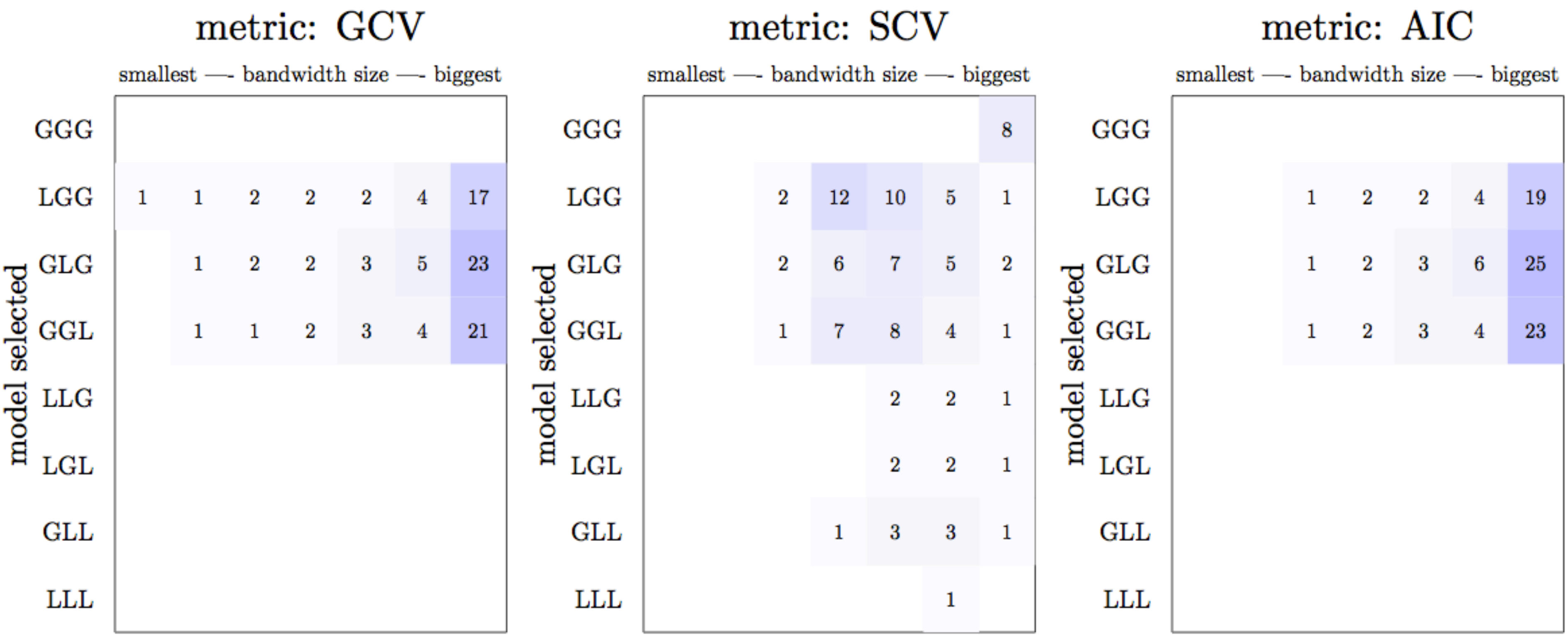
# Start Simple… True Model: GGG

| Model Selected | Metric: LOOCV | GCV | SCV | AIC | |
|---|---|---|---|---|---|
| GGG | 72 | 0 | 8 | 0 | 3/3 Correct |
| LGG | 7 | 28 | 29 | 28 | |
| GLG | 8 | 36 | 22 | 37 | 2/3 Correct |
| GGL | 8 | 33 | 22 | 34 | |
| LLG | 1 | 1 | 5 | 0 | |
| LGL | 2 | 1 | 5 | 1 | 1/3 Correct |
| GLL | 1 | 1 | 8 | 0 | |
| LLL | 0 | 0 | 1 | 0 | 0/3 Correct |
| | 100 | 100 | 100 | 100 | |

