Systems of equations

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Lets review the method you use in high-school to solve systems of equations - the elimination method - and then rewrite it under a new notation - the matrix and vector notation.

A few examples will be given.

Key concepts: *systems of equations in matrix-vector notation, elimination, pivot, rank, conditions for 1,0 or infinite solutions.*

0.1 Matrix-vector notation for a system

Consider the system with two equations, called l_1 and l_2:

$$\begin{cases} x - 4y = 2 \\ 2x - 6y = 5 \end{cases} \tag{1}$$

In Equation **??** we find two equations and two unknowns x and y. We want their values such that both equations are satisfied. (this system represents the interception of two lines)

Using the matrix-vector notation we can write Equation **??** as:

$$\begin{pmatrix} 1 & -4 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \tag{2}$$

Lets read Equation **??** in words: the 2 by 2 matrix of coefficients is multiplied by the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$, the result is $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. This is *one* equation with *one* unknown, the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$. Traditionally we write Equation **??** as $A\mathbf{x} = \mathbf{b}$.

How do we multiply a vector by a matrix?

Answer:

$$\overbrace{\begin{pmatrix} 1 & -4 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}^{\text{matrix-vector mult.}} = \overbrace{\begin{pmatrix} 1 \cdot x - 4 \cdot y \\ 2 \cdot x - 6 \cdot y \end{pmatrix}}^{\text{scalar mult.}}$$

Matrix times a vector on the lhs is just a super-compact way of writing the vector on the rhs. Moreover, notice the shapes of the matrix and vectors, this is very, very important. A 2 by 2 matrix times a 2 by 1 column vector yields a 2 by 1 column vector! If you understand this it should not be a problem to see what shapes are or not compatible, check this:

- $[2 \times 2][\ 3 \times 1] =$ Nonsense
- $[3 \times 2][\ 2 \times 1] = [3 \times 1]$
- $[2 \times 3][\ 3 \times 1] = [2 \times 1]$
- $[3 \times 2][\ 3 \times 1] =$ Nonsense
- $[1 \times 3][\ 3 \times 1] = [1 \times 1]$

0.2 Solving the system using the Elimination method

Lets solve the system Equation **??** using the traditional rules we already know from high-school, recall:

1. you can replace an equation by itself times some constant.
2. you can replace any one equation by the sum both equations.
3. you can isolate x or y in one equation and substitute in the other equation.

In other words, 1. and 2., just say this: you can replace l_1 or l_2 by some convenient combination $al_1 + bl_2$. Rule 3. is known as back-substitution.

Applying any one of these operations yields another and equivalent system of equations.

The central idea of the Elimination method is use linear combination of equations (1. and 2.) to eliminate variables and thus giving us an *equivalent* and *easier to solve* system. To eliminate variables we make clever use of rules 1 and 2. Once the system is simple enough we can use rule 3. How do you know what is or not a good combination? We'll see that with examples. But the guiding principle is to use the pivots.

This recaps what you know, now lets use these rules to solve the Equation **??** and in parallel see the corresponding matrix-vector version.

- **step 1:** Replace equation l_2 by, l_2 minus twice the equation l_1, i.e., make the new second equation l_2' into $l_2 - 2l_1$. This gives us:

$$\begin{cases} x - 4y = 2 \\ 2x - 6y = 5 \end{cases} \xrightarrow{l_2' = l_2 - 2l_1} \begin{cases} x - 4y = 2 \\ 2y = 1 \end{cases}$$

  Correspondingly we subtract from row l_2 twice the row l_1 in Equation **??**, giving us

$$\begin{pmatrix} 1 & -4 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \xrightarrow{l_2' = l_2 - 2l_1} \begin{pmatrix} 1 & -4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

  A good way to look at this is to focus on using the 1, to eliminate the 2. This entry of the matrix we focus on is called a pivot entry!

- **step 2:** Multiply equation l_2 by $1/2$:

$$\begin{cases} x - 4y = 2 \\ 2y = 1 \end{cases} \xrightarrow{l_2' = 1/2 l_2} \begin{cases} x - 4y = 2 \\ y = 1/2 \end{cases}$$

  In matrix-vector notation we find:

$$\begin{pmatrix} 1 & -4 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \xrightarrow{l_2' = 1/2 l_2} \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$$

- **step 3:** Focusing on the second pivot entry, we eliminate the entry, -4, by replacing l_1 by l_1 plus four times l_2:

$$\begin{cases} x - 4y = 2 \\ y = 1/2 \end{cases} \xrightarrow{l_1' = l_1 + 4l_2} \begin{cases} x = 4 \\ y = 1/2 \end{cases}$$

  In matrix-vector notation we find:

$$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix} \xrightarrow{l_1' = l_1 + 4l_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1/2 \end{pmatrix}$$

  From which we can read the final result $x = 4$ and $y = 1/2$.

Better notation: Going through the three steps again we notice we can improve our matrix-vector notation by suppressing from it the column $\begin{pmatrix} x \\ y \end{pmatrix}$ and writing instead the steps as:

$$\left(\begin{array}{cc|c} 1 & -4 & 2 \\ 2 & -6 & 5 \end{array}\right) \overset{l_2'=l_2-2l_1}{\longrightarrow} \left(\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 2 & 1 \end{array}\right) \overset{l_2'=1/2l_2}{\longrightarrow} \left(\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 1 & 1/2 \end{array}\right) \overset{l_1'=l_1+4l_2}{\longrightarrow} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1/2 \end{array}\right)$$

From now on we'll adopt this way of writing systems of equations, its called the extended matrix notation, because we appended a new column to right side of the 2×2 matrix. From the extended matrix we read the solution as follows

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1/2 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1/2 \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1/2 \end{pmatrix}$$

where in the last step we multiplied the vector by the matrix.

The vector

$$\begin{pmatrix} 4 \\ 1/2 \end{pmatrix}$$

is the solution of

$$\begin{pmatrix} 1 & -4 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Meaning, the solution of

$$\begin{cases} x - 4y = 2 \\ 2x - 6y = 5 \end{cases}$$

is:

$$\begin{cases} x = 4 \\ y = 1/2 \end{cases}$$