# True Model: GGG



# True Model: GGG



# True Model: GGG

| Model Selected | Coefficient RMSE $\widehat{\beta_0}$ | $\widehat{\beta_1}$ | $\widehat{\beta_2}$ |
|---|---|---|---|
| GGG | 6 | 6 | 7 |
| LGG | 3 | 23 | 24 |
| GLG | 22 | 4 | 24 |
| GGL | 23 | 23 | 3 |
| LLG | 5 | 3 | 33 |
| LGL | 5 | 36 | 2 |
| GLL | 32 | 4 | 4 |
| LLL | 4 | 2 | 3 |
| | 100 | 100 | 100 |

# True Model: GGG

### metric: B0RMSE

smallest —- bandwidth size —- biggest

| model selected | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGG | | | | | | | 6 |
| LGG | | | | | | 1 | 3 |
| GLG | 9 | 3 | 2 | 2 | 2 | 2 | 1 |
| GGL | 9 | 3 | 3 | 3 | 2 | 1 | 1 |
| LLG | | | | | | | 4 |
| LGL | | | | | | | 4 |
| GLL | 14 | 5 | 4 | 4 | 3 | 2 | 1 |
| LLL | | | | | | | 4 |

### metric: B1RMSE

smallest —- bandwidth size —- biggest

| model selected | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGG | | | | | | | 6 |
| LGG | 8 | 4 | 3 | 3 | 2 | 2 | 1 |
| GLG | | | | | | | 3 |
| GGL | 10 | 3 | 3 | 2 | 2 | 2 | 1 |
| LLG | | | | | | | 3 |
| LGL | 15 | 5 | 4 | 4 | 3 | 2 | 3 |
| GLL | | | | | | | 4 |
| LLL | | | | | | | 2 |

### metric: B2RMSE

smallest —- bandwidth size —- biggest

| model selected | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGG | | | | | | | 7 |
| LGG | 9 | 4 | 3 | 2 | 2 | 2 | 2 |
| GLG | 10 | 4 | 2 | 1 | 2 | 2 | 2 |
| GGL | | | | | | | 3 |
| LLG | 12 | 5 | 4 | 4 | 3 | 3 | 2 |
| LGL | | | | | | | 2 |
| GLL | | | | | | | 4 |
| LLL | | | | | | | 3 |

# True Model: GGG

### metric: B0RMSE

smallest —- bandwidth size —- biggest

| model selected | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGG | | | | | | | 6 |
| LGG | | | | | | 1 | 3 |
| GLG | 9 | 3 | 2 | 2 | 2 | 2 | 1 |
| GGL | 9 | 3 | 3 | 3 | 2 | 1 | 1 |
| LLG | | | | | | | 4 |
| LGL | | | | | | | 4 |
| GLL | 14 | 5 | 4 | 4 | 3 | 2 | 1 |
| LLL | | | | | | | 4 |

### metric: B1RMSE

smallest —- bandwidth size —- biggest

| model selected | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGG | | | | | | | 6 |
| LGG | 8 | 4 | 3 | 3 | 2 | 2 | 1 |
| GLG | | | | | | | 3 |
| GGL | 10 | 3 | 3 | 2 | 2 | 2 | 1 |
| LLG | | | | | | | 3 |
| LGL | 15 | 5 | 4 | 4 | 3 | 2 | 3 |
| GLL | | | | | | | 4 |
| LLL | | | | | | | 2 |

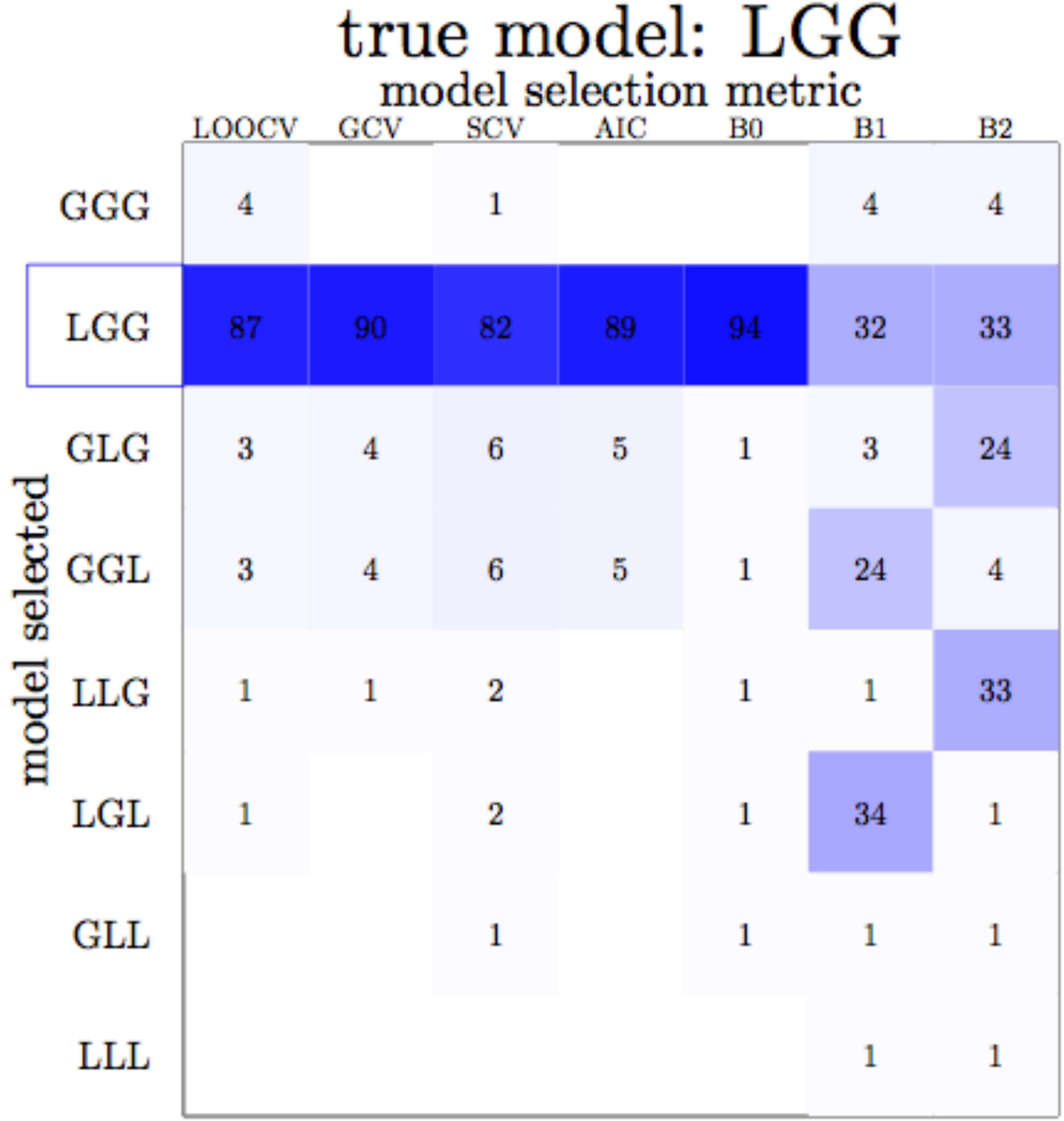
### metric: B2RMSE

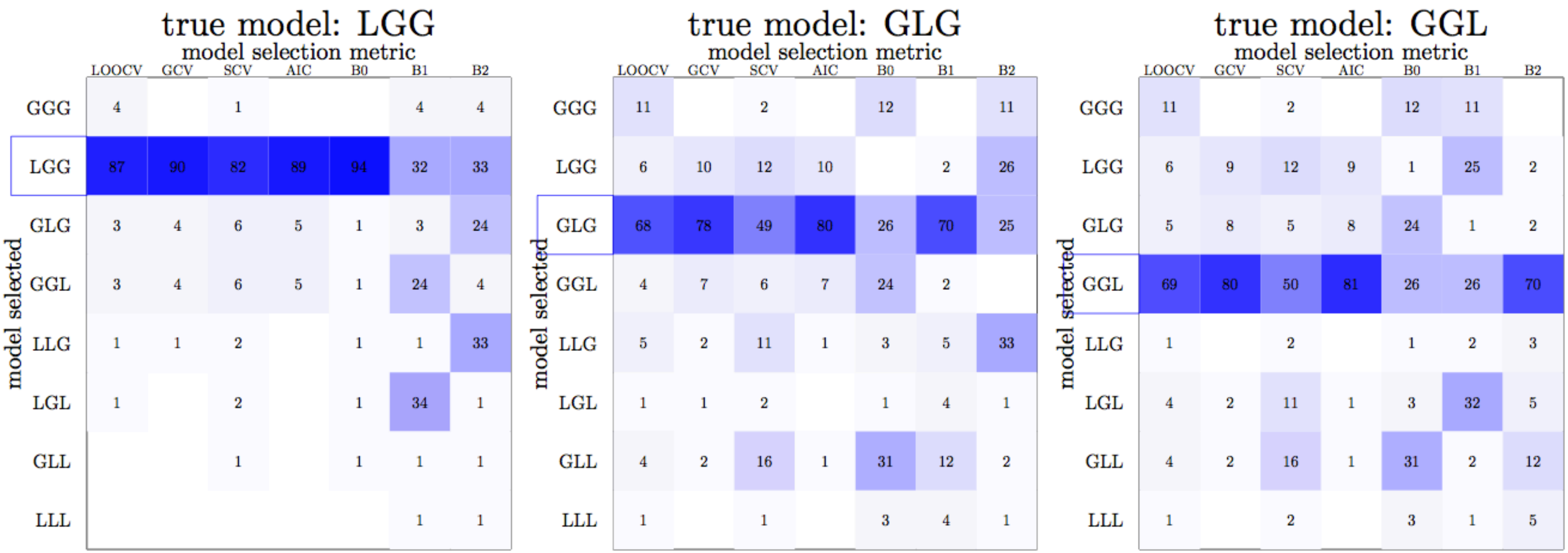
smallest —- bandwidth size —- biggest

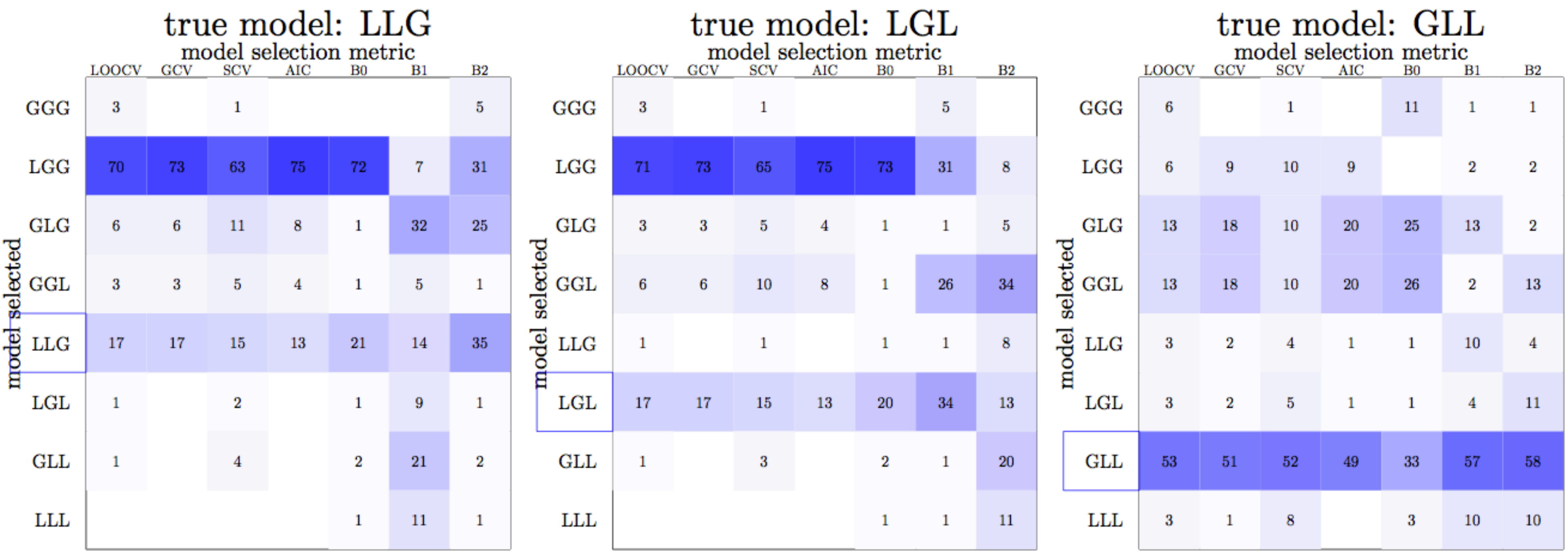
| model selected | | | | | | | |
|---|---|---|---|---|---|---|---|
| GGG | | | | | | | 7 |
| LGG | 9 | 4 | 3 | 2 | 2 | 2 | 2 |
| GLG | 10 | 4 | 2 | 1 | 2 | 2 | 2 |
| GGL | | | | | | | 3 |
| LLG | 12 | 5 | 4 | 4 | 3 | 3 | 2 |
| LGL | | | | | | | 2 |
| GLL | | | | | | | 4 |
| LLL | | | | | | | 3 |

Most accurate coefficient estimates:

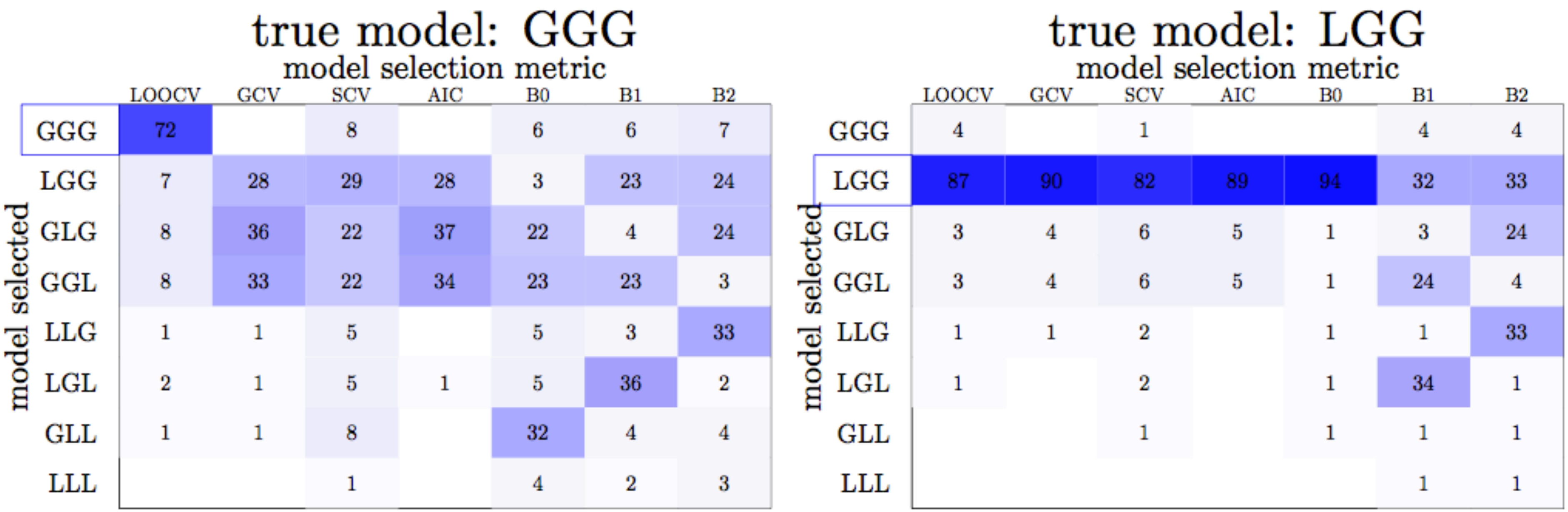
- Tend not to be the correct model
- Allow other coefficients to vary


true model: LGG
model selection metric
LOOCV
GCV
SCV
AIC
B0
B1
B2
model selected
GGG
LGG
GLG
GGL
LLG
LGL
GLL
LLL
4
1
4
4
87
90
82
89
94
32
33
3
4
6
5
1
3
24
3
4
6
5
1
24
4
1
1
2
1
1
33
1
2
1
34
1
1
1
1
1
1
1


true model: LGG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 4 1 4 4
LGG 87 90 82 89 94 32 33
GLG 3 4 6 5 1 3 24
GGL 3 4 6 5 1 24 4
LLG 1 1 2 1 1 33
LGL 1 2 1 34 1
GLL 1 1 1 1
LLL 1 1
true model: GLG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 11 2 12 11
LGG 6 10 12 10 2 26
GLG 68 78 49 80 26 70 25
GGL 4 7 6 7 24 2
LLG 5 2 11 1 3 5 33
LGL 1 1 2 1 4 1
GLL 4 2 16 1 31 12 2
LLL 1 1 3 4 1
true model: GGL
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 11 2 12 11
LGG 6 9 12 9 1 25 2
GLG 5 8 5 8 24 1 2
GGL 69 80 50 81 26 26 70
LLG 1 2 1 2 3
LGL 4 2 11 1 3 32 5
GLL 4 2 16 1 31 2 12
LLL 1 2 3 1 5



true model: LLG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 3 1 5
LGG 70 73 63 75 72 7 31
GLG 6 6 11 8 1 32 25
GGL 3 3 5 4 1 5 1
LLG 17 17 15 13 21 14 35
LGL 1 2 1 9 1
GLL 1 4 2 21 2
LLL 1 11 1
true model: LGL
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 3 1 5
LGG 71 73 65 75 73 31 8
GLG 3 3 5 4 1 1 5
GGL 6 6 10 8 1 26 34
LLG 1 1 1 1 8
LGL 17 17 15 13 20 34 13
GLL 1 3 2 1 20
LLL 1 1 11
true model: GLL
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 6 1 11 1 1
LGG 6 9 10 9 2 2
GLG 13 18 10 20 25 13 2
GGL 13 18 10 20 26 2 13
LLG 3 2 4 1 1 10 4
LGL 3 2 5 1 1 4 11
GLL 53 51 52 49 33 57 58
LLL 3 1 8 3 10 10


true model: GGG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 72 8 6 6 7
LGG 7 28 29 28 3 23 24
GLG 8 36 22 37 22 4 24
GGL 8 33 22 34 23 23 3
LLG 1 1 5 5 3 33
LGL 2 1 5 1 5 36 2
GLL 1 1 8 32 4 4
LLL 1 4 2 3
true model: LGG
model selection metric
LOOCV GCV SCV AIC B0 B1 B2
model selected
GGG 4 1 4 4
LGG 87 90 82 89 94 32 33
GLG 3 4 6 5 1 3 24
GGL 3 4 6 5 1 24 4
LLG 1 1 2 1 1 33
LGL 1 2 1 34 1
GLL 1 1 1 1
LLL 1 1

## true model: GLG

model selection metric

| model selected | LOOCV | GCV | SCV | AIC | B0 | B1 | B2 |
|---|---|---|---|---|---|---|---|
| GGG | 11 | | 2 | | 12 | | 11 |
| LGG | 6 | 10 | 12 | 10 | | 2 | 26 |
| GLG | 68 | 78 | 49 | 80 | 26 | 70 | 25 |
| GGL | 4 | 7 | 6 | 7 | 24 | 2 | |
| LLG | 5 | 2 | 11 | 1 | 3 | 5 | 33 |
| LGL | 1 | 1 | 2 | | 1 | 4 | 1 |
| GLL | 4 | 2 | 16 | 1 | 31 | 12 | 2 |
| LLL | 1 | | 1 | | 3 | 4 | 1 |

## true model: LLG

model selection metric

| model selected | LOOCV | GCV | SCV | AIC | B0 | B1 | B2 |
|---|---|---|---|---|---|---|---|
| GGG | 3 | | 1 | | | | 5 |
| LGG | 70 | 73 | 63 | 75 | 72 | 7 | 31 |
| GLG | 6 | 6 | 11 | 8 | 1 | 32 | 25 |
| GGL | 3 | 3 | 5 | 4 | 1 | 5 | 1 |
| LLG | 17 | 17 | 15 | 13 | 21 | 14 | 35 |
| LGL | 1 | | 2 | | 1 | 9 | 1 |
| GLL | 1 | | 4 | | 2 | 21 | 2 |
| LLL | | | | | 1 | 11 | 1 |

## true model: GGL

model selection metric

| model selected | LOOCV | GCV | SCV | AIC | B0 | B1 | B2 |
|---|---|---|---|---|---|---|---|
| GGG | 11 | | 2 | | 12 | 11 | |
| LGG | 6 | 9 | 12 | 9 | 1 | 25 | 2 |
| GLG | 5 | 8 | 5 | 8 | 24 | 1 | 2 |
| GGL | 69 | 80 | 50 | 81 | 26 | 26 | 70 |
| LLG | 1 | | 2 | | 1 | 2 | 3 |
| LGL | 4 | 2 | 11 | 1 | 3 | 32 | 5 |
| GLL | 4 | 2 | 16 | 1 | 31 | 2 | 12 |
| LLL | 1 | | 2 | | 3 | 1 | 5 |

## true model: LGL

model selection metric

| model selected | LOOCV | GCV | SCV | AIC | B0 | B1 | B2 |
|---|---|---|---|---|---|---|---|
| GGG | 3 | | 1 | | | 5 | |
| LGG | 71 | 73 | 65 | 75 | 73 | 31 | 8 |
| GLG | 3 | 3 | 5 | 4 | 1 | 1 | 5 |
| GGL | 6 | 6 | 10 | 8 | 1 | 26 | 34 |
| LLG | 1 | | 1 | | 1 | 1 | 8 |
| LGL | 17 | 17 | 15 | 13 | 20 | 34 | 13 |
| GLL | 1 | | 3 | | 2 | 1 | 20 |
| LLL | | | | | 1 | 1 | 11 |

## true model: GLL

model selection metric

| model selected | LOOCV | GCV | SCV | AIC | B0 | B1 | B2 |
|---|---|---|---|---|---|---|---|
| GGG | 6 | | 1 | | 11 | 1 | 1 |
| LGG | 6 | 9 | 10 | 9 | | 2 | 2 |
| GLG | 13 | 18 | 10 | 20 | 25 | 13 | 2 |
| GGL | 13 | 18 | 10 | 20 | 26 | 2 | 13 |
| LLG | 3 | 2 | 4 | 1 | 1 | 10 | 4 |
| LGL | 3 | 2 | 5 | 1 | 1 | 4 | 11 |
| GLL | 53 | 51 | 52 | 49 | 33 | 57 | 58 |
| LLL | 3 | 1 | 8 | | 3 | 10 | 10 |

## true model: LLL

model selection metric

| model selected | LOOCV | GCV | SCV | AIC | B0 | B1 | B2 |
|---|---|---|---|---|---|---|---|
| GGG | 2 | | | | 1 | 1 | |
| LGG | 63 | 65 | 55 | 69 | 61 | 8 | 8 |
| GLG | 5 | 5 | 7 | 7 | 2 | 16 | 6 |
| GGL | 4 | 5 | 7 | 6 | 2 | 5 | 17 |
| LLG | 8 | 8 | 8 | 5 | 11 | 14 | 10 |
| LGL | 8 | 8 | 8 | 6 | 11 | 9 | 14 |
| GLL | 1 | 1 | 8 | | 2 | 31 | 30 |
| LLL | 9 | 9 | 6 | 6 | 12 | 15 | 15 |

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Are coefficient estimates equally accurate across metrics?

What happens, ceteris paribus, with greater error variance?